DESCRIPTION

This document provides solution's to 3.25 to 3.43 from chapter 3 of statistical inference Book by Casella and Berger, Focusing on Exponential & location-scale families.

A few Problem's are yet to be solved, but I will update the document soon.

3.25 Suppose a r.v T is the length of life of an object. The hazard function $h_T(t)$ is

$$h_T(t) = \lim_{\delta \to 0} \frac{\mathbb{P}(t \le T < t+\delta \mid T \geq t)}{\delta}$$

$h_T(t) :=$ rate of change of Prob that the object survives a little past time t, given that the object survives to time t.

Show that if T is continuous r.v then

$$h_T(t) = \frac{f_T(t)}{1 - F_T(t)} = -\frac{d}{dt} \log\left(1 - F_T(t)\right)$$

Solution:

$$h_T(t) = \lim_{\delta \to 0} \frac{\mathbb{P}(t \le T < t+\delta \mid T \geq t)}{\delta}$$

$$\Rightarrow \mathbb{P}(t \leq T < t+\delta \mid T \geq t)$$

$$\Rightarrow \frac{\mathbb{P}(t \leq T < t+\delta \cap T \geq t)}{\mathbb{P}(T \geq t)}$$

The intersection is $\mathbb{P}(t \leq T < t+\delta)$

$$\Rightarrow \frac{\mathbb{P}(t \leq T < t+\delta)}{\mathbb{P}(T \geq t)}$$

therefore

$$h_T(t) = \lim_{\delta \to 0} \frac{\mathbb{P}(t \leq T < t+\delta)}{P(T \geq t) \cdot \delta}$$

$$= \lim_{\delta \to 0} \frac{F_T(t+\delta) - F_T(t)}{\delta \cdot (1 - F_T(t))}$$

$$= \frac{1}{1-F_T(t)} \lim_{\delta \to 0} \frac{F_T(t+\delta) - F_T(t)}{\delta}$$

$$= \frac{f_T(t)}{1-F_T(t)}$$

Hence

$$\boxed{h_T(t) = \frac{f_T(t)}{1-F_T(t)}}$$

$$-\frac{d}{dt} \log\left(1-F_T(t)\right) =$$

$$= -\frac{1}{1-F_T(t)} \frac{d}{dt}\left(1-F_T(t)\right)$$

$$= -\frac{1}{1-F_T(t)} \, -f_T(t) = \frac{f_T(t)}{1-F_T(t)}$$

Therefore

$$\boxed{h_T(t) = \frac{f_T(t)}{1-F_T(t)} = -\frac{d}{dt} \log\left(1-F_T(t)\right)}$$

3.26 Verify that the following pdf's have the Indicated hazard function's

(a) $T \sim \text{exponential}(\beta)$

$$h_T(t) = \frac{1}{\beta}$$

$$f_T(t) = \frac{1}{\beta} e^{-\frac{t}{\beta}}$$

$$F_T(t) = \int_0^t \frac{1}{\beta} e^{-\frac{x}{\beta}} \, dx$$

$$= \frac{1}{\beta} \cdot e^{-\frac{x}{\beta}} \cdot -\beta \Big]_0^t$$

$$= 1 - e^{-\frac{t}{\beta}}$$

$$h_T(t) = \frac{\frac{1}{\beta} \cdot e^{-\frac{t}{\beta}}}{1 - \left(1 - e^{-t/\beta}\right)} = \frac{1}{\beta}$$

$$\Rightarrow \boxed{h_T(t) = \frac{1}{\beta}}$$

(b) $T \sim \text{Weibull}(\gamma, \beta)$

$$h_T(t) = \left(\frac{\gamma}{\beta}\right) t^{\gamma-1}$$

Solution:

$$f_T(t) = \frac{\gamma}{\beta} t^{\gamma-1} \cdot e^{-\frac{t^\gamma}{\beta}}$$

$0 < t < \infty$

$\gamma > 0$

$\beta > 0$

$$F_T(t) = \frac{\gamma}{\beta} \int_0^t t^{\gamma-1} \cdot e^{-\frac{t^\gamma}{\beta}} \, dt$$

$$\frac{t^\gamma}{\beta} = u$$

$$\frac{\gamma}{\beta} \cdot t^{\gamma-1} \, dt = du$$

$$F_T(t) = \int e^{-u} \, du$$

$$= -e^{-\frac{t^\gamma}{\beta}} \Big]_0^t = 1 - e^{-\frac{t^\gamma}{\beta}}$$

$$h_T(t) = \frac{\frac{\gamma}{\beta} t^{\gamma-1} e^{-\cancel{\frac{t^\gamma}{\beta}}}}{e^{-\cancel{\frac{t^\gamma}{\beta}}}}$$

$$\Rightarrow \quad \boxed{h_T(t) = \frac{\gamma}{\beta} t^{\gamma-1}}$$

(c)

$T \sim \text{logistic}(\mu, \beta)$

$$F_T(t) = \frac{1}{1+e^{-\frac{(t-\mu)}{\beta}}}$$

$$\Rightarrow \quad f_t(t) = \frac{d}{dt} \frac{1}{1+e^{-\left(\frac{t-\mu}{\beta}\right)}}$$

$$= \frac{-1}{\left(1+e^{-\frac{(t-\mu)}{\beta}}\right)^2} e^{-\left(\frac{t-\mu}{\beta}\right)} \cdot \frac{-1}{\beta}$$

$$= \frac{1}{\beta} \cdot \left[\frac{e^{-\left(\frac{t-\mu}{\beta}\right)}}{\left(1+e^{-\left(\frac{t-\mu}{\beta}\right)}\right)^2} \right]$$

$$h_T(t) = \frac{1}{\beta} \cdot \left[\frac{e^{-\left(\frac{t-\mu}{\beta}\right)}}{\left(1+e^{-\left(\frac{t-\mu}{\beta}\right)}\right)^2} \right] \cdot \frac{1}{1-F_T(t)}$$

$$= \frac{1}{\beta} \cdot \left[\frac{\cancel{e^{-\left(\frac{t-\mu}{\beta}\right)}}}{\left(1+e^{-\left(\frac{t-\mu}{\beta}\right)}\right)^{\cancel{2}}} \right] \cdot \frac{1+\cancel{e^{-\left(\frac{t-\mu}{\beta}\right)}}}{\cancel{e^{-\left(\frac{t-\mu}{\beta}\right)}}}$$

$$= \frac{1}{\beta} \frac{1}{\left(1+e^{-\frac{t-\mu}{\beta}}\right)}$$

$$\Rightarrow \boxed{h_T(t) = \frac{1}{\beta} F_T(t)}$$

3.27 Show whether all the pdf's in the family are unimodal.

(a) uniform (a, b)

$\forall x \in (a, b)$

$$f(a) = f(x) = f(b)$$

$\Rightarrow$ all points are mode's

$\Rightarrow$ Unimodal.

(b) gamma (α, β)

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\beta}}$$

$0 < x < \infty$

$\alpha > 0$

$\beta > 0$

$$\frac{d}{dx} f_X(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \left[(\alpha-1) x^{\alpha-2} \cdot e^{-\frac{x}{\beta}} - \frac{1}{\beta} x^{\alpha-1} \cdot e^{-\frac{x}{\beta}} \right]$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \cdot x^{\alpha-2} \cdot e^{-\frac{x}{\beta}} \left[\alpha-1 - \frac{x}{\beta} \right]$$

The sign of derivative changes only once $\Rightarrow$ one mode

$$\frac{d}{dx} f_X(x) = 0 \Rightarrow \boxed{x_{mode} = \beta(\alpha-1)}$$

c) $N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\frac{d}{dx} f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot -\frac{1}{\sigma^2}(x-\mu)$$

Change in slope is only once $\Rightarrow$ unimodel.

$$\frac{d}{dx} f_X(x) = 0 \Rightarrow \boxed{X_{mode} = \mu}$$

(d) beta(α, β)

$$f_X(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1} \cdot (1-x)^{\beta-1}$$

$$\frac{d}{dx} f_X(x) = (\alpha-1) x^{\alpha-2} \cdot (1-x)^{\beta-1} - (\beta-1) x^{\alpha-1} \cdot (1-x)^{\beta-2}$$

$$= x^{\alpha-2} \cdot (1-x)^{\beta-2} \left((\alpha-1)(1-x) - (\beta-1)(x) \right)$$

unimodel

$$\frac{d}{dx} f_X(x) = 0 \Rightarrow (1-x)(\alpha-1) = x(\beta-1)$$

$$\Rightarrow \frac{x}{x-1} = \frac{\alpha-1}{-\beta+1}$$

$$\Rightarrow \frac{2x-1}{1} = \frac{\alpha-\beta}{\alpha+\beta-2}$$

$$\boxed{\Rightarrow X_{mode} = \frac{\alpha-1}{\alpha+\beta-2}}$$

3.28 Show that each of the following familites is an exponential family.

(a) normal family with either parameter μ or σ known

Solution: σ^2 known

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2} + \frac{x\mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2}\right)$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left(-\frac{x^2}{2\sigma^2}\right)}_{h(x)} \cdot \underbrace{\exp\left(-\frac{\mu^2}{2\sigma^2}\right)}_{c(\theta)} \cdot \exp\left(\frac{x\mu}{\sigma^2}\right)$$

$t_1(x) = x$

$w_1(\mu) = \frac{\mu}{\sigma^2}$

Hence Exponential familly.

μ known

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2} + \frac{x\mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2}\right)$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left(-\frac{\mu^2}{2\sigma^2}\right)}_{c(\sigma)} \cdot \exp\left(-\frac{x^2}{2\sigma^2} + \frac{x\mu}{\sigma^2}\right)$$

$t_1(x) = -\frac{x^2}{2}$ $\quad w_1(\sigma) = \frac{1}{\sigma^2}$

$t_2(x) = x\mu$ $\quad w_2(\sigma) = \frac{1}{\sigma^2}$

$h(x) = 1$ $\quad$ Hence Exponential family

(b) gamma family with either Parameter α or β or Both unknown

(i) Both unknown

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}$$

$$= \underbrace{\frac{1}{\Gamma(\alpha)\beta^\alpha}}_{c(\alpha,\beta)} \exp\left(\underbrace{(\alpha-1)}_{w_1(\alpha,\beta)} \underbrace{\log x}_{T_1(x)} - \underbrace{\frac{x}{\beta}}_{T_2(x) = x}\right)$$

$w_2(\alpha,\beta) = -\frac{1}{\beta}$

$h(x) = 1$

Hence exponential family.

(ii) β is known

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} e^{-\frac{x}{\beta}} \cdot \exp((\alpha-1)\log x)$$

$h(x) = e^{-\frac{x}{\beta}}$ $\qquad$ $w_1(\alpha) = \alpha - 1$

$c(\alpha) = \frac{1}{\Gamma(\alpha)\beta^\alpha}$ $\qquad$ $T_1(x) = \log x$

iii) α is known

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \exp((\alpha-1)\log x) \cdot \exp\left(-\frac{x}{\beta}\right)$$

$$c(\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}$$

$$h(x) = \exp((\alpha-1)\log x) = x^{\alpha-1}$$

$$w_1(\beta) = -\frac{1}{\beta}$$

$$T_1(x) = x$$

(c) beta family with either parameter α or β known or both unknown

Solution:

$$f(x \mid \alpha, \beta) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} \cdot (1-x)^{\beta-1}$$

$0 < x < 1$

$\alpha > 0, \beta > 0$

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

(i) α, β Both unknown

$$f(x) = B(\alpha,\beta) \cdot \exp(\alpha - 1 \cdot \log x + (\beta-1)\log(1-x))$$

$$\Rightarrow f(x) = I_{[0,1]}(x) \frac{1}{B(\alpha,\beta)} \cdot \exp((\alpha-1)\log x + (\beta-1)\log(1-x))$$

$$h(x) = I_{[0,1]}(x)$$

$$C(\alpha,\beta) = \frac{1}{B(\alpha,\beta)}$$

$$t_1(x) = \log x \qquad \omega_1(\alpha,\beta) = \alpha - 1$$

$$t_2(x) = \log(1-x) \qquad \omega_2(\alpha,\beta) = \beta - 1$$

(ii) β is known

$$f(x) = I_{[0,1]}x \; \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot (1-x)^{\beta-1} \cdot \exp((\alpha-1)\log x)$$

$$= I_{[0,1]}x\,(1-x)^{\beta-1}\,\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot\exp\left((\alpha-1)\log x\right)$$

$$h(x) = (1-x)^{\beta-1}\cdot I_{[0,1]}(x)$$

$$c(\alpha) = \frac{1}{B(\alpha,\beta)}$$

$$t_1(x) = \log x$$

$$w_1(\alpha) = \alpha-1$$

(iii) α is known

$$f(x) = x^{\alpha-1}\cdot I_{(0,1]}(x)\,\frac{1}{B(\alpha,\beta)}\cdot\exp\left((\beta-1)\log(1-x)\right)$$

$$h(x) = x^{\alpha-1}\,I_{[0,1]}(x)$$

$$c(\beta) = \frac{1}{B(\alpha,\beta)}$$

$$w_1(\beta) = \beta-1$$

$$T_1(x) = \log(1-x)$$

(d) Possion family

Solution:

$$f(x|\lambda) = \mathbb{P}(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$\Rightarrow f(x|\lambda) = e^{-\lambda} \cdot \frac{1}{x!} \cdot \exp(x \cdot \log\lambda)$$

$$= \underbrace{\frac{1}{x!} I_{\{0,1,2..\}}(x)}_{h(x)} \; \underbrace{e^{-\lambda}}_{c(\lambda)} \cdot \exp(\underbrace{x}_{t(x)} \cdot \underbrace{\log\lambda}_{w(\lambda)})$$

(e) negative binomial family with

r known, $0 < p < 1$

Solution:

$$\mathbb{P}(X=x) = \binom{r+x-1}{x} p^{r} (1-p)^{x}$$

$x = 0, 1, 2 \ldots$

$$= \underbrace{\binom{r+x-1}{x} I_{\{0,1,2,..\}}(x)}_{h(x)} \cdot \underbrace{p^{r}}_{c(p)} \exp(\underbrace{x}_{t(x)} \underbrace{\log(1-p)}_{w(p)})$$

(3.29) for each family in Exercise 3.28, describe the natural parameter space

(a) Normal

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-\mu^2}{2\sigma^2}\right) \exp\left(\frac{-x^2}{2\sigma^2} + \frac{\mu x}{\sigma^2}\right)$$

$$(\eta_1, \eta_2) = \left(\frac{-1}{2\sigma^2}, \frac{\mu}{\sigma^2}\right)$$

(b) gamma

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \exp\left((\alpha-1)\log x - \frac{x}{\beta}\right)$$

natural parameter's

$$(\eta_1, \eta_2) = \left(\alpha-1, -\frac{1}{\beta}\right)$$

(c) beta family $X \sim \text{beta}(\alpha,\beta)$

$$\Rightarrow f(x) = I_{[0,1]}(x) \frac{1}{B(\alpha,\beta)} \cdot \exp\left((\alpha-1)\log x + (\beta-1)\log(1-x)\right)$$

$$(\eta_1, \eta_2) = (\alpha-1, \beta-1)$$

(d) Poisson family

$$= \frac{1}{x!} I_{\{0,1,2..\}}(x) \; e^{-\lambda} . \exp(x . \log\lambda)$$

$$\eta_1 = \log\lambda$$

(e) negative binomial

$$\mathbb{P}(X=x) = \binom{r+x-1}{x} p^r (1-p)^x$$

$x = 0, 1, 2 \ldots$

r known

$$= \binom{r+x-1}{x} I_{\{0,1,2,..\}}(x) \cdot p^r \exp(x \log(1-p))$$

$$\eta_1 = \log(1-p)$$

(3.30) use the identities of theorem 3.4.2 to

(a) calculate the Var( ) of a binomial r.v

Solution:

$$\mathbb{P}(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$x = 0, 1, 2, \ldots n$

$$\Rightarrow \mathbb{P}(X=x) = \binom{n}{x} \exp\left(x \log p + (n-x) \log(1-p) \right)$$

$$= \binom{n}{x} \exp\left(x \log \frac{p}{1-p} + n \log(1-p) \right)$$

$$= I_{[0,1,2,\dots n]}(x) \binom{n}{x} (1-p)^n \cdot \exp\left(x \log \frac{p}{1-p} \right)$$

$$h(x) = I_{[0,1,2,\dots n]}(x) \binom{n}{x}$$

$$c(p) = (1-p)^n \Rightarrow$$

$$\frac{\partial}{\partial p} \log c(p) = \frac{-n}{1-p}$$

$$\frac{\partial^2}{\partial p^2} \log c(p) = \frac{-n}{(1-p)^2}$$

$$t_1(x) = x$$

$$w_1(p) = \log\left(\frac{p}{1-p} \right)$$

$$\frac{\partial w}{\partial p} = \frac{1-p}{p} \cdot \frac{(1-p)+p}{(1-p)^2} = \frac{1}{p(1-p)}$$

$$\frac{\partial^2 w}{\partial p^2} = \frac{\partial}{\partial p}\left(\frac{1}{p(1-p)} \right) = \frac{2p-1}{p^2(1-p)}$$

$p - p^2$

$1-2p$

$$\mathrm{Var}\left(\frac{\partial w(p)}{\partial p} \cdot t(x) \right)$$

$$= -\frac{\partial^2}{\partial p^2} \log c(\theta)$$

$$- \mathbb{E}\left[\frac{\partial^2 w(p)}{\partial p^2} t_1(x) \right]$$

$$\mathbb{E}\left[\frac{\partial w(p)}{\partial p}\cdot x\right] = -\frac{\partial}{\partial p}\log c(\theta)$$

$$\Rightarrow \quad \mathbb{E}\left[\frac{1}{p(1-p)}x\right] = \frac{n}{1-p}$$

$$\boxed{\Rightarrow \quad \mathbb{E}[x] = np}$$

$$\mathrm{Var}\left(\frac{2p-1}{p^2(1-p)^2}x\right) = \frac{n}{(1-p)^2} - \mathbb{E}\left[\frac{2p-1}{p^2(1-p)^2}x\right]$$

$$\Rightarrow \frac{(2p-1)^2}{p^4(1-p)^4}\mathrm{var}(x) = \frac{n}{(1-p)^2} - \frac{2p-1}{p^2(1-p)^2}\cdot np$$

$$\Rightarrow \frac{(2p-1)^2}{p^4(1-p)^2}\mathrm{var}(x) = n - \frac{(2p-1)\cdot n\cancel{p}}{p^{\cancel{2}}}$$

$$= \frac{np-2np+n}{p}$$

$$= \frac{-np+n}{p} = -n+\frac{n}{p}$$

$$\vdots$$

$$\mathrm{Var}(x) = np(1-p)$$

(b) calculate mean & var of beta(a,b) r.v

Solution:

$$f(x \mid \alpha, \beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1} \cdot (1-x)^{\beta-1}$$

$0 < x < 1$

$\alpha > 0, \beta > 0$

$$f(x) = \frac{1}{B(\alpha,\beta)} I_{[0,1]}(x) \cdot \exp\left((\alpha-1)\log x + (\beta-1)\log(1-x)\right)$$

$$h(x) = I_{[0,1]}(x)$$

$$c(\alpha,\beta) = \frac{1}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}$$

$w_1(\alpha) = \alpha - 1 \qquad t_1(x) = \log x$

$w_2(\beta) = \beta - 1 \qquad t_2(x) = \log(1-x)$

3.31 **Prove theorem 3.4.2**

(a) $$f(x|\theta) = h(x)\, c(\theta) \exp\left(\sum_{i=1}^{k} w_i(\theta) t_i(x)\right)$$

$$\int f(x|\theta) = \int h(x) c(\theta) \exp\left(\sum_{i=1}^{k} w_i(\theta)\, t_i(x)\right) dx = 1$$

derivate Both sides.

$$\frac{\partial}{\partial\theta} \int h(x) c(\theta) \exp\left(\sum_{i=1}^{k} w_i(\theta) t_i(x)\right) dx = 0$$

$$\int h(x) \frac{\partial c(\theta)}{\partial\theta} \cdot \exp\left(\sum_{i=1}^{k} w_i(\theta)\, t_i(x)\right) dx$$

$$+ \int h(x) c(\theta) \exp\left(\sum_{i=1}^{k} w_i(\theta) t_i(x)\right) \left[\sum_{i=1}^{k} \frac{\partial}{\partial\theta} w_i(\theta)\, t_i(x)\right] dx$$

$$= 0$$

$$\int h(x) \frac{\partial}{\partial\theta} C(\theta) \cdot \frac{C(\theta)}{C(\theta)} \cdot \exp\left(\sum_{i=1}^{k} \omega_i(\theta)\, t_i(x)\right) dx$$

$$+ \int f(x) \cdot \left[\sum_{i=1}^{k} \frac{\partial}{\partial\theta} \omega_i(\theta) \cdot t_i(x) \right] dx = 0$$

$$\Rightarrow \int h(x)\, C(\theta) \cdot \exp\left(\sum_{i=1}^{k} \omega_i(\theta)\, t_i(x) \right) \cdot \frac{\partial}{\partial\theta} \log C(\theta)\, dx$$

$$+ \mathbb{E}\left[\sum_{i=1}^{k} \frac{\partial}{\partial\theta} \omega_i(\theta) \cdot t_i(x) \right] = 0$$

$$= \frac{\partial}{\partial\theta} \log C(\theta) \int f(x)\, dx$$

$$+ \mathbb{E}\left[\sum_{i=1}^{k} \frac{\partial}{\partial\theta} \omega_i(\theta) \cdot t_i(x) \right] = 0$$

$$\boxed{\Rightarrow \mathbb{E}\left[\sum_{i=1}^{k} \frac{\partial}{\partial\theta} \omega_i(\theta) \cdot t_i(x) \right] = -\frac{\partial}{\partial\theta} \log C(\theta)}$$

(b) $$\int h(x)c(\theta)\exp\left(\sum_{i=1}^{k} w_i(\theta)\, t_i(x)\right)dx = 1$$

differential both sides twice

$$\frac{\partial^2}{\partial\theta^2}\int h(x)c(\theta)\exp\left(\sum_{i=1}^{k} w_i(\theta)\, t_i(x)\right)dx = 0$$

$$\frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial\theta}\log c(\theta)\int f(x)dx + \mathbb{E}\left[\sum_{i=1}^{k}\frac{\partial}{\partial\theta}w_i(\theta)\, t_i(x)\right]\right] = 0$$

$$\Rightarrow \frac{\partial^2}{\partial\theta^2}\log c(\theta) +$$

$$\frac{\partial}{\partial\theta}\log c(\theta)\left[\frac{\partial}{\partial\theta}\log c(\theta) + \mathbb{E}\left[\sum_{i=1}^{k}\frac{\partial}{\partial\theta}w_i(\theta)\, t_i(x)\right]\right]$$

$$+ \mathbb{E}\left[\sum_{i=1}^{k}\frac{\partial^2}{\partial\theta^2}w_i(\theta)\, t_i(x)\right] = 0$$

$\vdots$

Continued (Need to solve)