EXERCISES ON TRANSFORMATIONS

DESCription:

This document contain's solution's to Exercise's 2.1 to 2.10 from chapter 2 of Casella - Berger: Statistical inference, focusing on the transformation's of random-variables.

2.1 in each of the following find the Pdf of Y. show that the Pdf integrates to 1

(a) $Y = X^3$ and $f_X(x) = 42x^5(1-x)$
$0 < x < 1$

Solution:
Support set $\mathcal{X} = (0,1)$ & $Y = g(X) = X^3$

$\Rightarrow$ $\mathcal{Y} = (0,1)$

$$Y = X^3 \implies x = g^{-1}(y) = y^{1/3}$$

$$\frac{d}{dy} g^{-1}(y) = \frac{1}{3} \frac{1}{y^{2/3}}$$

$$\implies f_Y(y) = f_X(g^{-1}(y)) \cdot \left|\frac{d}{dy} g^{-1}(y)\right|$$

$$= 42 y^{\frac{5}{3}} (1 - y^{1/3}) \cdot \frac{1}{3} \cdot y^{-\frac{2}{3}}$$

$$\boxed{\implies f_Y(y) = 14y(1 - y^{1/3}) \qquad 0 < y < 1}$$

verifying it integrates to 1

$$\int_0^1 f_Y(y)\, dy = \int_0^1 14y\, dy - \int_0^1 14 y^{\frac{4}{3}} dy$$

$$= 7y^2\Big]_0^1 - 14 \cdot \frac{y^{\frac{7}{3}}}{7/3}\Bigg]_0^1$$

$$= 7 - \frac{14}{7} \cdot 3 = 1$$

$$= 1 \checkmark$$

2.1 b $\quad Y = 4X + 3$ and $f_X(x) = 7e^{-7x}$

$0 < x < \infty$

Solution:

$0 < x < \infty \quad \& \quad Y = 4X + 3$

$\Rightarrow \quad X = (0, \infty)$ and $Y = (3, \infty)$

$$Y = g(X) = 4X + 3 \Rightarrow X = g^{-1}(y)$$
$$= \frac{y-3}{4}$$

$$\Rightarrow \quad \frac{d}{dy} g^{-1}(y) = \frac{1}{4}$$

therefore

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$
$$= 7 \cdot e^{-7\left(\frac{y-3}{4}\right)} \cdot \frac{1}{4}$$

$$\boxed{f_Y(y) = \frac{7}{4} \cdot e^{-\frac{7}{4}(y-3)}}$$

verifying pdf integrates to 1

$$\int_3^\infty \frac{7}{4} e^{\frac{7}{4}(y-3)} \, dy = \frac{7}{4} e^{\frac{21}{4}} \int_3^\infty e^{-\frac{7}{4}y} \, dy$$

$$= \frac{7}{4} e^{\frac{21}{4}} \cdot \frac{e^{-\frac{7}{4}y}}{-\frac{7}{4}} \Bigg]_3^{\infty}$$

$$= -e^{\frac{21}{4}} \cdot e^{-\frac{7}{4}y} \Bigg]_3^{\infty}$$

$$= +e^{\frac{21}{4}} \cdot e^{-\frac{21}{4}}$$

$$= 1 \checkmark$$

2.1C

$Y = X^2$ and $f_X(x) = 30x^2(1-x)^2$

$0 < x < 1$

<u>solution:</u>

$$x = g^{-1}(y) = \sqrt{y}$$

$$X = (0,1) \quad , \quad Y = (0,1)$$

$$f_Y(y) = 30y(1-\sqrt{y})^2 \cdot \frac{1}{2\sqrt{y}}$$

$$\boxed{f_Y(y) = 15\sqrt{y}\,(1-\sqrt{y})^2 \quad , 0 < y < 1}$$

Verifying the Pdf integrates to 1

$$\int_0^1 f_Y(y)\,dy = \int_0^1 15\sqrt{y}\,(1+y-2\sqrt{y})\,dy$$

$$= \int_0^1 \left(15\sqrt{y} + 15y^{\frac{3}{2}} - 30y\right)dy$$

$$= \frac{15y^{\frac{3}{2}}}{3/2} + \frac{15y^{\frac{5}{2}}}{\frac{5}{2}} - \frac{30y^2}{2}\Big]_0^1$$

$$= 10y^{\frac{3}{2}} + 6y^{\frac{5}{2}} - 15y^2\Big]_0^1$$

$$= 10+6-15 = 1 \checkmark$$

2.2 in each of the following find the pdf of Y

ⓐ $Y = X^2$ and $f_X(x) = 1$, $0 < x < 1$

solution:

$$Y = X^2 \ , \ X = (0,1) \implies Y = (0,1)$$

$$X = g^{-1}(Y) = \sqrt{Y}$$

$$\frac{d}{dy} g^{-1}(y) = \frac{1}{2\sqrt{y}}$$

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$= 1 \cdot \frac{1}{2y^{1/2}} = \frac{1}{2} y^{-1/2}$$

$$\boxed{f_Y(y) = \frac{y^{-\frac{1}{2}}}{2} \ , 0 < y < 1}$$

(b) $Y = -\log X$ and X has pdf

$$f_X(x) = \frac{(n+m+1)!}{n!\,m!} x^n (1-x)^m \quad , 0 < x < 1$$

m, n Positive integer's

Solution:

$Y = -\log X$ (monotone decreasing function)

$X = (0,1) \implies Y = (0, \infty)$

$$x = g^{-1}(y) = e^{-y}$$

$$\frac{d}{dy} g^{-1}(y) = -e^{-y}$$

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$= \frac{(n+m+1)!}{n!\,m!} e^{-ny} \cdot (1-e^{-y})^m \cdot e^{-y}$$

$$\boxed{f_Y(y) = \frac{(n+m+1)!}{n!\,m!} e^{-y(n+1)} \cdot (1-e^{-y})^m \quad , 0 < y < \infty}$$

2.2 c $Y = e^X$ and X has Pdf

$$f_X(x) = \frac{1}{\sigma^2} x e^{-\left(\frac{x}{\sigma}\right)^2 \cdot \frac{1}{2}}$$

$0 < x < \infty$

Solution:

$$Y = e^X \Longrightarrow x = g^{-1}(y) = \log y$$

$$\frac{d}{dy} g^{-1}(y) = \frac{d}{dy} \log y = \frac{1}{y}$$

$$f_Y(y) = \frac{1}{\sigma^2} \log y \cdot e^{-\frac{(\log y)^2}{2\sigma^2}} \cdot \frac{1}{y}$$

$$\boxed{\Rightarrow f_Y(y) = \frac{1}{\sigma^2} \frac{\log y}{y} \cdot e^{-\frac{(\log y)^2}{2\sigma^2}} \quad 0 < y < \infty}$$

2.3 Suppose $X \sim$ geometric Pmf

$$\Rightarrow f_X(x) = \frac{1}{3}\left(\frac{2}{3}\right)^x, \quad x = 0, 1, 2, \ldots$$

Determine Pmf of $Y = \frac{X}{1+X}$

solution:

$$Y = \frac{X}{X+1} \implies y = \left\{1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \dots\right\}$$

and $Y = g(X)$ is one-to-one function

$$\mathbb{P}(Y=y) = \mathbb{P}\left(\frac{X}{1+X} = y\right)$$

$$= \mathbb{P}\left(X = \frac{y}{1-y}\right)$$

$$= \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{y}{1-y}}$$

$$\boxed{\implies \mathbb{P}(Y=y) = \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{y}{1-y}} \quad y \in \left\{1, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \dots\right\}}$$

2.4 Let $\lambda \in \mathbb{Z}^+$ and define the function by

$$f(x) = \begin{cases} \frac{1}{2}\lambda e^{-\lambda x} & \text{if } x \geq 0 \\ \frac{1}{2}\lambda e^{\lambda x} & \text{if } x < 0 \end{cases}$$

(a) verify that $f(x)$ is a PDF

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{0} \frac{1}{2}\lambda e^{\lambda x}\,dx + \int_{0}^{\infty} \frac{1}{2}\lambda e^{-\lambda x}\,dx$$

$$= \frac{1}{2}\lambda \frac{e^{\lambda x}}{\lambda}\Big]_{-\infty}^{0} + \frac{1}{2}\lambda \frac{e^{-\lambda x}}{-\lambda}\Big]_{0}^{\infty}$$

$$= \frac{1}{2}e^{\lambda x}\Big]_{-\infty}^{0} - \frac{1}{2}e^{-\lambda x}\Big]_{0}^{\infty}$$

$$= \frac{1}{2}[1-0) - \frac{1}{2}[0-1] = 1$$

$= 1$ (Hence Proved)

2.4 b find $\mathbb{P}(X<t)$ $\forall t$

Solution:

$$\mathbb{P}(X<t) = \begin{cases} \int_{-\infty}^{t} \frac{1}{2}\lambda e^{\lambda x}\,dx & t<0 \\ \frac{1}{2} + \int_{0}^{t} \frac{1}{2}\lambda e^{-\lambda x}\,dx & t\geq 0 \end{cases}$$

$$= \begin{cases} \left.\frac{1}{2}\lambda \frac{e^{\lambda x}}{\lambda}\right]_{-\infty}^{t} \\ \frac{1}{2} + \left.\frac{1}{2}\lambda \frac{e^{-\lambda x}}{-\lambda}\right]_{0}^{t} \end{cases}$$

$$\boxed{\mathbb{P}(X<t) = \begin{cases} \frac{e^{\lambda t}}{2} & t<0 \\ 1 - \frac{1}{2}e^{-\lambda t} & t\geq 0 \end{cases}}$$

2.4 c Find $\mathbb{P}(|X|<t)$ $\forall t$

$$\mathbb{P}(|X|<t) = \mathbb{P}(-t<X<t)$$

$$= \int_{-t}^{t} f_X(x)\,dx$$

$$= \int_{-t}^{0} \frac{1}{2}\lambda e^{\lambda x}\,dx + \int_{0}^{t} \frac{1}{2}\lambda e^{-\lambda x}\,dx$$

$$= \frac{1}{2}e^{\lambda x}\Big]_{-t}^{0} - \frac{1}{2}e^{-\lambda x}\Big]_{0}^{t}$$

$$= \frac{1}{2}(1-e^{-\lambda t}) - \frac{1}{2}(e^{-\lambda t}-1)$$

$$= 1-e^{-\lambda t}$$

$$\boxed{\mathbb{P}(|X|<t) = 1-e^{-\lambda t} \quad \forall t \geq 0}$$

2.5 Use Theorem 2.1.8 to find the pdf of Y in Example 2.1.2

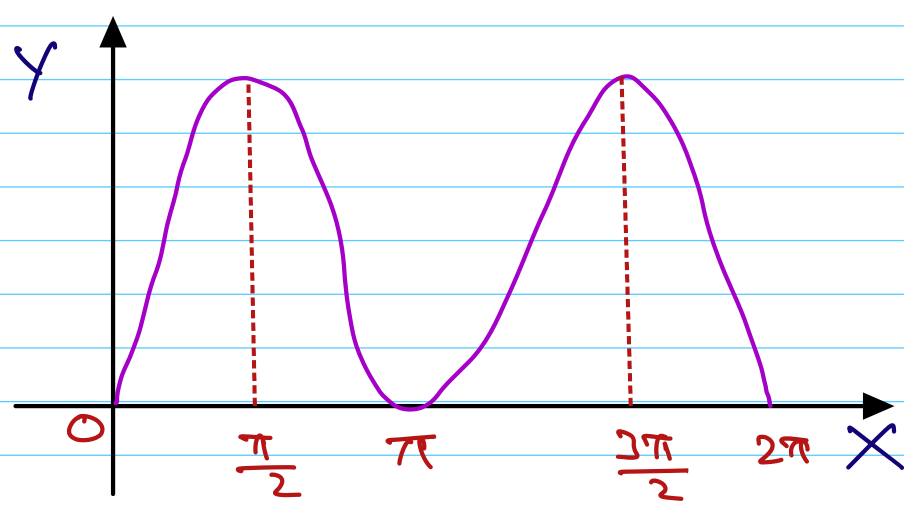
Solution:

$$X \sim \text{unif}(0, 2\pi)$$

$$f_X(x) = \begin{cases} \frac{1}{2\pi} & 0 < x < 2\pi \\ 0 & \text{o.w} \end{cases}$$

$$\mathcal{X} = (0, 2\pi) \implies \mathcal{Y} = (0, 1)$$

$$Y = g(X) = \sin^2(X)$$



divide the support set into 4 parts

$A_1 = (0, \frac{\pi}{2})$ $\quad g_1(x) = \sin^2 x \quad g_1^{-1}(y) = \sin^{-1}(\sqrt{y})$

$A_2 = (\frac{\pi}{2}, \pi)$ $\quad g_2(x) = \sin^2 x \quad g_2^{-1}(y) = \pi - \sin^{-1}(\sqrt{y})$

$A_3 = (\pi, \frac{3}{2}\pi)$ $\quad g_3(x) = \sin^2 x \quad g_3^{-1}(y) = \pi + \sin^{-1}(\sqrt{y})$

$A_4 = (\frac{3}{2}\pi, 2\pi)$ $\quad g_4(x) = \sin^2 x \quad g_4^{-1}(y) = 2\pi - \sin^{-1}(\sqrt{y})$

$$\left| \frac{d}{dy} g_i^{-1}(y) \right| = \frac{1}{\sqrt{1-y}} \cdot \frac{1}{2\sqrt{y}} \qquad \forall i = 1,2,3,4$$

$$f_Y(y) = \frac{1}{2\pi} \frac{1}{\sqrt{1-y}} \cdot \frac{1}{2\sqrt{y}} \cdot 4$$

$$\boxed{f_Y(y) = \frac{1}{\pi} \frac{1}{\sqrt{y(1-y)}} \qquad 0 < y < 1}$$

2.6 in each of the following find the Pdf of Y and show that the Pdf integrates to 1

(a)

$$f_X(x) = \frac{1}{2} e^{-|x|} \qquad -\infty < x < \infty$$

$$Y = |X|^3$$

<u>solution:</u>

$$A_1 = (-\infty, 0) \qquad g_1^{-1}(y) = -y^{1/3}$$

$$A_2 = (0, \infty) \qquad g_2^{-1}(y) = y^{1/3}$$

$$\left| \frac{d}{dy} g_i^{-1}(y) \right| = \frac{1}{3} y^{-\frac{2}{3}}$$

$$f_Y(y) = \frac{1}{2} e^{-y^{1/2}} \cdot \frac{1}{3} y^{-\frac{2}{3}} + \frac{1}{2} e^{-y^{1/3}} \cdot \frac{1}{3} y^{-\frac{2}{3}}$$

$$\boxed{f_Y(y) = \frac{1}{3} e^{-y^{1/3}} \cdot y^{-2/3} \qquad 0 < y < \infty}$$

verify pdf integrates to 1

$$\int_0^\infty \frac{1}{3} e^{-y^{1/3}} \cdot y^{-2/3} \, dy$$

$$y^{1/3} = v$$

$$\frac{1}{3} y^{-2/3} \, dy = dv$$

$$y : 0 \longrightarrow \infty \qquad v : 1 \longrightarrow \infty$$

$$\int_1^\infty e^{-v} dv = e^{-v} \Big]_1^\infty$$

$$= 1 \checkmark$$

(b)

$$f_X(x) = \frac{2}{8}(x+1)^2 \,, \quad -1 < x < 1 \,, \quad Y = 1 - X^2$$

Solution:

$$X = (-1, 1) \qquad Y = (0, 1)$$

$A_1 = (-1, 0) \qquad g_1^{-1}(y) = -\sqrt{1-y}$

$A_2 = (0, 1) \qquad g_2^{-1}(y) = \sqrt{1-y}$

$$\left| \frac{d}{dy} g_i^{-1}(y) \right| = \frac{1}{2\sqrt{1-y}}$$

$$f_Y(y) = \frac{3}{8}\left(1-\sqrt{1-y}\right)^2 \cdot \frac{1}{2\sqrt{1-y}}$$

$$+ \frac{3}{8}\left(1+\sqrt{1-y}\right)^2 \cdot \frac{1}{2\sqrt{1-y}}$$

$$= \frac{3}{8\sqrt{1-y}}\left(1+1-y-\cancel{2\sqrt{1-y}} + 2-y+\cancel{2\sqrt{1-y}}\right)$$

2.6C $f_X(x) = \frac{3}{8}(x+1)^2$, $-1 < x < 1$, $Y = 1 - X^2$ if $X \le 0$ and $Y = 1 - X$ if $X > 0$

<u>Solution:</u>

$A_1 = (-1, 0)$ $\quad g_1(x) = 1 - x^2$ $\quad g_1^{-1}(y) = -\sqrt{1-y}$

$A_2 = (0, 1)$ $\quad g_2(x) = 1 - x$ $\quad g_2^{-1}(y) = 1 - y$

$$f_Y(y) = \frac{3}{8}\left(1 - \sqrt{1-y}\right)^2 \cdot \frac{1}{2\sqrt{1-y}}$$

$$+ \frac{3}{8}(2-y)^2$$

$$\boxed{f_Y(y) = \frac{3}{16}\frac{\left(1-\sqrt{1-y}\right)^2}{\sqrt{1-y}} + \frac{3}{8}(2-y)^2 \qquad 0 < y < 1}$$

2.7 Let X have pdf $f_X(x) = \frac{2}{9}(x+1)$

$-1 \le x \le 2$

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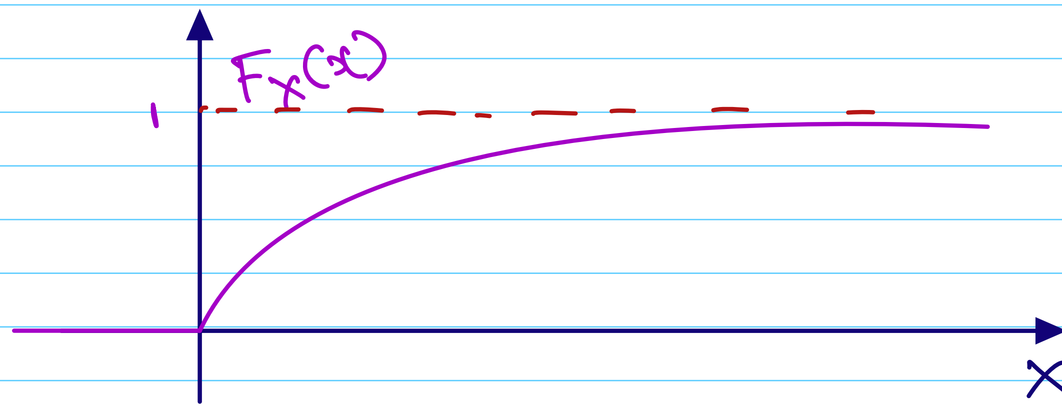
2.8 in each of the following show that the given function is a CDF and find $F_X^{-1}(y)$

(a)

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - e^{-x} & \text{if } x \geq 0 \end{cases}$$



$\lim_{x \to -\infty} F_X(x) = 0$ ✓

$\lim_{x \to \infty} F_X(x) = 1$ ✓

$F_X(x)$ continuous , right continuous ✓

$\frac{d}{dx} F_X(x) = e^{-x}$ (monotone increasing) ✓

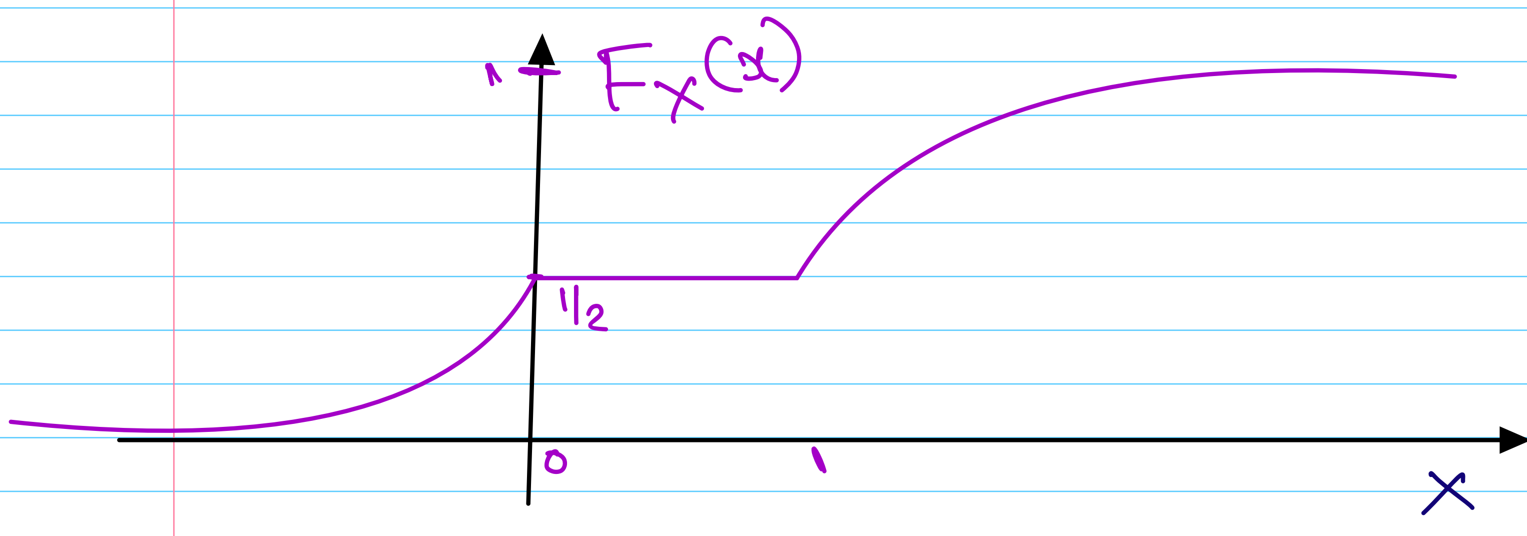
$F_X(x)$ is CDF

$$y = F_X(x) = 1 - e^{-x}$$

$$\Rightarrow \quad x = -\log(1-y) = F_X^{-1}(y)$$

$$\boxed{F_X^{-1}(y) = -\log(1-y) \qquad 0 < y < 1}$$

(b) $$F_X(x) = \begin{cases} e^{x/2} & \text{if } x < 0 \\ 1/2 & \text{if } 0 \le x < 1 \\ 1 - \dfrac{e^{1-x}}{2} & \text{if } 1 \le x \end{cases}$$



1. $\lim_{x \to -\infty} F_X(x) = 0$ & $\lim_{x \to +\infty} F_X(x) = 1$ ✓
2. right continuous

(2) $\frac{d}{dx} F_X(x) = \frac{e^x}{2}, 0, \frac{e^{1-x}}{2}$ (+ve)

(monotonous increasing)

$F_X(x)$ is CDF

(i) $F_X(x) = y \Rightarrow x = \log 2y \quad 0 < y < 1/2$

(ii) $\Rightarrow \quad y = 1 - \frac{e^{1-x}}{2}$

$\Rightarrow 2(1-y) = e^{1-x}$

$\Rightarrow 1-x = \log 2(1-y)$

$\Rightarrow x = 1 - \log 2(1-y) \quad \frac{1}{2} < y < 1$

$$\boxed{F_X^{-1}(y) = \begin{cases} \log 2y & 0 < y < \frac{1}{2} \\ 1 - \log 2(1-y) & \frac{1}{2} \leq y < 1 \end{cases}}$$

2.9

$$f(x) = \begin{cases} \frac{x-1}{2} & 1 < x < 3 \\ 0 & \text{o.w} \end{cases}$$

find monotonous function $u(x)$ s.t $Y = u(x)$ has a uniform $(0,1)$

<u>solution:</u>

$u(x) = F_x(x) \sim \text{unif}(0,1)$

$$F_x(x) = \int_1^x \left(\frac{x-1}{2}\right) dx$$

$$= \frac{(x-1)^2}{2}$$

$$\Rightarrow u(x) = \begin{cases} 0 & x < 1 \\ \frac{(x-1)^2}{2} & 1 < x \leq 3 \\ 1 & x \geq 3 \end{cases}$$