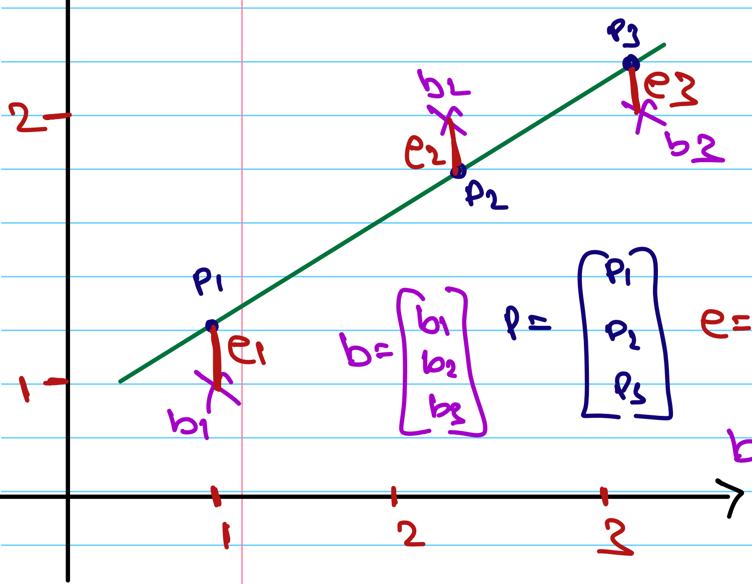
Lec 16: Projection Matrices & least squares

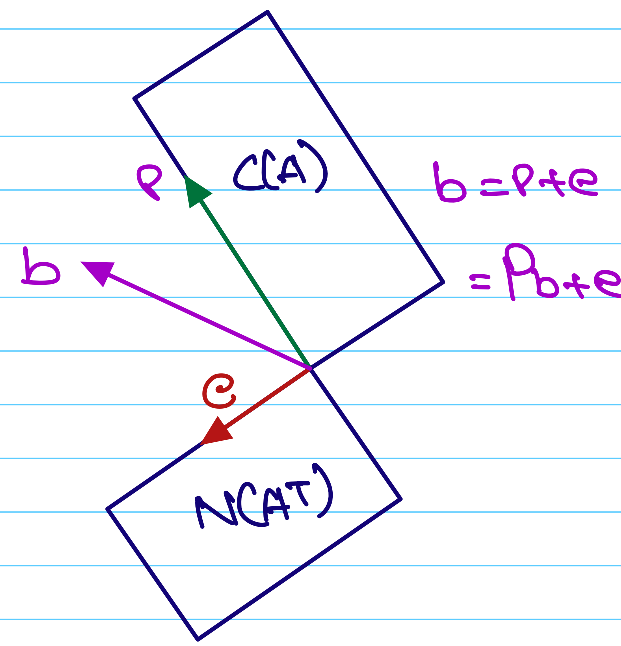
if b in column space $Pb = b$

if $b \perp$ column space $Pb = 0$

b has no component in the column space, its completely 90° to column space.

Least squares: Fitting by a Line





Roco Picture

Fit data points $(1,1)$, $(2,2)$ $(3,2)$

$b = C + Dt$

$$\Rightarrow \left.\begin{aligned} 1 &= C + D \\ 2 &= C + 2D \\ 2 &= C + 3D \end{aligned}\right\} \Rightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C \\ D \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}_{b}$$

$b = p + e$

$\Rightarrow b = Pb + e$

where P is a projection onto $C(A)$

e is also a projection onto $N(A^T)$

$\Rightarrow$ Projection matrix onto $N(A^T)$ is $I - P$

because $e = b - p$

$\Rightarrow e = b - Pb$

$\Rightarrow e = \underbrace{(I - P)}_{\text{Projection matrix onto } N(A^T)} b$

$\Rightarrow$ if P is projection, $I - P$ is projection

$\Rightarrow$ if P is symmetric $\Rightarrow I - P$ is symmetric

$\Rightarrow$ if $P^2 = P \Rightarrow (I - P)^2 = (I - P)$

minimize $\|Ax-b\|_2^2 = \|e\|^2$

Find $\hat{x} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$, P

$$A^T A \hat{x} = A^T b$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

$$\left.\begin{array}{l} 3C + 6D = 5 \\ 6C + 14D = 11 \end{array}\right\} \Rightarrow \text{Normal eq}^n$$

we can find these Normal eq^n from Calculus.

$$\|Ax-b\|_2^2 = (C+D-1)^2 + (C+2D-2)^2 + (C+3D-2)^2$$

$\min \|Ax-b\|_2^2$

$\Rightarrow \frac{\partial}{\partial C} \|Ax-b\|_2^2 = 0$

$\Rightarrow 3C + 6D = 5$

similliarly $\frac{\partial}{\partial D} \|Ax-b\|_2^2 = 0$

$\Rightarrow 6C + 14D = 11$

$C = \frac{2}{3}$

$D = 1/2$

Best line: $y = \frac{2}{3} + \frac{1}{2}t$

if A has independent column's then

A^TA is invertible.

Suppose $A^TAx = 0$ we need to prove

Trick: $x^TA^TAx = 0$ x must be 0

$\Rightarrow (Ax)^TAx = 0$

$\Rightarrow \|Ax\|_2^2 = 0 \Rightarrow Ax = 0$

But A has independent coloumn's.

$\Rightarrow x = 0$

Coloumn's are definitely independent if they are Perpendicular unit vectors

like $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\perp$ & independent.

Perpendicular unit vector's

$\Downarrow$

Orthonormal.