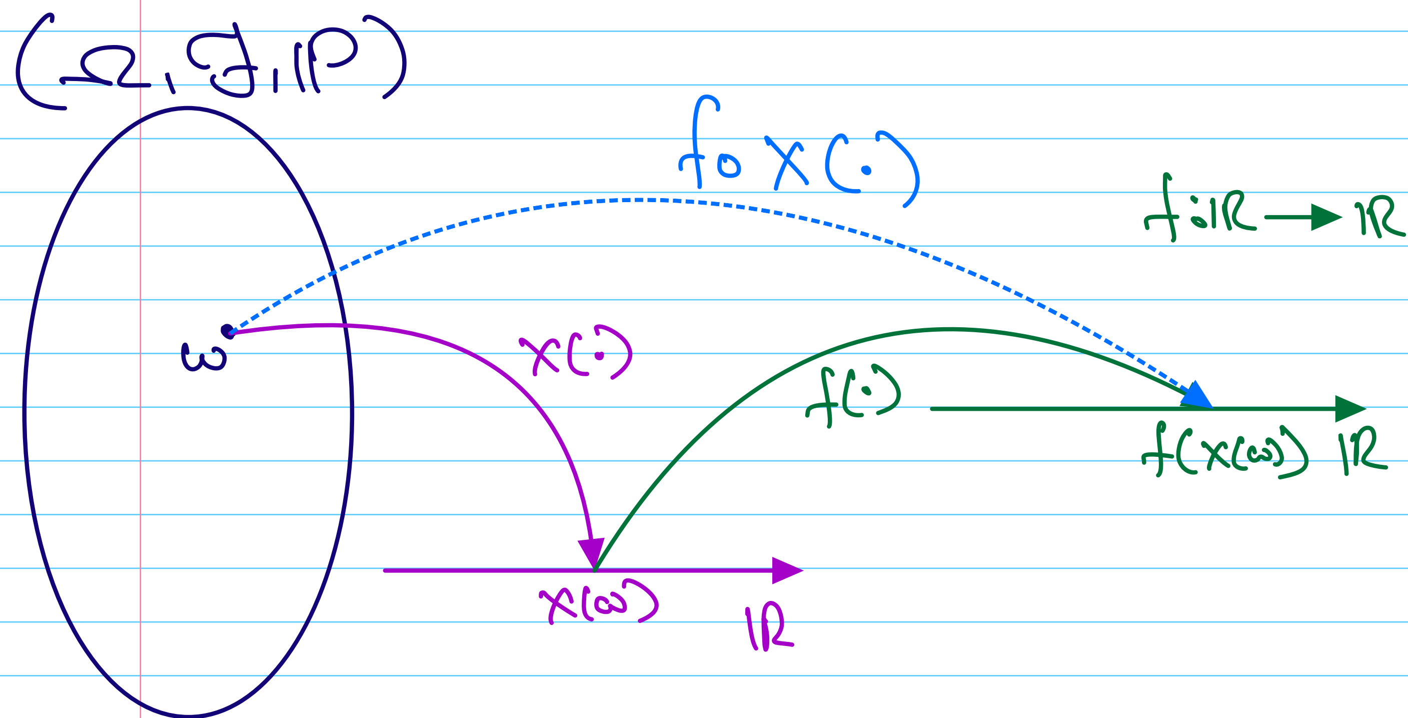
TRANSFORMATIONS



Suppose we are able to observe a r.v. in many practicle situation's, we may be more interested in some function of the obsererved r.v.

Consider a r.v $X : \Omega \to \mathbb{R}$ and let $g : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. Then $Y = g(X)$ is also a r.v and we wish to find the distribution of Y. Specifically, we are interested in finding the CDF of $F_Y(y)$

Given that CDF $F_X(x)$.

Formally, if we write $y = g(x)$, the function $g(x)$ defines mapping from the original sample space of X, $\mathcal{X}$, to a new sample space, $\mathcal{Y}$, The sample space of the random variable Y.

$$g(x) : \mathcal{X} \longrightarrow \mathcal{Y}$$

g^{-1}, mapping from subsets of $\mathcal{Y}$ to subsets of $\mathcal{X}$, defined by

$$g^{-1}(A) = \{x \in \mathcal{X}, g(x) \in A\}$$

Let $A \subset \mathcal{Y}$, then

$$P(Y \in A) = P(g(X) \in A)$$

$$= P(\{x \in \mathcal{X} : g(x) \in A\})$$

$$= P\left(X \in g^{-1}(A)\right)$$

X is a discrete random variable

$\Rightarrow$ The sample space of X is $\mathcal{X}$ is countable.

$\Rightarrow$ The sample space for $Y = g(X)$ is

$$\mathcal{Y} = \{y : y = g(x),\ x \in \mathcal{X}\}$$

which is also a countable set.

$\Rightarrow$ Y is a discrete r.v.

$\Rightarrow$ PMF of Y is

$$f_Y(y) = P(Y = y) = \sum_{x \in g^{-1}(y)} P(X = x)$$

$$= \sum_{x \in g^{-1}(y)} f_X(x)$$

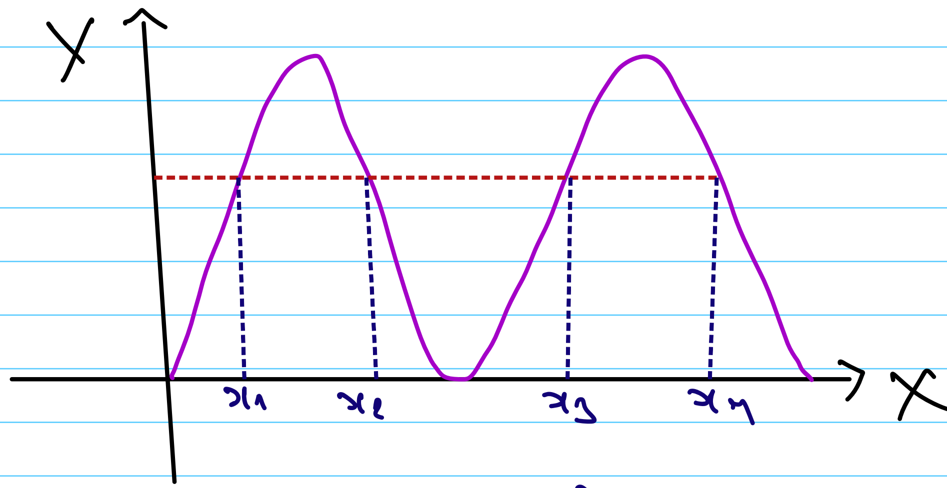
$\forall y \in \mathcal{Y}$

Example 2.1.2 (uniform transformation)

$$X \sim \text{unif}(0, 2\pi)$$

$$f_X(x) = \begin{cases} \frac{1}{2\pi} & 0 < x < 2\pi \\ 0 & \text{o.w} \end{cases}$$

$$Y = \sin^2(x)$$



The CDF of $Y = g(x)$ is

$$F_Y(y) = P(Y \leq y)$$

$$= \mathbb{P}(g(x) \leq y)$$

$$= \mathbb{P}(\{x \in X : g(x) \leq y\})$$

$$= \int_{\{x \in X : g(x) \leq y\}} f_X(x)\,dx$$

$$P(Y \le y) = \mathbb{P}(X \le x_1) + \mathbb{P}(x_2 \le X \le x_3) + \mathbb{P}(X \geq x_4)$$

$X \sim$ uniform distribution

$\Rightarrow \mathbb{P}(X \le x_1) = \mathbb{P}(X \geq x_4)$

$\Rightarrow \mathbb{P}(x_2 \le X \le x_3) = 2P(x_2 \le X \le \pi)$

$\vdots$

CDF of Y is not simple..

* $Y = g(X)$

$g : \mathcal{X} \longrightarrow \mathcal{Y}$

$\mathcal{X} = \{x : f_X(x) > 0\}$ support set.

$\mathcal{Y} = \{y : y = g(x)$ for some $x \in \mathcal{X}\}$

It is easiest to deal with functions $g(x)$ that are monotone, that is, those that satisfy either

$u > v \Rightarrow g(u) > g(v)$ (monotone increasing)

or

$u < v \Rightarrow g(u) > g(v)$ (monotone decreasing)

$\Rightarrow$ if the transformation $x \longrightarrow g(x)$ is monotone, then it is one-to-one, onto from $\mathcal{X} \longrightarrow \mathcal{Y}$.

$\Rightarrow$ That is, each x goes to only one y and each y comes from at most one x.

$\Rightarrow$ Transformation g uniquely pairs x's and y's.

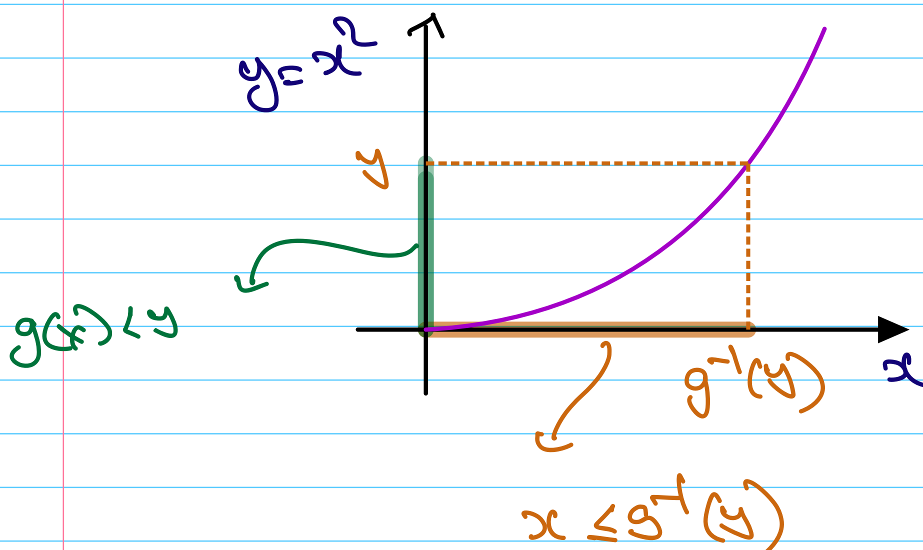
if $g(\cdot)$ is monotone increasing:

$$\{x \in \mathcal{X} : g(x) \le y\} = \{x \in \mathcal{X} : g^{-1}(g(x)) \le g^{-1}(y)\}$$

$$= \{x \in \mathcal{X} : x \le g^{-1}(y)\}$$

Ex:

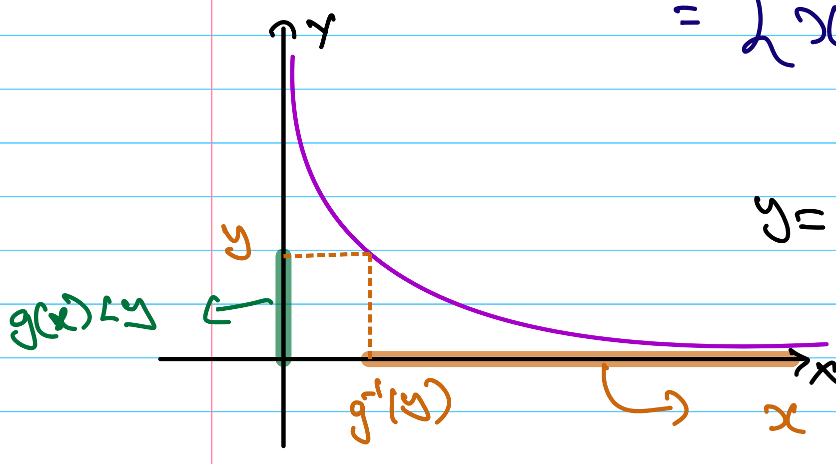
$g(x) = x^2 \qquad \mathcal{X} = (0, \infty)$



if $g(x)$ is monotone decreasing function:

$$\{x \in \mathcal{X} : g(x) \le y\} = \{x \in \mathcal{X} : g^{-1}(g(x)) \ge g^{-1}(y)\}$$

$$= \{x \in \mathcal{X} : x \ge g^{-1}(y)\}$$



$y = g(x) = e^{-x} \qquad x \in (0, \infty)$

$x > g^{-1}(y)$

CASE I if $g(x)$ is monotone increasing function

$$F_Y(y) = \int_{\{x \in \mathcal{X} : x \le g^{-1}(y)\}} f_X(x)\,dx = \int_{-\infty}^{g^{-1}(y)} f_X(x)\,dx$$

$$= F_X\left(g^{-1}(y)\right)$$

CASE II if $g(x)$ is monotone decreasing function

$$F_Y(y) = \int_{\{x \in \mathcal{X} : x > g^{-1}(y)\}} f_X(x)\,dx = \int_{g^{-1}(y)}^{\infty} f_X(x)\,dx$$

$$= 1 - F_X\left(g^{-1}(y)\right)$$

Theorem 2.1.3 :-

Let X have cdf $F_X(x)$, let $Y = g(X)$, and $\mathcal{X}$ and $\mathcal{Y}$ be defined as

$\mathcal{X} = \{x : f_X(x) > 0\}$ and $\mathcal{Y} = \{y : y = g(x) \text{ for some } x \in \mathcal{X}\}$

then

ⓐ if g is an increasing function on $\mathcal{X}$, then

$$F_Y(y) = F_X(g^{-1}(y)) \quad \text{for } y \in \mathcal{Y}$$

ⓑ if g is decreasing function on $\mathcal{X}$ and X is a continuous r.v, $F_Y(y) = 1 - F_X(g^{-1}(y))$ for some $y \in \mathcal{Y}$.

Example 2.1.4 :- (uniform- exponential relation)

$X \sim \text{Unif}(0,1)$

$$\Rightarrow f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{o.w} \end{cases}$$

$$\Rightarrow F_X(x) = \begin{cases} x & 0 < x < 1 \\ 1 & \text{o.w} \end{cases}$$

$\mathcal{X} = (0,1)$ support set

$$Y = g(x) = -\log x$$

$$\Rightarrow \quad \frac{d}{dx} g(x) = \frac{d}{dx} -\log x = -\frac{1}{x} \quad \text{for} \quad 0<x<1$$

$$\Rightarrow \quad \frac{d}{dx} g(x) = -\frac{1}{x} < 0 \quad \forall x \in (0,1)$$

$\Rightarrow$ $g(x)$ is a decreasing function.

$x \in (0,1) \Rightarrow y = g(x) = -\log x$

$y \in (0, \infty)$

$$x = g^{-1}(y) = e^{-y}$$

then $F_Y(y) = 1 - F_X(g^{-1}(y))$

$= 1 - F_X(e^{-y})$

$$F_Y(y) = 1 - e^{-y}$$

Theorem 2.1.5: Let X have pdf $f_X(x)$ and let $Y = g(X)$, where $g(x)$ is monotone function. Suppose that $f_X(x)$ is continuous function on $\mathcal{X}$ and that $g^{-1}(y)$ has a continuous derivative on $\mathcal{Y}$. Then the pdf of Y is given by

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| & y \in \mathcal{Y} \\ 0 & \text{o.w} \end{cases}$$

Proof: from theorem 2.1.3 we have,

if $g(x)$ is monotonous increasing function.

$$\Rightarrow \quad F_Y(y) = F_X(g^{-1}(y))$$

$$\Rightarrow \quad f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$= \frac{d}{dy} F_X(g^{-1}(y))$$

$$= F_X'(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

$$= f_X(g^{-1}(y)) \cdot \frac{d}{dy} g^{-1}(y)$$

if $g(x)$ is monotone decreasing

$$\Rightarrow f_Y(y) = \frac{d}{dy}\left(1 - F_X(g^{-1}(y))\right)$$

$$= - F_X'(g^{-1}(y)) \cdot \underbrace{\frac{d}{dy} g^{-1}(y)}_{\text{-ve as}}$$

$$= f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$

Example 2.1.6 inverted gamma Pdf

$X \sim$ gamma distribution.

$$f(x) = \frac{1}{(n-1)!\,\beta^n} x^{n-1} e^{-x/\beta}, \quad 0 < x < \infty$$

$\beta \in \mathbb{R}_+$, $n \in \mathbb{Z}^+$

find the Pdf of $g(x) = \frac{1}{x}$

$$x = (0, \infty) \qquad y = (0, \infty)$$

$$\Rightarrow y = g(x) = \frac{1}{x}$$

$$x = g^{-1}(y) = \frac{1}{y}, \quad \frac{d}{dy} g^{-1}(y) = -\frac{1}{y^2}$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$= \frac{1}{(n-1)!\,\beta^n} \left(\frac{1}{y}\right)^{n-1} e^{-\frac{1}{y\beta}} \cdot \frac{1}{y^2}$$

$$= \frac{1}{(n-1)!\,\beta^n} \frac{e^{-\frac{1}{y\beta}}}{y^{n+1}}$$

$$f_Y(y) = \frac{1}{(n-1)!\,\beta^n} \left(\frac{1}{y}\right)^{n+1} e^{-\frac{1}{y\beta}}$$

$$y = (0, \infty)$$

Theorem 2.1.8

Let X have pdf $f_X(x)$, let $Y = g(X)$, and define the sample space $\mathcal{X}$ as in (2.1.7), suppose there exists a partition, $A_0, A_1, \ldots A_k$ of $\mathcal{X}$ s.t $P(X \in A_0) = 0$ and $f_X(x)$ is continous on A_i. Further $\exists$ function's $g_1(x), g_2(x) \ldots g_k(x)$, defined on $A_1, \ldots A_k$, respectively, satisfying

(i) $g(x) = g_i(x) \quad \forall x \in A_i$

(ii) $g_i(x)$ is monotone on A_i

(iii) $g_i^{-1}(y)$ has continous derivative on $\mathcal{Y}$

Then

$$f_Y(y) = \begin{cases} \sum_{i=1}^{k} f_X(g_i^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| & y \in \mathcal{Y} \\ 0 & o.w \end{cases}$$

Example 7.1.9 (Normal Chi squared relationship)

$X \sim N(0,1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad -\infty < x < \infty$$

Now $Y = g(X) = X^2$

$\Rightarrow \quad \mathcal{Y} = (0, \infty)$

and $g(x) = x^2$ is monotone on $(-\infty, 0)$ and on $(0, \infty)$

$A_0 = \{0\}$;

$A_1 = (-\infty, 0)$, $g_1(x) = x^2$ $\quad g_1^{-1}(y) = -\sqrt{y}$

$A_2 = (0, \infty)$, $g_2(x) = x^2$ $\quad g_2^{-1}(y) = \sqrt{y}$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(-\sqrt{y})^2}{2}} \cdot \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{y})^2}{2}} \cdot \frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{y}} \cdot e^{-y/2} \qquad y \in (0, \infty)$$

Theorem 2.1.10 (Probability integral transformation)

$X \sim F_X(x)$ continuous and

$Y = F_X(X)$. Then Y is uniformly distributed on $(0,1)$

$\Rightarrow \quad P(Y \leq y) = y \quad 0 < y < 1$