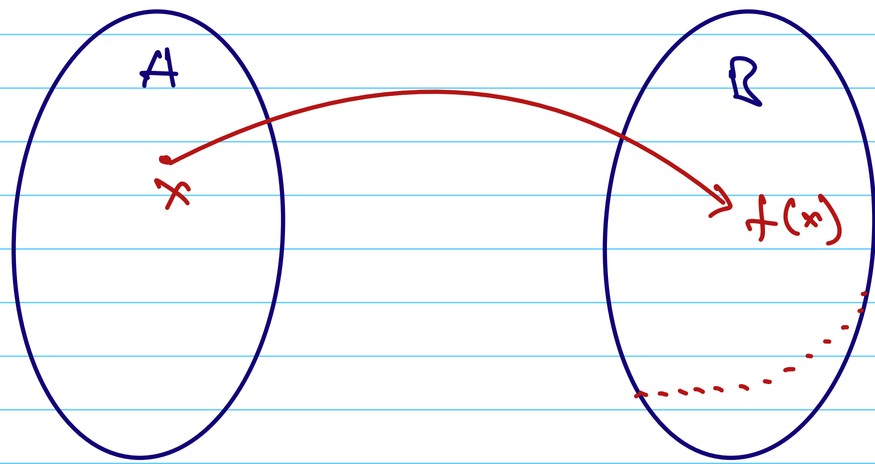
CARDINALITY & COUNTABILITY

function $f: A \longrightarrow B$

A function is a rule that associates every element of A with a unique element of B



Set A : Domain   Set B : Co-Domain

we cannot leave out any element in A, for every element in A there has to be unique element in B.

* $f(x)$ is the image of x
* x is the pre-image of $f(x)$

} terminology

(i) Set A is called Domain

(ii) set B is called co-Domain

(iii) Range , is to describe only those elements of B which take value of function.

Range of set B = $\left\{ y \in B \;\middle|\; \begin{array}{c} \exists x \in A \text{ for each} \\ f(x) = y \end{array} \right\}$

* we can have multiple element's from A mapping to one element in B But not vice-versa.

<u>Injective function:</u> (one-to-one)

Every element in R (Range) has a unique element in A

<u>SURJective Function:</u> (ONTO)

if Range = set B

* a function which is Both Surjective and Injective is called Bijective function.

Bijective = one-to-one + ONTO

= EAch element in the domain is Paired uniquely with an element in co-Domain

For Bijective function, the inverse map is also valid function

CARDINALity:

* IN informal terms, the cardinality of a set is the number of elements in that set.

Cardinality of set A := Number of elements in set A

:= A

* We are Interested in Comparing sizes of Different sets.
* if we are given a finite set's then we can easily calculate cardinality of sets and compare

Ex: $A = \{1, 2, *, \{\pi, e\}\} \Rightarrow |A| = 4$

$B = \{3, 4, 5, 6\} \Rightarrow |B| = 4$

$|A| = |B|$

* But if we go to infinite sets, this approach breaks down

Ex: $A = \mathbb{N}$, $B = \mathbb{Q}$

which set is bigger?

* To find a way for comparing infinite set's, Mathematician CANTOR decided that using the concept of Bijective function we can actually compare the sizes of infinite sets

<u>Def</u>: (i) two sets A and B are equicardinal ($|A| = |B|$) if there exists a Bijective function from A to B

(ii) B has Cardinality greater than or equal to that of A ($|B| \geq |A|$) if there exists an injective function from A to B

$|A| = |B|$ if $\exists\, f : A \longrightarrow B$ (Bijective)

$|B| \geq |A|$ if $\exists\, f : A \longrightarrow B$ (Injective)

(iii) $|B| > |A|$ (strictly greater)

if $\exists$ an injective function $f : A \rightarrow B$ But No bijective function from A to B

There are examples of infinite sets where one infinite set is Bigger than other.

$$|\mathbb{N}| < |\mathbb{R}|$$

* Both Natural number's $\mathbb{N}$ and Real Number's are infinite sets, But infinity of $\mathbb{R}$ is Bigger than infinity of $\mathbb{N}$
* There are sets even Bigger infinity than $\mathbb{R}$

Definition: Countability

A set E is said to be countably infinite if E and $\mathbb{N}$ are equicardinal.

+ And, a set is Said to be countable if it is either finite or countable infinite.

Countable set $\Rightarrow$ either finite or
countably infinite.
(i.e equicardinal with $\mathbb{N}$)

Ex:

$$f : \mathbb{N} \longrightarrow \mathbb{N}$$

$$f(n) = 2n$$

$\Rightarrow$ Set $A = \mathbb{N}$ (all Natural number's)

$\Rightarrow$ set B = even number's

$$\{1, 2, 3, 4, \ldots\} \longrightarrow \{2, 4, 6, 8, \ldots\}$$

We found a Bijection b/w set of Natural number's and even number's

$\Rightarrow$ $|\mathbb{N}| = |\text{set of even number's}|$
equicardinal.

$\Rightarrow$ set of even number's is countably infinite

Ex: Set of Rational number's $\mathbb{Q}$ is countably infinite.

The set of all rational number's b/w $[0, 1]$ i.e $\mathbb{Q} \cap [0, 1]$ is countably finite.

$\Rightarrow$ There are as many rational number's b/w $[0, 1]$ as Natural number's

$\Rightarrow$ Surprisingly all Rational number's $\mathbb{Q}$ is countably infinite, which is there are as many Rational number's as Natural number's.

$$|\mathbb{N}| = |\mathbb{Q}| = |\mathbb{Q} \cap [0, 1]|$$

How do we prove these? Find a Bijection.

Ex: $\mathbb{Q} \cap [0,1]$ rational's in $[0,1]$ is countable set.

$$\mathbb{Q} \cap [0,1] = \left\{ \frac{p}{q} \;\middle|\; \forall q \in \mathbb{N}, q \neq 0, \; 0 \leq p \leq q \text{ if } \frac{p}{q} \text{ is not already present} \right\}$$

$$\Rightarrow \mathbb{Q} \cap [0,1] = \left\{ \frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5} \ldots \right\}$$

$$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \cdots$$

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9 \ldots\}$$

$\Rightarrow$ although there seem's so many rationals b/w $[0,1]$ but it is actually is in Bijection with Natural number's.

$$\Rightarrow |\mathbb{N}| = |\mathbb{Q} \cap [0,1]|$$

Uncountable sets:

There are infinite set's which are truly bigger than $\mathbb{N}$, they are called uncountable sets

Theorem:

let I be a countable index set, and let E_i be collection of countable sets for each $i \in I$. Then $\bigcup_{i=I} E_i$ is countable.

(countable union of countable sets is countable)

Ex: $I = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (countable)

$E_i := \mathbb{Q} \cap [i, i+1]$ (countable infinite)

$\Rightarrow$ $E_1 = \mathbb{Q} \cap [0,1]$; $E_2 = \mathbb{Q} \cap [1,2] \ldots$

then the countable union

$\bigcup_{i=I} E_i$ is countable infinite

(Not proving in lecture)

Corollary:

Using this we can Prove set of all Rational number are countable

$$\mathbb{Q} = \bigcup_{i \in \mathbb{Z}} \mathbb{Q} \cap [i, i+1]$$

$\mathbb{Z}$: countable

$|\mathbb{Q} \cap [i, i+1]|$ = countabe infinite

$\Rightarrow$ countable union of countable sets $\Rightarrow$ countable

$\mathbb{Q}$ is countable set.

* There are only as many rational number's as natural number's

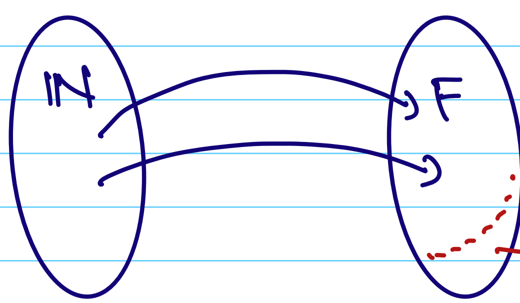
Uncountable sets:

There are certain infinite sets which are truely larger in cardinality than Natural number's.

Definite: A set F is countable if it has cardinality strictly larger than $\mathbb{N}$.

$$|F| > |\mathbb{N}|$$

$\Rightarrow$ $\exists$ an injective function from $\mathbb{N}$ to F But No Bijective.



there are few elements left.

Ex: $\mathbb{R}$ uncountable

$\mathbb{R} \setminus \mathbb{Q}$ uncountable

$[0, 1]$ uncountable

$\{0,1\}^\infty$ set of all infinite Binary string's. (uncountable)

$2^{\mathbb{N}}$ (uncountable)
(Power set)

(all subsets of Natural number's)

Theorem: $\{0,1\}^\infty$ is an uncountable set

Proof: it is easy to produce an injection from $\mathbb{N} \longrightarrow S$ but not Bijective.

for example lets define

$f(1) = 1\ 0\ 0\ 0\ 0\ 0\ 0 \ldots$

$f(2) = 0\ 1\ 0\ 0\ 0\ 0\ 0 \ldots$

$f(3) = 0\ 0\ 1\ 0\ 0\ 0\ 0 \ldots$

$f(4) = 0\ 0\ 0\ 1\ 0\ 0\ 0 \ldots$

$f(5) = 0\ 0\ 0\ 0\ 1\ 0\ 0 \ldots$

$\vdots$

in order to prove that no bijection exists
Assume the contrary. which mean's
assume a bijection exists and prove
contradiction.

therefore. assume that there exists a following
Listing where each $a_{ij} = 0$ or 1

$$
\begin{array}{cccccc}
a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\
a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\
a_{31} & a_{32} & a_{33} & a_{34} & \cdots \\
a_{41} & a_{42} & a_{43} & a_{44} & \cdots \\
a_{51} & a_{52} & a_{53} & a_{54} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \\
a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Assuming this listing complete's the
$\{0, 1\}^{\infty}$ and prove it wrong.

⟹ Let's produce a infinite binary string that cannot be contain in the above Lists

Consider diagonal Guys (string)

$$
\begin{array}{lccccc}
1st & a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\
2nd & a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\
3rd & a_{31} & a_{32} & a_{33} & a_{34} & \cdots \\
4th & a_{41} & a_{42} & a_{43} & a_{44} & \cdots \\
5th & a_{51} & a_{52} & a_{53} & a_{54} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \\
nth & a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Consider: $a_{11}\ a_{22}\ a_{33}\ a_{44} \cdots$ & Flip the Bits

$\Rightarrow \overline{a_{11}}\ \overline{a_{22}}\ \overline{a_{33}}\ \overline{a_{44}} \cdots$

Ex: the diagonal elements are

1 0 0 1 0 1 1 1

flip Now

$\Rightarrow$ 0 1 1 0 1 0 0 0

(this is a valid infinite string)

$\Rightarrow$ The key is in understanding this string is not contained in above List of String's.

$\Rightarrow$ Suppose this string is contained at mth position, then mth Bit is wrong (Because we Flipped)

Ex: this string is contained at 100th position. then $a_{100,100}$ is wrong

In order to Prove some other set is uncountable Ex: $[0,1]$, then find a bijection b/w $[0,1]$ & $\{0,1\}^\infty$

$\Rightarrow$ $\{0,1\}^\infty$ not in Bijection with $\mathbb{N}$

$\Rightarrow$ $[0,1]$ is uncountable.

Or find injection $f: \{0,1\}^\infty \longrightarrow$ set B

then $|B| \geq |\{0,1\}^\infty| \Rightarrow B$ is uncountable.

Ex: $[0,1]$ is uncountable

Do the Binary expansion.

$\Rightarrow$ $x \in [0,1]$ write x in binary expansion

$$x = 0.a_1a_2a_3\ldots$$

$$\Big\downarrow \text{bijection}$$

$$\{0,1\}^\infty$$

$\mathbb{R}$ is uncountable

$\Rightarrow$ $f(x) = \tan\left(\pi x - \frac{\pi}{2}\right)$
where $x \in [0,1]$

$f : [0,1] \longrightarrow \mathbb{R}$ is bijective

$\Rightarrow$ $[0,1]$ is uncountable
$\Rightarrow$ $\mathbb{R}$ is uncountable.

Ex: set of irrational number's

$$\underbrace{\mathbb{R}}_{\text{uncountable}} = (\mathbb{R} \setminus \mathbb{Q}) \cup \underbrace{\mathbb{Q}}_{\text{countable}}$$

suppose $\mathbb{R} \setminus \mathbb{Q}$ is countable

then $(\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q}$ is countable

But $\mathbb{R}$ is countable $\Longrightarrow$
$\mathbb{R} \setminus \mathbb{Q}$ is uncountable