Matrix Project On Coordinate Geometry

Sai Harsha Kottapalli and Abhishek Agarwal

February 14, 2019

Indian Institute of Technology Hyderabad

Index

Problem Statement

Steps to solve the problem

Solution

Figures

Problem Statement

Problem Statement

$\mathbf{P}$ and $\mathbf{Q}$ are two distinct points on the parabola

$$\mathbf{x}^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} - \begin{bmatrix} 4 \\ 0 \end{bmatrix} \mathbf{x} = 0$$

with parameters t and t_1 respectively. If the normal at $\mathbf{P}$ passes through $\mathbf{Q}$, then find the minimum value of t_1^2.

Steps to solve the problem

Steps

- Find the constants of the parabola

Steps

- Find the constants of the parabola
- Find Slope of tangent

Steps

- Find the constants of the parabola
- Find Slope of tangent
- Find slope of Normal

Steps

- Find the constants of the parabola
- Find Slope of tangent
- Find slope of Normal
- Write eqn of Normal using slope and parametric Point

Steps

- Find the constants of the parabola
- Find Slope of tangent
- Find slope of Normal
- Write eqn of Normal using slope and parametric Point
- Minimise as a function of parameter of point P

Solution

Solution

- Given equation:

$$\mathbf{x}^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} - \begin{bmatrix} 4 \\ 0 \end{bmatrix} \mathbf{x} = 0$$

Solution

- Given equation:

$$\mathbf{x}^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} - \begin{bmatrix} 4 \\ 0 \end{bmatrix} \mathbf{x} = 0$$

- Since, 3 entries of the 2x2 matrix is 0, the parabola is of the standard form (i.e.)

$$y^2 = 4ax$$

Solution

- Given equation:

$$\mathbf{x}^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} - \begin{bmatrix} 4 \\ 0 \end{bmatrix} \mathbf{x} = 0$$

- Since, 3 entries of the 2x2 matrix is 0, the parabola is of the standard form (i.e.)

$$y^2 = 4ax$$

- Coefficient of x = 4a

Solution

- Given equation:

$$\mathbf{x}^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} - \begin{bmatrix} 4 \\ 0 \end{bmatrix} \mathbf{x} = 0$$

- Since, 3 entries of the 2x2 matrix is 0, the parabola is of the standard form (i.e.)

$$y^2 = 4ax$$

- Coefficient of x = 4a
- a = 1

Solution Contd..

- Parametric form of points of parabola:

$$\begin{bmatrix} t^2 \\ 2t \end{bmatrix}$$

Solution Contd..

- Parametric form of points of parabola:

$$\begin{bmatrix} t^2 \\ 2t \end{bmatrix}$$

- Differentiate the equation and substitute $\frac{dy}{dx} = \text{m}$ (the slope of the tangent)

Solution Contd..

- Parametric form of points of parabola:

$$\begin{bmatrix} t^2 \\ 2t \end{bmatrix}$$

- Differentiate the equation and substitute $\frac{dy}{dx} = \mathrm{m}$ (the slope of the tangent)
- $\frac{d\mathbf{x}}{dx} = \begin{bmatrix} 1 & m \end{bmatrix}^T$

Solution Contd..

- $$\begin{bmatrix} 1 \\ m \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} + \mathbf{x}^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ m \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ m \end{bmatrix} = 0$$

Solution Contd..

- $$\begin{bmatrix} 1 \\ m \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} + \mathbf{x}^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ m \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ m \end{bmatrix} = 0$$
- Now, use the parametric form of $\mathbf{x}$ (i.e.)

$$\mathbf{x} = \begin{bmatrix} t^2 \\ 2t \end{bmatrix}$$

Solution Contd..

- $$\begin{bmatrix} 1 & m \end{bmatrix} \begin{bmatrix} 0 \\ 2t \end{bmatrix} + \begin{bmatrix} t^2 & 2t \end{bmatrix} \begin{bmatrix} 0 \\ m \end{bmatrix} - 4 = 0$$

Solution Contd..

- $$\begin{bmatrix} 1 & m \end{bmatrix} \begin{bmatrix} 0 \\ 2t \end{bmatrix} + \begin{bmatrix} t^2 & 2t \end{bmatrix} \begin{bmatrix} 0 \\ m \end{bmatrix} - 4 = 0$$
- 2mt + 2mt - 4 = 0

Solution Contd..

- $$\begin{bmatrix} 1 & m \end{bmatrix} \begin{bmatrix} 0 \\ 2t \end{bmatrix} + \begin{bmatrix} t^2 & 2t \end{bmatrix} \begin{bmatrix} 0 \\ m \end{bmatrix} - 4 = 0$$
- 2mt + 2mt - 4 = 0
- 4mt - 4 = 0

Solution Contd..

- $$\begin{bmatrix} 1 & m \end{bmatrix} \begin{bmatrix} 0 \\ 2t \end{bmatrix} + \begin{bmatrix} t^2 & 2t \end{bmatrix} \begin{bmatrix} 0 \\ m \end{bmatrix} - 4 = 0$$
- 2mt + 2mt - 4 = 0
- 4mt - 4 = 0
- m = $\frac{1}{t}$

Solution Contd..

- We know that, Two lines are perpendicular if the product of their slopes is -1.

Solution Contd..

- We know that, Two lines are perpendicular if the product of their slopes is -1.
- Slope of normal at point **p**: $m_N =$ -t

Solution Contd..

- We know that, Two lines are perpendicular if the product of their slopes is -1.
- Slope of normal at point **p**: $m_N =$ -t
- We calculate the equation of the line through P with slope $= m_N$

Solution Contd..

- Note: $\mathbf{p} = (x_1, y_1) = (t^2, 2t)$

Solution Contd..

- Note: $\mathbf{p} = (x_1, y_1) = (t^2, 2t)$
- $$\begin{bmatrix} -m & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -m & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

Solution Contd..

- Note: $\mathbf{p} = (x_1, y_1) = (t^2, 2t)$
- $$\begin{bmatrix} -m & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -m & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$
- $$\begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} t^2 \\ 2t \end{bmatrix}$$

Solution Contd..

- Note: $\mathbf{p} = (x_1, y_1) = (t^2, 2t)$
- $$\begin{bmatrix} -m & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -m & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$
- $$\begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} t^2 \\ 2t \end{bmatrix}$$
- $$\begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = t^3 + 2t$$

Solution Contd..

- Now, Let Q: $(t_1^2, 2t_1)$

Solution Contd..

- Now, Let Q: $(t_1^2, 2t_1)$
- Then, We have,

$$\begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} t_1^2 \\ 2t_1 \end{bmatrix} = t^3 + 2t$$

Solution Contd..

- Now, Let Q: $(t_1^2, 2t_1)$
- Then, We have,

$$\begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} t_1^2 \\ 2t_1 \end{bmatrix} = t^3 + 2t$$

- $$2t_1 + tt_1^2 = t^3 + 2t$$

Solution Contd..

- Now, Let Q: $(t_1^2, 2t_1)$
- Then, We have,

$$\begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} t_1^2 \\ 2t_1 \end{bmatrix} = t^3 + 2t$$

- $$2t_1 + tt_1^2 = t^3 + 2t$$

- $$-2(t - t_1) = t(t_1 + t)(t - t_1)$$

Solution Contd..

- Now, Let Q: $(t_1^2, 2t_1)$
- Then, We have,

$$\begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} t_1^2 \\ 2t_1 \end{bmatrix} = t^3 + 2t$$

- $$2t_1 + tt_1^2 = t^3 + 2t$$

- $$-2(t - t_1) = t(t_1 + t)(t - t_1)$$

- $$t_1 = -t - \frac{2}{t}$$

Solution Contd..

- Squaring the previous equation, we get,

$$t_1^2 = t^2 + \frac{4}{t^2} + 4$$

- Note: $t_1 \neq 0$ because, in this case the normal will never intersect the parabola.

Solution Contd..

- Squaring the previous equation, we get,

$$t_1^2 = t^2 + \frac{4}{t^2} + 4$$

- To solve above equation we use the property: Arithmetic Mean $\geq$ Geometric Mean on $t^2 + \frac{4}{t^2}$

- Note: $t_1 \neq 0$ because, in this case the normal will never intersect the parabola.

Solution Contd..

- Squaring the previous equation, we get,

$$t_1^2 = t^2 + \frac{4}{t^2} + 4$$

- To solve above equation we use the property: Arithmetic Mean $\geq$ Geometric Mean on $t^2 + \frac{4}{t^2}$
- A.M $\geq$ G.M. $\implies t^2 + \frac{4}{t^2} \geq 2\sqrt{4} = 4$

- Note: $t_1 \neq 0$ because, in this case the normal will never intersect the parabola.

Solution Contd..

- Squaring the previous equation, we get,

$$t_1^2 = t^2 + \frac{4}{t^2} + 4$$

- To solve above equation we use the property:
  Arithmetic Mean $\geq$ Geometric Mean on $t^2 + \frac{4}{t^2}$
- A.M $\geq$ G.M. $\implies t^2 + \frac{4}{t^2} \geq 2\sqrt{4} = 4$
- $t_1^2 \geq 4 + 4 = 8$
- Note: $t_1 \neq 0$ because, in this case the normal will never intersect the parabola.

Solution Contd..

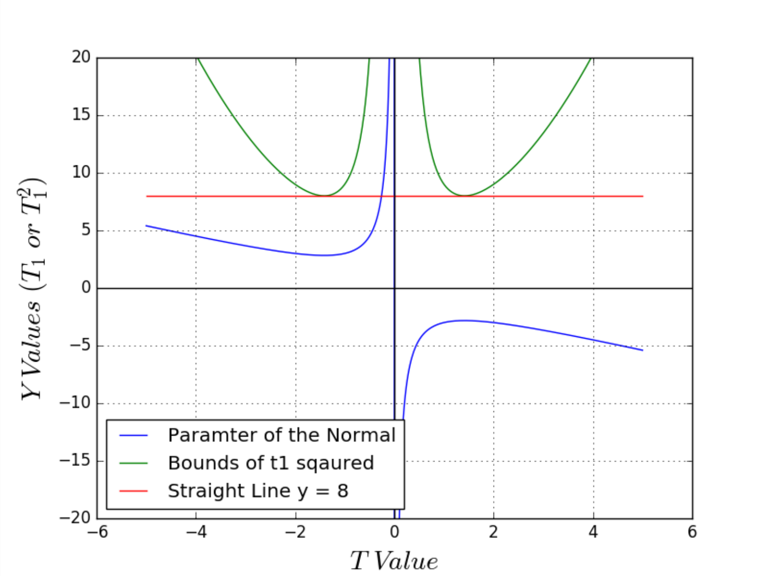
- Squaring the previous equation, we get,

$$t_1^2 = t^2 + \frac{4}{t^2} + 4$$

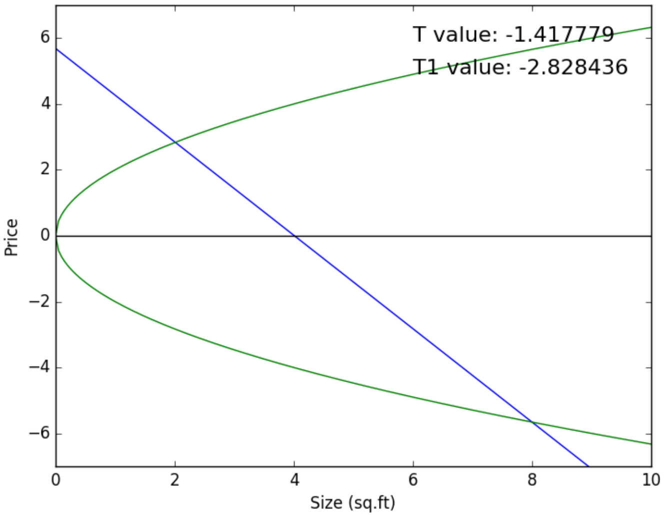
- To solve above equation we use the property:
  Arithmetic Mean $\geq$ Geometric Mean on $t^2 + \frac{4}{t^2}$
- A.M $\geq$ G.M. $\implies t^2 + \frac{4}{t^2} \geq 2\sqrt{4} = 4$
- $t_1^2 \geq 4 + 4 = 8$
- Note: $t_1 \neq 0$ because, in this case the normal will never intersect the parabola.
- $\therefore$ The Minimum possible value of t_1^2 is 8.

Figures

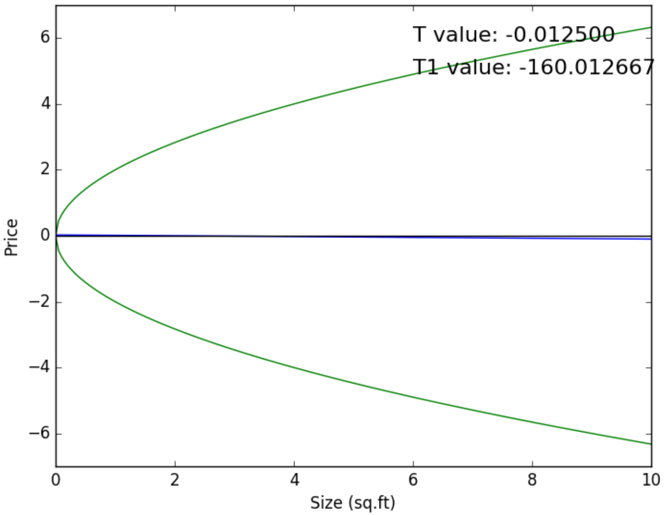
Bounds of t_1 and t_1^2



Normal at minimum t_1^2



$t_1^2 = 0$


T value: -0.012500
T1 value: -160.012667
Price
Size (sq.ft)
6
4
2
0
−2
−4
−6
0
2
4
6
8
10

The End