This document contain's solution's to Exercises 5.1 to 5.20 from chapter 5, of Statistical inference by Casella and Berger, focusing on Random Samples, Sum's of Random variables from a Random sample, Sampling from $N(\mu, \sigma^2)$

Some Problem's are still in Progress, and the document will be updated with their solution's soon.

5.1 Color Blindness appears in 1% of the people in a certain population. How large must a sample be if the probability of its containing a color-blind person is to be 0.95 or more?

Soln

Sample n people from population

then # color blind people follow Binomial

$$X \sim \text{Binomial}\left(n, \frac{1}{100}\right)$$

we want $\mathbb{P}(X>0) = 1 - \mathbb{P}(X=0)$

$$= 1 - \left(\frac{1}{100}\right)^0 \left(\frac{99}{100}\right)^n$$

$$\Rightarrow \mathbb{P}(X>0) = 1 - \left(\frac{99}{100}\right)^n > 0.95$$

$$\Rightarrow 0.05 > \left(\frac{99}{100}\right)^n$$

$$\Rightarrow \log(0.05) > n \cdot \log\left(\frac{99}{100}\right)$$

$$\Rightarrow \boxed{n \approx 299}$$

(5.2) Suppose $X_1, X_2, \ldots$ are jointly continuous and independent, each distributed with marginal $f(x)$, where each X_i represents annual rainfall at a given location.

(a) Find the distribution of the number of year's until the first year's rainfall, X_1, is exceeded for the first time.

Sol^n

(5.3) Let $X_1, X_2, \ldots X_n$ be iid random variables with continuous cdf F_X, and suppose $\mathbb{E}[X_i] = \mu$. Define a r.v's $Y_1, Y_2 \ldots Y_n$ by

$$Y_i = \begin{cases} 1 & \text{if } X_i > \mu \\ 0 & \text{if } X_i \leq \mu \end{cases}$$

Find the distribution of $\sum_{i=1}^{n} Y_i$

Sol^n

$$X_i \sim F_{X_i}(x_i)$$

Y_i is a Bernoulli distribution

$$\mathbb{P}(Y_i = 0) = \mathbb{P}(X_i \leq \mu) = F_X(\mu)$$

$$\mathbb{P}(Y_i = 1) = \mathbb{P}(X_i > \mu) = 1 - F_X(\mu)$$

$$\Rightarrow \quad Y_i \sim \text{Bernoulli}(F_X(\mu))$$

distribution of $\sum_{i=1}^{n} Y_i =$

$$\mathbb{P}\left(\sum_{i=1}^{n} Y_i = y\right) = \binom{n}{y} \left(1 - F_X(\mu)\right)^{y} F_X(\mu)^{n-y}$$

$$\sum_{i=1}^{n} Y_i \sim \text{Binomial}\left(n, 1 - F_X(\mu)\right)$$

5.4 The random variables $X_1, X_2, \dots X_n$ are exchangeable if any Permutation of any subset of them of size k ($k \le n$) has the same distribution. in this exercise we will see an example of r.v's that are exchangeble but not iid.

Let $X_i | P \overset{iid}{\sim}$ Bernoulli(P), $i = 1, \dots, n$ and let $P \sim$ uniform $(0,1)$

(a) Show that the marginal distribution of any k of the X's is the same as

$$\mathbb{P}(X_1 = x_1, X_2 = x_2, \dots, X_k = x_k) = \int_0^1 p^t (1-p)^{k-t} \, dp = \frac{t!\,(k-t)!}{(k+1)!}$$

where $t = \sum_{i=1}^{k} x_i$. Hence, the X's are Exchangeble.

Sol^n

$$X_i | P \sim \text{Bernoulli}(p)$$

$X_i|P$, $X_j|P$ are independent $\forall i \neq j$

$$\mathbb{P}(X_i) = \int_0^1 \mathbb{P}(X_i | P = p)\, \mathbb{P}(P = p)\, dp$$

$$\Rightarrow \mathbb{P}(X_i) = \int_0^1 p^{x_i} (1-p)^{1-x_i} \cdot 1 \cdot dp$$

$$\Rightarrow \mathbb{P}(X_i = x_i) = \int_0^1 p^{x_i} (1-p)^{1-x_i}\, dp$$

But $X_1, X_2, \ldots X_k$ are independent

$$\Rightarrow \mathbb{P}(X_1 = x_1, X_2 = x_2, \ldots X_k = x_k)$$

$$= \prod_{i=1}^{k} \int_0^1 p^{x_i} (1-p)^{1-x_i}\, dp$$

$$= \int_0^1 p^{x_1+x_2+\dots+x_k} (1-p)^{k-(x_1+x_2+\dots+x_k)} dp$$

$$= \underbrace{\int_0^1 p^{t} (1-p)^{k-t} dp}_{\text{Beta}(t+1,\ k-t+1)}$$

$$= \frac{\Gamma(t+1)\ \Gamma(k-t+1)}{\Gamma(t+1+k-t+1)}$$

$\Rightarrow$ t, k are integer's

$$\Rightarrow \frac{t!\ (k-t)!}{(k+1)!}$$ Hence Proved.

(b) Show that, marginally,

$$\mathbb{P}(X_1 = x_1, \dots X_n = x_n) \neq \prod_{i=1}^{n} \mathbb{P}(X_i = x_i)$$

so, that the distribution's of the X's is exchangeable but not iid.

(5.5) Let $X_1, X_2, \ldots X_n \overset{iid}{\sim} f_X(x)$ and let $\bar{X}$ denote the sample mean. Show that

$$f_{\bar{X}}(x) = n f_{X_1 + \cdots + X_n}(nx)$$

even if the mgf of X does not exists

Sol^n

$$Y = X_1 + X_2 + \cdots + X_n$$

if pdf of $Y \sim f_Y(y)$

then $\bar{X} = \dfrac{Y}{n}$

$$\Rightarrow \quad Y = n\bar{X} \qquad \frac{dY}{d\bar{X}} = n$$

$$\Rightarrow \quad f_{\bar{X}}(x) = f_Y(n\bar{x}) \cdot n$$

$$\boxed{\Rightarrow \quad f_{\bar{X}}(x) = n \cdot f_{X_1 + X_2 + \cdots + X_n}(nx)}$$

5.6 if X has pdf $f_X(x)$ and Y, independent of X, has pdf $f_Y(y)$, establish formulae, simillar to (5.2.3) for the random variable Z in each of the following situation's

5.2.3 is $\quad f_Z(z) = \int_{-\infty}^{\infty} f_X(w) f_Y(z-w)\,dw$

$Z = X + Y$

(a) $Z = X - Y$

Sol^n

$$\left.\begin{array}{l} Z = X - Y \\ W = X \end{array}\right\} \Rightarrow \left.\begin{array}{l} x = w \\ y = w - z \end{array}\right\} \Rightarrow J = \left|\begin{matrix} 0 & 1 \\ -1 & 1 \end{matrix}\right|$$

$|J| = 1$

$$\Rightarrow f_{Z,W}(z,w) = f_X(w)\, f_Y(w-z)$$

$$\Rightarrow \boxed{f_Z(z) = \int_w f_{Z,W}(z,w) = \int f_X(w)\, f_Y(w-z)\,dw}$$

(b) $Z = XY$

Sol^n

$$\left.\begin{array}{l} Z = XY \\ W = X \end{array}\right\} \quad \left.\begin{array}{l} X = W \\ Y = \frac{Z}{W} \end{array}\right\} \Rightarrow J = \begin{vmatrix} 0 & 1 \\ \frac{1}{W} & -\frac{Z}{W^2} \end{vmatrix}$$

$$|J| = \left|\frac{-1}{W}\right|$$

$$f_{Z,W}(z,w) = f_X(w)\, f_Y\left(\frac{z}{w}\right) \cdot \frac{1}{|-w|}$$

$$\boxed{f_Z(z) = \int_w f_X(w)\, f_X\left(\frac{z}{w}\right) \cdot \frac{1}{|-w|}\, dw}$$

(c) $Z = \frac{X}{Y}$

Sol^n $Z = \frac{X}{Y}$, $W = X$ $\Big\} \Rightarrow$

$$\left.\begin{array}{l} X = W \\ X = \frac{W}{Z} \end{array}\right\} J = \begin{vmatrix} 0 & 1 \\ -\frac{W}{Z^2} & \frac{1}{Z} \end{vmatrix}$$

$$|J| = \left|\frac{W}{Z^2}\right|$$

$$f_{Z,W}(z,w) = f_X(w)\, f_Y\left(\frac{w}{z}\right) \cdot \left|\frac{w}{z^2}\right|$$

$$\boxed{f_Z(z) = \int_w f_X(w)\, f_Y\left(\frac{w}{z}\right) \left|\frac{w}{z^2}\right| dw}$$

(5.7) in Example 5.2.10, a partial fraction decomposition is needed to derive the distribution of the sum of two independent Cauchy r.v's. This exercise provides the details that are skipped in that example.

(a) find the constants A, B, C, D that satisfy

$$\frac{1}{1+\left(\frac{\omega}{\sigma}\right)^2} \cdot \frac{1}{1+\left(\frac{z-\omega}{\tau}\right)^2}$$

$$= \frac{A\omega}{1+\left(\frac{\omega}{\sigma}\right)^2} + \frac{B}{1+\left(\frac{\omega}{\sigma}\right)^2} - \frac{C\omega}{1+\left(\frac{z-\omega}{\tau}\right)^2} - \frac{D}{1+\left(\frac{z-\omega}{\tau}\right)^2}$$

where A, B, C, D may depend on z but not on ω

Sol^n

$$\frac{1}{1+\left(\frac{\omega}{\sigma}\right)^2} \cdot \frac{1}{1+\left(\frac{z-\omega}{\sigma}\right)^2} =$$

$$A\omega\left(1+\left(\frac{z-\omega}{\sigma}\right)^2\right) + B\left(1+\left(\frac{z-\omega}{\sigma}\right)^2\right)$$

$$- C\omega\left(1+\left(\frac{\omega}{\sigma}\right)^2\right) - D\left(1+\left(\frac{\omega}{\sigma}\right)^2\right)$$

$$= 1$$

$$\boxed{B-D=1}$$

tedious factorization...

5.8 Let $X_1, X_2, \ldots X_n$ be a random sample, where $\bar{X}$ and S^2 are calculated in the usual way

(a) $$S^2 = \frac{1}{2n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (X_i - X_j)^2$$

Assume that X_i's have finite 4th moment and denote $\theta_1 = \mathbb{E}[X_i]$

$\theta_j = \mathbb{E}[(X_i - \theta_1)^j] \quad j = 2, 3, 4$

Sol^n

$$S^2 = \frac{1}{2n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (X_i - X_j)^2$$

$$= \frac{1}{2n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(X_i - \bar{X} - (X_j - \bar{X})\right)^2$$

$$= \frac{1}{2n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[(X_i - \bar{X})^2 + (X_j - \bar{X})^2 - 2(X_i - \bar{X})(X_j - \bar{X}) \right]$$

$$= \frac{1}{2n(n-1)} \sum_{i=1}^{n} \left[n(x_i-\bar{x})^2 + \sum_{j=1}^{n} (x_j-\bar{x})^2 - 2(x_i-\bar{x}) \cancel{\sum_{j=1}^{n} (x_j-\bar{x})}^{\,0} \right]$$

$$= \frac{1}{2n(n-1)} \left[n \sum_{i=1}^{n} (x_i-\bar{x})^2 + n \sum_{j=1}^{n} (x_j-\bar{x})^2 \right]$$

$$= \frac{1}{\cancel{2}\cancel{n}(n-1)} \left[\cancel{2}\cancel{n} \sum_{i=1}^{n} (x_i-\bar{x})^2 \right]$$

$$= \frac{1}{n-1} \sum_{i=1}^{n} (x_i-\bar{x})^2 = S^2$$

Hence Proved.

(b) Show that $Var(S^2) = \frac{1}{n}\left[\theta_4 - \frac{n-3}{n-1}\theta_2^2\right]$

$$\mathbb{E}[x_1] = \theta_1$$

$$\mathbb{E}[(X_1-\theta_1)^2] = \theta_2 = var(x)$$

$$\mathbb{E}[(X_1-\theta_1)^3] = \theta_3$$

$$\mathbb{E}[(X_1-\theta_4)^4] = \theta_4$$

$$S^2 = \frac{1}{2n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i - x_j)^2$$

$$S^2 = \frac{1}{2n(n-1)} \sum_{i=1}^{n} \left[(x_i - x_1)^2 + (x_i - x_2)^2 + \cdots + (x_i - x_n)^2 \right]$$

$$\Rightarrow S^2 = \frac{1}{2n(n-1)} \left[\begin{array}{l} (x_1 - x_1)^2 + (x_1 - x_2)^2 + \cdots + (x_1 - x_n)^2 \\ (x_2 - x_1)^2 + (x_2 - x_2)^2 + \cdots + (x_2 - x_n)^2 \\ \vdots \\ (x_n - x_1)^2 + (x_n - x_2)^2 + \cdots + (x_n - x_n)^2 \end{array} \right]$$

$$\Rightarrow S^2 = \frac{1}{2n(n-1)} \underbrace{\left[\begin{array}{l} 0 + (x_1 - x_2)^2 + \cdots + (x_1 - x_n)^2 \\ (x_2 - x_1)^2 + 0 + \cdots + (x_2 - x_n)^2 \\ \vdots \\ (x_n - x_1)^2 + (x_n - x_1)^2 + \cdots + 0 \end{array} \right]}_{\text{there are } n^2 \text{ terms of with } n \text{ are } 0}$$

$$\mathbb{E}[S^2] = \frac{1}{2n(n-1)} (n^2 - n)\, \mathbb{E}[(x_1 - x_2)^2]$$

$$= \frac{1}{2} \mathbb{E}[x_1^2 + x_2^2 - 2x_1 x_2]$$

$$= \frac{1}{2} \left(\mathbb{E}[x_1^2] + \mathbb{E}(x_2^2] - 2\,\mathbb{E}(x_1]\,\mathbb{E}[x_2] \right)$$

$$= \frac{1}{2}\left(\sigma^2 + \cancel{\mu^2} + \sigma^2 + \cancel{\mu^2} - 2\cancel{\mu^2}\right)$$

$$= \sigma^2 = \theta_2$$

$$\mathbb{E}[S^2] = \theta_2$$

$$\mathbb{E}[(S^2)^2] = \mathbb{E}[S^4]$$

$$= \mathbb{E}\left[\frac{1}{4n^2(n-1)^2}\left(\begin{matrix}(n^2-n) \text{ term's of which} \\ 2 \text{ are alike}\end{matrix}\right)^2\right]$$

$$= \frac{1}{4n^2(n-1)^2}\,\mathbb{E}\left[\left(2\underbrace{\left[\begin{matrix}(X_1-X_2)^2 + (X_1-X_3)^2 + \cdots + (X_1-X_n)^2 \\ + (X_2-X_3)^2 + \cdots + (X_2-X_n)^2 \\ + \cdots (X_{n-1}-X_n)^2\end{matrix}\right]}_{\frac{n(n-1)}{2} \text{ term's}}\right)^2\right]$$

$$= \frac{4}{4n^2(n-1)^2}\,\mathbb{E}\left[\left(\frac{n(n-1)}{2} \text{ term's}\right)^2\right]$$

$$= \frac{1}{n^2(n-1)^2}\,\mathbb{E}\left[\begin{matrix}(X_1-X_2)^4 + 2(X_1-X_2)^2(X_3-X_4)^2 \\ + 2(X_1-X_2)^2(X_1-X_3)^2 \\ \cdots\end{matrix}\right]$$

in $(x_1+x_2+\dots+x_n)^2 = x_1^2+x_2^2+\dots+x_n^2 + 2\sum_{1\le i\le j<n} x_i x_j$

$$n \quad + \quad \frac{n(n-1)}{2}$$

$$= \frac{n(n+1)}{2} \text{ term's}$$

in our case

we are having $\left(\frac{n(n-1)}{2} \text{ term's}\right)^2$

$$\Rightarrow \frac{\frac{n(n-1)}{2}\cdot\left(\frac{n(n-1)}{2}+1\right)}{2}$$

$$\Rightarrow \frac{\frac{n(n-1)}{2}\cdot\frac{n^2-n+2}{2}}{2}$$

$$\Rightarrow \frac{n(n-1)(n^2-n+2)}{8} \text{ term's.}$$

of which

$\frac{n(n-1)}{2}$ are of $(x_1-x_2)^4$ type term's

$$\frac{n(n-1)}{2} \text{ terms} \left\{ \begin{array}{llllll} (1,2) & (1,3) & (1,4) & \cdots & & (1,n) \\ (2,3) & (2,4) & \cdots (2,5) & \cdots & (2,n) & \\ (3,4) & (3,5) & (3,6) & \cdots & (3,n) & \\ \vdots & & & & & \\ (n-1,n) & & & & & \end{array} \right.$$

$((n-1)+n-2)$

$(2n-3)$

$(1,5)$ $(n+n-$

$(2n-1)$

of which

$\frac{n(n-1)}{2}$ term's will be $(x_1-x_2)^4$ type

$\frac{n(n-1)}{2} \cdot \frac{(2n-1)}{2}$ term's will be $2(x_1-x_2)^2(x_3-x_4)^2$

term's will be $2(x_1-x_2)^2(x_2-x_3)^2$

$$\frac{1}{2} \quad \frac{n\cdot(n-1)}{2} \cdot 2\cdot(n-2).$$

$(2,4)$ $\quad 4$

$(1,2)$ $(1,3)$ $(1,4)$ $(1,5)$

$(2,3)$ $(2,4)$ $(2,5)$

$(3,4)$ $(3,5)$

$(4,5)$

(5.10) Let $X_1, X_2, \dots X_n$ be a random sample from a $N(\mu, \sigma^2)$ population.

(a) Find expression's for $\theta_1, \dots \theta_4$

$$\mathbb{E}[X] = \mu = \theta_1$$

$$\mathbb{E}[(X-\mu)^2] = \sigma^2 = \theta_2$$

$$\mathbb{E}[(X-\mu)^3] = \mathbb{E}[(X-\mu)^2(X-\mu)]$$

<u>Stein Identity</u> $\mathbb{E}[g(X)(X-\theta)]$

$$= \sigma^2 \, \mathbb{E}[g'(X)]$$

$$\Rightarrow \mathbb{E}[(X-\mu)^3] = \sigma^2 \, \mathbb{E}[2(X-\mu)]$$

$$= \sigma^2 \, 2 \, \mathbb{E}[X-\mu] = 0 = \theta_3$$

$$\Rightarrow \mathbb{E}[(X-\mu)^4] = \mathbb{E}[(X-\mu)^3(X-\mu)]$$

$$= \sigma^2 \, \mathbb{E}[3(X-\mu)^2]$$

$$= 3\sigma^2 \cdot \sigma^2 = 3\sigma^4 = \theta_4$$

$$\Rightarrow \boxed{\theta_1 = \mu, \quad \theta_2 = \sigma^2, \quad \theta_3 = 0 \quad \theta_4 = 3\sigma^4}$$

5.10

(b) Use the result of Exercise 5.8, together, with the results of Part (a), to calculate $Var(s^2)$

Soln

$$Var(s^2) = \frac{1}{n}\left(\theta_4 - \frac{n-3}{n-1}\theta_2^2\right)$$

$$= \frac{1}{n}\left(3\sigma^4 - \frac{n-3}{n-1}\sigma^4\right)$$

$$= \frac{1}{n}\left(3 - \frac{n-3}{n-1}\right)\sigma^4$$

$$\Rightarrow \frac{1}{n}\left(\frac{3n-\cancel{3}-n+\cancel{3}}{n-1}\right)\sigma^4$$

$$\Rightarrow \frac{2}{n-1}\cdot\sigma^4$$

$$\boxed{\Rightarrow Var(s^2) = \frac{2}{n-1}\cdot\sigma^4}$$

5.10 c

Calculate $var(S^2)$ a completely different (and easier) way: use the fact that

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

Sol^n

$$var\left(\frac{(n-1)S^2}{\sigma^2}\right) = var\left(\chi^2_{n-1}\right)$$

$$\chi^2_{n-1} \sim gamma\left(\frac{n-1}{2}, 2\right)$$

$$\Rightarrow var\left(\chi^2_{n-1}\right) = \alpha\beta^2 = \left(\frac{n-1}{2}\right) \cdot 4$$

$$= 2(n-1)$$

$$\Rightarrow var\left(\frac{(n-1)S^2}{\sigma^2}\right) = 2(n-1)$$

$$\Rightarrow \frac{(n-1)^2}{\sigma^4} \cdot var(S^2) = 2(n-1)$$

$$\Rightarrow \boxed{var(S^2) = \frac{2}{n-1} \cdot \sigma^4}$$

5.11 Suppose $\bar{X}$ and S^2 are calculated from a random sample $X_1, \ldots X_n$ drawn from a population with finite variance σ^2. We know that $\mathbb{E}[S^2] = \sigma^2$. Prove that $\mathbb{E}[S] \leq \sigma$, and if $\sigma^2 > 0$, then $\mathbb{E}[S] < \sigma$

Soln

We can use Jenson's inequality

$$\mathbb{E}[g(x)] \geq g(E[x])$$

where $g(x)$ is a convex function

$$\Rightarrow \quad g(x) = S^2$$

$$\mathbb{E}[S^2] \geq g(E[x])$$

$$\Rightarrow \quad \sigma^2 \geq \mu^2$$

$$\Rightarrow \quad \boxed{\sigma \geq \mu}$$

5.12 Let $X_1, X_2, \ldots X_n$ be a random sample from a $n(0,1)$ population. Define

$$Y_1 = \left| \frac{1}{n} \sum_{i=1}^{n} X_i \right| , \quad Y_2 = \frac{1}{n} \sum_{i=1}^{n} |X_i|$$

calculate $E[Y_1]$ and $E[Y_2]$, and establish an inequality b/w them

Sol^n

$$Y_1 = \left| \frac{1}{n} \sum_{i=1}^{n} X_i \right|$$

$X_i \sim N(0,1)$

$\Rightarrow \quad \bar{X} \sim N(0, \frac{1}{n})$

$\Rightarrow \quad |\bar{X}| \sim |N(0, \frac{1}{n})| \Rightarrow$ Half-Normal

$Y_1 = |\bar{X}|$ $\qquad \bar{X} = -Y_1 \quad -\infty < \bar{X} < 0$

$\qquad \bar{X} = Y_1 \quad 0 < \bar{X} < \infty$

$\Rightarrow$

$|J| = 1$

$$\Rightarrow f_{Y_1}(y_1) = f_{\bar{X}}(g^{-1}(y)) \cdot |J|$$

$$f_{\bar{X}}(\bar{x}) = \frac{1}{\sqrt{2\pi \cdot \frac{1}{n}}} e^{-\frac{n}{2}\bar{x}^2}$$

$$\Rightarrow f_{Y_1}(y_1) = \frac{\sqrt{n}}{\sqrt{2\pi}} \cdot e^{-\frac{n}{2}y^2} \cdot 2$$

$$\Rightarrow f_Y(y) = \sqrt{\frac{2}{\pi}} \cdot \sqrt{n} \cdot e^{-\frac{n}{2}y^2}$$

$$\mathbb{E}[Y_1] = \mathbb{E}(|\bar{X}|)$$

$$= \int_{-\infty}^{0} -x \cdot \frac{\sqrt{n}}{\sqrt{2\pi}} \cdot e^{-\frac{n}{2}x^2} dx$$

$$+ \int_{0}^{\infty} x \cdot \frac{\sqrt{n}}{\sqrt{2\pi}} \cdot e^{-\frac{n}{2}x^2} dx$$

$$= \sqrt{\frac{2n}{\pi}} \int_{0}^{\infty} x \cdot e^{-\frac{n}{2}x^2} dx$$

$$\frac{n}{2}x^2 = u$$

$$\Rightarrow \quad nx\,dx = du$$

$$= \sqrt{\frac{2n}{\pi}} \cdot \frac{1}{n} \int_0^\infty e^{-u}\,du$$

$$= \sqrt{\frac{2n}{\pi}} \cdot \frac{1}{n} \quad -e^{-u}\Big]_0^\infty$$

$$= \sqrt{\frac{2n}{\pi}} \cdot \frac{1}{n} = \sqrt{\frac{2}{n\pi}}$$

$$\boxed{\Rightarrow \quad \mathbb{E}[X_i] = \sqrt{\frac{2}{n\pi}}}$$

$$Y_2 = \frac{1}{n} \sum_{i=1}^{n} |X_i|$$

$|X_i| \sim$ half-normal

$$|X_i| \sim \sqrt{\frac{2}{\pi}}\, e^{-\frac{x_i^2}{2}} \qquad 0 < |x_i| < \infty$$

Mgf of $V \sim$ Half normal

$$M_V(t) = \mathbb{E}[e^{tV}]$$

$$\Rightarrow \quad = \int_0^\infty e^{tv} \sqrt{\frac{2}{\pi}} \cdot e^{-\frac{v^2}{2}} \, du$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{v^2}{2} + tv} \, dv$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{1}{2}(v^2 - 2tv + t^2)} \cdot e^{\frac{t^2}{2}} \, dv$$

$$= \sqrt{\frac{2}{\pi}} e^{\frac{t^2}{2}} \int_0^\infty e^{-\frac{1}{2}(v-t)^2} \, dv$$

$v - t = u \quad \Rightarrow \quad dv = du$

$u : -t \to \infty$

$$= \sqrt{\frac{2}{\pi}} e^{\frac{t^2}{2}} \int_{-t}^\infty e^{-\frac{1}{2}u^2} \, du$$

$$= 2 \cdot e^{\frac{t^2}{2}} \frac{1}{\sqrt{2\pi}} \int_{-t}^{\infty} e^{-\frac{u^2}{2}} du$$

$$= 2e^{\frac{t^2}{2}} \left(1 - \Phi(-t)\right)$$

$$M_V(t) = 2e^{\frac{t^2}{2}} \Phi(t)$$

$$\Rightarrow \quad V_2 = \frac{1}{n}\left(|x_1| + |x_2| + |x_3| \cdots + |x_n|\right)$$

$$MGF(|x_1| + |x_2| \cdots + |x_n|)$$

$$= \left(2e^{\frac{t^2}{2}} \phi(t)\right)^n$$

this method is not working.

$$\mathbb{E}[Y_2] = \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n}|X_i|\right]$$

$$= \frac{1}{n}\cdot n\,\mathbb{E}[|X_1|]$$

$$= \mathbb{E}[|X_1|]$$

$$= \int_0^\infty x\cdot\sqrt{\frac{2}{\pi}}\cdot e^{-\frac{x^2}{2}}\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty x\cdot e^{-\frac{x^2}{2}}\,dx$$

$\frac{x^2}{2} = u$

$x\,dx = du$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-u}\,du = \sqrt{\frac{2}{\pi}}$$

$$\boxed{\Rightarrow \mathbb{E}[Y_2] = \sqrt{\frac{2}{\pi}}}$$

Therefore

$$\boxed{\begin{aligned}\mathbb{E}[Y_1] &= \sqrt{\frac{2}{n\pi}}\\ \mathbb{E}[Y_2] &= \sqrt{\frac{2}{\pi}}\end{aligned}}$$

lets use triangular inequality

$$|X_1 + X_2 + \cdots + X_n| \leq |X_1| + |X_2| \cdots + |X_n|$$

$$\Rightarrow \frac{1}{n}|X_1 + X_2 + \cdots + X_n| \leq \frac{1}{n}(|X_1| + |X_2| \cdots + |X_n|)$$

apply Expection

$$\Rightarrow \mathbb{E}\left[\frac{1}{n}|X_1 + X_2 + \cdots + X_n|\right] \leq \mathbb{E}\left[\frac{1}{n}(|X_1| + |X_2| \cdots + |X_n|)\right]$$

$$\Rightarrow \boxed{\mathbb{E}[\bar{X}_1] \leq \mathbb{E}[Y_2]}$$

5.13 Let $X_1, X_2, \ldots X_n$ be iid $n(\mu, \sigma^2)$.
Find a function of S^2, the sample variance, say $g(S^2)$ that satisfies $\mathbb{E}[g(S^2)] = \sigma$

(Hint: Try $g(S^2) = c\sqrt{S^2}$)

Sol^n

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

$$g(s^2) = c\sqrt{s^2}$$

$$\mathbb{E}\left[c\sqrt{s^2}\right] = c\,\mathbb{E}\left[\sqrt{s^2}\right]$$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} \sim \text{gamma}\left(\tfrac{n-1}{2}, 2\right)$$

$$\Rightarrow \quad S^2 \sim \frac{\sigma^2}{n-1} \chi^2_{n-1}$$

$$\Rightarrow \quad \sqrt{s^2} \sim \sqrt{\frac{\sigma^2}{n-1} \chi^2_{n-1}}$$

$$\mathbb{E}\left[\sqrt{s^2}\right] = \mathbb{E}\left[\sqrt{\frac{\sigma^2}{n-1} \chi^2_{n-1}}\right]$$

$$= \sqrt{\frac{\sigma^2}{n-1}}\, \mathbb{E}\left[\sqrt{\chi^2_{n-1}}\right]$$

$$= \sqrt{\frac{\sigma^2}{n-1}} \int_0^\infty \sqrt{x} \cdot \frac{1}{\Gamma\left(\frac{n-1}{2}\right) \cdot 2^{\frac{n-1}{2}}} \cdot x^{\frac{n-1}{2}-1} \cdot e^{-\frac{x}{2}} \, dx$$

$$= \sqrt{\frac{\sigma^2}{n-1}} \frac{1}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-1}{2}}} \underbrace{\int_0^\infty x^{\frac{n}{2}-1} \cdot e^{-\frac{x}{2}} \, dx}_{\text{gamma}\left(\frac{n}{2}, 2\right)}$$

$$= \sqrt{\frac{\sigma^2}{n-1}} \cdot \frac{1}{\Gamma\left(\frac{n-1}{2}\right) \cdot 2^{\frac{n-1}{2}}} \cdot \Gamma\left(\frac{n}{2}\right) \cdot 2^{\frac{n}{2}}$$

$$\mathbb{E}\left[\sqrt{s^2}\right] = \sqrt{\frac{2\sigma^2}{n-1}} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

$$\Rightarrow \quad c \cdot \mathbb{E}\left[\sqrt{s^2}\right] = \sigma$$

$$= \quad c \cdot \cancel{\sigma} \cdot \sqrt{\frac{2}{n-1}} \cdot \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} = \cancel{\sigma}$$

$$\Rightarrow \boxed{C = \sqrt{\frac{n-1}{2}} \cdot \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}}$$

5.14 (a) Prove that the statement of Lemma 5.3.3 follows from the spectial case of $\mu_i = 0$, and $\sigma_i^2 = 1$. That is, Show that if

$$X_j = \sigma_j Z_i + \mu_j \quad \text{and} \quad Z_j \sim n(0,1)$$

$j = 1, 2, \cdots n$, all independent

a_{ij}, b_{rj} are constants, and.

$$\operatorname{cov}\left(\sum_{j=1}^{n} a_{ij} Z_j, \sum_{j=1}^{n} b_{rj} Z_j\right) = 0$$

$$\Rightarrow \sum_{j=1}^{n} a_{ij} Z_j \text{ and } \sum_{j=1}^{n} b_{rj} Z_j \text{ are independent}$$

then

$$\operatorname{cov}\left(\sum_{j=1}^{n} a_{ij} X_j, \sum_{j=1}^{n} b_{rj} X_j\right) = 0 \Rightarrow \sum_{j=1}^{n} a_{ij} X_j \text{ and}$$

$$\sum_{j=1}^{n} b_{rj} X_j \text{ are independent}$$

5.15 Establish the following recursion relation's for mean's and variance. Let $\bar{x}_n$ and $\bar{S}_n$ be the mean and variance, respectively of $x_1, x_2, \dots x_n$. The suppose another observation x_{n+1}, becomes available, show that.

(a) $$\bar{X}_{n+1} = \frac{x_{n+1} + n\bar{x}_n}{n+1}$$

Soln

$$\bar{X}_{n+1} = \frac{X_1 + X_2 + \cdots + X_n + X_{n+1}}{n+1}$$

$$\Rightarrow (n+1)\bar{X}_{n+1} - X_{n+1} = n \cdot \bar{X}_n$$

$$\Rightarrow \boxed{\bar{X}_{n+1} = \frac{n\bar{X}_n + X_{n+1}}{n+1}}$$

5.15 b

$$(n-1) S_n^2 = (n-2) S_{n-1}^2 + \left(\frac{n-1}{n}\right)(X_n - \overline{X}_{n-1})^2$$

Proof of the Identity

$$(n-1) S_n^2 = \sum_{i=1}^{n} (X_i - \overline{X}_n)^2$$

$$\overline{X}_n = \frac{X_1 + X_2 + \cdots + X_{n-1} + X_n}{n}$$

$$\frac{n \overline{X}_n - X_n}{n-1} = \frac{X_1 + X_2 + \cdots + X_{n-1}}{n-1}$$

$$\Rightarrow \frac{n}{n-1} \overline{X}_n - \frac{X_n}{n-1} = \overline{X}_{n-1}$$

$$\Rightarrow \overline{X}_n = \frac{n-1}{n} \overline{X}_{n-1} + \frac{X_n}{n}$$

$$\Rightarrow (n-1) S_n^2 = \sum_{i=1}^{n} (x_i - \overline{x}_n)^2$$

$$= \sum_{i=1}^{n} \left(x_i - \frac{n-1}{n} \overline{x}_{n-1} - \frac{x_n}{n} \right)^2$$

$$= \sum_{i=1}^{n} \left(x_i - \frac{n-1}{n} \overline{x}_{n-1} - \frac{x_n}{n} \right)^2$$

$$= \sum_{i=1}^{n} \left(x_i - \overline{x}_{n-1} + \frac{\overline{x}_{n-1} - x_n}{n} \right)^2$$

$$= \sum_{i=1}^{n-1} \left(x_i - \overline{x}_{n-1} + \frac{\overline{x}_{n-1} - x_n}{n} \right)^2$$

$$+ \left(x_n - \overline{x}_{n-1} + \frac{\overline{x}_{n-1} - x_n}{n} \right)^2$$

$$= \sum_{i=1}^{n-1} \left[(x_i - \overline{x}_{n-1})^2 + \left(\frac{\overline{x}_{n-1} - x_n}{n} \right)^2 + 2 (x_i - \overline{x}_{n-1}) \left(\frac{\overline{x}_{n-1} - x_n}{n} \right) \right]$$

$$+ \left(1 - \frac{1}{n} \right)^2 (x_n - \overline{x}_{n-1})^2$$

$$= \sum_{i=1}^{n-1} (x_i - \bar{x}_{n-1}) + \sum_{i=1}^{n-1} \left(\frac{\bar{x}_{n-1} - x_n}{n} \right)^2$$

$$+ \quad 0 \quad + \left(1 - \frac{1}{n}\right)^2 (x_n - \bar{x}_{n-1})^2$$

$$= (n-2) S_{n-1}^2$$

$$+ \left(\frac{n-1}{n^2}\right) (x_n - \bar{x}_{n-1})^2$$

$$+ \left(1 - \frac{1}{n}\right)^2 (x_n - \bar{x}_{n-1})^2$$

$$= (n-2) S_{n-1}^2 + (x_n - \bar{x}_{n-1})^2 \left[\frac{1}{n} - \cancel{\frac{1}{n^2}} + 1 + \cancel{\frac{1}{n^2}} - \frac{2}{n} \right]$$

$$= (n-2) S_{n-1}^2 + \left(\frac{n-1}{n}\right) (x_n - \bar{x}_{n-1})^2$$

There fore

$$(n-1) S_n^2 = (n-2) S_{n-1}^2$$

$$+ \left(\frac{n-1}{n}\right) (x_n - \bar{x}_{n-1})^2$$

5.16 Let $X_i, i=1,2,3$ be independent with $N(i, i^2)$ distributions. For each of the following situations, use the X_i's to construct a statistic with the indicated distribution.

a Chi-squared with 3 d.o.f.

Soln

$$X_1 \sim N(1,1) \Rightarrow X_1 = 1 + Z_1$$
$$X_2 \sim N(2,4) \Rightarrow X_2 = 2 + 2Z_2$$
$$X_3 \sim N(3,9) \Rightarrow X_3 = 3 + 3Z_3$$

$$\chi_3^2 = Z_1^2 + Z_2^2 + Z_3^2$$

where $Z_i \sim N(0,1)$ independent.

$$\boxed{\Rightarrow \chi_3^2 = (X_1 - 1)^2 + \left(\frac{X_2 - 2}{2}\right)^2 + \left(\frac{X_3 - 3}{3}\right)^2}$$

5.16 b   t distribution with 2 d.o.f

Soln   Student-t distribution with $(n-1)$ d.o.f

can be formed using

$$\frac{\bar{X} - \mu}{\frac{S}{\sqrt{n-1}}}$$

i.e The statistic

$$\frac{\bar{X} - \mu}{\frac{S}{\sqrt{2}}}$$

$$\Rightarrow \frac{\sum_{i=1}^{3} \frac{x_i - i}{i}}{\sqrt{\frac{\sum_{i=1}^{3} \left(\frac{x_i - i}{i}\right)^2}{2}}}$$

(c) F distribution with 1 and 2 d.o.f