$$Cov(X,Y) = \mathbb{E}\left[(X-\mu_x)(Y-\mu_Y)\right]$$

Properties of Cov(X, Y) :

1. $Cov(X, c) = 0$ where c is constant

2. $Var(X) = Cov(X, X)$

3. $Cov(X, Y) = Cov(Y, X)$

4. $Cov(X+c, Y) = Cov(X, Y)$

5. $Cov(aX, cY) = ac\, Cov(X, Y)$

6. $Cov(X+Y, Z) = Cov(X, Z) + Cov(Y, Z)$

7. $Var(X+Y) = Var(X) + Var(Y) + 2\, Cov(X, Y)$

8. $Cov\left(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_i\right)$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_i \operatorname{cov}(x_i, y_i) b_j$$

(9) $$\operatorname{var}(ax+by) = a^2 \operatorname{var}(x) + b^2 \operatorname{var}(y) + 2ab \operatorname{cov}(x,y)$$

(10) $$\operatorname{cov}(ax+by, z) = a \operatorname{cov}(x,z) + b \operatorname{cov}(y,z)$$

Covariance matrix

$$\Rightarrow \operatorname{cov}\left(\sum_{i=1}^{m} a_i x_i , \sum_{j=1}^{n} b_j y_j \right)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_i \operatorname{cov}(x_i, y_j) b_j$$

$$= \underbrace{[a_1 \ a_2 \ \cdots \ a_m]}_{(1\times m)} \underbrace{C_{m\times n}}_{(m\times n)} \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_2 \end{bmatrix}}_{(n\times 1)}$$

$$= a^T C b$$

where $C = \begin{bmatrix} \mathrm{cov}(X_1, Y_1) & \mathrm{cov}(X_1, Y_2) & \cdots & \mathrm{cov}(X_1, Y_n) \\ \mathrm{cov}(X_2, Y_1) & \mathrm{cov}(X_2, Y_2) & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ \mathrm{cov}(X_m, Y_1) & \mathrm{cov}(X_m, Y_2) & \cdots & \mathrm{cov}(X_m, Y_n) \end{bmatrix}_{(m \times n)}$

Let $X_1, X_2, \ldots X_n$ be r.v's and

write

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}_{n \times 1}$$ random vector

$$\mathbb{E}[\vec{X}] = \begin{bmatrix} \mathbb{E}[X_1] \\ \mathbb{E}[X_2] \\ \vdots \\ \mathbb{E}[X_n] \end{bmatrix}_{n \times 1}$$

Def: The co-variance matrix of the random vector $\vec{X}$

$$\Sigma_X \triangleq \left(\operatorname{cov}(x_i, x_j) \right)$$

fact:

$$\Sigma_X = \mathbb{E}\left[[\vec{X} - \vec{\mu}][\vec{X} - \vec{\mu}]^T \right]$$

where $\vec{\mu} = \mathbb{E}[\vec{X}]$

$$\Sigma_X = \mathbb{E}\left[(x_i - \mu_i)(x_j - \mu_j)^T \right]_{n \times n}$$

$$= \operatorname{cov}(x_i, x_j)_{n \times n}$$

FACT: Let $\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ have a cov. matrix

Σ_X, then $\operatorname{Var}(a^T \vec{X})$ is

$$\mathrm{Var}(a^T \vec{x}) = \mathrm{Var}(a_1 x_1 + a_2 x_2 + \cdots + a_n x_n)$$

$$= \mathrm{Cov}\left(\sum_{i=1}^{n} a_i x_i , \sum_{j=1}^{n} a_j x_j\right)$$

$$= a^T \Sigma_x a$$