Lec 14 : orthonormal vector's and subspace.

Orthonormal vector's and subspace

Null space $\perp$ row space

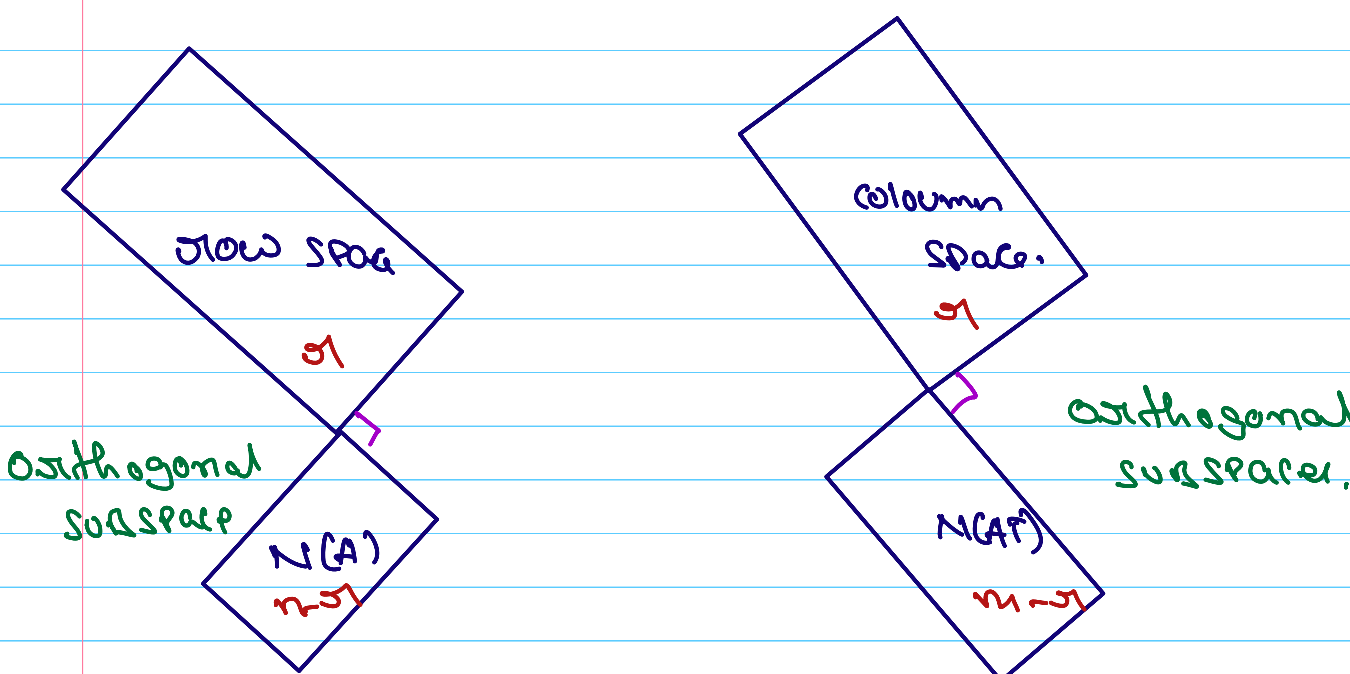
$$N(A^TA) = N(A)$$

Chapter about orthogonality. what it mean's to vector's to be orthogonal?

what it mean's subspace to be orthogonal?

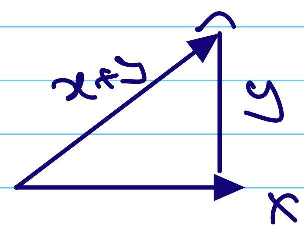
what it mean's BASIS to be orthogonal?

90° Chapter



the angle b/w these subspace is 90°.

Orthogonal vector's



$$\|x\|_2^2 + \|y\|_2^2 = \|x+y\|_2^2$$

$$\Rightarrow \cancel{x^Tx} + \cancel{y^Ty} = (x+y)^T(x+y)$$

$$= (x^T+y^T)(x+y)$$

$$= \cancel{x^Tx} + \cancel{y^Ty} + 2x^Ty$$

$$\Rightarrow x^Ty = 0$$

* Subspace S is orthogonal to subspace T

<u>means</u>: every vector in S is orthogonal to every vector in T

row space $\perp$ Null space

$Ax = 0 \qquad x \in N(A)$

$$\begin{bmatrix} \text{row } 1 \text{ of } A \\ \text{row } 2 \text{ of } A \\ \vdots \\ \text{row } m \text{ of } A \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\Rightarrow \quad (\text{row } 1)^T x = 0 \Rightarrow \text{row } 1 \perp x$

$(\text{row } 2)^T x = 0 \Rightarrow \text{row } 2 \perp x$

$\vdots \qquad\qquad \vdots$

$(\text{row } m)^T x = 0 \Rightarrow \text{row } m \perp x$

$\Rightarrow$ any linear combination of row's will give 0 as well

$$(c_1 \text{ row } 1 + c_2 \text{ row } 2 \cdots + c_m \text{ row } m)^T x = 0$$

$$c^T A x = 0$$

$$\dim(N(A)) + \dim(\text{row space}) = n$$

$N(A) \perp$ row space

row space is orthonormal compliment of Null space.

$\Rightarrow$ null space and row space are orthonormal complements in $\mathbb{R}^n$

$\Rightarrow$ Nullspace contain's all vector's $\perp$ to row space.

Coming:

$$Ax = b$$

we would like to solve this linear system of eq^n when there is no solution. $\Rightarrow$ $b \notin C(A)$

Solve when $m > n$

Matrix that play a key role.

A^TA

$A^TA \rightarrow$ square, symmetric
$(n \times n)$

$(A^TA)^T = A^T(A^T)^T = A^TA$ (symmetric)

$A^TA \rightarrow$ is it invertable? if not what its null space.

The good eqn we get is

$$A^TAx = A^Tb$$

Ex:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$A \quad x \quad = b$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$$

$$AB = C$$

row's of C is linear combination of row's of B

coloumn's of C are linear combination of coloumn's of A.

A^TA = (row's will be linear combination of row's of A)

(coloumn's will be linear combination of row's of A)

$$N(A^TA) = N(A)$$

Because $Ax = 0 \Rightarrow A^TAx = 0$

rank of A^TA = rank of A

$\Rightarrow$ A^TA is invertable exactly

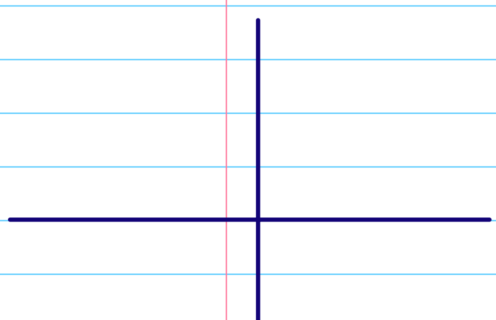
if A has independent coloumn's.

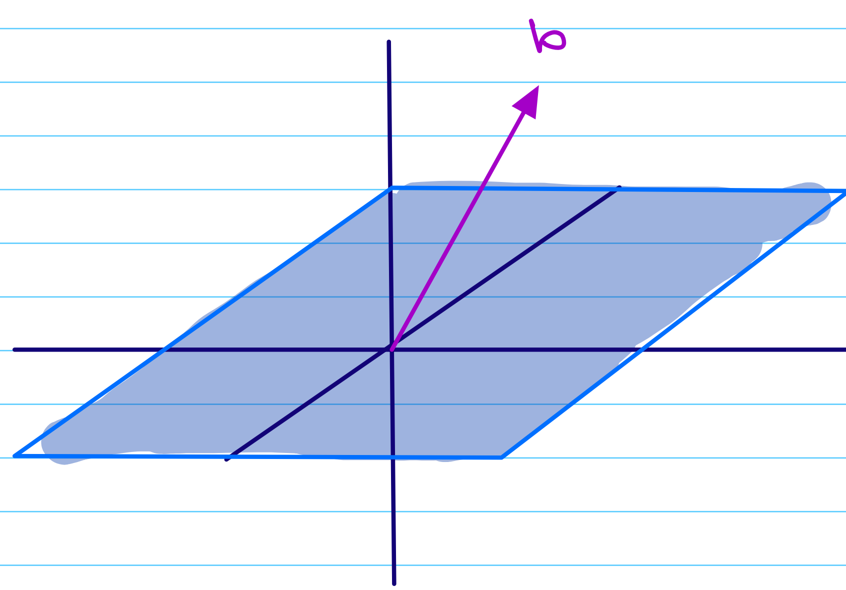
Let $A_{3\times 2}$ with rank 2

$\Rightarrow N(A) = \{0\}$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}_{3\times 2}$$

R^2





$Ax=b$ does not have

a solution becau $b \notin C(A)$

$$A^T A x = A^T b$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\Rightarrow \quad I x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\Rightarrow \quad x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}$

$$\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 2 & 5 \end{vmatrix} = i(3) - j(4)$$

$$k(2-1)$$

$$= 3\hat{i} - 4\hat{j} + \hat{k}$$

$$3x - 4y + z = 0$$

$$b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$