Problem set 4.2

① $x'' + 4x = 0; \quad x(0) = 5, \quad x'(0) = 0$

$\mathcal{L}(x'' + 4x) = \mathcal{L}(0)$

$s^2 X(s) - s x(0) - x'(0) + 4X(s) = 0$

$(s^2 + 4) X(s) = s x(0) + x'(0)$

$= 5s$

$X(s) = \dfrac{5s}{s^2 + 4}$

$x(t) = 5\cos 2t$

② $x'' + 9x = 0 \quad x(0) = 3 \quad x'(0) = 4$

$s^2 X(s) - s x(0) - x'(0) + 9X(s) = 0$

$(s^2 + 9) X(s) = 3s + 4$

$X(s) = \dfrac{3s + 4}{s^2 + 9}$

$x(t) = 3\cos 3t + \dfrac{4}{3}\sin 3t$

③ $x'' - x' - 2x = 0 \quad x(0) = 0 \quad x'(0) = 2$

$$s^2X(s) - sx(0) - x'(0) - sX(s) - x(0) - 2X(s) = 0$$

$$(s^2 - s - 2)X(s) = 2$$

$$X(s) = \frac{2}{s^2 - s - 2} = \frac{2}{(s+1)(s-2)}$$

$$X(s) = \frac{-\frac{2}{3}}{s+1} + \frac{\frac{2}{3}}{s-2}$$

$$x(t) = -\frac{2}{3}e^{-t} + \frac{2}{3}e^{2t}$$

④ $x'' + 8x' + 15x = 0 \quad x(0) = 2 \quad x'(0) = -3$

$$s^2X(s) - sx(0) - x'(0) + 8sX(s) - 8x(0) + 15X(s) = 0$$

$$(s^2 + 8s + 15)X(s) - sx(0) - x'(0) - 8x(0) = 0$$

$$(s+3)(s+5)\,X(s) = 2s - 13$$

$$X(s) = \frac{2s-13}{(s+3)(s+5)}$$

$$= \frac{\frac{7}{2}}{s+3} - \frac{\frac{3}{2}}{s+5}$$

$$x(t) = \frac{7}{2}e^{-3t} - \frac{3}{2}e^{-5t}$$

(5) $x'' + x = \sin 2t \qquad x(0) = 0 = x'(0)$

$$s^2 X(s) + X(s) = \frac{2}{s^2+4}$$

$$\Rightarrow \quad X(s) = \frac{2}{(s^2+4)(s^2+1)}$$

$$\Rightarrow \quad X(s) = \frac{-\frac{2}{3}}{s^2+4} + \frac{\frac{2}{3}}{s^2+1}$$

$$x(t) = -\frac{1}{3}\sin 2t + \frac{2}{3}\sin t$$

⑥ $x'' + 4x = \cos t \qquad x(0) = 0 = x'(0)$

$$(s^2+4)X(s) = \frac{s}{s^2+1}$$

$$X(s) = \frac{s}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$(As+B)(s^2+4) + (Cs+D)(s^2+1) = s$$

$$\Rightarrow \quad As^3 + 4As + Bs^2 + 4B + Cs^3 + Cs + Ds^2 + D = s$$

$$D = 0 \Leftarrow 4B + D = 0$$

$$A + C = 0 \Rightarrow A = -C$$

$$B = -D \Leftarrow B + D = 0$$

$$B = 0 \qquad 4A + C = 1$$

$$-3C = 1 \Rightarrow C = -\frac{1}{3}$$

$$A = \frac{1}{3}$$

$$X(s) = \frac{\frac{1}{3}s}{s^2+1} - \frac{\frac{1}{3}s}{s^2+4}$$

$$x(t) = \frac{1}{3}(\cos t - \cos 2t)$$

(7) $x'' + x = \cos 3t \qquad x(0) = 1 \quad x'(0) = 0$

$$s^2 X(s) - s + X(s) = \frac{s}{s^2+9}$$

$$(s^2+1) X(s) = \frac{s}{s^2+9} + s$$

$$X(s) = \frac{s}{(s^2+9)(s^2+1)} + \frac{s}{s^2+1}$$

$$X(s) = \frac{-\frac{1}{8} s}{s^2+9} + \frac{\frac{1}{8} s}{s^2+1} + \frac{s}{s^2+1}$$

$$x(t) = -\frac{1}{8} \cos 3t + \frac{1}{8} \cos t + \cos t$$

$$x(t) = \frac{1}{8} (9 \cos t - \cos 3t)$$

(8) $x'' + 9x = 1 \qquad x(0) = 0 = x'(0)$

$$s^2 X(s) + 9 X(s) = \frac{1}{s}$$

$$X(s) = \frac{1}{s} \cdot \frac{1}{s^2+9} = \frac{1/9}{s} + \frac{-\frac{1}{9} s}{s^2+9}$$

$$x(t) = 1/9 - \frac{1}{9} \cos 3t$$

(9) $x'' + 4x' + 3x = 1 \qquad x(0) = 0 = x'(0)$

$$X(s) = \frac{1}{s^2+4s+3} \cdot \frac{1}{s}$$

$$= \frac{1}{s(s+1)(s+3)}$$

$$X(s) = \frac{1/3}{s} + \frac{-1/2}{s+1} + \frac{1/6}{s+3}$$

$$x(t) = \frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t}$$

(10) $x'' + 3x' + 2x = t \qquad x(0) = 0 \qquad x'(0) = 2$

$$s^2X(s) - 2 + 3sX(s) + 2X(s) = \frac{1}{s^2}$$

$$X(s)(s^2+3s+2) = \frac{1}{s^2} + 2$$

$$X(s) = \frac{1}{s^2(s^2+3s+2)} + \frac{2}{s^2+3s+2}$$

$$X(s) = \frac{1}{s^2(s+1)(s+2)} + \frac{2}{(s+1)(s+2)}$$

(11) $x' = 2x + y \quad y' = 6x + 2y \quad x(0) = 1$

$y(0) = -2$

$$sX(s) - 1 = 2X(s) + Y(s)$$

$$sY(s) + 2 = 6X(s) + 2Y(s)$$

$$\Rightarrow (s-2)X(s) - Y(s) = 1$$

$$6X(s) - (s-2)Y(s) = 2$$

$$\Rightarrow 6(s-2)X(s) - 6Y(s) = 6$$

$$-6(s-2)X(s) + (s-2)(s-1)Y(s) = 2(s-2)$$

$$\left((s-3)(s-2) - 6\right)Y(s) = 6 - 2s + 4$$

$$Y(s) = \frac{10 - 2s}{(s-3)(s-2) - 6}$$

$$\vdots$$