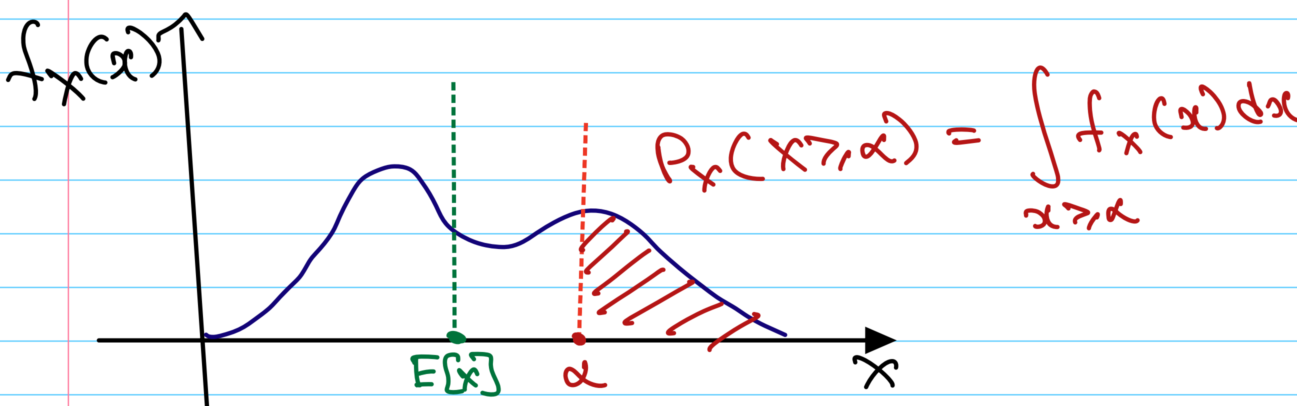
MARKOV Inequality:

$$\mathbb{P}_X(X \geq \alpha) \leq \frac{\mathbb{E}[X]}{\alpha}$$

* The Markov inequality is useful for bounding the probability of exceeding threshold values.
* It is defined only on non-negative random variables.

The purpose of markov inequality is to bound the prob of r.v exceed's some sort of threshold.



(1) define a new r.v $Z(\omega)$ which is discrete

$$Z(\omega) = \begin{cases} \alpha & ; X(\omega) \geq \alpha \\ 0 & ; \text{o.w} \end{cases}$$

(2)

$$\mathbb{E}[Z(\omega)] \leq \mathbb{E}[X(\omega)]$$

Order Preservation

$\left(\text{as } Z(\omega) \leq X(\omega) \; \forall \omega \in \Omega \right)$

(3) $\mathbb{E}[Z(\omega)]$

$$= \alpha P_X(X \geq \alpha) + 0 \, \mathbb{P}(X < \alpha)$$

(4)

$$\alpha P_X(X \geq \alpha) \leq \mathbb{E}[X]$$

$$\Rightarrow \quad P_X(X \geq \alpha) \leq \frac{\mathbb{E}[X]}{\alpha}$$

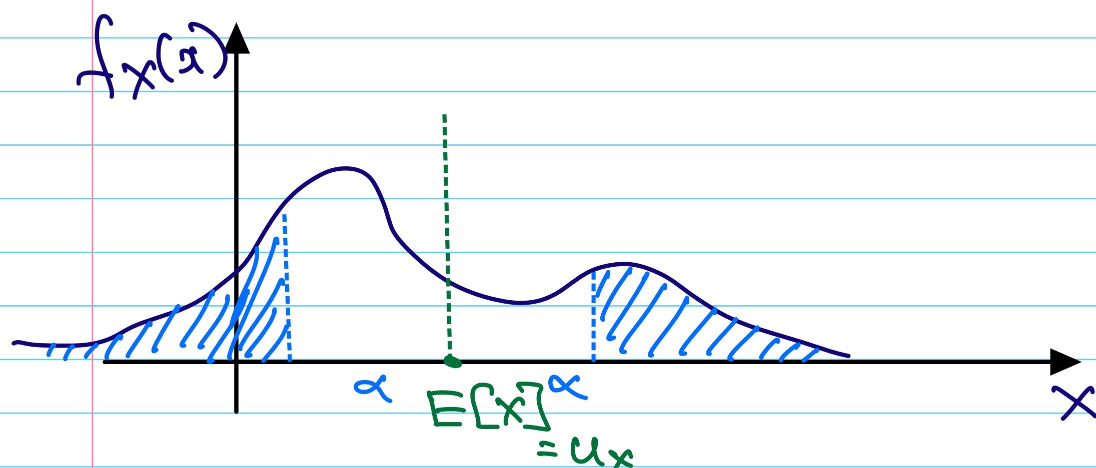
This is meaningfull only when $\alpha > E[X]$

$\Rightarrow$ if $E[X] > \alpha$ then $\frac{E[X]}{\alpha} > 1$ then

$\mathbb{P}(X > \alpha) \leq 1$ (which is always true)

Chebyshev Inequality

⇒ The Chebyshev inequality bounds the Probability of a random variable deviating from its mean μ_x



$$\mathbb{P}_X\left(|X-\mu_x| \geq \alpha \right) \leq \frac{\mathrm{var}(x)}{\alpha^2}$$

① $Y = \left(\frac{|X-\mu_x|}{\alpha}\right)^2$: Positive r.v

② use Markov inequality.

$$\mathbb{P}_Y(Y \geq 1) \leq \mathbb{E}[Y]$$

③ $\mathbb{P}\left(\left(\frac{|X-\mu_x|}{\alpha}\right)^2 \geq 1 \right) \leq \mathbb{E}\left[\left(\frac{X-\mu_x}{\alpha}\right)^2 \right]$

$$\Rightarrow \mathbb{P}\left((|X-\mu_x|)^2 \geq \alpha^2 \right) \leq \frac{\mathbb{E}\left[(X-\mu_x)^2\right]}{\alpha^2}$$

$$\Rightarrow \mathbb{P}\left(|X-\mu_x| \geq \alpha \right) \leq \frac{\text{var}(X)}{\alpha^2}$$

- if c is a big number. The Prob of being more than c away from the mean is going to be smaller number $\left(\frac{\sigma^2}{c^2}\right)$
- if the variance is small, The distribution is not very wide. and that when σ^2 is small, The Prob of being far away is small.

JENSON'S Inequality

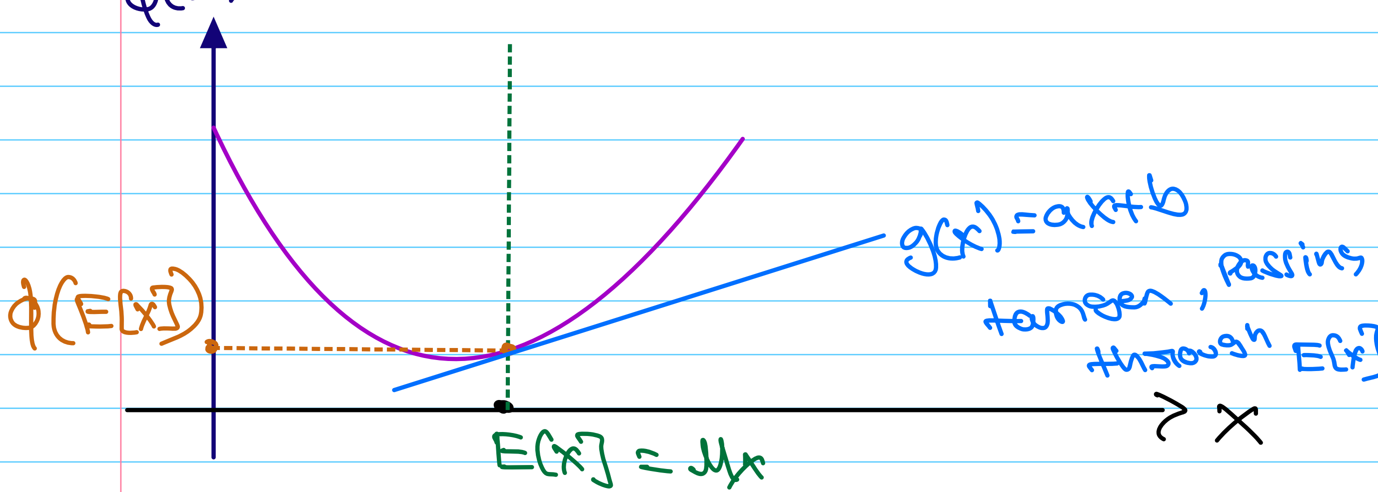
Jenson's inequality allows us to bound a function of the expectation by the expectation of a function

$$\phi(E[x]) \le \mathbb{E}[\phi(x)]$$

$\phi(\cdot)$ convex function

$$\lambda\,\phi(x) + (1-\lambda)\,\phi(y) \geq \phi(\lambda x + (1-\lambda) y)$$

$\phi(x)$: convex where $\lambda \in [0,1]$



① $g(x) = ax+b$ $\quad x^* = E[x]$

tangent to $\phi(x)$ @ x^*

(2) Since $\phi(x)$ is convex $\Rightarrow$ $g(x) \le \phi(x)$

$$\mathbb{E}[g(x)] \le \mathbb{E}[\phi(x)]$$

$$\mathbb{E}[ax+b] \le \mathbb{E}[\phi(x)]$$

$$a\mathbb{E}[x]+b \le \mathbb{E}[\phi(x)]$$

(3) Since $g(x)$ is Linear

$$\mathbb{E}[g(x)] = g(E[x])$$

$$\Rightarrow \quad g(E[x]) = a\mathbb{E}[x]+b$$

Since $g(x)$ is a tangent to $\phi(x)$ at $E[x]$

$$g(E[x]) = \phi(E[x])$$

$$\Rightarrow g(E[x]) = \mathbb{E}[g(x)] = \phi(E[x])$$

(4) and we know

$$\mathbb{E}[g(x)] \le \mathbb{E}[\phi(x)]$$

$$\boxed{\Rightarrow \phi(E[x]) \le \mathbb{E}[\phi(x)]}$$