Continuous Distribution

① Uniform Distribution:

$$f(x \mid a,b) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a,b] \\ 0 & \text{o.w} \end{cases}$$

$$\boxed{\mathbb{E}[x] = \frac{b+a}{2} \qquad \text{var}(x) = \frac{(b-a)^2}{12}}$$

② Gamma Distribution:

The gamma family distribution are flexible family of distribution's on $[0,\infty)$

if α is positive constant then

$$\int_0^\infty t^{\alpha-1} e^{-t}\, dt \quad \text{is finite.}$$

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$$

gamma function

$$\Gamma(\alpha+1) = \int_0^\infty t^\alpha \cdot e^{-t} \, dt$$

$$\Gamma(\alpha+1) = \underbrace{-t^{\alpha-1} \cdot e^{-t} \Big]_0^\infty}_{=0} + \int_0^\infty \alpha \cdot t^{\alpha-1} \cdot e^{-t} \, dt$$

$$\Rightarrow \quad \Gamma(\alpha+1) = \alpha \cdot \int_0^\infty t^{\alpha-1} \cdot e^{-t} \, dt$$

$$\Rightarrow \quad \boxed{\Gamma(\alpha+1) = \alpha \, \Gamma(\alpha)}$$

$$\Gamma(1) = \int_0^\infty t^0 \cdot e^{-t} = -e^{-t}\Big]_0^\infty$$

$$= 1$$

$\Rightarrow$ for an integer $n > 0$

$$\Gamma(n) = n \, \Gamma(n-1)$$

$$= n \cdot n-1 \, \Gamma(n-2)$$

$$= n \cdot n-1 \cdot n-2 \cdot \cdots \cdot 1 \cdot \Gamma(1)$$

$$\boxed{\Gamma(n) = n!}$$

$$f(t) = \frac{t^{\alpha-1} e^{-t}}{\Gamma(\alpha)}, \quad 0 < t < \infty$$

$$\int_0^\infty f(t) = 1$$

Therefore gamma (α, β) family,

$$f(x \mid \alpha, \beta) = \frac{1}{\Gamma(\alpha) \beta^\alpha} x^{\alpha-1} \cdot e^{-\frac{x}{\beta}}$$

$0 < x < \infty$
$\alpha > 0, \ \beta > 0$

α = shape parameter

it influence the peakedness of the distribution

β = scale parameter, spread of the distribution.

$$f(x \mid \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}$$

$$\mathbb{E}[x] = \int_0^\infty \frac{x}{\Gamma(\alpha)\beta^\alpha} \cdot x^{\alpha-1} e^{-\frac{x}{\beta}} \, dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \underbrace{\int_0^\infty x^\alpha \cdot e^{-\frac{x}{\beta}} \, dx}_{\text{Gamma}(\alpha+1, \beta)}$$

$$= \frac{\Gamma(\alpha+1)\beta^{\alpha+1}}{\Gamma(\alpha)\beta^\alpha} = \frac{\alpha \, \Gamma(\cancel{\alpha}) \cancel{\beta^\alpha} \cdot \beta}{\Gamma(\cancel{\alpha}) \cdot \cancel{\beta^\alpha}}$$

$$\boxed{\mathbb{E}[x] = \alpha\beta}$$

$$\mathrm{Var}(x) = \mathbb{E}[x^2] - \mathbb{E}[x]^2$$

$$\mathbb{E}[x^2] = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^2 \cdot x^{\alpha-1} e^{-\frac{x}{\beta}} \, dx$$

$$\mathbb{E}[x^2] = \frac{1}{\Gamma(\alpha)\beta^\alpha} \underbrace{\int_0^\infty x^{\alpha+1} \cdot e^{-\frac{x}{\beta}}\, dx}_{\text{gamma}(\alpha+2,\beta)}$$

$$= \frac{\Gamma(\alpha+2)\beta^{\alpha+2}}{\Gamma(\alpha)\beta^\alpha} = (\alpha+1)(\alpha)\beta^2$$

$$\Rightarrow \quad \mathbb{E}[x^2] = (\alpha+1)\alpha\beta^2$$

$$\text{Var}(x) = (\alpha+1)\alpha\beta^2 - \alpha^2\beta^2$$

$$= \alpha\beta^2$$

$$\boxed{\text{Var}(x) = \alpha\beta^2}$$

$$M_x(t) = \mathbb{E}[e^{tx}]$$

$$\mathbb{E}[e^{tx}] = \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty e^{tx} \cdot x^{\alpha-1} e^{-\frac{x}{\beta}}\, dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha-1} \cdot e^{-x\left(\frac{1}{\beta}-t\right)} dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha-1} \cdot e^{-\frac{x}{\left(\frac{\beta}{1-\beta t}\right)}} dx$$

$$= \frac{\Gamma(\alpha)\left(\frac{\beta}{1-\beta t}\right)^\alpha}{\Gamma(\alpha)\beta^\alpha}$$

$$\boxed{M_X(t) = \left(\frac{1}{1-\beta t}\right)^\alpha, \quad t < \frac{1}{\beta}}$$

Example 3.3.1 (Gamma - Possion relationship):-

if $X \sim \text{gamma}(\alpha, \beta)$

& α is an integer.

then for any x

$$\mathbb{P}(X \leq x) = P(Y \geq x)$$

where $Y \sim \text{Possion}(x/\beta)$

$$\mathbb{P}(X \le x) = \frac{1}{(\alpha-1)!\,\beta^\alpha} \int_0^x t^{\alpha-1} e^{-t/\beta}\, dt$$

$$Y \sim \text{Poisson}\,(x/\beta)$$

$$\mathbb{P}(Y \geq \alpha) = \sum_{y=\alpha}^{\infty} \frac{e^{-\lambda} (\lambda)^y}{y!}$$

here the possion variable is $\frac{x}{\beta}$

$$\Rightarrow \mathbb{P}(Y \geq \alpha) = \sum_{y=\alpha}^{\infty} \frac{e^{-\frac{x}{\beta}} \cdot \left(\frac{x}{\beta}\right)^y}{y!}$$

$$\mathbb{P}(X \le y) = \frac{1}{(\alpha-1)!\,\beta^\alpha} \int_0^x t^{\alpha-1} \cdot e^{-\frac{t}{\beta}}\, dt$$

integration by parts $\quad u = t^{\alpha-1} \quad v = e^{-\frac{t}{\beta}}$

$$\Rightarrow \mathbb{P}(X \le x) = \frac{1}{(\alpha-1)!\,\beta^\alpha}\left[\left. t^{\alpha-1} \cdot -\beta e^{-\frac{t}{\beta}} \right]_0^x + (\alpha-1)\beta \int_0^x t^{\alpha-2} e^{-\frac{t}{\beta}}\,dt\right.$$

$$\mathbb{P}(X \le x) = \frac{1}{(\alpha-1)!\,\beta^\alpha}\left[-x^{\alpha-1} \cdot \beta e^{-\frac{x}{\beta}} + (\alpha-1)\beta \cdot \Gamma(\alpha-1)\beta^{\alpha-1} \cdot \mathbb{P}(\right.$$

$\Downarrow$

Exercise 3.19

(3) Chi-square with p degrees of freedom:

in gamma distribution set

$\alpha = \frac{p}{2}$, where p is an integer

$\beta = 2$ $\qquad X \sim \text{gamma}\left(\frac{p}{2}, 2\right)$

$$\boxed{f(x|p) = \frac{1}{\Gamma(p/2)\, 2^{p/2}}\, x^{\frac{p}{2}-1} \cdot e^{-\frac{x}{2}} \qquad 0 < x < \infty}$$

Chi-square with p-d.o.f.

useful especially when sampling from a Normal distribution.

$$\mathbb{E}[X] = \alpha\beta = \frac{P}{2} \cdot 2 = P$$

$$Var(X) = \alpha\beta^2 = \frac{P}{2} \cdot 4 = 2P$$

$$M_X(t) = \left(\frac{1}{1-\beta t}\right)^{\alpha}, \quad t < \frac{1}{\beta}$$

$$\Rightarrow M_X(t) = \left(\frac{1}{1-2t}\right)^{P/2}, \quad t < \frac{1}{2}$$

4 Exponential pdf:

$$X \sim gamma(\alpha = 1, \beta) \qquad 0 < x < \infty$$

$$\boxed{f(x \mid \beta) = \frac{1}{\beta} e^{-\frac{x}{\beta}}, \quad 0 < x < \infty}$$

* Exponential distribution is analogulus to geometric distribution in Discrete
* it also share's Memoryless ness property

$$\mathbb{P}(X > s \mid X > t) = P(X > s-t)$$

$$\Rightarrow \quad \mathbb{P}(X > s \mid X > t) = \frac{\mathbb{P}(X > s, X > t)}{\mathbb{P}(X > t)}$$

$$= \frac{\mathbb{P}(X > s)}{\mathbb{P}(X > t)}$$

$$= \frac{\int_s^\infty \frac{1}{\beta} e^{-\frac{x}{\beta}} \, dx}{\int_t^\infty \frac{1}{\beta} e^{-\frac{x}{\beta}} \, dx}$$

$$= \frac{e^{-s/\beta}}{e^{-t/\beta}} = e^{-(s-t)/\beta}$$

$$= \mathbb{P}(X > s-t)$$

5 **Weibull distribution:**

$$X \sim \text{exponential}(\beta)$$

$$Y = X^{1/\gamma} \Rightarrow Y \sim \text{Weibull}(\gamma, \beta)$$

$$g(x) = x^{1/\gamma} \qquad x = g^{-1}(y) = y^{\gamma}$$

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$= \frac{1}{\beta} e^{-\frac{y^{\gamma}}{\beta}} \cdot \gamma \cdot y^{\gamma-1}$$

$$\boxed{f_Y(y) = \frac{\gamma}{\beta} y^{\gamma-1} \cdot e^{-\frac{y^{\gamma}}{\beta}} \qquad \gamma>0, \beta>0, 0<y<\infty}$$

6 Normal Distribution:

→ it Play's a Central role in a Large body of statistics.

Reason's

1. The Normal & distribution's associated with it are very tractable analytically.
2. Normal distribution has the familiar bell shape, whose symmetry makes it

an appealing choice for many population models.

(3) CLT show's that under mild condition's, the Normal distribution's can be used to approximate a large variety of distribution's in large samples.

$$X \sim N(\mu, \sigma^2)$$

$$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$-\infty < x < \infty$$

The r.v $Z = \frac{X-\mu}{\sigma}$ has a $N(0,1)$ distribution.

$$Z = \frac{X-\mu}{\sigma} = g(x) \Rightarrow X = g^{-1}(z) = \sigma Z + \mu$$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\sigma z + \mu - \mu)^2}{2\sigma^2}} \cdot \sigma$$

$$\Rightarrow \quad \boxed{f_Z(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{z^2}{2}}}$$

$$\mathbb{E}[Z] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \cdot e^{-\frac{z^2}{2}} \, dz$$

$$\frac{z^2}{2} = u \Rightarrow \quad z\,dz = du$$

$$= \frac{1}{\sqrt{2\pi}} \int e^{-u} \, du$$

$$= -\frac{1}{\sqrt{2\pi}} e^{-u} = -\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \Big]_{-\infty}^{+\infty}$$

$$= 0$$

$$\mathbb{E}[Z] = 0$$

$$\Rightarrow \quad \mathbb{E}[X] = \mathbb{E}[\sigma Z + \mu]$$

$$= \sigma \underbrace{\mathbb{E}[Z]}_{=0} + \mu = \mu$$

$$\Rightarrow \quad \mathbb{E}[X] = \mu$$

FACT: $$I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}$$

Proof:

$$I^2 = \int_0^{\infty} e^{-\frac{x^2}{2}} dx \cdot \int_0^{\infty} e^{-\frac{y^2}{2}} dy$$

$$= \int_0^{\infty}\int_0^{\infty} e^{-\frac{(x^2+y^2)}{2}}\, dx\, dy$$

Polar - co-ordinates

$$x = r\cos\theta$$
$$y = r\sin\theta \implies dx\,dy = r\,dr\,d\theta$$

$$= \int_0^{\infty}\int_0^{2\pi} e^{-\frac{r^2}{2}}\, r\, dr\, d\theta$$

$$= 2\pi \int_0^{\infty} r \cdot e^{-\frac{r^2}{2}}\, dr$$

$$\frac{r^2}{2} = u \Rightarrow r\,dr = du$$

$$= 2\pi \int e^{-u}\,du$$

$$= 2\pi \left[-e^{-\frac{x^2}{2}} \right]_0^\infty$$

$$= 2\pi$$

$$\Rightarrow \quad I^2 = 2\pi \Rightarrow I = \sqrt{2\pi}$$

$$\boxed{\Rightarrow \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}}$$

$$Z \sim N(0,1) \Rightarrow \mathbb{E}[Z] = 0 \quad Var(Z) = 1$$

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \; ; \; \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

$\mathbb{E}[X] = \mu \qquad \text{var}(X) = \text{var}(\sigma Z + \mu)$

$= \sigma^2 \text{var}(Z) = \sigma^2$

$$\Phi(-z) = 1 - \Phi(z)$$

FACT:

$$f_X(x) = f_Z\left(\frac{x-\mu}{\sigma}\right) \cdot \frac{1}{\sigma}$$

Proof:

$$F_X(x) = \mathbb{P}(X \le x)$$

$$= \mathbb{P}(\sigma Z + \mu \le x)$$

$$= \mathbb{P}\left(Z \le \frac{x-\mu}{\sigma}\right)$$

$$F_X(x) = F_Z\left(\frac{x-\mu}{\sigma}\right)$$

$$\boxed{\frac{d}{dx} F_X(x) = f_X(x) = f_Z\left(\frac{x-\mu}{\sigma}\right) \cdot \frac{1}{\sigma}}$$

Problem $X \sim N(\mu, \sigma^2)$ $X = \mu + \sigma Z$

$$\mathbb{P}(a < X < b) = \mathbb{P}\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

FACT:

$\mathbb{P}(|Z| \leq 1) \approx 0.6826$

$\mathbb{P}(|Z| \leq 2) \approx 0.9544$

$\mathbb{P}(|Z| \leq 3) \approx 0.9974$

if $X \sim N(\mu, \sigma^2)$

$$\mathbb{P}(|Z| \leq 1) \approx 0.6826$$

$$\Rightarrow \mathbb{P}(-1 \leq Z \leq 1) \approx 0.6826$$

$$\Rightarrow \mathbb{P}\left(-1 \leq \frac{X-\mu}{\sigma} \leq 1\right) \approx 0.6826$$

$$\Rightarrow \quad \mathbb{P}(-\sigma \leq X-\mu < \sigma) \approx 0.6826$$

Simillarly

$$\Rightarrow \quad \mathbb{P}(-2\sigma \leq X-\mu \leq 2\sigma) \approx 0.9544.$$

$$\Rightarrow \quad \mathbb{P}(-3\sigma \leq X-\mu \leq 3\sigma) \approx 0.9974.$$

$$\Rightarrow \quad \mathbb{P}(|X-\mu| \leq \sigma) = \mathbb{P}(|Z|<1) = 0.6826$$

$$\mathbb{P}(|X-\mu| \leq 2\sigma) = \mathbb{P}(|Z|<2) = 0.9544$$

$$\mathbb{P}(|X-\mu| \leq 3\sigma) = \mathbb{P}(|Z|<3) = 0.9974.$$

7

Beta Distribution:

The beta family is continuous family in $(0,1)$

$$X \sim \text{beta}(\alpha,\beta)$$

$$f_X(x \mid \alpha,\beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1}$$

$0 < x < 1$

$\alpha > 0$

$\beta > 0$

where

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1}\, dx$$

Moments of Beta distribution:

$$\mathbb{E}[x^n] = \frac{1}{B(\alpha,\beta)} \int_0^1 x^n \cdot x^{\alpha-1} (1-x)^{\beta-1} \, dx$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 x^{\alpha+n-1} (1-x)^{\beta-1} \, dx$$

$$\mathbb{E}[x^n] = \frac{B(\alpha+n, \beta)}{B(\alpha,\beta)}$$

$$\Rightarrow \quad \mathbb{E}[x^n] = \frac{\Gamma(\alpha+n)\,\cancel{\Gamma(\beta)}\;\;\Gamma(\alpha+\beta)}{\Gamma(\alpha+n+\beta)\,\Gamma(\alpha)\,\cancel{\Gamma(\beta)}}$$

$$\boxed{\mathbb{E}[x^n] = \frac{\Gamma(\alpha+n)\,\Gamma(\alpha+\beta)}{\Gamma(\alpha+n+\beta)\,\Gamma(\alpha)}}$$

$$\boxed{\mathbb{E}[x] = \frac{\Gamma(\alpha+1)\,\Gamma(\alpha+\beta)}{\Gamma(\alpha+1+\beta)\,\Gamma(\alpha)} = \frac{\alpha}{\alpha+\beta}}$$

$$\mathbb{E}[x^2] = \frac{\Gamma(\alpha+2)\,\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+2)\,\Gamma(\alpha)} = \frac{(\alpha+1)\alpha}{(\alpha+\beta+1)(\alpha+\beta)}$$

$$\Rightarrow \operatorname{var}(x) = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$= \frac{\alpha(\alpha+1)(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$\boxed{\operatorname{Var}(x) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$$

(8) Cauchy Distribution:

$$f(x \mid \theta) = \frac{1}{\pi} \frac{1}{1+(x+\theta)^2}$$

$-\infty < x < \infty$

$-\infty < \theta < \infty$

$$\mathbb{E}[|X|] = \infty$$

$\Rightarrow$ mean does not exists.

ratio of two standard normal's has a Cauchy distribution.

(9) Lognormal distribution:

$X \sim$ log normal

if $\log X \sim N(\mu, \sigma^2)$

$Y = \log X \sim N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2} \cdot \frac{1}{x}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \frac{1}{x} \cdot e^{-\frac{(\log x - \mu)^2}{2\sigma^2}}$$

$$\mathbb{E}[X] = \mathbb{E}[e^{\log X}] = \mathbb{E}[e^{Y}]$$

we know that MGF of a Normal distribution is

$$M_X(t) = \mathbb{E}[e^{tX}] = e^{\mu t + \sigma^2 t^2/2}$$

where $X \sim N(\mu, \sigma^2)$

There for log Normal $X \sim \log \text{Normal}$

$$\mathbb{E}[X] = \mathbb{E}[e^{\log X}]$$ wher $Y = \log X \sim N(\mu, \sigma^2)$

$$= \mathbb{E}[e^{Y}] = M_Y(1) = e^{\mu + \frac{\sigma^2}{2}}$$

$$\boxed{\mathbb{E}[X] = e^{\mu + \frac{\sigma^2}{2}}}$$

$$\mathbb{E}[X^2] = \mathbb{E}\left[e^{2\log X}\right] = \mathbb{E}\left[e^{2Y}\right]$$

$$= M_Y(2)$$

$$\mathbb{E}[X^2] = e^{2\mu + 2\sigma^2}$$

$$\text{Var}(X) = \mathbb{E}[X^2] - E[X]^2$$

$$= e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2}$$

$$\boxed{\text{Var}(X) = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2}}$$

10 Double Exponential Distribution :-

The double exponential distribution is formed by reflecting the exponential distribut around its mean.

$$f(x \mid \mu, \sigma) = \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}}$$

$-\infty < x < \infty$, $\sigma > 0$

$-\infty < \mu < \infty$

$E[X] = \mu$ $\quad$ $Var(X) = 2\sigma^2$