Kalman Filtering in Bayesian State Estimation

Bayesian State Estimation in Dynamical Systems

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Contents

1 Kalman Filtering as a Special Case of Gaussian Filtering

In this section, we derive the Kalman filtering equations as an exact and closed-form specialization of the Gaussian filtering framework.

We begin by restating the full Gaussian filtering equations. We then assume linear dynamics and linear observations, substitute these models into the Gaussian filtering recursion, and evaluate each integral explicitly. Unlike the Extended Kalman Filter, Unscented Kalman Filter, or other approximate Gaussian filters, the Kalman filter introduces no approximation errors. All integrals admit closed-form solutions due to linearity and Gaussianity.

1.1 Gaussian Filtering Equations

Gaussian filtering approximates the Bayesian filtering recursion by enforcing Gaussian predictive and filtering distributions:

$$\mathbb{P}(X_k \mid \mathcal{Y}_{k-1}) \approx \mathcal{N}(m_k^-, C_k^-), \qquad \mathbb{P}(X_k \mid \mathcal{Y}_k) \approx \mathcal{N}(m_k, C_k).$$

The Gaussian filtering recursion consists of the following steps.

Prediction:

$$m_k^- = \int \Phi(X_{k-1})\, \mathcal{N}(X_{k-1}; m_{k-1}, C_{k-1})\, dX_{k-1},$$

$$C_k^- = \int \big(\Phi(X_{k-1}) - m_k^-\big)\big(\Phi(X_{k-1}) - m_k^-\big)^T \mathcal{N}(X_{k-1}; m_{k-1}, C_{k-1})\, dX_{k-1} + \Sigma.$$

Update: Define

$$\mu = \int h(X_k)\, \mathcal{N}(X_k; m_k^-, C_k^-)\, dX_k,$$

$$U = \int (X_k - m_k^-)\big(h(X_k) - \mu\big)^T \mathcal{N}(X_k; m_k^-, C_k^-)\, dX_k,$$

$$S = \int \big(h(X_k) - \mu\big)\big(h(X_k) - \mu\big)^T \mathcal{N}(X_k; m_k^-, C_k^-)\, dX_k + \Gamma.$$

The filtering update is then

$$m_k = m_k^- + US^{-1}(y_k - \mu), \qquad C_k = C_k^- - US^{-1}U^T.$$

1.2 Linear Gaussian State-Space Model

We now assume the linear Gaussian system

$$X_{k+1} = AX_k + \xi_k, \qquad \xi_k \sim \mathcal{N}(0, \Sigma),$$
$$Y_k = HX_k + \eta_k, \qquad \eta_k \sim \mathcal{N}(0, \Gamma),$$

corresponding to

$$\Phi(X_k) = AX_k, \qquad h(X_k) = HX_k.$$

Under this model, all Gaussian filtering integrals can be evaluated exactly.

1.3 Prediction Step

1.3.1 Predicted Mean

Substituting $\Phi(X_{k-1}) = AX_{k-1}$,

$$m_k^- = \int AX_{k-1}\,\mathcal{N}(X_{k-1}; m_{k-1}, C_{k-1})\,dX_{k-1}.$$

Pulling the linear operator outside the integral,

$$m_k^- = A\int X_{k-1}\,\mathcal{N}(X_{k-1}; m_{k-1}, C_{k-1})\,dX_{k-1} = A\,\mathbb{E}[X_{k-1} \mid \mathcal{Y}_{k-1}] = Am_{k-1}.$$

1.3.2 Predicted Covariance

Substituting $\Phi(X_{k-1}) = AX_{k-1}$ and $m_k^- = Am_{k-1}$,

$$C_k^- = \int A(X_{k-1} - m_{k-1})(X_{k-1} - m_{k-1})^T A^T\,\mathcal{N}(X_{k-1}; m_{k-1}, C_{k-1})\,dX_{k-1} + \Sigma.$$

Pulling the linear operators outside the integral,

$$C_k^- = A\int (X_{k-1} - m_{k-1})(X_{k-1} - m_{k-1})^T\,\mathcal{N}(X_{k-1}; m_{k-1}, C_{k-1})\,dX_{k-1}A^T + \Sigma.$$

Recognizing the covariance,

$$C_k^- = A\ \mathrm{Cov}(X_{k-1} \mid \mathcal{Y}_{k-1})\,A^T + \Sigma = AC_{k-1}A^T + \Sigma.$$

1.4 Update Step

Assume

$$X_k \mid \mathcal{Y}_{k-1} \sim \mathcal{N}(m_k^-, C_k^-).$$

1.4.1 Predicted Measurement Mean

Substituting $h(X_k) = HX_k$,

$$\mu = \int HX_k\,\mathcal{N}(X_k; m_k^-, C_k^-)\,dX_k.$$

Pulling H outside,

$$\mu = H\int X_k\,\mathcal{N}(X_k; m_k^-, C_k^-)\,dX_k = H\,\mathbb{E}[X_k \mid \mathcal{Y}_{k-1}] = Hm_k^-.$$

1.4.2 Cross-Covariance

$$U = \int (X_k - m_k^-)(X_k - m_k^-)^T H^T\,\mathcal{N}(X_k; m_k^-, C_k^-)\,dX_k.$$

Recognizing the covariance,

$$U = \mathrm{Cov}(X_k \mid \mathcal{Y}_{k-1})H^T = C_k^- H^T.$$

1.4.3 Innovation Covariance

$$S = \int H(X_k - m_k^-)(X_k - m_k^-)^T H^T \mathcal{N}(X_k; m_k^-, C_k^-)\, dX_k + \Gamma.$$

Pulling linear operators outside,

$$S = H \int (X_k - m_k^-)(X_k - m_k^-)^T \mathcal{N}(X_k; m_k^-, C_k^-)\, dX_k H^T + \Gamma.$$

Thus,

$$S = H \ \mathrm{Cov}(X_k \mid \mathcal{Y}_{k-1})\, H^T + \Gamma = HC_k^- H^T + \Gamma.$$

1.5 Kalman Filtering Equations

Prediction Step

$$m_k^- = Am_{k-1},$$
$$C_k^- = AC_{k-1}A^T + \Sigma.$$

Update Step

$$\mu = Hm_k^-,$$
$$U = C_k^- H^T,$$
$$S = HC_k^- H^T + \Gamma.$$

$$m_k = m_k^- + US^{-1}(y_k - \mu),$$
$$C_k = C_k^- - US^{-1}U^T.$$

1.6 Conclusion

The Kalman filter is an exact realization of Gaussian filtering for linear systems with Gaussian noise. No approximations are introduced. All subsequent Gaussian filters, including the Extended Kalman Filter and Unscented Kalman Filter, arise by approximating the same Gaussian filtering integrals when linearity is lost.