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A minimal model for studying properties of the mode-coupling type instability in friction induced oscillations

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Abstract

A minimal two degree of freedom model is used to clarify from an intuitive perspective the physical mechanisms underlying the mode-coupling instability of self-excited friction induced oscillations. It is shown that simultaneous out-of-phase oscillations of friction force and displacement tangential to the friction force may lead to energy transfer from the frictional system to vibrational energy. Also it is shown that the friction force acts like a cross-coupling force linking motion normal to the contact surface to motion parallel to it and that a necessary condition for the onset of instability is that these friction-induced cross-coupling forces balance the corresponding structural cross-coupling forces of the system. Finally the origin and the role of phase shifts between oscillations normal and parallel to the contact surface is clarified with respect to the mode-coupling instability. It may be expected that the intuitive picture gained will be of considerable help for practical design purposes.

1. Introduction

Self-excited friction induced oscillations occur in a broad variety of engineering applications and are often the root cause of strong noise. Prominent examples are e.g. squealing railway wheels when the train is passing along a narrow curve, the squeaking of door hinges or the multitude of qualitatively different noise phenomena related to automotive brakes during braking action. A common aspect of all the phenomena under consideration is the instability of the steady sliding state against small perturbations, i.e. a linear instability in the sense of dynamical systems. Essentially four different instability mechanisms have been described until now: The most obvious possibility is of course the appearance of negative damping in the equations of motion rendering single structural modes of the system unstable. The second type of instability will in the following be called mode-coupling type (among the other names in literature are non-conservative

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displacement dependent forces and binary flutter). For this instability it is characteristic that the oscillation frequencies of two structural modes of an undamped system come—as a function of a control parameter—closer and closer together until they merge and a pair of an unstable and a stable mode results. As a third type sprag-slip (Spurr, 1961) has been identified as a kinematic instability and finally instability has been attributed to the fact that friction forces are follower forces. For further details review articles are available (Ibrahim, 1994; Wallaschek et al., 1999; Gaul and Nitsche, 2001). Some considerations have also already been given to the non-linearities finally leading to a saturation of the mode amplitudes growing exponentially in time due to the underlying instability (Allgaier et al., 1999). Also stick-slip and non-linear stiffness characteristics have been investigated already in detail (see e.g. Popp and Stelter, 1990). Summing up this short review of literature it may be concluded that quite a large amount of formal mathematical knowledge has already been gained on the subject of friction induced self-excited oscillations. Nevertheless a view on the related technical objects still shows that from a practical point of view avoiding friction induced noise seems still an unsolved design problem. One of the reasons for this clearly is the lack of an intuitively accessible picture for most of the instability mechanisms cited above. Although it certainly is clear from a mathematical point of view how to determine whether a technical system under consideration is stable or unstable, for the designer a physically intuitive picture is necessary to efficiently optimize his product. The goal of the present paper therefore is the investigation of the mode-coupling type instability—which has been chosen due to its major technical relevance—with respect to its underlying physical and intuitively accessible dynamics.

2. The model problem

To capture and investigate all the essential properties of the mode-coupling instability a minimal single mass two degree of freedom model possessing exactly two modes of vibration can be formulated as in Fig. 1 (cf. also to Hamabe et al. (1999)). A conveyor belt with constant velocity v_B is pushed with a constant normal force F_N against a block modelled as a point mass m. The block is held in position by two linear springs k_1 and k_2 and there is a linear spring k_3 which may be taken as a model for the normal contact stiffness between the block and the moving belt. To take into account sliding friction a Coulomb-type friction force F_F with a constant friction coefficient μ, is assumed.

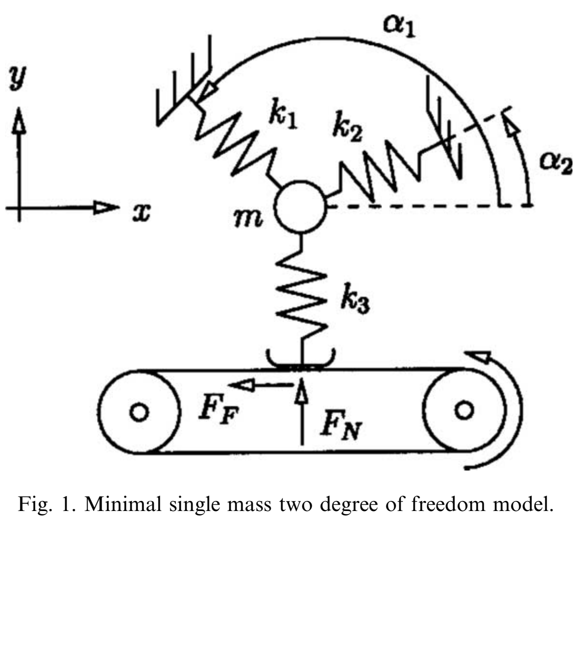

Fig. 1. Minimal single mass two degree of freedom model.

In a sense this model constitutes a two degree of freedom generalization of the one degree of freedom model well known from general studies on stick-slip. The main difference however is that in the present model displacements normal to the friction surface are possible. The equations of motion for the present model read:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F_{\mathrm{F}} \\ F_{\mathrm{N}} \end{pmatrix}, \tag{1}$$

where the coefficients of the stiffness matrix can be obtained from elementary considerations as

$$k_{11} = k_1 \cos^2 \alpha_1 + k_2 \cos^2 \alpha_2,$$

$$k_{12} = k_{21} = k_1 \sin \alpha_1 \cos \alpha_1 + k_2 \sin \alpha_2 \cos \alpha_2,$$

$$k_{22} = k_1 \sin^2 \alpha_1 + k_2 \sin^2 \alpha_2 + k_3.$$

In the following only small perturbations around the steady sliding state will be considered. Assuming that the mass of the conveyor belt system is much larger than the mass m of the block, it is assumed that the y-position of the belt does not show any significant change due to its inertia such that the friction force can well be approximated in terms of the contact spring's deformation state as $F_{\mathrm{F}} = \mu k_3 y$. The resulting system of equations may therefore now be written as a homogeneous system with non-symmetric stiffness matrix, which shows clearly that an eigenvalue problem has resulted:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} + \begin{bmatrix} k_{11} & k_{12} - \mu k_3 \\ k_{21} & k_{22} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0. \tag{2}$$

For simplifying notation in the following x and y—which now denote the deviation from the equilibrium steady sliding state—will also be called in-plane and out-of-plane displacements respectively.

3. A review on the generation of vibrational energy

A fundamental question with respect to the instability of a mechanical system addresses the source of the energy that allows the amplitude of an unstable mode to grow exponentially in time. Formally this question is of course answered by hinting at the non-conservativity of the frictional force which is expressed in the non-symmetry of the system's stiffness matrix. Although this answer is of course mathematically correct, it does not reveal the physics behind the instability, which is the question that will be addressed in the following. In our model the only non-conservative force is the in-plane frictional force. The work that this frictional force F_{F} performs during one cyclic displacement of the structural system is simply $W = \oint F_{\mathrm{F}}\, \mathrm{d}x$. From this two conditions for a non-zero energy gain of the structural vibration from frictional forces can directly be deduced: first, there have to be non-zero cyclic changes in both frictional force and corresponding in-plane displacement. Second, if those displacements are harmonic, a phase shift between the in-plane displacement and the corresponding friction force is necessary to obtain a non-vanishing energy transfer. Since in the minimal model considered out-of-plane displacements lead to changes in the frictional force via the change of the corresponding normal force these conditions translate into demanding that for a cyclic increase in vibrational energy simultaneous in- and out-of-plane oscillations which are out of phase are necessary. This sets the background for the organization of the following discussions which will establish the correspondence of this simple picture of out-of-phase in- and out-of-plane oscillations with the formal procedure of complex modal analysis: After a brief review of the general phenomenon of mode-coupling instability in terms of complex eigenvalue analysis an explanation for the instability will be given in terms of a balance of forces. Also the relation to the modal picture will be established and finally an investigation of the coupling structure will reveal the essential properties of the system's energy budget.

4. Complex eigenvalue analysis and cross-coupling force balance

Since the objective of the present work is not to investigate a specific system with respect to the parameter dependence of the stability characteristics (cf. e.g. Hamabe et al., 1999) but to reveal intuitively accessible reasons for the type of instability considered, the analysis is restricted to the special case of

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} + \begin{bmatrix} 2 & 1-\Delta \\ 1 & 2 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0, \tag{3}$$

which is e.g. obtained for $m = l$, $\alpha_1 = 150°$, $\alpha_2 = 30°$, $k_1 = \frac{2}{3}(2-\sqrt{3}) \approx 0.18$, $k_2 = \frac{2}{3}(2+\sqrt{3}) \approx 2.49$ and $k_3 = \frac{4}{3} \approx 1.33$ with $\Delta = \mu k_3$ as an abbreviation for the contribution of the friction in the stiffness matrix. As will be seen in the following, this case exhibits all characteristic dynamic properties to be investigated. Performing an eigenvalue analysis with the ansatz

$$\begin{pmatrix} x \\ y \end{pmatrix} = \exp(st), \tag{4}$$

where $s = \lambda + i\omega$ is the generally complex eigenvalue and x_0, y_0 are the generally complex components of the eigenvector, leads to

$$s_{1,2} = \pm\left[-2 \pm \sqrt{1-\Delta}\right]^{\frac{1}{2}} \tag{5}$$

as solutions of the characteristic equation. Fig. 2 shows the resulting real and imaginary part of s (growth rate and oscillation frequency) as a function of Δ. For $\Delta < 1$ there are two normal undamped modes with different frequencies. When Δ approaches 1 the frequencies coalesce and for $\Delta > 1$ a pair of an unstable and a stable mode results.

This is the well known picture for the mode-coupling type instability. Besides this formal result our simple model however also allows to gain further insight into the origin of the instability. For this purpose the equations may be written down with the inertial forces on the left-hand-side and the stiffness-forces on the right-hand-side respectively:

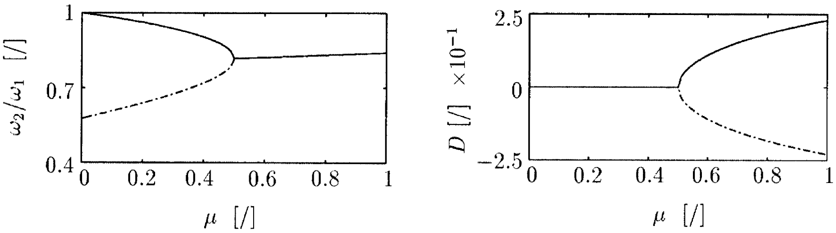

Fig. 2. Mode-coupling phenomenon in terms of the merging of natural frequencies and the appearance of non-zero growth rates for $\Delta > 1$.

$$\begin{aligned}\ddot{x} &= -2x - (1-\Delta)y,\\ \ddot{y} &= -x - 2y.\end{aligned} \tag{6}$$

The eigenvalue analysis has shown that instability sets in at $\Delta = 1$. A look at the equation reveals that this is exactly the configuration when the off-diagonal term $(1-\Delta)$ in the in-plane equation vanishes. An explanation in terms of forces is now easily at hand: Of course there are structural coupling terms between the in- and out-of-plane equations in the system of equations considered. But next to these structural coupling terms also the frictional force acts like a coupling; in contrast to the structural coupling however, which is symmetric between in- and out-of-plane motion, the frictional coupling acts on the in-plane equation only: out-of-plane displacements lead to in-plane forces, but not vice versa. This leads to the peculiar possibility, that when the frictional coupling term is increased, at some value ($\Delta = 1$) an exact balance between the structural and the frictional cross-coupling forces may result in the in-plane equation. In effect for $\Delta = 1$ the in-plane motion is uncoupled completely from the out-of-plane motion:

$$\begin{aligned}\ddot{x} &= -2x,\\ \ddot{y} &= -x - 2y.\end{aligned} \tag{7}$$

A look at the model geometry illustrates this behavior: when the mass is pushed towards the friction interface in general in-plane forces do result due to the springs k_1 and k_2. However also the normal force is increased by pushing the mass towards the friction interface, which in turn increases the in-plane frictional force. If these two in-plane forces just balance, the in-plane motion is in fact unaffected by out-of-plane displacements. The consequences for the dynamics of the system are at hand. Eq. (7) has two fundamental solutions:

$$\begin{aligned}x_1(t) &= x_{10}\cos(\sqrt{2}t), \quad x_2(t) = 0,\\ y_1(t) &= y_{10}t\sin(\sqrt{2}t), \quad y_2(t) = y_{20}\cos(\sqrt{2}t).\end{aligned} \tag{8}$$

The first one corresponds to an in-plane vibration with constant amplitude and an out-of-plane vibration with an amplitude increasing linearly with time. This is of course a plausible result since the x-equation is completely uncoupled from the y-equation which allows an undisturbed in-plane oscillation which then in turn acts like an external forcing to the out-of-plane motion at resonance. Here instability makes its first appearance, although in a comparatively weak linear growth. The second fundamental solution complements the picture in that it corresponds to the motion with vanishing in-plane contribution.

Further insight can be gained by plotting the right-hand-side of Eq. (6) as a restoring vector-field for subcritical, marginally critical and supercritical configurations ($\Delta < 1$, $\Delta = 1$ and $\Delta > 1$ respectively), see Fig. 3. It becomes clear that for subcritical configurations always two directions of displacement may be found for which the restoring forces are inversely proportional to the direction of displacement leading to linear motions in the x/y-plane. Of course these directions correspond to the two distinct real normal modes of the system. When friction becomes substantial (i.e. $\Delta \to 1$) the vector-field is increasingly distorted and these two directions come closer and closer together until they finally merge at the marginal point $\Delta = 1$. In fact here the eigenspace has degenerated, which corresponds closely to the pair of fundamental solutions given in Eq. (8), where only the second solution still corresponds to what is usually considered a mode. In the unstable regime ($\Delta > 1$) the distortion of the vector-field has reached such a strength that no directions of displacement have been left over where the restoring force would bring the mass back to the origin along the original displacement. Instead the resulting motion will exhibit a spiraling behavior corresponding to the formal appearance of complex modes.

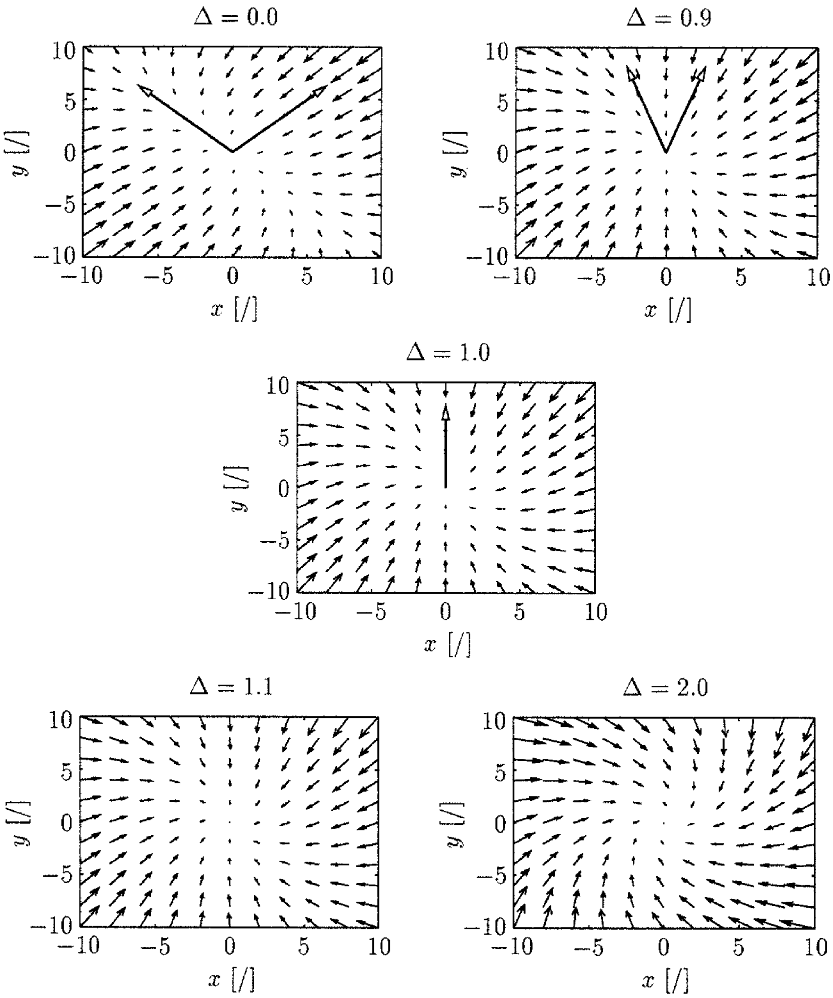

Fig. 3. Force vectorfields of the system for several values of Δ.

5. Coupling aspects: from beating to instability

The modal and force considerations given above have already shown fundamental properties of the type of instability considered. Especially it has become clear that coupling effects in the equations of motion play a crucial role. To further clarify the phenomena taking place some simple numerical integrations of Eq. (6) have been performed, see Fig. 4. When Δ is slightly below 1 there exist two normal modes with closely neighboring frequencies. Starting a time integration with general initial conditions therefore leads to a beating phenomenon. It should be mentioned briefly, that in contrast to the well known beating from conservatively coupled two-dimensional oscillator systems the total energy is not conserved here, due to the non-conservativity of the friction force: The motion consists of a periodical exchange of energy between in- and out-of-plane motion and the frictional system with the beating-frequency being half the frequency difference of the normal modes. The most important point in the observation of this beating phenomenon however is the appearance of phase shifts between in- and out-of-plane motion. Since energy has to be

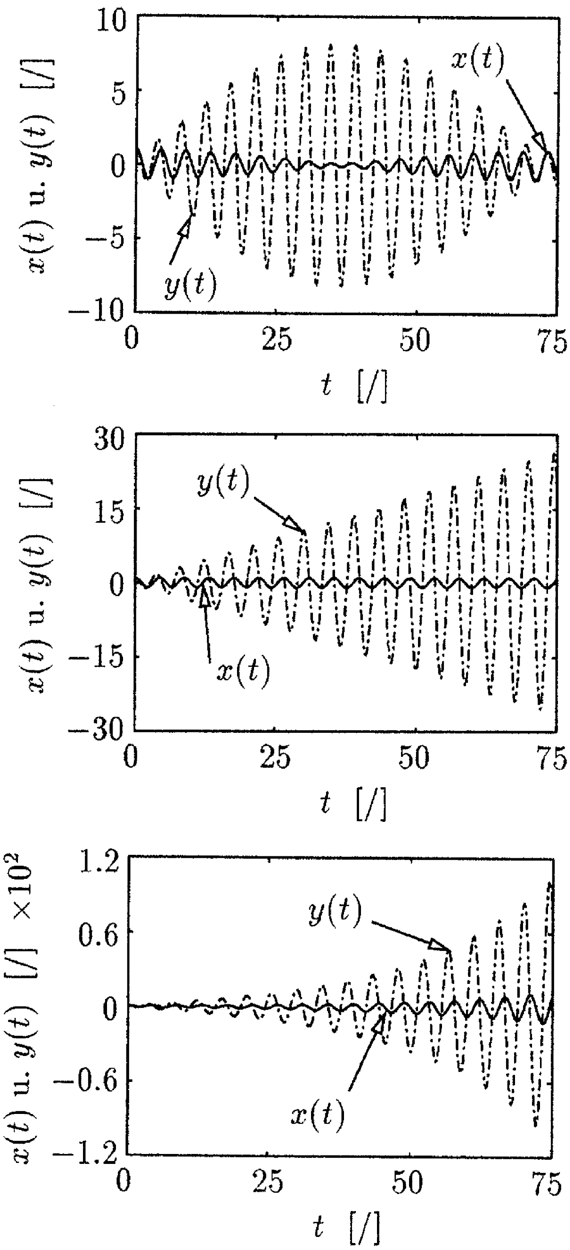

Fig. 4. Time series of the system for $\Delta = 0.9$ (top), $\Delta = 1$ (middle) and $\Delta = 1.1$ (bottom) showing the transition from beating to instability.

transferred back and forth between in- and out-of-plane motion during one beating cycle, these phase shifts are necessary from an energetic point of view: in fact the time-series in Fig. 4 shows nicely how the phase shift adjusts over time to allow for the corresponding energy transfer: when energy has to be transferred from in-plane to out-of-plane motion the in-plane oscillation lags behind the out-of-plane oscillation (due to the sign of the coupling term in Eq. (6); when the energy flow reverses, also the phase relation reverses, i.e. the out-of-plane oscillation lags behind the in-plane oscillation (cf. e.g. to Rossing et al., 1995 for an elementary discussion on this fundamental property of any beating oscillation).

Now when Δ gets closer to 1 the beating frequency decreases continuously and for $\Delta = 1$ a solution is reached (already written down analytically in Eq. (8)) that can be understood as the first moments of a beating being stretched out to infinity. Again a look at the equations of motions explains the behavior easily: when $\Delta = 1$ the cross-coupling term in the x-direction is zero, whereas the cross-coupling term in the y-direction has stayed unchanged. In a sense this means that for the y-equation a coupling still exists, while for the x-equation the coupling has vanished. Therefore the x-vibration is continuously feeding energy into the y-vibration, but the y-vibration cannot return energy into the x-vibration any more. When Δ exceeds 1 the system is unstable, both vibration components increase exponentially in time. Also this aspect can be understood from phase considerations: For $\Delta = 1$ the in-plane vibration excites the out-of-plane vibration, which is reflected in the in-plane oscillation lagging behind the out-of-plane oscillation; this is the general prerequisite for excitation, which can easily be explained by remembering the properties of the single degree of freedom harmonic oscillator and taking the signs of the present system of equations into account. When Δ slightly exceeds 1 the cross-coupling term in the x-equation also acts like an external forcing: the y-vibration starts forcing the x-vibration. The question now however is, whether this forcing adds energy to the x-vibration or takes energy out of it. We have already seen that in the case $\Delta < 1$ it takes energy out. But now note, that the sign of the cross-coupling-term in the x-equation has changed, which is equivalent to a phase shift of 180°. With this in mind have another look at the time series for $\Delta = 1$ in Fig. 4: the x-oscillation is lagging behind the linearly growing y-oscillation. If Δ is increased slightly beyond 1, due to the sign change of the coupling terms, for phase considerations not the y-oscillation itself has to be considered, but the negative y-oscillation; this negative y-oscillation on the other hand is then not ahead of the x-oscillation but behind it, which is just the right phase relationship to yield excitation with positive energy generation. In summary, although the arguments given above are without a doubt somewhat intricate, it has become clear that the destabilization of the system can be explained and understood both physically and intuitively by considering the interaction phenomena shown.

Finally it should be mentioned that the instability described is largely analogous to the so-called binary-flutter instability from aeroelasticity. Both instabilities are due to non-symmetric stiffness matrices and are of the coupling type. The main difference between the flutter instability and the type of instability considered here is the fact that in the case of flutter a physically caused additional skew-symmetric component of the stiffness matrix causes the instability. In contrast to that in the case of sliding friction only the in-plane equation is modified by components due to friction. Of course the problem is mathematically equivalent, since any non-symmetric matrix can be decomposed into a symmetric and a skew-symmetric part; in the case of flutter however this decomposition can directly be interpreted on a physical basis, whereas in the case of friction self-induced vibration this decomposition can hardly be attributed any physical meaning.

6. Conclusions

It has been shown how energy can be transferred from the frictional to the vibrational system due to a simple mechanism: Normal force oscillations lead to oscillations of the tangential frictional force. If there are simultaneously relative tangential displacements at the friction interface that are not exactly in phase with the force oscillations a cyclic growth of vibrational energy results. Also it has been shown that the friction force acts like a structural cross-coupling force linking out-of-plane motion to the in-plane motion and that instability results if these friction-induced cross-coupling forces balance the corresponding cross-coupling forces of the structural system. Finally the origin and the role of phase shifts between in- and out-of-plane oscillations with respect to the mode-coupling type instability have been clarified by elementary considerations. In total the results obtained form a clear and intuitive picture of the physical mechanisms underlying the mode-coupling type instability. It may be expected that, properly used in conjunction with the rather formal mathematical stability analysis, this intuitive picture will be of considerable help for practical design purposes.

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