Problem set 4.2

(1) Project the vector b onto the line through a. Check that e is perpendicular to a

(a) $b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ $\quad a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Projection matrix $P = \dfrac{aa^T}{a^Ta}$

$$\Rightarrow P = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$p = Pb = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{5}{3} \\ \frac{5}{3} \end{bmatrix}$$

$$e = b - e = \begin{bmatrix} 1 - \frac{5}{3} \\ 2 - \frac{5}{3} \\ 2 - \frac{5}{3} \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

$$a^T e = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} = 0$$

$$\Rightarrow \quad a \perp e$$

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a

$$b = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Sol^n

$$P = \frac{aa^T}{a^T a} = \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}}$$

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$b = \hat{x} a \Rightarrow \hat{x} = Pb$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta \\ 0 \end{bmatrix}$$

(b) $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $a = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$P = \frac{aa^T}{a^Ta} = \frac{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix}}{\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

$$= \frac{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}{2}$$

$$P = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\hat{x} = Pb = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad b \perp a$$

(4) $a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $a_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $P_2 = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$$P_1 + P_2 = \begin{bmatrix} 2.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

$$(P_1 + P_2)^2 = \begin{bmatrix} 2.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 2.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 6.25 & -1.5 \\ . & - \end{bmatrix}$$

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$$a_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \quad a_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

$$P_1 = \frac{a_1 a_1^T}{a_1^T a_1} = \frac{\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \end{bmatrix}}{\begin{bmatrix} -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}}$$

$$= \frac{\begin{bmatrix} +1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}}{9}$$

$$= \frac{1}{9} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$$

$$P_2 = \frac{\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 2 & -1 \end{bmatrix}}{\begin{bmatrix} 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}}$$

$$= \frac{1}{9} \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

$$P_1 P_2 = \frac{1}{81} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & +1 \end{bmatrix}$$

$$= \frac{1}{81} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$a_1^T a_2 = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = -2 + 4 - 2 = 0$$

Both $a_1 \perp a_2$

$\Rightarrow$ P_1 & P_2 are projection matrices onto orthonormal subspaces

$\Rightarrow$ any vector multiplied by both $P_1 P_2$ will be zero.

(6) Project $b = (1,0,0)$ onto the lines through a_1 and a_2 in Problems 5 and also onto $a_3 = (2,-1,2)$. and add up the three projection's $P_1 + P_2 + P$

$$P_1 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$P_1 = \frac{1}{9}\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} \qquad P_2 = \frac{1}{9}\begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix}$$

$$P_3 = \frac{a_3 a_3^T}{a_3^T a_3} = \frac{1}{9}\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}\begin{bmatrix} 2 & -1 & -2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 4 & -2 & -4 \\ -2 & 1 & 2 \\ -4 & 2 & 4 \end{bmatrix}$$

$$P_3 = \frac{1}{9}\begin{bmatrix} 4 \\ -2 \\ -4 \end{bmatrix}$$

$$P_1 + P_2 + P_3 = \frac{1}{9}\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix} + \frac{1}{9}\begin{bmatrix} 4 \\ -2 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$P_1 + P_2 + P_3 = a$$

(7) $P_1 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$

$P_2 = \frac{1}{9}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$

$P_3 = \frac{1}{9}\begin{bmatrix} 4 & -2 & -4 \\ -2 & 1 & 2 \\ -4 & 2 & 4 \end{bmatrix}$

$P_1 + P_2 + P_3 = I$

(8) Project the vector $b = (1,1)$ onto the lines through $a_1 = (1,0)$ and $a_2 = (1,2)$

$a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad P_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$a_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad P_2 = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ $\quad P_2 = \frac{1}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$P_1 + P_2 = \begin{bmatrix} \frac{6}{5} \\ \frac{2}{5} \end{bmatrix} \neq b$

this is because a_1, a_2 are not orthonormal.

9 a_1 a_2 are independent and BASIS for $\mathbb{R}^2$

$\Rightarrow$ $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ identity matrix.

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$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $P_2 = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$P_1 P_2 = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \neq 0$

$(P_1P_2)^T \neq P_1P_2 \Rightarrow P_1P_2$ is not Projection.