Expectation

$$\mathbb{E}(g(x)) = \begin{cases} \displaystyle\int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx & \text{if } X \text{ is continuous} \\ \displaystyle\sum_{x \in \mathcal{X}} g(x)\, f_X(x) & \text{if } X \text{ is discrete} \end{cases}$$

$$= \sum_{x \in \mathcal{X}} g(x)\, P(X = x)$$

if $\mathbb{E}[|g(x)|] = \infty \Rightarrow \mathbb{E}[g(x)]$ does not exists.

Example 2.2.2 (Exponential mean)

$$X \sim \text{Exponential}(\lambda)$$

$$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda} \qquad 0 \le x < \infty,\ \lambda > 0$$

$$\mathbb{E}[X] = \int_0^{\infty} x \cdot f_X(x)\, dx$$

$$= \int_0^{\infty} x \cdot \frac{1}{\lambda} e^{-\frac{x}{\lambda}}\, dx$$

$$= \frac{1}{\lambda} \int_0^\infty x \cdot e^{-\frac{x}{\lambda}} \, dx$$

$$= \frac{1}{\lambda} \left[\left[x \cdot \frac{e^{-\frac{x}{\lambda}}}{-\frac{1}{\lambda}} \right]_0^\infty - \int_0^\infty \frac{e^{-\frac{x}{\lambda}}}{-\frac{1}{\lambda}} \, dx \right]$$

$$= \frac{1}{\lambda} \left[\left[-x\lambda e^{-\frac{x}{\lambda}} \right]_0^\infty - \left[\lambda^2 e^{-\frac{x}{\lambda}} \right]_0^\infty \right]$$

$$= \frac{1}{\lambda} \left(0 + \lambda^2 \right) = \lambda$$

$$\mathbb{E}[X] = \lambda$$

<u>Example 2.2.3</u> (Binomial mean)

$$X \sim \text{Binomial}(n, p)$$

$$\mathbb{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$x = 0, 1, 2, \dots n$$

$$\mathbb{E}[X] = \sum_{x=0}^{n} x \cdot \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{(n-x)!\,x!} p^{x} (1-p)^{n-x}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(n-x)!\,(x-1)!} p^{x-1} (1-p)^{n-x}$$

$$= np \underbrace{\sum_{x=1}^{n-1} \binom{n-1}{x-1} p^{x-1} (1-p)^{n-x}}_{1 \quad \text{pdf}}$$

$$= np$$

Example 2.2.4 (cauchy mean)

$$f_X(x) = \frac{1}{\pi} \frac{1}{1+x^2} \qquad -\infty < x < \infty$$

$$\mathbb{E}[|X|] = \int_{-\infty}^{\infty} \frac{|x|}{\pi} \frac{1}{1+x^2} dx$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \frac{x}{1+x^2} dx$$

$1+x^2 = u$

$2x\,dx = du$

$$= \frac{1}{\pi} \int_0^\infty \frac{du}{u}$$

$$= \frac{1}{\pi} \log u \Big]_0^\infty = \infty$$

$\Rightarrow \mathbb{E}[X]$ does not exists.

Theorem 2.2.5

X is a r.v, a, b, c are constants.

Then $g_1(x)$ and $g_2(x)$ are function's

ⓐ $\mathbb{E}(a g_1(x) + b g_2(x) + c)$

$$= a\,\mathbb{E}[g_1(x)] + b\,\mathbb{E}[g_2(x)] + c$$

ⓑ if $g_1(x) \geq 0 \quad \forall x$

then $\mathbb{E}(g_1(x)) \geq 0$

(c) if $g_1(x) \geq g_2(x) \ \forall x$

then $\mathbb{E}[g_1(x)] \geq \mathbb{E}(g_2(x))$

(d) if $a \leq g_1(x) \leq b \ \forall x$, then

$$a \leq \mathbb{E}[g_1(X)] \leq b$$

Minimizing distance:

Suppose we measure the distance b/w a r.v X and a number b by $(X-b)^2$, the value of b that minimizes

$$\mathbb{E}[(X-b)^2]$$

$$\Rightarrow \mathbb{E}[(X-b)^2] = \mathbb{E}[(X - E[X] + E[X] - b)^2]$$

$$= E[(X-E[X])^2 + (E(X)-b)^2 + 2\underbrace{(X-E[X])}_{0}(E(X)-b)]$$

$$= \mathbb{E}[(X - E[X])^2] + (\mathbb{E}[X] - b)^2$$

$$\Rightarrow \mathbb{E}[(X-b)^2] = \mathbb{E}[(X - E[X])^2] + \underbrace{(\mathbb{E}[X] - b)^2}_{\text{constant.}}$$

for $\min_b \mathbb{E}[(X-b)^2] = \mathbb{E}[(X-E[X])^2]$

when $b = E[X]$

Moments and Moment generating function

Def: $\forall n \in \mathbb{N}$

n^{th} moment of X is $\mu_n' = \mathbb{E}[X^n]$

n^{th} central moment of X $\mu_n = \mathbb{E}[(X-\mu)^n]$

Variance = 2^{nd} central moment

$$\operatorname{var}(X) = \mathbb{E}\left[(X - E[X])^2\right]$$

Exponential variance:

$$\operatorname{Var}(X) = \mathbb{E}\left[(X-\lambda)^2\right]$$

$$= \int_0^\infty (x-\lambda)^2 \frac{1}{\lambda} e^{-x/\lambda}\, dx$$

$$= \int_0^\infty (x^2+\lambda^2-2x) \frac{1}{\lambda} e^{-x/\lambda}\, dx$$

$$= \lambda$$

Theorem 2.3.4

$$\operatorname{Var}(aX+b) = a^2 \operatorname{Var}(X)$$

$$\Rightarrow \quad \operatorname{var}(aX+b) = \mathbb{E}\left[(aX+b - E(aX+b))^2\right]$$

$$= \mathbb{E}\left[\left(aX + \cancel{b} - aE(X) - \cancel{b}\right)^2\right]$$

$$= a^2\, \mathbb{E}\left[(X - E[X])^2\right]$$

$$\operatorname{var}(aX+b) = a^2 \operatorname{var}(X)$$

$$\operatorname{var}(X) = \mathbb{E}[X^2] - E[X]^2$$

Def: Let X be a r.v with CDF F_X, The mgf of X denoted by $M_X(t)$

$$M_X(t) = \mathbb{E}[e^{tX}]$$

Provided that $\mathbb{E}$ exists for t in some neighborhood of 0.

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx$$

if X is continuous

$$M_X t = \sum_x e^{tx} \, \mathbb{P}(X = x) \quad \text{if } x \text{ is discrete.}$$

Theorem:

if X has mgf $M_X(t)$, then

$$\mathbb{E}[X^n] = M_X^{(n)}(0)$$

where

$$M_X^{(n)}(0) = \left. \frac{d^n}{dt^n} M_X(t) \right|_{t=0}$$

That is, n^{th} moment is equal to the n^{th} derivative of $M_X(t)$ evaluated at $t=0$.

Proof:

$$\frac{d}{dt} M_X(t) = \frac{d}{dt} \int_{-\infty}^{\infty} e^{tx} f_X(x) \, dx$$

$$= \int_{-\infty}^{\infty} \frac{d}{dt} e^{tx} \; f_X(x) \, dx$$

$$= \int_{-\infty}^{\infty} x e^{tx} f_X(x)\, dx$$

$$= \mathbb{E}[X e^{tX}]$$

$$\Rightarrow \frac{d}{dt} M_X(t) \Big|_{t=0} = \mathbb{E}[X e^{tX}] \Big|_{t=0}$$

$$= \mathbb{E}[X]$$

Simillarly

$$\frac{d^n}{dt^n} M_X(t) \Big|_{t=0} = \mathbb{E}[X^n e^{tX}] \Big|_{t=0}$$

$$= \mathbb{E}[X^n]$$