Lec17:

Orthogonal basis $q_1, \dots q_n$

Orthogonal matrix Q

Gram - Schmidt $A \rightarrow Q$

Orthonormal vector's

$$q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

* How does having orthonormal basis make things nice? it makes all the calculation's better, a whole lot of Numerical linear algebra built around working with orthonormal vector. they never overflow or underflow.

$$Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$$

$$Q^T Q = \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix} \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$$

$$= \begin{bmatrix} q_1^T q_1 & q_1^T q_2 & \cdots & q_1^T q_n \\ q_2^T q_1 & q_2^T q_2 & \cdots & q_2^T q_n \\ \vdots & & & \\ q_n^T q_1 & & & q_n^T q_n \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_{n \times n}$$

$\Rightarrow \quad Q^T Q = I_{n \times n}$

$\Rightarrow$ when we do $Q^T Q$ or $A^T A$ just asks for all the dot product's.

Now we have new Bunch of Important matrices :

we have seen

① triangular matrices

② Diagonal matrices D

③ Permutation matrices

④ row echelon form's

⑤ Projection matrices P

⑥ orthonormal matrix's Q

we only call orthonormal matrix Q when its square

$\Rightarrow$ if Q is square then $Q^TQ = I$

tell us that $Q^T = Q^{-1}$

Ex: Permutation matrix $Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$Q^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ & $Q^TQ = I$

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad Q^T Q = I$$

Suppose Q has orthonormal column's

Project onto its column space

$$P = Q (Q^T Q)^{-1} Q^T$$

$$P = Q (I)^{-1} Q^T$$

$$\Rightarrow \quad P = Q Q^T$$

$$P = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix}$$

$$\Rightarrow \quad P = q_1 q_1^T + q_2 q_2^T + \cdots + q_n q_n^T$$

(projection matrix)

if Q is square then $P = QQ^T = I$

Projection:

$$A^T A \hat{x} = A^T b$$

$$\Rightarrow Q^T Q \hat{x} = Q^T b$$

$$\Rightarrow \hat{x} = Q^T b$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} q_1^T b \\ q_2^T b \\ \vdots \\ q_n^T b \end{bmatrix} \Rightarrow \hat{x}_i = q_i^T b$$

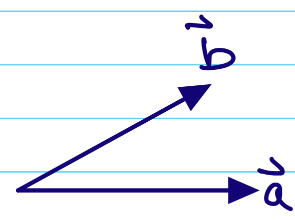
we don't start with orthogonal matrix

we just start with independent vector's

and make it orthogonal matrix

GRAM - Schmidt

independent vector's a, b



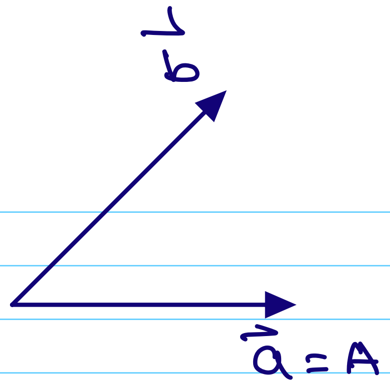
Goal is to get

two orthogonal vector's

A & B

and make it orthonormal

$q_1 = \frac{A}{||A||}$, $q_2 = \frac{B}{||B||}$



$$\vec{B} = b - Pb$$

$$= b - \left(\frac{AA^T}{A^TA}\right) b$$

$$\Rightarrow \vec{B} = \left(I - \frac{AA^T}{A^TA}\right) b$$

let there is another vector c

$$q_3 = c - \frac{AA^T}{A^TA} c - \frac{BB^T}{B^TB} c$$

$A = QR$

$$\begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} a_1^T q_1 & a_2^T q_1 \\ \underbrace{0}_{a_1^T q_2 = 0} & a_2^T q_2 \end{bmatrix}$$

A	=	Q	R
square invertable		orthonormal matrix	triangular matrix