DESCRIPTION

This document provides solution's to 3.44 to 3.50 from chapter 3 of statistical inference Book by Casella and Berger, Focusing on MARKOV, Chebyshev's inequalities and Identies.

3.44 For any random variable X for which $E[X^2]$ and $\mathbb{E}[|X|]$ exist, show that $\mathbb{P}(|X| \geq b)$ does not exceed either $\frac{E[X^2]}{b^2}$ or $\frac{\mathbb{E}|X|}{b}$, where b is a +ve constant.

if $f(x) = e^{-x}$ for $x > 0$, show that one bound is better when $b = 3$ and other when $b = \sqrt{2}$

Solution:

Markov inequality

$$\mathbb{P}(X \geq \alpha) \leq \frac{\mathbb{E}[X]}{\alpha}$$

for a r.v $X > 0$

$\Rightarrow$ for a random X, $|X|$, X^2 are +ve r.v

$$\Rightarrow \quad \mathbb{P}(|X| \geq b) \leq \frac{\mathbb{E}[|X|]}{b}$$

&

$$\mathbb{P}(X^2 \geq b^2) \leq \frac{\mathbb{E}[X^2]}{b^2} \quad \text{for } b > 0$$

$\Rightarrow$

$$\Rightarrow \quad \mathbb{P}(X \geq b, X \leq b) \leq \frac{\mathbb{E}[X^2]}{b^2}$$

$$\Rightarrow \quad \mathbb{P}(|X| \geq b) \leq \frac{\mathbb{E}[X^2]}{b^2}$$

Therefore:

$$\mathbb{P}(|X| \geq b) \leq \frac{\mathbb{E}[|X|]}{b}$$

$$\mathbb{P}(|X| \geq b) \leq \frac{\mathbb{E}[X^2]}{b^2}$$

for exponential distribution:

$X \sim \text{exponential}(1)$

$f(x) = e^{-x} \qquad 0 < x < \infty$

then $\mathbb{E}[|X|] = \mathbb{E}[X] = 1$

& $\mathbb{E}[X^2] = \text{var}(X) + E[X]^2$

$= 1 + 1 = 2$

for $b = 3$

$$\frac{E[|X|]}{b} = \frac{1}{3} \qquad \frac{\mathbb{E}[X^2]}{b^2} = \frac{2}{9}$$

$$\Rightarrow \quad \frac{\mathbb{E}[|X|]}{3} \geq \frac{\mathbb{E}[X^2]}{9}$$

for $b=\sqrt{2}$

$$\frac{\mathbb{E}[|X|]}{b} = \frac{1}{\sqrt{2}} \qquad \frac{\mathbb{E}(X^2)}{b^2} = \frac{2}{2} = 1$$

$$\Rightarrow \frac{\mathbb{E}[|X|]}{b} < \frac{\mathbb{E}[X^2]}{b^2} \quad \text{for } b=\sqrt{2}$$

3.45 X is a r.v with MGF $M_X(t)$, $-h<t<h$

(a) Prove that $\mathbb{P}(X \geq a) \leq e^{-at} M_X(t)$, $0<t<h$

Solution:

markov inequality $\mathbb{P}(X \geq b) \leq \frac{\mathbb{E}[X]}{b}$ for $X>0$

taking a +ve r.v e^{tX} and apply Markov

$$\Rightarrow \mathbb{P}(e^{tX} \geq b) \leq \frac{\mathbb{E}[e^{tX}]}{b}$$

for $t>0$ $(0<t<h)$

e^{tX} is increasing function

$$\Rightarrow \mathbb{P}(tX \geq \log b) \leq \frac{M_X(t)}{b}$$

$$\Rightarrow \mathbb{P}\left(X \geq \frac{\log b}{t}\right) \leq \frac{M_X(t)}{b}$$

taking $\frac{\log b}{t} = a$

$$\Rightarrow b = e^{at}$$

$$\boxed{\Rightarrow \quad \mathbb{P}(X \geq a) \leq e^{-at} M_X(t)}$$

(b) Simillary prove $\mathbb{P}(X \leq a) \leq e^{-at} M_X(t)$

for $t<0$ $(-h<t<0)$ e^{tx} is decreasing, so while apply log, the sign's revers.

$$\Rightarrow \quad \mathbb{P}(e^{tx} \geq b) \leq \frac{\mathbb{E}[e^{tx}]}{b}$$

$$\Rightarrow \quad \mathbb{P}(tx \leq \log b) \leq \frac{\mathbb{E}[e^{tx}]}{b}$$

$$\Rightarrow \quad \mathbb{P}\left(X \leq \frac{\log b}{t}\right) \leq \frac{M_X(t)}{b}$$

$$\boxed{\Rightarrow \quad \mathbb{P}(X \leq a) \leq e^{-at} M_X(t)}$$

3.46 Calculate $\mathbb{P}(|X-\mu_x| \geq k\sigma_x)$ for $X \sim$ Uniform $(0,1)$ and $X \sim$ exponential(λ) and compare your answer's to the bound from Chebychev's inequality

Solution:

$$\mathbb{P}(|X-\mu_x| > k\sigma_x) =$$

$$= \mathbb{P}((X-\mu_x) < -k\sigma_x) + \mathbb{P}(X-\mu_x > k\sigma_x)$$

$$= \mathbb{P}\left(x < \mu_x - k\sigma_x\right) + \mathbb{P}\left(X > k\sigma_x + \mu_x\right)$$

$$\mu_x = \frac{b+a}{2} = \frac{1}{2} \qquad \sigma_x^2 = \frac{(b-a)^2}{12}$$

$$\Rightarrow \sigma_x = \frac{1}{\sqrt{12}}$$

$$\Rightarrow \mathbb{P}\left(X < \frac{1}{2} - \frac{k}{\sqrt{12}}\right) + \mathbb{P}\left(X > \frac{k}{\sqrt{12}} + \frac{1}{2}\right)$$

$$\frac{1}{2} - \frac{k}{\sqrt{12}} > 0 \Rightarrow \frac{1}{2} > \frac{k}{\sqrt{12}} \qquad k < \frac{\sqrt{12}}{2}$$

$$\Rightarrow k < \sqrt{3}$$

for $k < \sqrt{3}$

$$\mathbb{P}\left(X < \frac{1}{2} - \frac{k}{\sqrt{12}}\right) + \mathbb{P}\left(X > \frac{1}{2} + \frac{k}{\sqrt{12}}\right)$$

$$= \int_0^{\frac{1}{2}-\frac{k}{\sqrt{12}}} f_X(x)\,dx + \int_{\frac{1}{2}+\frac{k}{\sqrt{12}}}^{1} f_X(x)\,dx$$

$$= \frac{1}{2} - \frac{k}{\sqrt{12}} + 1 - \frac{1}{2} - \frac{k}{\sqrt{12}}$$

$$= 1 - \frac{2k}{\sqrt{12}} = 1 - \frac{k}{\sqrt{3}}$$

for $k > \sqrt{3}$

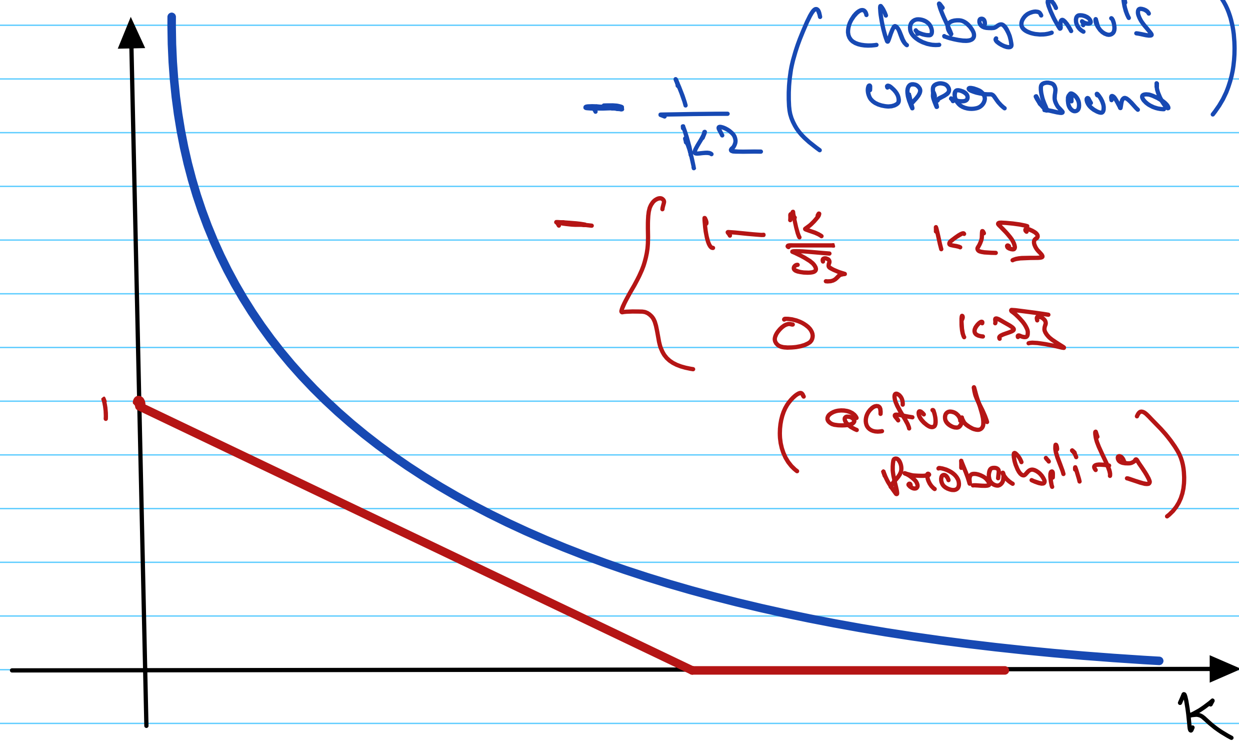
$$\underbrace{\mathbb{P}\left(X < \frac{1}{2} - \frac{k}{\sqrt{12}}\right)}_{=0} + \underbrace{\mathbb{P}\left(X > \frac{1}{2} + \frac{k}{\sqrt{12}}\right)}_{=0} = 0$$

$$\Rightarrow \quad \mathbb{P}\left(|X - \mu_X| \geq k\sigma_X\right) = \begin{cases} 1 - \frac{k}{\sqrt{3}} & k < \sqrt{3} \\ 0 & k > \sqrt{3} \end{cases}$$

Using Chebychev's inequality

$$\mathbb{P}(|X - \mu_x| \geq k\sigma_x) \leq \frac{1}{k^2}$$

$$\Rightarrow \quad \boxed{1 - \frac{k}{\sqrt{3}} \leq \frac{1}{k^2}}$$



$= \frac{1}{k^2}$ (Chebychev's upper bound)

$= \begin{cases} 1 - \frac{k}{\sqrt{3}} & k < \sqrt{3} \\ 0 & k > \sqrt{3} \end{cases}$ (actual probability)

For Exponential (λ)

$$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda} \qquad 0 < x < \infty$$
$$\lambda > 0$$

$$\mu_X = \lambda$$

$$\sigma_x^2 = \lambda^2 \Rightarrow \sigma_x = \lambda$$

$$\mathbb{P}(|X - \lambda| \geq k\lambda)$$

$$= \mathbb{P}(X - \lambda \leq -k\lambda) + \mathbb{P}(X - \lambda \geq k\lambda)$$

$$= \mathbb{P}(X \leq \lambda(1-k)) + \mathbb{P}(X \geq \lambda(1+k))$$

$$\lambda(1-k) > 0 \Rightarrow k < 1$$

For $k < 1$

$$\int_0^{\lambda(1-k)} f_X(x)\,dx + \int_{\lambda(1+k)}^{\infty} f_X(x)\,dx$$

$$= \int_0^{\lambda(1-k)} \frac{1}{\lambda} e^{-\frac{x}{\lambda}}\,dx + \int_{\lambda(1+k)}^{\infty} \frac{1}{\lambda} e^{-\frac{x}{\lambda}}\,dx$$

$$= -e^{-\frac{x}{\lambda}}\Big]_0^{\lambda(1-k)} - e^{-\frac{x}{\lambda}}\Big]_{\lambda(1+k)}^{\infty}$$

$$= 1 + e^{-(1+k)} - e^{k-1}$$

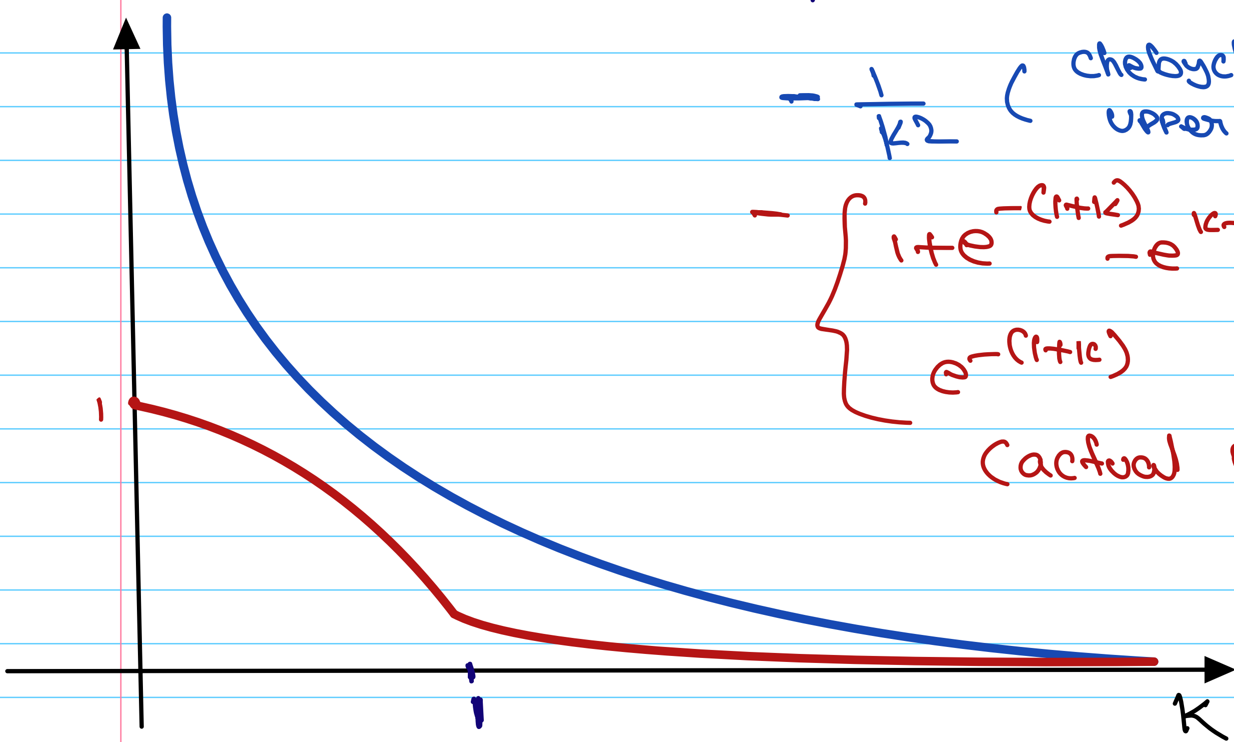
For $k>1$

$$\mathbb{P}(|X-\lambda| \geq k\lambda) = e^{-(1+k)}$$

$$\Rightarrow \mathbb{P}(|X-\lambda| \geq k\lambda) = \begin{cases} 1+e^{-(1+k)} - e^{k-1} & k<1 \\ e^{-(1+k)} & k\geq 1 \end{cases}$$

Using Chebychev's inequality

$$\mathbb{P}(|X-\lambda| \geq k\lambda) \leq \frac{1}{k^2}$$



— $\frac{1}{k^2}$ (Chebychev's upper bound)

— $\begin{cases} 1+e^{-(1+k)} - e^{k-1} & k<1 \\ e^{-(1+k)} & k\geq 1 \end{cases}$ (actual probability)

The Chebychev's inequality providing upper bound's on probabilities that is significantly larger than the actual probability

3.47 $Z \sim N(0,1)$, Prove this companion to the inequality in Example 3.6.3

$$\mathbb{P}(|Z| \geq t) \geq \sqrt{\frac{2}{\pi}} \frac{t}{1+t^2} e^{-\frac{t^2}{2}}$$

Solution:

Example 3.6.3

$$\mathbb{P}(|Z| \geq t) \leq \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{t^2}{2}}}{t} \quad \forall t > 0$$

(upper bound)

$\mathbb{P}(|Z| \geq t) \qquad t > 0$

$$= 2\mathbb{P}(Z > t)$$

$$= 2 \cdot \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-\frac{x^2}{2}} dx$$

for $t < x < \infty$ $\quad \frac{t}{x} < 1$

$$\geq 2 \cdot \frac{1}{\sqrt{2\pi}} \int_t^\infty \frac{t}{x} \cdot e^{-\frac{x^2}{2}} dx$$

$\vdots$

3.48 Derive recursion relation's, similar to the one given in (3.6.2) for the binomial, negative binomial, and hypergeometric distribution's.

Solution:

Binomial:

$$\mathbb{P}(X=x+1) = \binom{n}{x+1} p^{x+1} \cdot (1-p)^{n-x-1}$$

$$= \frac{n!}{(x+1)!\,(n-x-1)!}\, p^{x} \cdot (1-p)^{n-x} \cdot \frac{p}{1-p}$$

$$= \frac{(n-x)}{(x+1)} \binom{n}{x} p^{x} \cdot (1-p)^{n-x} \cdot \frac{p}{1-p}$$

$$\boxed{\mathbb{P}(X=x+1) = \frac{p}{1-p} \cdot \frac{n-x}{x+1} \cdot \mathbb{P}(X=x)}$$

negative binomial:

$$\mathbb{P}(X=x+1) = \binom{r+x+1-1}{x+1} p^{r} (1-p)^{x+1}$$

$$= \frac{(r+x)!}{(x+1)!\,(r-1)!}\, p^{r} \cdot (1-p)^{x} \cdot (1-p)$$

$$= \frac{(r+x)\,(r+x-1)!}{(x+1)\; x!\,(r-1)!}\, p^{r} \cdot (1-p)^{x}\, (1-p)$$

$$= (1-p) \cdot \frac{r+x}{x+1} \cdot \binom{r+x-1}{x} p^{r} (1-p)^{x}$$

$$\boxed{\Rightarrow \mathbb{P}(X=x+1) = (1-p)\, \frac{r+x}{x+1} \cdot \mathbb{P}(X=x)}$$

<u>hypergeometric distribution</u> :

$$\mathbb{P}(X=x) = \frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}}$$

for $\mathbb{P}(X=x+1) = \dfrac{\binom{M}{x+1}\binom{N-M}{K-x-1}}{\binom{N}{K}}$

$$= \frac{\frac{M-x}{x+1} \cdot \binom{M}{x} \frac{k-x}{N-M-k+x+1} \cdot \binom{N-m}{k-x}}{\binom{N}{k}}$$

$$\boxed{\mathbb{P}(X=x+1) = \frac{M-x}{x+1} \cdot \frac{k-x}{N-M-k+x+1} \cdot \mathbb{P}(X=x)}$$

Need to check Boundary condition's.

3.49 Prove the following analogs to Stein's lemma, assuming appropriate condition's on the function g

(a) $X \sim \text{gamma}(\alpha, \beta)$ then

$$E[g(x)(X-\alpha\beta)] = \beta E[Xg'(X)]$$

<u>solution.</u>

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} \quad 0<x<\infty$$

$$\mathbb{E}\left[g(x)(x-\alpha\beta)\right]$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty g(x)(x-\alpha\beta)\, x^{\alpha-1} \cdot e^{-\frac{x}{\beta}}\, dx$$

integration By Parts $u = g(x)$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \left[\left[g(x) \int (x-\alpha\beta)\, x^{\alpha-1} \cdot e^{-\frac{x}{\beta}}\, dx \right]_0^\infty - \int_0^\infty g'(x) \int (x-\alpha\beta)\, x^{\alpha-1} \cdot e^{-\frac{x}{\beta}}\, dx \right]$$

Solving : $\int x^\alpha \cdot e^{-\frac{x}{\beta}}\, dx$

$$\Rightarrow \int x^\alpha \cdot e^{-\frac{x}{\beta}} = x^\alpha \int e^{-\frac{x}{\beta}}\, dx - \int \alpha \cdot x^{\alpha-1} \int e^{-\frac{x}{\beta}}\, dx$$

$$= -\beta \cdot x^\alpha \cdot e^{-\frac{x}{\beta}} + \alpha\beta \int x^{\alpha-1} \cdot e^{-\frac{x}{\beta}}\, dx$$

$$\Rightarrow \int x^{\alpha} \cdot e^{-\frac{x}{\beta}} dx = -\beta \cdot x^{\alpha} \cdot e^{-\frac{x}{\beta}} + \alpha\beta \int x^{\alpha-1} \cdot e^{-\frac{x}{\beta}} dx$$

$$\boxed{\Rightarrow \int (x - \alpha\beta)\, x^{\alpha-1} \cdot e^{-\frac{x}{\beta}} dx = -\beta \cdot x^{\alpha} \cdot e^{-\frac{x}{\beta}}}$$

$$\mathbb{E}\left[g(x)\,(x-\alpha\beta)\right]$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \left[\left[g(x) \int (x-\alpha\beta)\, x^{\alpha-1} \cdot e^{-\frac{x}{\beta}} dx \right]_0^{\infty} - \int_0^{\infty} g'(x) \int (x-\alpha\beta)\, x^{\alpha-1} \cdot e^{-\frac{x}{\beta}} dx \right]$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \left[\left[-\beta g(x)\, x^{\alpha} \cdot e^{-\frac{x}{\beta}} \right]_0^{\infty} + \beta \int_0^{\infty} g'(x)\, x^{\alpha} \cdot e^{-\frac{x}{\beta}} dx \right]$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \left[0 - 0 \right] + \beta \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_0^{\infty} g'(x)\, x^{\alpha} \cdot e^{-\frac{x}{\beta}} dx$$

$$= \beta\,\mathbb{E}\left[g'(x)x \right]$$

$$\Rightarrow \boxed{\mathbb{E}\left[g(X)\,(X-\alpha\beta) \right] = \beta\,\mathbb{E}\left[X g'(X) \right]}$$

3.49 b $\quad X \sim \text{beta}(\alpha,\beta)$

$$\mathbb{E}\left[g(X)\left(\beta - (\alpha-1)\frac{(1-X)}{X} \right)\right.$$

$$= \mathbb{E}\left[(1-X)\, g'(X) \right]$$

Solution:

$$f_X(x) = \frac{1}{B(\alpha,\beta)} \cdot x^{\alpha-1}(1-x)^{\beta-1}$$

$$0 < x < 1$$
$$\alpha > 0$$
$$\beta > 0$$

$$\mathbb{E}\left[g(X)\left(\beta - (\alpha-1)\frac{(1-X)}{X} \right) \right]$$

$$= \frac{1}{B(\alpha,\beta)} \int_0^1 g(x)\left[\beta - (\alpha-1)\cdot\left(\frac{1-x}{x}\right) \right] x^{\alpha-1}\,(1-x)^{\beta-1}\,dx$$

Solving for $\int x^{\alpha-1} \cdot (1-x)^{\beta-1} \, dx$

$$= \int x^{\alpha-1} \cdot (1-x)^{\beta-1} \, dx = -x^{\alpha-1} \cdot \frac{(1-x)^{\beta}}{\beta} + \frac{\alpha-1}{\beta} \int x^{\alpha-2} \cdot (1-x)^{\beta} \, dx$$

$$\Rightarrow \int \left[x^{\alpha-1} \cdot (1-x)^{\beta-1} - \frac{\alpha-1}{\beta} \cdot x^{\alpha-2} \cdot (1-x)^{\beta} \right] dx$$

$$= -x^{\alpha-1} \cdot \frac{(1-x)^{\beta}}{\beta}$$

$$\Rightarrow \int x^{\alpha-1} \cdot (1-x)^{\beta-1} \left[1 - \frac{\alpha-1}{\beta} \, \frac{1-x}{x} \right] dx$$

$$= -\frac{x^{\alpha-1} \cdot (1-x)^{\beta}}{\beta}$$

$$\Rightarrow \int \left[\beta - (\alpha-1) \frac{1-x}{x} \right] x^{\alpha-1} \cdot (1-x)^{\beta-1} \, dx$$

$$= -x^{\alpha-1} \cdot (1-x)^{\beta}$$

$$\mathbb{E}\left[g(X)\left(\beta-(\alpha-1)\frac{(1-X)}{X}\right)\right]$$

$$=\frac{1}{B(\alpha,\beta)}\int_0^1 g(x)\left[\beta-(\alpha-1)\cdot\left(\frac{1-x}{x}\right)\right]x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$$=\frac{1}{B(\alpha,\beta)}\left[g(x)\int\left(\beta-(\alpha-1)\frac{1-x}{x}\right)x^{\alpha-1}\cdot(1-x)^{\beta-1}\,dx\right.$$
$$\left.-\int g'(x)\int\left[\beta-(\alpha-1)\frac{1-x}{x}\right]x^{\alpha-1}\cdot(1-x)^{\beta-1}\,dx\right]$$

$$=\frac{1}{B(\alpha,\beta)}\left[\Big[g(x)\cdot -x^{\alpha-1}(1-x)^{\beta}\Big]_0^1\right.$$
$$\left.+\int_0^1 g'(x)\,x^{\alpha-1}(1-x)^{\beta}\,dx\right]$$

$$=\frac{1}{B(\alpha,\beta)}\left[0+\int_0^1 g'(x)(1-x)\cdot x^{\alpha-1}\cdot(1-x)^{\beta-1}\,dx\right]$$

$$=\mathbb{E}\left[g'(x)\cdot 1-x\right]$$ Hence Proved.

3.50 Prove $X \sim$ negative binomial(r, p)

$$\mathbb{E}\left[(1-p)\,g(X)\right] = \mathbb{E}\left[\frac{X}{r+X-1}\,g(X-1)\right]$$

Solution:

$$P(X=x) = \binom{r+x-1}{x} p^r (1-p)^x$$

$x = 0, 1, 2, \ldots$

$$\mathbb{E}\left[(1-p)\,g(x)\right]$$

$$= \sum_{x=0}^{\infty} (1-p)\,g(x) \binom{r+x-1}{x} p^r (1-p)^x$$

$$= \sum_{x=0}^{\infty} g(x) \binom{r+x-1}{x} p^r (1-p)^{x+1}$$

$$= \sum_{x=0}^{\infty} g(x) \frac{(r+x-1)!}{x!\,(r-1)!} p^r (1-p)^{x+1}$$

$$= \sum_{x=0}^{\infty} g(x)\, (r+x-1)\, \frac{(r+x-1-1)!}{x\,(x-1)!\,(r-1)!}\, p^{r} (1-p)^{x+1}$$

$$= \sum_{x=1}^{\infty} g(x) \cdot \frac{x+1}{r+x} \cdot \binom{r+(x+1)-1}{x+1} p^{r} (1-x)^{x+1}$$

Put $x+1 = y$

$\Rightarrow x = y-1$

$$= \sum_{y=0}^{\infty} g(y-1)\, \frac{y}{r+y-1} \binom{r+y-1}{y} p^{r} (1-x)^{y}$$

$$= \mathbb{E}\left[\frac{X}{r+X-1}\, g(X-1) \right]$$

Hence Proved

$$\boxed{\mathbb{E}\left[(1-p)\, g(X)\right] = \mathbb{E}\left[\frac{X}{r+X-1} \cdot g(X-1) \right]}$$