Moment Generating function:

Def: The moment generating function (MGF) associated with a r.v X is a function $M_X : \mathbb{R} \longrightarrow [0, \infty]$ designed by

$$M_X(s) = \mathbb{E}[e^{sX}]$$

The Domain or region of convergence of M_X is the set

$$D_X = \{ s \mid M_X(s) < \infty \}$$

It is analogus to the Laplace transform.

if X is discrete with PMF $P_X(x)$, then

$$M_X(s) = \sum_x e^{sx} P_X(x)$$

if X is continuous with density $f_X(x)$

then $$M_X(s) = \int e^{sx} f_X(x)\, dx.$$

Example: Exponential r.v

$$f_X(x) = \mu e^{-\mu x} \qquad x \geq 0$$

$$M_X(s) = \int_0^\infty e^{sx} \mu e^{-\mu x}\, dx$$

$$= \int_0^\infty \mu e^{-(\mu - s)x}\, dx$$

$$= \frac{\mu}{\mu - s} \int_0^\infty (\mu - s) e^{-(\mu - s)x}\, dx$$

$$= \begin{cases} \dfrac{\mu}{\mu - s} & x \geq 0 \\ +\infty & \text{o.w} \end{cases}$$

Region of Convergence (ROC) =

$$\{s \mid M_X(s) < \infty\} \text{ i.e } \{s < \mu\}$$

Example 2 Std. Normal

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad x \in \mathbb{R}$$

$$M_X(s) = \mathbb{E}\left[e^{sX}\right]$$

$$= \int_{-\infty}^{\infty} e^{sx} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x^2 - 2sx + s^2)}{2}} \cdot e^{+\frac{s^2}{2}} \, dx$$

$$= e^{\frac{s^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-s)^2}{2}} \, dx$$

$$M_X(s) = e^{\frac{s^2}{2}} \qquad s \in \mathbb{R}$$

$$\text{ROC} = \mathbb{R}$$

Example Cauchy r.v

$$f_X(x) = \frac{1}{\pi(1+x^2)} \qquad x \in \mathbb{R}$$

$$M_X(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{sx} \frac{1}{1+x^2} \, dx$$

$$= \begin{cases} 1 & \text{if } s = 0 \\ +\infty & \text{o.w} \end{cases}$$

$\Rightarrow$ ROC $\Rightarrow$ $s = 0$

Theorem:

① Suppose $M_X(s)$ is finite in the interval $[-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$, then M_X uniquely determines the CDF of X

(2) if X and Y are two r.v s.t

$M_X(s) = M_Y(s) \quad \forall s \in [-\varepsilon, \varepsilon], \ \varepsilon > 0$

then X and Y have the same CDF.

Properties

(1) $M_X(0) = 1$

(2) suppose $M_X(s) < \infty$ for $s \in [-\varepsilon, \varepsilon]$, $\varepsilon > 0$ then

$$\frac{d}{ds} M_X(s) \Big|_{s=0} = \mathbb{E}[X]$$

more generally

$$\frac{d^m}{ds^m} M_X(s) \Big|_{s=0} = \mathbb{E}[X^m] \ ; \ m \geq 1$$