19 EHS 405: OPERATIONS RESEARCH(IDE - I)

B. Tech CSE - V Semester

Sections: I, J, O and P

Lecture Notes

In

OR

****

Prepared

by

S. Hemant Kumar

Assistant Professor
Department of Mechanical Engineering
GITAM School of Technology
GITAM (Deemed to be University)
Visakhapatnam

With
Sincere acknowledgements
to
Authors of the prescribed textbooks as per syllabus
and
Online content

2022-2023

19 EHS 405: OPERATIONS RESEARCH(IDE - I)

UNIT- I 10L

Basics of Operations Research:

- → History,
- → Definition,
- → Operations research models,
- → Phases of implementing operations research in practice.

Linear Programming:

- → Introduction,
- → Formulation,
- → Graphical solution,
- → Simplex method,
- → Artificial variable techniques:
- → Big m and two-phase methods,
- → Concept of duality, dual simplex method.

....

HISTORY OF OPERATIONS RESEARCH

Earlier **before civilization** started, all the available resources were used for fulfilling human necessities. The necessity of the utilizing the resources in limited way, for effective usage of the resources was felt only **after civilization** started. But there were no scientific methods adopted for effective utilization of the resources and decision making.

At the time of Second World War, United Kingdom felt the necessity of scientific methodologies for effective managing their limited military (Air and Land) resources and hence called upon a group of inter-disciplinary scientists from areas of Physiologists, Mathematical physicists, Astrophysicists, Army officers, surveyor and mathematicians forming first operation research (OR) team for bringing up the optimal scientific methodologies for usage and allocation of their limited military resources management. The scientist were not involved in the war, but acted as advisors for effectively (optimal) utilizing the military resources.

Later, the concept of optimal utilization of resource using scientific methodologies, provoked the United States, to develop and implement their own scientific optimal methodologies and so grouped OR teams with various names, like Operation Analysis, Operations research, System Analysis, System Research and Management Science etc. Their goal was to determine the operational effectiveness of weapons and equipment, Analyze the results of operations or exercises to determine the effectiveness of tactics, the influence of weapons on tactics and the tactics on weapons Predict the results of future operations, Analyze the efficiency of organizations or methods.

The Research activities in optimal utilization of resources for maximum benefit attracted the industries, after the Second World War and were in search for seeking solutions for complex executive problems, like allocation of resources, optimal financial expenditure, maximizing profits, minimizing the production cost etc. Today, OR is a burning topic in several areas, like Engineering, Management, Finance, Information Technologies, Supply chain management, Logistics, Transportation, Marketing and Natural resources management. In India, Operations Research started in 1949 at Regional Research Laboratory (RRL) at Hyderabad.

Definition of Operations Research

The subject OPERATIONS RESEARCH is a branch of mathematics - specially applied mathematics, used to provide a scientific base for management to take timely and effective decisions to their problems.

Operations Research (OR) is an "art of using interdisciplinary branches as algorithmic tools (like statistics, probability, game theory, decision analysis, simulation etc.) for optimizing the solutions of complex problems".

Typically, the objective function in industrial field is to find maxima or minima the objective functions like profit, performance, yield, loss, waiting time in queue, risk etc. In an organizational point of view OR is something that helps management achieve its goals using the scientific process. Several definitions have been aroused by eminent persons and argued their supporting factors.

Some of the definitions are highlighted below like:

Operations Research (OR) is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control. – Morse & Kimball

Operations research is a scientific approach to problem solving for executive management. –H.M. Wagner

Operations research is an aid for the executive in making this decision by providing him with the needed quantitative information based on the scientific method of analysis. – C. Kittel

Operations research is essentially a collection of mathematical techniques and tools which in conjunction with a system's approach, are applied to solve practical decision problems of an economic or engineering nature.

Operations research is the application of the methods of science to complex problems in the direction and management of large systems of men, machines, materials and money in industry, business, government and defence. The distinctive approach is to develop a scientific model of the system incorporating measurements of factors such as chance and risk, with which to predict

and compare the outcomes of alternative decisions, strategies or controls. The purpose is to help management in determining its policy and actions scientifically. – Operational Research Society, UK
And many more.......

Importance of operations research:

The field of operations research provides a more powerful approach to decision making than ordinary software and data analytics tools.
Employing operations research professionals can help companies achieve more complete datasets, consider all available options, predict all possible outcomes and estimate risk.
Additionally, operations research can be tailored to specific business processes or use cases to determine which techniques are most appropriate to solve the problem.

Characteristics of operations research

There are three primary characteristics of all operations research efforts:
Optimization- The purpose of operations research is to achieve the best performance under the given circumstances. Optimization also involves comparing and narrowing down potential options.
Simulation- This involves building models or replications in order to try out and test solutions before applying them.
Probability and statistics- This includes using mathematical algorithms and data to uncover helpful insights and risks, make reliable predictions and test possible solutions.

Scope of Operations Research

Operations Research addresses a wide variety of issues in transportation, inventory planning, production planning, communication operations, computer operations, financial assets, risk management, revenue management, and many other fields where improving business productivity is paramount

1. Manufacturing: OR's success in contemporary business pervades manufacturing and service operations, logistics, distribution, transportation, and telecommunication.
Operations research is used to for various activities which include scheduling, routing, workflow improvements, elimination of bottlenecks, inventory control, business process re-engineering, site selection, or facility and general operational planning.
OR helps in developing software for material flow analysis and design for flexible manufacturing facilities, using pattern recognition and graph theory algorithms. Further, approaches for the design of re-configurable manufacturing systems and progressive automation of discrete manufacturing systems are under development. Additional OR projects focus on the industrial deployment of computer-based methods for assembly line balancing, business process reengineering, capacity planning, pull scheduling, and setup reduction.
2. Revenue Management: The application of OR in revenue management entails

- → First to accurately forecasting the demand, and
- → Secondly to adjust the price structure over time to more profitably allocate fixed capacity.

3. Supply Chain Management: In the area of Supply Chain Management, OR helps in taking decisions that include the who, what, when, and where abstractions from purchasing and transporting raw materials and parts, through manufacturing actual products and goods, and finally distributing and delivering the items to the customers.
The primary objective here is to reduce overall cost while processing customer orders more efficiently than before. The power of utilizing OR methods allow examining a rather complex and convoluted chain in a comprehensive manner, and to search among a vast number of combinations for the resource optimization and allocation strategy that seem most effective, and hence beneficial to the operation.
4. Retailing: In supermarkets, data from store loyalty card schemes is analyzed by OR groups to advice on merchandising policies and profitability improvement. OR methods are also used to decide when and where new store developments should be made.
5. Financial Services: In financial markets, OR practitioners address issues such as portfolio and risk management and planning and analysis of customer service. They are widely employed in Credit Risk Management—a vital area for lenders needing to ensure that they find the optimum balance of risk and revenue. OR techniques are also applied in cash flow analysis and capital budgeting.
6. Marketing Management: OR helps marketing manager in making the apt selection of product mix. It helps them in making optimum sales resource allocation and assignments.
7. Human Resource Management: OR techniques are being applied widely in the functional area of Human Resource Management by helping the human resource managers in activities like manpower planning, resource allocation, staffing and scheduling of training programs.
8. General Management: OR helps in designing Decision Support System and management of information systems, organizational design and control, software process management and Knowledge Management.
9. Production systems: The area of operations research that concentrates on real-world operational problems is known as production systems. Production systems problems may arise in settings that include, but are not limited to, manufacturing, telecommunications, health-care delivery, facility location and layout, and staffing.

Applications of Operations Research:

Finance, Budgeting and investments

- → Cash flow analysis,
- → Capital requirement,
- → Investment analysis,
- → Dividend policies,
- → Credit policies,
- → Portfolio analysis, etc.

Purchasing & procurement

- → Quantity and timing of purchase,
- → Bidding policies,
- → Replacement policies, etc.

Production Management

- → Production scheduling,
- → Physical distribution,
- → Inventory control,
- → Manufacturing and facilities planning,
- → Maintenance policies,
- → Product-mix planning, etc.

Marketing Management

- → Product selection and competitive actions,
- → Advertising strategy,
- → Market research, etc.

Personnel Management

- → Recruitment policies,
- → Selection procedure,
- → Salary structure,
- → Bonus schemes,
- → Scheduling of training programmes, etc.

Research and Development Notes

- → Control of R&D projects,
- → Reliability and evaluation of alternative designs,
- → Determination of time and cost, etc.

Examples:

Managing consumer credit delinquency in the US economy: a multi-billion-dollar management science application: GE Capital provides credit card services for a consumer credit business exceeding $12 billion in total outstanding dollars. Its objective is to optimally manage delinquency by improving the allocation of limited collection resources to maximize net collections over multiple billing periods. GE developed a probabilistic Notes account flow model and statistically designed Programmes to provide accurate data on collection resource performance. A linear programming formulation produces optimal resource allocations that have been implemented across the business.

Control of the water distribution system under Irrigation scheme in Malaysia: A linear programming optimizations model was developed and adapted for daily operating decisions that would provide for a proper control of the water distribution system in real time for an Irrigation scheme in Malaysia.

Formulating insurance policies by Life Insurance Company in India: LIC uses OR to decide on the premium rate for its various policies and also how best the profits could be distributed in the case of profit policies.

Application of OR for optimum utilization of urban facilities: Increasingly, citizens are demanding more urban services, by type, quantity, and quality. The resulting pressure, between the demands for more and better services, on the one hand, and decreased revenue, on the other, has created a strong need for improved management decision making in urban services.

Forecasting: Using time series analysis to answer typical questions such as, how big will demand for products be? What are the sales patterns? How will this affect profits? Finance & Investment: How much capital do we need? Where can we get this? How much will it cost?

Manpower Planning & Assignment: How many employees do we need? What skills should they have? How long will they stay with us? Sequencing & Scheduling: What job is most important? In what order should we do jobs?

Location, Allocation, Distribution & Transportation: Where is the best location for operation? How big should facilities be? What resources are needed? Are there shortages? How can we set priorities? Reliability & Replacement Policy: How well is equipment working? How reliable is it? When should we replace it? Inventory Control and Stock out: How much stock should we hold? When do we order more? How much should we order?

Project Planning and Control: How long will a project take? What activities are most important? How should resources be used?

Queuing and Congestion: How long are queues? How many servers should we use? What service level are we giving?
This broad range of potential applications and a wide variety of OR techniques, which can be selected and combined for a multi-disciplinary approach, work together to make this course a dynamic and exciting one.

Models of Operations Research:

When a problem or process under investigation is simplified and represented with its typical features or characteristics, it is called as a model. The word 'model' has several meanings. All of which are relevant to Operations Research. For example, a model can act as a substitute for representing reality, such as small-scale model locomotive or may act as some sort of idealization, like a model plan for recruitment procedure, etc. Constructing a model helps in bringing the complexities and possible uncertainties into a logical framework required for comprehensive analysis. In fact, the model acts as a vehicle in arriving at a well-structured view of reality. An array of models can be seen in various business areas or industrial activities.

1. Physical Models:
    - (a) Iconic Models
    - (b) Analogue Models
2. Symbolic Models:
    - (a) Verbal Models
    - (b) Mathematical Models:
        - (i) Deterministic Models
        - (ii) Probabilistic Models
3. Combined Analogue and Mathematical Models
4. Function Models
5. Heuristic Models

1. Physical Models: To deal with specific types of problems, models like diagrams, charts, graphs and drawings are used, which are known as "Physical Models'. The schematic way of representation of significant factors and interrelationship may be in a pictorial form help in useful analysis. Moreover, they help in easy observation, description and prediction. However, they overrule any manipulation.

There are two types of Physical Models:
(i) Iconic Models: An image or likeness of an object or process is Icon. These models represent the system as it is by scaling it up or down. Even though use of these models in the area of management appears to be narrow, their usefulness is seen in the field of engineering and science. For example,

(a) In the field of R&D, prototype of the product is developed and tested to know the workability of the new product development.

(b) Photographs, portraits, drawings are the good example of iconic types. These models help in testifying the samples thus avoiding full scale designing and probable loss.

(ii) Analogue Models: These models are closely associated with iconic models. However, they are not the replicas of system or process. The analogue, in constructing these models, help in analyzing the issues and forces which are in the system or process. Because these models use 'one set of properties' which is 'analogous to another set of properties.' For example, kids, toys, rail-road models, etc.

2) Symbolic Models: Symbolic Models use letters, numbers, figures to represent decision variables of the system. There are two types of Symbolic Models—Verbal Models and Mathematical Models.
(a) Verbal Models: These models describe a situation in written or spoken language. Written sentences, books, etc., are examples of a verbal model.
(b) Mathematical Models: Mathematical symbols are used to represent a problem or a system under these types of models. Rules of mathematics enable the builder to make the models more abstract and precise.
There are two types of Mathematical
Models—Deterministic Models and Probabilistic Models.

(i) Deterministic Models: The exact statement of variables and their relationships are made under these models. The coefficients used for the mathematical formulation are known and are constant with certainty. So, to say, with a given set of data the answer will always be the same. For instance, determination of the break-even sales volume (BEP), the volume where the total cost equals the total sales revenue earned pertaining to a product.

(ii) Probabilistic Models: The risk involved, and the state of uncertainty are covered by these models. The decision variables take the form of a probability distribution and can assume more than single values. In the presence of risk and uncertainty, these models tend to yield different answers every time when attempted to. For example, uncertainty over acquisition of raw material to execute customer orders during a certain period. The purchaser has to consider both the sale and the delivery timing of the orders as they are variables. A probability distribution can be developed for the instant period for both sales delivery timings. However, the optimum selection is adhered to in accordance with the demand of the situation.

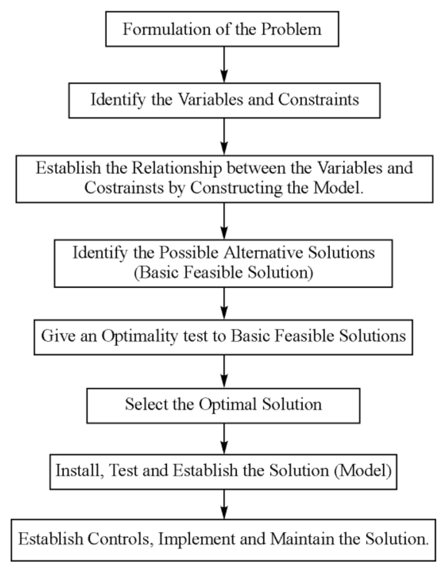
(3) Combined Analogue and Mathematical Models: When analogue models are interpreted with the use of mathematical symbols, they can be termed as physical-symbolic models. For instance, decision-makers tend to use mathematical symbols to represent their sales or profit figures.

(4) Function Models: Models which are used to represent a group of functions performed, are called function models.
Example: A monthly plan of processes to be carried on, a list of layouts, a calendar of events are the common examples of a functional model.

(5) Heuristic Models: When intuition guides a problem-solver to find solutions, heuristic models are developed. Even though he would not be able to find an optimum solution to the problem, with his past experience he arrives at the most advantageous solution. However, it depends on how intuitive and creative the decision-maker is.

Phases of implementing operations research in practice:

Step 1: Formulate the real practical problem by understanding the system situation and analyzing the data. The problem may be actual or abstract as it may involve current operations or proposed expansions/contraction. Estimate the values of the parameters that affect the organization's problem.
Step 2: Specify the real practical problem objectives and the parts of the problem (or system) that must be studied before the problem can be solved.
Step 3: Formulate an idealized simple mathematical model of the problem objective and constraints/restrictions based on the data of variables that have the functional relationships.
Step 4: Use any of the OR tools for generating the Algorithm by select a suitable Optimization technique (like Statistics, Probability theory, Game theory, Linear and non-linear programming, simulation, analytical tools, Dynamic programming, queuing theory, inventory theory, network theory etc) for generating alternative solutions for the given model and set of constraints
Step 5: Chooses the solution that best meets the objective of the real practical problem. In case of multiple optimal solutions, present them all to the decision-makers, or ask for more objectives or restrictions.
Step 6: Present the results and conclusions for approval of the recommendations.
Step 7: Implement and evaluate recommendation and monitor the system constantly for updating.



Linear Programming: Deterministic model.

Each and every organization aspires for optimal utilization of its limited scarce resources like men, money, materials, machines, methods and time to reach the targets. The results are generally measured in terms of profits, losses, return on money invested, etc. To achieve these results, the decision-maker has to have thorough knowledge about the tasks or jobs and the relationships among them. Among the popular techniques of Operations Research, Linear Programming deserves mention because it is one of the widely used techniques.

Linear Programming is used to allocate scarce resources in an optimal way so that the allocator can optimize the results either by maximizing the profits or minimizing the costs.

The general Linear Programming Problem calls for optimizing (maximizing/minimizing) a linear function for variables called the 'objective function' subject to a set of linear equations and/or inequalities called the 'constraints or restrictions.'

Terminology

The word **'linear'** is used to describe the relationship among two or more variables which are directly or precisely proportional.
Ex: If doubling the production of a product will exactly double the profit and required resources, then it is linear relationship.

'Programming' means the decisions which are taken systematically by adopting alternative courses of action.
'Linear Programming' indicates the planning of decision variables, which are directly proportional, to achieve the 'optimal' result considering the limitations within which the problem is to be solved.

Basic Requirements and their Relationships

1. Decision Variables and their Relationships: The decision variable refers to any candidate (person, service, projects, jobs, tasks) competing with other decision variables for limited resources. These variables are usually interrelated in terms of utilization of resources and need simultaneous solutions, i.e., the relationship among these variables should be linear.
2. Objective Function: The Linear Programming Problem must have a well-defined objective function to optimize the results. For instance, minimization of cost or maximization of profits. It should be expressed as linear function of decision variables ($Z = X1 + X2$), where Z represents the objective, i.e., minimization/maximization, X1 and X2 are the decision variables directly affecting the Z value).
3. Constraints: There would be limitations on resources which are to be allocated among various competing activities. These must be capable of being expressed as linear equalities or inequalities in terms of decision variables.
4. Alternative Courses of Action: There must be presence of alternative solutions for the purpose of choosing the best or optimum one.
5. Non-Negativity Restrictions: All variables must assume non-negative values. If any of the variable is unrestricted in sign, a tool can be employed which will enforce the negativity without changing the original information of a problem.
6. Linearity and Divisibility: All relationships (objective function and constraints) must exhibit linearity i.e., relationship among decision variables must be directly proportional. It is assumed that decision variables are continuous, i.e., fractional values of variables must be permissible in obtaining the optimum solution.
7. Deterministic: In Linear Programming it is assumed that all model coefficients are completely known. For example: profit per unit.

Application of Linear Programming

However, Linear Programming is exclusively used in the following areas:

1. Production Management
2. Personnel Management
3. Financial Management
4. Marketing Management

1. Production Management: In the area of production management, Linear Programming is used in the field of:
   - → Product planning
   - → Research and development
   - → Product portfolio management
   - → Line expansion and contraction decision
   - → Longevity of product life cycle.
2. Personnel Management: In this area, LP is used in the field of:
   - → Recruitment and staffing decisions
   - → Wage or salary management
   - → Job evaluation and allocation
   - → Employee benefits and welfare
   - → Overtime and related decisions.
3. Financial Management: In this area, LP is used in the field of:
   - → Portfolio decision
   - → Profit planning
   - → Alternative capital investment decisions
   - → Investment on inventories
   - → Allocation of funds to developmental activities.
4. Marketing Management: In the area of marketing management, it is used in the field of:
   - → Media planning and selection
   - → Travelling salesman problem
   - → Product development
   - → Ad and Pro budget
   - → Marketing mix decisions.

Advantages

1. It helps in proper and optimum utilization of the scarce resources
2. It helps in improving the quality of the decisions.

3. With the use of this technique, the decision-maker becomes more objective and less subjective.
4. It even helps in considering other constraints operating outside the problem.
5. Many a times it hints the manager about the hurdles faced during the production activities.

Limitations

1. The treatment of variables having non-linear relationships is the greatest limitation of this LP
2. It can come out with non-integer solutions too, which would be many a times meaningless.
3. It rules out effect of time and uncertainty conditions.
4. Generally, the objective set will be single and on the contrary, in the real life, there might be several objectives.
5. Large-scale problems tend to be unaccomodative to solve under LP

Steps for Formulating LPP

1. Identify the nature of the problem (maximization/minimization problem).
2. Identify the number of variables to establish the objective function.
3. Formulate the constraints.
4. Develop non-negativity constraints.

MATHEMATICAL MODEL OF LINEAR PROGRAMMING PROBLEM

Objective Function: Max/ Min $Z = C_1 X_1 + C_2 X_2 + C_3 X_3 + \ldots$

Constraints:

$$a_{11} X_1 + a_{12} X_2 + a_{13} X_3 \leq \text{ or } \geq b_1$$
$$a_{21} X_1 + a_{22} X_2 + a_{23} X_3 \leq \text{ or } \geq b_2$$
$$a_{31} X_1 + a_{32} X_2 + a_{33} X_3 \leq \text{ or } \geq b_3$$

Non-negative restrictions: $X_1, X_2, X_3 \geq 0$

Z: Objective

Ci: Cost coefficient

a_i : Technological Coefficient

X_i : Decision variable

b_i : availability or constant of the constraint.

Formulation of an LPP refers to translating the real-world problem into the form of mathematical equations which could be solved. It usually requires a thorough understanding of the problem.

Steps towards formulating a Linear Programming problem:

Step 1: Identify the 'n' number of decision variables which govern the behaviour of the objective function (which needs to be optimized).

Step 2: Identify the set of constraints on the decision variables and express them in the form of linear equations /inequations. This will set up our region in the n-dimensional space within which the objective function needs to be optimized. Don't forget to impose the condition of non-negativity on the decision variables i.e. all of them must be positive since the problem might represent a physical scenario, and such variables can't be negative.

Step 3: Express the objective function in the form of a linear equation in the decision variables.

Step 4: Optimize the objective function either graphically or mathematically.

Example 1: A firm manufactures 2 types of products A & B and sells them at a profit or ` 2 on type A & ` 3 on type B. Each product is processed on 2 machines G & H. Type A requires 1 minute of processing time on G and 2 minutes on H. Type B requires one minute on G &1 minute on H. The machine G is available for not more than 6 hrs. 40 mins., while machine H is available for 10 hrs. during any working day. Formulate the problem as LPP.

Machines	Time on Products (mins.)		Total time available (in minutes)
	Type A	Type B	
G	1	1	400
H	2	1	600
Profit Per Unit	₹ 2	₹ 3	

Solution:

Let x1 be the no. of products of type A

x2 be the no. of products of type B

Since the profit on type A is ` 2 per product, 2x1- will be the profit on selling x1 units of type A.

Similarly 3x2 will be the profit on selling x2 units of type B.

Hence the objective function will be, Maximize 'Z' = 2x1 + 3x2 is subject to constraints.

Since machine 'G' takes one minute on 'A' and one minute on 'B', the total number of minutes required is given by x1 + x2. Similarly, on machine 'H' 2x1 + x2. But 'G' is not available for more than 400 minutes. Therefore, $x1 + x2 \leq 400$ and H is not available for more than 600 minutes, therefore, $2x1 + x2 \leq 600$ and x1, x2, ≥ 0, i.e.,

$x1 + x2 \leq 400$ (Time availability constraints)
$2x1 + x2 \leq 600$
$x1, x2 \geq 0$ (Non-negativity constraints)

Example 2 : A retail store stocks two types of shirts A and B. These are packed in attractive cardboard boxes. During a week the store can sell a maximum of 400 shirts of type A and a maximum of 300 shirts of type B. The storage capacity, however, is limited to a maximum of 600 of both types combined. Type A shirt fetches a profit of Rs. 2/- per unit and type B a profit of Rs. 5/- per unit. How many of each type the store should stock per week to maximize the total profit? Formulate a mathematical model of the problem.

Solution: Here shirts A and B are problem variables. Let the store stock 'a' units of A and 'b' units of B. As the profit contribution of A and B are Rs.2/- and Rs.5/- respectively,
objective function is: Maximize $Z = 2a + 5b$ subjected to condition (s.t.)
Structural constraints are, stores can sell 400 units of shirt A and 300 units of shirt B and the storage capacity of both put together is 600 units. Hence the structural constraints are:
$1a + 0b \geq 400$ and $0a + 1b \leq 300$ for sales capacity and $1a + 1b \leq 600$ for storage capacity.
And non-negativity constraint is both a and b are ≥ 0. Hence the model is:
Maximize $Z = 2a + 5b$
s.t.
$1a + 0b \leq 400$
$0a + 1b \leq 300$
$1a + 1b \leq 600$ and
Both a and b are ≥ 0.

Example 3 : A ship has three cargo holds, forward, aft and center. The capacity limits are:
Forward 2000 tons, 100,000 cubic meters
Center 3000 tons, 135,000 cubic meters
Aft 1500 tons, 30,000 cubic meters.
The following cargoes are offered, the ship owners may accept all or any part of each commodity

Commodity	*Amount in tons.*	*Volume/ton in cubic meters*	*Profit per ton in Rs.*
A	6000	60	60
B	4000	50	80
C	2000	25	50

In order to preserve the trim of the ship the weight in each hold must be proportional to the capacity in tons. How should the cargo be distributed so as to maximize profit? Formulate this as linear programming problem.

Solution: Problem variables are commodities, A, B, and C. Let the shipping company ship 'a' units of
A and 'b' units of B and 'c' units of C. Then Objective function is:
Maximize $Z = 60a + 80b + 50c$
s.t.
Constraints are:
Weight constraint: $6000a + 4000b + 2000c \leq 6{,}500$ ($= 2000+3000+1500$)
The tonnage of commodity is 6000 and each ton occupies 60 cubic meters, hence there are 100
cubic meters capacity is available.
Similarly, availability of commodities B and C, which are having 80 cubic meter capacities each.
Hence capacity inequality will be:
$100a + 80b + 80c \leq 2{,}65{,}000$ ($= 100{,}000+135{,}000+30{,}000$).

Hence the LPP. Model is:
Maximise $Z = 60a+80b+50c$
s.t.
$100a = 6000/60 = 100$
$6000a + 4000b + 2000c \leq 6{,}500$ $80b = 4000/50 = 80$
$100a+80b+80c \leq 2{,}65{,}000$ and $80c = 2000/25 = 80$ etc.
a,b,c all ≥ 0

Example 4 : A patient consult a doctor to check up his ill health. Doctor examines him and advises him that he is having deficiency of two vitamins, vitamin A and vitamin D. Doctor advises him to consume vitamin A and D regularly for a period of time so that he can regain his health. Doctor prescribes tonic X and tonic Y, which are having vitamin A, and D in certain proportion. Also advises the patient to consume at least 40 units of vitamin A and 50 units of vitamin Daily. The cost of tonics X and Y and the proportion of vitamin A and D that present in X and Y are given in the table below. Formulate L.P.P. to minimize the cost of tonics.

Solution:

Let patient purchase x units of X and y units of Y.
Objective function: Minimize $Z = 5x + 3y$
Inequality for vitamin A is $2x + 4y \geq 40$ (Here at least word indicates that the patient can consume more than 40 units but not less than 40 units of vitamin A daily).
Similarly the inequality for vitamin D is $3x + 2y \geq 50$.
For non–negativity constraint the patient cannot consume negative units. Hence both x and y must be ≥ 0.
Now the l.p.p. model for the problem is:
Linear Programming Models (Resource Allocation Models)

Minimize $Z = 5x + 3y$
s.t.
$2x + 4y \geq 40$
$3x + 2y \geq 50$ and
Both x and y are ≥ 0

Example 5 : A city hospital has the following daily requirements of nurses at the minimal level:
Nurses report to the hospital at the beginning of each period and work for 8 consecutive hours. The wants to determine minimal number of nurses to be employed, so that there will be sufficient number of nurses available for each period. Formulate this as LP model by setting up appropriate constraints and objective function.

Period	Clock Time (24 Hours a Day)	Minimal no. of Nurses required
1	6 am- 10am	2
2	10am-2pm	7
3	2pm-6pm	15
4	6pm-10pm	8
5	10pm-2pm	20
6	2am-6am	6

Solution:

Let x1 be the no. of nurses working during period 1, x2 be the no. of nurses working during period 2, and x3, x4, x5 and x6 be the no. of nurses working during period 3,4,5, and 6 respectively.
Hence, the objective function is given by,

Minimize 'C' = $x1 + x2 + x3 + x4 + x5 + x6$

Subject to constraints:

$x1 + x2 \geq 2$
$x2 + x3 \geq 7$
$x3 + x4 \geq 15$
$x4 + x5 \geq 8$
$x5 + x6 \geq 20$
$x6 + x1 \geq 6$
$x1, x2, x3, x4, x5, x6 \geq 0$ (non-negativity constraints)

Example 6 : A company owns two flour mills A and B, which have different production capacities for high, medium and low quality flour. The company has entered a contract to supply flour to a firm every month with at least 8, 12 and 24 quintals of high, medium and low quality respectively. It costs the company Rs 2000 and Rs 1500 per day to run mill A and B respectively. In one day, Mill A produces 6, 2 and 4 quintals of high, medium and low quality flour respectively, Mill B produces 2, 4 and 12 quintals of high, medium and low quality flour respectively. Formulate the linear programming model to Minimize the cost.

Solution:

The contents of the statement of the problem can be summarized as follows

Mill	A	B	Min Required
high	6	2	8
medium	2	4	12
low	4	12	24

Decision variables
Here flour mills A and B are competing variables and flour quality are available resources.
Let x1 and x2 denotes mills A and B respectively
The LP Model
Here the objective is to minimize the cost

Minimize Z = 2000x1 + 1500x2
Subject to

$6x_1 + 2x_2 \geq 8$
$2x_1 + 4x_2 \geq 12$
$4x_1 + 12x_2 \geq 24$

Such that: $x_1, x_2 \geq 0$

Exercise Questions:

1. Leather products: Agra makes two kinds of leather belts. Belt A is a high-quality belt and Belt B is of lower quality. The respective profits are Rs. 4 and Rs. 3 per belt. Each belt of type A requires twice as much time as the belt of type B and if all belts are of type B, the company could make 1200 per day. The supply of leather is sufficient for only 900 belts per day (both A and B combined). Belt A requires a fancy buckle and only 500 per day available. Whereas for type belt B 700 buckles per day are available. Assuming that the supply of leather available per day is 10% defective and that only good leather can be used in manufacturing of the belts, formulate the problem as LPP and obtain the 0ptimal solution using graphical method.
2. Three grades of coal A, B and C contain phosphorus and ash as impurities. In a particular industrial process, fuel up to 100 tons is required which should contain ash not more than 3%. It is desired to maximize the project while satisfying these conditions. There is unlimited supply of each grade. The percentage of impurities and profits of grades are given as follows. Find the proportion in which the three grades be used.

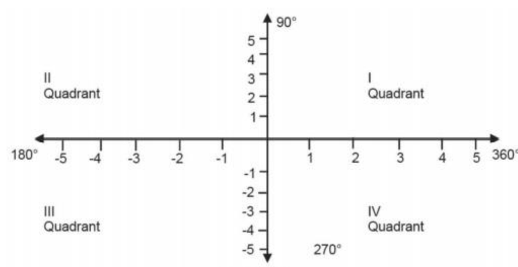
Coal	Phosphorus(%)	Ash(%)	Profit in `/ton
A	0.02	2	12/-
B	0.04	3	15/-
C	0.03	5	14/-

3. A resourceful home decorator manufactures two types of lamps say A and B. Both lamps go through two technicians first a courter, second a finisher. Lamp A requires 2 hours of the courters time and 1 hour of the finisher time. Lamp B requires 1 hour of the courter and 2 hours of the finisher time. The courter has 104 hours and finisher has 76 hours of time available each. Profit on one lamp A is `6/- and on lamp B is `11/-. Form the LPP and maximize

Graphical Solutions under Linear Programming:

Linear programming problems with two variables can be represented and solved graphically with ease. Though in real-life, the two variable problems are practiced very little, the interpretation of this method will help to understand the simplex method.

Finding an optimal solution to a linear programming problem means assigning values to the decision variables in such a way as to achieve a specified goal and conform to certain constraints. For a problem with n decision variables, any solution can be specified by a point $(x_1, x_2, \ldots, x_n)$. The feasible space (or feasible region) for the problem is the set of all such points that satisfy the problem constraints. The feasible space is therefore the set of all feasible solutions. An optimal feasible solution is a point in the feasible space that is as effective as any other point in achieving the specified goal.



Cartisian Plane

IMPORTANT DEFINITIONS

Solution: The set of values of decision variables x_j (j = 1, 2, . . ., n) that satisfy the constraints of an LP problem is said to constitute the solution to that LP problem.

Feasible solution: The set of values of decision variables x_j (j = 1, 2, . . ., n) that satisfy all the constraints and non-negativity conditions of an LP problem simultaneously is said to constitute the feasible solution to that LP problem.

Infeasible solution: The set of values of decision variables x_j (j = 1, 2, . . ., n) that do not satisfy all the constraints and non-negativity conditions of an LP problem simultaneously is said to constitute the infeasible solution to that LP problem.

Basic solution: For a set of m simultaneous equations in n variables (n > m) in an LP problem, a solution obtained by setting (n – m) variables equal to zero and solving for remaining m equations in m variables is called a basic solution of that LP problem. The (n – m) variables whose value did not appear in basic solution are called non-basic variables and the remaining m variables are called basic variables.

Basic feasible solution: A feasible solution to an LP problem which is also the basic solution is called the basic feasible solution. That is, all basic variables assume non-negative values.
Basic feasible solution is of two types:
(a) Degenerate A basic feasible solution is called degenerate if the value of at least one basic variable is zero.

(b) Non-degenerate A basic feasible solution is called non-degenerate if value of all m basic variables is non-zero and positive.

Optimum basic feasible solution: A basic feasible solution that optimizes (maximizes or minimizes) the objective function value of the given LP problem is called an optimum basic feasible solution.

Unbounded solution: A solution that can increase or decrease infinitely the value of the objective function of the LP problem is called an unbounded solution.

Extreme Point Solution Method

In this method, the coordinates of all corner (or extreme) points of the feasible region (space or area) are determined and then value of the objective function at each of these points is computed and compared.
The coordinates of an extreme point where the optimal (maximum or minimum) value of the objective function is found represent solution of the given LP problem.

The steps of the method are summarized as follows:
Step 1: Develop an LP model State the given problem in the mathematical LP model as illustrated in the previous chapter.
Step 2: Plot constraints on graph paper and decide the feasible region
(a) Replace the inequality sign in each constraint by an equality sign.
(b) Draw these straight lines on the graph paper and decide each time the area of feasible solutions according to the inequality sign of the constraint. Shade the common portion of the graph that satisfies all the constraints simultaneously drawn so far.
(c) The final shaded area is called the feasible region (or solution space) of the given LP problem. Any point inside this region is called feasible solution and this provides values of x1 and x2 that satisfy all the constraints.
Step 3: Examine extreme points of the feasible solution space to find an optimal solution
(a) Determine the coordinates of each extreme point of the feasible solution space.
(b) Compute and compare the value of the objective function at each extreme point.
(c) Identify the extreme point that gives optimal (max. or min.) value of the objective function.

Example 6:

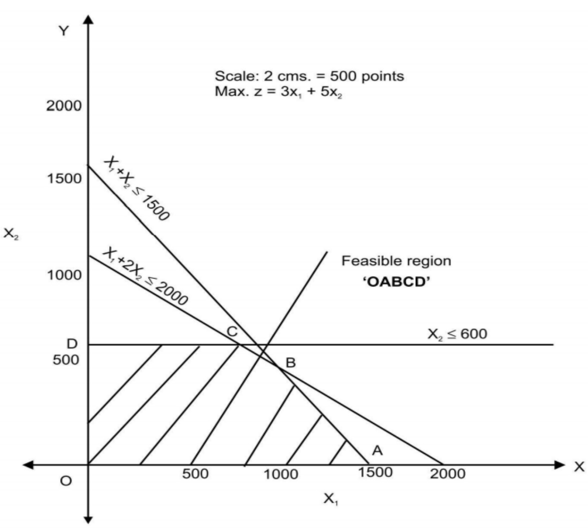
Maximize 'Z' = $3X_1 + 5X_2$ (Subject to constraints) $X_1 + 2X_2 \leq 2{,}000$, $X_1 + X_2 \leq 1{,}500$, $X_2 \leq 600$
Such that: X_1 , $X_2 \geq 0$

Solution:

Step 1: Convert the inequalities into equalities and find the divisibles of the equalities.

Equation	x_1	x_2
$x_1 + 2x_2 = 2{,}000$	2,000	1,000
$x_1 + x_2 = 1{,}500$	1,500	1,500
$x_2 = 600$	---	600

Step 2: Fix up the graphic scale.
Notes Maximum points = 2,000 Minimum points = 600, 2 cm = 500 points
Step 3: Graph the data
Step 4: Find the co-ordinates of the corner points
Step 5: Substitute the co-ordinates of corner points into the objective function.



Corner Points	X_1	X_2
O	0	0
A	1,500	0
B	1,000	500
C	800	600
D	0	600

Maximize Z = 3x1 + 5x2

At 'O', Z = 0 + 0 = 0
At 'A', Z = 3 (1,500) + 5 (0) = 4,500
At 'B', Z = 3 (1,000) + 5 (500) = 5,500
At 'C', Z = 3 (800) + 5 (600) = 5,400
At 'D', Z = 3 (0) + 5 (600) = 3,000

Inference: A maximum profit of ` 5,500 can be earned by producing 1,000 dolls of basic version and 500 dolls of deluxe version.

Example 7:

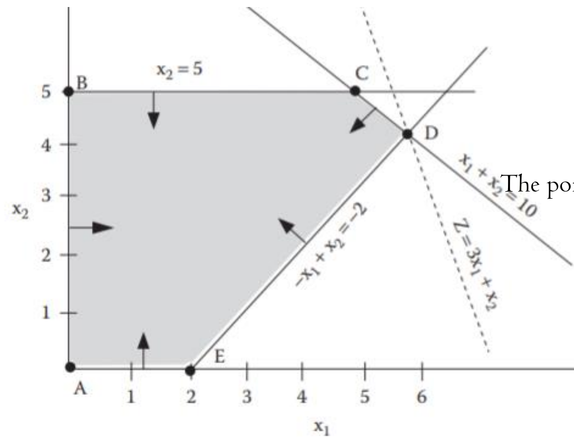
Maximize $Z = 3X_1 + X_2$

Subjected to: $X2 \leq 5$

$X_1 + X_2 \leq 10$

$-X_1 + X_2 \geq -2$

Such that: $X_1, X_2 \geq 0$



The points labeled

Solution:

$$z_A = z(0,0) = 3 \times 0 + 0 = 0$$

$$z_B = z(0,5) = 3 \times 0 + 5 = 5$$

$$z_C = z(5,5) = 3 \times 5 + 5 = 20$$

$$z_D = z(6,4) = 3 \times 6 + 4 = 22$$

$$z_E = z(2,0) = 3 \times 2 + 0 = 6$$

The optimal solution lies at extreme point D where x1 = 6 and x2 = 4, and the optimal value of the objective function is denoted by z* = 22.

Example:

A company owns two flour mills viz. A and B, which have different production capacities for high, medium and low-quality flour. The company has entered a contract to supply flour to a firm every month with at least 8, 12 and 24 quintals of high, medium and low quality respectively. It costs the company Rs.2000 and Rs.1500 per day to run mill A and B respectively. On a day, Mill A produces 6, 2 and 4 quintals of high, medium and low-quality flour, Mill B produces 2, 4 and 12 quintals of high, medium and low-quality flour respectively. How many days per month should each mill be operated in order to meet the contract order most economically?

Solution:

Let us define x1 and x2 are the mills A and B. Here the objective is to minimize the cost of the machine runs and to satisfy the contract order.

The linear programming problem is given by

Minimize $2000x1 + 1500x2$

Subject to:

$6x1 + 2x2 \geq 8$

$2x1 + 4x2 \geq 12$

$4x1 + 12x2 \geq 24$

Such that $x1, x2 \geq 0$

Example: Use the graphical method to solve the following LP problem.

Maximize $Z = 15x1 + 10x2$

subject to the constraints

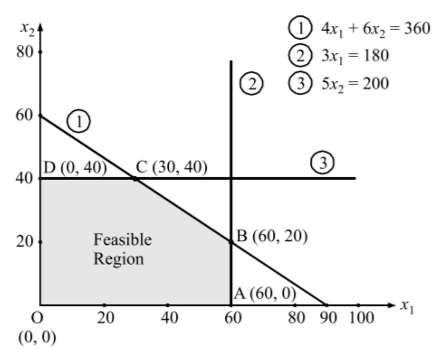
(i) $4x1 + 6x2 \leq 360$, (ii) $3x1 + 0x2 \leq 180$, (iii) $0x1 + 5x2 \leq 200$

and $x1, x2 \geq 0$.

Solution

1. The given LP problem is already in mathematical form.

2. Treat x1 as the horizontal axis and x2 as the vertical axis. Plot each constraint on the graph by treating it as a linear equation and it is then that the appropriate inequality conditions will be used to mark the area of feasible solutions.



Consider the first constraint $4x1 + 6x2 \leq 360$. Treat this as the equation $4x1 + 6x2 = 360$. For this find any two points that satisfy the equation and then draw a straight line through them. The two points are generally the points at which the line intersects the x1 and x2 axes.

For example, when $x1 = 0$ we get $6x2 = 360$ or $x2 = 60$.

Similarly, when $x2 = 0$, $4x1 = 360$, $x1 = 90$.

These two points are then connected by a straight line as shown in Fig. 3.1(a). But the question is: Where are these points satisfying $4x1 + 6x2 \leq 360$. Any point above the constraint line violates the inequality condition. But any point below the line does not violate the constraint. Thus, the inequality and non-negativity condition can only be satisfied by the shaded area (feasible region) as shown in Fig.

Similarly, the constraints $3x1 \leq 180$ and $5x2 \leq 200$ are also plotted on the graph and are indicated by the shaded area as shown in Fig. (b).

Since all constraints have been graphed, the area which is bounded by all the constraints lines including all the boundary points is called the feasible region (or solution space). The feasible region is shown in Fig. by the shaded area OABCD.

3. (i) Since the optimal value of the objective function occurs at one of the extreme points of the feasible region, it is necessary to determine their coordinates. The coordinates of extreme points of the feasible region are: O = (0, 0), A = (60, 0), B = (60, 20), C = (30, 40), D = (0, 40).

(ii) Evaluate objective function value at each extreme point of the feasible region as shown in the Table.

(iii) Since objective function Z is to be maximized, from Table we conclude that maximum value of Z = 1,100 is achieved at the point extreme B (60, 20). Hence the optimal solution to the given LP problem is: x1 = 60, x2 = 20 and Max Z = 1,100.

Extreme Point	*Coordinates* (x_1, x_2)	*Objective Function Value* $Z = 15x_1 + 10x_2$
O	(0, 0)	15(0) + 10(0) = 0
A	(60, 0)	15(60) + 10(0) = 900
B	(60, 20)	15(60) + 10(20) = 1,100
C	(30, 40)	15(30) + 10(40) = 850
D	(0, 40)	15(0) + 10(40) = 400

Example: Use the graphical method to solve the following LP problem.

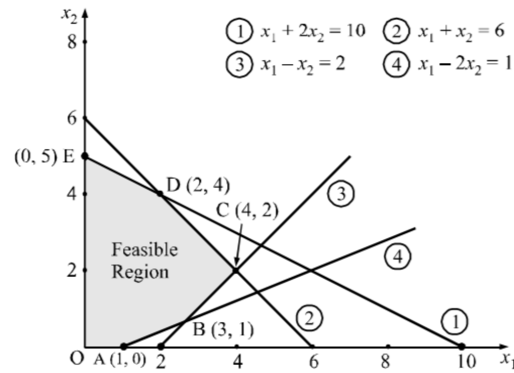
Maximize Z = 2x1 + x2

subject to the constraints

(i) x1 + 2x2 ≤ 10, (ii) x1 + x2 ≤ 6,

(iii) x1 – x2 ≤ 2, (iv) x1 – 2x2 ≤ 1

and x1, x2 ≥ 0.



Solution: Plot on a graph each constraint by first treating it as a linear equation. Then use inequality condition of each constraint to mark the feasible region by shaded area as shown in Fig. 3.2. It may be noted that we have not considered the area below the lines x1 – x2 = 2 and x1 – 2x2 = 1 for the negative values of x2. This is because of the non-negativity condition, x2 ≥ 0

The coordinates of extreme points of the feasible region are: O = (0, 0), A = (1, 0), B = (3, 1), C = (4, 2), D = (2, 4), and E = (0, 5). The value of objective function at each of these extreme points is shown in Table The maximum value of the objective function Z = 10 occurs at the extreme point (4, 2).

Hence, the optimal solution to the given LP problem is: x1 = 4, x2 = 2 and Max Z = 10.

Extreme Point	*Coordinates* (x_1, x_2)	*Objective Function Value* $Z = 2x_1 + x_2$
O	(0, 0)	2(0) + 1(0) = 0
A	(1, 0)	2(1) + 1(0) = 2
B	(3, 1)	2(3) + 1(1) = 7
C	(4, 2)	2(4) + 1(2) = 10
D	(2, 4)	2(2) + 1(4) = 8
E	(0, 5)	2(0) + 1(5) = 5

Example: Solve the following LP problem graphically: Maximize Z = – x1 + 2x2

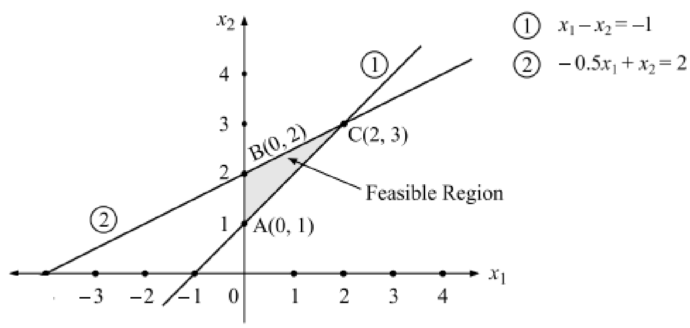
subject to the constraints

(i) x1 – x2 ≤ –1; (ii) – 0.5x1 + x2 ≤ 2 and x1, x2 ≥ 0.

Solution:

Since resource value (RHS) of the first constraint is negative, multiplying both sides of this constraint by –1, the constraint becomes: – x1 + x2 ≥ 1. Plot on a graph each constraint by first treating them as a linear equation and mark the feasible region as shown in Fig.

The value of the objective function at each of the extreme points A(0, 1), B(0, 2) and C(2, 3) is shown in Table



Extreme Point	*Coordinates* (x_1, x_2)	*Objective Function Value* $Z = -x_1 + 2x_2$
A	(0, 1)	$0 + 2 \times 1 = 2$
B	(0, 2)	$0 + 2 \times 2 = 4$
C	(2, 3)	$-1 \times 2 + 2 \times 3 = 4$

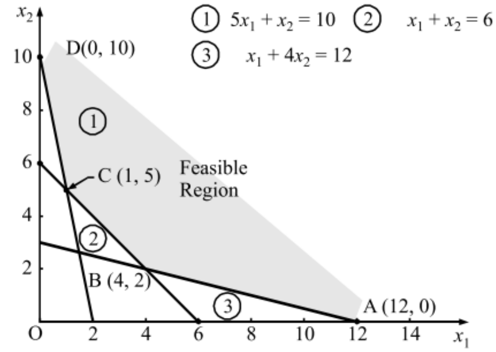
Multiple optimal solution

The maximum value of objective function Z = 4 occurs at extreme points B and C. This implies that every point between B and C on the line BC also gives the same value of Z.

Hence, problem has multiple optimal solutions: x1 = 0, x2 = 2 and x1 = 2, x2 = 3 and Max Z = 4.

Example:

Use the graphical method to solve the following LP problem.
Minimize Z = 3x1 + 2x2
subject to the constraints
(i) 5x1 + x2 ≥ 10, (ii) x1 + x2 ≥ 6, (iii) x1 + 4x2 ≥ 12 and x1, x2 ≥ 0.



Solution: Plot on a graph each constraint by first treating it as a linear equation. Then use inequality condition of each constraint to mark the feasible region by shaded area as shown in Fig. This region is bounded from below by extreme points A, B, C and D

The coordinates of the extreme points of the feasible region (bounded from below) are: A = (12, 0), B = (4, 2), C = (1, 5) and D = (0, 10). The value of objective function at each of these extreme points is shown in Table

Extreme Point	*Coordinates* (x_1, x_2)	*Objective Function Value* $Z = 3x_1 + 2x_2$
A	(12, 0)	3(12) + 2(0) = 36
B	(4, 2)	3(4) + 2(2) = 16
C	(1, 5)	3(1) + 2(5) = **13**
D	(0, 10)	3(0) + 2(10) = 20

The minimum (optimal) value of the objective function Z = 13 occurs at the extreme point C (1, 5).
Hence, the optimal solution to the given LP problem is: x1 = 1, x2 = 5, and Min Z = 13.

Example:

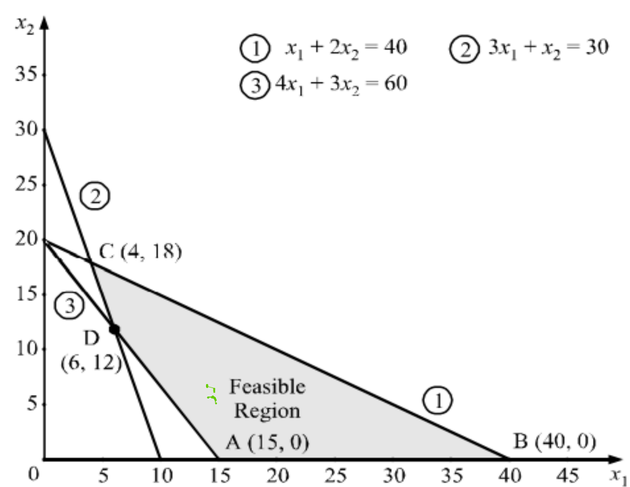
Mixed Constraints LP Problems

Use the graphical method to solve the following LP problem.
Minimize Z = 20x1 + 10x2
subject to the constraints
(i) x1 + 2x2 ≤ 40, (ii) 3x1 + x2 ≥ 30, (iii) 4x1 + 3x2 ≥ 60 and x1, x2 ≥ 0

Solution:
Plot on a graph each constraint by first treating it as a linear equation. Then use the inequality condition of each constraint to mark the feasible region by shaded area as shown in Fig.

The coordinates of the extreme points of the feasible region are:
A = (15, 0), B = (40, 0), C = (4, 18) and D = (6, 12).

The value of the objective function at each of these extreme points is shown Table



Extreme Point	*Coordinates* (x_1, x_2)	*Objective Function Value* $Z = 20x_1 + 10x_2$
A	(15,0)	20(15) + 10(0) = 300
B	(40,0)	20(40) + 10(0) = 800
C	(4,18)	20(4) + 10(18) = 260
D	(6,12)	20(6) + 10(12) = 240

The minimum (optimal) value of the objective function, Z = 240 occurs at the extreme point D (6, 12). Hence, the optimal solution to the given LP problem is: x1 = 6, x2 = 12 and Min Z = 240.

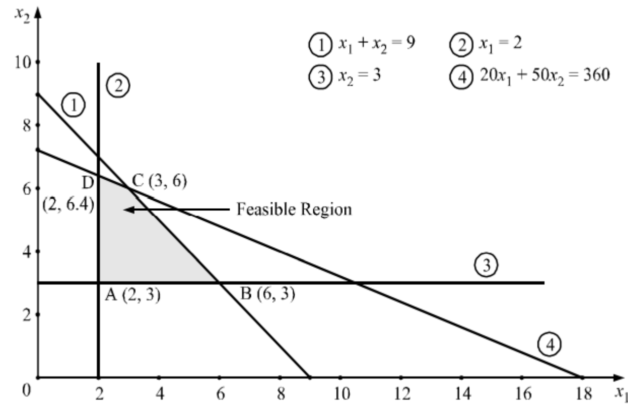
Example:

A firm makes two products X and Y and has a total production capacity of 9 tons per day. Both X and Y require the same production capacity. The firm has a permanent contract to supply at least 2 tons of X and at least 3 tons of Y per day to another company. Each ton of X requires 20 machine hours of production time and each ton of Y requires 50 machine hours of production time. The daily maximum possible number of machine hours is 360. All of the firm's output can be sold. The profit made is Rs 80 per ton of X and Rs 120 per ton of Y. Formulate this problem as an LP model and solve it by using graphical method to determine the production schedule that yields the maximum profit.

Solution: Let us define the following decision variables:
x1 and x2 = number of units (in tons) of products X and Y to be manufactured, respectively.
Then the LP model of the given problem can be written as
Maximize (total profit) Z = 80x1 + 120x2

subject to the constraints
(i) $x1 + x2 \leq 9$ (Production capacity)
(ii) $x1 \geq 2$; $x2 \geq 3$ (Supply)
(iii) $20x1 + 50x2 \leq 360$ (Machine hours) and $x1, x2 \geq 0$
For solving this LP problem graphically, plot on a graph each constraint by first treating it as a linear equation. Then use the inequality sign of each constraint to mark the feasible region as shown in Fig.

The coordinates of the extreme points of the feasible region are:
A = (2, 3), B = (6, 3), C = (3, 6), and
D = (2, 6.4).
The value of the objective function at each of these extreme points is shown in Table.



Extreme Point	Coordinates (x_1, x_2)	Objective Function Value $Z = 80x_1 + 120x_2$
A	(2, 3)	80(2) + 120(3) = 520
B	(6, 3)	80(6) + 120(3) = 840
C	(3, 6)	80(3) + 120(6) = **960**
D	(2, 6.4)	80(2) + 120(6.4) = 928

The maximum (optimal) value of the objective function, Z = 960 occurs at the extreme point C (3, 6).
Hence the company should produce, $x1 = 3$ tons of product X and $x2 = 6$ tons of product Y in order to yield a maximum profit of Rs 960.

Iso-profit (Cost) Function Line Method:

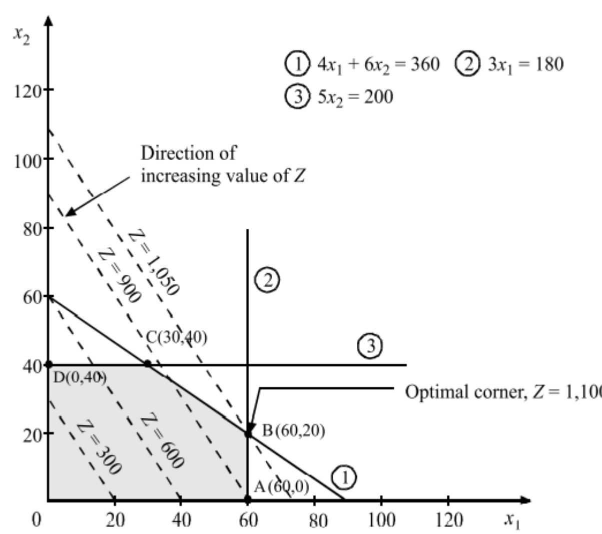
According to this method, the optimal solution is found by using the slope of the objective function line (or equation). An iso-profit (or cost) line is a collection of points that give solution with the same value of objective function. By assigning various values to Z, we get different profit (cost) lines. Graphically many such lines can be plotted parallel to each other.

The steps of iso-profit (cost) function method are as follows:

Step 1: Identify the feasible region and extreme points of the feasible region.
Step 2: Draw an iso-profit (iso-cost) line for an arbitrary but small value of the objective function without violating any of the constraints of the given LP problem. However, it is simple to pick a value that gives an integer value to x1 when we set x2 = 0 and vice-versa. A good choice is to use a number that is divided by the coefficients of both variables.
Step 3: Move iso-profit (iso-cost) lines parallel in the direction of increasing (decreasing) objective function values. The farthest iso-profit line may intersect only at one corner point of feasible region providing a single optimal solution. Also, this line may coincide with one of the boundary lines of the feasible area. Then at least two optimal solutions must lie on two adjoining corners and others will lie on the boundary connecting them. However, if the iso-profit line goes on without limit from the constraints, then an unbounded solution would exist. This usually indicates that an error has been made in formulating the LP model.
Step 4: An extreme (corner) point touched by an iso-profit (or cost) line is considered as the optimal solution point. The coordinates of this extreme point give the value of the objective function.

Example: Consider the LP problem. Maximize $Z = 15x1 + 10x2$
subject to the constraints
(i) $4x1 + 6x2 \leq 360$, (ii) $3x1 \leq 180$, (iii) $5x2 \leq 200$ and $x1, x2 \geq 0$.

Solution:
Plot each constraint on a graph first treating it as a linear equation. Then use inequality sign of each constraint to mark feasible solution region (shaded area) as shown in Fig.
A family of lines (shown by dotted lines) that represents various values of objective function is shown in Fig. Such lines are referred as iso-profit lines.



In Fig., a value of Z = 300 is arbitrarily selected. The iso-profit (objective) function equation then becomes: $15x1 + 10x2 = 300$. This equation is also plotted in the same way as other equality constraints plotted before. This line is then moved upward until it first intersects a corner (or corners) in the feasible region (corner B). The coordinates of corner point B can be read from the graph or can be computed as the intersection of the two linear equations.

The coordinates $x1 = 60$ and $x2 = 0$ of corner point B satisfy the given constraints and the total profit obtained is $Z = 1{,}100$.

Comparison of Two Graphical Solution Methods:

Extreme Point Method	Iso-Profit (or Cost) Method
Identify coordinates of each of the extreme (or corner) points of the feasible region by either drawing perpendiculars on the x-axis and the y-axis or by solving two intersecting equations.	Determine the slope (x1, x2) of the objective function and then join intercepts to identify the profit (or cost) line.
Compute the profit (or cost) at each extreme point by substituting that point's coordinates into the objective function	In case of maximization, maintain the same slope through a series of parallel lines, and move the line upward towards the right until it touches the feasible region at only one point. But in case of minimization, move downward towards left until it touches only one point in the feasible region
Identify the optimal solution at that extreme point with highest profit in a maximization problem or lowest cost in a minimization problem.	Compute the coordinates of the point touched by the iso-profit (or cost) line on the feasible region. Compute the profit or cost.

SPECIAL CASES IN LINEAR PROGRAMMING:

Alternative (or Multiple) Optimal Solutions:

We have seen that the optimal solution of any linear programming problem occurs at an extreme point of the feasible region and that the solution is unique, i.e., no other solution yields the same value of the objective function. However, in certain cases, a given LP problem may have more than one solution yielding the same optimal objective function value. Each of such optimal solutions is termed as alternative optimal solution.

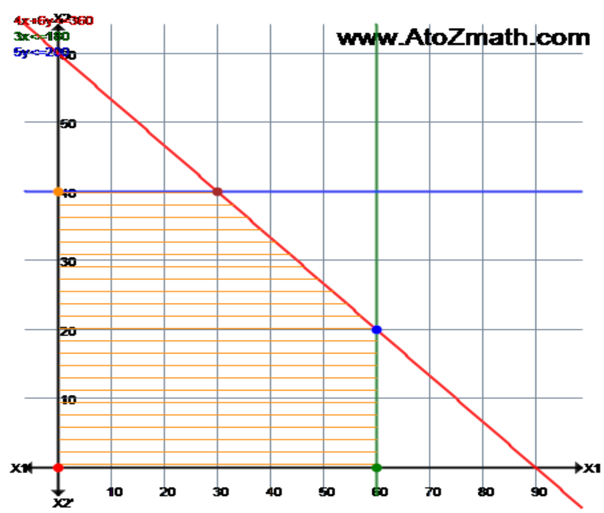
There are two conditions that should be satisfied for an alternative optimal solution to exist:

(i) The slope of the objective function should be the same as that of the constraint forming the boundary of the feasible solutions region, and

(ii) The constraint should form a boundary on the feasible region in the direction of optimal movement of the objective function. In other words, the constraint should be an active constraint.

Note: The constraint is said to be active (binding or tight), if at the point of optimality, the left-hand side of a constraint equals the right-hand side. In other words, an equality constraint is always active, and inequality constraint may or may not be active.

Find solution using graphical method
MAX Z = 15X1 + 10X2
subject to
4X1 + 6X2 <= 360
3X1 <= 180
5X2 <= 200
and X1,X2 >= 0



Solution:

Problem is

MAX $Z_x = 15X1 + 10X2$

subject to

$$\begin{aligned} 4X1 + 6X2 &\le 360 \\ 3X1 &\le 180 \\ 5X2 &\le 200 \end{aligned}$$

and $X1, X2 \ge 0$;

Hint to draw constraints

1. To draw constraint $4X1+6X2 \leq 360 \rightarrow (1)$

Treat it as $4X1+6X2=360$
When $X1=0$ then $X2=?$
$\Rightarrow 4(0) + 6X2=360 \Rightarrow X2=60$

When $X2=0$ then $X1=?$
$\Rightarrow 4X1+6(0)=360 \Rightarrow X1=90$
(x1, x2) = (90,60)

2. To draw constraint $3X1 \leq 180 \rightarrow (2)$
Treat it as $3X1=180$
$\Rightarrow X1=180/3=60$
Here line is parallel to Y-axis

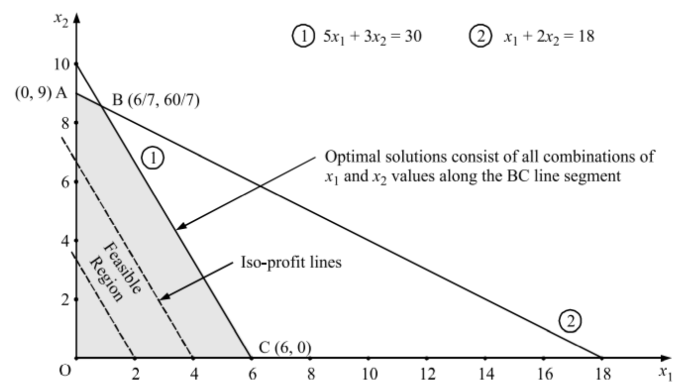
3. To draw constraint $5X2 \leq 200 \rightarrow (3)$
Treat it as $5X2=200$
$\Rightarrow X2=200/5=40$
Here line is parallel to X-axis

Extreme Point Coordinates (X1,X2)	Objective function value $Z=15X1+10X2$
O (0,0)	15(0) +10(0) = 0
A (60,0)	15(60) + 10(0) = 900
B (60,20)	15(60) + 10(20) = 1100
C (30,40)	15(30) + 10(40) = 850
D (0,40)	15(0) + 10(40) = 400

The value of the objective function at each of these extreme points is as follows:
The maximum value of the objective function $Z=1100$ occurs at the extreme point (60,20).
Hence, the optimal solution to the given LP problem is : $X1=60, X2=20$ and max $Z=1100$.

Example:

Use the graphical method to solve the following LP problem.
Maximize Z = 10x1 + 6x2
subject to the constraints
(i) 5x1 + 3x2 ≤ 30, (ii) x1 + 2x2 ≤ 18 and x1, x2 ≥ 0.



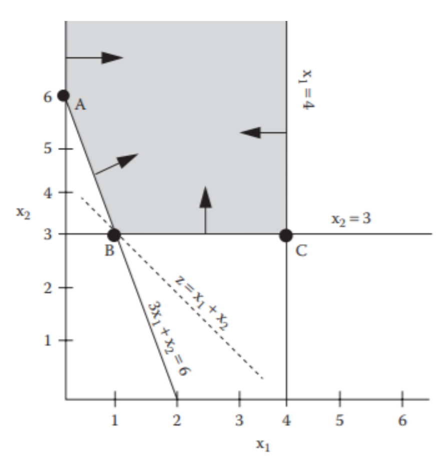
Solution:

The constraints are plotted on a graph by first treating these as equations and then their inequality signs are used to identify feasible region (shaded area) as shown in Fig.
The extreme points of the region are O, A, B and C.
Since objective function (iso-profit line) is parallel to the line BC (first constraint: 5x1 + 3x2 = 30), which also falls on the boundary of the feasible region. Thus, as the iso-profit line moves away from the origin, it coincides with the line BC of the constraint equation line that falls on the boundary of the feasible region.
This implies that an optimal solution of LP problem can be obtained at any point between B and C including extreme points B and C on the same line. Therefore, several combinations of values of x1 and x2 give the same value of objective function.
The value of variables x1 and x2 obtained at extreme points B and C should only be considered to establish that the solution to an LP problem will always lie at an extreme point of the feasible region.
The value of objective function at each of the extreme points is shown in Table
Since value (maximum) of objective function, Z = 60 at two different extreme points B and C is same, therefore two alternative solutions: x1 = 6/7, x2 = 60/7 and x1= 6, x2 = 0 exist

Extreme Point	*Coordinates* (x_1, x_2)	*Objective Function Value* $Z = 10x_1 + 6x_2$
O	(0, 0)	10(0) + 6(0) = 0
A	(0, 9)	10(0) + 6(9) = 54
B	(6/7, 60/7)	10(6/7) + 6(60/7) = 60
C	(6, 0)	10(6) + 6(0) = 60

Unbounded Solution:

Sometimes an LP problem may have an infinite solution. Such a solution is referred as an unbounded solution. It happens when value of certain decision variables and the value of the objective function (maximization case) are permitted to increase infinitely, without violating the feasibility condition. It may be noted that there is a difference between unbounded feasible region and unbounded solution to a LP problem. It is possible that for a particular LP problem the feasible region may be unbounded but LP problem solution may not be unbounded, i.e. an unbounded feasible region may yield some definite value of the objective function. In general, an unbounded LP problem solution exists due to improper formulation of the real-life problem.



Example 8:

Minimize Z = x1 + x2

St:

3x1 + x2 ≥ 30,

x2 ≥ 8

x1 ≤ 4

and x1, x2 ≥ 0.

The shaded area in Figure denotes the feasible region, which in this case is unbounded.

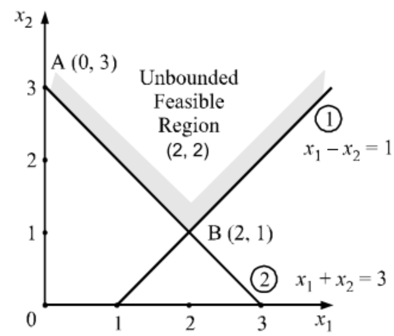
The minimal solution must occur at one of the extreme points A, B, or C. The objective function x1 + x2, with a slope of −1, is tangent to the feasible region at extreme point B. Therefore, the optimal solution occurs at x1 = 1 and x2 = 3, and the optimal objective function value at that point is z* = 4.

Example: Use graphical method to solve the following LP problem:

Maximize Z = 3x1 + 2x2

subject to the constraints

(i) x1 – x2 ≥ 1 (ii) x1 + x2 ≥ 3 and x1, x2 ≥ 0.



Solution: Plot on a graph each constraint by first treating it as a linear equation. Then use the inequality sign of each constraint to mark the feasible region (shaded area) as shown in Fig.

It may be noted that the shaded region (solution space) is unbounded from above. The two corners of the region are, A = (0, 3) and B = (2, 1). The value of the objective function at these corners is: Z(A) = 6 and Z(B) = 8.

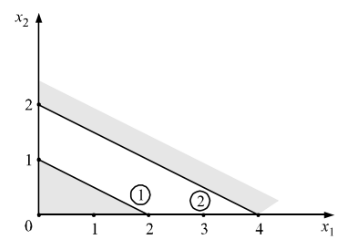
Since the given LP problem is of maximization, there exist a number of points in the shaded region

for which the value of the objective function is more than 8. For example, the point (2, 2) lies in the region and the objective function value at this point is 10 which is more than 8. Thus, as value of variables x1 and x2 increases arbitrarily large, the value of Z also starts increasing.

Hence, the LP problem has an unbounded solution

Infeasible Solution

An infeasible solution to an LP problem arises when there is no solution that satisfies all the constraints simultaneously. This happens when there is no unique (single) feasible region. This situation arises when a LP model that has conflicting constraints. Any point lying outside the feasible region violates one or more of the given constraints.



Example: Use the graphical method to solve the following LP problem:

Maximize Z = 6x1 – 4x2

subject to the constraints

(i) 2x1 + 4x2 ≤ 4 (ii) 4x1 + 8x2 ≥ 16 and x1, x2 ≥ 0

Solution:

The constraints are plotted on graph as usual as shown in Fig. 3.25. Since there is no unique feasible solution space, therefore a unique set of values of variables x1 and x2 that satisfy all the constraints cannot be determined. Hence, there is no feasible solution to this LP problem because of the conflicting constraints.

SIMPLEX PROCEDURE OF LPP:

Formulation of LPP
Convert the given real time situation in to
Objective Function: Max/ Min $Z = C_1 X_1 + C_2 X_2 + C_3 X_3 + \ldots$
Constraints:

$$a_{11} X_1 + a_{12} X_2 + a_{13} X_3 \leq \text{ or } \geq b_1$$
$$a_{21} X_1 + a_{22} X_2 + a_{23} X_3 \leq \text{ or } \geq b_2$$
$$a_{31} X_1 + a_{32} X_2 + a_{33} X_3 \leq \text{ or } \geq b_3$$

Non-negative restrictions: $X_1, X_2, X_3 \geq 0$
Z - Objective
C_i - Cost coefficient
a_i -Technological Coefficient
X_i - Decision variable
b_i - Availability or constant of the constraint.

Check: Verifying the suitability of applying Simplex method

- → Check that the Objective Function is Maximization or not
  - → If yes, then proceed with the procedure
  - → If no, then make it Maximise by multiplying both sides with '-1' then proceed
- → Check whether the availabilities (i.e., all) of constraint equations are positive or not.
  - → If yes, then proceed with the procedure
  - → If no, then make it positive by multiplying both sides with '-1' then proceed
- → Thereafter check whether all the constraint equations have less than or equal to ($\leq$) in-equality or not. If not, Simplex method cannot be adopted.

Step 1: Convert the inequalities into equality

- → Adding Slack variables

Add slack variables to convert all in-equality equations into equality equations. I.e., When the constraint equation has less than or equal to ($\leq$) in-equality, add a positive slack variable (S_i) to the LHS which represent left-over or unused capacity. Thereby the in-equality symbol of the constraint equations converts to equal (=) type symbol between LHS and RHS of the constraint equation.

Step 2: Hence, the modified objective function and constraint equations will be written as.

Objective Function:

$$\text{Max/ Min } Z = C_1 X_1 + C_2 X_2 + C_3 X_3 + 0{*}S_1 + 0{*}S_2 + 0{*} S_3$$

Subjected to
Constraints:

$$a_{11} X_1 + a_{12} X_2 + a_{13} X_3 + 1{*}S_1 + 0{*}S_2 + 0{*} S_3 = b_1$$
$$a_{21} X_1 + a_{22} X_2 + a_{23} X_3 + 0{*}S_1 + 1{*}S_2 + 0{*} S_3 = b_2$$
$$a_{31} X_1 + a_{32} X_2 + a_{33} X_3 + 0{*}S_1 + 0{*}S_2 + 1{*} S_3 = b_3$$

Non-negative restrictions: $X_1, X_2, X_3, S_1, S_2, S_3, b_1, b_2, b_3 \geq 0$

Step 3: Simplex problem Tabulation for finding the Initial Basic Feasible Solution

	Cost- Coefficient C_j→			C1	C2	C3	C4	C5	C6	
	Basic Variables (B.V)	CB	XB	Variables (Decision + Slack) →						Replacement Ratio=X_B/X_K For all X_k>0
				X1	X2	X3	S1	S2	S3	
				Body Matrix			Identity Matrix			
Row 1→	S1		b_1	a_{11}	a_{12}	a_{13}	1	0	0	
Row 2→	S2		b_2	a_{21}	a_{22}	a_{23}	0	1	0	
Row 3→	S3		b_3	a_{31}	a_{32}	a_{33}	0	0	1	
Row 4→	Non-Basic variables X1=X2=X3=0	Z=∑CB.XB		Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	

Simplex method

Find solution using Simplex method

MAX Z = 3x1 + 5x2 + 4x3

subject to

$2x1 + 3x2 \le 8$

$2x2 + 5x3 \le 10$

$3x1 + 2x2 + 4x3 \le 15$

and $x1, x2, x3 \ge 0$;

Solution:

Problem is

Max $Z = 3x1 + 5x2 + 4x3$

subject to

$2x1 + 3x2 \le 8$

$2x2 + 5x3 \le 10$

$3x1 + 2x2 + 4x3 \le 15$

and $x1, x2, x3 \ge 0$;

The problem is converted to canonical form by adding slack, surplus and artificial variables as appropiate

1. As the constraint-1 is of type '≤' we should add slack variable $S1$
2. As the constraint-2 is of type '≤' we should add slack variable $S2$
3. As the constraint-3 is of type '≤' we should add slack variable $S3$

After introducing slack variables

Max $Z = 3x1 + 5x2 + 4x3 + 0S1 + 0S2 + 0S3$

subject to

$2x1 + 3x2 + S1 = 8$

$2x2 + 5x3 + S2 = 10$

$3x1 + 2x2 + 4x3 + S3 = 15$

and $x1, x2, x3, S1, S2, S3 \ge 0$

Iteration-1		Cj	3	5	4	0	0	0	
BV	CB	XB	$x1$	$x2$	$x3$	$S1$	$S2$	$S3$	Min Ratio = $XB/x2$
$S1$	0	8	2	(3)	0	1	0	0	8/3=2.6667→
$S2$	0	10	0	2	5	0	1	0	10/2=5
$S3$	0	15	3	2	4	0	0	1	15/2=7.5
		$Zj-Cj$	-3	-5↑	-4	0	0	0	

Most Negative element in $Zj-Cj$ is -5 and its column index is 2. So, the entering variable is $x2$.

Minimum ratio is 2.6667 and its row index is 1. So, the leaving basis variable is $S1$.

∴ The pivot element is 3.

Entering $=x2$, Departing $=S1$, Key Element $=3$

R1(new)=R1(old)÷3

R2(new)=R2(old) - 2R1(new)

R3(new)=R3(old) - 2R1(new)

Iteration-2		Cj	3	5	4	0	0	0	
BV	CB	XB	$x1$	$x2$	$x3$	$S1$	$S2$	$S3$	Min Ratio = $XB/x3$
$x2$	5	2.6667	0.6667	1	0	0.3333	0	0	---

S2	0	4.6667	-1.3333	0	(5)	-0.6667	1	0	4.6667/5=0.9333→
S3	0	9.6667	1.6667	0	4	-0.6667	0	1	9.6667/4=2.4167
Z=13.3333		*Zj-Cj*	0.3333	0	-4↑	1.6667	0	0	

Negative minimum *Zj-Cj* is -4 and its column index is 3. So, the entering variable is *x3*.
Minimum ratio is 0.9333 and its row index is 2. So, the leaving basis variable is *S2*.
∴ The pivot element is 5.

Entering =*x3*, Departing =*S2*, Key Element =5
R2(new)=R2(old)÷5
R1(new)=R1(old)
R3(new)=R3(old) - 4R2(new)

Iteration-3		*Cj*	3	5	4	0	0	0	
BV	*CB*	*XB*	*x1*	*x2*	*x3*	*S1*	*S2*	*S3*	Min Ratio = *XB*/*x1*
x2	5	2.6667	0.6667	1	0	0.3333	0	0	2.6667/0.6667=4
x3	4	0.9333	-0.2667	0	1	-0.1333	0.2	0	---
S3	0	5.9333	(2.7333)	0	0	-0.1333	-0.8	1	5.9333/2.7333=2.1707→
Z=17.0667		*Zj-Cj*	-0.7333↑	0	0	1.1333	0.8	0	

Negative minimum *Zj-Cj* is -0.7333 and its column index is 1. So, the entering variable is *x1*.
Minimum ratio is 2.1707 and its row index is 3. So, the leaving basis variable is *S3*.
∴ The pivot element is 2.7333.

Entering =*x1*, Departing =*S3*, Key Element =2.7333
R3(new)=R3(old)÷2.7333
R1(new)=R1(old) - 0.6667R3(new)
R2(new)=R2(old) + 0.2667R3(new)

Iteration-4		*Cj*	3	5	4	0	0	0	
BV	*CB*	*XB*	*x1*	*x2*	*x3*	*S1*	*S2*	*S3*	*RR* = *XB*/ *XK*
x2	5	1.2195	0	1	0	0.3659	0.1951	-0.2439	
x3	4	1.5122	0	0	1	-0.1463	0.122	0.0976	
x1	3	2.1707	1	0	0	-0.0488	-0.2927	0.3659	
Z=18.6585		*Zj*	**3**	**5**	**4**	**1.0976**	**0.5854**	**0.2683**	
		Zj-Cj	0	0	0	1.0976	0.5854	0.2683	

Since all *Zj-Cj*≥0
Hence, optimal solution is arrived with value of variables as: *x1*=2.1707 ,*x2*=1.2195, *x3*=1.5122 and
Max *Z*=18.6585

Find solution using Simplex method
MAX Z = 3x1 + 5x2 + 4x3
subject to
2x1 + 3x2 <= 8
2x2 + 5x3 <= 10
3x1 + 2x2 + 4x3 <= 15
and x1,x2,x3 >= 0

Solution:
Problem is

Max $Z = 3x1 + 5x2 + 4x3$
subject to

$$
\begin{aligned}
2x1 + 3x2 \quad &\leq 8 \\
2x2 + 5x3 &\leq 10 \\
3x1 + 2x2 + 4x3 &\leq 15
\end{aligned}
$$

and $x1, x2, x3 \geq 0$;

The problem is converted to canonical form by adding slack, surplus and artificial variables as appropriate
1. As the constraint-1 is of type '≤' we should add slack variable $S1$
2. As the constraint-2 is of type '≤' we should add slack variable $S2$
3. As the constraint-3 is of type '≤' we should add slack variable $S3$

After introducing slack variables
Max $Z = 3x1 + 5x2 + 4x3 + 0S1 + 0S2 + 0S3$
subject to

$$
\begin{aligned}
2x1 + 3x2 + S1 &= 8 \\
2x2 + 5x3 + S2 &= 10 \\
3x1 + 2x2 + 4x3 + S3 &= 15
\end{aligned}
$$

and $x1, x2, x3, S1, S2, S3 \geq 0$

Iteration-1		Cj	3	5	4	0	0	0	
BV	***CB***	***XB***	***x1***	***x2***	***x3***	***S1***	***S2***	***S3***	**RR = *XB/X2***
S1	0	8	2	(3)	0	1	0	0	8/3=2.6667→
$S2$	0	10	0	2	5	0	1	0	10/2=5
$S3$	0	15	3	2	4	0	0	1	15/2=7.5
		Zj-Cj	-3	-5↑	-4	0	0	0	

Negative minimum Zj-Cj is -5 and its column index is 2. So, the entering variable is $x2$.
Minimum ratio is 2.6667 and its row index is 1. So, the leaving basis variable is $S1$.
∴ The pivot element is 3.

Entering $=x2$, Departing $=S1$, Key Element $=3$
R1(new)=R1(old)÷3
R2(new)=R2(old) - 2R1(new)
R3(new)=R3(old) - 2R1(new)

Iteration-2		Cj	3	5	4	0	0	0	
BV	***CB***	***XB***	***x1***	***x2***	***x3***	***S1***	***S2***	***S3***	**RR = *XB/X3***
$x2$	5	8/3	2/3	1	0	1/3	0	0	---
S2	0	14/3	-4/3	0	(5)	-2/3	1	0	(14/3)/5=14/15=0.9333→
$S3$	0	29/3	5/3	0	4	-2/3	0	1	(29/3)/4=29/12=2.4167
***Z*=40/3**		Zj-Cj	1/3	0	-4↑	5/3	0	0	

Negative minimum Zj-Cj is -4 and its column index is 3. So, the entering variable is $x3$.
Minimum ratio is 0.9333 and its row index is 2. So, the leaving basis variable is $S2$.
∴ The pivot element is 5.
Entering $=x3$, Departing $=S2$, Key Element $=5$

R2(new)=R2(old)÷5
R1(new)=R1(old)
R3(new)=R3(old) - 4R2(new)

Iteration-3		C_j	3	5	4	0	0	0	
BV	***CB***	***XB***	***x1***	***x2***	***x3***	***S1***	***S2***	***S3***	**RR = *XB/X1***
$x2$	5	8/3	2/3	1	0	1/3	0	0	(8/3)/(2/3)=4
$x3$	4	14/15	-4/15	0	1	-2/15	1/5	0	---
$S3$	0	89/15	(41/15)	0	0	-2/15	-4/5	1	(89/15)/(41/15)=89/41=2.1707→
Z=25615		Zj-Cj	-11/15↑	0	0	17/15	4/5	0	

Negative minimum Zj-Cj is -11/15 and its column index is 1. So, the entering variable is $x1$.
Minimum ratio is 2.1707 and its row index is 3. So, the leaving basis variable is $S3$.
∴ The pivot element is 41/15.
Entering =$x1$, Departing =$S3$, Key Element =41/15

R3(new)=R3(old) ×(15/41)
R1(new)=R1(old) – (2/3) R3(new)
R2(new)=R2(old) + (4/15) R3(new)

Iteration-4		C_j	3	5	4	0	0	0	
BV	***CB***	***XB***	***x1***	***x2***	***x3***	***S1***	***S2***	***S3***	**RR**
$x2$	5	50/41	0	1	0	15/41	8/41	-10/41	
$x3$	4	62/41	0	0	1	-6/41	5/41	4/41	
$x1$	3	89/41	1	0	0	-2/41	-12/41	15/41	
Z=765/41		Zj-Cj	0	0	0	45/41	24/41	11/41	

Since all Zj-Cj≥0
Hence, optimal solution is arrived with value of variables as : $x1$=89/41, $x2$=50/41, $x3$=62/41 and Max Z=765/41

Unbounded Solution Example: LPP

1. Maximize Z= 5x1 + 4x2
   subject to:
   x1 ≤ 7
   x1 - x2 ≤ 8
   x1, x2 ≥ 0.

2. Find solution using Two-Phase method
   MAX Z = -2x1 - 2x2
   Subject to
   x1 + x2 ≥ 2
   x2 ≤ 2
   and x1, x2 ≥0

TWO PHASE METHOD:

Find solution using Two-Phase method

Max Z= 3 $x1$ - $x2$ + 2 $x3$
subject to
$x1$ + 3 $x2$ + $x3$ ≤ 5
2 $x1$ - $x2$ + $x3$ ≥ 2
4 $x1$ + 3 $x2$ - 2 $x3$ = 5
and $x1$,$x2$,$x3$≥0;
Sol:
Max Z= 3 $x1$ - $x2$ + 2 $x3$

subject to

$x1 + 3\,x2 + \ \ x3 \le 5$

$2\,x1 - \ \ x2 + \ \ x3 \ge 2$

$4\,x1 + 3\,x2 - \ 2\,x3 = 5$

and $x1, x2, x3 \ge 0$;

-->Phase-1<--

The problem is converted to canonical form by adding slack, surplus and artificial variables as appropriate

1. As the constraint-1 is of type '≤' we should add slack variable $S1$
2. As the constraint-2 is of type '≥' we should subtract surplus variable $S2$ and add artificial variable $A1$
3. As the constraint-3 is of type '=' we should add artificial variable $A2$

Note: the coefficients of X, S are zeros and of A are -1 in Objective Function.

After introducing slack, surplus, artificial variables

Max $Z = \ - A1 - A2$

subject to

$x1 + 3\,x2 + \ x3 + S1 + S2 = 5$

$2\,x1 - \ x2 + \ x3 \ - \ S2 + A1 = 2$

$4\,x1 + 3\,x2 - \ 2\,x3 \ + A2 = 5$

and $x1, x2, x3, S1, S2, A1, A2 \ge 0$

Iteration-1		Cj	0	0	0	0	0	-1	-1	
BV	**CB**	**XB**	**x1**	**x2**	**x3**	**S1**	**S2**	**A1**	**A2**	RR=XB/XK **for All XK>0**
S1	0	5	1	3	1	1	0	0	0	5/1=5
A1	-1	2	(2)	-1	1	0	-1	1	0	2/2=1→
A2	-1	5	4	3	-2	0	0	0	1	5/4=1.25
Z=-7		Zj-Cj	-6↑	-2	1	0	1	0	0	

Negative minimum $Zj\text{-}Cj$ is -6 and its column index is 1. So, the entering variable is $x1$.
Minimum ratio is 1 and its row index is 2. So, the leaving basis variable is $A1$.
∴ The pivot element is 2.

Entering $=x1$, Departing $=A1$, Key Element $=2$
$R2$(new)=$R2$(old) ÷2
$R1$(new)=$R1$(old) - $R2$(new)
$R3$(new)=$R3$(old) - 4$R2$(new)

Iteration-2		*Cj*	0	0	0	0	0	-1	
BV	***CB***	***XB***	***x1***	***x2***	***x3***	***S1***	***S2***	***A2***	RR=XB/XK **for All XK>0**
S1	0	4	0	7/2	1/2	1	12	0	4/7/2=8/7=1.1429
x1	0	1	1	-1/2	1/2	0	-1/2	0	---
A2	-1	1	0	(5)	-4	0	2	1	1/5=0.2→
***Z*=-1**		*Zj-Cj*	0	-5↑	4	0	-2	0	

Negative minimum $Zj\text{-}Cj$ is -5 and its column index is 2. So, the entering variable is $x2$.
Minimum ratio is 0.2 and its row index is 3. So, the leaving basis variable is $A2$.
∴ The Key element is 5.
Entering $=x2$, Departing $=A2$, Key Element $=5$

$R3(\text{new})=R3(\text{old}) \div 5$
$R1(\text{new})=R1(\text{old}) - (7/2)R3(\text{new})$
$R2(\text{new})=R2(\text{old}) + (1/2)R3(\text{new})$

Iteration-3		*Cj*	0	0	0	0	0	
BV	***CB***	***XB***	***x*1**	***x*2**	***x*3**	***S*1**	***S*2**	RR=XB/XK **for All XK>0**
*S*1	0	33/10	0	0	33/10	1	-9/10	
*x*1	0	11/10	1	0	1/10	0	-3/10	
*x*2	0	1/5	0	1	-4/5	0	2/5	
***Z*=0**		*Zj-Cj*	0	0	0	0	0	

Since all $Zj\text{-}Cj \geq 0$
Hence, optimal solution is arrived with value of variables as : $x1=1110, x2=15, x3=0$ and Max $Z=0$

-->Phase-2<--

we eliminate the artificial variables and change the objective function for the original, Max $Z=3x1-x2+2x3+0S1+0S2$

Iteration-1		*Cj*	3	-1	2	0	0	
BV	***CB***	***XB***	***x*1**	***x*2**	***x*3**	***S*1**	***S*2**	RR=XB/XK **for All XK>0**
*S*1	0	33/10	0	0	33/10	1	-9/10	---
*x*1	3	11/10	1	0	1/10	0	-3/10	---
*x*2	-1	1/5	0	1	-4/5	0	(2/5)	1/5/2/5=1/2=0.5→
***Z*=31/10**		*Zj-Cj*	0	0	-9/10	0	-13/10↑	

Negative minimum $Zj\text{-}Cj$ is -13/10 and its column index is 5. So, the entering variable is $S2$.
Minimum ratio is 0.5 and its row index is 3. So, the leaving basis variable is $x2$.
∴ The pivot element is 2/5.

Entering $=S2$, Departing $=x2$, Key Element $=2/5$
$R3(\text{new})=R3(\text{old}) \times 5/2$
$R1(\text{new})=R1(\text{old}) + 9/10R3(\text{new})$
$R2(\text{new})=R2(\text{old}) + 3/10R3(\text{new})$

Iteration-2		*Cj*	3	-1	2	0	0	
B	***CB***	***XB***	***x*1**	***x*2**	***x*3**	***S*1**	***S*2**	RR=XB/XK **for All XK>0**
*S*1	0	15/4	0	9/4	(3/2)	1	0	15/4/3/2=5/2=2.5→
*x*1	3	5/4	1	3/4	-1/2	0	0	---
*S*2	0	1/2	0	5/2	-2	0	1	---
***Z*=15/4**		*Zj-Cj*	0	13/4	-7/2↑	0	0	

Negative minimum $Zj\text{-}Cj$ is -7/2 and its column index is 3. So, the entering variable is $x3$.
Minimum ratio is 2.5 and its row index is 1. So, the leaving basis variable is $S1$.
∴ The pivot element is 3/2.

Entering $=x3$, Departing $=S1$, Key Element $=3/2$
$R1(\text{new})=R1(\text{old}) \times 2/3$
$R2(\text{new})=R2(\text{old}) + 1/2R1(\text{new})$

$R3$(new)=$R3$(old) + 2$R1$(new)

Iteration-3		C_j	3	-1	2	0	0	
B	*CB*	*XB*	*x1*	*x2*	*x3*	*S1*	*S2*	RR=XB/XK **for All XK>0**
$x3$	2	5/2	0	3/2	1	2/3	0	
$x1$	3	5/2	1	3/2	0	1/3	0	
$S2$	0	11/2	0	11/2	0	4/3	1	
Z=25/2		Z_j-C_j	0	17/2	0	7/3	0	

Since all Z_j-C_j≥0 optimal solution is arrived with value of variables as : $x1$=52,$x2$=0,$x3$=52 and Max Z=25/2

Find solution using Two-Phase method
MAX Z = 3X1 + 2X2 + 2X3
subject to
5X1 + 7X2 + 4X3 <= 7
-4X1 + 7X2 + 5X3 >= -2
3X1 + 4X2 - 6X3 >= 29/7
and X1,X2,X3 >= 0

Solution:
Problem is

Max $Z = 3X1 + 2X2 + 2X3$

subject to

$5X1 + 7X2 + 4X3 \leq 7$

$-4X1 + 7X2 + 5X3 \geq -2$

Here $b2$ = -2 < 0, so multiply this constraint by -1 to make $b2$ > 0.

$4X1 - 7X2 - 5X3 \leq 2$

$3X1 + 4X2 - 6X3 \geq 29/7$

and $X1,X2,X3 \geq 0$;

-->Phase-1<--

The problem is converted to canonical form by adding slack, surplus and artificial variables as appropiate

1. As the constraint-1 is of type '≤' we should add slack variable $S1$
2. As the constraint-2 is of type '≤' we should add slack variable $S2$
3. As the constraint-3 is of type '≥' we should subtract surplus variable $S3$ and add artificial variable $A1$

After introducing slack, surplus, artificial variables

Max $Z = - A1$

subject to

$5X1 + 7X2 + 4X3 + S1 = 7$

$4X1 - 7X2 - 5X3 + S2 = 2$

$3X1 + 4X2 - 6X3 - S3 + A1 = 297$

and $X1,X2,X3,S1,S2,S3,A1 \geq 0$

Iteration-1		C_j	0	0	0	0	0	0	-1	
BV	*CB*	*XB*	*X1*	*X2*	*X3*	*S1*	*S2*	*S3*	*A1*	RR= *XB/X2*
S1	0	7	5	(7)	4	1	0	0	0	7/7=1→
S2	0	2	4	-7	-5	0	1	0	0	---

$A1$	-1	29/7	3	4	-6	0	0	-1	1	29/7/4=29/28=1.04
Z=-29/7		Zj-Cj	-3	-4↑	6	0	0	1	0	

Negative minimum Zj-Cj is -4 and its column index is 2. So, the entering variable is $X2$.
Minimum ratio is 1 and its row index is 1. So, the leaving basis variable is $S1$.
∴ The pivot element is 7.

Entering =$X2$, Departing =$S1$, Key Element =7
$R1$(new)=$R1$(old) ÷7
$R2$(new)=$R2$(old) + 7$R1$(new)
$R3$(new)=$R3$(old) - 4$R1$(new)

Iteration-2		Cj	0	0	0	0	0	0	-1	
BV	***CB***	***XB***	***X1***	***X2***	***X3***	***S1***	***S2***	***S3***	***A1***	**RR=*XB/X1***
$X2$	0	1	57	1	4/7	1/7	0	0	0	1/5/7=7/5=1.4
$S2$	0	9	9	0	-1	1	1	0	0	9/9=1
$A1$	-1	1/7	(1/7)	0	-58/7	-4/7	0	-1	1	1/7/1/7=1→
Z=-17		Zj-Cj	-17↑	0	58/7	4/7	0	1	0	

Negative minimum Zj-Cj is -1/7 and its column index is 1. So, the entering variable is $X1$.
Minimum ratio is 1 and its row index is 3. So, the leaving basis variable is $A1$.
∴ The pivot element is 1/7.

Entering =$X1$, Departing =$A1$, Key Element =1/7
$R3$(new)=$R3$(old) ×7
$R1$(new)=$R1$(old) - 57$R3$(new)
$R2$(new)=$R2$(old) - 9$R3$(new)

Iteration-3		Cj	0	0	0	0	0	0	
BV	***CB***	***XB***	***X1***	***X2***	***X3***	***S1***	***S2***	***S3***	**RR**
$X2$	0	2/7	0	1	42	3	0	5	
$S2$	0	0	0	0	521	37	1	63	
$X1$	0	1	1	0	-58	-4	0	-7	
Z=0		Zj-Cj	0	0	0	0	0	0	

Since all Zj-Cj≥0 Hence, optimal solution is arrived with value of variables as : $X1$=1,$X2$=27,$X3$=0 and Max Z=0

-->Phase-2<--

we eliminate the artificial variables and change the objective function for the original,
Max Z=3$X1$+2$X2$+2$X3$+0$S1$+0$S2$+0$S3$

Iteration-1		Cj	3	2	2	0	0	0	
BV	***CB***	***XB***	***X1***	***X2***	***X3***	***S1***	***S2***	***S3***	**RR=*XB/X3***
$X2$	2	27	0	1	42	3	0	5	2/7/42=1/147=0.01
$S2$	0	0	0	0	(521)	37	1	63	0/521=0→

*X*1	3	1	1	0	-58	-4	0	-7	---
Z=257		*Zj-Cj*	0	0	-92↑	-6	0	-11	

Negative minimum *Zj-Cj* is -92 and its column index is 3. So, the entering variable is *X*3.
Minimum ratio is 0 and its row index is 2. So, the leaving basis variable is *S*2.
∴ The pivot element is 521.

Entering =*X*3, Departing =*S*2, Key Element =521
*R*2(new)=*R*2(old) ÷521
*R*1(new)=*R*1(old) - 42*R*2(new)
*R*3(new)=*R*3(old) + 58*R*2(new)

Iteration-2		*Cj*	3	2	2	0	0	0	
BV	***CB***	***XB***	***X*1**	***X*2**	***X*3**	***S*1**	***S*2**	***S*3**	**RR**
*X*2	2	27	0	1	0	9/521	-42/521	-41/521	
*X*3	2	0	0	0	1	37/521	1/521	63/521	
*X*1	3	1	1	0	0	62/521	58/521	7/521	
***Z*=25/7**		*Zj-Cj*	0	0	0	0.53	0.18	0.12	

Since all *Zj-Cj*≥0 Hence, optimal solution is arrived with value of variables as : *X*1=1,*X*2=27,*X*3=0 and
Max *Z*=25/7

Big-M method:

Computational Procedure of Big - M Method, Charne's Penalty Method

If an LP has any > or = constraints, the Big M method or the two-phase simplex method may be used to solve the problem.

The Big M method is a version of the Simplex Algorithm that first finds a best feasible solution by adding "artificial" variables to the problem. The objective function of the original LP must, of course, be modified to ensure that the artificial variables are all equal to 0 at the conclusion of the simplex algorithm.

The iterative procedure of the algorithm is given below.

Step-1
Modify the constraints so that the RHS of each constraint is non-negative (This requires that each constraint with a negative RHS be multiplied by -1. Remember that if any negative number multiplies an inequality, the direction of the inequality is reversed). After modification, identify each constraint as a <, >, or = constraint.

Step-2
Convert each inequality constraint to standard form (If a constraint is a ≤ constraint, then add a slack variable '**Si**'; and if any constraint is a ≥ constraint, then subtract an excess variable '**Si**' , known as surplus variable).

Step-3
Add an artificial variable '**A**' to the constraints identified as '≥' or with '=' constraints at the end of Step2.
Also add the sign restriction Ai ≥ 0.

Step-4
Let '**M**' denote a very large positive number. Assign a high price (per unit penalty) to the artificial variables in the objective function. These large price will be designated through '**-M**' for maximization problems ('**+M**' for minimizing problem), where M > 0.

Step-5

Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex.

Now we solve the transformed problem by the simplex (In choosing the entering variable, remember that M is a very large positive number). If all artificial variables are equal to zero in the optimal solution, we have found the optimal solution to the original problem. If any artificial variables are positive in the optimal solution, the original problem is infeasible!!!

Note:

1. If at least one artificial variable is present in the basis with zero value, in such a case the current optimum basic feasible solution is a degenerate solution.

2. If at least one artificial variable is present in the basis with a positive value, then in such a case, the given LPP does not have an optimum basic feasible solution. The given problem is said to have a pseudo-optimum basic feasible solution.

Find solution using Simplex method (Big-M method)

MAX Z = 5x1 + 10x2 + 8x3

subject to

3x1 + 5x2 + 2x3 <= 60

4x1 + 4x2 + 4x3 >= 72

2x1 + 4x2 + 5x3 = 100

and x1, x2,x3 >= 0

Solution:

Problem is

Max $Z = 5x_1 + 10x_2 + 8x_3$

subject to

$3x_1 + 5x_2 + 2x_3 \leq 60$

$4x_1 + 4x_2 + 4x_3 \geq 72$

$2x_1 + 4x_2 + 5x_3 = 100$

and $x_1, x_2, x_3 \geq 0$;

The problem is converted to canonical form by adding slack, surplus and artificial variables as appropriate

1. As the constraint-1 is of type '$\leq$' we should add slack variable S_1
2. As the constraint-2 is of type '$\geq$' we should subtract surplus variable S_2 and add artificial variable A_1
3. As the constraint-3 is of type '$=$' we should add artificial variable A_2

After introducing slack, surplus, artificial variables

Max $Z = 5x_1 + 10x_2 + 8x_3 + 0S_1 + 0S_2 - MA_1 - MA_2$

subject to

$$
\begin{aligned}
3x_1 + 5x_2 + 2x_3 + S_1 &= 60 \\
4x_1 + 4x_2 + 4x_3 - S_2 + A_1 &= 72 \\
2x_1 + 4x_2 + 5x_3 + A_2 &= 100
\end{aligned}
$$

and $x_1, x_2, x_3, S_1, S_2, A_1, A_2 \geq 0$

Iteration-1		C_j	5	10	8	0	0	$-M$	$-M$	
B	C_B	X_B	x_1	x_2	x_3	S_1	S_2	A_1	A_2	RR=X_B/x_3
S_1	0	60	3	5	2	1	0	0	0	60/2=30
A_1	$-M$	72	4	4	(4)	0	-1	1	0	72/4=18→
A_2	$-M$	100	2	4	5	0	0	0	1	100/5=20
$Z=-172M$		Z_j-C_j	$-6M-5$	$-8M-10$	$-9M-8$↑	0	M	0	0	

Negative minimum Z_j-C_j is $-9M-8$ and its column index is 3. So, the entering variable is x_3.

Minimum ratio is 18 and its row index is 2. So, the leaving basis variable is A_1.

∴ The pivot element is 4.

Entering $=x_3$, Departing $=A_1$, Key Element $=4$

R_2(new)$=R_2$(old) $\div 4$

$R1$(new)=$R1$(old) - 2$R2$(new)
$R3$(new)=$R3$(old) - 5$R2$(new)

Iteration-2		Cj	5	10	8	0	0	-M	
B	CB	XB	$x1$	$x2$	$x3$	$S1$	$S2$	$A2$	RR=$XB/S2$
$S1$	0	24	1	3	0	1	1/2	0	24/1/2=48
$x3$	8	18	1	1	1	0	-1/4	0	---
$A2$	-M	10	-3	-1	0	0	(5/4)	1	10/5/4=8→
Z= -10M+144		Zj-Cj	3M+3	M-2	0	0	(-5M/4)-2↑	0	

Negative minimum Zj-Cj is –(5M/4)-2 and its column index is 5. So, the entering variable is $S2$.
Minimum ratio is 8 and its row index is 3. So, the leaving basis variable is $A2$.
∴ The pivot element is 54.
Entering =$S2$, Departing =$A2$, Key Element =5/4

$R3$(new)=$R3$(old) ×(4/5)
$R1$(new)=$R1$(old) – (1/2) $R3$(new)
$R2$(new)=$R2$(old) + (1/4)$R3$(new)

Iteration-3		Cj	5	10	8	0	0	
BV	CB	XB	$x1$	$x2$	$x3$	$S1$	$S2$	RR=$XB/x2$
$S1$	0	20	11/5	(17/5)	0	1	0	20/17/5=100/17=5.8824→
$x3$	8	20	2/5	4/5	1	0	0	20/4/5=25
$S2$	0	8	-12/5	-4/5	0	0	1	---
Z=160		Zj-Cj	-9/5	-18/5↑	0	0	0	

Negative minimum Zj-Cj is -18/5, and its column index is 2. So, the entering variable is $x2$.
Minimum ratio is 5.8824 and its row index is 1. So, the leaving basis variable is $S1$.
∴ The pivot element is 175.
Entering =$x2$, Departing =$S1$, Key Element =17/5

$R1$(new)=$R1$(old) ×5/17
$R2$(new)=$R2$(old) – (4/5) $R1$(new)
$R3$(new)=$R3$(old) + (4/5)$R1$(new)

Iteration-4		Cj	5	10	8	0	0	
BV	CB	XB	$x1$	$x2$	$x3$	$S1$	$S2$	RR
$x2$	10	100/17	11/17	1	0	5/17	0	
$x3$	8	260/17	-2/17	0	1	-4/17	0	
$S2$	0	216/17	-32/17	0	0	4/17	1	
Z=3080/17		Zj-Cj	9/17	0	0	18/17	0	

Since all Zj-$Cj \geq 0$
Hence, optimal solution is arrived with value of variables as : $x1$=0,$x2$=100/17,$x3$=260/17 and Max Z=3080/17

Find solution using Simplex method (Big-M method)
MIN Z = 5x1 + 3x2
subject to
2x1 + 4x2 <= 12
2x1 + 2x2 = 10
5x1 + 2x2 >= 10
and x1, x2 >= 0

Solution: Problem is

Min $Z = 5x1 + 3x2$

subject to

$2x1 + 4x2 \le 12$

$2x1 + 2x2 = 10$

$5x1 + 2x2 \ge 10$

and $x1, x2 \ge 0$;

∴ Max $Z = -5x1 - 3x2$

The problem is converted to canonical form by adding slack, surplus and artificial variables as appropriate
1. As the constraint-1 is of type '≤' we should add slack variable $S1$
2. As the constraint-2 is of type '=' we should add artificial variable $A1$
3. As the constraint-3 is of type '≥' we should subtract surplus variable $S2$ and add artificial variable $A2$

After introducing slack, surplus, artificial variables

Max $Z = -5x1 - 3x2 + 0S1 + 0S2 - MA1 - MA2$

subject to

$$\begin{aligned}
2x1 + 4x2 + S1 \quad &= 12\\
2x1 + 2x2 \quad + A1 &= 10\\
5x1 + 2x2 - S2 \quad + A2 &= 10
\end{aligned}$$

and $x1, x2, S1, S2, A1, A2 \ge 0$

Iteration-1		Cj	-5	-3	0	0	$-M$	$-M$	
BV	**CB**	**XB**	**$x1$**	**$x2$**	**$S1$**	**$S2$**	**$A1$**	**$A2$**	**RR=$XB/x1$**
$S1$	0	12	2	4	1	0	0	0	12/2=6
$A1$	$-M$	10	2	2	0	0	1	0	10/2=5
$A2$	$-M$	10	(5)	2	0	-1	0	1	10/5=2→
		Zj	**$-7M$**	**$-4M$**	**0**	**M**	**$-M$**	**$-M$**	
$Z=-20M$		$Zj-Cj$	$-7M+5$↑	$-4M+3$	0	M	0	0	

Negative minimum $Zj-Cj$ is $-7M+5$ and its column index is 1. So, the entering variable is $x1$.
Minimum ratio is 2 and its row index is 3. So, the leaving basis variable is $A2$.
∴ The pivot element is 5.

Entering $=x1$, Departing $=A2$, Key Element $=5$
$R3(\text{new})=R3(\text{old}) \div 5$
$R1(\text{new})=R1(\text{old}) - 2R3(\text{new})$
$R2(\text{new})=R2(\text{old}) - 2R3(\text{new})$

Iteration-2		Cj	-5	-3	0	0	$-M$	
BV	**CB**	**XB**	**$x1$**	**$x2$**	**$S1$**	**$S2$**	**$A1$**	**RR=$XB/x2$**
$S1$	0	8	0	(16/5)	1	25	0	8/16/5=5/2=2.5→

A_1	$-M$	6	0	65	0	25	1	6/6/5=5
x_1	-5	2	1	25	0	-15	0	2/2/5=5
$Z=-6M-10$		Z_j-C_j	0	$-(6M/5)+1$↑	0	$-(2M/5)+1$	0	

Negative minimum Z_j-C_j is $-6M5+1$ and its column index is 2. So, the entering variable is x_2.
Minimum ratio is 2.5 and its row index is 1. So, the leaving basis variable is S_1.

∴ The pivot element is 165.

Entering $=x_2$, Departing $=S_1$, Key Element $=165$
$R_1(\text{new})=R_1(\text{old})\times 516$
$R_2(\text{new})=R_2(\text{old}) - 65R_1(\text{new})$
$R_3(\text{new})=R_3(\text{old}) - 25R_1(\text{new})$

Iteration-3		C_j	-5	-3	0	0	$-M$	
BV	***CB***	***XB***	***x1***	***x2***	***S1***	***S2***	***A1***	**RR=*XB/S2***
x_2	-3	52	0	1	5/16	1/8	0	5/2/1/8=20
A_1	$-M$	3	0	0	-3/8	**(1/4)**	1	3/1/4=12→
x_1	-5	1	1	0	-1/8	-1/4	0	---
$Z=-3M-252$		Z_j-C_j	0	0	$(3M/8)-(5/16)$	$-(M/4)+(7/8)$↑	0	

Negative minimum Z_j-C_j is $-M4+78$ and its column index is 4. So, the entering variable is S_2.
Minimum ratio is 12 and its row index is 2. So, the leaving basis variable is A_1.
∴ The pivot element is 14.

Entering $=S_2$, Departing $=A_1$, Key Element $=14$

$R_2(\text{new})=R_2(\text{old})\times 4$
$R_1(\text{new})=R_1(\text{old}) - 18R_2(\text{new})$
$R_3(\text{new})=R_3(\text{old}) + 14R_2(\text{new})$

Iteration-4		C_j	-5	-3	0	0	
BV	***CB***	***XB***	***x1***	***x2***	***S1***	***S2***	RR
x_2	-3	1	0	1	1/2	0	
S_2	0	12	0	0	-3/2	1	
x_1	-5	4	1	0	-1/2	0	
$Z=-23$		Z_j-C_j	0	0	1	0	

Since all $Z_j-C_j\geq 0$
Hence, optimal solution is arrived with value of variables as : $x_1=4, x_2=1$
and ∴ Min $Z=23$