Lec 19 : introduction to the LAPLACE Transform

Laplace Transform from power series.

$$\sum_{0}^{\infty} a_n x^n = A(x)$$

$$\Rightarrow \quad \sum_{0}^{\infty} a[n] x^n = A(x)$$

Taking the discrete function $a[n]$

$\Big\downarrow$

Sum of power series

$a(n) \rightsquigarrow A(x)$

$a(n) = 1 \rightsquigarrow \sum_{0}^{\infty} x^n = \frac{1}{1-x}$, $|x| < 1$

$a(n) = \frac{1}{n!} \rightsquigarrow \sum_{0}^{\infty} \frac{x^n}{n!} = e^x$ $\forall x$

taking a discrete function defined for
+ve integer's $a[n]$ & doing this Process,
Out of it comes a continuous function in
x.

* what goes in is variable n, what
comes out is variable x.

* Suppose we made summation
continuous instead of discreet.

$\Rightarrow$ continuous analog

replace $n = 0, 1, 2, \ldots$

$$\downarrow$$

$t : 0 \le t < \infty$

$$\sum_{n=0}^{\infty} a[n]\, x^n \rightsquigarrow \int_{t=0}^{\infty} a(t)\, x^t \, dt$$

$= \text{function of } x$

$= A(x)$

we could leave in that form, But we want in the form of exponential.

$$x = e^{\ln x}$$

$$x^t = \left(e^{\ln x}\right)^t = e^{t \cdot \ln x}$$

we really want $0 \leq x < 1$ for Convergence

$$0 \leq x < 1$$

$$\Rightarrow -\infty < \ln x < 0$$

make $S = -\ln x$

$$\Rightarrow \quad 0 \leq S < \infty$$

$$\Rightarrow \quad x^t = \left(e^{\ln x}\right)^t$$

$$= e^{-st}$$

$$\Rightarrow \quad \boxed{\int_{t=0}^{\infty} f(t)\, e^{-st}\, dt = F(s)}$$

Continuous analog of summation of a power series.

This is called Laplace Transform

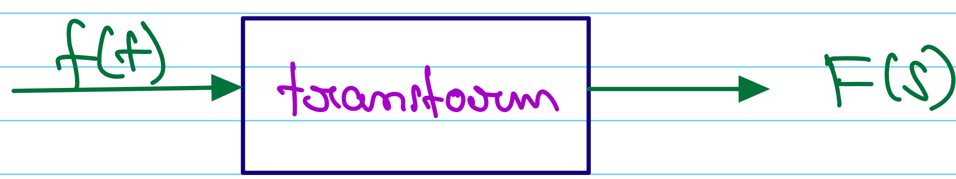
$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

Operation: Ex ① input x^2 output $2x$ D

② input x^2 output $\frac{x^3}{2}$ $\int$

No change of variable

Transform: Change of variable



* Laplace is a Linear transform:

* Notation: ① $\mathcal{L}(f(t)) = F(s)$

② $f(t) \rightsquigarrow F(s)$

<u>Linearity:</u>

$$\mathcal{L}(f+g) = \mathcal{L}f + \mathcal{L}g$$

$$\mathcal{L}(cf) = c\,\mathcal{L}(f)$$

<u>Laplace Transform's of familiar functions:</u>

$$1 \rightsquigarrow ?$$

① $\mathcal{L}(1) = \displaystyle\int_0^\infty 1 \cdot e^{-st}\,dt = \left.\frac{e^{-st}}{-s}\right]_0^\infty$

$$= \frac{1}{s} \qquad s > 0$$

② $$\mathcal{L}(e^{at}) = \int_0^\infty e^{at} e^{-st}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left. \frac{e^{-(s-a)}}{-(s-a)} \right]_0^\infty$$

$$= \frac{1}{s-a} \qquad s>0$$

<u>2^{nd} method:</u>

$$\mathcal{L}(e^{at} f(t)) = \int_0^\infty e^{at} f(t)\, e^{-st}\, dt$$

$$= \int_0^\infty f(t)\, e^{-(s-a)t}\, dt$$

$$= F(s-a)$$

$$\Rightarrow \quad \mathcal{L}(e^{at}\cdot 1) = F(s-a) = \frac{1}{s-a}, \quad s>a$$

on the left side we multiplied by an exponential, right side we translated.

Exponential – Shift formula

$$\mathcal{L}\left(e^{at} f(t)\right) = F(s-a)$$

can use also for $e^{(a+ib)t}$ as well

$$\mathcal{L}\left(e^{(a+ib)t} f(t)\right) = F(s-(a+ib))$$

$$\mathcal{L}\left(e^{(a+ib)t} \cdot 1\right) = \frac{1}{s-(a+ib)} \qquad s > a$$

$$\cos at = \frac{e^{iat} + e^{-iat}}{2}$$

$$\mathcal{L}(\cos at) = \mathcal{L}\left(\frac{e^{iat} + e^{-iat}}{2}\right)$$

$$= \frac{1}{2}\frac{1}{s-ia} + \frac{1}{2}\frac{1}{s+ia}$$

$$= \frac{1}{2} \frac{2s}{s^2+a^2} = \frac{s}{s^2+a^2}$$

$$\Rightarrow \mathcal{L}(\cos at) = \frac{s}{s^2+a^2} \quad s>0$$

$$\sin at = \frac{e^{iat} - e^{-iat}}{2i}$$

$$\mathcal{L}(\sin at) = \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right)$$

$$= \frac{1}{2i} \frac{2ia}{s^2+a^2}$$

$$= \frac{a}{s^2+a^2} \quad s>0$$

The hardest part is solving inverse laplace Transform.

$$\frac{1}{s(s+3)} = \frac{1/3}{s} + \frac{-\frac{1}{3}}{s+3}$$

$$= \frac{1}{3}\,\frac{1}{s} - \frac{1}{3}\,\frac{1}{s+3}$$

$$\frac{1}{3}\,\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{1}{3}\,\mathcal{L}^{-1}\left(\frac{1}{s+3}\right)$$

$$\Rightarrow \frac{1}{3}\cdot 1 - \frac{1}{3}\cdot e^{-3t}\cdot 1$$

$$\Rightarrow \frac{1}{3}\left(1 - e^{-3t}\right)$$

*

$$\mathcal{L}(t^n) = \int_0^\infty t^n e^{-st}\,dt$$

$$= t^n \int e^{-st}\,dt \Big]_0^\infty - \int_0^\infty n\cdot t^{n-1} \int e^{-st}\,dt\,dt$$

$$= t^n \cdot \frac{e^{-st}}{-s}\Big]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st}\,dt$$

$$= -\underbrace{\frac{t^n}{e^{st}}\Big]_0^\infty}_{0} + \frac{n}{s}\,\mathcal{L}(t^{n-1})$$

$$\mathcal{L}(t^n) = \frac{n}{s}\,\mathcal{L}(t^{n-1})$$

$$\mathcal{L}(t^n) = \frac{n}{s} \cdot \frac{n-1}{s} \cdots \frac{1}{s} \cdot \mathcal{L}(t^0)$$

$$= \frac{n!}{s^n} \cdot \frac{1}{s} = \frac{n!}{s^{n+1}}$$

$$\Rightarrow \boxed{\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}}$$