Lec08: Solving $Ax=b$: Row reduced form R

Complete solution of $Ax=b$

rank r $\quad x = x_p + x_n$

$$\left[\begin{array}{cccc|c} \boxed{1} & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right]$$

Augmented matrix

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 2 & b_1 \\ 0 & 0 & \boxed{2} & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{bmatrix}$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right]$$

$$0 = b_3 - b_2 - b_1$$

Condition for solvability.

$$b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} \quad \text{ok } b$$

$$b = \begin{bmatrix} 1 \\ 5 \\ 17 \end{bmatrix} \quad \text{Not ok } b$$

Solvability: Condition on b to Always have solution for $Ax=b$

$\Rightarrow$ in column space $Ax=b$ is solvable

$b \in C(A)$

If a combination of row's of A gives zero row

$\Rightarrow$ The same combination on components on b have to be zero.

To find complete sol'n to $Ax=b$

$\Rightarrow x_2=0 \quad x_4=0$

① $x_{particular}$: set all free variables to 0

Solve $Ax=b$ for pivot variables.

$$x_1 + 2x_3 = 1 \qquad x_1 = -2$$

$$2x_3 = 3 \qquad x_3 = 3/2$$

$$x_p = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$$

(2) add on $x_{nullspace}$

$\Rightarrow$ complete solution $= \underbrace{x_p + x_n}$

this show's up every where when we have linear eq^n.

$$Ax_p = b \qquad Ax_n = 0$$

$$\Rightarrow \quad A(x_p + x_n) = b$$

$$x_{complete} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

m by n matrix A of rank r

r = # number of Pivots.

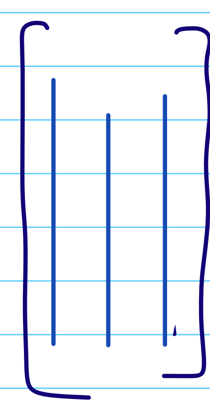
$\Rightarrow$ if we have m row's in the matrix and r Pivot's, we certainly know alway's $r \leq m$

$\Rightarrow$ Simillary, $r \leq n$

$\Rightarrow$ $r \leq m, n$

Full Coloumn rank:

$r = n$



$m \geq n = r$

$m \times n$

what does it tell about solution's? what does it tell about Noll space?

$\Rightarrow$ if th rank $r = n$, there is Pivot in every coloumn

$\Rightarrow$ How many pivot variables are there $= n$

$\Rightarrow$ How many free variables $= 0$

$\Rightarrow$ no free variable $\Rightarrow$ No Null space

Null space of $A = \{0\}$

$N(A) = \{0\}$

Solution's to $Ax = b$?

$x = x_p$ if exists

$Ax = b$ has unique (only only)

if it exists $\Rightarrow$ $b \in C(A$

$\Rightarrow$ there is 0 or 1 solution

when $r = n$ (Full coloumn rank)

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \longrightarrow \operatorname{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

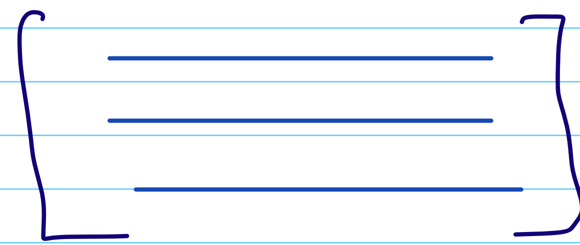
is there always a solution to $Ax=b$ in full column rank matrix.?

in above Ex: 4 eq^n, 2 unknown's

No. $b \in C(A)$ to have a solution.

Full Row rank:

$r=m$



$m \times n$

$r=m \le n$

$\Rightarrow$ every row has a pivot.

$\Rightarrow$ what about solvability? for which right hand side (b) we can solve $Ax=b$? for every b

because we won't be getting any zero row's, so there is no requirement on b. so we can solve $Ax=b$ for every b.

Solution always's exists

$\Rightarrow$ every row has a Pivot

$\Rightarrow$ $r = \#$ Pivots $= m$

Number of free variable $= n - m$

$\Rightarrow$ there will $n-m$ independent special solution's

Ex: $A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 6 & 5 \\ 0 & -5 & -17 & -14 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 6 & 5 \\ 0 & 1 & \frac{17}{5} & \frac{14}{5} \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} 1 & 0 & -\frac{4}{5} & -\frac{3}{5} \\ 0 & 1 & +\frac{17}{5} & \frac{14}{5} \end{bmatrix}}_{\text{rref}(A)}$$

$r = m = n$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ square, full rank, invertiable

$rref(A) = I_{2\times 2}$

$N(A) = \{0\}$

solution $Ax=b$ (1 unique) for all

$r=m=n$	$r=m<n$ (Full row rank)	$m>n=r$ (Full coloumn rank)
$R=I$	$R = [I \;\; F]$	$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$
1 solution	infinite nuber of solution's	(0 or 1) solution