Lec 29: Matrix exponential's

$$\boxed{\vec{x}' = A\vec{x}}$$

Fundamental matrix: for A

$\underline{X} = [\vec{x}_1 \quad \vec{x}_2]$ (2 independent sol^n)

Properties: of $\underline{X}$

① $|\underline{X}| \neq 0$ for any t

② $\underline{X}' = A\underline{X}$ (coloumn's are sol^n to system's)

One confusing thing about fundamental matrix is that, it is not unique.

There is no "the fundamental matrix", there is only "A fundamental matrix" write general solution to the system.

$$\vec{X}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

$$= \underline{\overline{X}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underline{\overline{X}}\, \vec{C}$$

$$\Rightarrow \vec{X}(t) = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Rightarrow \boxed{\vec{X}(t) = \underline{\overline{X}}\, \vec{C}}$$ general solution

* what do all fundamental sol^n look like?

* two column's are sol^n

$$\begin{bmatrix} \underbrace{\underline{\overline{X}}\, \vec{C}_1}_{sol^n} & \underbrace{\underline{\overline{X}}\, \vec{C}_2}_{sol^n} \end{bmatrix}$$

$$\Rightarrow \underline{\overline{X}} \begin{bmatrix} \vec{C}_1 & \vec{C}_2 \end{bmatrix}$$

$C_{2\times2}$ (constant's)

$$\Rightarrow \underline{\overline{X}}\, C$$

* Φ is a given fundamental matrix

* The most general fundamental matrix ΦC

general fundamental matrix $= \Phi C$

where $|C| \neq 0$

* in other word's fundamental matrix is not unique. But once we found one, all the other ones are found by multiplying ΦC

The solution is given by formula ——.

we will produce formula.

for one-by-one case

① $x' = ax$ soln is $x(t) = C e^{at}$

$\Rightarrow x(t) = x(t_0) e^{at}$

$$e^{at} := 1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + \cdots$$

$$\frac{d}{dt} e^{at} := a + \frac{2a^2 t}{2!} + \frac{3a^3t^2}{3!} + \cdots$$

$$= a + \frac{a^2 t}{1!} + \frac{a^3t^2}{2!} + \cdots$$

$$= a\left(1 + at + \frac{a^2t^2}{2!} + \cdots\right)$$

$$= ae^{at}$$

<u>Soln to $\vec{x}' = Ax$</u>

A Fundamental matrix for

$\underline{x}' = A\vec{x}$ is e^{At}

$$e^{At} := I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$\Rightarrow \frac{d}{dt} e^{At} = A + A^2 t + A^3 \frac{t^2}{2!} + \cdots$$

$$= A\left(I + At + \frac{A^2t^2}{2!} + \cdots\right)$$

or

$$= \left(I + At + \frac{A^2t^2}{2!} + \dots\right) A$$

$$\Rightarrow \quad \frac{d}{dt} e^{At} := A e^{At} = e^{At} A$$

is e^{At} a fundamental matrix?

1. $$\frac{d}{dt} e^{At} = A e^{At}$$

$$\Rightarrow \quad \bar{X}' = A\bar{X} \quad \checkmark$$

2. $$|\bar{X}(0)| = |e^{A0}| = I \neq 0$$

Hence e^{At} is fundamental matrix and unique.

IVP $$\bar{X}' = A\vec{X}$$

$$\bar{X}(0) = \vec{X}_0$$

general soln $x(t)$

$$\vec{x}(t) = e^{At}\vec{c}$$

$$\Rightarrow \vec{x}(0) = e^{A0}\vec{c}$$

$$\Rightarrow \vec{c} = \vec{x}(0)$$

$$\Rightarrow \vec{x}(t) = e^{At}\vec{x}_0$$

How to calculate e^{At} ?

$$e^{A-A} = e^{A} \cdot e^{-A} \Rightarrow I = e^{A}e^{-A}$$

$\Rightarrow$ Inverse of e^{A} is e^{-A}

$$(e^{A})^{-1} = e^{-A}$$

<u>Calculate e^{At}</u> 1 Series (too hard)

2 Symmetric matrix

Use exponential law $\longrightarrow$ $\begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$

③ $$\underbrace{\overline{X} \cdot \overline{X}(0)^{-1}} = e^{At}$$

① Fundamental matrix $\overline{X} \cdot C$

② value at 0

$$\overline{X}(0)\,\overline{X}(0)^{-1} = I$$

$\Rightarrow$ e^{At} has same two properties

$$\Rightarrow e^{At} = \overline{X}\,\overline{X}(0)^{-1}$$