BERNOULLI PROCESS

Probability is useful for developing the science of inference.

* we want to deal the phenomenon that evolve in time, random-process, or stocastic processes.

in the real world we don't just throw 2 r.v and go home, Rather the world goes on. so we generate a r.v and we get more random variable's, and thing's evolve in time.

Random Processes are supposed to be model's that capture the evolution of random phenomenon over time.

→ discrete time

→ Continuous time

Bernoulli Process → seq^n of coin flips.

LECTURE 13

The Bernoulli process

- **Readings:** Section 6.1

Lecture outline

- Definition of Bernoulli process
- Random processes
- Basic properties of Bernoulli process
- Distribution of interarrival times
- The time of the kth success
- Merging and splitting

The Bernoulli process

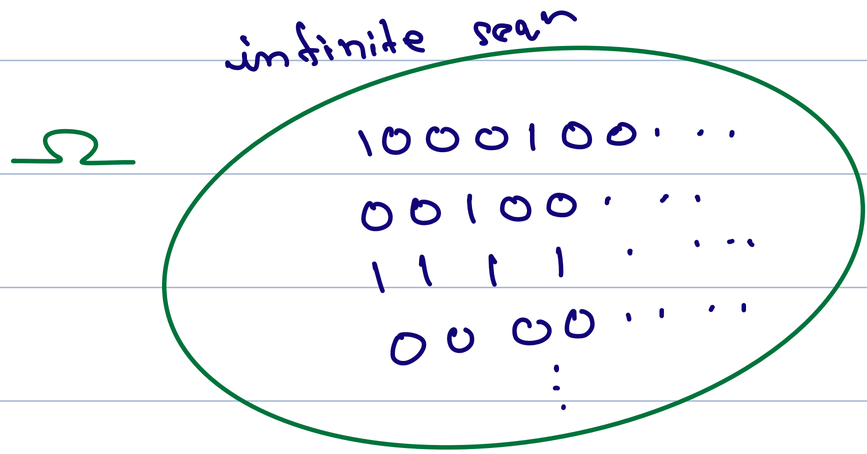
- A sequence of independent Bernoulli trials
- At each trial, i:
  - $\mathbf{P}(\text{success}) = \mathbf{P}(X_i = 1) = p$
  - $\mathbf{P}(\text{failure}) = \mathbf{P}(X_i = 0) = 1 - p$
- Examples:
  - Sequence of lottery wins/losses
  - Sequence of ups and downs of the Dow Jones
  - Arrivals (each second) to a bank
  - Arrivals (at each time slot) to server

Random processes

- First view: sequence of random variables $X_1, X_2, \ldots$
- $\mathbf{E}[X_t] = p \cdot 1 + (1-p) 0 = p$
- $\text{Var}(X_t) = (1-p)^2 p + (0-p)^2 (1-p) = p(1-p)$
- Second view: what is the right sample space?
- $\mathbf{P}(X_t = 1 \text{ for all } t) = 0$
- Random processes we will study:
  - Bernoulli process (memoryless, discrete time)
  - Poisson process (memoryless, continuous time)
  - Markov chains (with memory/dependence across time)

Number of successes S in n time slots

- $\mathbf{P}(S = k) = \binom{n}{k} p^k (1-p)^{n-k}$
- $\mathbf{E}[S] = np$
- $\text{Var}(S) = np(1-p)$



$$P(X_t = 1, \forall t) \leq P(X_1 = \cdots = X_{10} = 1) = p^{10}$$

$$\Rightarrow P(X_t = 1, \forall t) \leq P(X_1 \cdots = X_k = 1) = p^k$$

this is true $\forall k$

$$\Rightarrow \mathbb{P}(X_t = 1, \forall t) \leq \lim_{k \to \infty} p^k = 0$$

This Proves the Prob of seq^n of all 1's $= 0$

Bernoulli Process

* The Best to think about the model's correspond to the Bernoulli Process, ie in terms of arrivals of Jobs to a facility. And there are 2 types of Question's that we can Ask

① in a Given amount of time, How many Jobs arrived

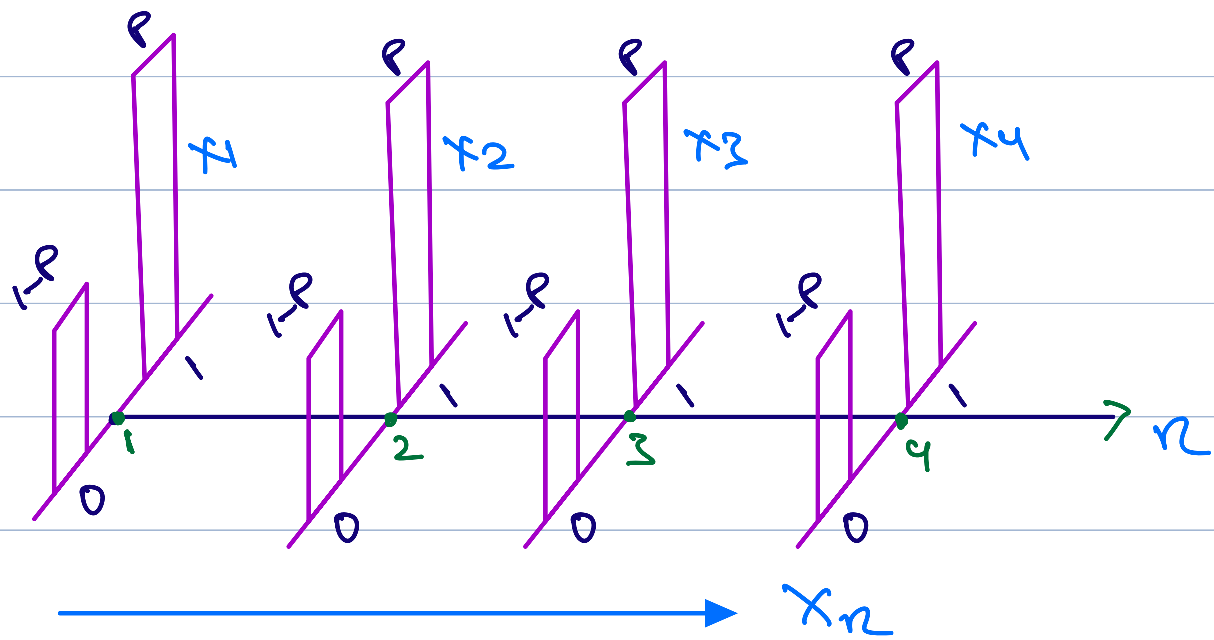
② or Conversely, for a given number of Jobs How much time did it take to arrive.

① for a given amount of time (n), number of Jobs arrived (K) $\Rightarrow$ Binomial r.V

$$\mathbb{P}(S=K) = \binom{n}{K} p^K (1-p)^{n-K} \qquad K=0,1,\dots n$$

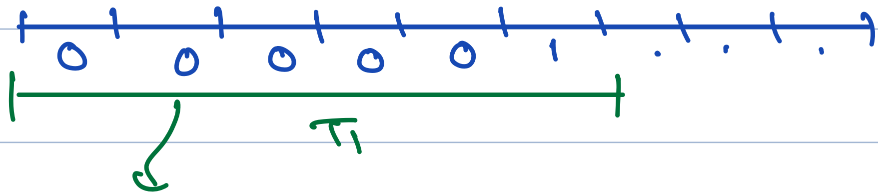
$$\mathbb{E}[S] = np$$

$$Var(S) = np(1-p)$$



② Now let us fix the number of arrivals and ask how much time did it take?

* Lets start with time until 1^{st} arrival.



Number of trails it took until to get 1, is T_1

T_1 = Time of the 1^{st} arrival.

$\Rightarrow$ T_1 = geometric distribution.

T_1 = number of trails until first success

$$\Rightarrow \mathbb{P}(T_1 = t) = (1-p)^{t-1} p \qquad t = 1, 2, \ldots$$

(geometric distribution)

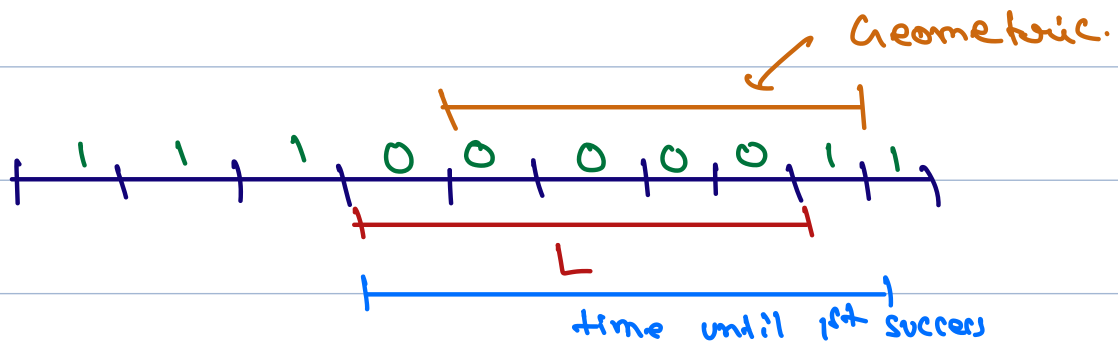
$$\mathbb{E}[T_1] = \frac{1}{p}$$ $\Rightarrow$ if p is small, it is expected to take longer time until the success.

$$\mathrm{var}(T_1) = \frac{(1-p)}{p^2}$$

Memoryless Property

* its essentially consequence of independence
* if I tell you the results of coin flips upto a certain time, Because of independence, this does not give you any information about the coin flips after that time

* if you Buy the lottery ticket everyday, what is the distribution of the length of the first loosing days?



$L+1$ = Geometric r.v

L cannot be zero. $\Rightarrow$ L+1 cannot be 1

$\Rightarrow$ L is not Geometric r.v

Interarrival times

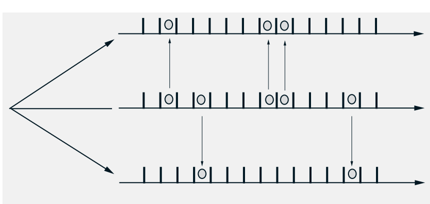
- T_1: number of trials until first success
  - $\mathbf{P}(T_1 = t) =$ $(1-p)^{t-1}p$
  - Memoryless property
  - $\mathbf{E}[T_1] =$ $1/p$
  - $\text{Var}(T_1) =$ $\frac{1-p}{p^2}$
- If you buy a lottery ticket every day, what is the distribution of the length of the first string of losing days?

Time of the kth arrival

- Given that first arrival was at time t i.e., $T_1 = t$: additional time, T_2, until next arrival
  - has the same (geometric) distribution
  - independent of T_1
- Y_k: number of trials to kth success
  - $\mathbf{E}[Y_k] =$
  - $\text{Var}(Y_k) =$
  - $\mathbf{P}(Y_k = t) =$

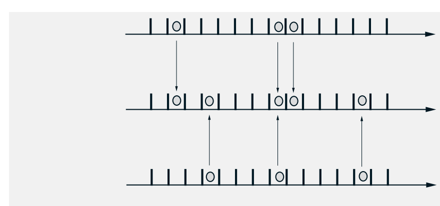
Splitting of a Bernoulli Process

(using independent coin flips)

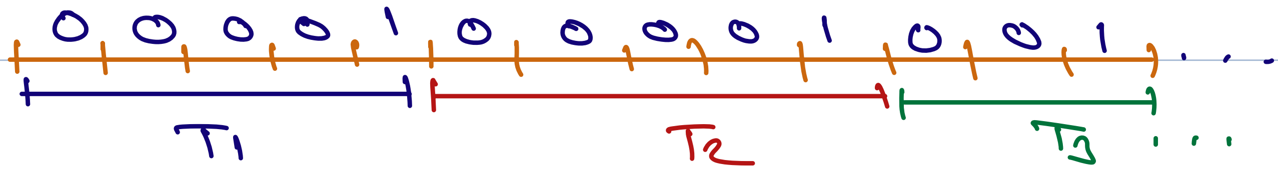


yields Bernoulli processes

Merging of Indep. Bernoulli Processes



yields a Bernoulli process
(collisions are counted as one arrival)



Time of the k^{th} arrival $= Y_k$

$$Y_t = T_1 + T_2 + T_3 + \cdots + T_t$$

each T_i = geometric r.v

each T_i r.v is independent geometric r.v

$Y_t = T_1 + T_2 + T_3 \cdots + T_t$ $T_i \sim$ geometric (p) independent

we can use convolution

using shortcut

$\Rightarrow \ \mathbb{P}(Y_t = k) =$ $(k-1)$ arrival's in $t-1$ slots

$\times$ k^{th} arrival in t^{th} slot

$= \mathbb{P}($ $k-1$ arrival's in $t-1$ slots, arrival at time t^{th} slot $)$

$$\boxed{\Rightarrow \ \mathbb{P}(Y_k = t) = \binom{t-1}{k-1} p^{k-1}(1-p)^{t-k} \cdot p \qquad t \geq k}$$

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discrete time :

fix n number of slots $\rightarrow$ i.e fix time

① number of success in n trails (n time slots)

$\Rightarrow$ Binomial(n, p)

$$\mathbb{P}(S=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

this is function of number of success

(discrete)

②

Fix number of success $= 1$

$\Rightarrow$ time taken for 1st success = Geometric (number of trails) Geometric (p)

$$\Rightarrow \quad \mathbb{P}(T_1 = t) = (1-p)^{t-1} \cdot p^t$$

(discrete)

③ time taken for kth success (k is fixed)

$$\mathbb{P}(Y_k = t) = \binom{t-1}{k-1} (1-p)^{k-1} p^{t-k} \cdot p$$

(Pascal) $t \geq k$

(discrete)