Problem set 4.1

① $f(t) = t$

$$L(t) = \int_0^\infty t \cdot e^{-st} \, dt$$

$$= t \int e^{-st} \Big]_0^\infty - \int_0^\infty 1 \cdot \int e^{-st} \, dt$$

$$= \frac{t \cdot e^{-st}}{-s} \Big]_0^\infty - \frac{1}{s^2} e^{-st} \Big]_0^\infty$$

$$= 0 - \frac{1}{s^2} \left[0 - 1 \right] = \frac{1}{s^2}$$

$$\mathcal{L}(t) = \frac{1}{s^2}$$

② $f(t) = t^2$

$$\mathcal{L}(t^2) = \int_0^\infty t^2 \cdot e^{-st} \, dt$$

$$\Rightarrow \quad t^2 \frac{e^{-st}}{-s}\Big]_0^\infty - \int_0^\infty \frac{2t\,e^{-st}}{-s}$$

$$\Rightarrow \quad 0 + \frac{2}{s}\int_0^\infty t e^{-st} = \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2}{s^3}$$

$$\mathcal{L}(t^2) = \frac{2}{s^3} \qquad s>0$$

(3) $f(t) = e^{3t+1}$

$$\mathcal{L}(e^{3t+1}) = e\int_0^\infty e^{3t}\cdot e^{-st}\,dt$$

$$= e\int_0^\infty e^{-(s-3)t}\,dt$$

$$= e\,\frac{e^{-(s-3)t}}{-(s-3)}\Big]_0^\infty$$

$$= \frac{-e}{s-3}\left[0-1\right] = \frac{1}{s-3}$$

$$\mathcal{L}(e^{3t+1}) = \frac{e}{s-3} \qquad s>3$$

④ $f(t) = \cos t$

$$\mathcal{L}(\cos t) = \mathcal{L}\left(\frac{e^{it} + e^{-it}}{2}\right)$$

$$= \frac{1}{2}\mathcal{L}(e^{it}) + \frac{1}{2}\mathcal{L}(e^{-it})$$

$$= \frac{1}{2}\left(\frac{1}{s-i} + \frac{1}{s+i}\right)$$

$$= \frac{1}{2}\left(\frac{2s}{s^2+1}\right) = \frac{s}{s^2+1}$$

⑤ $f(t) = \sinh t$

$$\mathcal{L}(\sinh t) = \mathcal{L}\left(\frac{e^{x} - e^{-x}}{2}\right)$$

$$= \frac{1}{2}\mathcal{L}(e^{x}) - \frac{1}{2}\mathcal{L}(e^{-x})$$

$$= \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) \quad s>1$$

$$= \frac{1}{s^2+1}$$

6 $f(t) = \sin^2 t$

$$\mathcal{L}(\sin^2 t) = \mathcal{L}\left(\frac{1}{2} - \frac{\cos 2x}{2} \right)$$

$$= \frac{1}{2s} - \frac{1}{2} \frac{s}{s^2+4}$$

$$= \frac{1}{2} \left[\frac{1}{s} - \frac{s}{s^2+4} \right]$$

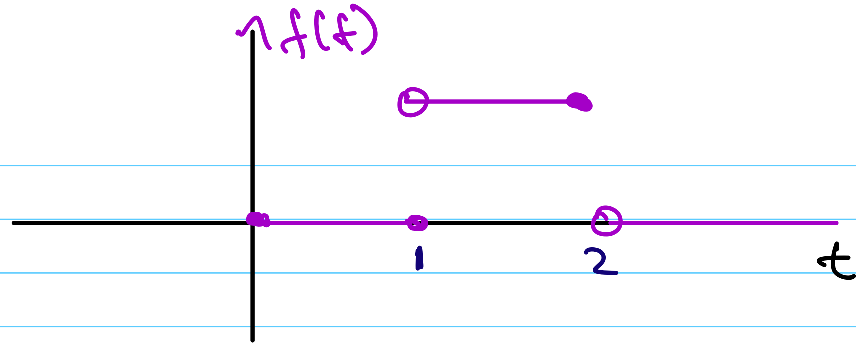
7 $f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{o.w} \end{cases}$

$$\mathcal{L}(f(t)) = \int_0^\infty f(t) e^{-st}\, dt$$

$$= \int_0^1 e^{-st}\, dt = \left. \frac{e^{-st}}{-s} \right]_0^1$$

$$= \frac{1}{s}\left[1 - e^{-s}\right] \qquad = -\frac{1}{s}\left[e^{-s} - 1\right]$$

⑧



$$f(t) = u_{12}(t) = u(t-1) - u(t-2)$$

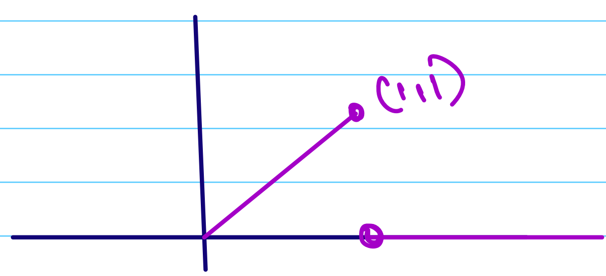
$$\mathcal{L}(u_{12}(t)) = \int_0^\infty e^{-st}\, u_{12}(t)\, dt$$

$$= \int_1^2 e^{-st}\, dt = \left.\frac{e^{-st}}{-s}\right]_1^2$$

$$= \frac{1}{s}\left[e^{-s} - e^{-2s}\right]$$

$s > 0$

⑨



$$\mathcal{L}(f(t)) = \int_0^1 t \cdot e^{-st}\, dt$$

$$= \left.\frac{t e^{-st}}{-s}\right]_0^1 - \int_0^1 \frac{e^{-st}}{-s}\, dt$$

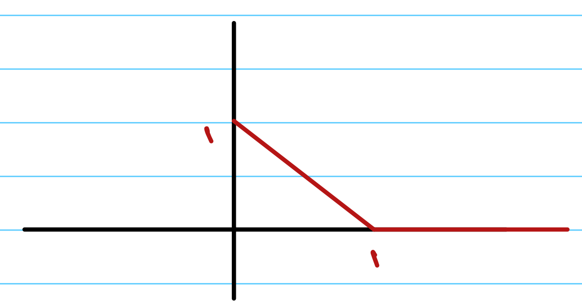
$$= \frac{e^{-s}}{-s} - \frac{e^{-st}}{s^2}\Big]_0^1$$

$$= \frac{e^{-s}}{-s} - \left[\frac{e^{-s}}{s^2} - \frac{1}{s^2}\right]$$

$$= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}$$

$$= \frac{1}{s^2}\left(s - e^{-s} - se^{-s}\right) \quad s>0$$

10



$$\mathcal{L}(f(t)) = \int_0^1 (1-t)\,e^{-st}\,dt$$

$$= \frac{(1-t)\,e^{-st}}{-s}\Big]_0^1 + \frac{e^{-st}}{s^2}\Big]_0^1$$

$$= \frac{1}{s} + \frac{e^{-s}}{s^2} - \frac{1}{s^2} \Rightarrow \frac{1}{s^2}\left(s-1+e^{-s}\right)$$

$s>0$

(11) $f(t) = \sqrt{t} + 3t$

$$f(t) = t^{1/2} + 3t$$

$$\mathcal{L}(f(t)) = \frac{\sqrt{\pi}}{s^{3/2}} + \frac{3}{s^2} \quad s>0$$

(12) $f(t) = 3t^{5/2} - 4t^3$

$$\mathcal{L}(f(t)) = 3\,\mathcal{L}(t^{5/2}) - 4\,\mathcal{L}(t^3)$$

$$= \frac{3\,\Gamma\left(\frac{7}{2}\right)}{s^{7/2}} - 4 \cdot \frac{3!}{s^4}$$

$$= \frac{3 \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \sqrt{\pi}}{s^{7/2}} - \frac{24}{s^4}$$

$$= \frac{45\sqrt{\pi}}{4s^{7/2}} - \frac{24}{s^4} \quad s>0$$

13) $f(t) = t - 2e^{3t}$

$$\mathcal{L}(f(t)) = \frac{1}{s^2} - \frac{2}{s-3} \qquad s>3$$

14) $f(t) = t^{3/2} - e^{-10t}$

$$\mathcal{L}(f(t)) = \frac{\Gamma(5/2)}{s^{5/2}} - \frac{1}{s+10} \qquad s>0$$

$$= \frac{\frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}}{s^{5/2}} - \frac{1}{s+10} \qquad s>0$$

$$= \frac{3\sqrt{\pi}}{4s^{5/2}} - \frac{1}{s+10} \qquad s>0$$

15) $f(t) = 1 + \cosh 5t$

$$\mathcal{L}(f(t)) = \frac{1}{s} + \frac{s}{s^2 - 25} \qquad s>5$$

(16) $f(t) = \sin 2t + \cos 2t$

$$\mathcal{L}(f(t)) = \frac{2}{s^2+4} + \frac{s}{s^2+4} = \frac{s+2}{s^2+4}$$

$s > 0$

(17) $f(t) = \cos^2 2t$

$\cos 2t = 2\cos^2 t - 1$

$$\mathcal{L}(f(t)) = \mathcal{L}\left(\frac{\cos 4t + 1}{2} \right)$$

$$= \frac{1}{2} \frac{1}{s} + \frac{s}{s^2+16} \qquad s > 0$$

(18) $f(t) = \sin 3t \cos 3t$

$$\mathcal{L}(f(t)) = \mathcal{L}\left(\frac{\sin 6t}{2} \right) = \frac{1}{2} \cdot \frac{6}{s^2+36}$$

$$= \frac{3}{s^2+36} \qquad s > 0$$

(19) $f(t) = (1+t)^3$

$$\mathcal{L}((1+t)^3) = \mathcal{L}(t^3 + 1 + 3t^2 + 3t)$$

$$= \frac{6}{s^4} + \frac{1}{s} + \frac{6}{s^3} + \frac{3}{s^2}$$

$s > 0$

20

$f(t) = te^t$

$$\mathcal{L}(e^t \cdot t) = \frac{1}{(s-1)^2} \quad s>1$$

21

$f(t) = t\cos 2t$

$$\mathcal{L}(t\cos 2t) = \mathcal{L}\left(t\left(\frac{e^{i2t}+e^{-i2t}}{2}\right)\right)$$

$$= \frac{1}{2}\mathcal{L}(te^{i2t}) + \frac{1}{2}\mathcal{L}(t\cdot e^{-i2t})$$

$$= \frac{1}{2}\frac{1}{(s-2i)^2} + \frac{1}{2}\left(\frac{1}{(s+2i)^2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{s^2-4-4i} + \frac{1}{s^2-4+4i}\right)$$

$$= \frac{1}{2}\left(\frac{2(s^2-4)}{(s^2-4)^2+16}\right)$$

$$= \frac{s^2-4}{(s^2-4)^2+16}$$

(23) $F(s) = \dfrac{3}{s^4}$

$$\mathcal{L}^{-1}\left(\frac{3}{s^4}\right) = \frac{1}{2}\,\mathcal{L}^{-1}\left(\frac{3!}{s^4}\right)$$

$$= \frac{t^3}{2}$$

(24) $F(s) = \dfrac{1}{s^{3/2}}$

$$\mathcal{L}^{-1}\left(\frac{1}{s^{3/2}}\right) = \mathcal{L}^{-1}\left(\frac{\Gamma(1/2+1)}{s^{1/2+1}} \cdot \frac{1}{\Gamma(1/2+1)}\right)$$

$$= \frac{t^{1/2}}{1}\,\frac{2}{\sqrt{\pi}}$$

$$= 2\sqrt{\frac{t}{\pi}}$$

(25) $F(s) = \dfrac{1}{s} - \dfrac{2}{s^{5/2}}$

$$\mathcal{L}^{-1}(F(s)) = t - \frac{2}{\Gamma(5/2)} \cdot t^{3/2}$$

$$= t - \frac{2}{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}\,t^{3/2} = t - \frac{8}{3\sqrt{\pi}}\,t^{3/2}$$

26 $F(s) = \dfrac{1}{s+5}$ $\quad f(t) = e^{-5t}$

27 $F(s) = \dfrac{3}{s-4}$ $\quad f(t) = 3e^{4t}$

28 $F(s) = \dfrac{3s+1}{s^2+4}$ $\quad f(t) = \dfrac{3}{2}\cos 2t + \dfrac{1}{2}\sin 2t$

29 $F(s) = \dfrac{5-3s}{s^2+9}$ $\quad f(t) = \dfrac{5}{3}\sin 3t - 3\cos 3t$

30 $F(s) = \dfrac{9+s}{4-s^2}$ $\Rightarrow f(t) = -\dfrac{9}{2}\sinh 2t - \cosh 2t$

31 $F(s) = \dfrac{10s-3}{25-s^2}$ $\quad f(t) = -10\cosh 5t + \dfrac{3}{5}\sinh 5t$

32 $F(s) = \dfrac{2}{s}e^{-3s}$ $\quad f(t) = 2u(t-3)$