Lec07: Solving $Ax=0$, Pivot variables, Special solution's.

Computing the nullspace ($Ax=0$)

Pivot variables - free variables.

Special solution's - rref (A) = R

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

while we are doing elimination, we are not changing Null Space,

we are solving $Ax=0$

after operation's $Rx=0$

$\Rightarrow$ we are not changing row space

we are changing column space.

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

echelon form $\rightarrow$

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

Pivot coloumn's

Free coloumn's

Number of Pivot's is 2, this Number we will call rank of the matrix

rank of A = # of Pivot's
$= 2$

Now

$Ux = 0$ (same null space)

* So for free coloumn's we can assign any number's freely. (x_2, x_4)

* and then solve for x_1, x_3

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$

$$2x_3 + 4x_4 = 0$$

$$x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad x = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Null space : $c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

Null space contain's exactly all the combination of the special solution's.

$\Rightarrow$ there is one special solution for every free variable

$\Rightarrow$ if r = rank = # Pivots

n coloumn's

$\Rightarrow$ $n - r$ = free coloum's

R = reduced row echelon form

$$Ux = 0$$

$$\begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

rref will have zero's above and below the pivot's.

$$\begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \overset{\text{rref}(A)}{\begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \text{Pivot col} & \text{free} & \text{Pivot col} & \text{Free} \end{array}$$

$$Ax = 0$$
$$\downarrow$$
$$Ux = 0$$
$$\downarrow$$
$$Rx = 0$$

Notice $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ sitting on pivot row's / column's

$$\begin{array}{cccc} & \text{Free} & & \\ & \downarrow & & \downarrow \end{array}$$

$$Rx=0 \quad \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Pivot coloumn's free coloumn's

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow I \quad \begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix} \rightarrow F$$

Row of zero's $\quad 0 \quad 0 \qquad 0 \quad 0$

$$\underbrace{\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}}_{\text{rref form}} \underbrace{\begin{bmatrix} -F \\ I \end{bmatrix}}_{\text{Null space matrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(matrix of special solution's)

$$Rx=0$$

$$\begin{bmatrix} I & F \end{bmatrix} \begin{bmatrix} x_{pivot} \\ x_{free} \end{bmatrix} = 0 \Rightarrow x_{pivot} = -F x_{free}$$

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & 0 \\ 0 & \boxed{2} & 2 \\ 0 & 4 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Free

$\Rightarrow \quad Rx = 0$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$r = 2$ again

number of Pivot coloumn's for A, A^T are same