EXPONENTIAL FAMILIES

— Casella - Berger Book

— 3.4 section

A family of PDf's, PMfs is called an exponential family if it can be expressed as

$$f(x|\theta) = h(x)\, c(\theta) \exp\left(\sum_{i=1}^{k} \omega_i(\theta)\, t_i(x)\right)$$

Here * $h(x) \geq 0$

* $t_1(x), \ldots t_k(x)$ are real-valued function's of observation x

* $c(\theta) \geq 0$

* $\omega_1(\theta), \ldots \omega_k(\theta)$ real-valued function's of θ

Example 3.4.1 (Binomial exponential family)

$$f(x|p) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \frac{n!}{(n-x)!\,x!} p^x (1-p)^{n-x}$$

$$= \binom{n}{x} \exp\left(x \log p + (n-x) \log(1-p) \right)$$

$$= \binom{n}{x} \exp\left(x \log \frac{p}{1-p} + n \log(1-p) \right)$$

$$= \binom{n}{x} (1-p)^n \exp\left(x \cdot \log \frac{p}{1-p} \right)$$

$$h(x) = \binom{n}{x}$$

$$c(\theta) = (1-p)^n$$

$$t_1(x) = x$$

$$w_1(\theta) = \log\left(\frac{p}{1-p}\right)$$

Theorem 3.4.2 :-

if X is a r.v with pdf or pmf of the exponential family form.

$$\mathbb{E}\left[\sum_{i=1}^{k} \frac{\partial w_i(\theta)}{\partial(\theta_j)} t_i(x)\right) = \frac{\partial}{\partial\theta_j} \log c(\theta)$$

$$\operatorname{var}\left[\sum_{i=1}^{k} \frac{\partial w_i(\theta)}{\partial(\theta_j)} t_i(x)\right) = -\frac{\partial^2}{\partial\theta_j^2} \log c(\theta)$$

$$-\mathbb{E}\left[\sum_{i=1}^{k} \frac{\partial^2 w_i(\theta)}{\partial\theta_j^2} t_i(x)\right]$$

Although these eq^n formidable, when applied to specific cases they can work out quite nicely.

(Proof is in Exercise 3.31 Check solution's I uploaded)

Example 3.4.3 (Binomial mean and variance)

$$f(x|p) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$h(x) = \binom{n}{x}$$

$$c(\theta) = (1-p)^n$$

$$w_1(p) = \log\left(\frac{p}{1-p}\right)$$

$$t_1(x) = x$$

$$\frac{\partial w_1(p)}{\partial p} = \frac{1-p}{p} \cdot \frac{(1-p)+p}{(1-p)^2}$$

$$= \frac{1}{p(1-p)}$$

$$-\frac{\partial}{\partial p} \log c(p) = \frac{+n}{1-p}$$

$$\mathbb{E}\left[\frac{\partial w_1(p)}{\partial p} t_1(x)\right] = \mathbb{E}\left[\frac{1}{p(1-p)} \cdot x\right]$$

$$= \frac{1}{p(1-p)} \mathbb{E}[x] = \frac{np}{p(1-p)}$$

$$\Rightarrow \boxed{\mathbb{E}\left[\frac{\partial w_1(p)}{\partial p} t_1(x)\right] = \frac{n}{1-p} = -\frac{\partial}{\partial p}\log c(p)}$$

Proved.

$$-\frac{\partial^2}{\partial p^2}\log c(p) = -\frac{\partial}{\partial p}\left(\frac{-n}{1-p}\right)$$

$$= n\frac{\partial}{\partial p}\left(\frac{1}{1-p}\right)$$

$$= \frac{-np}{(1-p)^2}$$

$p - p^2$

$$\frac{\partial^2 w(p)}{\partial p^2} = \frac{\partial}{\partial p}\left(\frac{1}{p(1-p)}\right)$$

$$= \frac{-(1-2p)}{p^2(1-p)^2} = \frac{2p-1}{p^2(1-p)^2}$$

$$\mathbb{E}\left(\frac{\partial^2 w(p)}{\partial p^2}\right) = \frac{n(2p-1)}{p(1-p)^2}$$

$$\Rightarrow -\frac{\partial^2}{\partial p^2}\log c(p) - \mathbb{E}\left[\frac{\partial^2 w(p)}{\partial p^2}\right]$$

$$= \frac{-np}{(1-p)^2} - \frac{n(2p-1)}{p(1-p)^2}$$

$$= \frac{-np^2 - n(2p-1)}{p(1-p)^2}$$

$$\vdots$$

Example 3.4.4 (Normal exponential family)

$$X \sim N(\mu, \sigma^2)$$

$$\theta = (\mu, \sigma)$$

$$-\infty < \mu < \infty, \; \sigma > 0$$

$$f(x \mid \sigma^2, \mu) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2} + \frac{x\mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2}\right)$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) \exp\left(x \cdot \frac{\mu}{\sigma^2} - x^2 \cdot \frac{1}{2\sigma^2}\right)$$

$$h(x) = 1$$

$$c(\theta) = \frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left(\frac{-\mu^2}{2\sigma^2}\right)$$

$$w_1(\theta) = \frac{\mu}{\sigma^2} \qquad t_1(x) = x$$

$$w_2(\theta) = \frac{1}{\sigma^2} \qquad t_2(x) = \frac{-x^2}{2}$$

Definition 3.4.5:- The Indicator function of a set A, most often denoted by $I_A(x)$ is the function,

$$I_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

An alternative notation is $I(x \in A)$

for $X \sim N(\mu, \sigma^2)$

$$f(x|\mu,\sigma^2) = h(x)c(\mu,\sigma) \cdot \exp\left(w_1(\mu,\sigma)t_1(x) + w_2(\mu,\sigma)t_2(x)\right) \cdot I_{(-\infty,\infty)}(x)$$

Location and scale Families

Theorem 3.5.1:

Let $f(x)$ be any pdf or pmf and let μ and $\sigma > 0$ be any given constants. Then the function

$$g(x \mid \mu, \sigma) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$$

is a pdf.

Proof:

$$\int_{-\infty}^{\infty} \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) dx$$

$$y = \frac{x-\mu}{\sigma}$$

$$dy = \frac{dx}{\sigma}$$

$$\Rightarrow \int_{-\infty}^{\infty} f(y)\, dy = 1$$

Theorem 3.5.6 :- Let $f(\cdot)$ be any Pdf. Let μ be any real number, and let σ be any positive real number. Then X is a r.v with Pdf $\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$ iff $\exists$ a r.v Z with Pdf $f(z)$ and $X = \sigma Z + \mu$

Proof:

$\Rightarrow$ $\quad g(z) = \sigma z + \mu = X$

$$\Rightarrow \quad z = g^{-1}(x) = \frac{x-\mu}{\sigma}$$

$$f_X(x) = f_Z(z) \cdot \left| \frac{d}{dx} g^{-1}(x) \right|$$

$$= f\left(\frac{x-\mu}{\sigma}\right) \cdot \frac{1}{\sigma}$$

$\Leftarrow$

$$\text{define} \quad g(x) = \frac{x-\mu}{\sigma} = z$$

$$x = g^{-1}(z) = \sigma z + \mu$$

$$f_Z(z) = f_X(\sigma z + \mu) \cdot \sigma$$

$$= \frac{1}{\sigma} f\left(\frac{\sigma z + \mu - \mu}{\sigma}\right) \cdot \sigma = f(z)$$

Theorem 3.5.7:-

$Z \sim$ r.v $f(z)$. Suppose $E(Z)$ and $Var(Z)$ exists

$$X \sim \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$$

Proof:

$$X = \sigma Z + \mu$$

$$E[X] = \mathbb{E}[\sigma Z + \mu] = \sigma \mathbb{E}[Z] + \mu$$

$$Var(X) = Var(\sigma Z + \mu)$$
$$= \sigma^2 Var(Z)$$

Probabilities for any member of a location-scale family may be computed in term's of the standard variable Z

$$\Rightarrow \mathbb{P}(X \leq x) = \mathbb{P}\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right)$$

$$= \mathbb{P}\left(Z \leq \frac{x-\mu}{\sigma}\right)$$