

Problem set 2.1

① $y'' - y = 0$ $y_1 = e^x$ $y_2 = e^{-x}$ $y(0) = 0$
 $y'(0) = 5$

Sol'n

$$y'' - y = 0$$

$$\alpha^2 - 1 = 0 \quad \alpha = \pm 1$$

$$\Rightarrow y_1 = e^x \quad y_2 = e^{-x}$$

$$y(x) = c_1 e^x + c_2 e^{-x}$$

$$0 = c_1 + c_2$$

$$5 = c_1 - c_2 \quad \Rightarrow c_1 = \frac{5}{2} \quad c_2 = -\frac{5}{2}$$

$$y(x) = \frac{5}{2} e^x - \frac{5}{2} e^{-x}$$

② $y'' - 9y = 0$ $y_1 = e^{3x}$ $y_2 = e^{-3x}$
 $y(0) = 1$ $y'(0) = 15$

Sol'n

$$\alpha^2 - 9 = 0 \quad \alpha = \pm 3$$

$$y(x) = c_1 e^{3x} + c_2 e^{-3x}$$

$$1 = c_1 + c_2$$

$$15 = c_1 - 3c_2$$

$$\Rightarrow 4c_1 = 16$$

$$c_1 = 4$$

$$c_2 = -3$$

$$y(x) = 4e^{3x} - 3e^{-3x}$$

②

$$y'' + 4y = 0$$

$$y_1 = \cos 2x$$

$$y_2 = \sin 2x$$

$$y(0) = 3 \quad y'(0) = 8$$

Soln

$$y'' + 4y = 0$$

$$y'' = -4y \Rightarrow y = \pm 2i$$

$$y(x) = e^{0x} (c_1 \cos 2x + c_2 \sin 2x)$$

$$\Rightarrow y(x) = c_1 \cos 2x + c_2 \sin 2x$$

$$3 = c_1$$

$$y'(x) = -2c_1 \sin 2x + 2c_2 \cos 2x$$

$$8 = 2c_2 \Rightarrow c_2 = 4$$

$$y(x) = 3 \cos 2x + 4 \sin 2x$$

④

$$y'' + 25y = 0$$

$$y_1 = \cos 5x \quad y(0) = 10$$

$$y_2 = \sin 5x \quad y'(0) = -10$$

$$\lambda^2 + 25 = 0 \Rightarrow \lambda = \pm 5i$$

$$y(x) = C_1 \cos 5x + C_2 \sin 5x$$

$$10 = C_1$$

$$y'(x) = -5C_1 \sin 5x + 5C_2 \cos 5x$$

$$-10 = 5C_2 \Rightarrow C_2 = -2$$

$$\Rightarrow y(x) = 10 \cos 5x - 2 \sin 5x$$

⑤

$$y'' - 3y' + 2y = 0 \quad y_1 = e^x \quad y_2 = e^{2x}$$

$$y(0) = 1 \quad y'(0) = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$x^2 - 2x - x + 2 = 0$$

$$\Rightarrow x(x-2) - 1(x-2) = (x-1)(x-2) = 0$$

$$x_1 = 2 \quad x_2 = 1$$

$$y(x) = C_1 e^{2x} + C_2 e^x$$

$$1 = C_1 + C_2$$

$$0 = 2C_1 + C_2 \Rightarrow \begin{aligned} C_1 &= -1 \\ C_2 &= 2 \end{aligned}$$

$$\Rightarrow y(x) = -e^{2x} + 2e^x$$

⑥ $y'' + y' - 6y = 0 \quad ; \quad y_1 = e^{2x} \quad y(0) = 7$
 $y_2 = e^{-3x} \quad y'(0) = -1$

$$x^2 + x - 6 = 0$$

$$x^2 + 3x - 2x - 6 = 0 \Rightarrow x(x+3) - 2(x+3) = 0$$

$$x_1 = 2, \quad x_2 = -3$$

$$y(x) = C_1 e^{2x} + C_2 e^{-3x}$$

$$C_1 + C_2 = 7$$

$$2C_1 - 3C_2 = -1$$

$$5C_1 = 20 \Rightarrow C_1 = 4$$

$$C_2 = 3$$

$$y(x) = 4e^{2x} + 3e^{-3x}$$

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$$y'' + y' = 0 \quad y_1 = 1$$

$$y_2 = e^{-x}$$

$$y(0) = -2$$

$$y'(0) = 8$$

$$\lambda^2 + \lambda = 0 \quad \lambda_1 = 0, \quad \lambda_2 = -1$$

$$y(x) = C_1 + C_2 e^{-x}$$

$$-2 = C_1 + C_2$$

$$C_1 = 6$$

$$8 = -C_2$$

$$C_2 = -8$$

$$y(x) = 6 - 8e^{-x}$$

8

$$y'' - 3y' = 0 \quad y_1 = 1, \quad y_2 = e^{3x}$$

$$y(0) = 4 \quad y'(0) = -2$$

$$\lambda^2 - 3\lambda = 0 \Rightarrow \lambda_1 = 0, \quad \lambda_2 = 3$$

$$y(x) = C_1 + C_2 e^{3x}$$

$$C_1 + C_2 = 4$$

$$3C_2 = -2$$

$$\Rightarrow C_2 = -\frac{2}{3}$$

$$C_1 = \frac{14}{3}$$

$$y(x) = \frac{14}{3} - \frac{2}{3} e^{3x}$$

(10)

$$y'' - 10y' + 25y = 0$$

$$y_1 = e^{5x}$$

$$y(0) = 3$$

$$y_2 = x e^{5x}$$

$$y'(0) = 13$$

Soln

$$\lambda^2 - 10\lambda + 25 = 0$$

$$(\lambda - 5)^2 = 0 \Rightarrow \lambda_{1,2} = 5$$

$$y(x) = (C_1 + C_2 x) e^{5x}$$

$$3 = C_1$$

$$y'(x) = 5(C_1 + C_2 x) e^{5x} + C_2 e^{5x}$$

$$13 = 5C_1 + C_2 \Rightarrow C_2 = -2$$

$$y(x) = (3 - 2x) e^{5x}$$

(11)

$$y'' - 2y' + 2y = 0$$

$$y_1 = e^x \cos x$$

$$y(0) = 0$$

$$y_2 = e^x \sin x$$

$$y'(0) = 2$$

Soln

$$\lambda^2 - 2\lambda + 2 = 0$$

$$(\lambda - 1)^2 + 1 = 0 \Rightarrow \lambda = 1 \pm i$$

$$y(x) = e^x (C_1 \cos x + C_2 \sin x)$$

$$C_1 = 0, \quad y'(x) = e^x (-C_1 \sin x + C_2 \cos x) + e^x (C_1 \cos x + C_2 \sin x)$$

$$S = C_2 + C_1 \Rightarrow C_2 = S$$

$$y(x) = S e^x \sin 2x$$

(12)

$$y'' + 6y' + 13y = 0$$

$$y_1 = e^{-3x} \cos 2x$$

$$y_2 = e^{-3x} \sin 2x$$

$$y(0) = 2 \quad y'(0) = 0$$

Soln

$$y'' + 6y' + 13y = 0$$

$$\lambda^2 + 6\lambda + 13 = 0$$

$$\lambda^2 + 6\lambda + 9 + 4 = 0$$

$$(\lambda + 3)^2 + 2^2 = 0$$

$$\lambda + 3 = \pm 2i \Rightarrow \lambda = -3 \pm 2i$$

$$y(x) = e^{-3x} (C_1 \cos 2x + C_2 \sin 2x)$$

$$y(0) = 2 = C_1$$

$$y'(0) = -3e^{-3x} (C_1 \cos 2x + C_2 \sin 2x)$$

$$+ e^{-3x} (-2C_1 \sin 2x + 2C_2 \cos 2x)$$

$$0 = -3(C_1) + 1(2C_2)$$

$$2C_2 = 3C_1 \Rightarrow 2C_2 = 6 \Rightarrow C_2 = 3$$

$$y(x) = e^{-3x}(2\cos 2x + 3\sin 2x)$$

(13)

$$x^2 y'' - 2xy' + 2y = 0$$

$$y_1 = x, \quad y_2 = x^2$$

$$y(1) = 1$$

$$y'(1) = 1$$

Soln

$$x^2 y'' - 2xy' + 2y = 0$$

(1)

$$y_1 = x \Rightarrow y'' = 0$$

$$y' = 1$$

$$-2x(1) + 2x = 0 \quad \checkmark$$

(2)

$$y_2 = x^2 \Rightarrow y' = 2x \quad y'' = 2$$

$$2x^2 - 4x^2 + 2x^2 = 0 \quad \checkmark$$

$$y(x) = C_1 x + C_2 x^2$$

$$C_1 + C_2 = 3$$

$$C_1 + 2C_2 = 1$$

\Rightarrow

$$C_2 = -1 \Rightarrow y(x) = 4x - x^2$$

$$C_1 = 4$$

(14)

$$x^2 y'' + 2xy' - 6y = 0$$

$$y_1 = x^2 \quad y_2 = x^{-3} \quad y(2) = 10 \quad y'(2) = 15$$

Soln

$$y_1 = x^2 \quad y'_1 = 2x \quad y''_1 = 2$$

$$(1) \quad 2x^2 + 4x(2x) - 6x^2 = 0 \quad \checkmark$$

$$(2) \quad y_2 = x^{-3} \quad y'_2 = -3x^{-4} \quad y''_2 = +12x^{-5}$$

$$12x^{-3} - 6x^{-3} - 6x^{-3} = 0 \quad \checkmark$$

$$y(x) = C_1 x^2 + C_2 x^{-3}$$

$$y(2) = 10 = 4C_1 + \frac{C_2}{8}$$

$$y'(2) = 2C_1 - 3C_2 x^{-4} = 15$$

$$4C_1 - \frac{3}{16} C_2 = 15$$

$$C_1 = 3 \quad C_2 = -16$$

(20)

$$f(x) = \pi \quad g(x) = \cos^2 x + \sin^2 x$$

$$W = \begin{vmatrix} \pi & \cos^2 x + \sin^2 x \\ 0 & -2\sin x \cos x + 2\sin x \cos x \end{vmatrix}$$

$$W = \begin{vmatrix} \pi & \cos^2 x + \sin^2 x \\ 0 & 0 \end{vmatrix} = 0$$

Linearly dependent

(21)

$$f(x) = x^3 \quad g(x) = x^2 |x|$$

$$\begin{vmatrix} x^3 & x^3 \\ 3x^2 & 3x^2 \end{vmatrix}_{x>0}, \quad \begin{vmatrix} x^3 & -x^3 \\ 3x^2 & -3x^2 \end{vmatrix}_{x<0}$$

linearly dependent

(22)

$$f(x) = 1+x \quad g(x) = 1+|x|$$

$$\begin{vmatrix} 1+x & 1-x \\ 1 & -1 \end{vmatrix}_{x<0}, \quad \begin{vmatrix} 1+x & 1+x \\ 1 & 1 \end{vmatrix}_{x>0}$$

$$\Rightarrow -1-x + 1-x = -2x \quad \begin{matrix} 0 & x > 0 \\ & x < 0 \end{matrix}$$

linearly independent.

(23)

$$f(x) = xe^x \quad g(x) = |x|e^x$$

$$\begin{vmatrix} xe^x & -xe^x \\ e^x + xe^x & -e^x - xe^x \end{vmatrix}_{x < 0} \quad \begin{vmatrix} xe^x & xe^x \\ e^x + xe^x & e^x + xe^x \end{vmatrix}_{x > 0}$$

$$-xe^{2x} - x^2e^{2x} + xe^{2x} + x^2e^{2x} = 0 \quad \begin{matrix} 1 \\ 0 \end{matrix}$$

linearly dependent.

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$$y_p = 1 \quad y_c = C_1 \cos x + C_2 \sin x.$$

$$y'' + y = 1 \quad y(0) = -1 = y'(0)$$

$$y(x) = 1 + C_1 \cos x + C_2 \sin x$$

$$y(0) = -1 = 1 + C_1$$

$$y'(0) = -1 = -C_1 \sin x + C_2 \cos x = C_2$$

$$\Rightarrow C_1 = -2$$

$$C_2 = -1$$

$$y(x) = 1 - 2 \cos x - 1 \sin x$$

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$$x^2 y'' - 4x y' + 6y = 0$$

$$y_1 = x^2 \quad y_2 = x^3$$

$$y(0) = y'(0) = 0$$

$$\Rightarrow y'' - \frac{4}{x} y' + \frac{6}{x^2} y = 0$$

$P(x) = -\frac{4}{x}$, $q(x) = \frac{6}{x^2}$ are not continuous at $x=0$

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a

$$y_1 = x^3 \quad y_2 = |x^3|$$

$$x^2 y'' - 3xy' + 3y = 0$$

Solⁿ

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}_x$$

$$\Rightarrow \begin{vmatrix} x^3 & -x^3 \\ 3x^2 & -3x^2 \end{vmatrix}_{x < 0} = 0, \quad \begin{vmatrix} x^3 & x^3 \\ 3x^2 & +3x^2 \end{vmatrix}_{x > 0} = 0$$

They Contradict because

$$P(x) = -\frac{3}{x}, \quad Q(x) = \frac{3}{x}, \quad \text{in}$$

not continuous at $x=0$

(32)

$$A(x) y'' + B(x) y' + C(x) y = 0$$

y_1, y_2 are independent solutions

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2$$

$$A(x) y'' + B(x) y' + C(x) y = 0$$

$$\begin{aligned} W'(y_1, y_2) &= y_1 y_2'' + \cancel{y_1' y_2'} \\ &\quad - y_1'' y_2 - \cancel{y_1' y_2'} \\ &= y_1 y_2'' - y_1'' y_2 \end{aligned}$$

$$\begin{aligned} A(x) W' &= A(x) y_1 y_2'' - A(x) y_1'' y_2 \\ &= y_1 (-B(x) y_2' - C(x) y_2) \\ &\quad - y_2 (-B(x) y_1' - C(x) y_1) \end{aligned}$$

$$\begin{aligned}
&= -B y_1 y_2' - C y_1 y_2 \\
&\quad + B y_1' y_2 + C y_1 y_2' \\
&= B(y_1' y_2 - y_1 y_2') + C(y_1 y_2' - y_1' y_2) \\
&= B w(x)
\end{aligned}$$

$$\Rightarrow A(x) \frac{dw}{dx} = -B(x) w(x)$$

$$\Rightarrow \frac{dw}{dx} = -\frac{B(x)}{A(x)} w(x)$$

$$w' + \frac{B}{A} w = 0$$

$$\circ \int \frac{B}{A} dx \quad w' + \frac{B}{A} \circ \int \frac{B}{A} dx \quad w = 0$$

$$\left(\circ \int \frac{B}{A} dx \quad w \right)' = 0$$

$$\Rightarrow w = C e^{-\int \frac{B}{A} dx}$$

$$\Rightarrow w(x) = C e^{-\int \frac{B(x)}{A(x)} dx}$$

(33)

$$y'' - 3y' + 2y = 0$$

$$x^2 - 3x + 2 = 0$$

$$x^2 - 2x - x + 2 = 0 \quad (x-2)(x-1) = 0$$

$$x = 2 \text{ or } 1$$

$$\Rightarrow y(x) = C_1 e^{2x} + C_2 e^x$$

(34)

$$y'' + 2y' - 15y = 0$$

$$x^2 + 2x - 15 = 0$$

$$x^2 + 5x - 3x - 15 = 0$$

$$(x-3)(x+5) = 0$$

$$y(x) = C_1 e^{-5x} + C_2 e^{3x}$$

(35)

$$y'' + 5y' = 0$$

$$r^2 + 5r = 0$$

$$r = 0, -5$$

$$y = C_1 + C_2 e^{-5x}$$

(36)

$$2y'' + 3y' = 0$$

$$2r^2 + 3r = 0$$

$$r = 0, -\frac{3}{2}$$

$$y(x) = C_1 + C_2 e^{-\frac{3}{2}x}$$

(37)

$$2y'' - y' - y = 0$$

$$r^2 - \frac{1}{2}r - \frac{1}{2} = 0$$

$$r^2 - r + \frac{1}{2}r - \frac{1}{2} = 0$$

$$r(r-1) + \frac{1}{2}(r-1) = 0$$

$$r = -\frac{1}{2}, 1$$

$$y(x) = C_1 e^{-\frac{1}{2}x} + C_2 e^x$$

(38)

$$4y'' + 8y' + 3y = 0$$

$$x^2 + 2x + \frac{3}{4} = 0$$

$$x^2 + 2x + 1 = \frac{1}{4}$$

$$(x+1)^2 = \frac{1}{4} \quad x+1 = \pm \frac{1}{2}$$

$$\Rightarrow x = -\frac{3}{2}, -\frac{1}{2}$$

$$y(x) = c_1 e^{-\frac{3}{2}x} + c_2 e^{-\frac{1}{2}x}$$

(39)

$$4y'' + 4y' + y = 0$$

$$y'' + y' + \frac{y}{4} = 0 \quad x^2 + x + \frac{1}{4} = 0$$

$$\left(x + \frac{1}{2}\right)^2 = 0$$

$$y(x) = (c_1 + c_2 x) e^{-\frac{1}{2}x}$$

(40)

$$9y'' - 12y' + 4y = 0$$

$$x^2 - \frac{4}{3}x + \frac{4}{9} = 0$$

$$\left(x - \frac{2}{3}\right)^2 = 0 \quad x = \frac{2}{3}, \frac{2}{3}$$

$$y(x) = (C_1 + C_2 x) e^{+\frac{2}{3}x}$$

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$$6y'' - 7y' - 20y = 0$$

$$x^2 - \frac{7}{6}x - \frac{20}{6} = 0$$

$$x = \frac{+\frac{7}{6} \pm \sqrt{\frac{49}{36} + 4 \cdot \frac{20}{6}}}{2}$$

$$x = \frac{7}{12} \pm \frac{1}{2} \sqrt{\frac{49 + 480}{36}}$$

$$\begin{array}{r} 80 \times 6 \\ 480 \end{array} \quad \frac{36}{12}$$

$$\begin{array}{r} 480 \\ 529 \end{array}$$

$$x = \frac{7}{12} \pm \frac{1}{2} \sqrt{\frac{529}{36}}$$

$$x = \frac{7}{12} \pm \frac{23}{12} = +\frac{5}{2}, +\frac{5}{3}$$

$$y(x) = C_1 e^{\frac{5}{2}x} + C_2 e^{-\frac{5}{3}x}$$

(43)

$$y(x) = C_1 + C_2 e^{-10x}$$

$$y(x+10) = 0$$

$$\Rightarrow x^2 + 10x = 0$$

$$y' + 10y = 0$$

(44)

$$y(x) = C_1 e^{10x} + C_2 e^{-10x}$$

$$(x+10)(x-10) = 0$$

$$x^2 - 100 = 0$$

$$y'' - 100y = 0$$

(45)

$$y(x) = C_1 e^{-10x} + C_2 x e^{-10x}$$

$$(x+10)^2 = 0$$

$$x^2 + 20x + 100 = 0$$

$$y'' + 20y' + 100y = 0$$

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$$y(x) = C_1 e^{10x} + C_2 e^{100x}$$

$$(x-10)(x-100)$$

$$\Rightarrow x^2 - 110x + 1000 = 0$$

$$y'' - 100y' + 1000y = 0$$