Matrix Project On Coordinate Geometry

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Index

Problem Statement

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$\mathbf{P}$ and $\mathbf{Q}$ are two distinct points on the parabola

$$\mathbf{x}^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} - \begin{bmatrix} 4 \\ 0 \end{bmatrix} \mathbf{x} = 0$$

with parameters t and t_1 respectively. If the normal at $\mathbf{P}$ passes through $\mathbf{Q}$, then find the minimum value of t_1^2.

Steps to solve the problem

Steps

- Find the constants of the parabola

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- Minimise as a function of parameter of point P

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- Coefficient of x = 4a
- a = -1

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- Differentiate the equation and substitute $\frac{dy}{dx} = \text{m}$ (the slope of the tangent)
- $\frac{d\mathbf{x}}{dx} = \begin{bmatrix} 1 & m \end{bmatrix}^T$

Solution Contd..

- $$\begin{bmatrix} 1 \\ m \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} + \mathbf{x}^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ m \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ m \end{bmatrix} = 0$$

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- Now, use the parametric form of $\mathbf{x}$ (i.e.)

$$\mathbf{x} = \begin{bmatrix} t^2 \\ -2t \end{bmatrix}$$

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- $$\begin{bmatrix} 1 & m \end{bmatrix} \begin{bmatrix} 0 \\ -2t \end{bmatrix} + \begin{bmatrix} t^2 & -2t \end{bmatrix} \begin{bmatrix} 0 \\ m \end{bmatrix} + 4 = 0$$

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- m $= \frac{1}{t}$

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- Slope of normal at point **p**: $m_N =$ -t
- We calculate the equation of the line through P with slope $= m_N$

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- $$\begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} t^2 \\ 2t \end{bmatrix}$$
- $$\begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = t^3 + 2t$$

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- $$t_1 = -t - \frac{2}{t}$$

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- $t_1^2 \geq 4 + 4 = 8$
- $\therefore$ The Minimum possible value of t_1^2 is 8.

The End