Lec03: Solving 1st order linear ODE's.

first order linear

x – independent variable.

$$\boxed{a(x)\, y' + b(x)\, y = c(x)}$$

why is it called linear eqn?

$\Rightarrow$ Because its linear in y, y' (variables)

$\Rightarrow$ Linear : $ay_1 + ay_2 = c$

Simillary linear in y, y'

if $(c=0)$ Homogenous diff eqn

$$a(x)\, y' + b(x)\, y = 0$$

(Homogenous eqn)

STANDARD Linear form

$$\boxed{y' + P(x)\, y = q(x)}$$

Not Standard form

$$\cancel{x}\, y' = P(x)\, y + q(x)$$

(Not a standard form)

What's so important with this diff eqⁿ $y' + P(x)\, y = q(x)$?

1. It can always be solved.
2. It also the eqⁿ which arises in variety of model's.

MODEL'S

1. Temp - concentration model
2. Mixing
3. Decay, Bank interest.
4. Some motion problem's. etc..

Ex: Temperature - concentration model.

Conduction: $\frac{dT}{dt} = k(T_e - T)$ $k>0$ (constant) Conductivity

$T(0) = T_0$

Diffusion: $\frac{dc}{dt} = k(c_e - c)$

Standard linear eqn

$y' + p(x)y = q(x)$

integrating factor $u(x)$

$u(x)$ is a function, we want to multiple the diff eqn by $u(x)$

$$u(x)y' + p(x)u(x)y = q(x)u(x)$$

$$\parallel$$

$$(\qquad)' = q(x)u(x)$$

$$\Rightarrow \quad uy' + Puy = qu$$

Pick u such that

$$\frac{du}{dx} = Pu$$

$$\Rightarrow \quad \frac{d}{dx} u(x) = P(x)\, u(x)$$

$$\Rightarrow \quad \frac{du}{u} = P(x)\, dx$$

$$\Rightarrow \quad \ln u = \int P(x)\, dx$$

$$\Rightarrow \quad \boxed{u(x) = e^{\int P(x)\, dx}} \quad \text{I.F}$$

we are not looking for every possible $u(x)$, which works, all we are looking is just one $u(x)$.

(No arbitrary constant, since only one $u(x)$ needed)

Method: $y' + Py = q$

(0) Standard Linear form

(1) Calculate $e^{\int P dx}$ I.F

(2) Multiply Both sides by $e^{\int P dx}$

(3) and then integrate

Ex: $xy' - y = x^3$

(0) $y' - \frac{1}{x} y = x^2$

(1) $e^{\int P dx} = e^{\int -\frac{1}{x} dx} = e^{-\ln x}$
$= \frac{1}{x}$

(2) $\frac{1}{x}\left(y' - \frac{1}{x} y\right) = \frac{1}{x} x^2$

$\Rightarrow \frac{y'}{x} - \frac{1}{x^2} y = x$

$\left(\frac{y}{x}\right)' = x$

(3) $\int d\left(\frac{y}{x}\right) = \int x \, dx$

$$\frac{y}{x} = \frac{x^2}{2} + C$$

$$\Rightarrow \boxed{y(x) = \frac{x^3}{2} + Cx}$$

Ex2 $\quad (1+\cos x)\, y' - \sin x \, y = 2x$

(0) $$y' - \frac{\sin x}{1+\cos x} y = \frac{2x}{1+\cos x}$$

(1) $$e^{\int \frac{-\sin x}{1+\cos x} dx} = e^{\ln(1+\cos x)}$$

$$= (1+\cos x)$$

(2) $$(1+\cos x)\, y' - \sin x \, y = 2x$$

$$\frac{d\big((1+\cos x)y\big)}{dx} = 2x$$

(3) $$(1+\cos x)\, y = x^2 + C$$

$$\Rightarrow \boxed{y = \frac{x^2 + C}{1+\cos x}}$$

if $y(0)=1 \Rightarrow y(0)=\frac{0+C}{1+1} \Rightarrow C=2$

$$\boxed{y(x)=\frac{x+2}{1+\cos x}}$$

Linear with k constant

Temp: $\frac{dT}{dt}+kT=kT_e$

$$I.F=e^{kt}$$

$$\Rightarrow e^{kt}\frac{dT}{dt}+ke^{kt}T=e^{kt}kT_e$$

$$\Rightarrow d(e^{kt}T)=kT_e e^{kt}dt$$

$$e^{kt}T=\int kT_e e^{kt}dt+C$$

$$T(t)=e^{-kt}\int kT_e(t)e^{kt}+Ce^{-kt}$$

if $T(0)=T_0$

$$T(t)=\underbrace{e^{-kt}\int_0^t kT_e(t_1)e^{kt_1}dt_1}_{\text{Steady state solution}}+\underbrace{T(0)e^{-kt}}_{\to 0 \text{ as } T\to\infty \text{ transient.}}$$

because $k>0$