Lec 23: Dirac Delta, Impulse Response

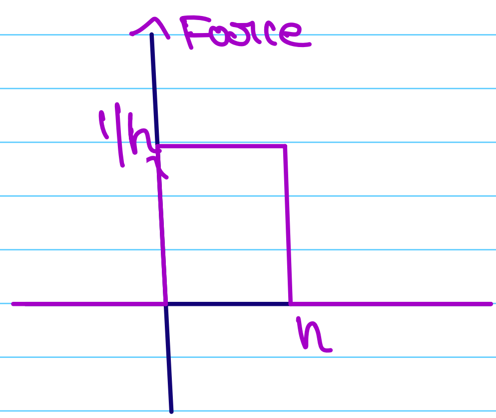
input is unit impulse

$f(t) =$ force

Impulse of $f(t) = \int_a^b f(t)\,dt$
over $[a,b]$

* if $f(t)$ is constant F

Impulse over time $= F \cdot (b-a)$



area = 1

Since Impulse is area under the curve,

if we want unit Impulse

Ex:

$$y'' + y = \frac{1}{h} u_{0h}(t)$$

what's the $\mathcal{L}\left(\frac{1}{h} u_{0h}(t)\right)$

$$\Rightarrow \frac{1}{h} \mathcal{L}\left(u(t) - u_h(t)\right)$$

$$\Rightarrow \frac{1}{h} \mathcal{L}\left(u(t) - u(t-h)\right)$$

$$\Rightarrow \frac{1}{h}\left(\frac{1}{s} - \frac{e^{-hs}}{s}\right)$$

$$\Rightarrow \mathcal{L}\left(\frac{1}{h} u_{0h}(t)\right) = \frac{1-e^{hs}}{hs}$$

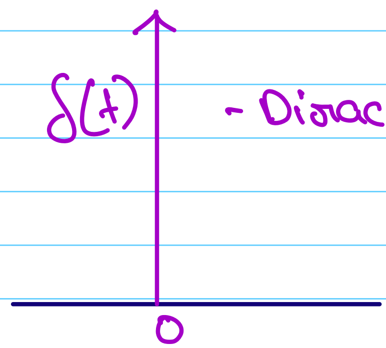
$$\lim_{h \to 0} \mathcal{L}\left(\frac{1}{h} u_{0h}(t)\right) = \lim_{h \to 0} \frac{1-e^{hs}}{hs}$$

$$= \lim_{u \to 0} \frac{1-e^{-u}}{u}$$

$$= \lim_{u \to 0} \frac{e^{-u}}{1} = 1$$

$$\frac{1}{h}\,\mathcal{U}_{0h}(t) \rightsquigarrow \frac{1-e^{hs}}{hs}$$

$$\Big\downarrow h \to 0 \qquad\qquad \Big\downarrow h \to 0$$



- Dirac delta

$$\rightsquigarrow$$

$$\mathcal{L}(\delta(t)) = 1$$

$$\delta(t) \rightsquigarrow 1$$

* $$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

* $$u(t)f(t) * \delta(t) \underset{\mathcal{L}}{\rightsquigarrow} F(s)\ F(\delta(t))$$

$$\Updownarrow$$

$$u(t)f(t) \underset{\mathcal{L}^{-1}}{\longleftarrow} = F(s)$$

$\Rightarrow$ $\delta(t)$ for the convolution operator is acting like an identity.

$$u'(t) = \delta(t)$$

Example:

$$y'' + y = A\,\delta\left(t - \frac{\pi}{2}\right) \qquad y(0) = 1 \quad y'(0) = 0$$

Spring is kicked with an impulse A at time $t = \frac{\pi}{2}$

* kick it mean's deliver that over an extremely short time interval, But in such a way kicked it sufficiently hard that the total impulse was A

$$y'' + y = A\,\delta\left(t - \frac{\pi}{2}\right)$$

$$\mathcal{L}(y'' + y) = \mathcal{L}\left(A\,\delta\left(t - \frac{\pi}{2}\right)\right)$$

$$s^2 Y(s) - s y(0) - y'(0)$$
$$+ Y(s)$$
$$= A e^{-\frac{\pi}{2}s}$$

$$(s^2+1)\, Y(s) - s = A e^{-\frac{\pi}{2}s}$$

$$Y(s) = \frac{s}{s^2+1} + \frac{A}{s^2+1} e^{-\frac{\pi}{2}s}$$

$$\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{s}{s^2+1} + \frac{A}{s^2+1} e^{-\frac{\pi}{2}s}\right)$$

$$= u(t)\cos t + A u\left(t-\tfrac{\pi}{2}\right)\sin\left(t-\tfrac{\pi}{2}\right)$$

$$y(t) = \begin{cases} \cos t & 0 \le t \le \frac{\pi}{2} \\ (1-A)\cos t & \text{o.w} \end{cases}$$

E(x) ÷

$$y'' + ay' + by = f(t)$$

↑ system ↑ input

$y(0) = 0$

$y'(0) = 0$

$$\mathcal{L}(y'' + ay' + by) = \mathcal{L}(f(t))$$

$$(s^2 + a + b)\,Y(s) = F(s)$$

$$Y(s) = F(s) \cdot \frac{1}{s^2 + as + b}$$

$\frac{1}{s^2+as+b}$ → It only depend's on Damping constant, spring constant

(Depend's only on system, Not on what Input going into it)

= Transfer function: $W(s)$ or $H(s)$

$$\mathcal{L}^{-1}(W(s)) = w(t)$$

= weight function of the system.

$$\Rightarrow \quad y(t) = \int_0^t f(u)\, w(t-u)\, du$$

in other word's the solution to differential eqn is presented as definite integral.

what is $w(t)$? really.

$$y'' + ay' + by = \delta(t)$$

$$y(0) = y'(0) = 0$$

kicking the system with unit impulse at $t=0$, unit Impulse.

$$\mathcal{L}(y'' + ay' + by) = \mathcal{L}(\delta(t))$$

$$\Rightarrow \quad Y(s)(s^2 + as + b) = 1$$

$$\Rightarrow \quad Y(s) = \frac{1}{s^2 + as + b}$$

$y(t) \rightsquigarrow$ weight function