Lec 09 : Solving Second-order Linear ODE's with constant co-efficient's.

$$y'' + Ay' + By = 0 \quad (\text{homogenous eq}^n)$$

Assume: general solution's

$$y = C_1 y_1 + C_2 y_2$$

where y_1 and y_2 are solution's.

* All we have to find is 2 solution's.

Initial condition's are satisfied by choosing C_1 and C_2.

To solve ODE's we need to find two independent solution's.

BASIC method: try $y = e^{rt}$

Plug in $e^{rt} = y$

$$r^2 e^{rt} + A \cdot r e^{rt} + B \cdot e^{rt} = 0$$

$$\Rightarrow \quad e^{rt}\left[r^2 + Ar + B\right) = 0$$

$$\Rightarrow \quad r^2 + Ar + B = 0 \quad \text{(characristic eqn)}$$

Case 1 root's $r_1 \neq r_2$ (real)

$$y_1 = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Case 2: Complex root's $r = a \pm bi$

we get complex solution $y = e^{(a \pm bi)t}$

Theorem: if $u + iv$ Cx soln to $y'' + Ay' + By = 0$

then u, v are two real solution's.

Proof: $(u+iv)'' + A(u+iv)' + B(u+iv) = 0$

$$\Rightarrow \quad (u'' + Au' + Bu) + i(v'' + Av' + Bv) = 0$$

$$\Rightarrow \quad u'' + Au' + Bu = 0$$

$$v'' + Av' + Bv = 0$$

$$y = e^{at+ibt} = e^{at} e^{ibt}$$

$$= e^{at}(\cos bt + i \sin bt)$$

$\Rightarrow$ $y = e^{at}\cos bt$, $e^{at}\sin bt$ are solution's to eqn

$$y = e^{at}(c_1 \cos bt + c_2 \sin bt)$$

Case II: Critically damped.

$r^2 + Ar + B = 0$ has 2 equal root's

$r = -a$ (root)

$(r+a)^2 = 0$ (characteristic eqn)

$$r^2 + 2ar + a^2 = 0$$

$\Rightarrow \quad y'' + 2ay' + a^2 y = 0$

Solution: $\quad y = e^{-at}$ (1st solution)

other sol^n?

if we know one sol^n to $y'' + py' + qy = 0$

then there is another of form

$$y = y_1 u$$

$a^2 \times \quad y = e^{-at} u$

$2a \times \quad y' = -ae^{-at} u + e^{-at} u'$

$1 \times \quad y'' = a^2 e^{-at} u - ae^{-at} u' - ae^{-at} u' + e^{-at} u''$

$\overset{\shortparallel}{0}$

$e^{-at} u'' = 0 \quad \Rightarrow \quad u'' = 0$

$u = c_1 t + c_2$

$$\boxed{y_2 = e^{-at} t}$$