

Bayesian Modeling of Dynamical Systems

Bayesian-Inference

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1 Bayesian Modeling of Dynamical Systems

1.1 Problem Statement in Robotics

In robotics (humanoid robots, quadrupeds, mobile robots), we deal with systems that evolve over time but whose internal states are not directly observable. Instead, we interact with the system through sensors that provide indirect, noisy measurements.

The central problem is not simulation, but **inference under uncertainty from partial and noisy observations**. Given sensor data, how do we infer the system's behavior and unknown parameters while respecting the temporal structure imposed by the dynamics?

This naturally leads to a probabilistic formulation in which uncertainty is modeled explicitly. Bayesian modeling provides a principled way to combine prior knowledge, system dynamics, sensor models, and observed data.

1.2 Notation

We use the following notation throughout:

- k : discrete time index, $k = 0, 1, \dots, n$
- X_k : latent system state at time k
- Y_k : observation random variable at time k
- y_k : realized measurement at time k
- $\Phi(\cdot; \theta_1)$: dynamics operator
- $h(\cdot; \theta_2)$: observation operator
- θ_1 : dynamics parameters
- θ_2 : observation (sensor) parameters
- ξ_k : process noise
- η_k : measurement noise

1.3 Dynamics Model

The evolution of the system state is modeled as

$$X_{k+1} = \Phi(X_k; \theta_1) + \xi_k, \quad \xi_k \sim \mathcal{N}(0, \Sigma).$$

Here:

- $\Phi(\cdot; \theta_1)$ represents the nominal physical model
- θ_1 contains unknown physical parameters such as mass, friction, inertia, contact parameters, or actuator gains
- ξ_k is **process noise**, capturing unmodeled dynamics, disturbances, and modeling error

The dynamics satisfy the Markov property:

$$P(X_{k+1} | X_{0:k}, \theta_1) = P(X_{k+1} | X_k, \theta_1).$$

1.4 Observation Model

The system state is not directly observed. Instead, measurements are generated by sensors according to

$$Y_k = h(X_k; \theta_2) + \eta_k, \quad \eta_k \sim \mathcal{N}(0, \Gamma).$$

Here:

- $h(\cdot; \theta_2)$ is the sensor model
- θ_2 represents unknown calibration parameters (bias, scale, extrinsics)
- η_k is **measurement noise**, capturing sensor corruption

1.5 What Is Uncertain and Why

The Bayesian formulation is motivated by the fact that, in real robotic systems, multiple components of the system are inherently uncertain. These uncertainties arise not because of poor engineering, but because of physical variability, limited sensing, and incomplete modeling.

We describe below each uncertain quantity and explain why it must be treated probabilistically.

(U1) Initial State X_0 The initial state of a robot is never known exactly. For example:

- A humanoid robot may be powered on with unknown joint offsets.
- A quadruped may start with uncertain base pose due to foot slippage.
- A mobile robot may have localization error at startup.

Even if encoders or IMUs provide measurements, these sensors themselves are noisy and biased. Therefore, the initial state X_0 must be modeled as a random variable rather than a fixed value.

(U2) Latent States X_1, X_2, \dots, X_n The system states at future times are uncertain because:

- The system evolves through nonlinear dynamics.
- External disturbances (contacts, terrain irregularities, wind) affect motion.
- States are not directly observable; they must be inferred through sensors.

For example, a quadruped's center-of-mass velocity or a humanoid's contact forces are not directly measurable. These states evolve over time and accumulate uncertainty, making the entire trajectory $X_{1:n}$ uncertain.

(U3) Dynamics Parameters θ_1 The dynamics parameters describe physical properties of the system, such as:

- Mass distribution and inertia
- Joint friction and damping
- Contact stiffness and friction coefficients

In practice, these parameters are only approximately known. Manufacturing tolerances, payload changes, wear-and-tear, and environment-dependent effects cause the true parameters to differ from nominal values. As a result, θ_1 must be inferred rather than assumed known.

(U4) Process Noise ξ_k Even with a well-structured dynamics model, no model captures reality exactly. Process noise accounts for:

- Unmodeled dynamics (flexibility, backlash, compliance)
- External disturbances (pushes, uneven terrain)
- Simplifications in the physics model

For example, a humanoid walking model may neglect joint compliance or ground deformation. Process noise ξ_k captures these discrepancies and allows the model to remain realistic.

(U5) Observation Parameters θ_2 Sensors are not perfectly calibrated. Observation parameters include:

- IMU bias and scale factors
- Camera extrinsics
- Encoder offsets

These parameters can drift over time or change due to temperature and mechanical stress. Therefore, the observation model itself contains unknown parameters that must be estimated.

(U6) Measurement Noise η_k Sensor measurements are corrupted by noise due to:

- Electronic noise
- Quantization effects
- Environmental interference

For instance, lidar returns may be noisy due to reflective surfaces, and force sensors may suffer from vibration-induced noise. Measurement noise η_k explicitly models this uncertainty.

Summary In summary, uncertainty arises from three fundamental sources:

1. Imperfect knowledge of the system state (U1, U2)
2. Imperfect knowledge of physical parameters (U3, U5)
3. Imperfect sensing and modeling (U4, U6)

Bayesian modeling provides a unified framework to represent and propagate all these uncertainties through time.

1.6 Bayesian Network Representation

The above assumptions define a Bayesian network with the following structure:

- Sequential state evolution:

$$X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_n$$

- Observations conditioned on states:

$$X_k \rightarrow Y_k$$

- Global dynamics parameters influencing all transitions:

$$\theta_1 \rightarrow X_0, X_1, \dots, X_n$$

- Global observation parameters influencing all measurements:

$$\theta_2 \rightarrow Y_1, Y_2, \dots, Y_n$$

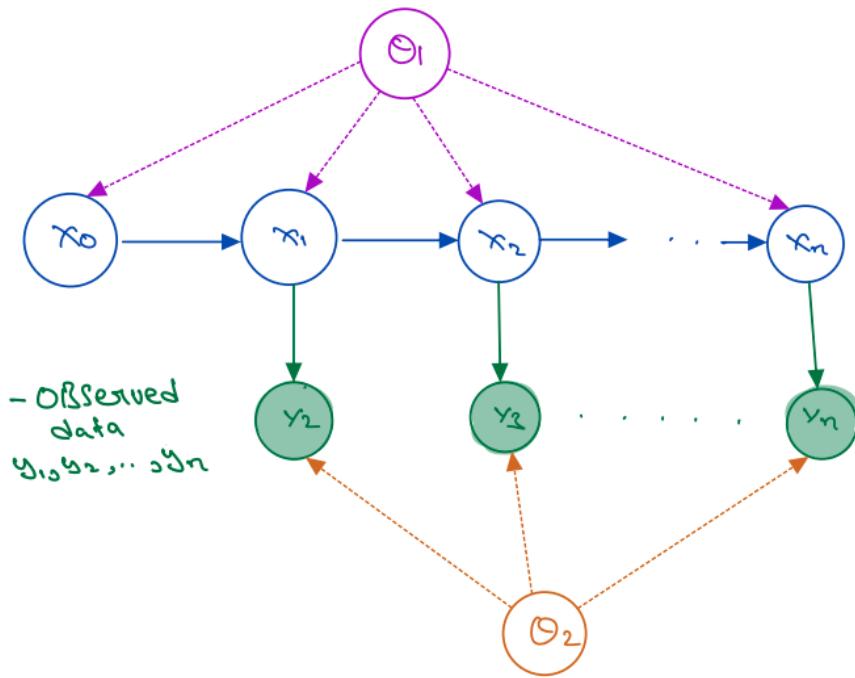


Figure 1: Bayesian network for a dynamical system with latent states, observations, and global parameters.

1.7 Joint Distribution

The Bayesian network implies the following factorization of the joint distribution:

$$\begin{aligned} P(X_0, X_1, \dots, X_n, Y_1, \dots, Y_n, \theta_1, \theta_2) &= P(\theta_1) P(\theta_2) P(X_0) \\ &\times \prod_{k=1}^n P(X_k | X_{k-1}, \theta_1) \prod_{k=1}^n P(Y_k | X_k, \theta_2). \end{aligned}$$

This expression encodes all modeling assumptions.

Interpretation The joint distribution formalizes how uncertainty propagates through the system over time. Uncertainty in the initial state and model parameters flows forward through the dynamics, while uncertainty in sensing limits how accurately latent states can be inferred from data.

The Bayesian network explicitly encodes these dependency relationships, and the joint distribution serves as the mathematical object from which all inference tasks are derived. Every posterior, likelihood, or marginal distribution considered later is obtained by conditioning or marginalizing this joint distribution.

1.8 Prior, Likelihood, and Posterior

Prior The prior captures knowledge before observing data:

$$P(\theta_1), \quad P(\theta_2), \quad P(X_0).$$

Likelihood Conditioning on latent states and parameters yields the likelihood:

$$P(Y_{1:n} | X_{0:n}, \theta_2) = \prod_{k=1}^n P(Y_k | X_k, \theta_2).$$

This factorization follows directly from the Bayesian network structure and reflects the conditional independence of measurements given the latent state trajectory and observation parameters.

Given observed data $y_{1:n}$, this becomes

$$P(y_{1:n} | X_{0:n}, \theta_2).$$

Posterior The posterior distribution is

$$P(X_{0:n}, \theta_1, \theta_2 | y_{1:n}),$$

which is the central object of inference.

1.9 Scope of This Report

This report focuses on:

- Defining dynamics and observation models
- Identifying sources of uncertainty
- Constructing the Bayesian network
- Writing the joint distribution
- Defining prior, likelihood, and posterior targets

No specific system is analyzed here.

The next report instantiates this framework for a concrete model (e.g., SIR dynamics or a robotic system) and performs Bayesian inference.

While this report focuses on formulation, the resulting posterior distributions are typically high-dimensional and analytically intractable, motivating approximate inference methods in subsequent work.