Lec 22

Diagonalizing a matrix $S^{-1}AS = \Lambda$

Power's of A | equation $U_{k+1} = AU_k$

* JOB1 is to find the eigenvalue & eigenvector's

* Now after we found them what do we do with them, The good way to see that is diagonalize the matrix. diagonalize matrix A

S = eigenvector matrix

There is S^{-1}, mean's we have to be able to invert eigenvector matrix $\Longrightarrow$ we need n independent eigenvalues.

$\Longrightarrow$ Suppose we have n independent eigenvector's. Put them in column's of S

$$AS = A\begin{bmatrix} x_1 & x_2, & \cdot & x_n \end{bmatrix}$$

$$= \begin{bmatrix} Ax_1 & Ax_2 & \cdot & \cdot & Ax_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdot & \cdot & \lambda_n x_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & \cdot & \cdot & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

$$\Rightarrow \quad AS = S\Lambda$$

$$\Rightarrow \quad S^{-1}AS = \Lambda \quad \text{or} \quad A = S\Lambda S^{-1}$$

Λ = diagonal eigenvalue's

$\Rightarrow$ we need n independent eigen-vectors, we could repeat eigen-values but not eigenvectors.

$\Rightarrow$ we want to be able to invert S.

$$\Rightarrow \quad S^{-1} A S = \Lambda$$

$$\Rightarrow \quad A = S \Lambda S^{-1}$$

(new factorization)

what are the eigenvalue & eigenvectors of A^2

$$Ax = \lambda x$$

Now $A^2 x = AAx = A\lambda x = \lambda Ax$
$= \lambda^2 x$

$$\Rightarrow \quad A^2 x = \lambda^2 x$$

$\Rightarrow$ A^2 has λ^2 eigenvalue, x as eigenvector.

$\Rightarrow$ eigen value of A^2 are λ^2

eigen vector's of A^2 stay same (x)

$$A = S \Lambda S^{-1}$$

$$A^2 = S \Lambda S^{-1} S \Lambda S^{-1}$$

$$= S \Lambda I \Lambda S^{-1}$$

$$\boxed{A^2 = S \Lambda^2 S^{-1}}$$

$$\boxed{A^k = S \Lambda^k S^{-1}}$$

* eigen values and eigenvector's give a great way to understand the Power's of matrix.

* eigen value's tell us about Power's of matrix.

* When do the Power's of matrix go to 0? (stable matrix)

$$A^k \longrightarrow 0 \quad \text{as} \quad k \to \infty$$

$$|\lambda_i| < 1 \quad \forall i$$

$\Rightarrow$ all eigenvalue's have to be $|\lambda_i|$ less than 1

* Again a pure eigenvalue eigenvector approach need's n independent eigenvector's.

Which matrices are diagonalizable?

A is sure to have n independent eigenvector's (and be diagonalizable) if all the λ's are different.

(No Repeated eigenvalue)

if some λ's are repeated then we have to look more closely.

$\Rightarrow$ we have to check has it got repeated eigenvalue's $\Rightarrow$ we may or may not have n independent eigenvector's.

$I_{10\times10} \Rightarrow$ all eigenvali's are $= 1$

But 10, independent eigenvector's

Ex:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \qquad \det(A-\lambda I) = \begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix}$$

$$\lambda = 2, 2$$

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

we don't have enough eigen value's.

$\Rightarrow$ we cannot diagonalizable.

1st order difference eq^n

Equation: $u_{k+1} = A u_k$

Start with given vector u_0

$$u_1 = A u_0$$

$$u_2 = A u_1 = A^2 u_0$$

$$u_3 = A u_2 = A^3 u_0$$

$$\vdots$$

$$u_k = A^k u_0$$

To really solve $u_k = A^k u_0$

write u_0 as a combination of eigen vector's

$$u_0 = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

$$A u_0 = c_1 \lambda_1 x_1 + c_2 \lambda_2 x_2 + \cdots + c_n \lambda_n x_n$$

$$\vdots$$

$$u_k = A^k u_0 = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + \cdots + c_n \lambda_n^k x_n$$

$$\Rightarrow \quad u_0 = Sc$$

$$A^k u_0 = \Lambda^{100} Sc$$

Fibonacci example: 0, 1, 1, 2, 3, 5, 8, 13..

$$F_{100} = ?$$

How fast are the Fibonacci numbers growing. The Answer lies in eigen values

$$F_{k+2} = F_{k+1} + F_k \qquad u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

$$u_{k+1} = \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}}_{u_k}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$$

$$\Rightarrow \quad \lambda^2 - \lambda - 1 = 0 \quad \Rightarrow \quad \lambda = \frac{1 \pm \sqrt{5}}{2}$$

$\lambda_1 = 1.618 \qquad \lambda_2 = -0.618$

$x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$

How fast are those Fibonacci number's increasing? They are increasing, Not doubling at every step. $\left(\lambda_1 = \frac{1+\sqrt{5}}{2}\right)$

$$F_{100} \approx c_1 \left(\frac{1+\sqrt{5}}{2}\right)$$

$$u_0 = c_1 x_1 + c_2 x_2$$

$$u_{100} = c_1 \lambda_1^{100} x_1 + \underbrace{c_2 \lambda_2^{100}}_{\text{Extremely small number}} x_2$$

* we are doing the Problem's that are evolving, we are doing dynamic instead $Ax=b$ (static Problem)
* we are now doing dynamic's $A, A^2, A^3, \ldots$ thing are evolving in time $\Rightarrow$ eigenvalues are crucial number's.