Optimal Proposal for Linear Measurement Model in Particle Filtering

Particle Filtering in Dynamical Systems

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1 Exact Optimal Proposal for Linear Measurement Models

We begin from the Gaussian-approximated optimal proposal recursion.

For each particle $x_{k-1}^{(i)}$, define

$$m_k^- = \Phi(x_{k-1}^{(i)}), \qquad C_k^- = \Sigma.$$

The Gaussian-approximated quantities are

$$\mu_k = \int h(x_k)\,\mathcal{N}(x_k; m_k^-, C_k^-)\,dx_k,$$

$$S_k = \int \big(h(x_k) - \mu_k\big)\big(h(x_k) - \mu_k\big)^T \mathcal{N}(x_k; m_k^-, C_k^-)\,dx_k + \Gamma,$$

$$U_k = \int (x_k - m_k^-)\big(h(x_k) - \mu_k\big)^T \mathcal{N}(x_k; m_k^-, C_k^-)\,dx_k.$$

The corresponding Gaussian optimal proposal is

$$X_k \mid x_{k-1}^{(i)}, y_k \sim \mathcal{N}(m_k^{(i)}, C_k^{(i)}),$$

with

$$m_k^{(i)} = m_k^- + U_k S_k^{-1}(y_k - \mu_k), \qquad C_k^{(i)} = C_k^- - U_k S_k^{-1} U_k^T.$$

We now specialize these expressions to a linear measurement model.

Linear Measurement Model

Assume

$$Y_k = HX_k + \eta_k, \qquad \eta_k \sim \mathcal{N}(0, \Gamma),$$

so that

$$h(x_k) = Hx_k.$$

Substituting into the predictive measurement mean,

$$\mu_k = \int Hx_k\,\mathcal{N}(x_k; m_k^-, C_k^-)\,dx_k = Hm_k^-.$$

Substituting into the innovation covariance,

$$S_k = \int H(x_k - m_k^-)(x_k - m_k^-)^T H^T \mathcal{N}(x_k; m_k^-, C_k^-)\,dx_k + \Gamma = HC_k^- H^T + \Gamma.$$

Substituting into the cross-covariance,

$$U_k = \int (x_k - m_k^-)(x_k - m_k^-)^T H^T \mathcal{N}(x_k; m_k^-, C_k^-)\,dx_k = C_k^- H^T.$$

Exact Optimal Proposal (Linear Measurement)

Collecting the results,

$$\mu_k = Hm_k^-, \qquad S_k = HC_k^- H^T + \Gamma, \qquad U_k = C_k^- H^T.$$

The optimal proposal is therefore

$$X_k \mid x_{k-1}^{(i)}, y_k \sim \mathcal{N}(m_k^{(i)}, C_k^{(i)}),$$

with

$$m_k^{(i)} = m_k^- + C_k^- H^T \left(HC_k^- H^T + \Gamma\right)^{-1} \left(y_k - Hm_k^-\right),$$

$$C_k^{(i)} = C_k^- - C_k^- H^T \left(HC_k^- H^T + \Gamma\right)^{-1} HC_k^-.$$

The corresponding weight update is

$$w_k^{(i)} = \bar{w}_{k-1}^{(i)} \mathcal{N}\big(y_k; Hm_k^-, \, HC_k^- H^T + \Gamma\big).$$

All expressions are exact and introduce no approximation.