Lec 10:- Characteristic roots, undamped and damped oscillation's.

2^{nd} order linear ODE

linear in y, y', y''

$$y'' + P(x) y' + q(x) y = 0 \quad \text{(Homogenous)}$$

The linearity of eqn that is the form in which appear's is gonna be the key idea today.

Solution method is to find two independent solution's y_1, y_2

$$y_2(t) \neq c\, y_1(t)$$

$$y_1(t) \neq c'\, y_2(t)$$

Then all solution's are

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

Q1) Why are all these are solution's?

Q2) Why are they all the solution's?

Q1 ⇒ Can be answered by superposition principle.

if $y_1(t)$, $y_2(t)$ are solution's to linear homogenou's ODE then

(Linear combination) $c_1 y_1(t) + c_2 y_2(t)$ is also a solution.

Proof:

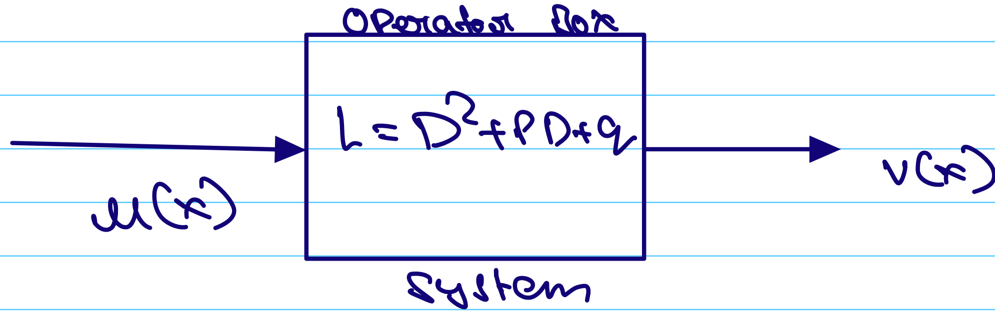
$$y'' + Py' + qy = 0$$

$$D^2 y + PDy + qy = 0$$

$$\underbrace{\left[D^2 + PD + q\right]}_{\text{Linear operator } L} y = 0$$

$$Ly = 0$$

$L = D^2 + pD + q$



* easy thing is put a function $u(x)$ and see what comes out $v(x)$

* Solving differential eq^n, is hard thing. Ask what should be the $u(x)$ such that $v(x) = 0$.
(inverse Problem)

$L = D^2 + pD + q$ Linear operator.

$\Rightarrow$ ① $L(u_1 + u_2) = Lu_1 + Lu_2$

$\Rightarrow$ ② $L(cu) = cLu$

c = constant.

__Ex:__ D is linear, $\int$ is Linear.

Proof of superposition:

$$ODE : \quad L = D^2 + pD + q$$

$$Ly = 0$$

if y_1, y_2 are two independent solution's.

$$\Rightarrow \quad L(c_1y_1 + c_2y_2)$$

$$\Rightarrow \quad c_1 Ly_1 + c_2 Ly_2 = 0$$ is also a solution to $Ly=0$

(i)2) why are these all the solution's?

Solving the IVP (fit initial values)

Then solve $\{c_1y_1 + c_2y_2\}$ these are all the solution's But we don't know yet

$\{c_1 y_1 + c_2 y_2\}$ is enough to Any initial condition. $\Rightarrow$ if we have any initial condition we can solve for c_1 & c_2.

Proof:

$$\left.\begin{aligned} y(x_0) &= a \\ y'(x_0) &= b \end{aligned}\right\} \text{ initial values}$$

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$\Rightarrow \quad y(x_0) = c_1 y_1(x_0) + c_2 y_2(x_0)$$
$$y'(x_0) = c_1 y_1'(x_0) + c_2 y_2'(x_0)$$

$$\Rightarrow \quad c_1 y_1(x_0) + c_2 y_2(x_0) = a$$
$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = b$$

$$\Rightarrow \quad \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

The Pair of simultaneous can be solved, i.e we can solve for c_1, c_2 only with one condition. A condition which gaurantees their solution exist's.

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}_{x_0}$$ Should be invertible

Solvable if matrix is invertable

$\Rightarrow$ det of matrix $\neq 0$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}_{x_0} \neq 0$$

This determinant is called Wronskian of y_1, y_2

$$W(y_1, y_2) = W(x)$$

function of x

$$W(y_1, y_2) = W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}_x$$

$\Rightarrow$ suppose y_1, y_2 are dependent

$$y_2 = c y_1$$

$$\Rightarrow \quad y_2' = c y_1'$$

$$\det \begin{vmatrix} y_1 & c y_1 \\ y_1' & c y_1' \end{vmatrix} = 0 \qquad \forall x$$

Theorem: if $y_1(x)$, $y_2(x)$ are the solution's to ODE, there are only two possibilites

either $\quad W(y_1, y_2) \equiv 0 \quad$ (for all x)

or $\quad W(y_1, y_2)$ is never zero for all x

$$\{c_1 y_1 + c_2 y_2\} = \{c_1' \tilde{y}_1 + c_2' \tilde{y}_2\}$$

$\tilde{y}_1$, $\tilde{y}_2$ another Pair of Linearly independent solution's.

Finding Normalized Sol^n (at 0)

$\bar{Y}_1$, $\bar{Y}_2$ they are one which

Satisfy initial Condition's

$$\bar{Y}_1 : \begin{cases} Y_1(0) = 1 \\ Y_1'(0) = 0 \end{cases} \qquad \bar{Y}_2 = \begin{cases} Y_2(0) = 0 \\ Y_2'(0) = 1 \end{cases}$$

if $\bar{Y}_1$, $\bar{Y}_2$ are Normalized Sol^n

Sol to IVP ODE + $\begin{cases} y(0) = a = y_0 \\ y'(0) = b = y_0' \end{cases}$

then solution is

$$y(t) = y_0 \bar{Y}_1 + y_0' \bar{Y}_2$$

Existance & uniqueness theorm:

$y'' + py' + qy = 0$ p, q are continuous for all x

then the theorem say's, there is one and only solution satisfying the IVP problem $y(0) = A$
$y'(0) = B$