Lec21 :- Convolution formula

Convolution:

Some combination of 2 function's to get 3rd function.

$f(t) * g(t)$ Convolution.

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$G(s) = \int_0^\infty e^{-st} g(t)\, dt$$

$$F(s) \cdot G(s) = \int_0^\infty e^{-st} \big(f(t) * g(t)\big)\, dt$$

Laplace transform is a continuous analog of Power series.

$$F(x) = \sum a_n x^n \qquad n \longrightarrow x \longrightarrow e^{-s}$$

$$G(x) = \sum b_n x^n$$

$$F(x)\,G(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right) \sum_{n=0}^{\infty} b_n x^n$$

$$F(x)\,G(x) = \sum_{n=0}^{\infty} c_n x^n$$

Ex:

$$F(x) = a_0 + a_1 x + a_2 x^2$$

$$G(x) = b_0 + b_1 x + b_2 x^2$$

$$F(x)\,G(x) = (a_0 + a_1 x + a_2 x^2)(b_0 + b_1 x + b_2 x^2)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)\,x$$
$$+ (a_0 b_2 + a_1 b_1 + a_2 b_0)\,x^2$$
$$+ (a_2 b_1 + b_2 a_1)\,x^3$$
$$(a_2 b_2)\,x^4$$

if $F(x) = \sum_{n=0}^{\infty} a_n x^n$

$G(x) = \sum_{n=0}^{\infty} b_n x^n$

$$F(x)\,G(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$c_n = \sum_{i=0}^{n} a_i \, b_{n-i}$$

$$\boxed{f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du}$$

$$t^2 * t = \int_0^t u^2 (t-u)\, du = t \cdot \frac{u^3}{3}\Big]_0^t - \frac{u^4}{4}\Big]_0^t$$

$$= \frac{t^4}{3} - \frac{t^4}{4} = \frac{t^4}{12}$$

$$\mathcal{L}(t^2) = \frac{2}{s^3}$$

$$\mathcal{L}(t) = \frac{1}{s^2} \Rightarrow \mathcal{L}(t^2 \cdot t) = \frac{2}{s^5} = \frac{1}{12}\,\frac{4!}{s^4}$$

$$\mathcal{L}^{-1}\left(\frac{2}{s^5}\right) = \frac{t^4}{12}$$

EX2

$$f(t) * 1$$

$$= \int_0^t f(u)\,du = \int_0^t f(u)\,du$$

Proof:

$$F(s)\,G(s) = \int_0^\infty e^{-su} f(u)\,du \int_0^\infty e^{-sv} g(v)\,dv$$

$$= \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u)\,g(v)\,du\,dv$$

$t = u+v$ $\quad v = t-u$

$u = u$

$$du\,dv = \frac{\partial(u,v)}{\partial(u,t)}\,du\,dt$$

$J = 1$

$$= \int_0^\infty \int_0^t e^{-st} f(u)\, g(t-u)\, du\, dt$$

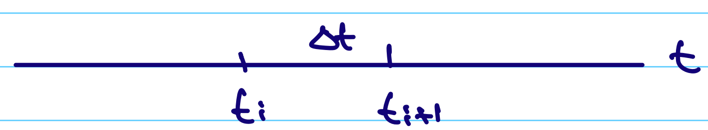
$$= \int_0^\infty e^{-st} \underbrace{\int_0^t f(u)\, g(t-u)\, du}_{f(t) * g(t) \quad \text{convolution}} \, dt$$

$$= \int_0^t e^{-st} (f*g)(t)\, dt$$

Example:

* a factory producing Nuclear material (a radio active waste) at a certain rate

+ Radio active waste in dumping rate $f(t)$
$t =$ years

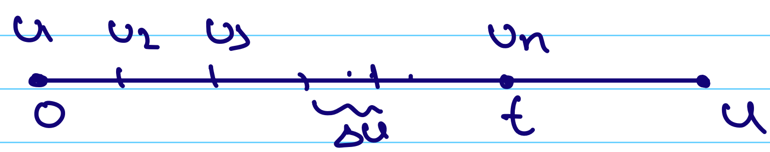


amount dumped b/w $[t_i, t_{i+1}] \approx f(t)\Delta t$

Problem: dumping started at $t = t_0$ at time $t = t$, How much radio active is in pile?

* A_0 initial amount of radio active

as time goes it decay as $\underbrace{A_0 e^{-kt}}_{\text{amount left at time } t}$



Amount dumped from $[u_i, u_{i+1}] \approx f(u_i)\Delta u$

By time t it has decayed to

$$= f(u_i)\,\Delta u\, e^{-k(t-u_i)}$$

length of time it had on pile

Total amount left at time t

$$\approx \sum_{i=1}^{n} f(u_i)\, e^{-k(t-u_i)}\, \Delta u$$

let $\Delta u \to 0$

$$= \int_0^t f(u)\, e^{-k(t-u)}\, du$$

$$= f(t) * e^{-kt}$$

Convolution of Dumping function and decay function.

Example:

① Dump Garbage $\Rightarrow$ No decay

$$\underset{\substack{\uparrow \\ \text{undecaying}}}{f(t) * 1} = \underbrace{\int_0^t f(u)\, du}_{\text{integration.}}$$

② $$\underset{\substack{\uparrow \\ \text{Production rate}}}{f(t)} * \underset{\substack{\uparrow \\ \text{grow's}}}{t} = \int_0^t f(u)\,(t-u)\, dt$$

(Linear growth of chicken's)