DESCRIPTION

This document provides solution's to 3.16 to 3.24 from chapter 3 of statistical inference Book by Casella and Berger, Focusing on continuous Probability distribution

3.16 verify these two identities regarding the gamma function that were given in the text:

a $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$

solution:

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\,dt$$

$$\Rightarrow \quad \Gamma(\alpha+1) = \int_0^\infty t^{\alpha} \cdot e^{-t}\,dt$$

integration bi-parts $u = t^{\alpha}$

$$\Gamma(\alpha+1) = t^{\alpha} \int e^{-t}\,dt \Big]_0^\infty - \int_0^\infty d \cdot t^{\alpha-1} \int e^{-t}\,dt\,dt$$

$$\Gamma(\alpha+1) = \underbrace{t^{\alpha} \cdot -e^{-t}\Big]_0^\infty}_{0} + \alpha \int_0^\infty t^{\alpha-1} \cdot e^{-t}\,dt$$

$$\Gamma(\alpha+1) = \alpha \int_0^\infty t^{\alpha-1} \cdot e^{-t}\,dt$$

$$\boxed{\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)}$$

(b) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \cdot e^{-t}\, dt$$

$$\Rightarrow \Gamma(1/2) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\, dt$$

take $t = \frac{z^2}{2}$

$$\Rightarrow \sqrt{2t} = z$$

$(2t)^{1/2}$

$$\Rightarrow d(\sqrt{2t}) = dz$$

$\frac{1}{2}(2t)^{-\frac{1}{2}} \cdot 2$

$$\Rightarrow \frac{1}{\cancel{2}} \frac{1}{\sqrt{2t}} \cdot \cancel{2}\, dt = dz$$

$$\Rightarrow \frac{dt}{\sqrt{2t}} = dz$$

$$\Rightarrow \Gamma(1/2) = \int_0^{\infty} \sqrt{2}\, e^{-\frac{z^2}{2}} dz$$

$$= \sqrt{2} \underbrace{\int_0^{\infty} e^{-\frac{z^2}{2}} dz}_{\text{standard Normal distribu.}}$$

$$= \frac{\sqrt{2}}{2} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz$$

$$= \frac{\sqrt{2}}{2} \sqrt{2\pi} = \sqrt{\pi}$$

$$\Rightarrow \boxed{\Gamma(1/2) = \sqrt{\pi}}$$

3.17 Establish the similar to (3.3.18) for the gamma distribution. if $X \sim \text{gamma}(\alpha, \beta)$ then for any positive constant ν,

$$\mathbb{E}[X^\nu] = \frac{\beta^\nu \, \Gamma(\nu+\alpha)}{\Gamma(\alpha)}$$

Solution:

$$X \sim \text{gamma}(\alpha, \beta)$$

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \cdot e^{-x/\beta}$$

$0 < x < \infty$

$\alpha > 0$

$\beta > 0$

3.3.18 in

$$E[X^n] = \frac{B(\alpha+n, \beta)}{B(\alpha, \beta)} = \frac{\Gamma(\alpha+n)\,\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+n)\cdot\Gamma(\alpha)}$$

for $X \sim \text{Beta}(\alpha, \beta)$

Expression for gamma

$$\mathbb{E}[X^\nu] = \int_0^\infty x^\nu \cdot \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \cdot e^{-x/\beta}\, dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha+\nu-1} \cdot e^{-\frac{x}{\beta}}\, dx$$

$$= \frac{\Gamma(\alpha+\vartheta) \cdot \beta^{\alpha+\vartheta}}{\Gamma(\alpha)\, \beta^{\alpha}}$$

$$= \frac{\Gamma(\alpha+\vartheta)}{\Gamma(\alpha)} \beta^{\vartheta}$$

Hence

$$\Rightarrow \boxed{E\{X^{\vartheta}\} = \frac{\beta^{\vartheta}\, \Gamma(\vartheta+\alpha)}{\Gamma(\alpha)}}$$

3.18

$Y \sim$ Negative Binomial (r, p)

Show that as $p \to 0$, the mgf of the random variable pY converges to that of a gamma distribution with parameters r, and 1

<u>Solution:</u>

$Y \sim$ negative Binomail (r, p)

$$\mathbb{P}(Y=y) = \binom{r+y-1}{y} p^r (1-p)^y$$

$y = 0, 1, 2, \ldots$

discrete r.v transformation

$$Z = g(Y) = pY$$

$\Rightarrow$ $Y = 0, 1, 2, \ldots$ (support set)

$Z = 0, p, 2p, 3p, \ldots$

$$\mathbb{P}(Z=z) = \mathbb{P}(pY=z)$$

$$= \mathbb{P}\left(Y = \frac{z}{p}\right)$$

$$\mathbb{P}(Z=z) = \binom{r+\frac{z}{p}-1}{\frac{z}{p}} p^r (1-p)^{\frac{z}{p}}$$

MGF of this r.v Z

$$M_Z(t) = \mathbb{E}\left[e^{tZ}\right]$$

$$= \sum_{Z=0,P,2P\cdots} \binom{r+\frac{Z}{P}-1}{\frac{Z}{P}} P^r (1-P)^{\frac{Z}{P}} \cdot e^{tZ}$$

$$= P^r \sum \binom{r+\frac{Z}{P}-1}{\frac{Z}{P}} \left(e^{Pt}(1-P)\right)^{\frac{Z}{P}}$$

$$= \frac{P^r}{\left[1-e^{Pt}(1-P)\right]^r}$$

$$M_Z(t) = \frac{P^r}{\left\{1-e^{Pt}(1-P)\right\}^r}$$

$$\lim_{P\to 0} M_Z(t) = \frac{P^r}{\left[1-e^{Pt}(1-P)\right]^r}$$

need to use L'Hopital rule.

$$\lim_{p \to 0} \left(\frac{p}{1 - e^{pt}(1-p)} \right)^{r}$$

$\Rightarrow$ use L' Hopital

$$\Rightarrow \lim_{p \to 0} \left(\frac{1}{-te^{pt}(1-p) + e^{pt}} \right)^{r}$$

$$\Rightarrow \left(\frac{1}{-te^{0t}(1-0) + e^{0t}} \right)^{r}$$

$$\Rightarrow \left(\frac{1}{1-t} \right)^{r}$$

$$\lim_{p \to 0} M_{pY}(t) = \left(\frac{1}{1-t} \right)^{r}$$

Mgf of gamma (α, β)

$$= \left(\frac{1}{1-\beta t} \right)^{r} \quad t < \frac{1}{\beta}$$

therefor

$$\boxed{\lim_{r\to\infty} M_{rY}(t) = \left(\frac{1}{1-t}\right)^{r} = \text{Mgf of gamma}(r,1)}$$

3.19 Show that

$$\int_x^\infty \frac{1}{\Gamma(\alpha)} z^{\alpha-1} \cdot e^{-z}\, dz = \sum_{y=0}^{\alpha-1} \frac{x^y e^{-x}}{y!}, \alpha = 1,2,3\ldots$$

Express this formula as a Probabilistic relationship b/w Poisson and gamma r.v.

<u>solution:</u>

$$\int_x^\infty \frac{1}{\Gamma(\alpha)} z^{\alpha-1} e^{-z}\, dz$$

$\alpha = 1,2,3,\ldots$

integer

$$\Rightarrow \quad \Gamma(\alpha) = (\alpha-1)!$$

$$\int_x^\infty \frac{1}{(\alpha-1)!} z^{\alpha-1} \cdot e^{-z} \, dz$$

integration by parts

$$= \frac{1}{(\alpha-1)!}\left[\left[z^{\alpha-1} \cdot -e^{-z} \right]_x^\infty + \int_x^\infty (\alpha-1) z^{\alpha-2} e^{-z} \, dz \right]$$

$$= \frac{1}{(\alpha-1)!}\left[x^{\alpha-1} \cdot e^{-x} + (\alpha-1) \int_x^\infty z^{\alpha-2} \cdot e^{-z} \, dz \right]$$

<u>sub-calculation</u>

$$\int_x^\infty z^{\alpha-2} \cdot e^{-z} \, dz = x^{\alpha-2} \cdot e^{-x} + (\alpha-2) \int_x^\infty z^{\alpha-3} \cdot e^{-z} \, dz$$

$$\Rightarrow \frac{1}{\Gamma(\alpha)} \int_x^\infty z^{\alpha-1} \cdot e^{-z}\, dz$$

$$= \frac{x^{\alpha-1} e^{-x}}{(\alpha-1)!} + \frac{1}{\Gamma(\alpha-1)} \int_x^\infty z^{\alpha-2} e^{-z}\, dz$$

$$= \frac{x^{\alpha-1} e^{-x}}{(\alpha-1)!} + \frac{x^{\alpha-2} e^{-x}}{(\alpha-2)!} + \frac{x^{\alpha-3} e^{-x}}{(\alpha-3)!}$$

$$+ \cdots + \underbrace{\frac{1}{\Gamma(\alpha-(\alpha-1))} \int_x^\infty z^{\alpha-\alpha} \cdot e^{-z}\, dz}_{e^{-x}}$$

$$= \sum_{y=1}^{\alpha} \frac{x^{y-1} e^{-x}}{(y-1)!}$$

$$\boxed{\Rightarrow \sum_{y=0}^{\alpha-1} \frac{x^{y} e^{-x}}{y!} \qquad \alpha = 1, 2, \ldots}$$

3.20 Let the random variable X have the pdf

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad 0 < x < \infty$$

(a) find the mean and variance of X

(folded normal distribution)

Solution:

$$\mathbb{E}[X] = \int_0^\infty x \cdot \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx$$

$$\frac{x^2}{2} = u$$

$$x\,dx = u$$

$$= \frac{2}{\sqrt{2\pi}} \int e^{-u} du$$

$$= \frac{2}{\sqrt{2\pi}} \left. -e^{-\frac{x^2}{2}} \right]_0^\infty$$

$$= \frac{2}{\sqrt{2\pi}}\left[0-(-1)\right] = \frac{2}{\sqrt{2\pi}}$$

$$\boxed{\mathbb{E}[X] = \frac{2}{\sqrt{2\pi}}}$$

$$\mathbb{E}[X^2] = \int_0^\infty \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} x^2 \, dx$$

$$= \frac{2}{\sqrt{2\pi}}\left[\int_0^\infty x^2 \cdot e^{-\frac{x^2}{2}} \, dx\right]$$

$$= \frac{2}{\sqrt{2\pi}}\left[x \cdot \int x \cdot e^{-\frac{x^2}{2}} dx \Big]_0^\infty - \int_0^\infty \int x \cdot e^{-\frac{x^2}{2}} \, dx\right]$$

$$\boxed{\int x \cdot e^{-\frac{x^2}{2}} \, dx = -e^{-\frac{x^2}{2}}}$$

$$= \frac{2}{\sqrt{2\pi}} \left[\underbrace{x \cdot -e^{-\frac{x^2}{2}} \Big]_0^{\infty}}_{\Rightarrow\, 0} + \underbrace{\int_0^{\infty} e^{-\frac{x^2}{2}}\, dx}_{\Rightarrow\, \text{pdf} \,\Rightarrow\, \frac{\sqrt{2\pi}}{2}} \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[0 + \frac{\sqrt{2\pi}}{2} \right] = 1$$

$$\text{var}(X) = \mathbb{E}[X^2] - E[X]^2$$

$$= \left(1 - \frac{2}{\pi} \right)$$

$$\Rightarrow \boxed{\text{var}(X) = 1 - \frac{2}{\pi}}$$

(b) if $X \sim$ folded Normal, find the transformation $g(X) = Y$ and values of α, and β so that $Y \sim \text{gamma}(\alpha, \beta)$

Solution:

$Y \sim gamma(\alpha, \beta)$

$$f(y) = \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta}$$

$0 < y < \infty$

$\alpha > 0$

$\beta > 0$

$X \sim$ folded Normal

$$f_X(x) = \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}} \qquad 0 < x < \infty$$

for $g(x) = Y$ pdf of Y is

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$\frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta} = \sqrt{\frac{2}{\pi}} e^{-\frac{(g^{-1}(y))^2}{2}} \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$

taking $g^{-1}(y) = \sqrt{y}$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} e^{-\frac{y}{2}} \cdot \frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{\pi}\sqrt{2}} \cdot y^{\frac{1}{2}-1} \cdot e^{-\frac{y}{2}}$$

$\boxed{\Gamma(1/2) = \sqrt{\pi}}$ $\quad \beta = 2$

$$= \frac{1}{\Gamma(1/2)(2)^{1/2}} \cdot y^{1/2-1} e^{-\frac{y}{2}}$$

$$= \text{gamma}(1/2, 2)$$

therefore $g(x) = Y = X^2$

and $\alpha = 1/2$, $\beta = 2$

3.21 write the integral that would define the mgf of the pdf

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2}$$

is the integral finite? (Do you expect it to be?)

Solution:

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{1}{1+x^2} e^{tx}\, dx$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{tx}}{1+x^2}\, dx$$

this is a Cauchy Distribution which mean and mgf does not exists.

3.22 For each of the following distribution's verify the formula's for $\mathbb{E}[X]$ and $Var(X)$ given in the text

(a) verify $Var(X)$ $X \sim Poisson(\lambda)$

$$\mathbb{E}[X(X-1)] = \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\lambda} \cdot \lambda^{x-2} \cdot \lambda^{2}}{(x-2)!}$$

$$= \lambda^2 \underbrace{\sum_{x=2}^{\infty} \frac{e^{-\lambda} \lambda^{x-2}}{(x-2)!}}_{\text{Pdf}}$$

$$= \lambda^2$$

$$\text{var}(X) = \mathbb{E}[X(X-1)] + \mathbb{E}[X] - E[X]^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$\boxed{\text{var}(X) = \lambda}$$

(b) verify $\text{var}(X)$ of $X \sim$ negative binomial (r, p)

<u>Solution:</u>

$$\mathbb{E}[X(X-1)] = \sum_{x=0}^{\infty} x(x-1) \binom{r+x-1}{x} p^{r} (1-p)^{x}$$

$$= \sum_{x=2}^{\infty} x(x-1) \frac{(r+x-1)!}{x!\,(r-1)!} p^{r} (1-p)^{x}$$

$$= \sum_{x=2}^{\infty} \frac{(r+x-1)!}{(x-2)!\,(r+1)!} p^{r+2} (1-p)^{x-2} \frac{(1-p)^2}{p^2} r(r+1)$$

$$= \left(\frac{1-p}{p}\right)^2 r(r+1) \cdot \underbrace{\sum_{x=2}^{\infty} \binom{r+x-1}{x-2} p^{r+2} (1-p)^{x-2}}_{-\text{ve Binomial } (r+2,\, p)}$$

$$\mathbb{E}[X(X-1)] = \frac{(1-p)^2}{p} r(r+1)$$

$$\operatorname{Var}(X) = \mathbb{E}[X(X-1)] + E[X] - E[X]^2$$

$$= \left(\frac{1-p}{p}\right)^2 \cdot r(r+1) + r\,\frac{1-p}{p} - r^2\left(\frac{1-p}{p}\right)^2$$

$$\boxed{\operatorname{Var}(X) = \frac{r(1-p)}{p^2}}$$

(c) verify $var(X)$, $X \sim gamma(\alpha, \beta)$

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$$

$0 < x < \infty$

$\alpha > 0, \beta > 0$

$$\mathbb{E}[x^2] = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^2 x^{\alpha-1} e^{-x/\beta} dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha+2-1} e^{-x/\beta} dx$$

$$= \frac{\Gamma(\alpha+2)\beta^{\alpha+2}}{\Gamma(\alpha)\beta^\alpha}$$

$$\mathbb{E}[x^2] = (\alpha+1)\alpha \cdot \beta^2$$

$$var(x) = \mathbb{E}[x^2] - \mathbb{E}[x]$$

$$= (\alpha+1)\alpha \cdot \beta^2 - \alpha^2\beta^2$$

$$= \alpha\beta^2$$

$$\boxed{var(x) = \alpha\beta^2}$$

verify

(d) $E[x]$ and $Var(x)$

$X \sim beta(\alpha, \beta)$

$$f_X(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

$\alpha > 0, \beta > 0$

$0 < x < 1$

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\mathbb{E}[x] = \frac{1}{B(\alpha,\beta)} \int_0^1 x \cdot x^{\alpha-1} \cdot (1-x)^{\beta-1}$$

$$= \frac{B(\alpha+1, \beta)}{B(\alpha,\beta)}$$

$$= \frac{\Gamma(\alpha+1)\,\cancel{\Gamma(\beta)}}{\Gamma(\alpha+\beta+1)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\cancel{\Gamma(\beta)}}$$

$$\boxed{\mathbb{E}[x] = \frac{\alpha}{\alpha+\beta}}$$

$$\mathbb{E}[x^2] = \frac{1}{B(\alpha,\beta)} \int_0^1 x^2 \cdot x^{\alpha-1} \cdot (1-x)^{\beta-1}\, dx$$

$$= \frac{B(\alpha+2, \beta)}{B(\alpha,\beta)}$$

$$= \frac{\Gamma(\alpha+2)\cancel{\Gamma(\beta)}}{\Gamma(\alpha+\beta+2)} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cancel{\Gamma(\beta)}}$$

$$\mathbb{E}[x^2] = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}$$

$$\mathrm{var}(x) = \mathbb{E}[x^2] - \mathbb{E}[x]^2$$

$$= \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$\Rightarrow \boxed{\mathrm{var}(x) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$$

(e) Verify $\mathbb{E}[X]$ and $Var(X)$

$X \sim$ double exponential (μ, σ)

Solution:

$$f_X(x) = \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}}$$

$-\infty < x < \infty$

$-\infty < \mu < \infty$

$\sigma > 0$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} \frac{x}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}} dx$$

$$\mathbb{E}[X] = \frac{1}{2\sigma}\left[\int_{-\infty}^{\mu} x \cdot e^{\frac{-(x-\mu)}{\sigma}} dx + \int_{\mu}^{\infty} x \cdot e^{\frac{x-\mu}{\sigma}} dx\right]$$

$$= \frac{1}{2\sigma}\left[e^{\frac{\mu}{\sigma}} \cdot \int_{-\infty}^{\mu} x \cdot e^{-\frac{x}{\sigma}} dx + e^{-\frac{\mu}{\sigma}} \cdot \int_{\mu}^{\infty} x \cdot e^{\frac{x}{\sigma}} dx\right]$$

$$= \frac{e^{\frac{\mu}{\sigma}}}{2\sigma} \cdot \int_{-\infty}^{\mu} x \cdot e^{-\frac{x}{\sigma}} dx + \frac{e^{-\frac{\mu}{\sigma}}}{2\sigma} \cdot \int_{\mu}^{\infty} x \cdot e^{\frac{x}{\sigma}} dx$$

3.23 Pareto distribution

$$f_X(x) = \frac{\beta \alpha^{\beta}}{x^{\beta+1}} \qquad \alpha < x < \infty$$

$$\alpha > 0$$

$$\beta > 0$$

ⓐ verify that $f_X(x)$ is Pdf.

<u>Solution:</u>

$$\int_{\alpha}^{\infty} \frac{\beta \alpha^{\beta}}{x^{\beta+1}}\, dx = \beta \alpha^{\beta} \int_{\alpha}^{\infty} x^{-(\beta+1)}\, dx$$

$$= \beta \alpha^{\beta} \left. \frac{x^{-\beta}}{-\beta} \right]_{\alpha}^{\infty}$$

$$= 0 + \beta \alpha^{\beta} \cdot \frac{\alpha^{-\beta}}{\beta}$$

$$= 1$$

$f_X(x)$ is Pdf.

(b) Derive $\mathbb{E}[x]$ & $Var(x)$

<u>solution:</u>

$$\mathbb{E}[x] = \int_{\alpha}^{\infty} x \cdot \frac{\beta \alpha^{\beta}}{x^{\beta+1}}$$

$$= \beta \alpha^{\beta} \int_{\alpha}^{\infty} \frac{1}{x^{\beta}} dx$$

$$= \frac{\beta \alpha^{\beta}}{(\beta-1)\alpha^{\beta-1}} \underbrace{\int_{\alpha}^{\infty} \frac{(\beta-1)\alpha^{\beta-1}}{x^{\beta-1+1}} dx}_{\text{Pdf} = 1}$$

$$\boxed{\mathbb{E}[x] = \frac{\alpha \beta}{\beta-1}}$$

$$\mathbb{E}[x^2] = \int_{\alpha}^{\infty} x^2 \cdot \frac{\beta \alpha^{\beta}}{x^{\beta+1}} dx$$

$$= \frac{\beta \alpha^{\beta}}{(\beta-2)\alpha^{\beta-2}} = \frac{\alpha^2 \beta}{\beta-2}$$

$$\boxed{var(X) = \frac{\alpha^2\beta}{\beta-2} - \frac{\alpha^2\beta^2}{\beta-1}}$$

3.24 Many named distribution's are special cases of the more common distribution's already discussed. For each of the following named distribution's derive the form of the pdf, verify that it is a pdf, and calculate mean & variance.

(a) if $X \sim$ exponential (β), then $Y = X^{1/\gamma}$ has the weibull (γ, β) distribution, where $\gamma > 0$ is a constant.

Solution:

$$X \sim \text{exponential } (\beta)$$

$$f_X(x) = \frac{1}{\beta} e^{-x/\beta} \qquad 0 < x < \infty$$

$$Y = g(X) = X^{1/\gamma} \qquad 0 < y < \infty$$

$\gamma > 0$

$\beta > 0$

$$\Rightarrow \quad x = g^{-1}(y) = y^{\gamma}$$

$$\frac{d}{dy} g^{-1}(y) = \gamma \cdot y^{\gamma - 1}$$

$$f_Y(y) = f_X\left(g^{-1}(y)\right) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$= \frac{1}{\beta} e^{-\frac{y^{\gamma}}{\beta}} \cdot \gamma \cdot y^{\gamma-1}$$

$$f_Y(y) = \frac{\gamma}{\beta} y^{\gamma-1} \cdot e^{-\frac{y^{\gamma}}{\beta}}$$

$$\Rightarrow \quad Y \sim \text{Weibull}(\gamma, \beta)$$

Verify it is a pdf

$$\Rightarrow \quad \int_0^{\infty} f_Y(y)\, dy = \int_0^{\infty} \frac{\gamma}{\beta} y^{\gamma-1} e^{-\frac{y^{\gamma}}{\beta}} dy$$

$$= \frac{\gamma}{\beta} \int_0^{\infty} y^{\gamma-1} \cdot e^{-\frac{y^{\gamma}}{\beta}} dy$$

$$\frac{y^{\gamma}}{\beta} = u$$

$$\frac{\gamma}{\beta} y^{\gamma-1} dy = du$$

$$= \int_0^\infty e^{-u} \, du = -e^{-u}\Big]_0^\infty$$

$$= 1 \checkmark$$

$$E[Y] = \int_0^\infty y \cdot \frac{\gamma}{\beta} y^{\gamma-1} \cdot e^{-\frac{y^\gamma}{\beta}} \, dy$$

$$= \frac{\gamma}{\beta} \int_0^\infty y^\gamma \cdot e^{-\frac{y^\gamma}{\beta}} \, dy$$

$$= \frac{\gamma}{\beta} \left[y \int y^{\gamma-1} \cdot e^{-\frac{y^\gamma}{\beta}} \Big]_0^\infty - \int_0^\infty \int y^{\gamma-1} e^{-\frac{y^\gamma}{\beta}} \, dy \, dy \right]$$

$$= y \cdot -e^{-\frac{y^\gamma}{\beta}} \Big]_0^\infty + \int_0^\infty e^{-\frac{y^\gamma}{\beta}} \, dy$$

$$= 0 + \int_0^\infty e^{-\frac{y^\gamma}{\beta}} \, dy$$

$$\frac{y^\gamma}{\beta} = u \implies y = (\beta)^{1/\gamma} \cdot u^{1/\gamma}$$

$$dy = \beta^{1/\gamma} \cdot \frac{1}{\gamma} u^{\frac{1}{\gamma}-1} \, du$$

$$= \int_0^\infty \frac{\beta^{1/\gamma}}{\gamma} \cdot u^{\frac{1}{\gamma}-1} e^{-u}\, du$$

$$= \frac{\beta^{1/\gamma}}{\gamma} \underbrace{\int_0^\infty u^{\frac{1}{\gamma}-1} \cdot e^{-u}\, du}_{\text{gamma}\left(\frac{1}{\gamma},\, 1\right)}$$

$$= \frac{\beta^{1/\gamma}}{\gamma} \cdot \Gamma\left(\frac{1}{\gamma}\right)$$

$$\boxed{\Rightarrow \quad \mathbb{E}[X] = \frac{\beta^{\frac{1}{\gamma}} \cdot \Gamma\left(\frac{1}{\gamma}\right)}{\gamma} = \beta^{1/\gamma}\, \Gamma\left(\frac{1}{\gamma}+1\right)}$$

$$\mathbb{E}[y^2] = \int_0^\infty y^2 \cdot \frac{\gamma}{\beta} y^{\gamma-1} \cdot e^{-\frac{y^\gamma}{\beta}}$$

$$= \int_0^\infty \frac{\gamma}{\beta} \cdot y^{\gamma+1} \cdot e^{-\frac{y^\gamma}{\beta}}$$

$$\frac{y^\gamma}{\beta} = u$$

$$\Rightarrow \quad \gamma \cdot \frac{y^{\gamma-1}}{\beta} dy = du$$

$$= \int_0^\infty y^2 \cdot e^{-u} \, du$$

$$= \int_0^\infty \beta^{\frac{2}{\gamma}} \cdot u^{2/\gamma} \cdot e^{-u} \, du$$

$$= \beta^{\frac{2}{\gamma}} \underbrace{\int_0^\infty u^{\frac{2}{\gamma}+1-1} \cdot e^{-u} \, du}_{\text{gamma}\left(\frac{2}{\gamma}+1,\ 1\right)}$$

$$= \beta^{\frac{2}{\gamma}} \cdot \Gamma\left(\frac{2}{\gamma}+1\right)$$

$$\operatorname{Var}(y) = \mathbb{E}[y^2] - \mathbb{E}[y]^2$$

$$= \beta^{\frac{2}{\gamma}} \cdot \Gamma\left(\frac{2}{\gamma}+1\right) - \beta^{\frac{2}{\gamma}} \cdot \Gamma\left(\frac{1}{\gamma}+1\right)^2$$

$$\boxed{\operatorname{Var}(x) = \beta^{2/\gamma} \left[\Gamma\left(\frac{2}{\gamma}+1\right) - \Gamma\left(\frac{1}{\gamma}+1\right)^2 \right]}$$

(b) $X \sim$ exponential (β)

$Y = \left(\frac{2X}{\beta}\right)^{1/2}$ has a Rayleigh distribution

solution:

$$Y = g(x) = \left(\frac{2x}{\beta}\right)^{1/2} \quad y \in (0, \infty)$$

$$x = g^{-1}(y) = \frac{\beta}{2} y^2$$

$$f_Y(y) = \frac{1}{\beta} e^{-\frac{\beta}{2}\frac{y^2}{\beta}} \cdot \beta y$$

$$f_Y(y) = y \cdot e^{-\frac{y^2}{2}} \qquad 0 < y < \infty$$

$Y \sim$ Rayleigh distribution.

Verify its a pdf

$$\int_0^\infty y \cdot e^{-\frac{y^2}{2}} \, dy$$

$$\frac{y^2}{2} = u \Rightarrow \qquad y\,dy = du$$

$$\int_0^\infty e^{-u} du = -e^{-u} \Big]_0^\infty = 1 \checkmark$$

$\mathbb{E}[Y]$, $Var[Y] = ?$

$$\mathbb{E}[Y] = \int_0^\infty y^2 \cdot e^{-\frac{y^2}{2}} \, dy$$

$\frac{y^2}{2} = u \Rightarrow y\,dy = du$

$$= \int_0^\infty y \cdot e^{-u} \, du$$

$$= \underbrace{\int_0^\infty \sqrt{2} \cdot u^{3/2 - 1} \cdot e^{-u} \, du}_{gamma(3/2, 1)}$$

$$= \sqrt{2} \cdot \Gamma(3/2)$$

$$= \sqrt{2} \cdot \frac{1}{2} \Gamma(1/2)$$

$$= \frac{1}{\sqrt{2}} \cdot \sqrt{\pi} = \sqrt{\frac{\pi}{2}}$$

$$\boxed{\mathbb{E}[Y] = \sqrt{\frac{\pi}{2}}}$$

$$\mathrm{var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}(Y)^2$$

$$\mathbb{E}[Y^2] = \int_0^\infty y^3 e^{-\frac{y^2}{2}} dy \qquad \frac{y^2}{2} = u \qquad y\,dy = du$$

$$= \int_0^\infty y^2 e^{-u} du$$

$$= \underbrace{\int_0^\infty 2u e^{-u} du}_{\text{gamma}(2,1)}$$

$$= 2 \cdot \Gamma(2) = 2$$

$$\boxed{\mathrm{var}(Y) = 2 - \frac{\pi}{2}}$$

(c) $X \sim \text{gamma}(a,b)$ $\quad Y = \frac{1}{X}$ has a inverted Gamma $IG(a,b)$

<u>solution:</u>

$$X \sim \text{gamma}(a,b)$$

$$f_X(x) = \frac{1}{\Gamma(a)\, b^a} x^{a-1} e^{-\frac{x}{b}}$$

$0 < x < \infty$
$a > 0$
$b > 0$

$$Y = g(x) = \frac{1}{x} \Rightarrow X = g^{-1}(y) = \frac{1}{y}$$

$$\frac{d}{dy} g^{-1}(y) = -\frac{1}{y^2}$$

$$f_Y(y) = \frac{1}{\Gamma(a)\, b^a} \frac{1}{y^{a-1}} \cdot e^{-\frac{1}{yb}} \cdot +\frac{1}{y^2}$$

$$\boxed{f_Y(y) = \frac{1}{\Gamma(a) b^a} y^{-(a+1)} \cdot e^{-\frac{1}{yb}}}$$

verify its pdf

$$\int_0^{\infty} \frac{1}{\Gamma(a) b^a} y^{-(a+1)} \cdot e^{-\frac{1}{yb}} \, dy$$

$$\frac{1}{yb} = u$$

$$y = \frac{1}{ub}$$

$$dy = \frac{1}{b} \cdot -\frac{1}{u^2} \, du$$

$$\int_0^\infty \frac{1}{\Gamma(a)b^a}(ub)^{a+1} \cdot e^{-u} \cdot \frac{1}{b} \cdot \frac{+1}{u^2} du$$

$$= \int_0^\infty \frac{1}{\Gamma(a)b^a} b^a \cdot u^{a-1} e^{-u} du$$

$$= \frac{\cancel{b^a}}{\Gamma(a)\cancel{b^a}} \underbrace{\int_0^\infty u^{a-1} e^{-u} du}_{\text{gamma}(a,1)}$$

$$= \frac{\Gamma(a)}{\Gamma(a)} = 1 \checkmark$$

$$\mathbb{E}[Y] = \int_0^\infty y \cdot \frac{1}{\Gamma(a)b^a} y^{-(a+1)} \cdot e^{-\frac{1}{yb}} dy$$

$$= \int_0^\infty \frac{1}{\Gamma(a)b^a} y^{-(a-1+1)} \cdot e^{-\frac{1}{yb}} dy$$

$$= \frac{1}{\Gamma(a)b^a} \underbrace{\int_0^\infty y^{-(a-1+1)} \cdot e^{-\frac{1}{yb}} dy}_{IG(a-1,b)}$$

$$= \frac{\Gamma(a-1) \cdot b^{\cancel{a}-1}}{\Gamma(a)\, \cancel{b^{a}}} = \frac{1}{b(a-1)}$$

$$\boxed{\mathbb{E}[x] = \frac{1}{b(a-1)}}$$

$$\mathbb{E}[x^2] = \frac{1}{\Gamma(a)\, b^a} \int_0^\infty y^{-(a-2+1)} \cdot e^{-\frac{1}{yb}} dy$$

$$= \frac{\Gamma(a-2)\, b^{a-2}}{\Gamma(a)\, b^a}$$

$$= \frac{1}{b^2 (a-1)(a-2)}$$

$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2$$

$$= \frac{1}{b^2 (a-1)(a-2)} - \frac{1}{b^2 (a-1)^2}$$

$$= \frac{1}{b^2 (a-1)} \left[\frac{1}{a-2} - \frac{1}{a-1} \right]$$

$$= \frac{1}{b^2(a-1)}\left[\frac{1}{(a-2)(a-1)}\right]$$

$$\boxed{Var(Y) = \frac{1}{b^2(a-1)^2(a-2)}}$$

d) $X \sim \text{gamma}\left(\frac{3}{2}, \beta\right)$

$Y = \left(\frac{X}{\beta}\right)^{1/2}$ has the maxwell distribution.

<u>Solution:</u>

$$f_X(x) = \frac{1}{\Gamma(3/2)\,\beta^{3/2}}\, x^{1/2} \cdot e^{-\frac{x}{\beta}}$$

$$Y = \left(\frac{X}{\beta}\right)^{1/2} = g(x) \qquad y = (0, \infty)$$

$$x = g^{-1}(y) = \beta y^2$$

$$f_Y(y) = \frac{1}{\Gamma(3/2)\,\cancel{\beta^{3/2}}} \cdot \cancel{\beta^{1/2}} \cdot y \cdot e^{-y^2} \cdot 2\cancel{\beta} y$$

$$f_Y(y) = \frac{2}{\Gamma(3/2)} \cdot y^2 \cdot e^{-y^2}$$

<u>verify its pdf</u>

$$\int_0^\infty \frac{2}{\Gamma(3/2)} \cdot y^2 \cdot e^{-y^2} \, dy$$

$$= \frac{2}{\Gamma(3/2)} \int_0^\infty y^2 \cdot e^{-y^2} \, dy$$

$y^2 = u$

$2y\,dy = du$

$$= \frac{1}{\Gamma(3/2)} \int_0^\infty u^{\frac{3}{2}-1} \cdot e^{-u} \, du$$

$$= \frac{\Gamma(3/2)}{\Gamma(3/2)} = 1$$

$\mathbb{E}[Y]$ and $\text{var}(Y)$

$$\mathbb{E}[Y] = \frac{2}{\Gamma(3/2)} \int_0^\infty y^3 \cdot e^{-y^2} \, dy$$

$$u = y^2$$
$$du = 2y\,dy$$

$$\mathbb{E}(Y) = \frac{1}{\Gamma(3/2)} \underbrace{\int_0^\infty u e^{-u} \, du}_{\text{gamma}(2,1)}$$

$$\boxed{\mathbb{E}[Y] = \frac{\Gamma(2)}{\Gamma(3/2)} = \frac{1}{\Gamma(3/2)}}$$

$$\mathbb{E}[Y^2] = \frac{2}{\Gamma(3/2)} \int_0^\infty y^4 \cdot e^{-y^2} \, dy$$

$$y^2 = u \Rightarrow 2y\,dy = du$$

$$= \frac{1}{\Gamma(3/2)} \underbrace{\int_0^\infty u^{\frac{3}{2}} e^{-u} \, du}_{\text{gamma}(\frac{5}{2},1)}$$

$$= \frac{\Gamma(\frac{5}{2})}{\Gamma(3/2)} = \frac{3}{2}$$

$$\Rightarrow \quad Var(Y) = \frac{3}{2} - \left(\frac{1}{\Gamma(3/2)} \right)^2$$

$$= \frac{3}{2} - \left(\frac{1}{\frac{1}{2}\sqrt{\pi}} \right)^2$$

$$\boxed{Var(Y) = \frac{3}{2} - \frac{4}{\pi}}$$