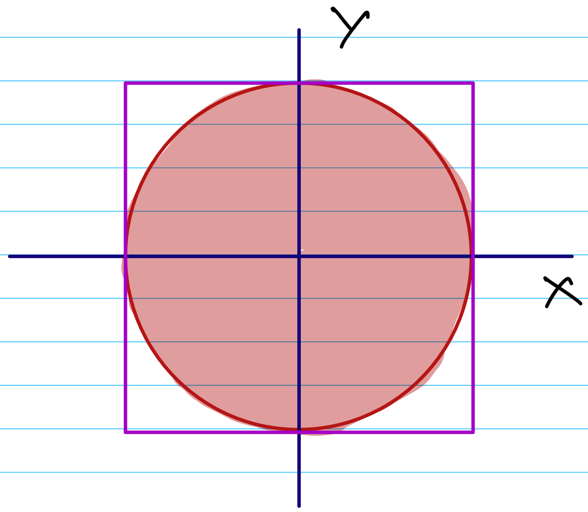
DESCRIPTION

This document provides solution's to 4.1 to 4.14 from chapter 4 of statistical inference Book by Casella and Berger, Focusing on Joint & Marginal Distribution, Conditional Distribution's & Independence

A few Problem's are yet to be solved, but I will update the document soon.

(4.1) A random vector (X, Y) is distributed uniformly on the square with vertices $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$. That is, the joint pdf is $f(x,y) = \frac{1}{4}$. Determine the prob of the following events

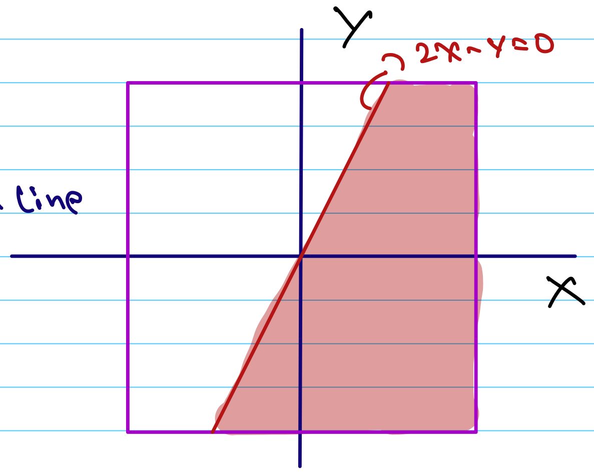
(a) $x^2 + y^2 < 1$



The circle $x^2+y^2<1$ will have area $\pi \cdot r^2 = \pi$

$$\Rightarrow \quad \mathbb{P}(x^2+y^2<1) = \frac{\text{area of circle}}{\text{area of square}} = \pi$$

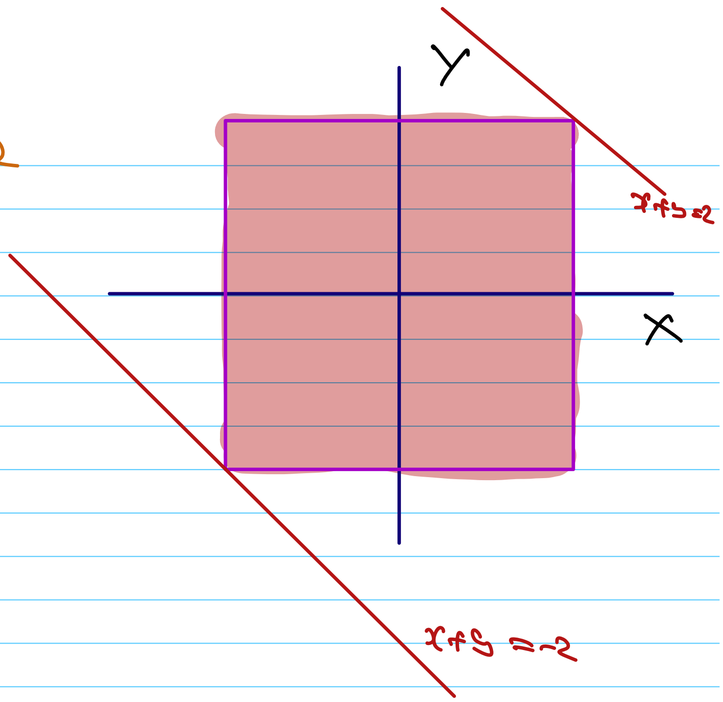
(b) $2x - y > 0$



$\mathbb{P}(2x-y>0) =$ area under line

$= \frac{1}{2}$

(c) $|x+y| < 2$



$\Rightarrow$ $-2 < x+y < 2$

Total area = 1

(4.2) Prove.

(a) $\mathbb{E}[a g_1(x,y) + b g_2(x,y) + c]$

$= a E[g_1(x,y)] + b E[g_2(x,y)] + c$

<u>solution:</u>

$\mathbb{E}[a g_1(x,y) + b g_2(x,y) + c]$

$= \mathbb{E}[a g_1(x,y)] + \mathbb{E}[b g_2(x,y) + E[c]$

$= a \mathbb{E}[g_1(x,y)] + b E[g_2(x,y)] + c$

(b) if $g_1(x,y) \geq 0$ then $\mathbb{E}(g_1(x,y)) \geq 0$

$$\mathbb{E}[g_1(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{g_1(x,y)}_{+ve}\, \underbrace{f_{x,y}(x,y)}_{+ve}\, dx\,dy$$

$$\Rightarrow \mathbb{E}[g_1(x,y)] \geq 0$$

(c) $g_1(x,y) \geq g_2(x,y)$

$$\Rightarrow \mathbb{E}(g_1(x,y)) \geq \mathbb{E}\, g_2(x,y))$$

Solution:

$$g_1(x,y) \geq g_2(x,y) \quad \forall x,y$$

$$\Rightarrow g_1(x,y)\, f_{x,y}(x,y) \geq g_2(x,y)\, f_{x,y}(x,y)$$

$$(\because f_{x,y}(x,y) \geq 0)$$

$$\Rightarrow \iint g_1(x,y) f(x,y) \geq \iint g_2(x,y) f(x,y)$$

$$\Rightarrow \mathbb{E}[g_1(x,y)] \geq \mathbb{E}[g_2(x,y)]$$

(d) $a \leq g_1(x,y) \leq b \Rightarrow a \leq E[g_1(x,y)) \leq b$

Solution:

$$a \leq g_1(x,y) \leq b$$

$\Rightarrow \quad a f(x,y) \le g_1(x,y) f(x,y) \le b f(x,y)$

$(\because f(x,y) \geq 0)$

$\Rightarrow \iint a f(x,y) \le \iint g(x,y) f(x,y) \le \iint b f(x,y)$

$\Rightarrow a \iint f(x,y) \le \mathbb{E}[g_1(x,y)] \le b \iint f(x,y)$

$\Rightarrow \quad \boxed{a \le \mathbb{E}[g_1(x,y)] \le b}$

4.3 Example 4.1.5

$f(0,0) = f(0,1) = \frac{1}{6}$

$f(1,0) = f(1,1) = \frac{1}{3}$

$f(x,y) = 0 \quad \forall$ remainin (x,y)

⋮

4.4

$$f(x,y) = \begin{cases} c\,(x+2y) & \text{if } 0<y<1 \ \& \\ & 0<x<2 \\ 0 & \text{o.w} \end{cases}$$

(a) find the value of c

Solution:

$$\int_0^2 \int_0^1 c(x+2y)\,dy\,dx = 1$$

$$\Rightarrow \int_0^2 c(xy+y^2)\Big]_0^1 dx = 1$$

$$c = 1/4$$

(b) $f_X(x) = ?$

$$f_X(x) = \int_{y=0}^{y=1} \frac{1}{4}(x+2y)\,dy$$

$$= \frac{1}{4}xy + \frac{y^2}{4}\Big]_0^1 = \frac{1}{4}(x+1)$$

(c) Joint CDF of x, Y

$$F_{XY}(x,y) = \mathbb{P}(X \le x, Y \le y)$$

$$= \int_0^x \int_0^y \frac{1}{4}(x+2y)\, dy\, dx$$

$$= \frac{1}{4}\int_0^x \left. xy+y^2 \right]_0^y dx = \frac{1}{4}\int_0^x xy+y^2$$

$$= \frac{1}{4}\left(\frac{x^2 y}{2} + y^2 x\right)$$

$$\boxed{F_{X,Y}(x,y) = \frac{x^2 y}{8} + \frac{y^2 x}{4}}$$

(d) $Z = \dfrac{9}{(X+1)^2}$ $\quad f_Z(z)$?

<u>sol^n</u>:

$$f_X(x) = \frac{1}{4}(x+1) \qquad 0 < x < 2$$

$$z = \frac{9}{(x+1)^2} \Rightarrow (x+1)^2 = \left(\frac{3}{\sqrt{z}}\right)^2$$

$$x+1 = -\frac{3}{\sqrt{z}} \quad \text{or} \quad x+1 = \frac{3}{\sqrt{z}}$$

$\backslash\backslash$ Not possible $\qquad x = \frac{3}{\sqrt{z}} - 1$

$$x = \frac{3}{\sqrt{z}} - 1 \qquad z \in (0,9)$$

$$\frac{d g^{-1}(z)}{dz} = -\frac{1}{2} \frac{3}{z^{3/2}}$$

$$f_Z(z) = \frac{1}{4}\left(\frac{3}{\sqrt{z}} - 1 + 1\right) \cdot +\frac{1}{2} \frac{3}{z^{3/2}}$$

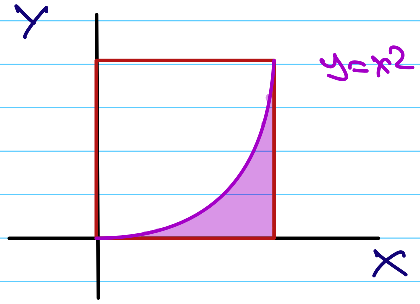
$$= \frac{9}{8z^2}$$

$$\boxed{\Rightarrow \quad f_Z(z) = \frac{9}{8z^2} \qquad 0 < z < 9}$$

45 ⓐ Find $\mathbb{P}(X > \sqrt{Y})$ if X and Y are Jointly distributed with Pdf

$$f(x,y) = x+y \;, \quad 0 \le x \le 1$$
$$0 \le y \le 1$$

solution:



$$\mathbb{P}(X > \sqrt{Y}) = \int_{y=0}^{y=1} \int_{x=\sqrt{y}}^{x=1} (x+y)\,dx\,dy$$

$$= \int_{y=0}^{y=1} \frac{x^2}{2} + xy \Big]_{\sqrt{y}}^{1} dy$$

$$= \int_{y=0}^{y=1} \left(\frac{1}{2} + y - \frac{y}{2} - y^{3/2}\right) dy$$

$$= \int_{y=0}^{y=1} \left(\frac{1}{2} + \frac{y}{2} - y^{3/2}\right) dy = \frac{1}{2}y + \frac{y^2}{4} - y^{5/2} \cdot \frac{2}{5}\Big]_0^1$$

$$= \frac{1}{2} + \frac{1}{4} - \frac{2}{5}$$

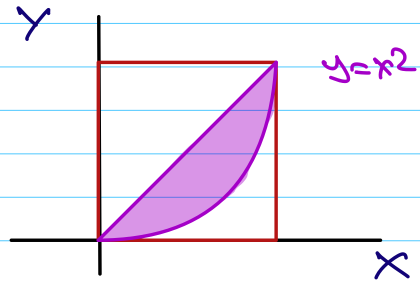
$$= \frac{3}{4} - \frac{2}{5} = \frac{7}{20}$$

$$\Rightarrow \boxed{\mathbb{P}(X > \sqrt{Y}) = \frac{7}{20}}$$

(b) $\mathbb{P}(X^2 < Y < X)$ if X and Y are Jointly distributed with pdf $f(x,y) = 2X \quad 0 \le x \le 1 \quad 0 \le y \le 1$

Solution:

$$\mathbb{P}(X^2 < Y < X) = \iint f(x,y)\,dx\,dy$$



$$= \int_{x=0}^{1}\int_{y=x^2}^{x} 2x \, dy \, dx = \int_{x=0}^{1} 2xy\Big]_{x^2}^{x} dx$$

$$= \int_{x=0}^{1} (2x^2 - 2x^3)\, dx$$

$$= \frac{2x^3}{3} - \frac{2x^4}{4}\Big]_0^1$$

$$= \frac{2}{3} - \frac{2}{4} = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

$$\boxed{\mathbb{P}(x^2 < y < x) = \frac{1}{6}}$$

4.6 A and B agree to meet at a certain place b/w 1PM and 2PM. Suppose they arrive at the meeting place independently and randomly during the hour. Find the distribution of the length of time that A waits for B. If B arrives before A, define A's waiting time = 0

1PM ———————— 2PM

<u>Solution:</u>

$$X \sim \text{Unif}(0,1) \quad (\text{A arrive})$$

$$Y \sim \text{Unif}(0,1) \quad (\text{B arrive})$$

$$(X, Y) \sim \text{Unif}\big((0,1), (0,1)\big)$$

$T =$ Length of time A waits for B

$$T = \begin{cases} t = Y - X & \text{if } y > x \\ 0 & \text{if } y \leq x \end{cases}$$

$$\mathbb{P}(T=0) = \int_{x=0}^{1}\int_{y=0}^{1} 1 \, dx\, dy = \frac{1}{2}$$

$$F_T(t) = \mathbb{P}(T < t) = 1 - \mathbb{P}(T > t)$$

$y - x = t$

$x = y - t$

$$= 1 - \int_{y=t}^{y=1}\int_{x=0}^{y-t} 1 \, dx\, dy$$

$$= 1 - \int_{y=t}^{y=1} (y-t)\,dy$$

$$= 1 - \left[\frac{y^2}{2} - ty\right]_{y=t}^{y=1}$$

$$= 1 - \left(\frac{1}{2} - t - \frac{t^2}{2} + t^2\right)$$

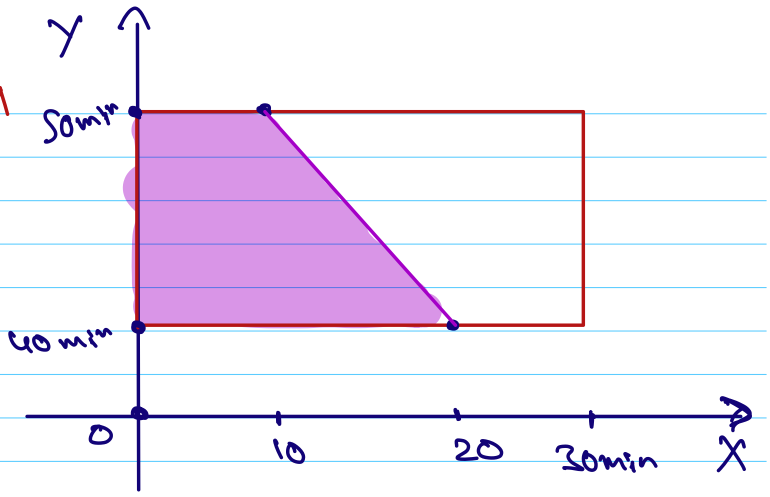
$$= 1 - \frac{1}{2} + t - \frac{t^2}{2}$$

$$\boxed{F_T(t) = \frac{1}{2} + t - \frac{t^2}{2}}$$

4.7

A women leaves for work b/w 8AM & 8:30AM and takes b/w 40 to 50 min to get there. Let the r.v X denote her time of departure, and the r.v Y the tavel time. Assuming these are independent and uniformly distributed, find the Probability that women arrives at work before 9AM

Solution:



- Starts at 8:00AM will reach by 9AM for sure

- Starts at 8:10AM need to travel not more than 50min : always reaches

- Starts at 8:15AM, need to travel not more than 45 min

$$A = \{ x + y \leq 60 \mid 0 < x < 30,\ 40 < y < 50 \}$$

$$\mathbb{P}(x + y \leq 60) = \int_{y=40}^{y=50} \int_{x=0}^{60-y} \frac{1}{30} \cdot \frac{1}{10} \, dx \, dy$$

$$= \frac{1}{300}\int_{y=40}^{y=50} (60-y)\,dy$$

$$= \frac{1}{300}\left[60y - \frac{y^2}{2}\right]_{40}^{50}$$

$$= \frac{1}{300}\left(60\times 50 - \frac{50^2}{2} - 60\times 40 + \frac{40^2}{2}\right)$$

$$= \left(60\times 10 - \left(\frac{50^2}{2} - \frac{40^2}{2}\right)\right)\frac{1}{300}$$

$$= 600 - \frac{90\cdot 10}{2} = 600-450 = \frac{150}{300} = \frac{1}{2}$$

$$\boxed{\mathbb{P}(X+Y \leq 60\text{ min}) = \frac{1}{2}}$$

4.8

4.10 A random pair (X,Y) has the distribution

$Y \backslash X$	1	2	3
2	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
3	$\frac{1}{6}$	0	$\frac{1}{6}$
4	0	$\frac{1}{3}$	0

(a) Show that X and Y are dependent.

<u>Solution:</u>

$\mathbb{P}(X=1)=\frac{1}{4}$ $\quad \mathbb{P}(X=2)=\frac{1}{2}$ $\quad \mathbb{P}(X=3)=\frac{1}{4}$

$\mathbb{P}(Y=2)=\frac{1}{3}$ $\quad \mathbb{P}(X=3)=\frac{1}{3}$ $\quad \mathbb{P}(Y=4)=\frac{1}{3}$

$$\mathbb{P}(X=1, Y=4) = 0 \neq \mathbb{P}(X=1)\cdot\mathbb{P}(Y=4)$$

$$0 \neq \frac{1}{12}$$

So X, Y are dependent.

4.11 U = # of trails needed to get the 1st head
V = # of trails needed to get 2 heads
in repeated tosses of a fair coin.

Are U & V are independent.

Solution:

$U \sim$ geometric (p)

$V \sim$ negative binomial

$U = \{0, 1, 2, \ldots \}$

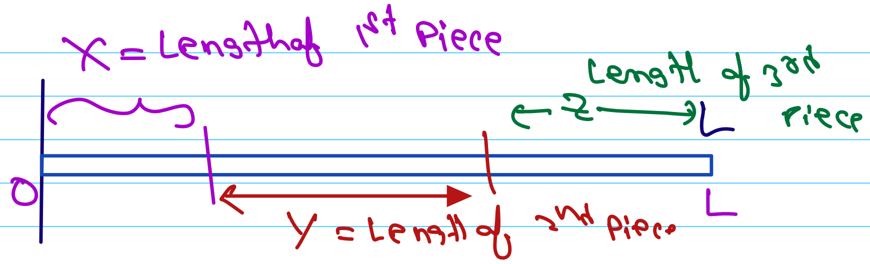
$V = \{U+1, U+2, U+3, \ldots \}$

The random variable V depends on U

$\Longrightarrow$ U, V are dependent.

4.12 if a stick is broken at random into three pieces, what is the probability that the pieces can be put together in a triangle?

Solution:



x = Length of 1st piece

y = Length of 2nd piece

z = Length of 3rd piece.

$$X \sim \text{unif}(0, L)$$

$$Y \sim \text{Unif}(0, L-X)$$

$$Z \sim \text{Unif}(0, L-X-Y)$$

To form a triangle

$$X + Y \geq Z$$

$$Y + Z \geq X$$

$$Z + X \geq Y$$

4.13 Let X and Y are r.v with finite means

(a) Show that

$$\min_{g(x)} \mathbb{E}\left[(Y-g(X))\right]^2 = \mathbb{E}\left[Y-\mathbb{E}[Y|X]\right]^2$$

where $g(x)$ ranges over all function's.

Solution:

$$\mathbb{E}\left[Y-g(x)^2\right]$$

$$= \mathbb{E}\left[\left(Y- E[Y|X] + E[Y|X]-g(x)\right)^2\right]$$

$$= \mathbb{E}\left[\left(Y-\mathbb{E}[Y|X]\right)^2\right]$$

$$+ \mathbb{E}\left[\left(\mathbb{E}[Y|X]-g(x)\right)^2\right]$$

$$+ 2\mathbb{E}\left[\left(Y-E[Y|X]\right)\left(E[Y|X]-g(X)\right)\right]$$