TRANS form's

1. Probability Generating function (PGF) (Z-transform)
2. Moment Generating function (MGF) (Laplace)
3. Characteristic Generating fun (CGF) (Fourier)

These transform techniques are at some level, we can think of some frequency domain techniques.

* Just like in signals and system's, we have time domain technique's, and frequency domain technique's, so we can think of all our PDF's and PMF's are some kind of time domain function's and these transform's as some frequency domain function's.

(Z- Transform)

Probability Generating function (PGF)

Used for discrete r.v's.

Let X be an integer valued r.v. then

$$G_X(z) \triangleq \mathbb{E}[z^X] = \sum_i z^i \, \mathbb{P}(X=i)$$

Generally z would be complex number.

Convergence:

For an non-negative valued r.v.

$\exists R$, possible $+\infty$, s.t the PGF converges for $\forall z$ s.t $|z| < R$, and diverges for $|z| > R$.

Example:

$X \sim$ Possion r.v

$$\mathbb{P}(X=i) = \frac{e^{-\lambda} \lambda^i}{i!}, \quad i \geq 0$$

then $$G_X(z) = \sum_{i=1}^{\infty} \frac{z^i \lambda^i e^{-\lambda}}{i!}$$

$$= \sum_{i=1}^{\infty} \frac{(z\lambda)^i e^{-\lambda z} e^{-\lambda} \cdot e^{\lambda z}}{i!}$$

$$= \sum_{i=1}^{\infty} \frac{(z\lambda)^i e^{-\lambda z}}{i!} \cdot e^{\lambda(z-1)}$$

$$= 1 \cdot e^{\lambda(z-1)}$$

$$= e^{\lambda(z-1)} \quad , \forall c \in \mathbb{C}$$

Example:

$X \sim$ geometric r.v

$$\mathbb{P}(X = i) = (1-p)^{i-1} p \quad , i \geq 1$$

then

$$G_X(z) = \sum_{i=1}^{\infty} (1-p)^{i-1} p \cdot z^i$$

$$= \sum_{i=1}^{\infty} ((1-p)z)^{i-1} pz$$

$$= \sum_{i=1}^{\infty} \frac{(z - zp)^{i-1}(1 - z - zp)pz}{(1 - z(1-p))}$$

$$= \frac{pz}{1 - z(1-p)} \quad \forall\ 0 < z - zp < 1$$

$$\Rightarrow \quad |z| < \frac{1}{1-p}$$

Properties of PGF

1. $G_X(1) = 1$

2. $\left.\frac{d}{dz} G_X(z)\right|_{z=1} = \mathbb{E}[X]$

Proof: $G_X(z) = \mathbb{E}[z^X]$

$$= \sum_i z^i \, \mathbb{P}(X=i)$$

$$\frac{d}{dz} G_X(z) = \frac{d}{dz} \sum_i z^i \, \mathbb{P}(X=i)$$

$$= \sum_i \frac{d}{dz} z^i \, \mathbb{P}(X=i)$$

$$= \sum_i i z^{i-1} \, \mathbb{P}(X=i)$$

thus $\displaystyle \frac{d}{dz} G_X(z) \Big|_{z=1} = \sum_i i \, \mathbb{P}(X=i)$

$$= \mathbb{E}[X]$$

(3) $$\frac{d^k G_X(z)}{dz^k} \Big|_{z=1} = \mathbb{E}\left[X(X-1)(X-2)\cdots(X-k+1)\right]$$

(4) if X and Y are independent

and $Z = X + Y$ then

$$G_Z(z) = G_X(z) \, G_Y(z)$$

Proof:

$$G_Z(z) = \mathbb{E}\left\{ z^Z \right\}$$

$$= \mathbb{E}\left[z^{X+Y}\right]$$

$$= \mathbb{E}\left[z^{X} \cdot z^{Y}\right]$$

$$= \mathbb{E}\left[z^{X}\right] \cdot \mathbb{E}\left[z^{Y}\right]$$

$$= G_X(z) \cdot G_Y(z)$$