DESCRIPTION

This document provides solution's to 3.1 to 3.15 from chapter 3 of statistical inference Book by Casella and Berger, Focusing on discrete Probability distribution

4, 5, 6, 7, 8

$$\frac{1}{8-4+1}$$

3.1

$X \sim \text{Unif}(N_0, N_1)$

where $X = N_0, N_1, \ldots N_1$

$N_0 \leq N_1$ & integer's

$\mathbb{E}[X]$, $\text{var}(X)$?

$$\mathbb{P}(X=x) = \begin{cases} \dfrac{1}{N_1 - N_0 + 1} & \text{if } x \in \{N_0, N_1 \ldots N_1\} \\ 0 & \text{ow} \end{cases}$$

$$\mathbb{E}[X] = \sum_{x=N_0}^{N_1} x \cdot \mathbb{P}(X=x)$$

$$= \frac{1}{N_1 - N_0 + 1} \cdot \sum_{x=N_0}^{N_1} x$$

$$= \frac{1}{N_1 - N_0 + 1} \cdot \left(\sum_{x=0}^{N_1} - \sum_{x=0}^{N_0 - 1} x \right)$$

$$= \frac{1}{N_1 - N_0 + 1} \cdot \left(\frac{N_1(N_1+1)}{2} - \frac{(N_0-1)N_0}{2} \right)$$

$$= \frac{1}{N_1 - N_0 + 1} \left(\frac{N_1^2 + N_1 - N_0^2 + N_0}{2} \right)$$

$$= \frac{1}{N_1 - \cancel{N_0 + 1}} \; \frac{(N_1 - \cancel{N_0 + 1})(N_1 + N_0)}{2}$$

$$\boxed{\mathbb{E}[X] = \frac{N_1 + N_0}{2}}$$

$$\mathbb{E}[X^2] = \sum_{x = N_0}^{N_1} x^2 \, \mathbb{P}(X = x)$$

$$= \frac{1}{N_1 - N_0 + 1} \left[\sum_{x=0}^{N_1} x^2 - \sum_{x=0}^{N_0 - 1} \right]$$

$$= \frac{1}{N_1 - N_0 + 1} \left[\frac{N_1 (N_1 + 1)(2N_1 + 1)}{6} - \frac{(N_0 - 1)(N_0)(2N_0 - 1)}{6} \right)$$

$$\mathrm{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X^2]$$

= substituting these we get

$$\boxed{\mathrm{var}(X) = \frac{(N_1 - N_0)(N_1 - N_0 + 2)}{12}}$$

3.2 A manufacturer recieves a lot of 100 Parts from a vendor. The lot will be unacceptable if more than 5 of the Parts are defective. The manufacturer is going to select randomly K Parts from the lot for inspection and the lot will be accepted if no defective Parts are found in the sample.

a) How Large does K have to be to ensure that the Prob that the manufacturer accepts an unacceptable lot is less than 0.10?

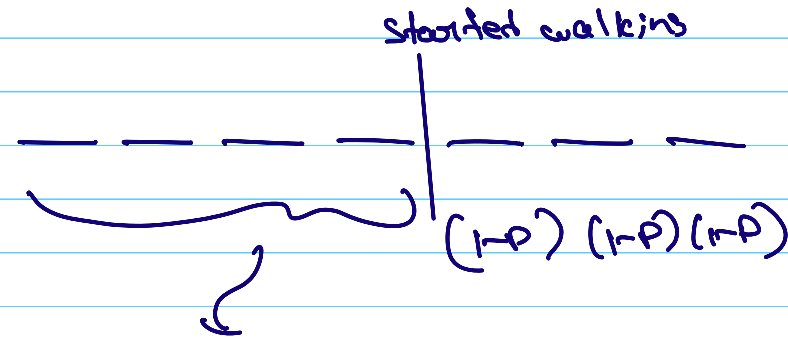
Solution :

$n =$ number of defective Parts

$\Rightarrow$ for a lot to be unacceptable $n \geq 6$

3.3 Prob of a car passing during any given second is a const p and there is no interaction b/w the passing of car's at different seconds.

Solution:



only way that he will wait for 1st 4 seconds is (atleast one car pass in 1st 3 seconds i.e $(1-p^3)$ and a car pass in 4th secon's.

$$\Rightarrow \quad (1-p^3)\, p\, (1-p)^3$$

3.4 A man with n key's wants to open his door and tries the key at random. Exactly one key will open the door. Find the mean Number of trails if.

ⓐ unsuccessfull key's are not eliminated from further secection's.

<u>Solution:</u>

$$\mathbb{P}(\text{selecting right key out of } n) = \frac{1}{n} = p$$

Occurance of 1st success without eleminating the unsuccesful key's

$X =$ # of trail's

$(x-1)$ uncceesful, x^{th} trail succes

$X \sim \text{geometric}(p$

$$\mathbb{P}(X=x) = (1-p)^{x-1} \cdot p$$

$$= \left(\frac{99}{100}\right)^{x-1} \cdot p$$

$$\mathbb{E}[X] = \frac{1}{p} = 100$$

(b) unsuccessful keys are eliminated

X = number trails

$= (x-1)$ unsuccessful, x^{th} success

$$\mathbb{P}(X=1) = \frac{1}{100} = \frac{1}{100}$$

$$\mathbb{P}(X=2) = \frac{99}{100} \cdot \frac{1}{99} = \frac{1}{100}$$

$$\mathbb{P}(X=3) = \frac{99}{100} \cdot \frac{98}{99} \cdot \frac{1}{98} = \frac{1}{100}$$

$$\mathbb{P}(X=4) = \frac{99}{100} \cdot \frac{98}{99} \cdot \frac{97}{98} \cdot \frac{1}{97} = \frac{1}{100}$$

$$\Rightarrow \quad \mathbb{P}(X=x) = \frac{1}{100} \quad x \in \{1, 2, \dots 100\}$$

$$\boxed{\mathbb{E}[X] = \frac{N+1}{2} = \frac{101}{2} = 50.5}$$

3.5 A standard drug is known to be effective in 80% of the cases in which it is used. A new drug is tested on 100 patients and found to be effective in 85 cases. Is the new drug superior?

Solution:

$\mathbb{P}$ (success when testes on a patient) $= 0.8$

$X \sim$ Binomial $(100, 0.8)$

where x is number of success when tested on 100 patients

then $\mathbb{P}(X \geq 85) =$

$$\sum_{x=85}^{100} \binom{100}{x} (0.8)^{x} (0.2)^{100-x}$$

$\mathbb{P}(X \geq 85) = 0.1285$

this Probability is for old drug.

* the Probability that the old drug works on more than 85 = 0.1285

=> which mean's there is a significant (0.1285 is not small) time's old works on 85 patients

=> if the new drug work on 85 Patients out of 100, we cannot conclude that it is good

3.6

A Large number of insects are expected to be attracted to a certain variety of Plants. A commercial insecticide is advertised as being 99% effective. Suppose 2000 insects infest a rose garden where the insecticide has been applied, and X = number of surviving insects.

(a) What Prob dist might provide a reasonable model for this experiment?

$$X \sim \text{Binomial}\left(2000, \frac{1}{100}\right)$$

(b) Prob that fewer than 100 insects survive

$$\mathbb{P}(X \le 100) = \sum_{x=0}^{100} \binom{2000}{x} \left(\frac{1}{100}\right)^{x} \left(\frac{99}{100}\right)^{200-x}$$

(c) evaluate an approximation to the probability in part (b)

Solution:

Out of 2000, Expected number of surviving $= \mathbb{E}[X] = np = 20$

we can approximate this with Poisson

with $\lambda = 20$

$$\mathbb{P}(X = x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

$$\Rightarrow \quad \mathbb{P}(X \leq 100) = \sum_{x=0}^{100} \frac{e^{-20} \cdot 20^{x}}{x!}$$

$$= 0.99$$

3.7 Let the number of chocolate chips in a certain type of cookie have a Poisson distribution. We want the probability that a randomly choosen cookie has at least two chocolate chips to be greater than 0.99. Find the smallest value of the mean of the distribution that ensures this probability.

Solution:

$\mathbb{P}($randomly choosen cookie has at least two choclate chips$) > 0.99$

$X \sim$ Number of choclate chips on cookies

$$\mathbb{P}(X=x) = \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

$$\mathbb{P}(X \geq 2) \geq 0.99$$

$\Rightarrow$ Probability that atleast 2 choclate chips on a cookie

$$\Rightarrow \quad 1 - \mathbb{P}(X=0) - \mathbb{P}(X=1) \geq 0.99$$

$$\Rightarrow \quad 1 - e^{-\lambda} - e^{-\lambda} \cdot \lambda \geq 0.99$$

$$\Rightarrow \quad 0.01 \geq e^{-\lambda} \cdot (1+\lambda)$$

Using trail and error $\boxed{\lambda = 6.63835}$

3.8 (a) Probability of turning away a customer (because of a full house) is less than 1%

Solution:

Probability of a person coming to choosing a theater $= \frac{1}{2}$

$X \sim \text{Binomial}(1000, \frac{1}{2})$

$$\mathbb{P}(X > N) = \sum_{x=N+1}^{1000} \binom{1000}{x} \left(\frac{1}{2}\right)^{x} \left(\frac{1}{2}\right)^{1000-x} < 0.01$$

$$\Rightarrow \sum_{x=N+1}^{1000} \binom{1000}{x} \left(\frac{1}{2}\right)^{1000} < 0.01$$

Solve this inequality. ($N = 537$ around)

(b)

$X \sim \text{Binomial}(1000, \frac{1}{2})$

$\mathbb{E}[X] = np = 500$ $\quad$ $\text{Var}(X) = np(1-p)$
$= 250$

$Y \sim N(500, 250)$

$$Y \sim N(500, \sqrt{250})$$

$$\mathbb{P}(Y > N) < 0.01$$

$$\Rightarrow \quad \mathbb{P}(\sqrt{250}\, Z + 500 > N) < 0.01$$

$$\Rightarrow \quad P\left(Z > \frac{N-500}{\sqrt{250}} \right) < 0.01$$

from Normal table

$$\mathbb{P}(Z > 2.33) \approx 0.099 < 0.01$$

$$\Rightarrow \quad \frac{N-500}{\sqrt{250}} = 2.33$$

$$\Rightarrow \quad \boxed{N = 537}$$

3.9

(a) $X \sim \text{Binomial}\left(60, \frac{1}{90}\right)$

$$\mathbb{P}(X \geq 5) = \sum_{x=5}^{60} \binom{60}{x} \left(\frac{1}{90}\right)^{x} \left(\frac{89}{90}\right)^{60-x}$$

$$= 0.006$$

(b) 62 Counties, and 5 schools in each county

$\Rightarrow 62 \times 5 = 310$

$X \sim$ Number of school's with 5 or more twin's

$X \sim \text{Binomial}(310, 0.0006)$

then $\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X=0)$

$= 1 - \binom{310}{0}\left(\frac{89}{90}\right)^{310}$

3.12

$X \sim$ binomial (n, p)

$Y \sim$ negative binomial (r, p)

Show that $F_X(r-1) = 1 - F_Y(n-r)$

solution:

$$F_X(r-1) = \sum_{x=0}^{r-1} \binom{n}{x} p^x (1-p)^{n-x}$$

$$F_Y(n-r) = \sum_{y=0}^{n-r} \binom{r+y-1}{y} p^r (1-p)^y$$

$$1 - F_X(r-1) = \sum_{x=r}^{n} \binom{n}{x} p^x (1-p)^{n-x}$$

(3.13) A truncated discrete distribution is one in which a particular class cannot be observed and is eliminated from the sample space. In particular, if X has range $0, 1, 2, \ldots$ and the 0 class cannot be observed (as is usually the case), the 0-truncated r.v X_T has pmf

$$\mathbb{P}(X_T = x) = \frac{P(X = x)}{P(X > 0)}, \quad x = 1, 2, \ldots$$

Find the PMF, mean, and variance of the 0-truncated r.v.

$$\mathbb{P}(X_T = x) = \mathbb{P}(X = x \mid X > 0)$$

$$= \frac{\mathbb{P}(X = x \cap X > 0)}{\mathbb{P}(X > 0)}$$

The event $X = x$ and $X > 0$ will have the intersection with $X = x$

i.e $\quad \mathbb{P}(X=x \cap X>0) = \mathbb{P}(X=x)$

therefore

$$\mathbb{P}(X_T=x) = \mathbb{P}(X=x \mid X>0)$$

$$= \frac{\mathbb{P}(X=x \cap X>0)}{\mathbb{P}(X>0)}$$

$$= \frac{\mathbb{P}(X=x)}{\mathbb{P}(X>0)}$$

(a) $X \sim \text{Poisson}(\lambda)$

$$\mathbb{P}(X=x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!} \qquad x=0,1,2,\ldots$$

$$\mathbb{P}(X_T=x) = \frac{\mathbb{P}(X=x)}{\mathbb{P}(X>0)} \qquad x=1,2,\ldots$$

$(\mathbb{P}(X>0) = 1-\mathbb{P}(X=0))$

$$= \frac{\frac{e^{-\lambda} \cdot \lambda^x}{x!}}{1-e^{-\lambda}}$$

PMF

$$\boxed{\mathbb{P}(X_T = x) = \frac{e^{-\lambda}}{1-e^{-\lambda}} \cdot \frac{\lambda^x}{x!} \quad x = 1, 2, \ldots}$$

$$\mathbb{E}[X_T] = \sum_{x=1}^{\infty} x \cdot \mathbb{P}(X_T = x)$$

$$= \sum_{x=1}^{\infty} x \cdot \frac{\mathbb{P}(X=x)}{\mathbb{P}(X>0)}$$

$$= \frac{1}{\mathbb{P}(X>0)} \sum_{x=1}^{\infty} x \cdot \mathbb{P}(X=x)$$

$$= \frac{1}{\mathbb{P}(X>0)} \cdot \mathbb{E}[X] = \frac{\lambda}{1-e^{-\lambda}}$$

$$\boxed{\mathbb{E}[X] = \frac{\lambda}{1-e^{-\lambda}}}$$

$$\mathbb{E}[X_T(X_T-1)] = \sum_{x=1}^{\infty} \frac{x(x-1) \cdot \mathbb{P}(X=x)}{\mathbb{P}(X>0)}$$

$$= \frac{1}{\mathbb{P}(X>0)} \sum_{x=1}^{\infty} x(x-1)\,\mathbb{P}(X=x)$$

$$= \frac{1}{\mathbb{P}(X>0)} \sum_{x=2}^{\infty} x(x-1) \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

$$= \frac{\lambda^2}{1-e^{-\lambda}} \cdot \underbrace{\sum_{x=2}^{\infty} \frac{e^{-\lambda} \cdot \lambda^{x-2}}{(x-2)!}}_{\text{PMF} = 1}$$

$$\mathbb{E}[X_T(X_T-1)] = \frac{\lambda^2}{1-e^{-\lambda}}$$

$$\Rightarrow \operatorname{Var}(X_T) = \mathbb{E}[X_T(X_T-1)] + E[X_T] - E[X_T]^2$$

$$= \frac{\lambda^2}{1-e^{-\lambda}} + \frac{\lambda}{1-e^{-\lambda}} - \frac{\lambda^2}{(1-e^{-\lambda})^2}$$

$$= \frac{1}{(1-e^{-\lambda})^2} \left[\lambda^2(1-e^{-\lambda}) + \lambda(1-e^{-\lambda}) - \lambda^2 \right]$$

$$= \frac{1}{(1-e^{-\lambda})^2}\left[-\lambda^2 e^{-\lambda} + \lambda - \lambda e^{-\lambda}\right]$$

$$\boxed{Var(X_T) = \frac{\lambda}{(1-e^{-\lambda})^2} - \frac{e^{-\lambda}}{(1-e^{-\lambda})^2}(\lambda^2+\lambda)}$$

(b) $X \sim$ negative binomial (r, p)

$$\mathbb{P}(X=x) = \binom{r+x-1}{x} p^r (1-p)^x$$

$x = 0, 1, 2, \ldots$

<u>solution:</u>

$$\mathbb{P}(X=0) = \binom{r-1}{0} p^r$$

$$\mathbb{P}(X>0) = 1 - \mathbb{P}(X=0)$$

$$= 1 - p^r$$

$$\Rightarrow \mathbb{P}(X_T = x) = \frac{\mathbb{P}(X=x)}{\mathbb{P}(X>0)}$$

PMF

$$\boxed{\mathbb{P}(X_T = x) = \frac{1}{1-p^r} \binom{r+x-1}{x} p^r (1-p)^x \qquad x = 1, 2, \dots}$$

$$\mathbb{E}[X_T] = \frac{1}{\mathbb{P}(X>0)} \sum_{x=1}^{\infty} x \cdot \mathbb{P}(X=x)$$

$$= \frac{\mathbb{E}[X]}{\mathbb{P}(X>0)} = \frac{1}{1-p^r} \frac{r(1-p)}{p}$$

$$\Rightarrow \boxed{\mathbb{E}[X_T] = r \cdot \frac{1-p}{p(1-p^r)}}$$

$$\mathbb{E}[X_T(1-X_T)] = \sum_{x=2}^{\infty} \frac{\mathbb{P}(X=x) \cdot x(1-x)}{\mathbb{P}(X>0)}$$

$$= \frac{1}{\mathbb{P}(X>0)} \sum_{x=2}^{\infty} x(1-x)\, \mathbb{P}(X=x)$$

$$= \frac{1}{\mathbb{P}(X>0)} \cdot \mathbb{E}[X(X-1)]$$

$$\mathbb{E}[X_T(X_T-1)] = \frac{1}{\mathbb{P}(X>0)} \sum_{x=2}^{\infty} x(x-1) \binom{r+x-1}{x} p^r (1-p)^x$$

$$= \frac{1}{\mathbb{P}(X>0)} \sum_{x=2}^{\infty} x(x-1) \cdot \frac{(r+x-1)!}{x!\,(r-1)!} p^r (1-p)^x$$

$$= \frac{1}{\mathbb{P}(X>0)} \sum_{x=2}^{\infty} \frac{(r+x-1)!}{(x-2)!\,(r+1)!} p^{r+2} \cdot (1-p)^{x-2} \cdot \frac{r(r+1)(1-p)^2}{p^2}$$

$$= \frac{1}{\mathbb{P}(X>0)} \frac{r(r+1)(1-p)^2}{p^2} \cdot \underbrace{\sum_{x=2}^{\infty} \binom{r+x-1}{x-2} p^{r+2} \cdot (1-p)^{x-2}}_{\text{negative Binomial }(r+2,\,p)\ =1}$$

$$= \frac{1}{(1-p^r)} \cdot \left(\frac{1-p}{p}\right)^2 \cdot r(r+1)$$

$$\mathrm{Var}(X_T) = \mathbb{E}[X_T(X_T-1)] + \mathbb{E}[X_T] - \mathbb{E}[X_T]^2$$

$$= \frac{1}{(1-p^r)} \left(\frac{1-p}{p}\right)^2 \cdot r(r+1) + \frac{1}{(1-p^r)} \cdot \left(\frac{1-p}{p}\right) r$$

$$- \frac{1}{(1-pr)^2}\left(\frac{1-p}{p}\right)^2 r^2$$

$$\boxed{Var(X_T) = \frac{1}{(1-p^r)} \cdot \left(\frac{1-p}{p}\right) \cdot r \left[\frac{1-p}{p} \cdot (r+1) - 1\right] - \frac{1}{(1-pr)^2}\left(\frac{1-p}{p}\right)^2 r^2}$$

3.14 Starting from the 0-truncated negative binomial (refer to Exercise 3.13), if we let $r \to 0$, we get an interesting distribution, the logarithmic series distribution. A r.v X has a logarithmic Series Distribution with parameter's p if

$$\mathbb{P}(X=x) = \frac{-(1-p)^x}{x \log p}, \quad x = 1, 2, \ldots$$

$$0 < p < 1$$

(a) Verify that this defines a legitimate probability function.

$$\mathbb{P}(X=x) = \frac{-(1-p)^x}{x \log p}, \quad x = 1, 2, \ldots$$

$$0 < p < 1$$

$$\sum_{x=1}^{\infty} \mathbb{P}(X=x) = \sum_{x=1}^{\infty} \frac{-(1-p)^x}{x \log p}$$

$$= \frac{-1}{\log p} \sum_{x=1}^{\infty} \frac{(1-p)^x}{x}$$

taylor series expansion of $\log x$

$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^2 + \cdots$$

here $\ln(p)$ expansion where $p < 1$

$$\Rightarrow \ln(p) = (p-1) - \frac{1}{2}(p-1)^2 + \frac{1}{3}(p-1)^3 - \frac{1}{4}(p-1)^2 + \cdots$$

$$= -(1-p) - \frac{1}{2}(1-p)^2 - \frac{1}{3}(1-p)^3$$

$$- \frac{1}{4}(1-p)^2 - \frac{1}{5}(1-p)^5 \cdots$$

$$= - \sum_{x=1}^{\infty} \frac{(1-p)^x}{x}$$

$$\Rightarrow \sum_{x=1}^{\infty} \mathbb{P}(X=x) = \frac{-1}{\log p} \sum_{x=1}^{\infty} \frac{(1-p)^x}{x}$$

$$= \frac{-1}{\log p} \cdot -\log p$$

$$= 1$$

$$\boxed{\Rightarrow \sum_{x=1}^{\infty} \mathbb{P}(X=x) = 1}$$

and

$$\mathbb{P}(X=x) \geq 0 \quad \forall \; x = 1, 2, \cdots \infty$$

There fore this is legitimate PMF

(b) Find the mean and variance of X

$$\mathbb{E}[X] = \sum_{x=1}^{\infty} \frac{-(1-p)^x}{x \log p} \cdot x$$

$$= \frac{-1}{\log p} \sum_{x=1}^{\infty} (1-p)^x$$

$$= \frac{-1}{\log p} \left(\sum_{x=0}^{\infty} (1-p)^x - 1 \right)$$

$$= \frac{-1}{\log p} \left(\frac{1}{1-(1-p)} - 1 \right) = \frac{-1}{\log p} \left(\frac{1-p}{p} \right)$$

$$\boxed{\mathbb{E}[X] = \frac{-1}{\log p} \left(\frac{1-p}{p} \right)}$$

$$\mathbb{E}[X^2] = \frac{-1}{\log p} \sum_{x=1}^{\infty} \frac{x^2 \cdot (1-p)^x}{x}$$

$$= \frac{-1}{\log p} \sum_{x=1}^{\infty} x (1-p)^x$$

identity:

$$\sum_{k=0}^{\infty} a r^k = \frac{a}{1-r} \quad \text{if } |r| < 1$$

Now differentiate Both side's with r

$$\frac{d}{dr} \sum_{k=0}^{\infty} a r^k = \frac{d}{dr} \frac{a}{1-r}$$

$$\Rightarrow \sum_{k=0}^{\infty} k \cdot a r^{k-1} = \frac{a}{(1-r)^2}$$

for $k=0$ the summatio $=0$

$$\Rightarrow \sum_{k=1}^{\infty} a \cdot k \cdot r^{k-1} = \frac{a}{(1-r)^2}$$

$$\mathbb{E}[x^2] = \frac{-1}{\log P} \sum_{x=1}^{\infty} x (1-P)^x = \frac{-(1-P)}{\log P} \left(\sum_{x=1}^{\infty} x (1-P)^{x-1} \right)$$

$$= \frac{-(1-P)}{\log P} \cdot \frac{1}{P^2}$$

$$\operatorname{var}(X) = \mathbb{E}[x^2] - E[X]^2$$

$$= \frac{-(1-P)}{\log P} \cdot \frac{1}{P^2} - \frac{1}{(\log P)^2} \frac{(1-P)^2}{P^2}$$

$$= \frac{-(1-P)}{\log P \cdot P} = \left(\frac{1}{P} + \frac{1-P}{\log P} \right)$$

$$\boxed{Var(X) = -\frac{(1-p)}{p} \cdot \frac{1}{\log p}\left(\frac{1}{p} + \frac{1-p}{\log p}\right)}$$

3.15 In section 3.2 it was claimed that the Poisson(λ) distribution is the limit of the negative binomial (r, p) distribution as $r \to \infty$, $p \to 1$, and $r(1-p) \to \lambda$. Show that under these condition's the mgf of the negative binomial converges to that of the Poisson.

<u>Solution:</u>

$$X \sim \text{Poisson}(\lambda), \quad M_X(t) = e^{\lambda(e^t - 1)}$$

$X \sim$ negative binomial (r, p)

$$\mathbb{P}(X = x) = \binom{r+x-1}{x} p^r (1-p)^r$$

$x = 0, 1, 2, \ldots$

$$M_x(t) = \mathbb{E}\left[e^{tx}\right]$$

$$= \sum_{x=0}^{\infty} e^{tx} \binom{r+x-1}{x} p^r (1-p)^x$$

$$= \sum_{x=0}^{\infty} \binom{r+x-1}{x} p^r \left(e^t(1-p)\right)^x$$

$$= \sum_{x=0}^{\infty} \binom{r+x-1}{x} p^r \frac{\left(1-\left(e^t(1-p)\right)\right)^r \left[e^t(1-p)\right]^x}{\left[1-e^t(1-p)\right]^r}$$

$$= \frac{p^r}{\left[1-e^t(1-p)\right]^r} \underbrace{\sum_{x=0}^{\infty} \binom{r+x-1}{x} \left(1-e^t(1-p)\right)^r \left(e^t(1-p)\right)^x}_{\text{negative binomial}\left(r,\ e^t(1-p)\right) = 1}$$

$$M_x(t) = \frac{p^r}{\left[1-e^t(1-p)\right]^r}$$

$$\lim_{\substack{r \to \infty \\ p \to 1 \\ r(1-p) \to \lambda}} M_X(t) = \lim_{\substack{r \to \infty \\ p \to 1 \\ r(1-p) \to \lambda}} \frac{p^r}{\{1 - e^t(1-p)\}^r}$$