Matrix Project On Coordinate Geometry

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Problem Statement

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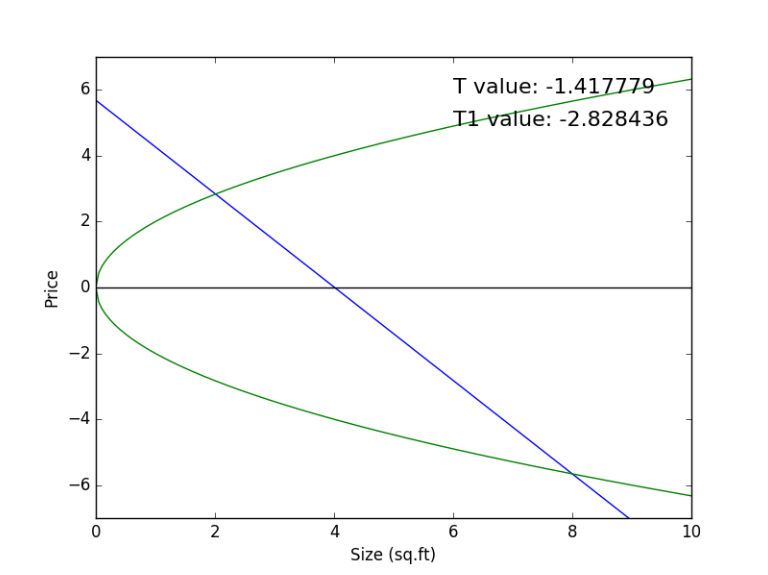
P and **Q** are two distinct points on the parabola

$$\mathbf{x}^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} - \begin{bmatrix} 4 \\ 0 \end{bmatrix} \mathbf{x} = 0$$

with parameters t and t_1 respectively. If the normal at **P** passes through **Q**, then find the minimum value of t_1^2.

To understand relation between t and t_1



Steps to solve the problem

Steps

- Find the constants of the parabola

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- Write eqn of Normal using slope and parametric Point
- Minimise as a function of parameter of point P

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- Differentiate the matrix equation the slope of the tangent as a direction vector
- Direction vector of the tangent

$$\begin{bmatrix} 1 & -m \end{bmatrix}^T$$

Solution Contd..

- $$\begin{bmatrix} 1 \\ -m \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} + \mathbf{x}^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -m \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ -m \end{bmatrix} = 0$$

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- Now, use the parametric form of $\mathbf{x}$ (i.e.)

$$\mathbf{x} = \begin{bmatrix} t^2 \\ 2t \end{bmatrix}$$

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- $$\begin{bmatrix} 1 \\ -m \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{t} \end{bmatrix}$$

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- Hence, the direction vector of the normal is :

$$\begin{bmatrix} t & 1 \end{bmatrix}$$

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- Solving this, we get:

$$t_1 = -t - \frac{2}{t}$$

Solution Contd..

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$$t_1^2 = t^2 + \frac{4}{t^2} + 4$$

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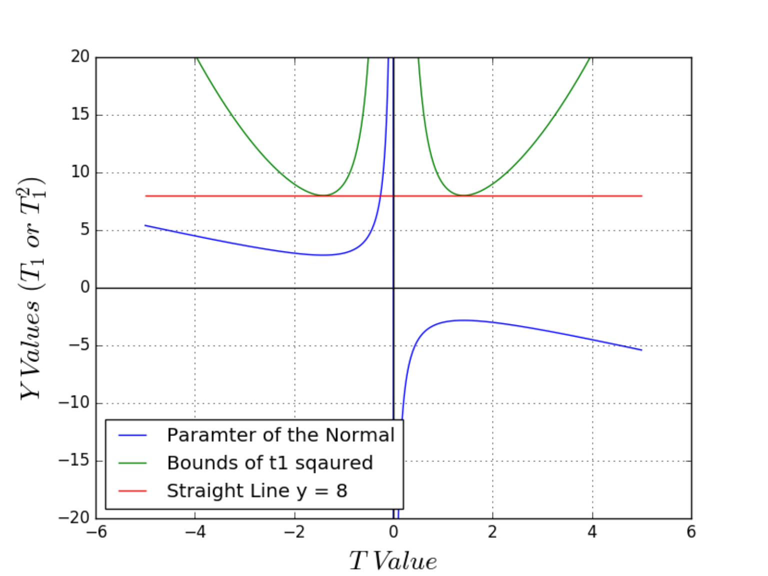
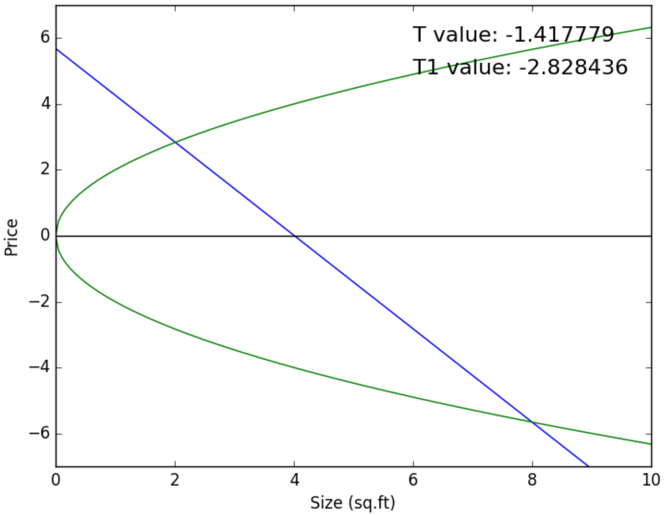
- $t_1^2 \geq 4 + 4 = 8$
- Note: $t_1 \neq 0$ because, in this case the normal will never intersect the parabola.
- $\therefore$ The Minimum possible value of t_1^2 is 8.

Figures

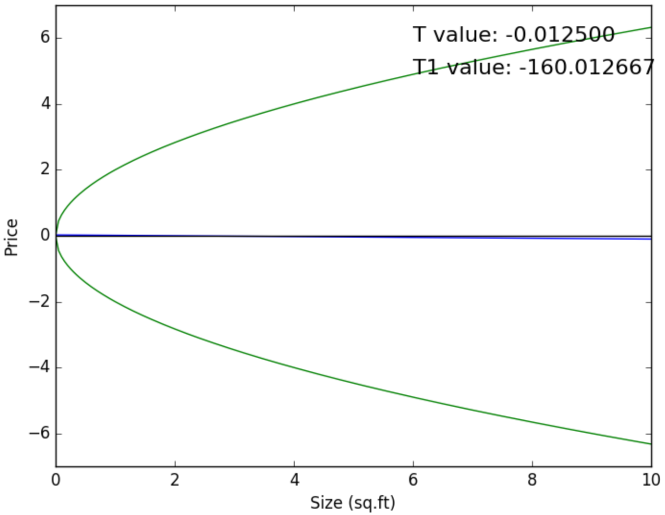
Bounds of t_1 and t_1^2



Normal at minimum t_1^2



$t_1^2 = 0$



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T1 value: -160.012667
Price
Size (sq.ft)
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4
2
0
−2
−4
−6
0
2
4
6
8
10

The End