5.5 Convergence Concept's

Def: A seq^n of r.v's $(X_n)_{n \in \mathbb{N}}$ converges in Probability to a r.v X if, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} P(|X_n - X| \geq \varepsilon) = 0$$

or

$$\lim_{n \to \infty} \mathbb{P}(|X_n - X| < \varepsilon) = 1$$

Theorem 5.5.2: (Weak Law of Large Number's)

Let $X_1, X_2, \ldots \overset{iid}{\sim} X$

$\mathbb{E}[X_i] = \mu$ and $var(X_i) = \sigma^2 < \infty$

Define $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$. Then

$\forall \varepsilon > 0$, $\lim_{n \to \infty} \mathbb{P}(|\bar{X}_n - \mu| < \varepsilon) = 1$

that is $\overline{X}_n$ converges in Probability to μ.

<u>Proof:</u> Apply Chebychev's inequality.

$$\mathbb{P}(|\overline{X}-\mu| \geq \varepsilon)$$

$$\Rightarrow \mathbb{P}(|\overline{X}-\mu|^2 \geq \varepsilon^2) \leq \frac{\mathbb{E}[(\overline{X}-\mu)^2]}{\varepsilon^2}$$

$$\Rightarrow \mathbb{P}(|\overline{X}-\mu|^2 \geq \varepsilon) \leq \frac{Var(\overline{X})}{\varepsilon^2}$$

$$\Rightarrow \mathbb{P}(|\overline{X}-\mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2}$$

$$\Rightarrow \mathbb{P}(|\overline{X}-\mu| < \varepsilon) \geq 1 - \frac{\sigma^2}{n\varepsilon^2}$$

$$\lim_{n\to\infty} \mathbb{P}(|\overline{X}-\mu| < \varepsilon) \geq 1$$

$$\Rightarrow \quad \lim_{n\to\infty} \mathbb{P}(|\bar{X}-\mu| < \varepsilon) = 1$$

Ex 5.5.3

$$S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$$

$$\mathbb{E}[S_n^2] = \sigma^2$$

$$\mathbb{P}(|S_n^2 - \sigma^2| \geq \varepsilon) \leq \frac{\mathbb{E}[(S^2 - \sigma^2)^2]}{\varepsilon^2}$$

$$\Rightarrow \quad \mathbb{P}(|S_n^2 - \sigma^2| \geq \varepsilon) \leq \frac{\mathrm{var}(S^2)}{\varepsilon^2}$$

$$\lim_{n\to\infty} \mathbb{P}(|S_n^2 - \sigma^2| \geq \varepsilon) \leq \frac{\lim_{n\to\infty} \mathrm{var}(S^2)}{\varepsilon^2}$$

$$= 0$$

$$\Rightarrow \quad S_n^2 \longrightarrow \sigma^2$$

Theorem 5.5.4 :

Suppose that $X_1, X_2, \ldots$ converges in Prob to a random variable X and that h is a Continuous function. Then

$h(X_1), h(X_2) \ldots$ converges in Prob to $h(X)$

Proof: Since h is continuous,

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|X_n - X| < \delta \Rightarrow |h(X_n) - h(X)| < \varepsilon$$

We know

$$\lim_{n\to\infty} \mathbb{P}(|X_n - X| < \delta) = 1$$

$$\Rightarrow \lim_{n\to\infty} \mathbb{P}(|h(X_n) - h(X)| < \varepsilon) = 1$$

Almost Sure Convergence

Def 5.5.6 :-

A seq^n of r.v's $X_1, X_2, \dots$ converges almost surely to a r.v' X if , $\forall \varepsilon > 0$

$$\mathbb{P}\left(\lim_{n\to\infty} |X_n - X| < \varepsilon\right) = 1$$

Ex 5.5.7

$S = [0,1]$ sample space.

with uniform probability.

$$X_n(s) = s + s^n$$

$$X(s) = s$$

$\Rightarrow$ $\forall s \in [0,1)$, $s^n \to 0$ as $n \to \infty$

$$X_n(s) \longrightarrow s = X(s) \quad \forall s \in [0,1)$$

But $X_n(1) = 2 \neq X(1) = 1$

But we know

$$\mathbb{P}(X = 1) = 0$$

i.e $\mathbb{P}([0,1)) = 1$

$$\Rightarrow \quad X_n \xrightarrow{a.s} X$$

Theorem 5.5.9 (Strong Law of Large numbers)

let $X_1, X_2 \ldots$ be iid rv's with $\mathbb{E}[X_i] = \mu$ and $Var(X_i) = \sigma^2 < \infty$, and define

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \quad , \text{ Then, } \quad \forall \varepsilon > 0$$

$$\mathbb{P}\left(\lim_{n \to \infty} |\bar{X}_n - \mu| < \varepsilon \right) = 1$$

$$\bar{X}_n \xrightarrow{a.s} \mu$$

Def 5.5.10:

A seq of r.v $X_1, X_2, \dots$ converges in distribution to a r.v X if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

at all points x where $F_X(x)$ is continuous.

Theorem 5.5.14 (Central Limit theorem)

let $X_1, X_2, \dots$ be a seq^n of iid r.v's whose mgf's exists in a neighbourhood of 0

Let $E[X_i] = \mu$ $\quad Var(X_i) = \sigma^2 > 0$

(Both μ, σ^2 are finite)

then $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$. Let $G_n(x)$

denote the cdf of $\sqrt{n}\left(\frac{\bar{X}_n - \mu}{\sigma}\right)$

$$E[X] = \mu \qquad var(X) = \sigma^2$$

$$E[\bar{X}] = \mu \qquad var(\bar{X}) = \frac{\sigma^2}{n}$$

$$E[\bar{X} - \mu] = 0 \qquad var(\bar{X} - \mu) = \frac{\sigma^2}{n}$$

$$\mathbb{E}\left[\frac{\bar{X} - \mu}{\sigma}\right] = 0 \qquad var\left(\frac{\bar{X} - \mu}{\sigma}\right] = \frac{1}{n}$$

$$\mathbb{E}\left[\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma}\right] = 0 \qquad var\left[\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma}\right] = 1$$

$\Rightarrow$ $G_n(x)$ is cdf of $\sqrt{n}\left(\frac{\bar{X} - \mu}{\sigma}\right)$

where its mean $= 0$, var $= 1$

then

$$\lim_{n \to \infty} G_n(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy.$$

Proof: we will show for $|t| < h$

the mgf of $\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}$ is

$e^{\frac{t^2}{2}}$, the mgf of $N(0,1)$

define

$$Y_i = \frac{X_i - \mu}{\sigma}$$

$\Rightarrow$ M_{Y_i} are common mgf of Y_i

$$\Rightarrow \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}$$

$$\Rightarrow \frac{1}{\sqrt{n}}\left(\frac{X_1+X_2+\cdots+X_n-n\mu}{\sigma}\right)$$

$$\Rightarrow \frac{1}{\sqrt{n}}\left(\frac{X_1-\mu}{\sigma}+\frac{X_2-\mu}{\sigma}+\cdots+\frac{X_n-\mu}{\sigma}\right)$$

$$= \frac{1}{\sqrt{n}}\left(Y_1+Y_2+Y_3+\cdots+Y_n\right)$$

$$M_{\frac{\sum y_i}{\sqrt{n}}}(t) = \mathbb{E}\left[e^{t \cdot \frac{\sum y_i}{\sqrt{n}}}\right]$$

$$= \mathbb{E}\left[e^{\frac{t}{\sqrt{n}} y_1}\right] \mathbb{E}\left[e^{\frac{t}{\sqrt{n}} y_2}\right] \cdots$$

$$= \left(\mathbb{E}\left[e^{\frac{t}{\sqrt{n}} y_1}\right]\right)^n$$

$$= \left[M_Y\left(\frac{t}{\sqrt{n}}\right)\right]^n$$

taylor series expansion around a

$$f(x) = f(a) + f'(a)\frac{(x-a)}{1} + f''(a)\frac{(x-a)^2}{2!} + f'''(x)\frac{(x-a)^3}{3!} + \cdots$$

$$M_Y\left(\frac{t}{\sqrt{n}}\right) = M_Y(0) + \frac{M_Y'(0)\cdot\left(\frac{t}{\sqrt{n}}\right)}{1!} + \frac{M_Y''(0)\cdot\left(\frac{t}{\sqrt{n}}\right)^2}{2!}$$

$$= \sum_{k=0}^{\infty} \frac{M_Y^{(k)}(0)\cdot}{k!}\left(\frac{t}{\sqrt{n}}\right)^k$$

we have $M_Y(0) = E[1] = 1$

$M'_Y(0) = \mathbb{E}[Y] = 0$

$M''_Y(0) = \mathrm{var}(Y) = 1$

$$\Rightarrow M_Y\left(\frac{t}{\sqrt{n}}\right) = 1 + 0 + \left(\frac{t}{\sqrt{n}}\right)^2 \cdot \frac{1}{2!} + \underbrace{R_Y\left(\frac{t}{\sqrt{n}}\right)}_{\approx 0}$$

$$\Rightarrow M_Y\left(\frac{t}{\sqrt{n}}\right) = 1 + \frac{t^2}{2n} + R_Y\left[\frac{t}{\sqrt{n}}\right]$$

$$\Rightarrow M_{\frac{\Sigma x}{\sqrt{n}}} = \left[M_X\left(\frac{t}{\sqrt{n}}\right) \right]^n$$

$$\lim_{n \to \infty} M_{\frac{\Sigma x}{\sqrt{n}}} = \lim_{n \to \infty} \left[1 + \frac{1}{n}\left(\frac{t^2}{2} + n R_X\left(\frac{t}{\sqrt{n}}\right)\right)\right]^n$$

$$= \lim_{n \to \infty} \left[1 + \frac{t^2}{2} \right]^n$$

$$= e^{\frac{t^2}{2}}$$

$$= M_Z(t)$$

Example 5.5.16 (Normal approximation of Binomial distribution)

Sol^n

$$X_1, X_2, \ldots \overset{iid}{\sim} X$$

$$X \sim \text{binomial}(r, p)$$

$$\mathbb{P}(X = x) = \binom{r}{x} p^x (1-p)^{r-x}$$

$$x = 0, 1, 2, \ldots r$$

$$E[X] = \frac{r(1-p)}{p}$$

$$var(X) = \frac{r(1-p)}{p^2}$$

$$\sqrt{n}\left(\frac{\overline{X} - \mu}{\sigma}\right) \sim N(0,1)$$

$$r = 10, \quad p = \frac{1}{2}, \quad n = 30.$$

$$\mathbb{P}(\overline{X} < 11) = \mathbb{P}\left(\frac{X_1 + X_2 + \cdots + X_{30}}{30} < 11\right)$$

$$= \mathbb{P}\left(\sum x_i < 330\right)$$

$$= \sum_{x=0}^{330} \binom{300+x-1}{x} \left(\frac{1}{2}\right)^{300} \left(\frac{1}{2}\right)^{x}$$

very difficult

$$\mathbb{E}[X] = \frac{r(1-p)}{p} = \frac{10 \cdot \frac{1}{2}}{1/2} = 10$$

$$\operatorname{var}(X) = \frac{r(1-p)}{p^2} = \frac{10 \cdot \frac{1}{2}}{1/4} = 20$$

$$\Rightarrow \mathbb{P}(\overline{X} < 11) = \mathbb{P}\left(\frac{\overline{X} - 10}{\sqrt{20}} < \frac{11-10}{\sqrt{20}}\right)$$

$$= \mathbb{P}\left(\sqrt{30}\left(\frac{\overline{X}-10}{\sqrt{20}}\right) < \frac{1\sqrt{30}}{\sqrt{20}}\right)$$

$$= \mathbb{P}\left(\sqrt{30}\left(\frac{\overline{X}-10}{\sqrt{20}}\right) < 1.2247\right)$$

$$\approx \mathbb{P}(Z < 1.2247)$$

Theorem 5.5.17 (Slutsky's theorem)

if $X_n \longrightarrow X$ in distribution

and $Y_n \longrightarrow a$, a is constant,

in probability, then

(a) $Y_n X_n \longrightarrow aX$ in distribution

(b) $X_n + Y_n \longrightarrow X + a$ in distribution.

The Delta Method

Ex 5.5.19:-

$X_1, X_2, \ldots \overset{iid}{\sim} X$

$X \sim$ Bernoulli (p)

* Parameter of interest $= p$

another popular parameter $= \frac{p}{1-p}$

ie $p = \frac{2}{3}$ then $2:1$ (ratio)

to estimate $\frac{p}{1-p}$ we might start with

$$\hat{p} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

and then calculat

$$\frac{\hat{p}}{1-\hat{p}}$$

so, what's the sampling distribution of $\frac{\hat{p}}{1-\hat{p}}$, and its variance?