BIVARIATE TRANSFORMATION

Let (X,Y) be a bivariate random vector with a known probability distribution. Now consider a new bivariate random vector (U,V) defined by $U = g_1(X,Y)$ and $V = g_2(U,V)$

if $B \subset \mathbb{R}^2$, then $(U,V) \in B$

$$\Updownarrow$$

$(X,Y) \in A$, where $A = \{(x,y) : (g_1(x,y), g_2(x,y)) \in B\}$

$$\Rightarrow \quad \mathbb{P}((U,V) \in B) = \mathbb{P}((X,Y) \in A)$$

if $(X,Y) \sim$ discrete Bivariate random vector, then there are only countable set of possible values for the discrete random vector (U,V).

<u>define</u>,

$(u,v) \in B$, $A_{uv} = \{(x,y) \in A : g_1(x,y) = u,\ g_2(x,y) = v\}$

the $f_{U,V}(u,v) = \mathbb{P}(U=u, V=v) = \mathbb{P}((X,Y) \in A_{uv})$

$$= \sum_{(x,y)\in A_{uv}} f_{X,Y}(x,y)$$

Example 4.3.1:-

$X \sim \text{Poisson}(\theta)$
$Y \sim \text{Poisson}(\lambda)$
} independent

$$f_{X,Y}(x,y) = \frac{\theta^x e^{-\theta}}{x!} \frac{\lambda^y e^{-\lambda}}{y!} \qquad x = 0,1,2,\ldots \quad y = 0,1,2,\ldots$$

$$A = \{(x,y) \mid x = 0,1,2,\ldots, \; y = 0,1,2,\ldots\}$$

$U = X + Y$ and $V = Y$

$\Rightarrow \quad U = g_1(X,Y) = X + Y$

$\qquad V = g_2(X,Y) = Y$

B = Set of possible (u,v) values

$U = X + Y$

$V = Y$

$\Rightarrow$ set of possible values of $U = 0, 1, 2, \dots$

$\Rightarrow$ set of possible value of $U = V, V+1, V+2 \dots$

$B = \{(U, V) \mid V = 0, 1, 2, 3 \dots, U = V, V+1, V+2, \dots\}$

$U = g_1(x, y)$

$V = g_2(x, y)$ is a one-to one

$(x, y) = (U - V, V)$

$$\Rightarrow f_{U,V}(U, V) = f_{X,Y}(U - V, V)$$

$$= \frac{e^{-\theta} \cdot \theta^{U-V}}{(U-V)!} \cdot \frac{e^{-\lambda} \cdot \lambda^{V}}{V!}$$

$V = 0, 1, 2, \dots$

$U = V, V+1, V+2 \dots$

<u>Marginal U</u> :

for a fixed U, V can be $0, 1, 2, \dots U$

$$\Rightarrow f_U(u) = \sum_{v \in \{0,1,2,\dots,u\}} f_{U,V}(u,v)$$

$$= \sum_{v=0}^{u} \frac{e^{-\theta} \cdot \theta^{u-v}}{(u-v)!} \frac{e^{-\lambda} \cdot \lambda^{v}}{v!}$$

$$= e^{-(\theta+\lambda)} \cdot \theta^{u} \sum_{v=0}^{u} \frac{1}{(u-v)!\,v!} \left(\frac{\lambda}{\theta}\right)^{v}$$

$$= \frac{e^{-(\theta+\lambda)}}{u!} \sum_{v=0}^{u} \frac{u!}{(u-v)!\,v!} \lambda^{v} \cdot \theta^{u-v}$$

$$= \frac{e^{-(\theta+\lambda)}}{u!} \sum_{v=0}^{u} \binom{u}{v} \lambda^{v} \cdot \theta^{u-v}$$

$$= \frac{e^{-(\theta+\lambda)}}{u!} (\lambda+\theta)^{u}$$

$$\Rightarrow \quad U \sim \text{Poisson}(\lambda+\theta)$$

$$u = 0,1,2,\dots$$

Theorem: if $X \sim$ Poisson(θ) and $Y \sim$ Poisson(λ) and X and Y are independent,

$$X + Y \sim \text{Poisson}(\theta + \lambda)$$

Continuous case

(X, Y) are continuous random vector with joint pdf $f_{X,Y}(x,y)$

$$A = \{(x,y) \mid f_{X,Y}(x,y) > 0\} \text{ support set}$$

$$B = \{(u,v) \mid u = g_1(x,y),\ v = g_2(x,y) \text{ for some } (x,y) \in A\}$$

(i) Assuming $g_1(x,y) = u$, $v = g_2(x,y)$ is a one-to-one

$$\Rightarrow (u,v) = (g_1(x,y), g_2(x,y))$$

$$\Rightarrow \quad x = h_1(u,v), \quad y = h_2(u,v)$$

Then $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$$J = \begin{vmatrix} \frac{\partial h_1(x,y)}{\partial u} & \frac{\partial h_1(x,y)}{\partial v} \\ \frac{\partial h_2(x,y)}{\partial u} & \frac{\partial h_2(x,y)}{\partial v} \end{vmatrix}$$

Then

$$f_{U,V}(u,v) = f_{X,Y}(h_1(u,v), h_2(u,v))\,|J|$$

<u>Example 4.3.3</u>

$$\left.\begin{array}{l} X \sim beta(\alpha,\beta) \\ Y \sim beta(\alpha+\beta,\gamma) \end{array}\right\} \text{independent}$$

$$f_{X,Y}(x,y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

$$\times \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)} y^{\alpha+\beta-1}(1-y)^{\gamma-1}$$

$U = XY, \quad V = X$

Solution:

$X \in (0,1)$

$Y \in (0,1)$

$$(x,y) = ((0,1) \times (0,1))$$

$$V = x \implies V \in (0,1)$$

$$U = XY \implies U = VY$$

$$\implies U \in (0,V)$$

$$A = \{(0,1) \times (0,1)\}$$

$$B = \{(u,v) \mid 0 < u < v < 1\}$$

$$\implies \left.\begin{array}{l} u = xy \\ v = x \end{array}\right\} \implies \begin{array}{ll} x = v & = h_1(u,v) \\ y = \frac{u}{v} & = h_2(u,v) \end{array}$$

$$J = \begin{vmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_1}{\partial v} \end{vmatrix}$$

$$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} \Rightarrow |J| = \frac{1}{v}$$

$$f_{U,V}(u,v) = f_{X,Y}\left(h_1(u,v), h_2(u,v)\right) \cdot |J|$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \cdot v^{\alpha-1} \cdot (1-v)^{\beta-1} \cdot \left(\frac{u}{v}\right)^{\alpha+\beta-1} \left(1-\frac{u}{v}\right)^{\gamma-1} \cdot \frac{1}{v}$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \cdot u^{\alpha+\beta-1} \cdot \left(1-\frac{u}{v}\right)^{\gamma-1} \cdot \frac{(1-v)^{\beta-1}}{v^{\beta}} \cdot \frac{1}{v}$$

Marginal pdf of u

for a fixed u, $u < v < 1$

$$\Rightarrow f_U(u) = \int_{v=u}^{1} f_{U,V}(u,v)$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_{v=u}^{1} v^{\alpha-1} \cdot (1-v)^{\beta-1} \cdot \left(\frac{u}{v}\right)^{\alpha+\beta-1} \left(1-\frac{u}{v}\right)^{\gamma-1} \cdot \frac{1}{v}$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \cdot u^{\alpha-1} \int_{v=u}^{1} (u-uv)^{\beta-1} \cdot \left(1-\frac{u}{v}\right)^{\gamma-1} \cdot \frac{1}{v^{\beta+1}}\, dv$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \cdot u^{\alpha-1} \int_{v=u}^{1} \left(\frac{u}{v}-u\right)^{\beta-1} \left(1-\frac{u}{v}\right)^{\gamma-1} \frac{u}{v^2}\, dv$$

$$y = \frac{\frac{u}{v}-u}{1-u}$$

$$1 - y(1-u)+u$$

$$\Rightarrow \quad \Rightarrow (1-u) - y(1-u)$$

$$\Rightarrow (1-y)(1-u)$$

$$dy = \frac{1}{1-u}\left(-\frac{u}{v^2}\right) dv$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \cdot u^{\alpha-1} \int_{0}^{1} y^{\beta-1} \cdot (1-u)^{\beta-1} (1-y)^{\gamma-1} (1-u)^{\gamma-1} (1-u)\, dy$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \cdot u^{\alpha-1} \cdot (1-u)^{\beta+\gamma-1} \int_{0}^{1} y^{\beta-1} (1-y)^{\gamma-1}\, dy$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\,\Gamma(\cancel{\beta})\,\Gamma(\cancel{\gamma})} \cdot u^{\alpha-1} \cdot (1-u)^{\beta+\gamma-1} \cdot \frac{\Gamma(\cancel{\beta})\,\Gamma(\cancel{\gamma})}{\Gamma(\beta+\gamma)}$$

$$f_U(u) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\,\Gamma(\beta+\gamma)}\, u^{\alpha-1} \cdot (1-u)^{\beta+\gamma-1}$$

$$\Rightarrow \quad U \sim \text{Beta}(\alpha, \beta+\gamma)$$

Example 4.3.4

Sum and difference of normal variables.

$$X \sim N(0,1) \qquad Y \sim N(0,1)$$

independent

$$U = X + Y = g_1(x,y)$$

$$V = X - Y = g_2(x,y)$$

$$\Rightarrow \quad x = \frac{u+v}{2} = h_1(u,v)$$

$$y = \frac{1}{2}(u-v) = h_2(u,v)$$

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$f_{U,V}(u,v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{u+v}{2}\right)^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{u-v}{2}\right)^2}{2}} \cdot \frac{1}{2}$$

$$= \frac{1}{2\cdot 2\pi} e^{-\frac{1}{8}\left(u^2+v^2+2\cancel{u}v + u^2+v^2-2\cancel{u}v\right)}$$

$$= \frac{1}{2\cdot 2\pi} e^{-\frac{1}{4}\left(u^2+v^2\right)}$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u}{\sqrt{2}}\right)^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{v^2}{\sqrt{2}}}\right)$$

$$= \frac{1}{\sqrt{2\pi}\cdot\sqrt{2}} e^{-\frac{1}{2}\left(\frac{u}{\sqrt{2}}\right)^2} \cdot \frac{1}{\sqrt{2\pi}\cdot\sqrt{2}} e^{-\frac{1}{2}\left(\frac{v}{\sqrt{2}}\right)^2}$$

$$\Rightarrow \quad f_U(u) = \frac{1}{\sqrt{2\pi}\cdot\sqrt{2}} e^{-\frac{1}{2}\left(\frac{u}{\sqrt{2}}\right)^2}$$

$$f_V(v) = \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{1}{2}\left(\frac{v}{\sqrt{2}}\right)^2}$$

Theorem 4.3.5 :-

Let X, Y be independent r.v. Let $g(x)$ be a function only of x and $h(y)$ be a function only of y. Then the r.v $U = g(X)$ and $V = h(Y)$ are independent.

* if $g_1(x,y)$, $g_2(x,y)$ is not one-to-one then use

$$f_{U,V}(u,v) = \sum_{i=1}^{k} f_{X,Y}\left(h_{1i}(u,v), h_{2i}(u,v)\right) |J_i|$$

Example 4.3.6 : Distribution of the ratio of normal variables.

$X \sim N(0,1)$ $\quad Y \sim N(0,1)$ $\Rightarrow$ independent.

$U = \dfrac{X}{Y}$ and $V = |Y|$

$\Rightarrow$ This is not one-to-one

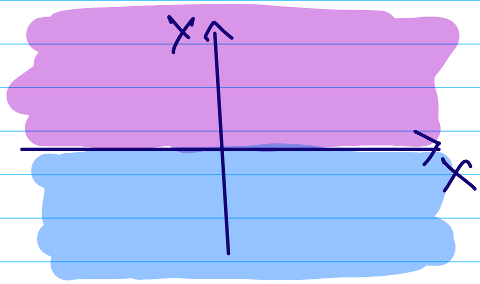
Example for $(x,y) = (1,1)$ & $(-1,-1)$

$(u,v) = (1,1)$ & $(1,1)$

$A_0 = \{(x,y) \mid y = 0\}$

$A_1 = \{(x,y) \mid y < 0\}$

$A_2 = \{(x,y) \mid y > 0\}$



$$\left.\begin{array}{l} U = g_{11}(x,y) = \dfrac{X}{Y} \\ V = g_{21}(x,y) = -Y \end{array}\right\} \Rightarrow \begin{array}{l} X = h_{11}(u,v) = -UV \\ Y = h_{21}(u,v) = -V \end{array}$$

$$J_1 = \begin{vmatrix} -v & -u \\ 0 & -1 \end{vmatrix} = v$$

$$U = g_{12}(x,y) = \frac{X}{Y} \qquad X = h_{12}(u,v) = uv$$

$$V = g_{22}(x,y) = +y \qquad Y = h_{22}(u,v) = v$$

$$J_2 = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

$$f_{U,V}(u,v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2v^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} \cdot v \cdot 2$$

$$B = \{(u,v) \mid \infty < u < +\infty,\ 0 < v < \infty\}$$

$$\Rightarrow f_{U,V}(u,v) = \frac{v}{\pi} e^{-\frac{v^2}{2}(u^2+1)}$$

Marginal Pdf of U

$$\Rightarrow f_U(u) = \int_{v=0}^{\infty} \frac{v}{\pi} e^{-\frac{v^2}{2}(u^2+1)}$$

$$\frac{v^2(u^2+1)}{2} = y$$

$$\Rightarrow \quad v(u^2+1) = dy$$

$$= \frac{1}{\pi(u^2+1)} \int_{y=0}^{\infty} e^{-y} \, dy$$

$$= \frac{1}{\pi(u^2+1)} \left[-e^{-y} \right]_0^{\infty}$$

$$\boxed{f_U(u) = \frac{1}{\pi(u^2+1} \qquad -\infty < u < +\infty}$$