19 EHS 405: OPERATIONS RESEARCH (IDE - I)

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UNIT- I 10L

Basics of Operations Research: History, definition, operations research models, phases of implementing operations research in practice.

Linear Programming: Introduction, formulation, graphical solution, simplex method, artificial variable techniques – Big M and Two-Phase methods, concept of duality, dual simplex method.

UNIT- II 8L

Transportation Model: Formulation, methods for initial feasible solution, optimal solution – MODI method, unbalanced transportation problems, degeneracy in transportation problems.

Assignment Model: Formulation, optimal solution, Hungarian method, travelling salesman problem.

UNIT- III 8L

Queuing Models: Introduction, Kendall's notation, classification of queuing models, single server and multi-server models, Poisson arrival, exponential service, infinite population

Sequencing Models: Introduction, assumptions, processing n-jobs through two machines, n-jobs through three machines, n-jobs through m-machines, graphic solution for processing 2 jobs through n machines with differentorder of sequence

UNIT- IV 9L

Replacement Models: Introduction, replacement of items that deteriorate with time - value of money unchanging and changing, simple probabilistic model for replacement of items that fail completely.

Game Theory: Introduction, game with pure strategies, game with mixed strategies, dominance principle, graphical method for 2xn and mx2 games, linear programming approach for game theory.

UNIT- V 9L

Inventory Models: Introduction, inventory costs, Economic Order Quantity (EOQ) and Economic Batch Quantity (EBQ) models with and without shortages, inventory models with quantity discounts

Project Management: Introduction, phases of project management, network construction, numbering the events-Fulkerson's rule, Critical Path Method (CPM), Programme Evaluation and Review Technique (PERT)

Text Book(s):

1. Gupta P K. & Hira D.S., Operation Research, 6/e, S Chand Publishers, 2006.
2. Paneerselvam R., Operations Research, 2/e, Prentice Hall of India, 2010.

UNIT- II

Transportation Model

Formulation,
Methods for initial feasible solution,
Optimal solution – MODI method,
Unbalanced transportation problems,
Degeneracy in transportation problems.

Assignment Model

Formulation,
Optimal solution,
Hungarian method,
Travelling salesman problem.

Transportation Problem

Transportation problem is a special kind of Linear Programming Problem (LPP) in which goods are transported from a set of sources to a set of destinations subject to the supply and demand of the sources and destination respectively such that the total cost of transportation is minimized. It is also sometimes called as Hitchcock problem.

The transportation problem is a special type of linear programming problem where the objective consists in minimizing transportation cost of a given commodity from a number of sources or origins (e.g., factory, manufacturing facility) to a number of destinations (e.g., warehouse, store). Each source has a limited supply (i.e., maximum number of products that can be sent from it) while each destination has a demand to be satisfied (i.e., minimum number of products that need to be shipped to it). The cost of shipping from a source to a destination is directly proportional to the number of units shipped.

Basic Notation:

m = number of sources ($i = 1 \dots m$)
n = number of destinations ($j = 1 \dots n$)
$c_{i,j}$ = unit cost of shipping from source i to destination j
$x_{i,j}$ = amount shipped from source i to destination j
a_i = supply at source i
b_j = demand at destination j

Sources are represented by rows while destinations are represented by columns. In general, a transportation problem has m rows and n columns. The problem is solvable if there are exactly (m+n-1) basic variables.

Types of Transportation problems:

Balanced: When both supplies and demands are equal then the problem is said to be a balanced transportation problem.

Unbalanced: When the supply and demand are not equal then it is said to be an unbalanced transportation problem. In this type of problem, either a dummy row or a dummy column is added according to the requirement to make it a balanced problem. Then it can be solved similar to the balanced problem.

Solution of the transportation problem:

Stage I: Finding an initial basic feasible solution.
Stage II: Checking for optimality

Existence of Feasible Solution:

A necessary and sufficient condition for the existence of a feasible solution to the general transportation problem is that

Total supply = Total demand

The number of basic variables of the general transportation problem at any stage of feasible solution must be $(m + n - 1)$. Now degenerate basic feasible solution (a feasible solution) involving exactly $(m + n - 1)$ positive variables is known as non-degenerate basic feasible solution otherwise it is said to be degenerate basic feasible. These allocations should be independent positions in case of non-degenerate basic feasible solutions.

Optimum Solution: A feasible solution is said to be optimal, if it minimizes the total transportation cost.
Unbalance TP If total supply is not equal to total demand, then it balance with dummy source or destination.

Methods to Solve:

To find the initial basic feasible solution there are five methods:

$: Read more https://www.gatexplore.com/transportation-problem-study-notes/

1. North -West Corner Cell Method.
2. Least Call Cell Method.
3. Vogel's Approximation Method (VAM).
4. Row – Minima Method
5. Column – Minima Method

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1. Steps for North-West Corner Method

I. Allocate the maximum amount allowable by the supply and demand constraints to the variable x11 (i.e. the cell in the top left corner of the transportation tableau).

II. If a column (or row) is satisfied, cross it out. The remaining decision variables in that column (or row) are non-basic and are set equal to zero. If a row and column are satisfied simultaneously, cross only one out (it does not matter which).

III. Adjust supply and demand for the non-crossed out rows and columns.

IV. Allocate the maximum feasible amount to the first available non-crossed out element in the next column (or row).

V. When exactly one row or column is left, all the remaining variables are basic and are assigned the only feasible allocation.

	D1	D2	D3	D4	Supply
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8	70	20	18
Demand	5	8	7	14	

Solution:

TOTAL number of supply constraints: 3
TOTAL number of demand constraints: 4
Problem Table is
The rim values for $S1=7$ and $D1=5$ is compared.
The smaller of the two i.e., min $(7,5) = 5$ is assigned to $S1$ $D1$
This meets the complete demand of $D1$ and leaves 7 - 5=2 units with $S1$

Table-1

	$D1$	$D2$	$D3$	$D4$	Supply
$S1$	19(5)	30	50	10	2
$S2$	70	30	40	60	9
$S3$	40	8	70	20	18
Demand	0	8	7	14	

The rim values for $S1$=2 and $D2$=8 are compared.
The smaller of the two i.e. min(2,8) = **2** is assigned to $S1$ $D2$
This exhausts the capacity of $S1$ and leaves 8 - 2=6 units with $D2$

Table-2

	$D1$	$D2$	$D3$	$D4$	Supply
$S1$	19(5)	30(2)	50	10	0
$S2$	70	30	40	60	9
$S3$	40	8	70	20	18
Demand	0	6	7	14	

The rim values for $S2$=9 and $D2$=6 are compared.
The smaller of the two i.e. min(9,6) = **6** is assigned to $S2$ $D2$
This meets the complete demand of $D2$ and leaves 9 - 6=3 units with $S2$

Table-3

	$D1$	$D2$	$D3$	$D4$	Supply
$S1$	19(5)	30(2)	50	10	0
$S2$	70	30(6)	40	60	3
$S3$	40	8	70	20	18
Demand	0	0	7	14	

The rim values for $S2$=3 and $D3$=7 are compared.
The smaller of the two i.e. min(3,7) = **3** is assigned to $S2$ $D3$
This exhausts the capacity of $S2$ and leaves 7 - 3=4 units with $D3$

Table-4

	$D1$	$D2$	$D3$	$D4$	Supply
$S1$	19(5)	30(2)	50	10	0
$S2$	70	30(6)	40(3)	60	0
$S3$	40	8	70	20	18
Demand	0	0	4	14	

The rim values for $S3$=18 and $D3$=4 are compared.
The smaller of the two i.e. min(18,4) = **4** is assigned to $S3$ $D3$
This meets the complete demand of $D3$ and leaves 18 - 4=14 units with $S3$

Table-5

	$D1$	$D2$	$D3$	$D4$	Supply
$S1$	19(5)	30(2)	50	10	0
$S2$	70	30(6)	40(3)	60	0
$S3$	40	8	70(4)	20	14

Demand	0	0	0	14	

The rim values for $S3$=14 and $D4$=14 are compared.
The smaller of the two i.e. min(14,14) = 14 is assigned to $S3$ $D4$

Table-6

	$D1$	$D2$	$D3$	$D4$	Supply
$S1$	19(5)	30(2)	50	10	0
$S2$	70	30(6)	40(3)	60	0
$S3$	40	8	70(4)	20(14)	0
Demand	0	0	0	0	

Initial

IBFS	$D1$	$D2$	$D3$	$D4$	Supply
$S1$	19 (5)	30 (2)	50	10	7
$S2$	70	30 (6)	40 (3)	60	9
$S3$	40	8	70 (4)	20 (14)	18
Demand	5	8	7	14	

feasible solution is
The minimum total transportation cost $=19\times5+30\times2+30\times6+40\times3+70\times4+20\times14=1015$
Here, the number of allocated cells = 6 is equal to m + n - 1 = 3 + 4 - 1 = 6
∴ This solution is non-degenerate

2. Steps for Least Cost Method.

I. Assign as much as possible to the cell with the smallest unit cost in the entire tableau. If there is a tie, then choose arbitrarily.
II. Cross out the row or column which has satisfied supply or demand. If a row and column are both satisfied, then cross out only one of them.
III. Adjust the supply and demand for those rows and columns which are not crossed out.
IV. When exactly one row or column is left, all the remaining variables are basic and are assigned the only feasible allocation.

	D1	D2	D3	D4	Supply
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8	70	20	18
Demand	5	8	7	14	

Solution:
TOTAL number of supply constraints: 3
TOTAL number of demand constraints : 4
The smallest transportation cost is 8 in cell $S3D2$
The allocation to this cell is min(18,8) = 8.
This satisfies the entire demand of $D2$ and leaves 18 - 8=10 units with $S3$

Table-1

	$D1$	$D2$	$D3$	$D4$	Supply
$S1$	19	30	50	10	7
$S2$	70	30	40	60	9
$S3$	40	8(8)	70	20	10
Demand	5	0	7	14	

The smallest transportation cost is 10 in cell *S1D4*
The allocation to this cell is min (7,14) = 7.
This exhausts the capacity of *S1* and leaves 14 - 7=7 units with *D4*

Table-2

	D1	*D2*	*D3*	*D4*	Supply
S1	19	30	50	10(7)	0
S2	70	30	40	60	9
S3	40	8(8)	70	20	10
Demand	5	0	7	7	

The smallest transportation cost is 20 in cell *S3D4*
The allocation to this cell is min(10,7) = 7.
This satisfies the entire demand of *D4* and leaves 10 - 7=3 units with *S3*

Table-3

	D1	*D2*	*D3*	*D4*	Supply
S1	19	30	50	10(7)	0
S2	70	30	40	60	9
S3	40	8(8)	70	20(7)	3
Demand	5	0	7	0	

The smallest transportation cost is 40 in cell *S2D3*
The allocation to this cell is min(9,7) = 7.
This satisfies the entire demand of *D3* and leaves 9 - 7=2 units with *S2*

Table-4

	D1	*D2*	*D3*	*D4*	Supply
S1	19	30	50	10(7)	0
S2	70	30	40(7)	60	2
S3	40	8(8)	70	20(7)	3
Demand	5	0	0	0	

The smallest transportation cost is 40 in cell *S3D1*
The allocation to this cell is min(3,5) = 3.
This exhausts the capacity of *S3* and leaves 5 - 3=2 units with *D1*

Table-5

	D1	*D2*	*D3*	*D4*	Supply
S1	19	30	50	10(7)	0
S2	70	30	40(7)	60	2

S3	40(3)	8(8)	70	20(7)	0
Demand	2	0	0	0	

The smallest transportation cost is 70 in cell $S2D1$
The allocation to this cell is min(2,2) = 2.

Table-6

	D1	D2	D3	D4	Supply
S1	19	30	50	10(7)	0
S2	70(2)	30	40(7)	60	0
S3	40(3)	8(8)	70	20(7)	0
Demand	0	0	0	0	

IBFS	D1	D2	D3	D4	Supply
S1	19	30	50	10 (7)	7
S2	70 (2)	30	40 (7)	60	9
S3	40 (3)	8 (8)	70	20 (7)	18
Demand	5	8	7	14	

Initial feasible solution is
The minimum total transportation cost $=10\times7+70\times2+40\times7+40\times3+8\times8+20\times7=814$
Here, the number of allocated cells = 6 is equal to m + n - 1 = 3 + 4 - 1 = 6
∴ This solution is non-degenerate

3. Find Solution using Vogel's Approximation method.

Step-1: Find the cells having smallest and next to smallest cost in each row and write the difference (called penalty) along the side of the table in row penalty.
Step-2: Find the cells having smallest and next to smallest cost in each column and write the difference (called penalty) along the side of the table in each column penalty.
Step-3: Select the row or column with the maximum penalty and find cell that has least cost in selected row or column. Allocate as much as possible in this cell.
If there is a tie in the values of penalties, then select the cell where maximum allocation can be possible
Step-4: Adjust the supply & demand and cross out (strike out) the satisfied row or column.
Step-5: Repeat this steps until all supply and demand values are 0.

	D1	D2	D3	D4	Supply
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8	70	20	18
Demand	5	8	7	14	

Solution:
TOTAL number of supply constraints : 3
TOTAL number of demand constraints : 4
Problem Table is

Table-1

	D1	D2	D3	D4	Supply	Row Penalty
S1	19	30	50	10	7	9=19-10

$S2$	70	30	40	60	9	10=40-30
$S3$	40	8	70	20	18	12=20-8
Demand	5	8	7	14		
Column Penalty	21=40-19	22=30-8	10=50-40	10=20-10		

The maximum penalty, 22, occurs in column $D2$.
The minimum cij in this column is $c32$=8.
The maximum allocation in this cell is min(18,8) = 8.
It satisfy demand of $D2$ and adjust the supply of $S3$ from 18 to 10 (18 - 8=10).

Table-2

	$D1$	$D2$	$D3$	$D4$	Supply	Row Penalty
$S1$	19	30	50	10	7	9=19-10
$S2$	70	30	40	60	9	20=60-40
$S3$	40	8(8)	70	20	10	20=40-20
Demand	5	0	7	14		
Column Penalty	21=40-19	--	10=50-40	10=20-10		

The maximum penalty, 21, occurs in column $D1$.
The minimum cij in this column is $c11$=19.
The maximum allocation in this cell is min(7,5) = 5.
It satisfy demand of $D1$ and adjust the supply of $S1$ from 7 to 2 (7 - 5=2).

Table-3

	$D1$	$D2$	$D3$	$D4$	Supply	Row Penalty
$S1$	19(5)	30	50	10	2	40=50-10
$S2$	70	30	40	60	9	20=60-40
$S3$	40	8(8)	70	20	10	50=70-20
Demand	0	0	7	14		
Column Penalty	--	--	10=50-40	10=20-10		

The maximum penalty, 50, occurs in row $S3$.
The minimum cij in this row is $c34$=20.
The maximum allocation in this cell is min(10,14) = 10.
It satisfy supply of $S3$ and adjust the demand of $D4$ from 14 to 4 (14 - 10=4).

Table-4

	$D1$	$D2$	$D3$	$D4$	Supply	Row Penalty
$S1$	19(5)	30	50	10	2	40=50-10
$S2$	70	30	40	60	9	20=60-40

S3	40	8(8)	70	20(10)	0	--
Demand	0	0	7	4		
Column Penalty	--	--	10=50-40	50=60-10		

The maximum penalty, 50, occurs in column *D4*.
The minimum *cij* in this column is *c*14=10.
The maximum allocation in this cell is min(2,4) = 2.
It satisfy supply of *S1* and adjust the demand of *D4* from 4 to 2 (4 - 2=2).

Table-5

	D1	*D2*	*D3*	*D4*	Supply	Row Penalty
S1	19(5)	30	50	10(2)	0	--
S2	70	30	40	60	9	20=60-40
S3	40	8(8)	70	20(10)	0	--
Demand	0	0	7	2		
Column Penalty	--	--	40	60		

The maximum penalty, 60, occurs in column *D4*.
The minimum *cij* in this column is *c*24=60.
The maximum allocation in this cell is min(9,2) = 2.
It satisfy demand of *D4* and adjust the supply of *S2* from 9 to 7 (9 - 2=7).

Table-6

	D1	*D2*	*D3*	*D4*	Supply	Row Penalty
S1	19(5)	30	50	10(2)	0	--
S2	70	30	40	60(2)	7	40
S3	40	8(8)	70	20(10)	0	--
Demand	0	0	7	0		
Column Penalty	--	--	40	--		

The maximum penalty, 40, occurs in row *S2*.
The minimum *cij* in this row is *c*23=40.
The maximum allocation in this cell is min(7,7) = 7.
It satisfy supply of *S2* and demand of *D3*.

Initial feasible solution is

<table>
<tr><th></th><th>D1</th><th>D2</th><th>D3</th><th>D4</th><th>Supply</th><th>Row Penalty</th></tr>
<tr><td>S1</td><td>19(5)</td><td>30</td><td>50</td><td>10(2)</td><td>7</td><td>9 | 9 | 40 | 40 | -- | -- |</td></tr>
<tr><td>S2</td><td>70</td><td>30</td><td>40(7)</td><td>60(2)</td><td>9</td><td>10 | 20 | 20 | 20 | 20 | 40 |</td></tr>
<tr><td>S3</td><td>40</td><td>8(8)</td><td>70</td><td>20(10)</td><td>18</td><td>12 | 20 | 50 | -- | -- | -- |</td></tr>
</table>

Demand	5	8	7	14		
Column Penalty	21 21 -- -- -- --	22 -- -- -- -- --	10 10 10 10 40 40	10 10 10 50 60 --		

The minimum total transportation cost $=19\times5+10\times2+40\times7+60\times2+8\times8+20\times10=779$

Here, the number of allocated cells = 6 is equal to $m + n - 1 = 3 + 4 - 1 = 6$

∴ This solution is non-degenerate

3. Find Solution using Vogel's Approximation method

	D1	D2	D3	D4	Supply
S1	11	13	17	14	250
S2	16	18	14	10	300
S3	21	24	13	10	400
Demand	200	225	275	250	

Solution:

TOTAL number of supply constraints : 3

TOTAL number of demand constraints : 4

Problem Table is

Table-1

	$D1$	$D2$	$D3$	$D4$	Supply	Row Penalty
$S1$	11	13	17	14	250	2=13-11
$S2$	16	18	14	10	300	4=14-10
$S3$	21	24	13	10	400	3=13-10
Demand	200	225	275	250		
Column Penalty	5=16-11	5=18-13	1=14-13	0=10-10		

The maximum penalty, 5, occurs in column $D1$.

The minimum c_{ij} in this column is $c_{11}=11$.

The maximum allocation in this cell is min(250,200) = 200.

It satisfy demand of $D1$ and adjust the supply of $S1$ from 250 to 50 (250 - 200=50).

Table-2

	$D1$	$D2$	$D3$	$D4$	Supply	Row Penalty
$S1$	11(200)	13	17	14	50	1=14-13
$S2$	16	18	14	10	300	4=14-10

S3	21	24	13	10	400	3=13-10
Demand	0	225	275	250		
Column Penalty	--	5=18-13	1=14-13	0=10-10		

The maximum penalty, 5, occurs in column *D2*.
The minimum *cij* in this column is *c*12=13.
The maximum allocation in this cell is min(50,225) = 50.
It satisfy supply of *S1* and adjust the demand of *D2* from 225 to 175 (225 - 50=175).

Table-3

	D1	*D2*	*D3*	*D4*	Supply	Row Penalty
S1	11(200)	13(50)	17	14	0	--
S2	16	18	14	10	300	4=14-10
S3	21	24	13	10	400	3=13-10
Demand	0	175	275	250		
Column Penalty	--	6=24-18	1=14-13	0=10-10		

The maximum penalty, 6, occurs in column *D2*.
The minimum *cij* in this column is *c*22=18.
The maximum allocation in this cell is min (300,175) = 175.
It satisfy demand of *D2* and adjust the supply of *S2* from 300 to 125 (300 - 175=125).

Table-4

	D1	*D2*	*D3*	*D4*	Supply	Row Penalty
S1	11(200)	13(50)	17	14	0	--
S2	16	18(175)	14	10	125	4=14-10
S3	21	24	13	10	400	3=13-10
Demand	0	0	275	250		
Column Penalty	--	--	1=14-13	0=10-10		

The maximum penalty, 4, occurs in row *S2*.
The minimum *cij* in this row is *c*24=10.
The maximum allocation in this cell is min (125,250) = 125.
It satisfy supply of *S2* and adjust the demand of *D4* from 250 to 125 (250 - 125=125).

Table-5

	D1	*D2*	*D3*	*D4*	Supply	Row Penalty
S1	11(200)	13(50)	17	14	0	--
S2	16	18(175)	14	10(125)	0	--
S3	21	24	13	10	400	3=13-10

Demand	0	0	275	125		
Column Penalty	--	--	13	10		

The maximum penalty, 13, occurs in column $D3$.
The minimum cij in this column is $c33=13$.
The maximum allocation in this cell is min (400,275) = 275.
It satisfy demand of $D3$ and adjust the supply of $S3$ from 400 to 125 (400 - 275=125).

Table-6

	$D1$	$D2$	$D3$	$D4$	Supply	Row Penalty
$S1$	11(200)	13(50)	17	14	0	--
$S2$	16	18(175)	14	10(125)	0	--
$S3$	21	24	13(275)	10	125	10
Demand	0	0	0	125		
Column Penalty	--	--	--	10		

The maximum penalty, 10, occurs in row $S3$.
The minimum cij in this row is $c34=10$.
The maximum allocation in this cell is min(125,125) = 125.
It satisfy supply of $S3$ and demand of $D4$.

Initial feasible solution is

	$D1$	$D2$	$D3$	$D4$	Supply	Row Penalty
$S1$	11(200)	13(50)	17	14	250	2 | 1 | -- | -- | -- | -- |
$S2$	16	18(175)	14	10(125)	300	4 | 4 | 4 | 4 | -- | -- |
$S3$	21	24	13(275)	10(125)	400	3 | 3 | 3 | 3 | 3 | 10 |
Demand	200	225	275	250		
Column Penalty	5 -- -- -- -- --	5 5 6 -- -- --	1 1 1 1 13 --	0 0 0 0 10 10		

The minimum total transportation cost $=11\times200+13\times50+18\times175+10\times125+13\times275+10\times125=12075$

Here, the number of allocated cells = 6 is equal to m + n - 1 = 3 + 4 - 1 = 6
∴ This solution is non-degenerate

MODI Method Steps:

Step-1: Find an initial basic feasible solution using any one of the three methods NWCM, LCM or VAM.

Step-2: Find **ui** and **vj** for rows and columns.
To start:
a. assign '0' to **ui** or **vj** where maximum number of allocations in a row or column respectively.

b. Calculate other **ui**'s and **vj**'s using **cij = ui+vj**, for all occupied cells.

Step-3: For all unoccupied cells, calculate **dij = cij-(ui+vj)**

Step-4: Check the sign of **dij**

a. If **dij > 0**, then current basic feasible solution is optimal and stop this procedure.

b. If **dij = 0** then alternative solution exists, with different set allocation and same transportation cost. Now stop this procedure.

c. If **dij < 0**, then the given solution is not an optimal solution and further improvement in the solution is possible.

Step-5: Select the unoccupied cell with the largest negative value of **dij**, and include in the next solution.

Step-6: Draw a closed path (or loop) from the unoccupied cell (selected in the previous step). The right angle (90^0) turn in this path is allowed only at occupied cells and at the original unoccupied cell. Mark (+) and (-) sign alternatively at each corner, starting from the original unoccupied cell.

Step-7: a. Select the minimum value from cells marked with (-) sign of the closed path.

b. Assign this value to selected unoccupied cell (So unoccupied cell becomes occupied cell).

c. Add this value to the other occupied cells marked with (+) sign.

d. Subtract this value to the other occupied cells marked with (-) sign.

Step-8: Repeat Step-2 to step-7 until optimal solution is obtained. This procedure stops when all dij ≥ 0 for unoccupied cells.

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1. Find the optimal solution for the following problem using MODI method.

Allocation Table is

	P	*Q*	*R*	Supply
A	19 (5)	30	50 (2)	7
B	70	30	40 (8)	8
S3	40	8 (8)	70 (2)	10
Demand	5	8	12	

Iteration-1 of optimality test

1. Find u_i and v_j for all occupied cells (i, j), where $c_{ij}=u_i+v_j$

1. Substituting, $v_3=0$, we get
2. $c_{13}=u_1+v_3 \Rightarrow u_1=c_{13}-v_3 \Rightarrow u_1=50-0 \Rightarrow u_1=50$
3. $c_{11}=u_1+v_1 \Rightarrow v_1=c_{11}-u_1 \Rightarrow v_1=19-50 \Rightarrow v_1=-31$
4. $c_{23}=u_2+v_3 \Rightarrow u_2=c_{23}-v_3 \Rightarrow u_2=40-0 \Rightarrow u_2=40$
5. $c_{33}=u_3+v_3 \Rightarrow u_3=c_{33}-v_3 \Rightarrow u_3=70-0 \Rightarrow u_3=70$
6. $c_{32}=u_3+v_2 \Rightarrow v_2=c_{32}-u_3 \Rightarrow v_2=8-70 \Rightarrow v_2=-62$

	P	*Q*	*R*	Supply	u_i
A	19 (5)	30	50 (2)	7	$u_1=50$
B	70	30	40 (8)	8	$u_2=40$
S3	40	8 (8)	70 (2)	10	$u_3=70$
Demand	5	8	12		
v_j	$v_1=-31$	$v_2=-62$	$v_3=0$		

2. Find d_{ij} for all unoccupied cells(i,j), where $d_{ij}=c_{ij}-(u_i+v_j)$

1. $d_{12}=c_{12}-(u_1+v_2)=30-(50-62)=42$
2. $d_{21}=c_{21}-(u_2+v_1)=70-(40-31)=61$
3. $d_{22}=c_{22}-(u_2+v_2)=30-(40-62)=52$
4. $d_{31}=c_{31}-(u_3+v_1)=40-(70-31)=1$

	P	Q	R	Supply	u_i
A	19 (5)	30 [42]	50 (2)	7	$u_1=50$
B	70 [61]	30 [52]	40 (8)	8	$u_2=40$
S_3	40 [1]	8 (8)	70 (2)	10	$u_3=70$
Demand	5	8	12		
v_j	$v_1=-31$	$v_2=-62$	$v_3=0$		

Since all $d_{ij} \geq 0$.

So final optimal solution is arrived.

	P	Q	R	Supply
A	19 (5)	30	50 (2)	7
B	70	30	40 (8)	8
S_3	40	8 (8)	70 (2)	10
Demand	5	8	12	

The minimum total transportation cost $=19\times5+50\times2+40\times8+8\times8+70\times2=719$

https://cbom.atozmath.com/example/CBOM/Transportation.aspx?q=modi&q1=E1
Example: Optimality test using modi method...
Initial feasible solution is

	D_1	D_2	D_3	D_4	Supply	Row Penalty
S_1	19(5)	30	50	10(2)	7	9 | 9 | 40 | 40 | -- | -- |
S_2	70	30	40(7)	60(2)	9	10 | 20 | 20 | 20 | 20 | 40 |
S_3	40	8(8)	70	20(10)	18	12 | 20 | 50 | -- | -- | -- |
Demand	5	8	7	14		
Column Penalty	21 21 -- -- -- --	22 -- -- -- -- --	10 10 10 10 40 40	10 10 10 50 60 --		

The minimum total transportation cost $=19\times5+10\times2+40\times7+60\times2+8\times8+20\times10=779$
Optimality test using modi method...
Allocation Table is

	D_1	D_2	D_3	D_4	Supply

$S1$	19 (5)	30	50	10 (2)	7
$S2$	70	30	40 (7)	60 (2)	9
$S3$	40	8 (8)	70	20 (10)	18
Demand	5	8	7	14	

	$D1$	$D2$	$D3$	$D4$	Supply	ui
$S1$	19 (5)	30 [32]	50 [60]	10 (2)	7	$u1=10$
$S2$	70 [1]	30 [-18]	40 (7)	60 (2)	9	$u2=60$
$S3$	40 [11]	8 (8)	70 [70]	20 (10)	18	$u3=20$
Demand	5	8	7	14		
vj	$v1=9$	$v2=-12$	$v3=-20$	$v4=0$		

3. Now choose the minimum negative value from all dij (opportunity cost) = $d22$ = [-18] and draw a closed path from $S2D2$.
Closed path is $S2D2 \rightarrow S2D4 \rightarrow S3D4 \rightarrow S3D2$
Closed path and plus/minus sign allocation...

	$D1$	$D2$	$D3$	$D4$		Supply	ui
$S1$	19 (5)	30 [32]	50 [60]	10 (2)		7	$u1=10$
$S2$	70 [1]	30 [-18] (+)	40 (7)	60 (2) (-)		9	$u2=60$
$S3$	40 [11]	8 (8) (-)	70 [70]	20 (10) (+)		18	$u3=20$
Demand	5	8	7	14			
vj	$v1=9$	$v2=-12$	$v3=-20$	$v4=0$			

4. Minimum allocated value among all negative position (-) on closed path = 2
Substract 2 from all (-) and Add it to all (+)

	$D1$	$D2$	$D3$	$D4$	Supply
$S1$	19 (5)	30	50	10 (2)	7
$S2$	70	30 (2)	40 (7)	60	9
$S3$	40	8 (6)	70	20 (12)	18
Demand	5	8	7	14	

5. Repeat the step 1 to 4, until an optimal solution is obtained.

	D1	D2	D3	D4	Supply	ui
S1	19 (5)	30	50	10 (2)	7	$u_1=0$
S2	70	30 (2)	40 (7)	60	9	$u_2=32$
S3	40	8 (6)	70	20 (12)	18	$u_3=10$
Demand	5	8	7	14		
vj	$v_1=19$	$v_2=-2$	$v_3=8$	$v_4=10$		

	D1	D2	D3	D4	Supply	ui
S1	19 (5)	30 [32]	50 [42]	10 (2)	7	$u_1=0$
S2	70 [19]	30 (2)	40 (7)	60 [18]	9	$u_2=32$
S3	40 [11]	8 (6)	70 [52]	20 (12)	18	$u_3=10$
Demand	5	8	7	14		
vj	$v_1=19$	$v_2=-2$	$v_3=8$	$v_4=10$		

Since all $d_{ij} \geq 0$.

So final optimal solution is arrived.

	D1	D2	D3	D4	Supply
S1	19 (5)	30	50	10 (2)	7
S2	70	30 (2)	40 (7)	60	9
S3	40	8 (6)	70	20 (12)	18
Demand	5	8	7	14	

The minimum total transportation cost $=19\times5+10\times2+30\times2+40\times7+8\times6+20\times12=743$

2. Find the optimal solution for the following problem using MODI method.

Allocation Table is

	D1	D2	D3	D4	Supply
S1	11 (200)	13 (50)	17	14	250
S2	16	18 (175)	14	10 (125)	300
S3	21	24	13 (275)	10 (125)	400
Demand	200	225	275	250	

Iteration-1 of optimality test

1. Find ui and vj for all occupied cells(i,j), where cij=ui+vj

1. Substituting, $u1=0$, we get
2. $c11=u1+v1 \Rightarrow v1=c11-u1 \Rightarrow v1=11-0 \Rightarrow v1=11$
3. $c12=u1+v2 \Rightarrow v2=c12-u1 \Rightarrow v2=13-0 \Rightarrow v2=13$
4. $c22=u2+v2 \Rightarrow u2=c22-v2 \Rightarrow u2=18-13 \Rightarrow u2=5$
5. $c24=u2+v4 \Rightarrow v4=c24-u2 \Rightarrow v4=10-5 \Rightarrow v4=5$
6. $c34=u3+v4 \Rightarrow u3=c34-v4 \Rightarrow u3=10-5 \Rightarrow u3=5$
7. $c33=u3+v3 \Rightarrow v3=c33-u3 \Rightarrow v3=13-5 \Rightarrow v3=8$

	D1	D2	D3	D4	Supply	ui
S1	11 (200)	13 (50)	17	14	250	u1=0
S2	16	18 (175)	14	10 (125)	300	u2=5
S3	21	24	13 (275)	10 (125)	400	u3=5
Demand	200	225	275	250		
vj	v1=11	v2=13	v3=8	v4=5		

2. Find dij for all unoccupied cells(i,j), where $d_{ij}=c_{ij}-(u_i+v_j)$
1.$d13=c13-(u1+v3)=17-(0+8)=9$
2.$d14=c14-(u1+v4)=14-(0+5)=9$
3.$d21=c21-(u2+v1)=16-(5+11)=0$
4.$d23=c23-(u2+v3)=14-(5+8)=1$
5.$d31=c31-(u3+v1)=21-(5+11)=5$
6.$d32=c32-(u3+v2)=24-(5+13)=6$

	D1	D2	D3	D4	Supply	ui
S1	11 (200)	13 (50)	17 [9]	14 [9]	250	u1=0
S2	16 [0]	18 (175)	14 [1]	10 (125)	300	u2=5
S3	21 [5]	24 [6]	13 (275)	10 (125)	400	u3=5
Demand	200	225	275	250		
vj	v1=11	v2=13	v3=8	v4=5		

	D1	D2	D3	D4	Supply
S1	11 (200)	13 (50)	17	14	250
S2	16	18 (175)	14	10 (125)	300
S3	21	24	13 (275)	10 (125)	400
Demand	200	225	275	250	

Since all $d_{ij} \geq 0$.
So final optimal solution is arrived.
The minimum total transportation cost $=11\times200+13\times50+18\times175+10\times125+13\times275+10\times125=12075$
Notice alternate solution is available with unoccupied cell S2D1.

3. Find Solution using Least Cost method, also find optimal solution using MODI method,

	P	Q	R	Supply
A	16	20	12	200
B	14	8	18	160
C	26	24	16	90
Demand	180	120	150	

Solution:

Initial feasible solution is

	P	Q	R	Supply
A	16 (50)	20	12 (150)	200
B	14 (40)	8 (120)	18	160
$S3$	26 (90)	24	16	90
Demand	180	120	150	

The minimum total transportation cost $=16\times50+12\times150+14\times40+8\times120+26\times90=6460$

Here, the number of allocated cells = 5 is equal to m + n - 1 = 3 + 3 - 1 = 5

∴ This solution is non-degenerate

Optimality test using MODI method...Allocation Table is

	P	Q	R	Supply
A	16 (50)	20	12 (150)	200
B	14 (40)	8 (120)	18	160
$S3$	26 (90)	24	16	90
Demand	180	120	150	

Iteration-1 of optimality test

1. Find u_i and v_j for all occupied cells(i, j), where $c_{ij}=u_i+v_j$

1. Substituting, $v_1=0$, we get
2. $c_{11}=u_1+v_1\Rightarrow u_1=c_{11}-v_1\Rightarrow u_1=16-0\Rightarrow u_1=16$
3. $c_{13}=u_1+v_3\Rightarrow v_3=c_{13}-u_1\Rightarrow v_3=12-16\Rightarrow v_3=-4$
4. $c_{21}=u_2+v_1\Rightarrow u_2=c_{21}-v_1\Rightarrow u_2=14-0\Rightarrow u_2=14$
5. $c_{22}=u_2+v_2\Rightarrow v_2=c_{22}-u_2\Rightarrow v_2=8-14\Rightarrow v_2=-6$
6. $c_{31}=u_3+v_1\Rightarrow u_3=c_{31}-v_1\Rightarrow u_3=26-0\Rightarrow u_3=26$

	P	Q	R	Supply	u_i
A	16 (50)	20	12 (150)	200	$u_1=16$
B	14 (40)	8 (120)	18	160	$u_2=14$
$S3$	26 (90)	24	16	90	$u_3=26$
Demand	180	120	150		
v_j	$v_1=0$	$v_2=-6$	$v_3=-4$		

2. Find d_{ij} for all unoccupied cells(i,j), where $d_{ij}=c_{ij}-(u_i+v_j)$

1. $d_{12}=c_{12}-(u_1+v_2)=20-(16-6)=10$
2. $d_{23}=c_{23}-(u_2+v_3)=18-(14-4)=8$
3. $d_{32}=c_{32}-(u_3+v_2)=24-(26-6)=4$
4. $d_{33}=c_{33}-(u_3+v_3)=16-(26-4)=-6$

	P	Q	R	Supply	u_i
A	16 (50)	20 [10]	12 (150)	200	$u_1=16$
B	14 (40)	8 (120)	18 [8]	160	$u_2=14$

$S3$	26 (90)	24 [4]	16 [-6]	90	$u3=26$
Demand	180	120	150		
vj	$v1=0$	$v2=-6$	$v3=-4$		

3. Now choose the minimum negative value from all dij (opportunity cost) = $d33$ = [-6]
and draw a closed path from $S3R$.
Closed path is $S3R \to S3P \to AP \to AR$
Closed path and plus/minus sign allocation...

	P	Q	R	Supply	ui
A	16 (50) (+)	20 [10]	12 (150) (-)	200	$u1=16$
B	14 (40)	8 (120)	18 [8]	160	$u2=14$
$S3$	26 (90) (-)	24 [4]	16 [-6] (+)	90	$u3=26$
Demand	180	120	150		
vj	$v1=0$	$v2=-6$	$v3=-4$		

4. Minimum allocated value among all negative position (-) on closed path = 90
Subtract 90 from all (-) and Add it to all (+)

	P	Q	R	Supply
A	16 (140)	20	12 (60)	200
B	14 (40)	8 (120)	18	160
$S3$	26	24	16 (90)	90
Demand	180	120	150	

5. Repeat the step 1 to 4, until an optimal solution is obtained.

Iteration-2 of optimality test

1. Find ui and vj for all occupied cells (i, j), where $cij=ui+vj$

1. Substituting, $u1=0$, we get
2. $c11=u1+v1 \Rightarrow v1=c11-u1 \Rightarrow v1=16-0 \Rightarrow v1=16$
3. $c21=u2+v1 \Rightarrow u2=c21-v1 \Rightarrow u2=14-16 \Rightarrow u2=-2$
4. $c22=u2+v2 \Rightarrow v2=c22-u2 \Rightarrow v2=8+2 \Rightarrow v2=10$
5. $c13=u1+v3 \Rightarrow v3=c13-u1 \Rightarrow v3=12-0 \Rightarrow v3=12$
6. $c33=u3+v3 \Rightarrow u3=c33-v3 \Rightarrow u3=16-12 \Rightarrow u3=4$

	P	Q	R	Supply	ui
A	16 (140)	20	12 (60)	200	$u1=0$
B	14 (40)	8 (120)	18	160	$u2=-2$
$S3$	26	24	16 (90)	90	$u3=4$
Demand	180	120	150		
vj	$v1=16$	$v2=10$	$v3=12$		

2. Find d_{ij} for all unoccupied cells(I ,j), where $d_{ij}=c_{ij}-(u_i+v_j)$

1.$d_{12}=c_{12}-(u_1+v_2)=20-(0+10)=10$
2.$d_{23}=c_{23}-(u_2+v_3)=18-(-2+12)=8$
3.$d_{31}=c_{31}-(u_3+v_1)=26-(4+16)=6$
4.$d_{32}=c_{32}-(u_3+v_2)=24-(4+10)=10$

	P	Q	R	Supply	ui
A	16 (140)	20 [10]	12 (60)	200	$u_1=0$
B	14 (40)	8 (120)	18 [8]	160	$u_2=-2$
S3	26 [6]	24 [10]	16 (90)	90	$u_3=4$
Demand	180	120	150		
vj	$v_1=16$	$v_2=10$	$v_3=12$		

Since all $d_{ij}\geq 0$.
So final optimal solution is arrived.

	P	Q	R	Supply
A	16 (140)	20	12 (60)	200
B	14 (40)	8 (120)	18	160
S3	26	24	16 (90)	90
Demand	180	120	150	

The minimum total transportation cost $=16\times140+12\times60+14\times40+8\times120+16\times90=5920$

EXAMPLE: Optimality test using Modi method... Allocation Table is

	D1	D2	D3	D4	D5	Supply
S1	8 (200)	10 (50)	12	17	15	250
S2	15	13 (175)	18	11 (75)	9 (50)	300
S3	14	20	6 (275)	10 (125)	13	400
S4	13	19	7	5 (50)	12	50
Demand	200	225	275	250	50	

Iteration-1 of optimality test

1. Find u_i and v_j for all occupied cells(i,j), where $c_{ij}=u_i+v_j$

1. Substituting, $u_2=0$, we get
2.$c_{22}=u_2+v_2\Rightarrow v_2=c_{22}-u_2\Rightarrow v_2=13-0\Rightarrow v_2=13$
3.$c_{12}=u_1+v_2\Rightarrow u_1=c_{12}-v_2\Rightarrow u_1=10-13\Rightarrow u_1=-3$
4.$c_{11}=u_1+v_1\Rightarrow v_1=c_{11}-u_1\Rightarrow v_1=8+3\Rightarrow v_1=11$
5.$c_{24}=u_2+v_4\Rightarrow v_4=c_{24}-u_2\Rightarrow v_4=11-0\Rightarrow v_4=11$
6.$c_{34}=u_3+v_4\Rightarrow u_3=c_{34}-v_4\Rightarrow u_3=10-11\Rightarrow u_3=-1$
7.$c_{33}=u_3+v_3\Rightarrow v_3=c_{33}-u_3\Rightarrow v_3=6+1\Rightarrow v_3=7$
8.$c_{44}=u_4+v_4\Rightarrow u_4=c_{44}-v_4\Rightarrow u_4=5-11\Rightarrow u_4=-6$
9.$c_{25}=u_2+v_5\Rightarrow v_5=c_{25}-u_2\Rightarrow v_5=9-0\Rightarrow v_5=9$

	$D1$	$D2$	$D3$	$D4$	$D5$	Supply	ui
$S1$	8 (200)	10 (50)	12	17	15	250	$u1$=-3
$S2$	15	13 (175)	18	11 (75)	9 (50)	300	$u2$=0
$S3$	14	20	6 (275)	10 (125)	13	400	$u3$=-1
$S4$	13	19	7	5 (50)	12	50	$u4$=-6
Demand	200	225	275	250	50		
vj	$v1$=11	$v2$=13	$v3$=7	$v4$=11	$v5$=9		

2. Find dij for all unoccupied cells(i,j), where $dij=cij-(ui+vj)$

1.$d13=c13-(u1+v3)$=12-(-3+7)=8
2.$d14=c14-(u1+v4)$=17-(-3+11)=9
3.$d15=c15-(u1+v5)$=15-(-3+9)=9
4.$d21=c21-(u2+v1)$=15-(0+11)=4
5.$d23=c23-(u2+v3)$=18-(0+7)=11
6.$d31=c31-(u3+v1)$=14-(-1+11)=4
7.$d32=c32-(u3+v2)$=20-(-1+13)=8
8.$d35=c35-(u3+v5)$=13-(-1+9)=5
9.$d41=c41-(u4+v1)$=13-(-6+11)=8
10.$d42=c42-(u4+v2)$=19-(-6+13)=12
11.$d43=c43-(u4+v3)$=7-(-6+7)=6

12.$d45=c45-(u4+v5)$=12-(-6+9)=9

	$D1$	$D2$	$D3$	$D4$	$D5$	Supply	ui
$S1$	8 (200)	10 (50)	12 [8]	17 [9]	15 [9]	250	$u1$=-3
$S2$	15 [4]	13 (175)	18 [11]	11 (75)	9 (50)	300	$u2$=0
$S3$	14 [4]	20 [8]	6 (275)	10 (125)	13 [5]	400	$u3$=-1
$S4$	13 [8]	19 [12]	7 [6]	5 (50)	12 [9]	50	$u4$=-6
Demand	200	225	275	250	50		
vj	$v1$=11	$v2$=13	$v3$=7	$v4$=11	$v5$=9		

Since all $dij \geq 0$.

So final optimal solution is arrived.

	$D1$	$D2$	$D3$	$D4$	$D5$	Supply
$S1$	8 (200)	10 (50)	12	17	15	250
$S2$	15	13 (175)	18	11 (75)	9 (50)	300
$S3$	14	20	6 (275)	10 (125)	13	400
$S4$	13	19	7	5 (50)	12	50
Demand	200	225	275	250	50	

The minimum total transportation cost $=8\times200+10\times50+13\times175+11\times75+9\times50+6\times275+10\times125+5\times50=8800$

Example: Optimality test using modi method...
Allocation Table is

	$D1$	$D2$	$D3$	$D4$	Supply
$S1$	6	1 (35)	9	3 (35)	70
$S2$	11 (5)	5	2 (50)	8	55
$S3$	10 (80)	12	4	7 (10)	90
Demand	85	35	50	45	

Iteration-1 of optimality test
1. Find u_i and v_j for all occupied cells(i,j), where $c_{ij}=u_i+v_j$

1. Substituting, $u_1=0$, we get
2.$c_{12}=u_1+v_2 \Rightarrow v_2=c_{12}-u_1 \Rightarrow v_2=1-0 \Rightarrow v_2=1$
3.$c_{14}=u_1+v_4 \Rightarrow v_4=c_{14}-u_1 \Rightarrow v_4=3-0 \Rightarrow v_4=3$
4.$c_{34}=u_3+v_4 \Rightarrow u_3=c_{34}-v_4 \Rightarrow u_3=7-3 \Rightarrow u_3=4$
5.$c_{31}=u_3+v_1 \Rightarrow v_1=c_{31}-u_3 \Rightarrow v_1=10-4 \Rightarrow v_1=6$
6.$c_{21}=u_2+v_1 \Rightarrow u_2=c_{21}-v_1 \Rightarrow u_2=11-6 \Rightarrow u_2=5$
7.$c_{23}=u_2+v_3 \Rightarrow v_3=c_{23}-u_2 \Rightarrow v_3=2-5 \Rightarrow v_3=-3$

	$D1$	$D2$	$D3$	$D4$	Supply	u_i
$S1$	6	1 (35)	9	3 (35)	70	$u_1=0$
$S2$	11 (5)	5	2 (50)	8	55	$u_2=5$
$S3$	10 (80)	12	4	7 (10)	90	$u_3=4$
Demand	85	35	50	45		
v_j	$v_1=6$	$v_2=1$	$v_3=-3$	$v_4=3$		

2. Find d_{ij} for all unoccupied cells(i,j), where $d_{ij}=c_{ij}-(u_i+v_j)$
1.$d_{11}=c_{11}-(u_1+v_1)=6-(0+6)=0$
2.$d_{13}=c_{13}-(u_1+v_3)=9-(0-3)=12$
3.$d_{22}=c_{22}-(u_2+v_2)=5-(5+1)=-1$
4.$d_{24}=c_{24}-(u_2+v_4)=8-(5+3)=0$
5.$d_{32}=c_{32}-(u_3+v_2)=12-(4+1)=7$
6.$d_{33}=c_{33}-(u_3+v_3)=4-(4-3)=3$

	$D1$	$D2$	$D3$	$D4$	Supply	u_i
$S1$	6 [0]	1 (35)	9 [12]	3 (35)	70	$u_1=0$
$S2$	11 (5)	5 [-1]	2 (50)	8 [0]	55	$u_2=5$
$S3$	10 (80)	12 [7]	4 [3]	7 (10)	90	$u_3=4$
Demand	85	35	50	45		

v_j	$v_1=6$	$v_2=1$	$v_3=-3$	$v_4=3$		

3. Now choose the minimum negative value from all d_{ij} (opportunity cost) = d_{22} = [-1]
and draw a closed path from S_2D_2.
Closed path is $S_2D_2 \rightarrow S_2D_1 \rightarrow S_3D_1 \rightarrow S_3D_4 \rightarrow S_1D_4 \rightarrow S_1D_2$
Closed path and plus/minus sign allocation...

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	6 [0]	1 (35) (-)	9 [12]	3 (35) (+)	70	$u_1=0$
S_2	11 (5) (-)	5 [-1] (+)	2 (50)	8 [0]	55	$u_2=5$
S_3	10 (80) (+)	12 [7]	4 [3]	7 (10) (-)	90	$u_3=4$
Demand	85	35	50	45		
v_j	$v_1=6$	$v_2=1$	$v_3=-3$	$v_4=3$		

4. Minimum allocated value among all negative position (-) on closed path = 5
Substract 5 from all (-) and Add it to all (+)

	D_1	D_2	D_3	D_4	Supply
S_1	6	1 (30)	9	3 (40)	70
S_2	11	5 (5)	2 (50)	8	55
S_3	10 (85)	12	4	7 (5)	90
Demand	85	35	50	45	

5. Repeat the step 1 to 4, until an optimal solution is obtained.

Iteration-2 of optimality test
1. Find u_i and v_j for all occupied cells(i,j), where $c_{ij}=u_i+v_j$
1. Substituting, $u_1=0$, we get
2.$c_{12}=u_1+v_2 \Rightarrow v_2=c_{12}-u_1 \Rightarrow v_2=1-0 \Rightarrow v_2=1$
3.$c_{22}=u_2+v_2 \Rightarrow u_2=c_{22}-v_2 \Rightarrow u_2=5-1 \Rightarrow u_2=4$
4. $c_{23}=u_2+v_3 \Rightarrow v_3=c_{23}-u_2 \Rightarrow v_3=2-4 \Rightarrow v_3=-2$
5.$c_{14}=u_1+v_4 \Rightarrow v_4=c_{14}-u_1 \Rightarrow v_4=3-0 \Rightarrow v_4=3$
6.$c_{34}=u_3+v_4 \Rightarrow u_3=c_{34}-v_4 \Rightarrow u_3=7-3 \Rightarrow u_3=4$
7.$c_{31}=u_3+v_1 \Rightarrow v_1=c_{31}-u_3 \Rightarrow v_1=10-4 \Rightarrow v_1=6$

	D_1	D_2	D_3	D_4	Supply	u_i
S_1	6	1 (30)	9	3 (40)	70	$u_1=0$
S_2	11	5 (5)	2 (50)	8	55	$u_2=4$
S_3	10 (85)	12	4	7 (5)	90	$u_3=4$
Demand	85	35	50	45		
v_j	$v_1=6$	$v_2=1$	$v_3=-2$	$v_4=3$		

2. Find d_{ij} for all unoccupied cells(i,j), where $d_{ij}=c_{ij}-(u_i+v_j)$
1.$d_{11}=c_{11}-(u_1+v_1)=6-(0+6)=0$
2.$d_{13}=c_{13}-(u_1+v_3)=9-(0-2)=11$

3. $d_{21}=c_{21}-(u_2+v_1)=11-(4+6)=1$
4. $d_{24}=c_{24}-(u_2+v_4)=8-(4+3)=1$
5. $d_{32}=c_{32}-(u_3+v_2)=12-(4+1)=7$
6. $d_{33}=c_{33}-(u_3+v_3)=4-(4-2)=2$

	$D1$	$D2$	$D3$	$D4$	Supply	u_i
$S1$	6 [0]	1 (30)	9 [11]	3 (40)	70	$u_1=0$
$S2$	11 [1]	5 (5)	2 (50)	8 [1]	55	$u_2=4$
$S3$	10 (85)	12 [7]	4 [2]	7 (5)	90	$u_3=4$
Demand	85	35	50	45		
v_j	$v_1=6$	$v_2=1$	$v_3=-2$	$v_4=3$		

Since all $d_{ij} \geq 0$.
So final optimal solution is arrived.

	$D1$	$D2$	$D3$	$D4$	Supply
$S1$	6	1 (30)	9	3 (40)	70
$S2$	11	5 (5)	2 (50)	8	55
$S3$	10 (85)	12	4	7 (5)	90
Demand	85	35	50	45	

The minimum total transportation cost $=1\times30+3\times40+5\times5+2\times50+10\times85+7\times5=1160$
Notice alternate solution is available with unoccupied cell $S1D1$: d_{11} = [0], but with the same optimal value.

Assignment Model: Formulation, optimal solution, Hungarian method, travelling salesman problem.

An assignment problem is a particular case of transportation problem where the **objective is to assign a number of resources to an equal number of activities so as to minimise total cost** or maximize total profit of allocation.

The problem of assignment arises because available resources such as men, machines etc. have varying degrees of efficiency for performing different activities, therefore, cost, profit or loss of performing the different activities is different.

Its goal consists in assigning m resources (usually workers) to n tasks (usually jobs) on a one-to-one basis while minimizing assignment costs. As a general rule, all jobs must be performed by exactly one worker and every worker must be assigned exclusively to one job. Any worker can be assigned to perform any job, incurring in some cost that may vary depending on the work-job assignment.

Basic Notation:

- m = number of worker ($i = 1 \ldots m$)
- n = number of jobs ($j = 1 \ldots n$)
- c_{ij} = unit cost of assigning worker i to job j
- x_{ij} = worker i assigned to job j (1 if assigned, 0 otherwise)

Note: m (number of workers) must be equal to n (number of jobs).

Formulation:

Hungarian Method Steps (Rule)

Step-1: If number of rows is not equal to number of columns, then add dummy rows or columns with cost '0', to make it a balance matrix.

Step-2: a. Identify the minimum element in each row and subtract it from each element of that row.
b. Identify the minimum element in each column and subtract it from every element of that column.

Step-3: Make assignment in the opportunity cost table

a. Identify rows with exactly one unmarked '0'. Make an assignment to this single '0' by marking a square ([0]) around it and cross off all other '0' in the same column.

b. Identify columns with exactly one unmarked '0'. Make an assignment to this single '0' by making a square ([0]) around it and cross off all other '0' in the same rows.

c. If a row and/or column has two or more unmarked '0' and one cannot be chosen by inspection, then choose the cell arbitrarily.

d. Continue this process until all '0' in rows/columns are either assigned or cross off (Ø).

Step-4: a. If the number of assigned cells = the number of rows, then an optimal assignment is found and in case you have chosen a "0" cell arbitrarily, then there may be an alternate optimal solution exists.

b. If the solution is not optimal, then go to Step-5.

Step-5: Draw a set of horizontal and vertical lines to cover all the 0

a. Tick (✓) mark all the rows in which no assigned '0'.

b. Examine Tick (✓) marked rows, if any '0' cell occurs in that row, then tick (✓) mark that column.

c. Examine Tick (✓) marked columns, if any assigned '0' exists in that columns, then tick (✓) mark that row.

d. Repeat this process until no more rows or columns can be marked.

e. Draw a straight line for each unmarked rows and marked columns.

f. If the number of lines is equal to the number of rows, then the current solution is the optimal, otherwise go to step 6

Step-6: Develop the new revised opportunity cost table

a. Select the minimum element, say 'k', from the cells not covered by any line,

b. Subtract 'k' from each element not covered by a line.

c. Add 'k' to each intersection element of two lines.

d. Uncovered elements remains unchanged.

Step-7: Repeat steps 3 to 6 until an optimal solution is arrived.

1. Find Solution of Assignment problem using Hungarian method-1 (MIN case)

job\person	1	2	3	4	5
A	10	5	13	15	16
B	3	9	18	13	6
C	10	7	2	2	2
D	7	11	9	7	12
E	7	9	10	4	12

Solution:

The number of rows = 5 and columns = 5, Here given problem is balanced.

Step-1: Find out each row minimum element and subtract it from that row

job\person	1	2	3	4	5
A	5	0	8	10	11
B	0	6	15	10	3
C	8	5	0	0	0
D	0	4	2	0	5

E	3	5	6	0	8

Step-2: Find out each column minimum element and subtract it from that column.

	1	2	3	4	5
A	5	0	8	10	11
B	0	6	15	10	3
C	8	5	0	0	0
D	0	4	2	0	5
E	3	5	6	0	8

Iteration-1 of steps 3 to 6
Step-3: Make assignment in the opportunity cost table

a. Identify rows with exactly one unmarked 0. Make an assignment to this single 0 by marking a square ([0]) around it and cross off all other 0 in the same column.
b. Identify columns with exactly one unmarked 0. Make an assignment to this single 0 by marking a square ([0]) around it and cross off all other 0 in the same rows.
c. If a row and/or column has two or more unmarked 0 and one cannot be chosen by inspection, then choose the cell arbitrarily.
d. Continue this process until all 0 in rows/columns are either assigned or cross off($\cancel{0}$).

Step-3: Make assignment in the opportunity cost table
(1) Row wise cell (*A*,2) is assigned
(2) Row wise cell (*B*,1) is assigned, so column wise cell (*D*,1) crossed off.
(3) Row wise cell (*D*,4) is assigned, so column wise cell (*C*,4),(*E*,4) crossed off.
(4) Column wise cell (*C*,3) is assigned, so row wise cell (*C*,5) crossed off.

Row wise & column wise assignment shown in table

	1	2	3	4	5
A	5	[0]	8	10	11
B	[0]	6	15	10	3
C	8	5	[0]	$\cancel{0}$	$\cancel{0}$
D	$\cancel{0}$	4	2	[0]	5
E	3	5	6	$\cancel{0}$	8

Row wise & column wise assignment shown in table
Step-4: Number of assignments = 4, number of rows = 5
Which is not equal, so solution is not optimal.

Step-5: Draw a set of horizontal and vertical lines to cover all the 0
Cover the 0 with minimum number of lines
(1) Mark(✓) row *E* since it has no assignment
(2) Mark(✓) column 4 since row *E* has 0 in this column
(3) Mark(✓) row *D* since column 4 has an assignment in this row *D*.
(4) Mark(✓) column 1 since row *D* has 0 in this column
(5) Mark(✓) row *B* since column 1 has an assignment in this row *B*.

(6) Since no other rows or columns can be marked, therefore draw straight lines through the unmarked rows A, C and marked columns 1,4

Tick mark not allocated rows and allocated columns

	1	2	3	4	5	
A	5	[0]	8	10	11	
B	[0]	6	15	10	3	✓ (5)
C	8	5	[0]	$\cancel{0}$	$\cancel{0}$	
D	$\cancel{0}$	4	2	[0]	5	✓ (3)
E	3	5	6	$\cancel{0}$	8	✓ (1)
	✓ (4)			✓ (2)		

Step-6: Develop the new revised table by selecting the smallest element, among the cells not covered by any line (say k = 2)
Subtract k = 2 from every element in the cell not covered by a line.
Add k = 2 to every element in the intersection cell of two lines.

	1	2	3	4	5	
A	7	0	8	12	11	
B	0	4	13	10	1	
C	10	5	0	2	0	
D	0	2	0	0	3	
E	3	3	4	0	6	

Repeat steps 3 to 6 until an optimal solution is obtained.
Iteration : 1
Iteration-2 of steps 3 to 6
Step-3: Make assignment in the opportunity cost table
(1) Row wise cell $(A,2)$ is assigned
(2) Row wise cell $(B,1)$ is assigned, so column wise cell $(D,1)$ crossed off.
(3) Row wise cell $(E,4)$ is assigned, so column wise cell $(D,4)$ crossed off.
(4) Row wise cell $(D,3)$ is assigned, so column wise cell $(C,3)$ crossed off.
(5) Row wise cell $(C,5)$ is assigned

Row wise & column wise assignment shown in table

	1	2	3	4	5
A	7	[0]	8	12	11
B	[0]	4	13	10	1
C	10	5	$\cancel{0}$	2	[0]
D	$\cancel{0}$	2	[0]	$\cancel{0}$	3
E	3	3	4	[0]	6

Step-4: Number of assignments = 5, number of rows = 5, Which is equal, so solution is optimal
Optimal assignments are

	I	2	3	4	5
A	7	[0]	8	I2	II
B	[0]	4	I3	I0	I
C	I0	5	~~0~~	2	[0]
D	~~0~~	2	[0]	~~0~~	3
E	3	3	4	[0]	6

Optimal solution is

Job	Person	Cost
A	2	5
B	I	3
C	5	2
D	3	9
E	4	4
	Total	Rs 23/-

Example:

An airline company has drawn up a new flight schedule involving five flights. To assist in allocating five pilots to the flights, it has asked them to state their preference scores by giving each flight a number out of I0. The higher the number, the greater is the preference. Certain of these flights are unsuitable to some pilots owing to domestic reasons. These have been marked with a -.

Flight Number						
Pilot		I	II	III	IV	V
	A	8	2	-	5	4
	B	I0	9	2	8	4
	C	5	4	9	6	-
	D	3	6	2	8	7
	E	5	6	I0	4	3

What should be the allocation of the pilots to flights in order to meet as many preferences as possible?

Sol: Assignment problem using Hungarian method-I (MIN case)

pilot\flight	I	II	III	IV	V
A	8	2	x	5	4
B	I0	9	2	8	4

C	5	4	9	6	x
D	3	6	2	8	7
E	5	6	10	4	3

Solution:

The number of rows = 5 and columns = 5

	I	II	III	IV	V
A	8	2	M	5	4
B	10	9	2	8	4
C	5	4	9	6	M
D	3	6	2	8	7
E	5	6	10	4	3

Here given problem is balanced.

Step-1: Find out the each row minimum element and subtract it from that row

	I	II	III	IV	V	
A	6	0	M	3	2	(-2)
B	8	7	0	6	2	(-2)
C	1	0	5	2	M	(-4)
D	1	4	0	6	5	(-2)
E	2	3	7	1	0	(-3)

Step-2: Find out the each column minimum element and subtract it from that column.

	I	II	III	IV	V
A	5	0	M	2	2
B	7	7	0	5	2
C	0	0	5	1	M
D	0	4	0	5	5
E	1	3	7	0	0
	(-1)	(-0)	(-0)	(-1)	(-0)

Iteration-1 of steps 3 to 6

Step-3: Make assignment in the opportunity cost table

Step-3: Make assignment in the opportunity cost table

(1) Row wise cell (*A*,*B*) is assigned, so column wise cell (*C*,*B*) crossed off.

(2) Row wise cell (*B*,*C*) is assigned, so column wise cell (*D*,*C*) crossed off.

(3) Row wise cell (*C*,*A*) is assigned, so column wise cell (*D*,*A*) crossed off.

(4) Column wise cell (E,D) is assigned, so row wise cell (E,E) crossed off.

Row wise & column wise assignment shown in table

	I	II	III	IV	V
A	5	[0]	M	2	2
B	7	7	[0]	5	2
C	[0]	0	5	1	M
D	0	4	0	5	5
E	1	3	7	[0]	0

Step-4: Number of assignments = 4, number of rows = 5
Which is not equal, so solution is not optimal.

Step-5: Draw a set of horizontal and vertical lines to cover all the 0
Step-5: Cover the 0 with minimum number of lines
(1) Mark(✓) row D since it has no assignment
(2) Mark(✓) column A since row D has 0 in this column
(3) Mark(✓) column C since row D has 0 in this column
(4) Mark(✓) row C since column A has an assignment in this row C.
(5) Mark(✓) row B since column C has an assignment in this row B.
(6) Mark(✓) column B since row C has 0 in this column
(7) Mark(✓) row A since column B has an assignment in this row A.
(8) Since no other rows or columns can be marked, therefore draw straight lines through the unmarked rows E and marked columns A,B,C

Tick mark for not allocated rows and allocated columns

	I	II	III	IV	V	
A	5	[0]	M	2	2	✓
B	7	7	[0]	5	2	✓
C	[0]	0	5	1	M	✓
D	0	4	0	5	5	✓
E	1	3	7	[0]	0	
	✓	✓	✓			

Step-6: Develop the new revised opportunity cost table
Step-6: Develop the new revised table by selecting the smallest element, among the cells not covered by any line (say k = 1)
Subtract k = 1 from every element in the cell not covered by a line.
Add k = 1 to every element in the intersection cell of two lines.

	I	II	III	IV	V

A	5	0	M	I	I
B	7	7	0	4	I
C	0	0	5	0	M
D	0	4	0	4	4
E	2	4	8	0	0

Repeat steps 3 to 6 until an optimal solution is obtained.

Iteration : I
Iteration-2 of steps 3 to 6
Step-3: Make assignment in the opportunity cost table
(1) Row wise cell (*A*,*B*) is assigned, so column wise cell (*C*,*B*) crossed off.
(2) Row wise cell (*B*,*C*) is assigned, so column wise cell (*D*,*C*) crossed off.
(3) Row wise cell (*D*,*A*) is assigned, so column wise cell (*C*,*A*) crossed off.
(4) Row wise cell (*C*,*D*) is assigned, so column wise cell (*E*,*D*) crossed off.
(5) Row wise cell (*E*,*E*) is assigned

Row wise & column wise assignment shown in table

	I	II	III	IV	V
A	5	[0]	M	I	I
B	7	7	[0]	4	I
C	0	0	5	[0]	M
D	[0]	4	0	4	4
E	2	4	8	0	[0]

Step-4: Number of assignments = 5, number of rows = 5
Which is equal, so solution is optimal

Optimal assignments are as follows

Pilot	Flight
A	II
B	III
C	IV
D	I
E	V

Exercise:

1. In the modification of a plant layout of a factory four new machines M1, M2, M3 and M4 are to be installed in a machine shop. There are five vacant places A, B, C, D and E available. Because of limited space, machine M2 cannot be placed at C and M3 cannot be placed at A. The cost of locating a machine at a place (in hundred rupees) is as follows.

Location						
		A	B	C	D	E
Machine	M1	9	11	15	10	11
	M2	12	9	--	10	9
	M3	--	11	14	11	7
	M4	14	8	12	7	8

Find the optimal assignment schedule.

2. A department has five employees with five jobs to be performed. The time (in hours) each man will take to perform each job is given in the effectiveness matrix.

Employees						
		I	II	III	IV	V
Jobs	A	10	5	13	15	16
	B	3	9	18	13	6
	C	10	7	2	2	2
	D	7	11	9	7	12
	E	7	9	10	4	12

How should the jobs be allocated, one per employee, so as to minimize the total man-hours?

3. A dairy plant has five milk tankers I, II, III, IV & V. These milk tankers are to be used on five delivery routes A, B, C, D, and E. The distances (in kms) between dairy plant and the delivery routes are given in the following distance matrix

	I	II	III	IV	V
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

How the milk tankers should be assigned to the chilling centers so as to minimize the distance travelled?

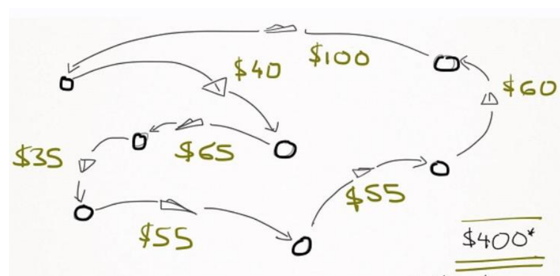
4. A plant manager has four subordinates, and four tasks to be performed. The subordinates differ in efficiency and the tasks differ in their intrinsic difficulty. This estimate of the times each man would take to perform each task is given in the effectiveness matrix below.

	I	II	III	IV
A	8	26	17	11
B	13	28	4	26
C	38	19	18	15
D	19	26	24	10

How should the tasks be allocated, one to a man, so as to minimize the total man hours?

Travelling Salesman Problem Rule

A travelling salesman plans to visit **'n'** cities. He wishes to visit each city only once, and again arriving back to his home city from where he started. So that the total travelling distance is minimum.



If there are **'n'** cities, then there are (n - 1)! possible ways for his tour. For example, if the number of cities to be visited is 4, then there are 3! = 6 different combination is possible. Such type of problems can be solved by Hungarian method, branch and bound method, penalty method, nearest neighbor method.

Find Solution of Travelling salesman problem (MIN case)

City\City	A	B	C	D
A	x	5	8	4
B	5	x	7	4
C	8	7	x	8
D	4	4	8	x

Solution:

The number of rows = 4 and columns = 4

Step-I: Find out each row minimum element and subtract it from that row

City\City	*A*	*B*	*C*	*D*
A	M	1	4	0
B	1	M	3	0
C	1	0	M	1
D	0	0	4	M

Step-2: Find out each column minimum element and subtract it from that column.

City\City	*A*	*B*	*C*	*D*
A	M	I	I	0
B	I	M	0	0
C	I	0	M	I
D	0	0	I	M

Iteration-1 of steps 3 to 6

Step-3: Make assignment in the opportunity cost table

(1) Row wise cell (*A*,*D*) is assigned, so column wise cell (*B*,*D*) crossed off.

(2) Row wise cell (*B*,*C*) is assigned

(3) Row wise cell (*C*,*B*) is assigned, so column wise cell (*D*,*B*) crossed off.

(4) Row wise cell (*D*,*A*) is assigned

Row wise & column wise assignment shown in table

City\City	*A*	*B*	*C*	*D*
A	M	I	I	[0]
B	I	M	[0]	0
C	I	[0]	M	I
D	[0]	0	I	M

Step-4: Number of assignments = 4, number of rows = 4

The solution gives the sequence: $A \to D$, $D \to A$

The above solution is not a solution to the travelling salesman problem as he visits each city only once.

Iteration-2 of steps 3 to 6

The next best solution can be obtained by bringing the minimum non-zero element, i.e., I into the solution.
The cost I occurs at 6 places. We will consider all the cases separately until the acceptable solution is obtained.

Case: I of 6 for minimum non-zero element I

Make the assignment in the cell (*A*, *B*) and repeat Step-3.

Step-3: Make assignment in the opportunity cost table

(1) Row wise cell (*A*, *B*) is assigned, so column wise cell (*C*, *B*), (*D*, *B*) crossed off. and row wise cell (*A*, *C*), (*A*, *D*) crossed off.

(2) Row wise cell (*C*, *A*) is assigned, so column wise cell (*B*, *A*), (*D*, *A*) crossed off. and row wise cell (*C*, *D*) crossed off.

(3) Row wise cell (*D*, *C*) is assigned, so column wise cell (*B*, *C*) crossed off.

(4) Row wise cell (*B*, *D*) is assigned

Row wise & column wise assignment shown in table

City\City	*A*	*B*	*C*	*D*
A	M	[I]	I	0
B	I	M	0	[0]

C	[I]	0	M	I
D	0	0	[I]	M

Step-4: Number of assignments = 4, number of rows = 4
The solution gives the sequence: $A \to B$, $B \to D$, $D \to C$, $C \to A$

So solution is optimal

Optimal assignments are

City\City	*A*	*B*	*C*	*D*
A	M	[I]	I	0
B	I	M	0	[0]
C	[I]	0	M	I
D	0	0	[I]	M

Optimal solution is

City	City	Cost
A	*B*	5
B	*D*	4
C	*A*	8
D	*C*	8
TTC		25

Find Solution of Travelling salesman problem (MIN case)

City\City	A	B	C	D
A	x	5	8	4
B	5	x	7	4
C	8	7	x	8
D	4	4	8	x

Solution:

The number of rows = 4 and columns = 4

Step-1: Find out each row minimum element and subtract it from that row

City\City	A	B	C	D
A	M	I	4	0
B	I	M	3	0
C	I	0	M	I
D	0	0	4	M

Step-2: Find out each column minimum element and subtract it from that column.

City\City	A	B	C	D
A	M	I	I	0
B	I	M	0	0
C	I	0	M	I
D	0	0	I	M

Iteration-1 of steps 3 to 6

Step-3: Make assignment in the opportunity cost table

Step-3: Make assignment in the opportunity cost table

(1) Row wise cell (A,D) is assigned, so column wise cell (B,D) crossed off.

(2) Row wise cell (B,C) is assigned

(3) Row wise cell (C,B) is assigned, so column wise cell (D,B) crossed off.

(4) Row wise cell (D,A) is assigned

Row wise & column wise assignment shown in table

City\City	A	B	C	D
A	M	I	I	[0]
B	I	M	[0]	0
C	I	[0]	M	I
D	[0]	0	I	M

Step-4: Number of assignments = 4, number of rows = 4

The solution gives the sequence: $A \to D$, $D \to A$

The above solution is not a solution to the travelling salesman problem.

Iteration-2 of steps 3 to 6

The next best solution can be obtained by bringing the minimum non-zero element, i.e., I into the solution.

The cost I occurs at 6 places. We will consider all the cases separately until the acceptable solution is obtained.

Case: I of 6 for minimum non-zero element I
Make the assignment in the cell (A, B) and repeat Step-3.

Step-3: Make assignment in the opportunity cost table
(1) Row wise cell (A, B) is assigned, so column wise cell (C, B), (D, B) crossed off. and row wise cell (A, C), (A, D) crossed off.
(2) Row wise cell (C, A) is assigned, so column wise cell (B, A),(D,A) crossed off. and row wise cell (C, D) crossed off.
(3) Row wise cell (D, C) is assigned, so column wise cell (B, C) crossed off.
(4) Row wise cell (B, D) is assigned

Row wise & column wise assignment shown in table

City\City	*A*	*B*	*C*	*D*
A	M	[I]	I	0
B	I	M	0	[0]
C	[I]	0	M	I
D	0	0	[I]	M

Step-4: Number of assignments = 4, number of rows = 4
The solution gives the sequence: $A \rightarrow B$, $B \rightarrow D$, $D \rightarrow C$, $C \rightarrow A$

So solution is optimal

Optimal assignments are

City\City	A	B	C	D
A	M	[I]	I	0
B	I	M	0	[0]
C	[I]	0	M	I
D	0	0	[I]	M

Optimal solution is

City	City	Cost
A	B	5
B	D	4
C	A	8
D	C	8
	Total	25

Find Solution of Travelling salesman problem (MIN case)

City \City	A	B	C	D	E
A	x	5	8	4	5
B	5	x	7	4	5
C	8	7	x	8	6
D	4	4	8	x	8
E	5	5	6	8	x

Solution:

The number of rows = 5 and columns = 5

	A	*B*	*C*	*D*	*E*
A	M	5	8	4	5
B	5	M	7	4	5
C	8	7	M	8	6
D	4	4	8	M	8
E	5	5	6	8	M

Step-1: Find out each row minimum element and subtract it from that row

	A	*B*	*C*	*D*	*E*
A	M	1	4	0	1
B	1	M	3	0	1
C	2	1	M	2	0
D	0	0	4	M	4
E	0	0	1	3	M

Step-2: Find out each column minimum element and subtract it from that column.

	A	*B*	*C*	*D*	*E*
A	M	1	3	0	1
B	1	M	2	0	1
C	2	1	M	2	0
D	0	0	3	M	4
E	0	0	0	3	M

	A	*B*	*C*	*D*	*E*

A	M	1	3	[0]	1
B	1	M	2	0	1
C	2	1	M	2	[0]
D	[0]	0	3	M	4
E	0	0	[0]	3	M

	A	*B*	*C*	*D*	*E*	
A	M	1	3	[0]	1	✓(3)
B	1	M	2	0	1	✓(1)
C	2	1	M	2	[0]	
D	[0]	0	3	M	4	
E	0	0	[0]	3	M	
				✓ (2)		

Step-6: Develop the new revised opportunity cost table

	A	*B*	*C*	*D*	*E*
A	M	0	2	0	0
B	0	M	1	0	0
C	2	1	M	3	0
D	0	0	3	M	4
E	0	0	0	4	M

Repeat steps 3 to 6 until an optimal solution is arrived.

	A	*B*	*C*	*D*	*E*
A	M	[0]	2	0	0
B	0	M	1	[0]	0
C	2	1	M	3	[0]
D	[0]	0	3	M	4
E	0	0	[0]	4	M

Step-4: Number of assignments = 5, number of rows = 5
The solution gives the sequence : $A \rightarrow B, B \rightarrow D, D \rightarrow A$

Step-3: Make assignment in the opportunity cost table

(1) Row wise cell (C,E) is assigned, so column wise cell (A,E),(B,E) crossed off.
(2) Column wise cell (E,C) is assigned, so row wise cell (E,A),(E,B) crossed off.
(3) Row wise cell (A,D) is assigned, so column wise cell (B,D) crossed off. and row wise cell (A, B) crossed off.
(4) Row wise cell (B,A) is assigned, so column wise cell (D,A) crossed off.
(5) Row wise cell (D,B) is assigned

Row wise & column wise assignment shown in table

	A	B	C	D	E
A	M	0	2	[0]	0
B	[0]	M	1	0	0
C	2	1	M	3	[0]
D	0	[0]	3	M	4
E	0	0	[0]	4	M

Step-4: Number of assignments = 5, number of rows = 5
The solution gives the sequence: $A \to D, D \to B, B \to A$
The above solution is not a solution to the travelling salesman problem.

Iteration-3 of steps 3 to 6

The next best solution can be obtained by bringing the minimum non-zero element, i.e., 1 into the solution.
The cost 1 occurs at 2 places. We will consider all the cases separately until the acceptable solution is obtained.

Case: 1 of 2 for minimum non-zero element 1
Make the assignment in the cell (B,C) and repeat Step-3.

Step-3: Make assignment in the opportunity cost table
(1) Row wise cell (B,C) is assigned, so column wise cell (E,C) crossed off. and row wise cell (B,A),(B,D),(B,E) crossed off.
(2) Column wise cell (A,D) is assigned, so row wise cell (A,B),(A,E) crossed off.
(3) Column wise cell (C,E) is assigned, so row wise cell (C,B) crossed off.
(4) Row wise cell (D,A) is assigned, so column wise cell (E,A) crossed off. and row wise cell (D,B) crossed off.
(5) Row wise cell (E,B) is assigned

Row wise & column wise assignment shown in table

	A	B	C	D	E
A	M	0	2	[0]	0
B	0	M	[1]	0	0
C	2	1	M	3	[0]
D	[0]	0	3	M	4
E	0	[0]	0	4	M

Step-4: Number of assignments = 5, number of rows = 5
The solution gives the sequence : $A \to D, D \to A$

Step-3: Make assignment in the opportunity cost table
(1) Row wise cell (B,C) is assigned, so column wise cell (E,C) crossed off. and row wise cell (B,A),(B,D),(B,E) crossed off.
(2) Column wise cell (A,D) is assigned, so row wise cell (A,B),(A,E) crossed off.
(3) Column wise cell (C,E) is assigned, so row wise cell (C,B) crossed off.
(4) Row wise cell (D,B) is assigned, so column wise cell (E,B) crossed off. and row wise cell (D,A) crossed off.
(5) Row wise cell (E,A) is assigned

Row wise & column wise assignment shown in table

	A	B	C	D	E
A	M	0	2	[0]	0
B	0	M	[1]	0	0
C	2	1	M	3	[0]
D	0	[0]	3	M	4
E	[0]	0	0	4	M

Step-4: Number of assignments = 5, number of rows = 5
The solution gives the sequence: $A \to D, D \to B, B \to C, C \to E, E \to A$

So solution is optimal

Optimal assignments are

	A	B	C	D	E	
A	M	0	2	[0]	0	
B	0	M	[1]	0	0	
C	2	1	M	3	[0]	
D	0	[0]	3	M	4	
E	[0]	0	0	4	M	

Optimal solution is

City	City	Cost
A	D	4
B	C	7
C	E	6
D	B	4
E	A	5
Total transportation cost		26