DESCRIPTION

This document provides solution's to 4.39 to 4.50 from chapter 4 of statistical inference Book by Casella and Berger, Focusing on covariance and correlation and multivariate Distribution's.

A few Problem's are yet to be solved, but I will update the document soon.

4.39 Let $(X_1, X_2, \ldots X_n)$ have a multinomial distribution with m trails and cell probabilities $p_1, \ldots p_n$

Show that for every i and j

$$X_i | X_j = x_j \sim \text{binomial}\left(m - x_j, \frac{p_i}{1-p_j}\right)$$

$$X_j \sim \text{binomial}(m, p_j)$$

and that $Cov(X_i, X_j) = -m p_i p_j$

Solution:

Let $x_1, x_2, \ldots \quad x_n$

$p_1, p_2, \ldots \quad p_n$

for every trail the outcome can be x_i with Prob p_i

$X_i = \#$ number of x_i outcomes in m trails.

$X_i | X_j \Rightarrow$ X_i number of x_i outcome's given that there are X_j outcome's of x_j's in m trails.

$$f(x_1, x_2, x_3 \ldots x_{j-1}, x_{j+1} \ldots x_n \mid x_j)$$

$$= \frac{f(x_1, x_2, \ldots x_{j-1}, x_j, x_{j+1} \ldots, x_n)}{f(x_j)}$$

marginal distribution of x_j

$$f(x_j) = \sum_{x_n=0}^{m} \ldots \sum_{x_{j+1}=0}^{m} \sum_{x_{j-1}=0}^{m} \ldots \sum_{x_3=0}^{m} \sum_{x_2=0}^{n} \ldots \sum_{x_1=0}^{m} f(x_1, x_2, \ldots x_j \ldots x_n)$$

$$= \sum_{(x_1, \ldots x_{j-1}, x_n)} \frac{m! \; (p_1)^{x_1} (p_2)^{x_2} \ldots p_n^{x_n}}{x_1! \cdot x_2! \cdots x_{j-1}! \; x_j! \cdots x_n!}$$

$$= \sum \frac{m!}{x_1! \cdots x_n!} p_1^{x_1} \cdot p_2^{x_2} \cdots p_n^{x_n} \cdot \frac{(m-x_j)!}{(m-x_j)!} \cdot \frac{(1-p_j)^{m-x_j}}{(1-p_j)^{m-x_j}}$$

$$= \binom{m}{x_j} p_j^{x_j} \cdot (1-p_j)^{m-x_j} \times$$

$$\sum \frac{(m-x_j)!}{x_1!\, x_2! \cdots x_{j-1}!\, x_{j+1}! \cdots x_n!} \times \frac{p_1^{x_1} \cdot p_2^{x_2} \cdots p_{j-1}^{x_{j-1}}\, p_{j+1}^{x_{j+1}} \cdots p_n}{(1-p_j)^{m-x_j}}$$

$$= \binom{m}{x_j} p_j^{x_j} (1-p_j)^{m-x_j} \times$$

$$\underbrace{\sum \frac{(m-x_j)!}{x_1!\, x_2! \cdots x_{j-1}!\, x_{j+1}! \cdots x_n!} \left(\frac{p_1}{1-p_j}\right)^{x_1} \left(\frac{p_2}{1-p_j}\right)^{x_2} \cdot \left(\frac{p_{j-1}}{1-p_j}\right)^{x_{j-1}} \left(\frac{p_{j+1}}{1-p_j}\right)^{x_{j+1}} \cdots \left(\frac{p_n}{1-p_j}\right)^{x_n}}_{1}$$

$$= \binom{m}{x_j} p_j^{x_j} (1-p_j)^{m-x_j}$$

$$\Rightarrow \quad X_j \sim \text{binomial}\,(m, p_j)$$

$$\mathbb{P}(X_i \mid X_j) = f(x_i \mid x_j)$$

$$= \frac{f(x_i, x_j)}{f(x_j)}$$

$$= \frac{\dfrac{\cancel{m!}}{x_i!\,\cancel{x_j!}\,(m-x_i-x_j)!}\; p_i^{x_i}\, \cancel{p_j^{x_j}}\, (1-p_i-p_j)^{m-x_i-x_j}}{\dfrac{\cancel{m!}}{\cancel{x_j!}\,(m-x_j)!}\; \cancel{p_j^{x_j}}\, (1-p_j)^{m-x_j}}$$

$$= \frac{(m-x_j)!}{x_i!\,(m-x_i-x_j)!} \; \frac{p_i^{x_i} \cdot (1-p_i-p_j)^{m-x_i-x_j}}{(1-p_j)^{m-x_j}}$$

$$= \binom{m-x_j}{x_i} \frac{p_i^{x_i} \cdot (1-p_i-p_j)^{m-x_i-x_j}}{(1-p_j)^{m-x_j}}$$

$$= \binom{m-x_j}{x_i} \frac{p_i^{x_i}}{(1-p_j)^{x_i}} \left(\frac{1-p_i-p_j}{1-p_j}\right)^{m-x_i-x_j}$$

$$= \binom{m-x_j}{x_i} \left(\frac{p_i}{1-p_j}\right)^{x_i} \left(1 - \frac{p_i}{1-p_j}\right)^{m-x_j-x_i}$$

$$\Rightarrow \quad X_i \mid X_j \sim \text{binomial}\left(m-x_j, \frac{p_i}{1-p_j}\right)$$

(b) $\text{cov}(X_i, X_j) = -m p_i p_j$

<u>Solution:</u>

$$X_i + X_j \sim \text{binomial}(m, p_i + p_j)$$

$$\text{var}(X_i + X_j) = m(p_i + p_j)(1 - p_i - p_j)$$

$$\Rightarrow \text{var}(X_i + X_j) = \text{var}(X_i) + \text{var}(X_j) + 2\,\text{cov}(X_i, X_j)$$

$$\Rightarrow m(p_i + p_j)(1 - p_i - p_j)$$

$$= m p_i (1 - p_i) + m p_j (1 - p_j) + 2\,\text{cov}(X_i, X_j)$$

$$\Rightarrow m p_i \cancel{(1-p_i)} - m p_i p_j + m \cancel{p_j}(1-p_j) - m p_j p_i$$

$$= m p_i \cancel{(1-p_i)} + m \cancel{p_j}(1-p_j) + 2\,\mathrm{cov}(X_i, X_j)$$

$$\boxed{\Rightarrow \quad \mathrm{cov}(X_i, X_j) = -m p_i p_j}$$

4.40 A Generalization of beta distribution is the Dirichlet Distribution. In its bivariate version, (X, Y) have pdf,

$$f(x,y) = C x^{a-1} y^{b-1} (1-x-y)^{c-1}$$

$0 < x < 1$ where $a > 0$, $b > 0$

$0 < y < 1$ $c > 0$

$0 < y < 1-x < 1$ are constants.

(a) Show that $C = \dfrac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}$

Solution:

$$\int_0^1 \int_0^1 f(x,y)\, dx\, dy = 1$$

$$= \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} C \; x^{a-1} \, y^{b-1} \, (1-x-y)^{c-1} \, dy\, dx$$

$$= C \int_0^1 \int_0^{1-x} x^{a-1} \, y^{b-1} \, (1-x-y)^{c-1} \, dy\, dx$$

$$= C \int_0^1 x^{a-1} \cdot (1-x)^{b+c-2} \int_0^{1-x} \frac{y^{b-1}}{(1-x)^{b-1}} \cdot \left(\frac{1-x-y}{1-x} \right)^{c-1} dy\, dx$$

$$= C \int_0^1 x^{a-1} \cdot (1-x)^{b+c-2} \int_0^{1-x} \left(\frac{y}{1-x} \right)^{b-1} \cdot \left(1 - \frac{y}{1-x} \right)^{c-1} dy\, dx$$

$$\frac{y}{1-x} = u \qquad dy = (1-x)\, du$$

$$= C \int_0^1 x^{a-1} \cdot (1-x)^{b+c-1} \int_0^1 u^{b-1} (1-u)^{c-1} \, du \, dx$$

$$= C \int_0^1 x^{a-1} \cdot (1-x)^{b+c-1} \cdot \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)} \, dx$$

$$= C \cdot \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)} \cdot \int_0^1 x^{a-1} \cdot (1-x)^{b+c-1} \, dx$$

$$1 = C \cdot \frac{\Gamma(\cancel{b+c})}{\Gamma(b)\Gamma(c)} \cdot \frac{\Gamma(a+b+c)}{\Gamma(a)\,\Gamma(\cancel{b+c})}$$

$$\Rightarrow \boxed{C = \frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c)}{\Gamma(a+b+c)}}$$

4.40b Show that marginally, both X and Y are beta

$$f_X(x) = \int_y f(x,y)\, dy$$

$$= \int_0^{x-1} C \cdot x^{a-1} y^{b-1} (1-x-y)^{c-1} dy$$

$$= C \cdot x^{a-1} \cdot (1-x)^{b+c-1} \int_0^{x-1} \left(\frac{y}{1-x}\right)^{b-1} \left(1-\frac{y}{1-x}\right)^{c-1} \cdot \frac{dy}{1-x}$$

$$u = \frac{y}{1-x} \Rightarrow du = \frac{dy}{1-x}$$

$$u: 0 \to 1$$

$$f_X(x) = C \cdot x^{a-1} \cdot (1-x)^{b+c-1} \int_0^1 u^{b-1} \cdot (1-u)^{c-1} du$$

$$f_X(x) = C \cdot x^{a-1} \cdot (1-x)^{b+c-1} \cdot \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)}$$

$$f_X(x) = \frac{\Gamma(a+b+c)}{\Gamma(b+c)\,\Gamma(a)}\, x^{a-1}(1-x)^{b+c-1}$$

$0<x<1$

$$X \sim beta(a, b+c)$$

Simillarly

$$f_Y(y) = C \cdot \int_{x=0}^{1-y} x^{a-1} \cdot y^{b-1} (1-x-y)^{c-1}\, dx$$

$$= C \cdot y^{b-1}(1-y)^{a+c-2} \int_{x=0}^{1-y} \left(\frac{x}{1-y}\right)^{b-1} \left(1-\frac{x}{1-y}\right)^{c-1} dx$$

$$= C \cdot y^{b-1}(1-y)^{a+c-1} \int_{u=0}^{1} u^{b-1}(1-u)^{c-1}\, du$$

$$f_X(y) = \frac{\Gamma(a+b+c)}{\Gamma(b)\,\Gamma(a+c)}\, y^{b-1}(1-y)^{a+c-1}$$

$$\Rightarrow \quad Y \sim beta(b, a+c)$$

4.40c

Find the conditional distribution of $Y|X=x$, and show that

$$\frac{Y}{1-X} \text{ is } beta(b, c)$$

soln

$$f(Y|X=x) = \frac{f(x, Y)}{f(X=x)}$$

$$= \frac{\Gamma(\cancel{a+b+c})}{\Gamma(\cancel{a})\Gamma(b)\Gamma(c)} \cdot \frac{\cancel{x^{a-1}} y^{b-1} \cdot (1-x-y)^{c-1}}{\frac{\Gamma(\cancel{a+b+c})}{\Gamma(\cancel{a})\Gamma(b+c)} \cdot \cancel{x^{a-1}} \cdot (1-x)^{b+c-1}}$$

$$f(Y|X=x) = \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)} \cdot \left(\frac{y}{1-x}\right)^{b-1}\left(1-\frac{y}{1-x}\right)^{c-1}$$

$$\Rightarrow \quad Y|X=x \sim beta(b, c)$$

take $\frac{Y}{1-x} = u$

$$\Rightarrow f_U(u) = f\left(Y = u_0(1-x) \mid X = x\right)$$

$$= \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)} u^{b-1} \cdot (1-u)^{c-1}$$

$$\Rightarrow \quad U \sim beta(b, c)$$

$$\boxed{\Rightarrow \quad \frac{Y}{1-x} \sim beta(b, c)}$$

(4.40d) Show that $\mathbb{E}[XY] = \dfrac{ab}{(a+b+c+1)(a+b+c)}$

and find $Cov(X, Y)$

Sol^n

$$\mathbb{E}[XY] = \int_0^1 \int_0^{1-x} C \cdot x^a y^b (1-x-y)^{c-1} \, dy \, dx$$

$$\Rightarrow \mathbb{E}[XY] = C(a, b, c) \cdot \frac{1}{C(a+1, b+1, c)}$$

$$= \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(\cancel{c})} \cdot \frac{\Gamma(a+1)\,\Gamma(b+1)\,\cancel{\Gamma(c)}}{\Gamma(a+b+c+2)}$$

$$= \frac{\Gamma(\cancel{a+b+c})}{\cancel{\Gamma(a)}\,\cancel{\Gamma(b)}} \cdot \frac{a\,\cancel{\Gamma(a)}\;\; b\,\cancel{\Gamma(b)}}{(a+b+c+1)\,(a+b+c)\cdot\Gamma(\cancel{a+b+c})}$$

$$\mathbb{E}[XY] = \frac{ab}{(a+b+c+1)\,(a+b+c)}$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}[X,Y] - \mathbb{E}[X]\,\mathbb{E}[Y]$$

$$= \frac{ab}{(a+b+c+1)\,(a+b+c)} - \frac{a}{a+b+c} \cdot \frac{b}{a+b+c}$$

$$\boxed{\mathrm{Cov}(X,Y) = \frac{ab}{a+b+c}\left[\frac{1}{a+b+c+1} - 1\right]}$$

4.41 Show that any random variable is un co-related with a variable.

Solution:

$$\mathrm{Cov}(a, X) = \mathbb{E}[aX] - E[a]\,E[X]$$

$$= aE[X] - a\,E[X]$$

$$\boxed{\mathrm{Cov}(a, X) = 0}$$

4.42 $X, Y \sim$ independent r.v with mean's μ_x, μ_y and variance σ_x^2, σ_y^2.

find the expression for the co-rrelation of XY and Y in term's of these mean's and variances.

Solution:

$$\rho_{XY,Y} = \frac{\mathrm{Cov}(XY, Y)}{\sqrt{\mathrm{var}(XY)\cdot \mathrm{var}(Y)}}$$

$$= \frac{E[XY^2] - E[XY]E[Y]}{\sigma_Y \cdot \sqrt{E[X^2Y^2] - E[XY]^2}}$$

$$= \frac{\mu_X E[Y^2] - \mu_X \mu_Y^2}{\sigma_X \sqrt{E[X^2]E[Y^2] - \mu_X^2 \mu_Y^2}}$$

$$= \frac{\mu_X(\sigma_Y^2 + \mu_Y^2) - \mu_X \mu_Y^2}{\sigma_Y \sqrt{(\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - \mu_X^2 \mu_Y^2}}$$

$$\boxed{Cov(XY, Y) = \frac{\mu_X \sigma_Y}{\sqrt{\sigma_X^2 \sigma_Y^2 + \sigma_X^2 \mu_Y^2 + \sigma_Y^2 \mu_X^2}}}$$

4.43 Let X_1, X_2, and X_3 be uncorrelated r.v, each with mean μ and variance σ^2. Find, in terms of μ and σ^2, $Cov(X_1 + X_2, X_2 + X_3)$ and

$\text{cov}(X_1+X_2,\ X_1-X_2)$.

Solution:

$$\text{cov}(X_1+X_2,\ X_1-X_2)$$

$$= \mathbb{E}[(X_1+X_2)(X_1-X_2)] - \mathbb{E}[X_1+X_2]\,\mathbb{E}[X_1-X_2]$$

$$= \mathbb{E}[X_1^2 - X_2^2] - 0$$

$$= \mathbb{E}[X_1^2] - \mathbb{E}[X_2^2]$$

$$= \sigma^2 + \mu^2 - \sigma^2 - \mu^2 = 0$$

$$\text{cov}(X_1+X_2,\ X_2+X_3)$$

$$= \mathbb{E}[X_1X_2 + X_1X_3 + X_2^2 + X_2X_3]$$

$$- \mathbb{E}[X_1+X_2]\,\mathbb{E}[X_2+X_3]$$

$$= \mu^2 + \mu^2 + \sigma^2 + \mu^2 + \mu^2 - 4\mu^2$$

$$= \sigma^2$$

$$\boxed{\text{cov}(X_1+X_2,\ X_2+X_3) = \sigma^2}$$

(4.44) Prove the generalization of Theorem 4.5.6. For any random variable $(X_1, X_2, \ldots X_n)$

$$\mathrm{var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{var}(X_i) + 2\sum_{1\le i<j\le n} \mathrm{Cov}(X_i, X_j)$$

Solution:

$$\mathrm{var}(X_1 + X_2 + \cdots + X_n)$$

$$= \mathbb{E}\left[\left(X_1 + X_2 + \cdots + X_n - E[X_1] - E[X_2] \cdots E[X_n]\right)^2\right]$$

$$= \mathbb{E}\left[\left((X_1 - \mu_1) + (X_2 - \mu_2) + \cdots + (X_n - \mu_n)\right)^2\right]$$

$$= \mathbb{E}\left[\sum_{i=1}^{n} (X_i - \mu_i)^2 + \sum_{\substack{1\le i\le n \\ 1\le j\le n \\ i\ne j}} (X_i - \mu_i)(X_j - \mu_j)\right]$$

$$= \sum_{i=1}^{n} \mathbb{E}\left[(X_i-\mu_i)^2\right]$$

$$+ \sum_{\substack{1\le i\le n \\ 1\le j\le n \\ i\ne j}} \mathbb{E}\left[(X_i-\mu_i)(X_j-\mu_j)\right]$$

$$\boxed{= \sum_{i=1}^{n} \operatorname{var}(X_i) + 2\sum_{1\le i\le j<n} \operatorname{cov}(X_i, X_j)}$$

4.45 Show that if $(X,Y) \sim$ bivariate normal $(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$ then the following are true.

(a) The marginal distribution of X is $N(\mu_X, \sigma_X^2)$ and the marginal distribution of Y is $N(\mu_Y, \sigma_Y^2)$

Sol^n

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \times \exp\left(\frac{-1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right\}\right)$$

$$f_X(x) = \int_{y=-\infty}^{\infty} f(x,y)\,dy$$

$$u = \frac{x-\mu_x}{\sigma_x}$$

$$v = \frac{y-\mu_y}{\sigma_y}$$

$$dy = \sigma_y\,dv$$

$$f_X(x) = \frac{1\cdot\sigma_y}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2(1-\rho^2)}\left(u^2+v^2-2\rho uv\right)\right) dv$$

$$= \frac{1}{2\pi\sigma_x\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2(1-\rho^2)}\left(v^2-2\rho uv+\rho u^2-\rho^2u^2+u^2\right)\right) dv$$

$$= \frac{\exp\left(-\frac{u^2}{2}\right)}{2\pi\sigma_x\cdot\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2(1-\rho^2)}(v-\rho u)^2\right) du$$

$$\Downarrow$$

$$N(\rho v,\ 1-\rho^2)$$

$$f_X(x) = \frac{\exp\left(\frac{-u^2}{2}\right)}{2\pi\sigma_x \cdot \sqrt{1-\rho^2}} \cdot \sqrt{2\pi}\sqrt{1-\rho^2}$$

$$\Rightarrow \quad f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left(\frac{-u^2}{2}\right)$$

$$\Rightarrow \quad f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left(-\frac{1}{2\sigma_x^2}(x-\mu_x)^2\right)$$

$$X \sim N(\mu_x, \sigma_x^2)$$

similar proof for $Y \sim N(\mu_y, \sigma_y^2)$

4.45b The conditional distribution of Y given $X = x$ is

$$N\left(\mu_Y + \rho\left(\frac{\sigma_Y}{\sigma_X}\right)(x-\mu_X),\ \sigma_Y^2(1-\rho^2)\right)$$

Solution:

$$f(Y \mid X=x) = \frac{f(x,y)}{f(X=x)}$$

$$= \frac{\dfrac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \cdot \exp\left(\dfrac{-1}{2(1-\rho^2)}\left(\left(\dfrac{x-\mu_x}{\sigma_x}\right)^2 + \left(\dfrac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\dfrac{x-\mu_x}{\sigma_x}\right)\left(\dfrac{y-\mu_y}{\sigma_y}\right)\right]\right.}{\dfrac{1}{\sqrt{2\pi}\sigma_x} \cdot \exp\left(-\dfrac{1}{2}\left(\dfrac{x-\mu_x}{\sigma_x}\right)^2\right)}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y(\sqrt{1-\rho^2})} \cdot \exp\left(\frac{-1}{2(1-\rho^2)}\left(\left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{x-\mu_x}{\sigma_x}\right)^2 \cdot \rho^2\right)\right)$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y(\sqrt{1-\rho^2})} \exp\left(\frac{-1}{2(1-\rho^2)}\left(\frac{y-\mu_y}{\sigma_y} - \rho\left(\frac{x-\mu_x}{\sigma_x}\right)\right)^2\right)$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y(\sqrt{1-\rho^2})} \exp\left(\frac{-1}{2\sigma_y(1-\rho^2)}\left[y - \left[\mu_y + \frac{\sigma_y}{\sigma_x} \cdot \rho(x-\mu_x)\right]\right]\right.$$

This is a normal pdf

$$\Rightarrow \boxed{Y \mid_{X=x} \sim N\left(\mu_y + \frac{\sigma_y}{\sigma_x} \cdot \rho(x-\mu_x) \,,\, \sigma_y^2(1-\rho^2)\right)}$$

4.4.5 C for any constants a and b, the Distribution of $aX+bY$ is $N(a\mu_X + b\mu_Y \,,\, a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y)$

Sol^n

$$\mathbb{E}[aX+bY] = a\mu_X + b\mu_Y$$

$$\mathrm{Var}(aX+bY) = \mathbb{E}\left[(aX+bY-a\mu_X-b\mu_Y)^2\right]$$

$$= \mathbb{E}\left[a^2(X-\mu_X)^2\right] + \mathbb{E}\left[b^2(Y-\mu_Y)^2\right] + 2ab\,\mathbb{E}\left[(X-\mu_X)(Y-\mu_Y)\right]$$

$$\mathrm{var}(aX+bY) = a^2\,\mathrm{var}(X) + b^2\,\mathrm{var}(Y) + 2ab\,\rho\,\mathrm{cov}(X,Y)$$

distribution of $aX + bY$?

$$\begin{aligned} U &= aX+bY \\ V &= Y \end{aligned} \Rightarrow \left\{ \begin{aligned} X &= \frac{U-bV}{a} \\ Y &= V \end{aligned} \right.$$

$$|J| = \begin{vmatrix} \frac{1}{a} & -\frac{b}{a} \\ 0 & 1 \end{vmatrix} = \frac{1}{a}$$

$$\frac{X-\mu_X}{\sigma_X} = \frac{\frac{U-bV}{a} - \mu_X}{\sigma_X} = \frac{U-bV-a\mu_X}{\sigma_X a}$$

$$\frac{Y-\mu_Y}{\sigma_Y} = \frac{V-\mu_Y}{\sigma_Y}$$

$$\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right) = \left(\frac{U-bV-a\mu_X}{\sigma_X a}\right)\left(\frac{V-\mu_Y}{\sigma_Y}\right)$$

$$\left(\frac{x-u_x}{\sigma_x}\right)^2+\left(\frac{y-u_y}{\sigma_y}\right)^2-2e\left(\frac{x-u_x}{\sigma_x}\right)\left(\frac{x-u_y}{\sigma_y}\right)$$

$$=\left(\frac{u-au_x-bv}{a\sigma_x}\right)^2+\left(\frac{v-u_y}{\sigma_y}\right)^2$$

$$-2e\left(\frac{u-au_x-bv}{a\sigma_x}\right)\left(\frac{v-u_y}{\sigma_y}\right)$$

$$=\left(\frac{u-au_x}{a\sigma_x}-\frac{bv}{a\sigma_x}\right)^2+\left(\frac{v-u_y}{\sigma_y}\right)^2$$

$$-2e\left(\frac{u-au_x}{a\sigma_x}-\frac{bv}{a\sigma_x}\right)\left(\frac{v-u_y}{\sigma_y}\right)$$

$\vdots$

Simplify...

4.46 $\quad Z_1, Z_2 \sim N(0,1)$ independent

$$X = a_x Z_1 + b_x Z_2 + C_x$$
$$Y = a_y Z_1 + b_y Z_2 + C_y$$

(a) Show that

$E[X] = C_x \qquad var(X) = a_x^2 + b_x^2$

$E[Y] = C_y \qquad var(Y) = a_y^2 + b_y^2$

$Cov(X,Y) = a_x a_y + b_x b_y$

Sol^n

$$\mathbb{E}[X] = \mathbb{E}[a_x Z_1 + b_x Z_2 + C_x]$$

$$= a_x \underbrace{\mathbb{E}[Z_1]}_{=0} + b_x \underbrace{\mathbb{E}[Z_2]}_{=0} + \mathbb{E}[C_x]$$

$$\mathbb{E}[X] = C_x$$

Similarly $\quad \mathbb{E}[Y] = C_y$

$$var(a_x Z_1 + b_x Z_2 + C_x) = var(a_x Z_1) + var(b_x Z_2) + var(C_x)$$

$$= a_x^2 + b_x^2 + 0$$

Similarly for $\mathrm{var}(Y) = a_y^2 + b_y^2$

$$\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

$$= \mathbb{E}\Big[a_x a_y z_1^2 + b_x b_y z_2^2 + c_x c_y + a_x b_y z_1 z_2 + a_x c_y z_1 + b_x a_y z_2 z_1\Big] \dots$$

$$= a_x a_y + b_x b_y$$

4.46

(b) if we define the constants a_x, b_x, c_x, a_y, b_y and c_y by

$$a_x = \sqrt{\frac{1+\rho}{2}}\,\sigma_x \qquad b_x = \sqrt{\frac{1-\rho}{2}}\,\sigma_x \qquad c_x = \mu_x$$

$$a_y = \sqrt{\frac{1+\rho}{2}}\,\sigma_y \qquad b_y = \sqrt{\frac{1-\rho}{2}}\,\sigma_y, \quad c_y = \mu_y$$

Show that $\mathbb{E}[X] = \mu_x$, $\mathrm{var}(X) = \sigma_x^2$

$$\mathrm{var}(x) = a_x^2 + b_x^2$$

$$= \left(\frac{1+\rho}{2}\right)\sigma_x^2 + \left(\frac{1-\rho}{2}\right)\sigma_x^2$$

$$\mathrm{var}(x) = \sigma_x^2$$

similar $\quad \mathrm{var}(y) = \sigma_y^2$

$$\mathrm{cov}(x, y) = a_x a_y + b_x b_y$$

$$= \left(\frac{1+\rho}{2}\right)\sigma_x\sigma_y - \left(\frac{1-\rho}{2}\right)\sigma_x\sigma_y$$

$$= \sigma_x\sigma_y\left(\frac{1+\rho-1+\rho}{2}\right)$$

$$\mathrm{cov}(x, y) = \sigma_x\sigma_y\rho$$

$$\boxed{\rho_{xy} = \frac{\mathrm{cov}(x, y)}{\sigma_x\sigma_y} = \rho}$$

4.46 c

Show that (X, Y) has the bivariate normal pdf with parameters $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$ and ρ.

Solution:

$$X = a_{x1} Z_1 + b_x Z_2 + c_x$$

$$Y = a_y Z_1 + b_y Z_2 + c_y$$

$$\Rightarrow \quad a_x Z_1 + b_x Z_2 = X - c_x$$

$$a_y Z_1 + b_y Z_2 = Y - c_y$$

$$\Rightarrow \quad \begin{bmatrix} a_x & b_x \\ a_y & b_y \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} X - c_x \\ Y - c_Y \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \frac{1}{a_x b_y - a_y b_x} \begin{bmatrix} b_y & -b_x \\ -a_y & a_x \end{bmatrix} \begin{bmatrix} X - c_X \\ Y - c_x \end{bmatrix}$$

$$\text{Det} = D = a_x b_y - a_y b_x$$

$$z_1 = \frac{b_y(x - c_x) - b_x(y - c_y)}{D}$$

$$z_2 = \frac{-a_y(x - c_x) + a_x(y - c_y)}{D}$$

$$z_1 = \frac{-\sqrt{\frac{1-e}{2}}\,\sigma_y\,(x - \mu_x) - \sqrt{\frac{1-e}{2}}\,\sigma_x\,(y - \mu_y)}{D}$$

$$D = a_x b_y - a_y b_x$$

$$= -\sqrt{\frac{1-e^2}{4}}\,\sigma_x\sigma_y - \sqrt{\frac{1-e^2}{4}}\,\sigma_x\sigma_y$$

$$D = -\sqrt{1-e^2}\,\sigma_x\sigma_y$$

$$z_1 = \frac{-\sqrt{1-e}\,\sigma_y\,(x - \mu_x) - \sqrt{1-e}\,\sigma_x\,(y - \mu_y)}{-\sqrt{2}\,\sqrt{1-e^2}\,\sigma_x\sigma_y}$$

$$Z_1 = \frac{x-\mu x}{\sqrt{2(1+\rho)}\,\sigma x} + \frac{y-\mu y}{\sqrt{2(1+\rho)}\,\sigma y}$$

$$Z_2 = \frac{-a_y(x-c_x) + a_x(y-c_y)}{D}$$

$$= \frac{-\sqrt{\frac{1+\rho}{2}}\,\sigma_y\,(x-\mu x) + \sqrt{\frac{1+\rho}{2}}\,\sigma_x\,(y-\mu x)}{-\sqrt{1-\rho^2}\,\sigma x\,\sigma y}$$

$$Z_2 = \frac{x-\mu x}{\sqrt{2}\sqrt{1-\rho}\cdot\sigma x} - \frac{y-\mu y}{\sqrt{2(1-\rho)}\cdot\sigma y}$$

$$J = \begin{vmatrix} \frac{1}{\sqrt{2(1+\rho)}\,\sigma x} & \frac{1}{\sqrt{2(1+\rho)}\,\sigma y} \\ \frac{1}{\sqrt{2(1-\rho)}\,\sigma x} & -\frac{1}{\sqrt{2(1-\rho)}\,\sigma y} \end{vmatrix}$$

$$J = \frac{-1}{2\sqrt{1-\rho^2}}\cdot\frac{1}{\sigma x\,\sigma y} - \frac{1}{2\sqrt{1-\rho^2}\,\sigma x\,\sigma y}$$

$$|J| = \frac{1}{\sqrt{1-\rho^2}\,\sigma_x \sigma_y}$$

$$f_{X,Y}(x,y) = f_{Z_1 Z_2}(x,y)\,|J|$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}(z_1^2+z_2^2)} \cdot \frac{1}{\sqrt{1-\rho^2}\,\sigma_x \sigma_y}$$

Simpilifying this gives

$$\frac{z_1^2+z_2^2}{2} = \frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]$$

therefo

$$\boxed{f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}\,\sigma_x \sigma_y} \exp\left[-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]\right]}$$

4.47 Let X and Y be independent $N(0,1)$ r.v, and define a new r.v Z by

$$Z = \begin{cases} X & \text{if } XY > 0 \\ -X & \text{if } XY < 0 \end{cases}$$

a) Show that Z has a normal distribution.

Solution:

$$X \sim N(0,1)$$
$$Y \sim N(0,1)$$

$$XY > 0 \implies \{(x,y) : x > 0,\ y > 0\} + \{(x,y) : x < 0,\ y < 0\}$$

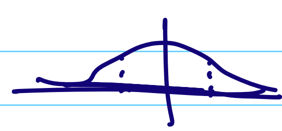
$$xy < 0 \implies \{(x,y) : x > 0,\ y < 0\} + \{(x,y) : x < 0,\ y > 0\}$$

$$F_Z(z) = \mathbb{P}(Z < z)$$

$XY > 0$ & $XY < 0$ are disjoint sets

apply Law of total Probability

$$
\begin{aligned}
F_Z(z) &= \mathbb{P}(Z < z) \\
&= \mathbb{P}(Z < z \cap XY > 0) + \mathbb{P}(Z < z \cap XY < 0) \\
&= \mathbb{P}(Z < z \mid XY > 0) \cdot \mathbb{P}(XY > 0) \\
&\qquad + \mathbb{P}(Z < z \mid XY < 0)\, \mathbb{P}(XY < 0) \\
&= \mathbb{P}(Z < z \mid X > 0, Y > 0)\, \mathbb{P}(X > 0, Y > 0) \\
&\qquad + \mathbb{P}(Z < z \mid X < 0, Y < 0)\, \mathbb{P}(X < 0, Y < 0) \\
&\qquad + \mathbb{P}(Z < z \mid X > 0, Y < 0)\, P(X > 0, Y < 0) \\
&\qquad + \mathbb{P}(Z < z \mid X < 0, Y > 0)\, \mathbb{P}(X < 0, Y > 0) \\
&= \mathbb{P}(X < z) \cdot \frac{1}{4} + \mathbb{P}(-X < z) \cdot \frac{1}{4} \\
&\qquad + P(X < z) \cdot \frac{1}{4} + \mathbb{P}(-X < z) \cdot \frac{1}{4} \\
&= \mathbb{P}(X < z) \cdot \frac{1}{2} + \mathbb{P}(-X < z) \cdot \frac{1}{2} \\
&= \mathbb{P}(X < z) \cdot \frac{1}{2} + \mathbb{P}(X > -z) \cdot \frac{1}{2}
\end{aligned}
$$



$$\mathbb{P}(X<z) = \mathbb{P}(X>-z)$$

$$\Rightarrow \quad \mathbb{P}(X<z)\cdot\frac{1}{2}\cdot 2$$

$$\Rightarrow \quad \mathbb{P}(Z<z) = \mathbb{P}(X<z)$$

$$\boxed{\Rightarrow \quad Z \sim X \sim N(0,1)}$$

4.47 b Show that the joint distribution of Z and Y is not bivariate normal (Hint: Show that Z and Y always have the same sign)

solution:

$$X \sim N(0,1)$$
$$Y \sim N(0,1)$$
$$Z \sim N(0,1)$$

$Z>0 \Longrightarrow$ ① $X\cdot Y>0$, $X>0 \Rightarrow Y>0$

② $XY<0$, $X<0 \Rightarrow Y>0$

Simillarly for $Z<0$

$Z<0 \Rightarrow$ ① $XY>0$, $X<0 \Rightarrow Y<0$

② $XY<0$, $X>0 \Rightarrow Y<0$

$\Rightarrow$ Z, Y are not independent.

4.48 Gelman and Meng (1991) give an example of a bivariate family of Distribution that are not bivariate normal but have normal Conditionals. Define the joint pdf of (X, Y) as.

$$f(x,y) \propto \exp\left\{-\frac{1}{2}\left[Ax^2y^2 + x^2 + y^2 - 2Bxy - 2Cx - 2Dy\right]\right\}$$

Where A, B, C, D are constants

ⓐ Show that the distribution of $X \mid Y=y$ is normal with mean $\frac{By+C}{Ay^2+1}$ and variance $\frac{1}{Ay^2+1}$. Derive a corresponding result for the distribution of $Y | X = x$.

Solution

$$f(x,y) \propto \exp\left\{ -\frac{1}{2} \left[Ax^2y^2 + x^2 + y^2 - 2Bxy - 2Cx - 2Dy \right] \right\}$$

$$f(X | Y = y) = \frac{f(x,y)}{f(Y=y)}$$

marginal distribution of Y

$$f_Y(y) = \int_x f(x,y)\, dx$$

$$\propto \int \exp\left\{ -\frac{1}{2} \left[Ax^2 + x^2 + y^2 - 2Bxy - 2Cx - 2Dy \right] \right\}$$

$$\propto \int_{-\infty}^{\infty} \exp\left\{ -\frac{1}{2} \left[(Ay^2+1)x^2 - 2(By+C)x - 2Dy + y^2 \right] \right\} dx$$

$$\propto e^{-\frac{1}{2}(y^2-2Dy)} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}(Ay^2+1)\left[x^2 - 2\,\frac{(By+C)}{Ay^2+1}\,x\right]\right. dx$$

$$\propto e^{-\frac{1}{2}(y^2-2Dy)} \cdot e^{\frac{1}{2}\frac{(By+C)^2}{Ay^2+1}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}(Ay^2+1)\left[x^2 - 2\left(\frac{By+C}{Ay^2+1}\right)x + \left(\frac{By+C}{Ay^2+1}\right)^2\right]\right.$$

$$\propto e^{-\frac{1}{2}\left(y^2-2Dy-\left(\frac{By+C}{Ay^2+1}\right)^2\right)} \underbrace{\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}(Ay^2+1)\left[x - \frac{By+C}{Ay^2+1}\right]^2\right)}_{\text{Normal distribution with}}$$

Normal distribution with

$$\mu = \frac{By+C}{Ay^2+1}, \quad \sigma = \frac{1}{Ay^2+1}$$

$$f_Y(y) \propto \exp\left(-\frac{1}{2}\left[y^2 - 2Dy - \frac{(By+C)^2}{Ay^2+1}\right]\right)$$

$$\times \frac{\sqrt{2\pi}}{Ay^2+1}$$

therefore

$$f(X \mid y=y) = \frac{f(x,y)}{f(y=y}$$

$$= \frac{\exp\left[-\frac{1}{2}\left(Ax^2y^2 + x^2 + y^2 - 2Dxy - 2Cx - 2Dy\right)\right]}{\frac{\sqrt{2\pi}}{Ay^2+1} \exp\left(-\frac{1}{2}\left\{y^2 - 2Dy - \frac{(By+C)^2}{Ay^2+1}\right]\right)}$$

$$= \frac{1}{\sqrt{2\pi} \cdot \frac{1}{Ay^2+1}} \exp\left(-\frac{1}{2}\left(Ax^2y^2 + x^2 - 2Bxy - 2Cx + \frac{(By+C)^2}{Ay^2+1}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}\cdot\frac{1}{Ay^2+1}} \exp\left(-\frac{1}{2}\left((Ay^2+1)x^2 - 2(By+C)x + \frac{(By+C)^2}{Ay^2+1}\right)\right)$$

$$= \frac{1}{\sqrt{2\pi}\cdot\frac{1}{Ay^2+1}} \exp\left(-\frac{1}{2}(Ay^2+1)\left[x^2 - 2\frac{By+C}{Ay^2+1}x + \left(\frac{By+C}{Ay^2+1}\right)^2\right]\right)$$

$$= \frac{1}{\sqrt{2\pi}\cdot\frac{1}{A^2+1}} \exp\left(-\frac{1}{2\cdot\frac{1}{Ay^2+1}}\left[x - \frac{By+C}{Ay^2+1}\right]^2\right)$$

$$\boxed{\Rightarrow \quad X \mid Y=y \sim N\left(\frac{By+C}{Ay^2+1}, \frac{1}{Ay^2+1}\right)}$$

Similarly

$$Y \mid X=x \sim N\left(\frac{Bx+D}{Ax^2+1}, \frac{1}{Ax^2+1}\right)$$

4.48b

A most interesting configuration is $A=1$, $B=0$, $C=D=8$. Show that this joint distribution is bimodal.

Solution:

$$f(x,y) \propto \exp\left\{-\frac{1}{2}\left[x^2y^2 + x^2 + y^2 - 16x - 16y\right]\right\}$$

A bivariate PDF $f(x,y)$ is bimodal if there exists at least two distinct points (x_1, y_1) and (x_2, y_2) where

① $\nabla f(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x}(x,y) \\ \frac{\partial f}{\partial y}(x,y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ critical points

② Hessian matrix H is Negative definite at critical points.

$$\frac{d}{dx} f(x,y) \propto f(x,y) \cdot \underbrace{\left[-\frac{1}{2}\left[2xy^2 + 2x - 16\right]\right]}_{=0}$$

$$\Rightarrow \qquad 2xy^2 + 2x = 16$$

$$\frac{d}{dy} f(x,y) \propto f(x,y) \cdot \underbrace{\left[-\frac{1}{2}\left[2yx^2 + 2y - 16\right]\right]}_{=0}$$

$\Rightarrow \quad 2yx^2 + 2y = 16$

therefor $\quad \Rightarrow \quad 2x(y^2+1) = 16$

$$2y(x^2+1) = 16$$

$\Rightarrow \quad x(y^2+1) = 8$ x, y are symmetric

$$y(x^2+1) = 8$$

4.49 Behboodian (1990) illustrates, how to construct bivariate random variables that are uncorrelated but dependent.

Suppose that f_1, f_2, g_1, g_2 are univariate densities with mean's $\mu_1, \mu_2, \xi_1, \xi_2$ respectively, and the bivariate r.v (X, Y) has density,

$$(X, Y) \sim a f_1(x) g_1(y) + (1-a) f_2(x) g_2(y)$$

where $0 < a < 1$ is known

(a) Show that marginal distribution are given by $f_X(x) = a f_1(x) + (1-a) f_2(x)$ and

$$f_Y(y) = a g_1(y) + (1-a) g_2(y)$$

Solution:

$$f_{X,Y}(x,y) = a f_1(x) g_1(y) + (1-a) f_2(x) g_2(y)$$

$$f_X(x) = \int_y a f_1(x) g_1(y) + (1-a) f_2(x) g_2(y)\, dy$$

$$= a f_1(x) \int_y g_1(y) dy + (1-a) f_2(x) \int_y g_2(y) dy$$

$$f_X(x) = a f_1(x) + (1-a) f_2(x)$$

Simillarly $\quad f_Y(y) = a g_1(x) + (1-a) g_2(x)$

4.49 (b) Show that X and Y are independent

$\Longleftrightarrow$ if $[f_1(x) - f_2(x)][g_1(y) - g_2(y)] = 0$

Solution:

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

if X, Y are independent

$$a f_1(x) g_1(y) + (1-a) f_2(x) g_2(y)$$

$$= \left[a f_1(x) + (1-a) f_2(x)\right]\left[a g_1(y) + (1-a) g_2(y)\right]$$

$$a f_1(x) g_1(y) + (1-a) f_2(x) g_2(y)$$

$$= a^2 f_1(x) g_1(y) + a(1-a) f_1(x) g_2(y)$$
$$+ a(1-a) f_2(x) g_1(y)$$
$$+ (1-a)^2 f_2(x) g_2(y)$$

$$a(1-a) f_1(x) g_1(y) + a(1-a) f_2(x) g_2(y)$$
$$= a(1-a) f_1(x) g_2(y)$$
$$+ a(1-a) f_2(x) g_1(y)$$

$$\Rightarrow f_1(x) g_1(y) + f_2(x) g_2(y)$$
$$- f_1(x) g_2(y) - f_2(x) g_1(y)$$
$$= 0$$

$$\boxed{\Rightarrow \left[f_1(x) - f_2(x)\right]\left[g_1(y) - g_2(y)\right] = 0}$$

4.49C Show that $\mathrm{Cov}(X, Y) = a(1-a)[\mu_1 - \mu_2][\varepsilon_1 - \varepsilon_2]$ and thus explain how to construct depend -ent uncorrelated random variables.

Solution:

$$\operatorname{cov}(X,Y) = \mathbb{E}[XY] - E[X]E[Y]$$

$$E[X] = a\mu_1 + (1-a)\mu_2$$

$$E[Y] = a\xi_1 + (1-a)\xi_2$$

$$\mathbb{E}[XY] = \iint a f_1(x) g_1(y)\, x y \, dx\, dy$$

$$+ \iint (1-a) f_2(x) g_2(y)\, x y \, dx\, dy$$

$$= \int a f_1(x)\, x\, \xi_1 \, dx + \int (1-a) f_2(x)\, x\, \xi_2\, dx$$

$$= a\mu_1\xi_1 + (1-a)\mu_2\xi_2$$

$$\Rightarrow \operatorname{cov}(X,Y) = a\mu_1\xi_1 + (1-a)\mu_2\xi_2$$

$$- \left[a\mu_1 + (1-a)\mu_2\right]\left[a\xi_1 + (1-a)\xi_2\right]$$

$$= a\mu_1\varepsilon_1 + (1-a)\mu_2\varepsilon_2 - a^2\mu_1\varepsilon_1$$
$$- a(1-a)\mu_1\varepsilon_2$$
$$- a(1-a)\mu_2\varepsilon_1$$
$$- (1-a)^2\mu_2\varepsilon_2$$

$$= a(1-a)\mu_1\varepsilon_1 + a(1-a)\mu_2\varepsilon_2$$
$$- a(1-a)\mu_1\varepsilon_1 - a(1-a)\mu_2\varepsilon_1$$

$$\Rightarrow a(1-a)\left[(\mu_1-\mu_2)(\varepsilon_1-\varepsilon_2)\right]$$

$$\boxed{Cov(X,Y) = a(1-a)(\mu_1-\mu_2)(\varepsilon_1-\varepsilon_2)}$$

dependent $\Rightarrow \left[f_1(x) - f_2(x)\right]\left[g_1(y) - g_2(y)\right] \neq 0$

un-correlated $\Rightarrow \mu_1 = \mu_2$ or $\varepsilon_1 = \varepsilon_2$ or $a = 1$

4.49d

independent $\Rightarrow$

4.50 if (X, Y) has the bivariate normal pdf

$$f(x,y) = \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left(\frac{-1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right)$$

Show that $\operatorname{Corr}(X, Y) = \rho$. and $\operatorname{Corr}(X^2, Y^2) = \rho^2$ (conditional expectation will simplify calculation)

Solution:

$$\operatorname{cov}(X, Y) = \frac{\mathbb{E}[XY] - E[X]E[Y]}{\sigma_x \sigma_y}$$

we know that $E[X] = 0$ $E[Y] = 0$

$\sigma_x = 1$, $\sigma_y = 1$

$$\Rightarrow \operatorname{cov}(X, Y) = \mathbb{E}[XY]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left(\frac{-1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right)$$

$$= \frac{1}{2\pi(1-\rho^2)^{1/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot \exp\left(\frac{-1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right) dx\, dy$$

$$= \frac{1}{2\pi(1-\rho^2)^{1/2}} \int\int xy \cdot \exp\left(\frac{-1}{2(1-\rho^2)}\left[(x-\rho y)^2 + y^2(1-\rho^2)\right]\right) dx\, dy$$

$$= \frac{1}{2\pi(1-\rho^2)^{1/2}} \int y \int x \cdot \exp\left(\frac{-1}{2(1-\rho^2)}(x-\rho y)^2 - \frac{y^2}{2}\right) dx\, dy$$

$$= \frac{1}{2\pi(1-\rho^2)^{1/2}} \int y \cdot e^{-\frac{y^2}{2}} \underbrace{\int x \cdot \exp\left(\frac{-1}{2(1-\rho^2)}(x-\rho y)^2\right) dx}_{\text{Normal}\,(\rho y,\ 1-\rho^2)}$$

$$\Downarrow$$

$$\rho y \cdot \sqrt{2\pi} \cdot \sqrt{1-\rho^2}$$

$$= \frac{1}{2\pi(1-\rho^2)^{1/2}} \int_{-\infty}^{\infty} \rho y^2 \cdot e^{-\frac{y^2}{2}} \cdot \sqrt{2\pi} \cdot \sqrt{1-\rho^2}\, dy$$

$$= \frac{\rho}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} y^2 \cdot e^{-\frac{y^2}{2}}}_{N(0,1)}$$

$$E[y^2] = var(y) + E[y]^2 = 1$$

$$= \rho \cdot 1$$

$$\boxed{\Rightarrow \quad \rho_{xy} = \rho}$$

$$Corr(x^2, y^2) = \frac{cov(x^2, y^2)}{\sigma_{x^2}\sigma_{y^2}}$$

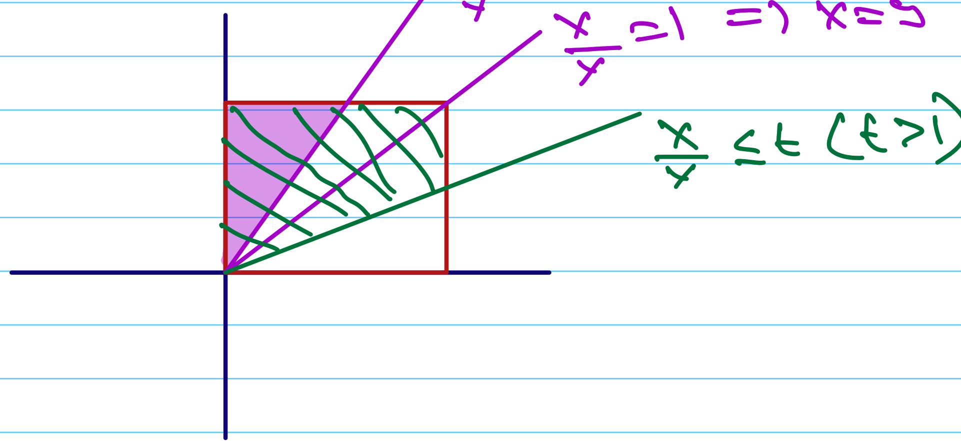
$$= \frac{E[x^2 y^2] - E[x^2]E[y^2]}{\sqrt{var(x^2)\, var(y^2)}}$$

$\vdots$

4.51 Let X, Y, and Z be independent uniform (0,1) r.v

ⓐ Find $\mathbb{P}\left(\frac{X}{Y} \le t\right)$ and $\mathbb{P}(XY \le t)$

Solution:

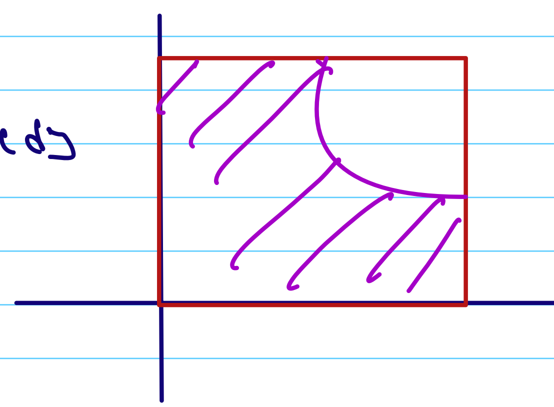


$$\mathbb{P}\left(\frac{X}{Y} \le t\right) = \frac{1}{2}t \quad \text{if } t<1$$

$$\mathbb{P}\left(\frac{X}{Y} \le t\right) = 1 - \frac{1}{2t}$$

$$\Rightarrow \quad \mathbb{P}\left(\frac{X}{Y} \le t\right) = \begin{cases} \frac{t}{2} & \text{if } t \le 1 \\ 1-\frac{1}{2t} & \text{if } t>1 \end{cases}$$

$$\mathbb{P}(XY \le t) = 1 - \int_{y=t}^{1} \int_{x=\frac{t}{y}}^{1} 1 \, dx\, dy$$



$$= 1 - \int_{t}^{1} \left(1 - \frac{t}{y}\right) dy$$

$$= 1 - \left. y - t \cdot \log y \right]_t^1$$

$$= 1 - [1 - 0 - t - t \cdot \log t)]$$

$$= t - t \log t$$

$$\mathbb{P}(XY \leq t) = \begin{cases} t - t\log t & 0 < t < 1 \\ 1 & t > 1 \end{cases}$$

4.51 b find $\mathbb{P}\left(\frac{XY}{Z} \leq t\right)$

$$\Rightarrow \mathbb{P}\left(\frac{XY}{Z} \leq t\right) = \mathbb{P}(XY \leq Zt)$$

$\Downarrow$

eliminate Z

$$= \mathbb{P}(XY \leq Zt) = \int_0^1 \mathbb{P}(XY \leq zt)\, dz$$

$$= \int_0^1 (zt - zt \log zt\, dz$$

$$= \frac{t}{2} - \int_0^1 zt \cdot \log zt\, dz$$

$$\frac{1}{zt} = \alpha$$