# EXERCISES on $\mathbb{E}[\cdot]$, $M_X(t)$, var( )

## DEScription:

This document contain's solution's to Exercise's 2.11 to 2.34 from Chapter 2 of Casella - Berger: Statistical inference, focusing on the Expectation's, variance, moment Generating function's.

**2.11** Let $X$ have the standard normal pdf $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

(a) Find $\mathbb{E}[X^2]$, and then by using the pdf of $X^2$ from Example 2.1.7 and calculating $\mathbb{E}[Y]$

Solution:

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

integration By Parts

$$u = x \qquad V = \frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= x \int \frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \Big]_{-\infty}^{\infty}$$

$$- \int_{-\infty}^{\infty} 1 \int \frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \, dx$$

Calculating $\int \frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

$$u = \frac{x^2}{2} \Rightarrow du = x dx$$

$$\int \frac{1}{\sqrt{2\pi}} e^{-u} du = \frac{1}{\sqrt{2\pi}} - e^{-u}$$

$$= -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$= x \cdot \frac{-1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Big]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= 0 + 1$$

$$\Rightarrow \quad E[X^2] = 1$$

Now let $Y = X^2$ (Normal-chi squared)

$A_0 = \{0\}$

$A_1 = (-\infty, 0) \quad g_1(x) = x^2 \quad g_1^{-1}(y) = -\sqrt{y}$

$A_2 = (0, \infty) \quad g_2(x) = x^2 \quad g_2^{-1}(y) = \sqrt{y}$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \cdot \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \frac{1}{2\sqrt{y}}$$

$$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} e^{-\frac{y}{2}} \qquad 0 < y < \infty}$$

$$\mathbb{E}[Y] = \int_0^\infty y \cdot \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} e^{-\frac{y}{2}} \, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \sqrt{y} \cdot e^{-\frac{y}{2}} \, dy$$

$$= \frac{1}{\sqrt{2\pi}} \left[ \frac{\sqrt{y} \cdot e^{-\frac{y}{2}}}{-\frac{1}{2}} \Big|_0^\infty - \int_0^\infty \frac{1}{2} \frac{1}{\sqrt{y}} \cdot \frac{e^{-\frac{y}{2}}}{-1/2} \, dy \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[ 0 + \underbrace{\int_0^\infty \frac{1}{\sqrt{y}} e^{-\frac{y}{2}} \, dy}_{\text{Pdf} \cdot \sqrt{2\pi}} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[ 0 + \sqrt{2\pi} \right] = 1$$

$$\boxed{\Rightarrow \mathbb{E}[X^2] = \mathbb{E}[Y] = 1}$$

(b) Find the Pdf of $Y = |X|$, and find its mean and variance.

<u>Solution:</u>

$Y = |X|$

$A_1 = (-\infty, 0) \quad g_1(x) = -X \quad g_1^{-1}(y) = -y$

$A_2 = (0, \infty) \quad g_2(x) = X \quad g_2^{-1}(y) = y$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \cdot 1 + \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \cdot 1$$

$$\boxed{f_Y(y) = \sqrt{\frac{2}{\pi}} e^{-\frac{y^2}{2}} \qquad 0 < y < \infty}$$

$$\mathbb{E}[Y] = \int_0^\infty y \cdot \sqrt{\frac{2}{\pi}} e^{-\frac{y^2}{2}} \, dy$$

$$\frac{y^2}{2} = u$$

$u: 0 \to \infty$

$$y\,dy = du$$

$$= \int_0^\infty \sqrt{\frac{2}{\pi}} \cdot e^{-u} \, du$$

$$= \sqrt{\frac{2}{\pi}} \; \frac{e^{-u}}{-1} \Big]_0^\infty$$

$$\mathbb{E}[Y] = \sqrt{\frac{2}{\pi}}$$

$$\operatorname{var}(Y) = \mathbb{E}[Y^2] - E[Y]^2$$

$$\mathbb{E}[Y^2] = \int_0^\infty y^2 \cdot \sqrt{\frac{2}{\pi}} \, e^{-\frac{y^2}{2}} \, dy$$

integration by parts $u = y$ $\quad v = y \cdot \sqrt{\frac{2}{\pi}} e^{-\frac{y^2}{2}} dy$

$$= y \int y \cdot \sqrt{\frac{2}{\pi}} e^{-\frac{y^2}{2}} \, dy \Big]_0^\infty$$

$$- \int_0^\infty 1 \cdot \int y \cdot \sqrt{\frac{2}{\pi}} \, e^{-\frac{y^2}{2}} \, dy \, dy$$

Evaluating $\int y \cdot \sqrt{\frac{2}{\pi}} \cdot e^{-\frac{y^2}{2}} \, dy$

$$\Rightarrow \quad u = \frac{y^2}{2}$$

$$du = y\,dy$$

$$\Rightarrow \int \sqrt{\frac{2}{\pi}} \, e^{-u} \, du$$

$$\Rightarrow \sqrt{\frac{2}{\pi}} \, \frac{e^{-u}}{-1} = -\sqrt{\frac{2}{\pi}} \, e^{-\frac{y^2}{2}}$$

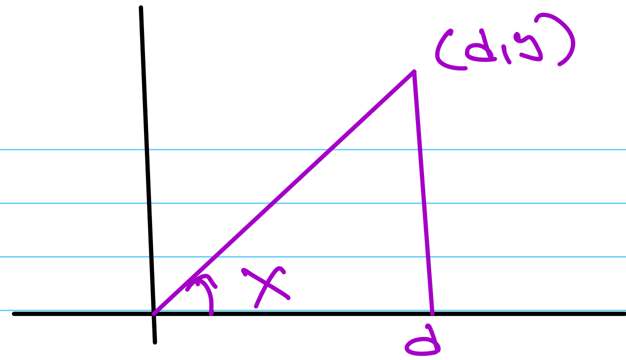
$$= -y\sqrt{\frac{2}{\pi}} \, e^{-\frac{y^2}{2}} \Big]_0^{\infty} + \underbrace{\int_0^{\infty} \sqrt{\frac{2}{\pi}} \, e^{-\frac{y^2}{2}} \, dy}_{\text{pdf} = 1}$$

$$= 0 + 1 = 1$$

$E[Y^2] = 1 \qquad E[Y] = \sqrt{\frac{2}{\pi}}$

$$\boxed{\operatorname{var}(Y) = 1 - \frac{2}{\pi}}$$

2.21




Let X be a random angle whose distribution is uniform on $(0, \frac{\pi}{2})$

Here $Y =$ height of the random triangle for a fixed constant d. find the distribution of Y and $E[Y]$

Solution:

$$X \sim \text{Unif}\left(0, \frac{\pi}{2}\right)$$

$$f_X(x) = \begin{cases} \frac{2}{\pi} & 0 < x < \frac{\pi}{2} \\ 0 & \text{o.w} \end{cases}$$

$$Y = g(X) = d \tan X$$

$$\mathcal{X} = \left(0, \frac{\pi}{2}\right) \qquad y \in (0, \infty)$$

$g(x) = d\tan x$ is a monotone increasing function.

$$x = g^{-1}(y) = \tan^{-1}\left(\frac{y}{d}\right)$$

$$\frac{d}{dy} g^{-1}(y) = \frac{1}{1+\left(\frac{y}{d}\right)^2} \cdot \frac{1}{d}$$

therefore

$$f_Y(y) = f_X\left(g^{-1}(y)\right) \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$= \frac{2}{\pi} \frac{1}{1+\left(\frac{y}{d}\right)^2} \cdot \frac{1}{d}$$

$$\boxed{\Rightarrow \quad f_Y(y) = \frac{2}{\pi d} \cdot \frac{1}{1+\left(\frac{y}{d}\right)^2} \qquad 0 < y < \infty}$$

cauchy distribution

$$\mathbb{E}[Y] = \int_0^\infty y \cdot \frac{2}{\pi d} \frac{1}{1+\left(\frac{y}{d}\right)^2}$$

$\mathbb{E}|Y| = \infty \Rightarrow$ mean does not exists.

2.13 Consider a seq$^n$ of independent coin flips, each of which has probability p of being heads. Define a r.v X as the length of the run (of either heads or tails) started by the first trial. (For Example, X=3 if either TTTH or HHHT is observed)

Find the distribution of X, and find $\mathbb{E}[X]$

Solution:

for $n+1$ coin Flips

$\mathbb{P}(X=n)$ = 1st Tail and next $n-1$ tail and $(n+1)^{th}$ Head

+ 1st Head and next $n-1$ Heads and $(n+1)^{th}$ Tail

$$\Rightarrow \mathbb{P}(X=n) = (1-p)(1-p)^{n-1} \cdot p + p \cdot p^{n-1} \cdot (1-p)$$

$$\Rightarrow \mathbb{P}(X=n) = (1-p)^{n} \cdot p + p^{n}(1-p)$$

$n = 1, 2, \ldots$

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} n \cdot \left[(1-p)^{n} \cdot p + p^{n}(1-p)\right]$$

$$= p(1-p) \sum_{n=1}^{\infty} n\left[(1-p)^{n-1} + p^{n-1}\right]$$

identity:

$$\sum_{k=1}^{\infty} a \cdot k \cdot r^{k-1} = \frac{a}{(1-r)^2}$$

$$= p(1-p)\left[\sum_{n=1}^{\infty} n \cdot (1-p)^{n-1} + \sum_{n=1}^{\infty} n \cdot p^{n}\right]$$

$$= p(1-p)\left[\frac{1}{(1-(1-p))^2} + \frac{1}{(1-p)^2}\right]$$

$$\boxed{E[X] = p(1-p)\left(\frac{1}{p^2} + \frac{1}{(1-p)^2}\right)}$$

identity:

$$\sum_{k=0}^{\infty} a r^{k} = \frac{a}{1-r} \quad \text{if } |r|<1$$

Now differentiate Both side's with $r$

$$\frac{d}{dr} \sum_{k=0}^{\infty} a r^{k} = \frac{d}{dr} \frac{a}{1-r}$$

$$\Rightarrow \sum_{k=0}^{\infty} k \cdot a r^{k-1} = \frac{a}{(1-r)^2}$$

for $k=0$ the summatio $=0$

$$\Rightarrow \sum_{k=1}^{\infty} a \cdot k \cdot r^{k-1} = \frac{a}{(1-r)^2}$$

2.14

(a) Let $X$ be a continuous, nonnegative r.v [$f(x)=0$ for $x<0$]. Show that

$$\mathbb{E}[X] = \int_0^{\infty} (1 - F_X(x)] dx$$

Where $F_X(x)$ is the CdF of $X$

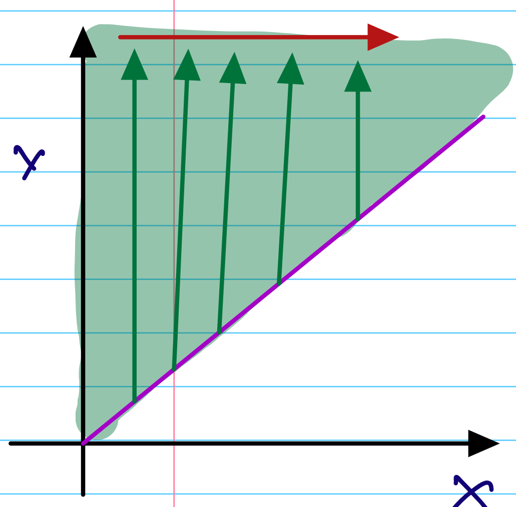
Sootion:

$$\int_0^{\infty} [1 - F_X(x)] dx$$

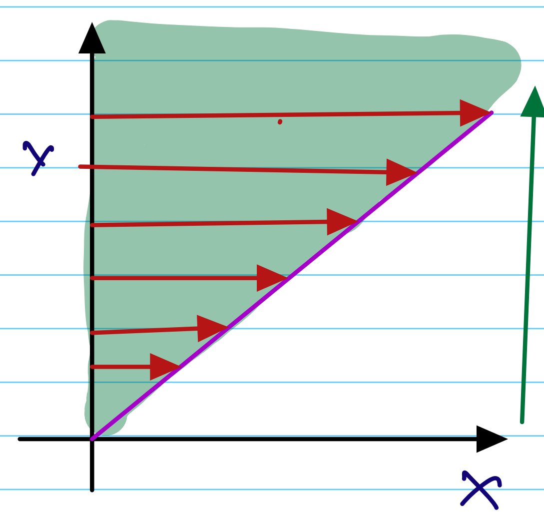
$$= \int_{x=0}^{x=\infty} \left[ 1 - \int_{y=0}^{y=x} f_Y(y)\, dy \right] dx$$

$$= \int_{x=0}^{x=\infty} \left[ \int_{y=x}^{y=\infty} f_Y(y)\, dy \right] dx$$

Now let's do change of order of integral's.




Before




After change of order of integration

$$= \int_{x=0}^{x=\infty} \left[ \int_{y=x}^{y=\infty} f_Y(y)\, dy \right] dx$$

$$= \int_{y=0}^{y=\infty} \int_{x=0}^{x=y} f_Y(y)\, dx\, dy$$

$$= \int_{y=0}^{y=\infty} f_X(y) \left[ \int_{x=0}^{x=y} dx \right] dy$$

$$= \int_{y=0}^{y=\infty} f_X(y)\, y\, dy = \mathbb{E}[X]$$

2.14 b

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} \left(1 - F_X(k)\right)$$

$$= \sum_{k=0}^{\infty} \left(1 - \sum_{n=0}^{k} \mathbb{P}(X=n)\right)$$

$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \mathbb{P}(X=n)$$

Same change of integral's

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \mathbb{P}(X=n)$$

$$= \sum_{n=0}^{\infty} \mathbb{P}(X=n) \sum_{k=0}^{n} 1$$

$$= \sum_{n=0}^{\infty} \mathbb{P}(X=n) \cdot n = \mathbb{E}[X]$$

2.15 Let $X, Y$ be any two r.v's and define

$$X \wedge Y = \min(X, Y)$$

$$X \vee Y = \max(X, Y)$$

s.t $\mathbb{E}[X \vee Y] = \mathbb{E}[X] + \mathbb{E}[Y] - \mathbb{E}[X \wedge Y]$

<u>Solution:</u>

After realizing $X, Y$ we can write

$$X + Y = \max(X, Y) + \min(X, Y)$$

$$\Rightarrow \max(X, Y) = X + Y - \min(X, Y)$$

$$\Rightarrow \mathbb{E}[\max(X, Y)] = \mathbb{E}[X] + \mathbb{E}[Y] - \mathbb{E}[\min(X, Y)]$$

2.16 Using 2.14 to find the mean duration of certain telephone calls, where we assume that the duration, $T$, of a particular call can be described probabilitically by

$$\mathbb{P}(T>t) = ae^{-\lambda t} + (1-a)e^{-\mu t}$$

where $0<a<1$, $\lambda>0$, $\mu>0$

Solution:

$$\mathbb{E}[T] = \int_0^\infty (1 - F_T(t))\, dt$$

$$= \int_0^\infty \int_t^\infty f_T(t)\, dt$$

$$= \int_0^\infty \mathbb{P}(T>t)\, dt$$

$$= \int_0^\infty [ae^{-\lambda t} + (1-a)e^{-\mu t}]\, dt$$

$$= \frac{a}{-\lambda} e^{-\lambda t} + \frac{(1-a)}{-\mu} e^{-\mu t} \Big]_0^{\infty}$$

$$\boxed{\mathbb{E}[T] = \frac{a}{\lambda} + \frac{(1-a)}{\mu}}$$

2.17 A median of a distribution is a value $m$ s.t $\mathbb{P}(X \leq m) \geq \frac{1}{2}$ and

$\mathbb{P}(X \geq m) \geq \frac{1}{2}$.

if $X$ is continuous, $m$ satisfy

$$\int_{-\infty}^{m} f(x)\, dx = \int_{m}^{\infty} f(x)\, dx = \frac{1}{2}$$

Find the median of the following distribution's

(a) $f(x) = 3x^2, \quad 0 < x < 1$

Solution: $\int_0^m 3x^2\,dx = \frac{1}{2}$

$$\Rightarrow \left. x^3 \right]_0^m = \frac{1}{2}$$

$$\Rightarrow m = \left(\frac{1}{2}\right)^{1/3}$$

(b) $f(x) = \dfrac{1}{\pi(1+x^2)} \qquad -\infty < x < \infty$

Solution:

Cauchy distribution is symmetric about 0. so $m = 0$

2.18 S.T if $X$ is continuous r.v, then

$$\min_a \mathbb{E}|X-a| = \mathbb{E}[X-m]$$

Where $m$ is the median of $X$.

$$\mathbb{E}[|X-a|] = \int_{-\infty}^{a} (a-x)\, f_X(x)\, dx$$

$$+ \int_{a}^{\infty} (x-a)\, f_X(x)\, dx$$

for $\min_a \mathbb{E}[|X-a|]$ set

$$\frac{d}{da}\mathbb{E}[|X-a|] = 0$$

$$\Rightarrow \frac{d}{da}\left[\int_{-\infty}^{a} (a-x) f_X(x)\, dx + \int_{a}^{\infty} (x-a) f_X(x)\, dx\right]$$

using Leibnitz's rule

$$\Rightarrow \int_{-\infty}^{a} \frac{d}{da}(a-x) f_X(x)\, dx + \int_{a}^{\infty} \frac{d}{da}(x-a) f_X(x)\, dx$$

$$\Rightarrow \int_{-\infty}^{a} f_X(x)\, dx - \int_{a}^{\infty} f_X(x)\, dx = 0$$

$$\Rightarrow \int_{-\infty}^{a} f_X(x)\,dx = \int_{a}^{\infty} f_X(x)\,dx$$

$$\Rightarrow \quad a = \text{median} = m$$

To verify Let's do double derivate and check its +ve.

$$\frac{d}{da}\left[\frac{d}{da}\mathbb{E}\left[|X-a|\right]\right]$$

$$\Rightarrow \frac{d}{da}\left(\int_{-\infty}^{a} f_X(x)\,dx - \int_{a}^{\infty} f_X(x)\,dx\right)$$

$$\Rightarrow \quad f(a) + f(a) = 2f(a) > 0$$

2.19 Prove that $\frac{d}{da}\mathbb{E}[(X-a)^2] = 0$

$\iff \mathbb{E}[X] = a$

Solution:

$$\frac{d}{da}\int_{-\infty}^{\infty}(x-a)^2 f_X(x)\,dx = 0$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{d}{da}(x-a)^2 f_X(x)\,dx = 0$$

$$\Rightarrow \int_{-\infty}^{\infty}-2(x-a) f_X(x)\,dx = 0$$

$$\Rightarrow \int_{-\infty}^{\infty}(x-a) f_X(x)\,dx = 0$$

$$\boxed{\Rightarrow \quad a = \int_{-\infty}^{\infty}x f_X(x)\,dx = \mathbb{E}[X]}$$

Let's verify its min By checking

$$\frac{d^2}{da^2} \mathbb{E}((X-a)^2)$$

$$\Rightarrow \frac{d}{da}\left( \frac{d}{da} \mathbb{E}((X-a)^2) \right)$$

$$\Rightarrow \frac{d}{da}\left( -2\int_{-\infty}^{\infty} (x-a)\, f_X(x)\, dx \right)$$

$$\Rightarrow -2 \cdot \int_{-\infty}^{\infty} \frac{d}{da} (x-a)\, f_X(x)\, dx$$

$$\Rightarrow -2 \int_{-\infty}^{\infty} -f_X(x)\, dx = 2 > 0$$

Hence its minimum

2.20 A couple decides to continue to have children until a daughter is born. What is the expected number of children of this couple?

Solution:

this is a Geometric Pmf

$\Rightarrow$ $X$ = number of children

$$\mathbb{P}(X=x) = (1-p)^{x-1} p$$

Assuming probability of giving birth to son & daughter = $\frac{1}{2}$

$$\mathbb{P}(X=x) = \left(\frac{1}{2}\right)^{x}$$

$$\mathbb{E}[X] = \sum_{x=1}^{\infty} \mathbb{P}(X=x) \cdot x$$

$$= \sum_{x=1}^{\infty} \left(\frac{1}{2}\right)^{x} \cdot x$$

$$= \frac{1}{2} \sum_{x=1}^{\infty} \left(\frac{1}{2}\right)^{x-1} \cdot x$$

$$= \frac{1}{2} \frac{1}{\left(1-\frac{1}{2}\right)^2} = \frac{4}{2} = 2$$

Expected number of children = 2

2.21

Prove the "two-way" rule for expectations, Eqn 2.2.5, which say's $\mathbb{E}[g(X)] = \mathbb{E}[Y]$ where $Y = g(X)$. Assume that $g(x)$ is a monotone function.

<u>Solution:</u>

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} y \, f_Y(y) \, dy$$

$$= \int_{-\infty}^{\infty} y \quad f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| dy$$

$$y = g(x) \implies x = g^{-1}(y)$$

$$x = g^{-1}(y)$$

$$\frac{dx}{dy} = \frac{d}{dy} g^{-1}(y)$$

$$dx = \frac{d}{dy} g^{-1}(y) \, dy$$

$$= \int_{-\infty}^{\infty} g(x) \, f_X(x) \underbrace{\left| \frac{d}{dy} g^{-1}(y) \right| dy}_{dx}$$

$$= \int_{-\infty}^{\infty} g(x) \, f_X(x) \, dx = \mathbb{E}[X]$$

2.22 Let $X$ is a r.v with Pdf

$$f(x) = \frac{4}{\beta^3\sqrt{\pi}} x^2 e^{-\frac{x^2}{\beta^2}}$$

$$0 < x < \infty, \quad \beta > 0$$

(a) Verify that $f(x)$ is a Pdf

Solution:

(i) $f(x) > 0 \quad \forall x \in (0, \infty)$

(ii) $$\int_0^\infty \frac{4}{\beta^3\sqrt{\pi}} x^2 e^{-\frac{x^2}{\beta^2}} dx$$

$$= \frac{4}{\beta^3\sqrt{\pi}} \int_0^\infty x^2 \cdot e^{-\frac{x^2}{\beta^2}} dx$$

$u = x \qquad v = x \cdot e^{-\frac{x^2}{\beta^2}}$

Integration By parts

$$= \frac{4}{\beta^3\sqrt{\pi}}\left[ x \int x \cdot e^{-\frac{x^2}{\beta^2}} dx \Big]_0^\infty - \int_0^\infty \int x e^{-\frac{x^2}{\beta^2}} dx \right]$$

$$\int x \cdot e^{-\frac{x^2}{\beta^2}} dx \qquad \frac{x^2}{\beta^2} = u$$

$$\Rightarrow \quad \frac{2x}{\beta^2} dx = du$$

$$\Rightarrow \frac{\beta^2}{2} \int \frac{2x}{\beta^2} e^{-\frac{x^2}{\beta^2}} dx$$

$$\Rightarrow \quad \frac{\beta^2}{2} \int e^{-u} du$$

$$= \quad -\frac{\beta^2}{2} e^{-\frac{x^2}{\beta^2}}$$

$$= \frac{4}{\beta^3\sqrt{\pi}} \left[ \underbrace{-\frac{\beta^2}{2} e^{-\frac{x^2}{\beta^2}} \cdot x \Big]_0^\infty}_{0} + \frac{\beta^2}{2} \int_0^\infty e^{-\frac{x^2}{\beta^2}} dx \right]$$

$$= \quad \frac{2}{\beta\sqrt{\pi}} \underbrace{\int_0^\infty e^{-\frac{x^2}{\beta^2}} dx}_{} = \frac{\sqrt{\pi}\,\beta}{2}$$

$$= \frac{2}{\beta\sqrt{\pi}} \cdot \frac{\sqrt{\pi}\,\beta}{2} = 1 \checkmark$$

Hence $f_X(x)$ is Pdf

(b) find $E[X]$ and $var(X)$

2.23 Let x have the pdf

$$f(x) = \frac{1}{2}(1+x) \quad , -1<x<1$$

ⓐ find the pdf of $Y = X^2$

Solution: $\mathcal{Y} = (0,1)$

$A_1 = (-1,0) \quad g_1(x) = x^2 \quad g_1^{-1}(y) = -\sqrt{y}$

$A_2 = (0,1) \quad g_2(x) = x^2 \quad g_2^{-1}(y) = \sqrt{y}$

$$f_Y(y) = \frac{1}{2}(1-\sqrt{y}) \cdot \frac{1}{2\sqrt{y}}$$

$$+ \frac{1}{2}(1+\sqrt{y}) \cdot \frac{1}{2\sqrt{y}}$$

$$= \frac{1}{4\sqrt{y}}(1-\sqrt{y}+1+\sqrt{y})$$

$$\boxed{f_Y(y) = \frac{1}{2\sqrt{y}} \quad , \quad 0<y<1}$$

(b) $\mathbb{E}[Y]$ and $\mathrm{var}(Y)$

Solution:

$$\mathbb{E}\{Y\} = \int_0^1 y \cdot \frac{1}{2\sqrt{y}}\, dy = \frac{1}{3}$$

$$\mathbb{E}[Y^2] = \int_0^1 \frac{y^2}{2\sqrt{y}}\, dy$$

$$= \frac{1}{5}$$

$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - E[Y]^2$$

$$= \frac{1}{5} - \frac{1}{9} = \frac{4}{45}$$

(2.24) Compute the $\mathbb{E}[X]$ and $\mathrm{var}(X)$ for each of the following probability distribution's

(a) $$f_X(x) = a \cdot x^{a-1} \quad \begin{matrix} 0 < x < 1 \\ a > 0 \end{matrix}$$

$$\mathbb{E}[X] = \int_0^1 a \cdot x^a\, dx = \frac{a\, x^{a+1}}{a+1}\Big]_0^1$$

$$= \frac{a}{a+1}$$

$$\mathbb{E}[X^2] = \int_0^1 a \cdot x^{a+1}\, dx$$

$$= a \left. \frac{x^{a+2}}{a+2} \right]_0^1$$

$$= \frac{a}{a+2}$$

$$\text{Var}(X) = \frac{a}{a+2} - \left(\frac{a}{a+1}\right)^2$$

(b) $f_X(x) = \frac{1}{n} \quad x = 1, 2, \ldots n, \; n > 0$

<u>Solution:</u>

$$\mathbb{E}[X] = \sum_{x=1}^{n} \frac{x}{n} = \frac{1}{n} \sum_{x=1}^{n} x$$

$$= \frac{1}{\cancel{n}} \frac{\cancel{n}(n+1)}{2}$$

$$\mathbb{E}[X] = \frac{n+1}{2}$$

$$\mathbb{E}[X^2] = \frac{1}{n} \sum_{x=1}^{n} x^2 = \frac{(n+1)(2n+1)}{6}$$

$$\text{Var}(X) = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$\boxed{\text{Var}(X) = \frac{n^2+1}{12}}$$

(c) $f_X(x) = \frac{3}{2}(x-1)^2 \quad , 0<x<2$

<u>solution:</u>

$$\mathbb{E}[X] = \int_0^2 \frac{3}{2}(x-1)^2 x \, dx$$

$$= \int_0^2 \frac{3}{2}\left(x^3 + x - 2x^2\right) dx$$

$$= \frac{3}{2}\left(\frac{x^4}{4} + \frac{x^2}{2} - \frac{2}{3}x^3\right)\Big]_0^2$$

$$\boxed{E(X) = 1}$$

$$E[X^2] = \int_0^2 x^2 \frac{3}{2}(x-1)^2 \, dx$$

$$= \frac{3}{2}\int_0^2 \left(x^4 + x^2 - 2x^3\right) dx$$

$$= \frac{3}{2}\left(\frac{x^5}{5} + \frac{x^3}{3} - \frac{2x^4}{4}\right]_0^2 = \frac{8}{5}$$

$$\mathrm{var}(X) = \frac{8}{5} - 1^2 = \frac{3}{5}$$

2.25 Suppose that the pdf $f_X(x)$ of a r.v $X$ is a even function.

$\Rightarrow \quad f_X(x) = f_X(-x) \quad \forall x$

Show that

(a) $X$ and $-X$ are identically distributed

Solution:

$$Y = g(X) = -X$$

$$g^{-1}(y) = -y$$

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

$$= f_X(-y) \cdot 1$$

$$= f_X(-y)$$

$$f_Y(y) = f_X(y)$$

Both their pdf are same

$\Rightarrow$ identically distributed.

(b) $M_x(t)$ is symmetric about 0

we need to show

$$M_x(0+\varepsilon) = M_x(0-\varepsilon)$$

(2.28) Let $\mu_n$ denote the $n^{th}$ central moment of a r.v $X$. Two quantities of interest, in addition to the mean and variance, are

$$\alpha_3 = \frac{\mu_3}{(\mu_2)^{3/2}} \quad \text{and} \quad \alpha_4 = \frac{\mu_4}{\mu_2^2}$$

(Skewness) (Kurtosis)

The Skewness $\alpha_3$ measures the lack of symmetry in the pdf.

The kurtosis, although harder to interpret, measures the peakedness or flatness of the pdf.

(a) S.t if a pdf is symmetric about a point $a$, then $\alpha_3 = 0$

<u>Solution:</u>

$\mu_3$ = 3rd central moment

Assuming $f_X(x)$ is symmetric around $a$ $\Rightarrow$ $\mathbb{E}[X] = a$

$$\mu_3 = \int_{-\infty}^{\infty} (x-a)^3 f_X(x)\,dx$$

$$= \int_{-\infty}^{a} (x-a)^3 f_X(x)\,dx + \int_{a}^{\infty} (x-a)^3 f_X(x)\,dx$$

take $y = x - a$

$\Rightarrow$ $dy = dx$

$$= \int_{-\infty}^{0} y^3 f_X(y+a)\,dy + \int_{0}^{\infty} y^3 f_X(y+a)\,dy$$

$$f(y+a) = f(-y+a)$$

$$= \int_{-\infty}^{0} y^3 f_X(-y+a)\,dy + \int_{0}^{\infty} y^3 f_X(y+a)\,dy$$

$$= \int_0^\infty -y^3 f_x(-y+a)\,dy + \int_0^\infty y^3 f_x(y+a)\,dy$$

$$= 0$$

(b) Calculate $\alpha_3$ for $f(x) = e^{-x}$

$x \geq 0$, a pdf that is skewed to the Right.

__Soln__ $\quad f_x(x) = e^{-x}$

$$\mu_1 = \mathbb{E}[x] = \int_0^\infty e^{-x} = 1$$

$$\mu_2 = \mathbb{E}[(x-1)^2] = \int_0^\infty (x-1)^2 e^{-x}$$

$$= 1$$

$$\alpha_3 = \mu_3$$

$$= \int_0^\infty (x-1)^3 e^{-x}\,dx$$

$$= \int_0^{\infty} (x^3 - 3x^2 + 3x - 1) e^{-x} \, dx$$

(simplify)

$$= 3$$

2.29 ⓐ calculate the factorial moment $\mathbb{E}[X(X-1)]$ for the binomial and posion distribution.

Solution:

Binomial:

$$\mathbb{E}[X(X-1)] =$$

$$\sum_{x=0}^{n} \binom{n}{x} p^{x} (1-p)^{n-x} \quad x(x-1)$$

$$= \sum_{x=0}^{n} \frac{n!}{(n-x)!\, x!} p^{x} (1-p)^{n-x} x(x-1)$$

$$= \sum_{x=0}^{n} \frac{n\cdot(n-1)(n-2)!}{(n-x)!\,(x-2)!} p^{x} (1-p)^{n-x}$$

$$= n(n-1)\cdot p^{2} \sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2} (1-p)^{n-x}$$

$$= n(n-1)p^2 \underbrace{\sum_{y=0}^{n-2} \binom{n-2}{y} p^y (1-p)^{n-2-y}}_{\text{Pdf } Bin(n-2,p)}$$

$$\boxed{\mathbb{E}[X(1-X)] = n(n-1)p^2}$$

## Poisson distribution:

$$\mathbb{E}[X(X-1)] = \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x-2} \cdot \lambda^2}{(x-2)!}$$

$$= \underbrace{\sum_{y=0}^{\infty} \frac{e^{-\lambda} \cdot \lambda^y}{y!}}_{\text{Pdf}} \cdot \lambda^2$$

$$\boxed{\mathbb{E}[X(X-1)] = \lambda^2}$$

2.29 b Use the results of Part (a) to Calculate the variances of the Binomial and Possion distributions.

Solution:

$$E[X(X-1)] = \mathbb{E}[X^2] - \mathbb{E}[X]$$

so $$\text{var}(X) = \mathbb{E}[X^2] - E[X]^2$$
$$= \mathbb{E}[X(X-1)] + E[X] - E[X]^2$$

for Binomial:

$$\text{var}(X) = \mathbb{E}[X(X-1)] + E[X] - E[X]^2$$
$$= n(n-1)p^2 + np - (np)^2$$
$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$\boxed{\text{var}(X) = np(1-p)}$$

Possion:

$$\text{var}(X) = \mathbb{E}[X(X-1)] + E[X] - E[X]^2$$
$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$\boxed{\text{var}(X) = \lambda}$$

2.29 c $$P(Y=y) = \frac{a(y+a)\binom{n}{y}\binom{a+b-1}{a}}{\binom{n+a+b-1}{y+a}}$$

$n, a, b$ are integer's

$y = 0, 1, 2, \ldots, n$

use factorial moment's to calculate the variance of the beta-binomial.

<u>solution:</u>

$$\mathbb{E}[Y(Y-1)] =$$

$$= \sum_{y=2}^{n} y(y-1) \frac{a(y+a)\binom{n}{y}\binom{a+b-1}{a}}{\binom{n+a+b-1}{y+a}}$$

for $y=0,1$ it's 0

$$= \sum_{y=2}^{n} \frac{n(n-1)\, a(y+a)\binom{n-2}{y-2}\binom{a+b-1}{a}}{\binom{n+a+b-1}{y+a}}$$

$$= \frac{n(n-1)\,a}{a+2} \sum_{y=2}^{n} \frac{(a+2)(y-2+a+2)\binom{n-2}{y-2}\binom{a+b-1}{a}}{\binom{n-2+a+2+b-1}{y-2+a+2}}$$

$$\binom{a+b-1}{a} = \frac{(a+b-1)!}{(b-1)!\,a!}$$

$$= \frac{(a+b-1)!}{(b-1)!\,a!} \cdot \frac{(a+1)(a+2)}{(a+1)(a+2)} \cdot \frac{(a+b)(a+b+1)}{(a+b)(a+b+1)}$$

$$= \frac{(a+2+b-1)!}{(b-1)!\,(a+2)!} \; \frac{(a+1)(a+2)}{(a+b)(a+b+1)}$$

$$= \binom{a+2+b-1}{a+2} \frac{(a+1)(a+2)}{(a+b)(a+b+1)}$$

$$= \frac{n(n-1)a}{(a+2)} \cdot \frac{(a+1)(a+2)}{(a+b)(a+b+1)}$$

$$\cdot \sum_{y=2}^{n} \frac{(a+2)(y-2+a+2)\binom{n-2}{y-2}\binom{a+2+b-1}{a+2}}{\binom{n-2+a+2+b-1}{y-2+a+2}}$$

Put $y-2=x$

$$= \frac{n(n-1)\,a\,(a+1)}{(a+b)(a+b+1)} \underbrace{\sum_{x=0}^{n-2} \frac{(a+2)}{(x+a+2)} \frac{\binom{n-2}{x}\binom{a+2+b-1}{a+2}}{\binom{n-2+a+2+b-1}{x+a+2}}}_{\substack{\text{pdf with, } n-2,\ a+2,\ b \\ =1}}$$

$$\mathbb{E}[Y(Y-1)] = \frac{n(n-1)\,a\,(a+1)}{(a+b)(a+b+1)}$$

Calculating $\mathbb{E}[Y]$

$$\mathbb{E}[Y] = \sum_{y=1}^{n} y \cdot \frac{\frac{a}{(y+a)}\binom{n}{y}\binom{a+b-1}{a}}{\binom{n+a+b-1}{y+a}}$$

for $y=0$, the value $=0$

$$= \frac{a}{a+1} \cdot n \sum_{y=1}^{n} \frac{(a+1)(y+a)\binom{n-1}{y-1}\binom{a+b-1}{a}}{\binom{n+a+b-1}{y+a}}$$

$$\binom{a+b-1}{a} = \binom{a+1+b-1}{a+1} \frac{a+1}{a+b}$$

$$= \frac{n \cdot a}{a+b} \sum_{y=1}^{n} \frac{(a+1)(y+a)\binom{n-1}{y-1}\binom{a+1+b-1}{a+1}}{\binom{n+a+b-1}{y+a}}$$

put $y-1=x$

$$= \frac{n \cdot a}{a+b} \sum_{x=0}^{n-1} \underbrace{\frac{(a+1)(x+a+1)\binom{n-1}{x}\binom{a+1+b-1}{a+1}}{\binom{n+a+b-1}{x+a+1}}}_{\text{Pdf with } n-1,\ a+1,\ b \ = 0}$$

$$\mathbb{E}[Y] = \frac{na}{a+b}$$

$$\operatorname{Var}(Y) = \mathbb{E}[Y(Y-1)] + \mathbb{E}[X] - \mathbb{E}[X]^2$$

$$= \frac{n(n-1)a(a+1)}{(a+b)(a+b+1)} + \frac{na}{a+b} - \frac{n^2a^2}{(a+b)^2}$$

$$= \frac{n(n-1)a(a+1)(a+b) + na(a+b)(a+b+1) - n^2a^2(a+b+1)}{(a+b)^2(a+b+1)}$$

$$= \frac{na\left((n-1)(a+1)(a+b) + (a+b)(a+b+1) - na(a+b+1)\right)}{(a+b)^2(a+b+1)}$$

$$= \frac{nab(n+a+b)}{(a+b)^2(a+b+1)}$$

2.30 Find the MGF corresponding to

(a) $f(x) = \frac{1}{c} \qquad 0 < x < c$

Solution:

$$M_X(t) = \mathbb{E}[e^{tX}]$$

$$= \int_0^c \frac{1}{c} \cdot e^{tx}\, dx$$

$$= \frac{1}{c} \frac{e^{tx}}{t} \Big]_0^c$$

$$= \frac{1}{tc} \left[ e^{tc} - 1 \right)$$

(b) $f(x) = \frac{2x}{c^2} \qquad 0 < x < c$

Solution:

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_0^c \frac{2x}{c^2} \cdot e^{tx}\, dx$$

$$= \frac{2}{c^2} \int_0^c x \cdot e^{tx} \, dx$$

$$= \frac{2}{c^2} \left[ \left. x \cdot \frac{e^{tx}}{t} \right]_0^c - \left. \frac{e^{tx}}{t^2} \right]_0^c \right]$$

$$= \frac{2}{c^2} \left[ \frac{c}{t} e^{tc} - \frac{e^{tc}}{t^2} + \frac{1}{t^2} \right]$$

$$= \frac{2}{(tc)^2} \left( ct\, e^{tc} - e^{tc} + 1 \right)$$

(c) $f(x) = \frac{1}{2\beta} e^{-\frac{|x-\alpha|}{\beta}}$ , $-\infty < x < \infty$ , $-\infty < \alpha < \infty$ , $\beta > 0$

<u>solution:</u>

$$E[e^{tx}] = \int_{-\infty}^{\infty} \frac{1}{2\beta} e^{-\frac{(x-\alpha)}{\beta}} \cdot e^{tx} \, dx$$

$$= \int_{-\infty}^{\alpha} \frac{1}{2\beta} e^{\frac{x-\alpha}{\beta} + tx} \, dx + \int_{\alpha}^{+\infty} \frac{1}{2\beta} e^{-\frac{(x-\alpha)}{\beta} + tx} \, dx$$

$$= \int_{-\infty}^{\alpha} \frac{1}{2\beta} e^{\frac{x - \frac{\alpha}{1+t\beta}}{\frac{\beta}{1+t\beta}}} \, dx + \int_{\alpha}^{+\infty} \frac{1}{2\beta} e^{\frac{\frac{\alpha}{1-t\beta} - x}{\frac{\beta}{1-t\beta}}} \, dx$$

$$= \int_{-\infty}^{\alpha} \frac{1}{2\beta} e^{x\left(\frac{1}{\beta}+t\right)} \cdot e^{-\frac{\alpha}{\beta}} \, dx + \int_{\alpha}^{+\infty} \frac{1}{2\beta} e^{\frac{\alpha}{\beta}} \cdot e^{-x\left(\frac{1}{\beta}-t\right)} \, dx$$

$$= \frac{e^{-\frac{\alpha}{\beta}}}{2\beta} \int_{-\infty}^{\alpha} e^{x(1/\beta+t)} \, dx + \frac{1}{2\beta} e^{\frac{\alpha}{\beta}} \int_{\alpha}^{+\infty} e^{-x\left(\frac{1}{\beta}-t\right)} \, dx$$

$$= \frac{e^{-\frac{\alpha}{\beta}}}{2\beta} \cdot e^{x\left(\frac{1}{\beta}+t\right)} \cdot (1/\beta+t) \Big]_{-\infty}^{\alpha}$$

$$+ \left. \frac{e^{\frac{\alpha}{\beta}}}{2\beta} \cdot e^{-\alpha\left(\frac{1}{\beta}-t\right)} \cdot \left(\frac{1}{\beta}-t\right) \right]_{\alpha}^{+\infty}$$

$$= \frac{e^{-\frac{\alpha}{\beta}}}{2\beta} \cdot e^{\alpha\left(\frac{1}{\beta}+t\right)} \cdot \frac{1}{\left(\frac{1}{\beta}+t\right)}$$

$$- \frac{e^{\frac{\alpha}{\beta}}}{2\beta} \cdot e^{-\alpha\left(\frac{1}{\beta}-t\right)} \frac{1}{\left(\frac{1}{\beta}-t\right)}$$

$$= \frac{1}{2\beta} e^{\alpha t} \cdot \frac{1}{\left(\frac{1}{\beta}+t\right)} - \frac{1}{2\beta} e^{\alpha t} \cdot \frac{1}{\left(\frac{1}{\beta}-t\right)}$$

$$= \frac{1}{2\beta} e^{\alpha t} \left( \frac{\beta}{1+\beta t} - \frac{\beta}{1-\beta t} \right)$$

2.30 d $\quad \mathbb{P}(X=x) = \binom{r+x-1}{x} p^r (1-p)^x$

$x = 0, 1, \ldots \qquad 0 < p < 1 \quad , r > 0$ an integer.

<u>Solution:</u>

$$\mathbb{E}[e^{tX}] =$$

$$= \sum_{x=0}^{\infty} e^{tx} \binom{r+x-1}{x} p^r (1-p)^x$$

$$= \sum_{x=0}^{\infty} \binom{r+x-1}{x} p^r \left(e^t(1-p)\right)^x$$

$$= \sum_{x=0}^{\infty} \binom{r+x-1}{x} p^r \left(\underbrace{e^t - e^t p}_{\displaystyle \| \atop \displaystyle [1-(1-e^t(1-p))]}\right)^x$$

$$= \sum_{x=0}^{\infty} \binom{r+x-1}{x} \frac{\left(1-e^{t}(1-p)\right)^{r}}{\dfrac{\left(1-e^{t}(1-p)\right)^{r}}{p^{r}}} \left(1-(1-e^{t}(1-p)\right]^{x}$$

$$= \frac{p^{r}}{\left(1-e^{t}(1-p)\right)^{r}} \underbrace{\sum_{x=0}^{\infty} \binom{r+x-1}{x} \left(1-e^{t}(1-p)\right)^{r} \left(1-(1-e^{t}(1-p)\right)^{x}}_{\text{pdf } = 1}$$

$$\mathbb{E}\left[e^{tx}\right] = \left(\frac{p}{1-e^{t}(1-p)}\right)^{r}$$

$$\Rightarrow \quad M_{x}(t) = \left[\frac{p}{1-e^{t}(1-p)}\right]^{r}$$

such that $\left|\dfrac{p}{1-e^{t}(1-p)}\right| < 1$

2.31 Does a distribution exists for which

$M_X(t) = \dfrac{t}{1-t}$ , $|t| < 1$ ? if yes,

find it. if no Prove it.

**Proof:**

$$M_X(0) = \mathbb{E}[e^{0x}] = \mathbb{E}[1] = 1$$

$$\Rightarrow \quad M_X(0) = \frac{0}{1-0} = 0 \Rightarrow \text{it is}$$

not MGF

(2.32) Let $M_X(t)$ be the MGF. and define

$$S(t) = \log(M_X(t)) \quad \text{s.t}$$

$$\frac{d}{dt} S(t)\Big|_{t=0} = \mathbb{E}[X]$$

and

$$\frac{d^2 S(t)}{dt^2}\Big|_{t=0} = \text{Var}(X)$$

Solution:

$$\frac{d}{dt} S(t) = \frac{d}{dt} \log(M_X(t))$$

$$= \frac{d}{dt} \log(\mathbb{E}[e^{tX}])$$

$$= \frac{1}{\mathbb{E}[e^{tX}]} \cdot \frac{d}{dt} \mathbb{E}[e^{tX}]$$

$$= \frac{1}{\mathbb{E}[e^{tX}]} \cdot \mathbb{E}[Xe^{tX}]\Big|_{t=0}$$

$$= \frac{1}{\mathbb{E}[1]} \cdot \mathbb{E}[X]$$

$$\boxed{\frac{d}{dt} S(t)\Big|_{t=0} = \mathbb{E}[X]}$$

$$\frac{d^2}{dt^2} S(t) = \frac{d}{dt}\left(\frac{d}{dt} S(t)\right)$$

$$= \frac{d}{dt}\left(\frac{\mathbb{E}[X \cdot e^{tX}]}{\mathbb{E}[e^{tX}]}\right)$$

$$= \frac{\mathbb{E}[e^{tX}] \cdot \mathbb{E}[X^2 e^{tX}] - \mathbb{E}[Xe^{tX}]\,\mathbb{E}[Xe^{tX}]}{\mathbb{E}[e^{tX}]^2}$$

$$= \frac{\mathbb{E}[e^{tX}]\,\mathbb{E}[X^2 e^{tX}] - \mathbb{E}[Xe^{tX}]^2}{\mathbb{E}[e^{tX}]^2}$$

$\Rightarrow$ at $t = 0$

$$= \frac{\mathbb{E}[1] \cdot \mathbb{E}[X^2] - \mathbb{E}[X]^2}{\mathbb{E}[1]^2}$$

$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \text{var}(X)$$

$$\boxed{\Rightarrow \frac{d^2}{dt^2} S(t) \Big|_{t=0} = \text{var}(X)}$$

2.33 ⓐ $\mathbb{P}(X=x) = \dfrac{e^{-\lambda} \lambda^x}{x!}$

<u>Solution:</u>

$$M_X(t) = \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!} \cdot e^{tx}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}\,(e^{t}\cdot\lambda)^{x}}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}}{e^{-\lambda\cdot e^{t}}}\; \frac{e^{-\lambda\cdot e^{t}}\,(e^{t}\cdot\lambda)^{x}}{x!}$$

$$= e^{\lambda(e^{t}-1)} \underbrace{\sum_{x=0}^{\infty} \frac{e^{-\lambda\cdot e^{t}}\cdot(e^{t}\cdot\lambda)^{x}}{x!}}_{\text{Possion distributio} = 1}$$

$$\boxed{M_x(t) = e^{\lambda(e^{t}-1)}}$$

$$\mathbb{E}[X] = M_x^{(1)}(0)$$

$$= e^{\lambda(e^{t}-1)}\cdot\lambda\cdot e^{t}\Big|_{t=0}$$

$$\boxed{\mathbb{E}[X] = \lambda}$$

$$\mathbb{E}[X^2] = M_x^{(2)}(0)$$

$$= \lambda\left( e^t e^{\lambda(e^t-1)} + e^t \left(e^{\lambda(e^t-1)} \cdot \lambda \cdot e^t\right)\right)$$

$$= \lambda e^t e^{\lambda(e^t-1)} \left[1 + e^t \cdot \lambda\right] \Big|_{t=0}$$

$$= \lambda\left[1+\lambda\right]$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$= \lambda + \lambda^2 - \lambda^2 = \lambda$$

$$\boxed{\mathrm{Var}(X) = \lambda}$$

(b) $\mathbb{P}(X=x) = p(1-p)^x$

$x = 0, 1, \ldots \qquad 0 < p < 1$

<u>Solution:</u>

$$M_X(t) = \mathbb{E}\left[e^{tX}\right] = \sum_{x=0}^{\infty} p(1-p)^x \cdot e^{tx}$$

$$= \sum_{x=0}^{\infty} p \left[(1-p)e^t\right]^x$$

$$= \frac{p}{\left(1-(1-p)e^t\right)} \underbrace{\sum_{x=0}^{\infty} \left(1-(1-p)e^t\right)\left((1-p)e^t\right)^x}_{\text{Pdf} \quad =0}$$

$$\boxed{M_x(t) = \frac{p}{1-(1-p)e^t}}$$

$$\mathbb{E}[x] = \frac{d}{dt} M_x(t) \Big|_{t=0}$$

$$= \frac{-p\left(-(1-p)e^t\right)}{\left(1-(1-p)e^t\right)^2} \Bigg|_{t=0}$$

$$= \frac{+p(1-p)}{p^2} = \frac{1-p}{p}$$

$$\boxed{\mathbb{E}[x] = \frac{1-p}{p}}$$

$$\mathbb{E}[x^2] = \frac{d^2}{dt^2} M_x(t) \Big|_{t=0}$$

$$= \frac{d}{dt}\left( \frac{p(1-p)e^t}{(1-(1-p)e^t)^2} \right)$$

$$= \frac{(1-(1-p)e^t)^2 \, p(1-p)e^t + p(1-p)^2 e^{2t} \cdot 2 \cdot (1-(1-p)e^t)}{(1-(1-p)e^t)^4}$$

at $t=0$

$$= \frac{p^2 \, p(1-p) + p(1-p)^2 \cdot 2 \cdot p}{p^4}$$

$$= \frac{1-p}{p} + 2 \cdot \left(\frac{1-p}{p}\right)^2$$

$$Var(x) = \mathbb{E}[x^2] - E[x]^2$$

$$= \left(\frac{1-p}{p}\right) + 2 \cdot \left(\frac{1-p}{p}\right)^2 - \left(\frac{1-p}{p}\right)^2$$

$$= \frac{1-p}{p} + \left(\frac{1-p}{p}\right)^2 = \frac{1-p}{p^2}$$

$$\boxed{Var(x) = \frac{1-p}{p^2}}$$

2.33 C $\quad f_X(x) = \dfrac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}$

Soution:

$$M_X(t) = \mathbb{E}[e^{tX}]$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{\frac{-(x^2+\mu^2-2x\mu-2\sigma^2 tx)}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x^2-2\cdot x(\mu+\sigma^2 t)+(\mu+\sigma^2 t)^2)}{2\sigma^2}} \cdot e^{-\frac{1}{2\sigma^2}(\mu^2-(\mu+\sigma^2 t)^2)}$$

$$= \frac{e^{-\frac{1}{2\sigma^2}(\cancel{\mu^2}-\cancel{\mu^2}-\sigma^4 t^2-2\mu\sigma^2 t)}}{\sqrt{2\pi}\,\sigma} \cdot \int_{-\infty}^{\infty} e^{-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}}\,dx \Big\}\, pdf = 1$$

$$= e^{\frac{\sigma^2 t^2 + 2\mu t}{2}}$$

$$\boxed{M_X(t) = e^{\frac{\sigma^2 t^2}{2} + \mu t}}$$

$$\mathbb{E}[X] = \frac{d}{dt} M_X(t) \Big|_{t=0}$$

$$= e^{\frac{\sigma^2 t^2}{2} + \mu t} \left( \sigma^2 t + \mu \right) \Big|_{t=0}$$

$$\boxed{\mathbb{E}[X] = \mu}$$

$$\mathbb{E}(X^2) = \frac{d^2}{dt^2} M_X(t) \Big|_{t=0}$$

$$= \frac{d}{dt} \left( e^{\frac{\sigma^2 t^2}{2} + \mu t} \cdot (\sigma^2 t + \mu) \right]$$

$$= e^{\frac{\sigma^2 t^2}{2} + \mu t} (\sigma^2 t + \mu)^2$$

$$+ e^{\frac{\sigma^2 t^2}{2} + \mu t} (\sigma^2) \Big|_{t=0}$$

$$= \mu^2 + \sigma^2$$

$$\operatorname{Var}(X) = \mathbb{E}[X^2] - E[X]^2$$

$$= \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$

$$\boxed{\operatorname{Var}(X) = \sigma^2}$$

2.34 A distribution cannot be uniquely determined by a finite collection of moments. as this example from Romano and Siegel (1986) shows.

Let $X \sim N(0,1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad -\infty < x < \infty$$

Define a discrete random variable. $Y$ by

$$\mathbb{P}(Y = \sqrt{3}) = P(Y = -\sqrt{3}) = \frac{1}{6}$$

$$\mathbb{P}(Y = 0) = \frac{2}{3}$$

Show that $\mathbb{E}[X^r] = E[Y^r]$ for $r = 1, 2, 3, 4$

Solution:

$$M_X(t) = e^{t^2/2} \quad \text{for} \quad X \sim N(0,1)$$

$$M_Y(t) = \sum_y e^{ty} \cdot P(Y=y)$$

$$= e^{\sqrt{3}t} \cdot \frac{1}{6} + e^{-\sqrt{3}t} \cdot \frac{1}{6}$$

$$+ \frac{2}{3}$$

$$= \frac{1}{3}\left(\frac{e^{\sqrt{3}t} + e^{-\sqrt{3}t}}{2}\right) + \frac{2}{3}$$

$$E[Y] = \frac{d}{dt} M_X(t) \Big|_{t=0}$$

$$= \frac{1}{6}\left(\sqrt{3}e^{\sqrt{3}t} - \sqrt{3}e^{-\sqrt{3}t}\right)$$

$$E[Y] = \frac{\sqrt{3}}{6}\left(e^{\sqrt{3}t} - e^{-\sqrt{3}t}\right)\Big|_{t=0}$$

$$= 0$$

$$\mathbb{E}[Y^2] = \frac{3}{6}\left(e^{\sqrt{3}t} + e^{-\sqrt{3}t}\right)\Big|_{t=0}$$

$$= 1$$

$$\mathbb{E}[Y^3] = 0$$

$$\mathbb{E}[Y^4] = \frac{9}{6}\left(e^{\sqrt{3}t} + e^{-\sqrt{3}t}\right)\Big|_{t=0}$$

$$= 3$$

$\vdots$

Simillary $\mathbb{E}[X] = 0$

$\mathbb{E}[X^2] = 1$

$\mathbb{E}[X^3] = 0$

$\mathbb{E}[X^4] = 3$

All of them are equal