Problem set 4.1

① Construct any 2×3 matrix of rank one.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

② $$A_{3\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

row space dim = 2

$N(A) = \{0\}$ zero vector

$C(A) = 2$

$N(A^T) = 1$

③ Construct a matrix with the required property or say why that is impossible.

(a) Coloumn space contain's $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$, Nullspace contain's $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Sol^n this should be 3×3 matrix with rank 2

$$1 \cdot \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} + 1 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{array}{l} 1+2+x=0 \\ 2-3+y=0 \\ -3+5+z=0 \end{array}\right\} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix}$$

(b) Row space contain's $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$

Null space contain's $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Row space $\perp$ $N(A)$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = 0 \quad \checkmark$$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} = 4 \quad \times$$

Both row vector & Null vector are not orthonormal

$\Rightarrow$ Impossible to construct

(c) $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has a solution

and $A^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Sol^n $\quad Ax = \begin{bmatrix}1\\1\\1\end{bmatrix}$ has a solution

$$\Rightarrow \begin{bmatrix}1\\1\\1\end{bmatrix} \in C(A)$$

$$A^T \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1\\0\\0\end{bmatrix} \in N(A^T)$$

But $\quad C(A) \perp N(A^T)$

$$\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = 1 \neq 0$$

$\Rightarrow$ Impossible to construct.

(d) Every row is orthonormal to every column (A is not the zero matrix)

Sol^n $\quad$ Need's to be square matrix

$$C(A) \perp C(A^T)$$

coloumn space $\perp$ rowspace

$$\Rightarrow \quad C(A) = N(A)$$

$$\& \quad C(A^T) = N(A^T)$$

Let $A = 3 \times 3$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

row's need's to be $\perp$ coloumn's

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = 0$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix} \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

$\Rightarrow \quad AA = 0$

(4) if $AB=0$ then the coloumn's of B are in the __________ of A. The row's of A are in the __________ of B.

with $AB=0$, why can't A and B be 3×3 matrix of rank 2?

Sol^n

if $AB = 0$

the coloumn's of $B \in N(A)$

the row's of $A \in N(A^T)$

$$A_{3\times3}\, B_{3\times3} = 0$$

where A, B are rank 2

the coloumn's of (Dim 2) need to belong $\in \underbrace{N(A)}$

this is rank 1

becau $A_{3\times3}$ rank 2

$\Rightarrow$ it Impossible to construct.

5 a if $Ax=b$ has a solution $A^Ty=0$, is $y^Tx=0$ or $y^Tb=0$?

Sol^n

$$Ax=b \Rightarrow b \in C(A)$$
$$x \in C(A^T)$$
$$A^Ty=0 \Rightarrow y \in N(A^T)$$
$$\Rightarrow y^Tb=0 \quad \text{because}$$
$$C(A) \perp N(A^T)$$

b if $A^Ty=(1,1,1)$ has a solution and $Ax=0$, then.

$$A^T y = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in C(A^T)$$

$$Ax = 0 \Rightarrow x \in N(A)$$

$$\Rightarrow x^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0 \quad \text{(orthonormal)}$$

6 This system of eq^n $Ax = b$ has no solution (they lead to $0 = 1$)

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 9 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 5 \\ 2 & 2 & 3 & 5 \\ 3 & 4 & 5 & 9 \end{array}\right]$$

$$= \begin{bmatrix} 1 & 2 & 2 & 5 \\ 0 & -2 & -1 & -5 \\ 0 & -2 & -1 & -6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 & 5 \\ 0 & -2 & -1 & -5 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

unable to follow solvability

$\Rightarrow$ there is no solution for this system of eqn

$\Rightarrow$ $Ax = b$ when $b \notin C(A)$

Let's find y_1, y_2 y_3

$$[y_1 \; y_2 \; y_3] \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [y_1 \; y_2 \; y_3] \begin{bmatrix} 5 \\ 5 \\ 9 \end{bmatrix}$$

To make $y^T A = 0$

$$\Rightarrow \quad y \in N(A^T)$$

but $\quad b \notin C(A)$

$$\Rightarrow \quad b = \underbrace{p}_{\text{vector in } C(A)} + e$$

$(e \neq 0)$

vector in $N(A^T)$

$\&$ $\quad y \in N(A^T)$

$$y^T b = y^T p + y^T e$$
$$= 0 + y^T e$$

$$\Rightarrow \quad y^T b = y^T e \qquad (e \neq 0)$$

without loss of generality we can make $y^T b = y^T e = 1$

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 9 \end{bmatrix}$$

$$y_1 + 2y_2 + 3y_3 = 0$$

$$2y_1 + 2y_2 + 4y_3 = 0$$

$$2y_1 + 3y_2 + 5y_3 = 0$$

$$5y_1 + 5y_2 + 9y_3 = 1$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 2 & 3 & 5 \\ 5 & 5 & 9 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 2 & 4 & 0 \\ 2 & 3 & 5 & 0 \\ 5 & 5 & 9 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -5 & -6 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & -6 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$y_1 + y_3 = 0$$

$$y_2 + y_3 = 0$$

$$y_3 = 1$$

$\Rightarrow$ $(-1, -1, 1)$ solution.

8. How do we know that $Ax_r = Ax$?

How do we know that this vector is in column space?

if $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

what is x_r?

soln

$$Ax = Ax_r \quad \text{when}$$

$$x = x_r + x_n \quad \& \quad x_n \in N(A)$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \qquad x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$x_r = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad x_n = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

(9) if $A^TAx=0$ then $Ax=0$. Reason
Ax is in the Nullspace of A^T and also
in the ______________ of A and
those spaces are ___________.
Conclusion: A^TA has the same Nullspace
of A . This key fact is repeated
in the Next section.

Soln $A^TAx=0 \Rightarrow A^T(Ax)=0$

$Ax \in N(A^T)$

but $Ax \in C(A)$

$\Rightarrow Ax=0$ (common vector)

intersection.

(10) $A^T=A$ (symmetric)

(a) Why its column space is $\perp$
Nullspace?

$A = A^T$

$\Rightarrow \quad Ax = A^T x$

$\Rightarrow \quad Ax \in C(A) \quad A^T x \in C(A^T)$

$\Rightarrow \quad C(A) = C(A^T)$

$\Rightarrow \quad C(A^T) \perp N(A)$

$\Rightarrow \quad C(A) \perp N(A)$

(b) if $Ax = 0$ and $Az = 5z$, which subsets contain these "eigenvectors" x and z? Symmetric matrices has $\perp$ eigenvector's $x^T z = 0$

Sol^n

$Ax = 0 \Rightarrow x \in N(A)$

$Az = 5z \Rightarrow z \in C(A)$

$z \in C(A) \Rightarrow z \in C(A^T)$

$\Rightarrow \quad x^T z = 0$

13

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

$$W = \begin{bmatrix} w_1 & w_2 & \cdots & w_m \end{bmatrix}$$

Sol^n

$V \perp W$ (orthogonal subspace)

$$V^T W = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} \begin{bmatrix} w_1 & w_2 & \cdots & w_m \end{bmatrix}$$

$$= \begin{bmatrix} v_1^T w_1 & v_1^T w_2 & v_1^T w_3 \cdots & v_1^T w_m \\ v_2^T w_1 & v_2^T w_2 & \cdots & v_2^T w_m \\ \vdots & & & \\ v_n^T w_1 & v_n^T w_2 & \cdots & v_n^T w_m \end{bmatrix}$$

all these vectors are $\perp$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & \cdots & & 0 \end{bmatrix} = 0 \text{ matrix}$$

14

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 4 \\ 6 & 3 \\ 5 & 1 \end{bmatrix}$$

$$Ax = B\hat{x}$$

$$\Rightarrow \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ -\hat{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 5 & 4 \\ 1 & 3 & 6 & 3 \\ 1 & 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 5 & 4 \\ 1 & 3 & 6 & 3 \\ 1 & 2 & 5 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \boxed{1} & 2 & 5 & 4 \\ 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix} \Rightarrow \begin{bmatrix} \boxed{1} & 0 & 3 & 6 \\ 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \boxed{1} & 0 & 3 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

$$x_N = \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$x = \begin{pmatrix} -3 \\ -1 \end{pmatrix} \qquad \hat{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

15 $\quad p + q > n$

16 Prove that every y in $N(A^T) \perp$ to every Ax in the $C(A)$.

Sol^n

$$y \in N(A^T) \Rightarrow A^T y = 0$$

vector $\quad b \in C(A) \Rightarrow b = Ax$

$$x^T A^T y = 0$$

$$\Rightarrow (Ax)^T y = 0$$

$$\Rightarrow Ax \perp y$$

$$C(A) \perp N(A^T)$$

(17) S subspace in $\mathbb{R}^3$ containing only the zero vector. what is $S^\perp$?

$\Rightarrow \quad S^\perp = \mathbb{R}^3$

if S is spanned by $(1,1,1)$, what is $S^\perp$?

$$(0, 1, -1), \quad (1, 0, -1)$$

(19)

$$L^\perp \perp L$$

$$(L^\perp)^\perp \perp L^\perp$$

$$(L^\perp)^\perp = L$$

(20) if $V = \mathbb{R}^4$ then $V^\perp = \{0\}$

$(V^\perp)^\perp = \mathbb{R}^4 \qquad (V^\perp)^\perp = V$