Problem set 4.6

① $x'' + 4x = \delta(t) \qquad x(0) = x'(0) = 0$

$$(s^2 + 4)\, X(s) = 1$$

$$X(s) = \frac{1}{s^2+4} \Rightarrow x(t) = \frac{1}{2}\sin 2t$$

② $x'' + 4x = \delta(t) + \delta(t-\pi) \qquad x(0) = x'(0) = 0$

$$(s^2+4)\, X(s) = 1 + e^{-\pi s}$$

$$X(s) = \frac{1 + e^{-\pi s}}{s^2+4}$$

$$\Rightarrow \quad x(t) = \frac{1}{2}\cos 2t + \frac{1}{2}u(t-\pi)\sin 2(t-\pi)$$

$$\Rightarrow x(t) = \frac{1}{2}\cos 2t + \frac{1}{2}u(t-\pi)\sin 2t$$

③ $x'' + 4x' + 4x = 1 + \delta(t-2) \qquad x(0) = x'(0) = 1$

$$(s^2 + 4s + 4) X(s) = \frac{1}{s} + e^{-2s}$$

$$X(s) = \frac{1}{s(s+2)^2} + \frac{e^{-2s}}{(s+2)^2}$$

$$X(s) = \frac{1}{4s} - \frac{1}{4(s+2)} - \frac{1}{2(s+2)^2} + \frac{e^{-2s}}{(s+2)^2}$$

$$x(t) = \frac{1}{4} - \frac{1}{4} e^{-2t} - \frac{1}{2} e^{-2t} t + u(t-2)\, e^{-2(t-2)} (t-2)$$

$$x(t) = \frac{1}{4}\left(1 - e^{-2t} - 2e^{-2t} t\right) + u(t-2)\,(t-2)\, e^{-2(t-2)}$$

③ $x'' + 2x' + x = t + \delta(t) \qquad x(0) = 0 \quad x'(0) = 1$

$$(s^2 + 2s + 1) X(s) = \frac{1}{s^2} + 2$$

$$X(s) = \frac{1}{s^2(s+1)^2} + \frac{2}{(s+1)^2}$$

$$X(s) = \frac{-2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{1}{(s+1)^2} + \frac{2}{(s+1)^2}$$

$$x(t) = -2 + t + 2e^{-t} + 3te^{-t}$$

⑤ $x'' + 2x' + 2x = 2\delta(t-\pi) \quad x(0) = x'(0) = 0$

$$(s^2 + 2s + 2)X(s) = 2e^{-\pi s}$$

$$X(s) = \frac{2e^{-\pi s}}{(s+1)^2 + 1^2}$$

$$x(t) = 2 \cdot u(t-\pi)\, e^{-(t-\pi)} \sin(t-\pi)$$

⑥ $x'' + 9x = \delta(t-3\pi) + 10\sin 3t \quad x(0) = x'(0) = 0$

$$(s^2 + 9)X(s) = e^{-3\pi s} + \frac{s}{s^2+9}$$

$$X(s) = \frac{e^{-3\pi s}}{s^2+9} + \frac{s}{(s^2+9)^2}$$

(7) $x'' + 4x' + 5x = \delta(t-\pi) + \delta(t-2\pi)$

$x(0) = 0$

$x'(0) = 2$

$$(s^2 + 4s + 5)X(s) - 2 = e^{-\pi s} + e^{-2\pi s}$$

$$X(s) = \frac{e^{-\pi s}}{(s+2)^2+1^2} + \frac{e^{-2\pi s}}{(s+2)^2+1} + \frac{2}{(s+2)^2+1^2}$$

$$x(t) = u(t-\pi)\, e^{-2(t-\pi)} \sin(t-\pi)$$

$$u(t-2\pi)\, e^{-2(t-2\pi)} \sin(t-2\pi)$$

$$+ 2e^{-2t} \sin t$$

(8) $x'' + 2x' + x = \delta(t) - \delta(t-2)$ $\quad x(0) = x'(0) = 2$

$$s^2 X(s) - 2s - 2 + 2sX(s) - 4 + X(s) = 1 - e^{-2s}$$

$$X(s)\left(s^2+2s+1\right) = 7+2s-e^{-2s}$$

$$X(s) = \frac{7}{(s+1)^2} + \frac{2s}{(s+1)^2} - \frac{e^{-2s}}{(s+1)^2}$$

(9) $x'' + 4x = f(t) \qquad x(0) = x'(0) = 0$

$$x'' + 4x = \delta(t)$$

$$W(s) = \frac{1}{s^2+4} \Rightarrow w(t) = \frac{1}{2}\sin 2t$$

$$x(t) = \int_0^t f(t-y)\,\frac{1}{2}\sin 2y\,dy$$

(10) $x'' + 6x' + 9x = f(t) \qquad x(0) = x'(0) = 0$

$$W(s) = \frac{1}{s^2+6s+9} = \frac{1}{(s+3)^2}$$

impulse response $\Rightarrow w(t) = e^{-3t}\,t$

$$x(t) = \int_0^t f(t-y)\,e^{-3y}\,y\,dy$$

(11) $x'' + 6x' + 8x = f(t) \qquad x(0) = x'(0) = 0$

$$W(s) = \frac{1}{s^2 + 6s + 8} = \frac{1}{(s+2)(s+4)}$$

$$= \frac{1/2}{s+2} + \frac{-1/2}{s+4}$$

$$w(t) = \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t}$$

$$x(t) = \int_0^t f(t-y)\left[\frac{1}{2}e^{-2y} - \frac{1}{2}e^{-4y}\right] dy$$

(12) $x'' + 4x' + 8x = f(t) \qquad x(0) = x'(0) = 0$

$$W(s) = \frac{1}{s^2 + 4s + 8}$$

$$W(s) = \frac{1}{(s+2)^2 + 2^2}$$

$$w(t) = \frac{1}{2}e^{-2t}\sin 2t$$

$$x(t) = \int_0^t f(t-y)\,\frac{1}{2}e^{-2y}\sin 2y\, dy$$