Conditional Probability & Independence

$(\Omega, \mathcal{F}, \mathbb{P})$ Probability space

Let $B \in \mathcal{F}$ be an event s.t $\mathbb{P}(B) > 0$

Def: The conditional Probability of $A|B$ is defined as ; where $A \in \mathcal{F}$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Caution: we cannot condition on sets of zero Probability. For example $(\Omega, \mathcal{B}, \lambda)$ we cannot condition on sets of rational

Theorem: Let $B \in \mathcal{F}$ and $\mathbb{P}(B) > 0$. Then $\mathbb{P}(\cdot|B) : \mathcal{F} \longrightarrow [0,1]$ is a Probability measure on $(\Omega, \mathcal{F})$

Proof: ① $\mathbb{P}(\phi|B) = \dfrac{\mathbb{P}(\phi \cap B)}{\mathbb{P}(B)}$

$$= \frac{0}{\mathbb{P}(B)} = 0$$

② $\mathbb{P}(\Omega|B)$

$$= \frac{\mathbb{P}(\Omega \cap B)}{\mathbb{P}(B)}$$

$$= \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1$$

③ Countable additivity Property.

Let $A_1, A_2, A_3 \ldots$ be disjoint.

we need to show

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i | B\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i|B)$$

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i \mid B\right) = \frac{\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i \cap B\right)}{\mathbb{P}(B)}$$

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

$$= \frac{\mathbb{P}\left(\bigcup_{i=1}^{\infty} (A_i \cap B)\right)}{\mathbb{P}(B)}$$

A_i's are disjoint, so $A_i \cap B$ is also disjoint

$$= \frac{\sum_{i=1}^{\infty} \mathbb{P}(A_i \cap B)}{\mathbb{P}(B)}$$

$$= \sum_{i=1}^{\infty} \frac{\mathbb{P}(A_i \cap B)}{\mathbb{P}(B)}$$

$$= \sum_{i=1}^{\infty} \mathbb{P}(A_i \mid B)$$

Properties of Conditional Probability

① The Law of Total Probability

Let $A \in \mathcal{F}$, and let $B_i \in \mathcal{F}$, $\forall i \in \mathbb{N}$ be events that partition Ω

$$\Rightarrow \bigcup_{i \in \mathbb{N}} B_i = \Omega, \quad B_i \cap B_j = \phi \quad \forall i \neq j$$

with $\mathbb{P}(B_i) > 0$, $\forall i$, then

$$\mathbb{P}(A) = \sum_{i=1}^{\infty} \mathbb{P}(A \mid B_i) \cdot \mathbb{P}(B_i)$$

in particular if $B \in \mathcal{F}$ and $0 < P(B) < 1$ then

$$\mathbb{P}(A) = \mathbb{P}(A \mid B) P(B) + \mathbb{P}(A \mid B^c) \mathbb{P}(B^c)$$

Proof:

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} A \cap B_i \right)$$

$$= \sum_{i=1}^{\infty} \mathbb{P}(A \cap B_i)$$

$$= \sum_{i=1}^{\infty} \mathbb{P}(A | B_i)\, P(B_i)$$

② Baye's Rule:

Let $A \in \mathcal{F}$, with $P(A) > 0$ and B_i, $\forall i \in \mathbb{N}$ be a Partition on Ω s.t $\mathbb{P}(B_i) > 0$ $\forall i$, Then we have

$$\mathbb{P}(B_i / A) = \frac{\mathbb{P}(A|B_i)\,\mathbb{P}(B_i)}{\sum_{j=1}^{\infty} \mathbb{P}(A|B_j)\,\mathbb{P}(B_j)}$$

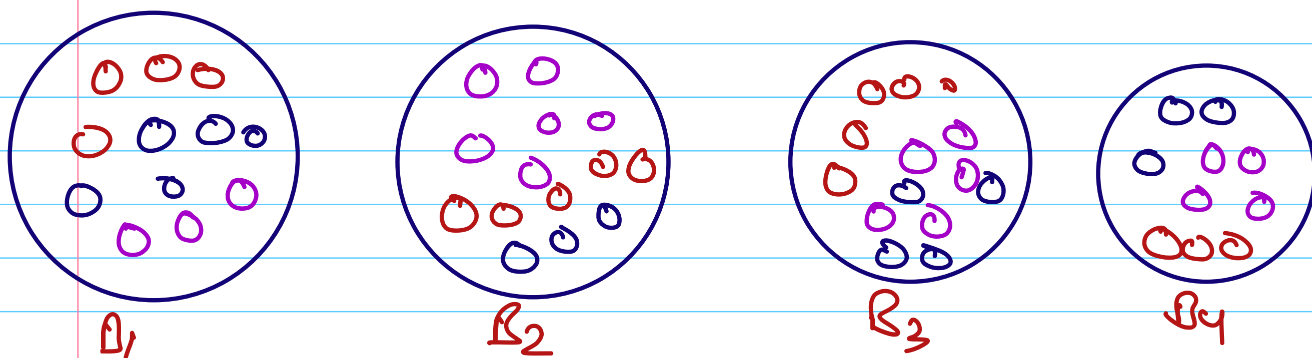
Proof:

$$\mathbb{P}(B_i | A) = \frac{\mathbb{P}(A \cap B_i)}{\mathbb{P}(A)}$$

$$= \frac{\mathbb{P}(A \cap B_i)}{\sum_{j=1}^{\infty} \mathbb{P}(A|B_j)\, P(B_j)}$$

$$= \frac{\mathbb{P}(A \mid B_j)\,\mathbb{P}(B_j)}{\sum_{j=1}^{\infty} \mathbb{P}(A \mid B_j)\, P(B_j)}$$

Ex:



$\mathbb{P}(A_1) =$ Probability of picking red ball

$\mathbb{P}(A_2) =$ " " Blue Ball

$\mathbb{P}(A_3) =$ " " Purple Ball

$\mathbb{P}(A_1 \mid B_3) =$ Probability of picking red ball, Given the we are taking from B_3

$\mathbb{P}(B_3 \mid A_1) =$ Given that we got a red ball, what is the Prob that it came from B_3.

(3) for a seq^n of events $\{A_i\}$ we have

$$\mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = \mathbb{P}(A_1) \prod_{i=2}^{\infty} \mathbb{P}(A_i \mid A_1 \cap A_2 \cdots \cap A_{i-1})$$

as long as conditional probabilities are well defined.

Ex:

$$\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1)\,\mathbb{P}(A_2 \mid A_1)\,\mathbb{P}(A_3 \mid A_1 \cap A_2)$$

Independence: of events

$(\Omega, \mathcal{F}, \mathbb{P})$ Probability space.

Two events A and B are said to be independent under $\mathbb{P}$ if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$$

Note: if $\mathbb{P}(B) > 0$ and A, B are independent, then we have

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

$$= \frac{\mathbb{P}(A)\,\mathbb{P}(B)}{\mathbb{P}(B)}$$

$$= \mathbb{P}(A)$$

Example:- Can disjoint sets be independent at all?

Let $A, B \in \mathcal{F}$ be two disjoint sets.

$$\Rightarrow \mathbb{P}(A \cap B) = \mathbb{P}(\phi)$$

$$= 0$$

for independence we need

$$\mathbb{P}(A)\,\mathbb{P}(B) = \mathbb{P}(A \cap B) = 0$$

$\Rightarrow$ This can only happen when either $\mathbb{P}(A)$ or $\mathbb{P}(B) = 0$

$\Rightarrow$ Therefore, in general, two disjoint events are independent $\Longleftrightarrow$ at least one of them has zero probability.

Def: $A_1, A_2, \ldots A_n$ are independent if $\forall$ non-empty $I_0 \subseteq \{1, 2, \ldots n\}$ we have

$$\mathbb{P}\left(\bigcap_{i \in I_0} A_i\right) = \prod_{i \in I_0} A_i$$