DESCRIPTION

This document provides solution's to 4.32 to 4.39 from chapter 4 of statistical inference Book by Casella and Berger, Focusing on Hierarchical Models and Mixture distribution's.

A few Problem's are yet to be solved, but I will update the document soon.

4.32 (a) for the hierarchical model

$$Y \mid \Lambda \sim \text{Poisson}(\lambda) \quad \text{and}$$

$$\Lambda \sim \text{gamma}(\alpha, \beta)$$

find the marginal distribution, mean and variance of Y. Show that the marginal distribution of Y is a negative binomial if α is an integer.

solution:

$$\mathbb{P}(Y=y) = \int_0^\infty \mathbb{P}(Y=y, \Lambda=\lambda)$$

$$= \int_0^\infty \mathbb{P}(Y=y \mid \Lambda=\lambda)\, \mathbb{P}(\Lambda=\lambda)\, d\alpha$$

$$= \int_0^\infty \frac{e^{-\lambda} \cdot \lambda^y}{y!} \cdot \frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{\alpha-1} \cdot e^{-\frac{\lambda}{\beta}}\, d\lambda$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha} \cdot y!} \int_0^{\infty} e^{-\lambda} \cdot \lambda^{y} \cdot \lambda^{\alpha-1} \cdot e^{-\frac{\lambda}{\beta}} \, d\lambda$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha} y!} \int_0^{\infty} e^{-\lambda\left(1+\frac{1}{\beta}\right)} \cdot \lambda^{\alpha+y-1} \, d\lambda$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha} y!} \int_0^{\infty} \lambda^{(\alpha+y-1)} \cdot e^{-\frac{\lambda}{\frac{\beta}{1+\beta}}} \, d\lambda$$

$$= \frac{\Gamma(\alpha+y)\left(\frac{\beta}{1+\beta}\right)^{\alpha+y}}{\Gamma(\alpha)\beta^{\alpha} y!}$$

$$\Rightarrow f_Y(y) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \cdot \frac{1}{y!} \Gamma(\alpha+y) \cdot \left(\frac{\beta}{1+\beta}\right)^{\alpha+y}$$

$$y = 0, 1, 2, \ldots$$

if α is an integer

$$f_Y(y) = \frac{1}{(\alpha-1)!\,\beta^\alpha} \cdot \frac{1}{y!} \cdot (\alpha+y-1)! \left(\frac{\beta}{1+\beta}\right)^{\alpha+y}$$

$$f_Y(y) = \binom{\alpha+y-1}{y} \left(\frac{\beta}{1+\beta}\right)^{y} \cdot \left(1-\frac{\beta}{1+\beta}\right)^{\alpha}$$

$$\boxed{Y \sim \text{Negative Binomial}\,(\alpha, p)}$$

$$\mathbb{E}[Y] = \mathbb{E}\left[E[Y|\Lambda]\right]$$

$$= \mathbb{E}[\Lambda]$$

$$\boxed{E[X] = \alpha\beta}$$

$$\text{var}(Y) = \mathbb{E}[\text{var}(Y|\Lambda)] + \text{var}(E[Y|\Lambda])$$

$$= \mathbb{E}[\Lambda] + \text{var}(\Lambda)$$

$$\boxed{\text{var}(X) = \alpha\beta + \alpha\beta^2}$$

4.32 b Show that the 3-stage model

$Y|N \sim \text{Binomial}(N, P)$

$N|\Lambda \sim \text{Poisson}(\Lambda)$

$\Lambda \sim \text{gamma}(\alpha, \beta)$ Leads to

the Same Marginal (unconditional)

distribution of Y

Solution:

$$\mathbb{P}(Y=y|\lambda) = \sum_{N=y}^{\infty} \mathbb{P}(Y|N,\Lambda)\, \mathbb{P}(N|\Lambda)\, \mathbb{P}(\Lambda)$$

$$= \sum_{N=y}^{\infty} \binom{n}{y} p^y \cdot (1-p)^{n-y} \cdot \frac{e^{-\lambda} \cdot \lambda^n}{n!}$$

$$= \left(\frac{p}{1-p}\right)^y \cdot e^{-\lambda} \sum_{n=y}^{\infty} \frac{\cancel{n!}}{(n-y)!\, y!} \frac{\left[(1-p)\lambda\right]^n}{\cancel{n!}}$$

$$= \frac{\left(\frac{p}{1-p}\right)^y \cdot e^{-\lambda}}{y!} \sum_{n=y}^{\infty} \frac{\left[(1-p)\lambda\right]^n}{(n-y)!}$$

$$= \frac{(P\lambda)^y e^{-P\lambda}}{y!}$$

$$\Rightarrow \quad Y \mid \Lambda \sim \text{Poisson}(\Lambda)$$

$$\text{and} \quad \Lambda \sim \text{gamma}(\alpha, \beta)$$

4.34 (a) For the hierarchy in Example 4.4.6, show that the marginal distribution of X is given by the beta-binomial distribution.

$$\mathbb{P}(X=x) = \binom{n}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(x+\alpha)\Gamma(n-x+\beta)}{\Gamma(\alpha+\beta+n)}$$

Sol^n Example 4.4.6

$$X \mid P \sim \text{binomial}(P)$$

$$i = 1, 2, \ldots n$$

$$P \sim \text{beta}(\alpha, \beta)$$

$$\mathbb{P}(X=x) = \int_{p=0}^{1} \mathbb{P}(X=x \mid P=p)\, \mathbb{P}(P=p)\, dp$$

$$= \int_{p=0}^{1} \binom{n}{x} p^{x} (1-p)^{n-x} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1} (1-p)^{\beta-1} dp$$

$$= \binom{n}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_{p=0}^{1} p^{\alpha+x-1} \cdot (1-p)^{n+\beta-x-1}$$

$$\boxed{P(X=x) = \binom{n}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(x+\alpha)\Gamma(n-x+\beta)}{\Gamma(\alpha+\beta+n)}}$$

4.34 b

A variation on the hierarchical model in Part (a) is

$X \mid P \sim$ negative binomial (r, P)

$P \sim$ beta (α, β)

find marginal pmf of X, mean, var

Solution:

$$\mathbb{P}(X=x) = \int_{p=0}^{1} \binom{r+x-1}{x} p^{r} (1-p)^{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1} (1-p)^{\beta-1} dp$$

$$= \binom{r+x-1}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_{p=0}^{1} p^{r+\alpha-1} \cdot (1-p)^{\beta+x-1} dp$$

$$\boxed{P(X=x) = \binom{r+x-1}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(r+\alpha)\Gamma(\beta+x)}{\Gamma(r+\alpha+\beta+x)}}$$

$$\mathbb{E}[X] = \mathbb{E}\left[\mathbb{E}[X|P]\right]$$

$$= \mathbb{E}\left[\frac{r(1-P)}{P}\right] = r\,\mathbb{E}\left[\frac{1-P}{P}\right]$$

$$\mathbb{E}\left[\frac{1-p}{p}\right] = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \frac{1-p}{p}\, p^{\alpha-1} \cdot (1-p)^{\beta-1}\, dp$$

$$= \frac{\Gamma(\cancel{\alpha+\beta})}{\Gamma(\alpha)\,\Gamma(\beta)} \cdot \frac{\Gamma(\alpha-1)\,\Gamma(\beta+1)}{\Gamma(\cancel{\alpha+\beta})}$$

$$= \frac{\beta}{\alpha-1}$$

$$\Rightarrow \boxed{\mathbb{E}\{X\} = \frac{r\beta}{\alpha-1}}$$

$$\mathrm{var}(X) = \mathbb{E}\{\mathrm{var}(X|P)\} + \mathrm{var}(\mathbb{E}[X|P])$$

$$= \mathbb{E}\left[\frac{r(1-p)}{p^2}\right] + \mathrm{var}\left(r\,\frac{(1-p)}{p}\right)$$

$$= r\,\mathbb{E}\left\{\frac{1-p}{p^2}\right\} + r\,\mathrm{var}\left(\frac{1-p}{p}\right)$$

$$\mathbb{E}\left(\frac{1-p}{p^2}\right) = \frac{1}{B(\alpha,\beta)} \int_0^1 p^{\alpha-2-1} \cdot (1-p)^{\beta+1-1} \, dp$$

$$= \frac{B(\alpha-2, \beta+1)}{B(\alpha,\beta)} = \frac{(\beta+1)(\alpha+\beta)}{\alpha(\alpha-1)}$$

$$\operatorname{var}\left(\frac{1-p}{p}\right) = \mathbb{E}\left[\left(\frac{1-p}{p}\right)^2\right] - \mathbb{E}\left[\frac{1-p}{p}\right]^2$$

$$\mathbb{E}\left[\left(\frac{1-p}{p}\right)^2\right] = \frac{1}{B(\alpha,\beta)} \int_0^1 p^{\alpha-2-1} \cdot (1-p)^{\beta+2-1} \, dp$$

$$= \frac{B(\alpha-2, \beta+2)}{B(\alpha,\beta)}$$

$$= \frac{(\beta+1)\beta}{(\alpha-1)(\alpha+2)}$$

$$\boxed{\operatorname{var}(X) = r\frac{(\beta+1)\beta}{(\alpha-1)(\alpha+2)} - r\left(\frac{\beta}{\alpha-1}\right)^2}$$

4.35 For a hierarchy in Example 4.4.6, show that the variance of X can be written

$$\mathrm{Var}(X) = n E P(1-EP) + n(n-1) \mathrm{Var}$$

Example 4.4.6 :- $X|P \sim \text{binomial}(P)$

$i=1,2,\ldots n$

$P \sim \text{beta}(\alpha,\beta)$

Sol^n

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X|P)] + \mathrm{Var}(E[X|P])$$

$$\Rightarrow \mathrm{Var}(X) = \mathbb{E}[nP(1-P)] + \mathrm{Var}(nP)$$

$$= n\mathbb{E}[P(1-P)] + n^2 \mathrm{Var}(P)$$

$$= n\,\mathbb{E}[P-P^2] + n^2 \mathrm{Var}(P)$$

$$= n(EP - EP^2] + n^2 \mathrm{Var}(P)$$

$$= n(EP - E[P]^2 + E[P]^2 - \mathbb{E}[P^2]) + n^2 \mathrm{Var}(P)$$

$$= n\left(EP - E[P]^2\right) - n\,\mathrm{var}(P) + n^2\,\mathrm{var}(P)$$

$$\boxed{E[X] = nE[P](1-E[P]) + n(n-1)\,\mathrm{var}(P)}$$

4.35b For the hierarchy in Exercise 4.32, show that the variance of Y can be written

$$\mathrm{Var}\,Y = E[\Lambda] + \mathrm{Var}(\Lambda) = \mu + \frac{1}{\alpha}\mu^2$$

<u>Exercise 4.32:-</u>

$$Y \mid \Lambda \sim \mathrm{Poisson}(\Lambda)$$

$$\Lambda \sim \mathrm{gamma}(\alpha, \beta)$$

<u>Solution:</u>

$$\mathrm{var}(Y) = \mathbb{E}[\mathrm{var}(Y \mid \Lambda)] + \mathrm{var}(E[Y \mid \Lambda])$$

$$\operatorname{var}(Y) = \mathbb{E}[\Lambda] + \operatorname{var}(\Lambda)$$

$$= \alpha\beta + \alpha\beta^2$$

if $E[\Lambda] = \mu = \alpha\beta$

then $\boxed{\operatorname{var}(Y) = \mu + \frac{\mu^2}{\alpha}}$

(4.36) one generalization of the Bernoulli trails hierarchy in Example 4.4.6 is to allow the success prob to vary from trail to trial, keeping the trails independent. A standard model for the situation is

$$X_i \mid P_i \sim \text{Bernoulli}(P_i)$$
$$i = 1, 2, \dots n$$

$$P_i \sim \text{beta}(\alpha, \beta)$$

This model might be appropriate, for example, if we are measuring the success of a drug on n patients and, because the patients are different, we are reluctant to assume that the success prob are constant. (This can be thought of as an empirical Bayes model;)

A r.v of interest is $Y = \sum_{i=1}^{n} Y_i$ the total number of success.

(a) Show that $E[Y] = \frac{n\alpha}{\alpha+\beta}$

Solution:

$$\mathbb{E}[Y] = \sum_{i=1}^{n} E[X_i] =$$

$$= \sum_{i=1}^{n} \mathbb{E}\left[\mathbb{E}[X_i \mid P_i]\right]$$

$$= \sum_{i=1}^{n} \mathbb{E}[P_i]$$

$$= \sum_{i=1}^{n} \frac{\alpha}{\alpha+\beta} = \frac{n\alpha}{\alpha+\beta}$$

(b) Show that $\operatorname{var}(Y) = \frac{n\alpha\beta}{(\alpha+\beta)^2}$, and hence

Y has the same mean and variance as

a Binomial $\left(n, \frac{\alpha}{\alpha+\beta}\right)$ r.v. What is the distribution of Y?

Solution:

$$\mathrm{Var}(Y) = \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right)$$

$$= \sum_{i=1}^{n} \mathrm{Var}(X_i)$$

$$= \sum_{i=1}^{n} \mathbb{E}\left[\mathrm{Var}(X_i \mid P_i)\right] + \mathrm{Var}\left[E(X_i \mid P_i)\right]$$

$$= \sum_{i=1}^{n} \mathbb{E}\left[P_i(1-P_i)\right] + \mathrm{Var}(P_i)$$

$$= \sum_{i=1}^{n} \mathbb{E}[P_i] - \mathbb{E}[P_i^2] + \mathrm{Var}(P_i)$$

$$= \sum_{i=1}^{n} \mathbb{E}[P_i] - \mathbb{E}\cancel{[P_i^2]} + \cancel{E}[P^2] - E[P_i]^2$$

$$= \sum_{i=1}^{n} \mathbb{E}[P_i] - \mathbb{E}[P_i]^2$$

$$= \sum_{i=1}^{n} \frac{\alpha}{\alpha+\beta} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$= n\left(\frac{\alpha(\alpha+\beta) - \alpha^2}{(\alpha+\beta)^2}\right)$$

$$\boxed{\operatorname{Var}(Y) = \frac{n\alpha\beta}{(\alpha+\beta)^2}}$$

4.36 (c) Suppose now that the model is

$$X_i \mid P_i \sim \text{binomial}(n_i, P_i) \quad i=1,2,\ldots k$$

$$P_i \sim \text{beta}(\alpha, \beta)$$

Show that for $Y = \sum_{i=1}^{k} X_i$, 1

$E[Y] = \frac{\alpha}{\alpha+\beta} \sum_{i=1}^{k} n_i$ and $\operatorname{Var}(Y) = \sum_{i=1}^{k} \operatorname{Var}(X_i)$

where

$$\operatorname{Var}(X_i) = n_i \frac{\alpha\beta(\alpha+\beta+n_i)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Solution:

$$Y = \sum_{i=1}^{K} x_i$$

$$\mathbb{E}(Y) = \sum_{i=1}^{K} \mathbb{E}[x_i]$$

$$= \sum_{i=1}^{K} \mathbb{E}\{\mathbb{E}\{x_i \mid P_i\}\}$$

$$= \sum_{i=1}^{K} \mathbb{E}[n_i P_i]$$

$$= \sum_{i=1}^{K} n_i \, \mathbb{E}[P_i] = \sum_{i=1}^{K} n_i \frac{\alpha}{\alpha+\beta}$$

$$\boxed{E[X] = \frac{\alpha}{\alpha+\beta} \sum_{i=1}^{K} n_i}$$

$$\mathrm{var}(Y) = \sum_{i=1}^{K} \mathrm{var}(x_i)$$

$$= \sum_{i=1}^{K} \mathbb{E}[\mathrm{var}(X_i \mid P_i)] + \mathrm{var}[E[X_i \mid P_i]]$$

$$= \sum_{i=1}^{K} \mathbb{E}\left[n_i p_i (1-p_i)\right] + \mathrm{Var}(n_i p_i)$$

$$= \sum_{i=1}^{K} n_i \mathbb{E}[p_i] - n_i \mathbb{E}[p_i^2] + n_i^2 \mathrm{Var}(p_i)$$

$$= \sum_{i=1}^{K} n_i \mathbb{E}[p_i] - n_i \left\{ \mathrm{Var}(p_i) + \mathbb{E}[p_i]^2 \right\} + n_i^2 \mathrm{Var}(p_i)$$

$$= \sum_{i=1}^{K} n_i \mathbb{E}[p_i] - n_i \mathbb{E}[p_i]^2 + n_i (n_i - 1) \mathrm{Var}(p_i)$$

$$= \sum_{i=1}^{K} n_i \frac{\alpha}{\alpha+\beta} - n_i \frac{\alpha^2}{(\alpha+\beta)^2} + n_i (n_i-1) \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \sum_{i=1}^{K} \frac{n_i (\alpha+\beta)(\alpha+\beta+1)}{\beta} - \frac{n_i \alpha}{\beta} (\alpha+\beta+1) + n_i (n_i - 1)$$

$$= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \sum_{i=1}^{K} \left[n_i (\alpha+\beta+1) + n_i (n_i - 1) \right]$$

$$= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \sum_{i=1}^{k} n_i(\alpha+\beta+n_i)$$

$$\Rightarrow \quad Var(X_i) = \frac{\alpha\beta \cdot n_i\,(\alpha+\beta+n_i)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

4.38 The gamma as a mixture of exponentials.

Show that in certain cases, the gamma distribution can be written as a scale mixture of exponential's, an identity suggested by different analysis of the same data. Let $f(x)$ be a gamma (α, λ) pdf.

(a) Show that if $\alpha \le 1$, then $f(x)$ can be written

$$f(x) = \int_0^{\lambda} \frac{1}{v} e^{-x/v} p_\lambda(v)\, dv$$

Where $P_\lambda(v) = \dfrac{1}{\Gamma(\gamma)\Gamma(1-\gamma)} \dfrac{v^{\gamma-1}}{(\lambda-v)^{\gamma}}, \quad 0 < v < \lambda$

Solution:

$$f(x) = \int_0^\lambda \frac{1}{v} e^{-\frac{x}{v}} P_\lambda(v)\, dv$$

$$f(x) = \int_0^\lambda \frac{1}{v} e^{-\frac{x}{v}} \frac{1}{\Gamma(\gamma)\Gamma(1-\gamma)} \frac{v^{\gamma-1}}{(\lambda-v)^{\gamma}} dv$$

$\gamma \leq 1$

$0 < v < \lambda$

$$f(x) = \frac{1}{\Gamma(\gamma)\Gamma(1-\gamma)} \int_0^\lambda \frac{1}{v} e^{-\frac{x}{v}} \frac{v^{\gamma-1}}{(\lambda-v)^{\gamma}} dv$$

$$u = \frac{x}{v} - \frac{x}{\lambda} \Rightarrow \quad du = -\frac{x}{v^2} dv \qquad u : \infty \to 0$$

$$\frac{u}{x} = \frac{1}{v} - \frac{1}{\lambda} \Rightarrow \frac{u}{x} = \frac{\lambda - v}{\lambda v} \Rightarrow \frac{v}{\lambda - v} = \frac{x}{\lambda u}$$

$$f(x) = \frac{1}{\Gamma(\gamma)\Gamma(1-\gamma)} \int_0^\infty \frac{1}{x} \cdot e^{-u-\frac{x}{\lambda}} \cdot \frac{x^\alpha}{\lambda^\alpha \cdot u^\alpha}\, du$$

$$f(x) = \frac{1}{\Gamma(\gamma)\Gamma(1-\gamma)} \frac{x^{\alpha-1} \cdot e^{-\frac{x}{\lambda}}}{\lambda^\alpha} \int_0^\infty \frac{e^{-u}}{u^\alpha}\, du$$

$$f(x) = \frac{1}{\Gamma(\gamma)\Gamma(1-\gamma)} \frac{x^{\alpha-1} \cdot e^{-\frac{x}{\lambda}}}{\lambda^\alpha} \underbrace{\int_0^\infty u^{1-\alpha-1} \cdot e^{-u}\, du}_{\text{gamma}(1-\alpha,1)}$$

$$f(x) = \frac{1}{\Gamma(\gamma)\cancel{\Gamma(1-\gamma)}} \frac{x^{\alpha-1} \cdot e^{-\frac{x}{\lambda}}}{\lambda^\alpha} \cdot \cancel{\Gamma(1-\alpha)}$$

$$\boxed{\Rightarrow \quad f(x) = \frac{1}{\Gamma(\gamma)\lambda^\alpha} x^{\alpha-1} \cdot e^{-\frac{x}{\lambda}} \qquad X \sim \text{gamma}(\alpha, \lambda)}$$

4.38 b Show that $P_\lambda(v)$ is a pdf, for $r \le 1$ by showing that

$$\int_0^\lambda P_\lambda(v)\,dv = 1$$

Solution:

$$\int_0^\lambda P_\lambda(v)\,dv = \frac{1}{\Gamma(\gamma)\,\Gamma(1-\gamma)} \int_0^\lambda \frac{v^{r-1}}{(\lambda-v)^{r}}\,dv$$

$$= \frac{1}{\Gamma(r)\,\Gamma(1-r)} \int_0^\lambda v^{r-1} \cdot (\lambda-v)^{-r}\,dv$$

$$u = \frac{v}{\lambda} \Rightarrow du = \frac{dv}{\lambda}$$

$$= \frac{1}{\Gamma(r)\,\Gamma(1-r)} \int_0^1 \lambda^{r-1} \cdot u^{r-1} \cdot \lambda^{-r} \cdot (1-u)^{-r}\,\lambda \cdot du$$

$$= \frac{1}{\Gamma(\gamma)\,\Gamma(1-\gamma)} \underbrace{\int_0^1 u^{\gamma-1} \cdot (1-u)^{-\gamma}\, du}_{\text{Beta}(\gamma,\,1-\gamma)}$$

$$= \frac{\Gamma(\gamma)\,\Gamma(1-\gamma)}{\Gamma(\gamma)\,\Gamma(1-\gamma)} = 1$$

$$\boxed{\text{Hence} \quad \int_0^{\infty} P_X(u)\, du = 1}$$

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