Lec 23 Differential eqn

$$\frac{du}{dt} = Au$$

Exponential e^{At} of a matrix

This section is about how to solve system of 1st order, 1st derivative constant coeff linear eqn

Ex:

$$\frac{du_1}{dt} = -u_1 + 2u_2$$

$$\frac{du_2}{dt} = u_1 - 2u_2$$

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\frac{du}{dt} = A \quad u$$

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$$ Singular

$\Rightarrow \lambda_1 = 0 \quad x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\Rightarrow \lambda_2 = -3 \quad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Now it's exponential.

$$\frac{du_1}{dt} = -u_1 + 2u_2 \qquad u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{du_2}{dt} = u_1 - 2u_2$$

at start its all u_1

* As time goes on $\frac{du_2}{dt} = +ve$ flow will move into u_2 component, and $\frac{du_1}{dt} = -ve$ so flow go out in u_1 component.

$\Rightarrow$ So we follow that movement as time goes forward, By looking at eigenvalue & eigenvector's.

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$$

Singular matrix

$\Rightarrow \lambda_1 = 0 \ , \lambda_2 = -3$

$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} , x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

<u>solution:</u>

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$$

$$u(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad u(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\begin{aligned} 2c_1 + c_2 &= 1 \\ c_1 - c_2 &= 0 \end{aligned} \Rightarrow c_1 = c_2 = \frac{1}{3}$$

Steady stat $\quad c(\infty) = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

When do we get stability : $u(t) \to 0$

$\Rightarrow \lambda_1, \lambda_2 < 0$ (-ve eigenvalue)

$Re(\lambda) < 0$

$$c_1 e^{\lambda_1 t} x_1 + c_1 e^{\lambda_2 t} x_2 = u(0)$$

$$\Rightarrow \begin{bmatrix} e^{\lambda_1 t} x_1 & e^{\lambda_2 t} x_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = u(0)$$

at $t=0$ $$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = u(0)$$

$$S \quad c = u(0)$$

$$\Rightarrow \quad Sc = u(0)$$

$\Rightarrow$ we are finding how much of each pure exponential is in the solution, by getting it right at the start, then it stay's right forever.

$\Rightarrow$ each pure exponential $e^{\lambda_1 t} x_1$, $e^{\lambda_2 t} x_2$ goes on its own way once you start from u_0. So we start it by figuring out how much each one pure exponential present in $u(0)$ and then off they go.

$\Rightarrow$ $\dot{x} = Ax$ system of two eqn and two unknown's coupled. The matrix sort of couple u_1 & u_2 and eigen value's & eigenvector's uncouples them, diagonalize them.

Solution in term's of S and λ

$$\frac{du}{dt} = Au \quad (A \text{ couples them})$$

the whole point of eigen value's, & eigenvector's is to uncouple them.

$$\frac{du}{dt} = Au \quad \Rightarrow \quad u = Sv$$

$$S\frac{dv}{dt} = ASv \quad \Rightarrow \quad \frac{dv}{dt} = S^{-1}ASv$$

$$\Rightarrow \quad \frac{dv}{dt} = \Lambda v$$

$$\Rightarrow \quad \left.\begin{aligned} \frac{dv_1}{dt} &= \lambda_1 v_1 \\ \frac{dv_2}{dt} &= \lambda_2 v_2 \end{aligned}\right\} \text{decoupled}$$

$$v(t) = e^{\Lambda t}\, w(0)$$

$$\Rightarrow \quad v_1 = c_1 e^{\lambda_1 t}, \quad v_2 = c_2 e^{\lambda_2 t}$$

$$u(t) = S e^{\Lambda t} S^{-1} u(0)$$

$$e^{At} = S e^{\Lambda t} S^{-1}$$

<u>Matrix exponential:</u> e^{At}

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots + \frac{A^nt^n}{n!}$$

$$\frac{d}{dt}e^{At} = A + A^2t + \frac{A^3t^2}{2!} + \dots + A^{n+1}\frac{t^n}{n!} + \dots$$

$$= Ae^{At} = e^{At}A$$

$$e^{At} = I + At + A^2\frac{t^2}{2!} + \cdots + A^n\frac{t^n}{n!} + \cdots$$

$$e^{At} = SS^{-1} + S\Lambda S^{-1}t + S\Lambda^2 S^{-1}\frac{t^2}{2!} + \cdots + S\Lambda^n S^{-1}\frac{t^n}{n!}$$

$$= S\left[I + \Lambda t + \Lambda^2\frac{t^2}{2!} + \cdots\right]S^{-1}$$

$$= S e^{\Lambda t} S^{-1}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & \ddots & e^{\lambda_n t} \end{bmatrix}$$

$Re(\lambda_i) < 0$ for stable.