Lec 28: Theory, Fundamental matrix variation of Parameter's

Theory:

$$\boxed{\vec{x}' = A\vec{x}}$$

$n \times 2$ homogenou's

(A constant)

3 theorem we need to understand

$\boxed{A}$ General solution to $\vec{x}' = A\vec{x}$

$$\vec{x} = C_1\vec{x}_1 + C_2\vec{x}_2$$

Linear combination of two independent solution's.

Where $\vec{x}_1$, $\vec{x}_2$ two linearly independent solution's (neither constant multiple of other) if $\vec{x}_1$ and $\vec{x}_2$ are two independent solution then $c_1\vec{x}_1 + c_2\vec{x}_2$ is also a solution

$$\left.\begin{aligned}\vec{x}_1' &= A\vec{x}_1 \\ \vec{x}_2' &= A\vec{x}_2\end{aligned}\right\} \Rightarrow \left(c_1\vec{x}_1 + c_2\vec{x}_2\right)'$$

$$\Rightarrow c_1\vec{x}_1' + c_2\vec{x}_2'$$

$$\Rightarrow c_1 A x_1 + c_2 A x_2$$

$$\Rightarrow A(c_1 x_1 + c_2 x_2)$$

* Linearity & Superposition.

EASY:- All these are solution's (Linearity + Superposition)

HARD:- These are all the solution's (There are no other solution's)

$\boxed{B}$ Wronskian of two solⁿ

$$W(\vec{x}_1, \vec{x}_2) := \left| \vec{x}_1 \;\; \vec{x}_2 \right|$$

(determinant)

Theorem :-

either $W(x_1, x_2) = W(t)$

$\rightarrow \equiv 0 \;\; \forall t$ (if x_1, x_2 are dependent)

$\rightarrow$ never zero $\forall t$ ($\vec{x}_1(t)$, $\vec{x}_2(t)$ are independent)

Fundamental matrix for $\vec{x}\,' = A\vec{x}$

matrix whose two coloumn's are two solution's.

$$\underline{X} := \begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix}$$

$\vec{x}_1$ & $\vec{x}_2$ are two independent solution's.

Properties:

① $|\underline{X}| \neq 0$ for any t
(Wronskian)

② $\underline{X}' = AX$

$\Updownarrow$

$$[\vec{x}_1{}' \quad \vec{x}_2{}'] = A[\vec{x}_1 \quad \vec{x}_2]$$

$$= [A\vec{x}_1 \quad A\vec{x}_2]$$

$$= [\vec{x}_1{}' \quad \vec{x}_2{}']$$

* Solving inhomogenous system's

$$x' = ax + by + r_1(t)$$

$$y' = cx + dy + r_2(t)$$

$$\Rightarrow \boxed{\vec{x}' = A\vec{x} + \vec{r}(t)}$$

theorem

$\boxed{C}$

$$\vec{X}_{gen} = \vec{X}_c + \vec{X}_p$$

general sol^n to $\vec{x}' = Ax$ Particular solution.

(Linearity & superposition)

we need to find a Particular sol^n

Ex:

$$x' = -3x + 2y + 5e^{-t}$$
$$y' = 3x - 4y + 0$$

$$\vec{x}' = \begin{bmatrix} -3 & 2 \\ 3 & -4 \end{bmatrix} \vec{x} + \begin{bmatrix} 5e^{-t} \\ 0 \end{bmatrix}$$

* Method to solve $\vec{X}' = A\vec{X} + \vec{r}(t)$

find $\vec{X}_p$

Variation of Parameter's:

$$\vec{X}_p = V_1(t)\,\vec{X}_1 + V_2(t)\,\vec{X}_2$$

look for a solution of that form

$$\vec{x}_p = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

$$\Rightarrow \quad \vec{x}_p = \underline{\overline{X}} \; \vec{v}(t) = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$$

Substitute into system

$$\vec{x}_p' = A x_p + \vec{r}(t)$$

$$\Rightarrow \quad \underline{\overline{X}}' v(t) + \underline{\overline{X}} \, \vec{v}'(t) = A x_p + \vec{r}(t)$$

$$\Rightarrow \quad \cancel{A \underline{\overline{X}} \vec{v}(t)} + \underline{\overline{X}} \, \vec{v}'(t) = \cancel{A \underline{\overline{X}} \vec{v}(t)} + \vec{r}(t)$$

$$\Rightarrow \quad \underline{\overline{X}} \, \vec{v}'(t) = \vec{r}(t)$$

$$\Rightarrow \quad \vec{v}'(t) = \underline{\overline{X}}^{-1} \vec{r}(t)$$

$$\vec{v}(t) = \int_0^t \underline{\overline{X}}^{-1} \vec{r}(s) \, ds$$

Particular sol^n is $\vec{x}_p = \underline{\overline{X}} \, \vec{v}(t)$

$$\boxed{\Rightarrow \quad \vec{x}_p = \underline{\overline{X}} \int_0^t \underline{\overline{X}}^{-1} \vec{r}(t) \, dt}$$