# Assignment

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## Theorems and a summary of notes learned from course

- Any two bases should have same cardinality

    We prove this by expressing each vector in one basis in terms of the vectors in the other basis, and using this result to show the cardinalities must be equal, by also using the fact that the "other" basis must span and be linearly independent.

- Any vector should contain orthonormal basis
- Unique-Representation Lemma
- Rank nullity theorem
- The total number of independent rows in A and $A^T$ are the same.
- In a given matrix, the total number of independent rows is equal to the total number of independent columns
- Optimum should be a vertex in a linear programming problem.
- If all the rows and columns are independent then inverse exists.
- Separating hyperplane theorem
- Point is vertex iff total number of columns in a tight constraint matrix
- A point is a vertex if and only if rank(matrix) equals the number of columns in the tight constraint matrix
- Approximation algorithm for vertex cover problem
- The rows of the matrix are orthogonal to the vectors in the null space of the matrix.

### Graphs

- Matching: A matching in a graph is a set of edges, no two of which share an end-point.
- the cycles will have even length
- Paths are characterised by which matching the rst and last edges lie in.
- An iterative algorithm to find the maximum cardinality matching in given graph G(modified BFS).

- The discussed algorithm is of $O(m * n^2)$. It can be improved to $O(n^{0..5} * m)$
- An Integer Linear Program is a linear program where the variables are constrained to take integer values only
- A template for designing combinatorial algorithms is discussed.
- ILP algorithm for minimum spanning trees is discussed.

**Definition 1** A path in G will be called alternating (w.r.t. M) if its edges alternate between those in M and those not in M.

**Definition 2** A vertex in G will be called unmatched or free if there are no edges from M incident on it.

 **Definition 3** An augmenting path is an alternating path that starts from and ends on free (unmatched) vertices.

**Theorem 1** A matching M is maximum if and only if the resulting graph does not contain any augmenting path.

**Lemma** The vertex ui will appear in the i th level of the BFS tree. And vertices uj ; j > i will not.

**Simplex Theorem** If the cost does not increase along any of the columns of $-(A^T)^{-1}$ then $x_0$ is optimal.

## Simplex Algorithm

The simplex algorithm is the classical method to solve the optimization problem of linear programming. We used it to traverse from one extreme point of a polyhedral to another extreme point such that the cost function is maximized. The LP has two cases:

**Non-degenerate case**

- The number of tight equations of any vertex is not greater than the dimension.

**Algorithms** ✄

### Initial feasible point

We know that $Ax \leq B$ and $\forall k \, x_k \geq 0$.

- In B, $\exists \, B_i$ such that $B_i < 0$

Model a new linear programming problem such that:

- // Let n be dimension.
- Introduce variable j into the LP by adding j to all constraints.
- The dimension now becomes   n + 1.
- Let $B_m$ be the minimum value $\forall$ $B_i \in B$.
- Introduce the constraint $j \geq B_m$.
- A feasible point for the constructed LP is $(0, 0, 0, ..., B_m)$.

Solution to the above simplified LP gives us the initial feasible point required

- In B, $\forall$ $B_i$ such that $B_i \geq 0$

We can take origin as the required initial feasible point as origin acts here is a tight point for those with $x_i \geq 0$. Together with the given condition $\forall$ $B_i : B_i \geq 0$, all the given constraints are satisfied.

**Optimal vertex**

Repeat until we find the optimal point:

- From the initial(current) feasible point, find any direction vector such that travelling along it can convert some tight rows to loose and some loose rows to tight where any n of them stay tight.
- We can hence represent **c** as a linear combination of those n rows which are tight as all of these rows are linearly independent.
- Let the corresponding coefficients of the above step be $(a_1, a_2, , ..., a_n)$. Check the following conditions:
    - If $\exists$ $a_i < 0$:
        - From the matrix formed by the n tight rows, find the inverse.
        - Choose j such that $a_j < 0$. The direction we travel now is the negative of $j^{th}$ column of the inverse matrix obtained.

- This direction is orthogonal to rest n - 1 planes and travelling along it will maximize the cost function as this had $a_i < 0$ previously.
  - Else:
    - the cost function can't be increased anymore as it would go out of bounds of the polyhedral. Hence, the current point is the optimal point.
    - So, exit

## Degenerate case

- The number of tight equations of some vertex is greater than the dimension.

## Algorithm

### Initial feasible point

Repeat until we find the point:

- Choose a infinitesimally small positive value and add it to the RHS of all the constraint equations.
- If the modified LP still has more than n tight constraint then:
  - Repeat the current algorithm.
- Else:
  - Find a solution to the modified LP (which has only n tight constraints) using the above discussed algo.
  - Use the n planes corresponding to these constraints from the unmodified LP to find intersection of these n planes to obtain a solution.
  - If solution is feasible:
    - We found the required solution. So, exit
  - Else:
    - Repeat the current algorithm.

# Duality

- Original problem of LLP is called primal.
- Dual of dual gives us primal.
- The values of the optimal solutions to the primal and dual are always equal.

- For finding the dual, signs of all constraints in primal must be(if not converted to):

    $\leq$ , if the goal is maximization

    $\geq$ , if the goal is minimization

- The goal of the original problem is flipped that is:

| Primal | | Dual |
|---|---|---|
| Maximization | -> | Minimization |
| Minimization | -> | Maximization |

- If primal has j number of variables, dual has j number of constraints.
- If dual has i number of constraints, primal has j number of variables.
- If $x_j$ is unrestricted in sign then the corresponding jth constraint will be of = type
- If ith constraint is of = type then the corresponding ith variable will be unrestricted in sign

**Primal is equal to dual**

*dual is feasible*

We have to show that there exists y such that $A^Ty = c$, that is the cost function can be represented as a linear combination of rows of A where coefficients are non-negative.

Rows of A are also the outward normals to the hyperplanes in $Ax \leq b$

We know that at an optimal point, the cost vector can be written as a linear combination of the normals to the corresponding hyperplane.

As mentioned above, these coefficients of non-negative linear combination yield a feasible point in the dual. Hence, for each point in primal we can find a feasible point in dual.

Therefore, dual is feasible as the set of points in dual has at least the optimal point.

*cost of optimums of both primal and dual are equal*

If $x_0$ is the optimal point in primal. Let y be a feasible point in dual.

The cost of $y$ in the dual t is $y^T b \geq y^T A x_0 = c^T x_0$. so it has a bound, so dual has finite optimum. The dual point associated with the optimal point in primal, is indeed feasible while also having the same cost. Hence, this point has to be the optimal for dual too.

**Additional Notes and theorems in references shared:**

**Theorem (Weak duality)** If $x1, \ldots, xn$ and $y1, \ldots, ym$ are feasible solutions for the primal and dual, respectively, and the primal problem is a maximization problem, then $c1x1 + \ldots + cnxn \leq b1y1 + \ldots + bmym$.

> Basically, any feasible solution to the dual corresponds to an upper bound on any solution to the primal by definition.

> In fact, equality occurs at the optimal solution(Strong duality).

**Theorem (Strong duality)** If $x1, \ldots, xn$ and $y1, \ldots, ym$ are optimal solutions for the primal and dual, respectively, then $c1x1 + \ldots + cnxn = b1y1 + \ldots + bmym$. Moreover, if either the primal or the dual has an optimal solution, then so does the other.

> Here, we assume an optimal solution exists.

> Else, if primal(dual) is unbounded then the dual (primal) has no feasible solution. (if not dual of dual would put a bound on primal which is contradiction)

- **Theorem** If the primal is feasible and the cost is bounded, then the dual is feasible and its cost is also bounded. Moreover, their optimum values coincide.
- **Theorem** Consider an $x0$ and $y0$, feasible in the primal and dual respectively. Then both are optimum i
  $c^T x_0 = y_0^T b$.

**Theorem (Complementary slackness)** Suppose that $x1, \ldots, xn$ and $y1, \ldots, ym$ are feasible solutions for the primal and dual, respectively, and let $w1, \ldots, wm$ and $z1, \ldots, zn$ be the corresponding slacks for the primal and dual, respectively. Then $x1, \ldots, xn$ and $y1, \ldots, ym$ are both optimal if and only if the following conditions hold:

1. for all i, $wiyi = 0$, and 2. for all j, $zjxj = 0$