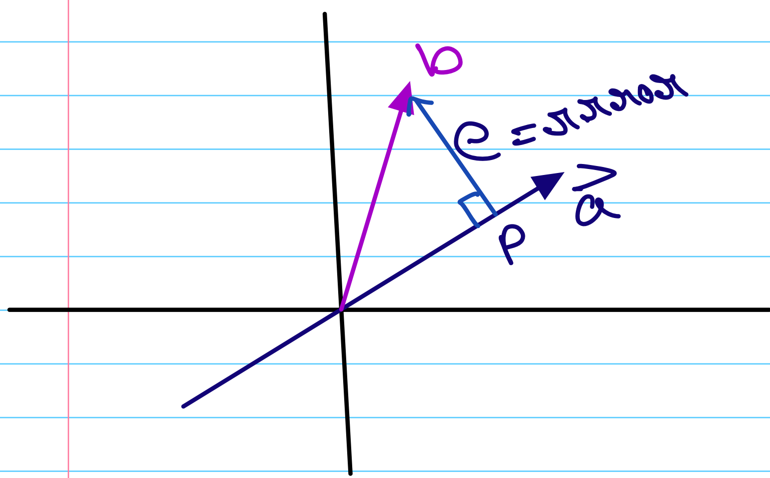
lec 15: Projection's onto subspaces



Projection p onto the line $\vec{a}$

$e = b - p$

$\Rightarrow e \perp a$

we know p is some multiple of a

$\Rightarrow \quad p = xa$

$a \perp e = b - p$

$\Rightarrow \quad a \perp b - xa$

$$\Rightarrow \quad a^T(b - x a) = 0$$

$$\Rightarrow \quad a^T b - x a^T a = 0$$

$$\Rightarrow \quad x = \frac{a^T b}{a^T a}$$

in calculus

$$x = \frac{a^T b}{a^T a} = \frac{\cancel{\|a\|_2} \|b\|_2 \cos\theta}{\|a\|_2 \cancel{\|a\|_2}}$$

$$x = \frac{\|b\|_2}{\|a\|_2} \cos\theta$$

let $A = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}$

$$A^T b = \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} \begin{bmatrix} b \end{bmatrix}$$

$$= \begin{bmatrix} a_1^T b \\ a_2^T b \end{bmatrix}$$

$$A^T A x = A^T b$$

$$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix} x = A^T b$$

$$\begin{bmatrix} a_1^T a_1 & a_1^T a_2 \\ a_2^T a_1 & a_2^T a_2 \end{bmatrix} x = \begin{bmatrix} a_1^T b \\ a_2^T b \end{bmatrix}$$

$$x = \frac{a^T b}{a^T a} \qquad P = ax$$

$$\Rightarrow \quad P = a \cdot \frac{a^T b}{a^T a}$$

$$\Rightarrow \quad P = \frac{a a^T}{a^T a} \cdot b$$

$$P = \frac{a a^T}{a^T a}$$ (projection matrix)

* what's the column space of Projection matrix?

if we multiply that matrix by any vector we always get into the column space?

And The column space of a matrix is when we multiply any vector by that matrix - any vector b, we always land in the column space.

$\Rightarrow$ we always land in column space.

if $b \in C(A)$ then $Pb = b$

$\Rightarrow$ rank of Projection matrix is rank of column space.

① is P symmetric? $P = \dfrac{aa^T}{a^Ta}$

$$P^T = \left(\frac{aa^T}{a^Ta}\right)^T = \frac{aa^T}{a^Ta}$$ yes

(2) $P^2 = ?$

let $p = Pb$ where p is the Projected vector of b

the $Pp = p$ again becaus $p \in C(A)$

$\Rightarrow P^2 b = Pp = Pb$

$\Rightarrow P^2 = P$

$\Rightarrow$ Projection matrix $P = P^T$

$P^2 = P$

Why Project?

Because $Ax = b$ may have no solution. $(m > n = r)$

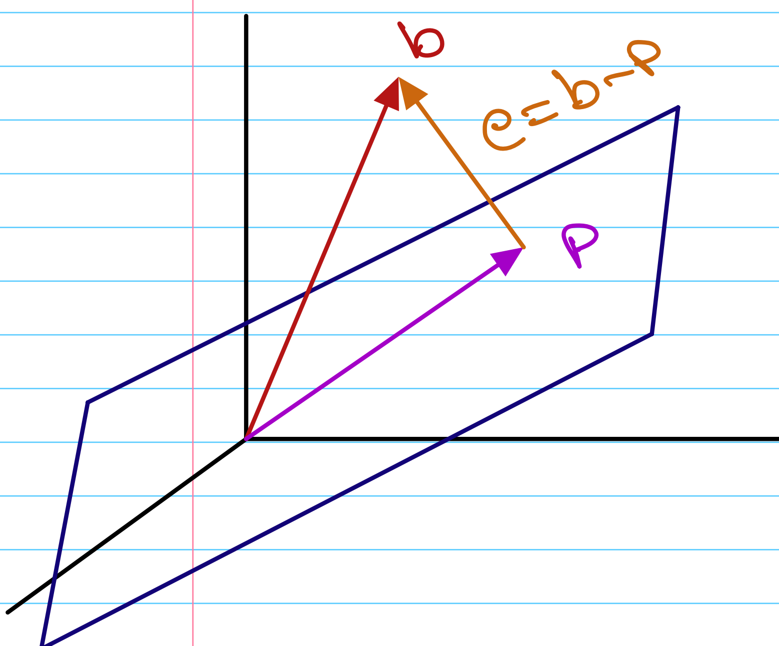
* what should we do, let's solve the closest problem.

* The Problem is $Ax \in C(A)$ & b Probably not in $C(A)$

* Let's change b to the closest vector in $C(A)$

* Now solve $Ax = p$

* where p is the projection of b onto $C(A)$



a_1, a_2 two vectors on the plane

& plane in coloumn space of A

$$A = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}$$

$e = b - p \perp$ to plane

to $C(A)$

Projection vector

$$p = x_1 a_1 + x_2 a_2$$

$$p = A\hat{x}$$

$p = A\hat{x}$ find $\hat{x}$

Key:

$e \perp a_1 \Rightarrow a_1^T e = 0 \Rightarrow a_1^T(b - A\hat{x}) = 0$

$e \perp a_2 \Rightarrow a_2^T e = 0 \Rightarrow a_2^T(b - A\hat{x}) = 0$

$e \perp C(A)$

$\Updownarrow$

$e \in N(A^T)$

$$\Rightarrow \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} (b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow A^T(b - A\hat{x}) = 0$$

$$\Rightarrow A^T b - A^T A\hat{x} = 0$$

$$\Rightarrow A^T A\hat{x} = A^T b$$

$$\hat{x} = (A^T A)^{-1} A^T b$$

all the coloumn's need to be independent.

$$p = A\hat{x} = \underbrace{A(A^T A)^{-1} A^T}_{\text{Projection matrix}} b$$

matrix $P = A(A^T A)^{-1} A^T$

$$P^T = \left(A(A^TA)^{-1}A^T\right)^T$$

$$= A\left((A^TA)^{-1}\right)^T A^T$$

$$= A(A^TA)^{-1}A^T = P$$

$$\Rightarrow P^T = P$$

$$P^2 = A(A^TA)^{-1}A^T \cdot A(A^TA)^{-1}A^T$$

$$= A(A^TA)^{-1}\underbrace{A^TA(A^TA)^{-1}}_{I}A^T$$

$$= A(A^TA)^{-1}A^T = P$$