PROBABILITY SPACES

Anything that happen's in life is uncertain, there is uncertanity everywhere, so whatever you try to do you need to have some way of dealing or thinking about the uncertanity.

two undefined entities

1. Random Experiment
2. Outcome

There is a random experiment, this is a Experiment whose outcome is random. And there is a outcome everytime we conduct this experiment.

SAMPLE SPACE (Ω)

Set of all possible outcomes of a random experiment.

* An outcome (or elementary outcome) of the random experiment is usually denoted by ω (small omega)

* Thus when a random experiment is Performed, the outcome $\omega \in \Omega$ is Picked by the Goddess of Chance or Mother Nature or your favourite genie

Ω: The List (set of all Possible outcome's) must be

(i) <u>Mutually Exclusive</u>:

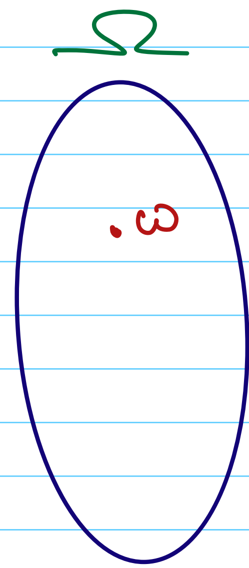
if one outcome happen's the remaining outcome cannot happen

(ii) <u>Collectively exhaustive</u>: No matter what happen's in the experiment, you are going get one of the outcome from Ω

⟹ An outcome is denoted by ω (small omega)

⟹ Choosing ω is what is random, this is the thing we don't have control over

⟹ This is the source of randomness.



EVENTS:

informally, subsets of Ω which are of "interest" are called events

Ex: Toss a coin thrice, $\Omega = \{H, T\}^3$

Event of interest : at least 2 heads

⟹ $\{HHH, HHT, THH, HTH\}$

$\Rightarrow$ we want to Built subsets of interest of Ω

$\Rightarrow$ if A is an event of interest, then not A is also of interest

$\Rightarrow$ if an occurence of event is of interest then its non-occurance is also of interest.

$\Rightarrow$ if A, B are events, then $A \cup B$ is of interest

$\Rightarrow$ Ω is of my interesting events.

Def: A event $A \subseteq \Omega$ is said to occur if $\omega \in A$ is outcome of random experiment.

Algebra, $\mathcal{F}_0$

A collection $\mathcal{F}_0$ of subsets of Ω is called an algebra if

(i) $\phi \in \mathcal{F}_0$

(ii) if $A \in \mathcal{F}_0$ then $A^c \in \mathcal{F}_0$

(iii) $A \in \mathcal{F}_0$ and $B \in \mathcal{F}_0$, then $A \cup B \in \mathcal{F}_0$

Notation Classification:

(i) Single element is represented by small letter

(ii) Collection of elements (i.e set) are represented by capital letter's

(iii) Collection of elements, then we use scripted Letter's.

$$x \in A$$

(element) (collection of elements)

$$A \in \mathcal{F}_0$$

(collection of sets)

$$A \subseteq B$$

(They must be of same type) sets

Ex: toss a coin twice

$$\Omega = \{H, T\}^2$$

$$\Rightarrow \Omega = \{HH, TT, HT, TH\}$$

Alegbra $\mathcal{F}_0$

$$= \{\phi, \Omega, \{HH\}, \{TT, HT, TH\}\}$$

Exercise:

Let $A_1, A_2, \ldots A_n \in \mathcal{F}_0$, Show that

$$\bigcup_{i=1}^{n} A_i \in \mathcal{F}_0 \ \& \ \bigcap_{i=1}^{n} A_i \in \mathcal{F}_0$$

Finite union's & Finite intersection's are elements of $\mathcal{F}_0$

Ex: $A_1, A_2, A_3 \in \mathcal{F}_0$

$\Rightarrow$ from Axiom's of algebra

$A_1 \cup A_2 \in \mathcal{F}_0$, $A_1 \cup A_3 \in \mathcal{F}_0$, $A_2 \cup A_3 \in \mathcal{F}_0$

if $A_1 \cup A_2 \in \mathcal{F}_0$, $A_3 \in \mathcal{F}_0$

$$\Downarrow$$

$A_1 \cup A_2 \cup A_3 \in \mathcal{F}_0$ (Proof By Induction)

* $\bigcap_{i=1}^{n} A_i \in \mathcal{F}_0$

use De Morgan's law:

$$(A \cup B)^c = A^c \cap B^c$$

$\Rightarrow$ $A_1, A_2, \dots A_n \in \mathcal{F}_0$

$\Rightarrow$ $A_1^c, A_2^c, \dots A_n^c \in \mathcal{F}_0$

$\Rightarrow$ $A_1^c \cup A_2^c \cup \dots \cup A_n^c \in \mathcal{F}_0$

$\Rightarrow$

$$\bigcup_{i=1}^{n} A_i^c \in \mathcal{F}_0$$

$$\Rightarrow \left(\bigcup_{i=1}^{n} A_i^c \right)^c \in \mathcal{F}_0$$

$$\Rightarrow \bigcap_{i=1}^{n} A_i \in \mathcal{F}_0 \quad \checkmark$$

<u>Algebra:</u> Collection of subsets closed under complementation and finite union's

Algebra do not require closure under countable union's or intersection's.
– only finite one's.

* in order to study the events of interest in day-to-day modelling of probability, closure under finite union's is little bit short on what we need

* we need little more structure

Ex: Toss until the first head show's.

$\Omega = \{H, TH, TTH, TTTH, \dots\}$

$|\Omega| \coloneqq$ countable infinite.

Event = Total number of tosses.

* if we include all these elementary outcome's and include in Algebra
* then we need to include countable infinite union's (Algebra does not support)

we can never build $\mathcal{F}_0$ by taking finite union and complementary, because the sample space is countable infinite

σ - Algebra , $\mathcal{F}$

A collection of $\mathcal{F}$ of subsets of Ω is called σ-algebra if

(i) $\phi \in \mathcal{F}$

(ii) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

(iii) if $A_1, A_2, A_3, \dots$ is a countable collection of subsets in $\mathcal{F}$, then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F} \quad \text{or} \quad \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{F}$$

set of all elements contained in atleast one of the A_i's. Not interpreted as some Limit.

we can also show if $\{A_1, A_2 \dots\}$ are subsets in $\mathcal{F}$, then

$$\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$$

A σ-algebra is also a Algebra
(converse is not true)

subsets in $\mathcal{F}$ are called $\mathcal{F}$-measurable sets

Ex: (i) $\mathcal{F} = \{\phi, \Omega\}$

(ii) $\mathcal{F} = \{\phi, A, A^c, \Omega\}$

(iii) $\Omega = \mathbb{N}$

$\mathcal{F} = 2^{\Omega}$, set of all possible subsets of the sample space

$|\Omega| = \mathbb{N}$ = countable infinite

$|\mathcal{F}| = 2^{\infty}$ = uncountabe

But we are only Imposing closure under countabe infinite union's.

* So, if I choose $\mathcal{F} = 2^{\Omega}$, I am saying all subsets are interesting to me.

* When a sample space Ω is finite or countably infinite then we can actually afford to take $\mathcal{F} = 2^{\Omega}$ include all subsets of Ω, and still assign probabilities to them.

* But if Ω is uncountable then the power set of uncountable is too larger set to assign probabilities.

<u>Ex:</u> $\Omega = [0,1]$ uncountable

$\mathcal{F} = 2^{[0,1]}$ (too Large)

$\Rightarrow$ when Ω is uncountable, we have to settel for a sigma algebra which is smaller.

(we cannot take union of uncountable number of sets)

$(\Omega, \mathcal{F})$ is called measurable space.

where Ω = collection's of outcomes

$\mathcal{F}$ = collection of subsets

also, every member of σ-algebra $\mathcal{F}$ is called $\mathcal{F}$-measurable set.

in the context of measure theory.

Measure:

Def: Let $(\Omega, \mathcal{F})$ is a measurable space. A measure on $(\Omega, \mathcal{F})$ is a function $\mu : \mathcal{F} \longrightarrow [0, \infty]$ s.t

(i) $\mu(\phi) = 0$

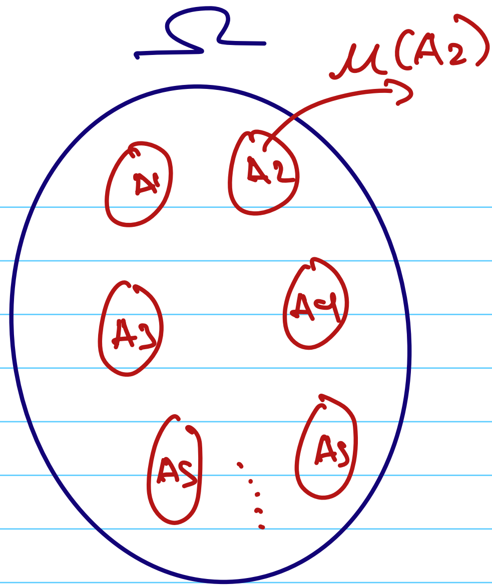
(ii) if $A_1, A_2, A_3 \cdots$ is a countable collection of disjoint sets, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

(countable Additivity)

* measure can only be assigned to elements of $\mathcal{F}$.

* measure space: something we can put a measure on.



$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

$(\Omega, \mathcal{F})$ measurable space.

The triplet $(\Omega, \mathcal{F}, \mu)$ is called measure space.

* if $\mu(\Omega) < \infty$ (finite measure)
* if $\mu(\Omega) = \infty$ (infinite measure)
* if $\mu(\Omega) = 1$ Probability measure

Probability MEASURE

A Probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \longrightarrow [0,1]$ s.t

(i) $\mathbb{P}(\phi) = 0$

(ii) $\mathbb{P}(\Omega) = 1$

(iii) (Countable Additivity)

if $A_1, A_2, A_3, \ldots$ is a seq^n of disjoint sets in $\mathcal{F}$, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Probability space.

Understanding the Union: $\bigcup_{i=1}^{\infty} A_i$

The infinite union $\bigcup_{i=1}^{\infty} A_i$ is not a seq^n of finite union's getter "Bigger and Bigger". It Represents the set of all elements that belong to at least one of the sets. Formally

$$\bigcup_{i=1}^{\infty} A_i = \{ \omega / \omega \in A_i \text{ for some } i \in \mathbb{N} \}$$

This is well defined as a single set and its Probability $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right)$ exists as long as $\{A_i\}$ are measurable.

2. ## The summation $\sum_{i=1}^{\infty} \mathbb{P}(A_i)$

The infinite summation $\sum_{i=1}^{\infty} \mathbb{P}(A_i)$ is defined as the limit of partial sums

$$\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \lim_{n \to \infty} \sum_{i=1}^{n} \mathbb{P}(A_i)$$

+ in measure space (Not Probability space)

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) \quad \text{the}$$

Summation $\sum_{i=1}^{\infty} \mu(A_i)$ can be infinite or finite

⇒ we don't want it to be "<u>Undefined</u>"

⇒ Like Ex: $\sum_{i=1}^{\infty} (-1)^i$ = Keeps Jumping (Not defined)

⇒ But such a situation does not Arise here. why not?

Because $\mu(A_i) \geq 0$ (always Non-negative)

$\lim_{n \to \infty} \sum_{i=1}^{\infty} \mathbb{P}(A_i)$ this as a seq in n

$$\left(\sum_{i=1}^{n} \mathbb{P}(A_i)\right)_{n \in \mathbb{N}}$$ this is a monotonically non-decresing seq^n, has to either converge or goes off to ∞.

$\Rightarrow$ for a $(\Omega, \mathcal{F}, \mathbb{P})$ Probability measure the $\mathcal{F}$-measurable sets are called events

$\Rightarrow$ we can only assign Probabilities to $\mathcal{F}$-measurable sets, not all subsets of Ω.

$\Rightarrow$ in other words, Probability measure's are assigned to only events.

$\Rightarrow$ even when we speak about the $\mathbb{P}$ of a elementary outcome ω, it is

$\mathbb{P}(\{\omega\})$

$\Rightarrow$ unless $\mathcal{F} = 2^{\Omega}$ we will not be assigning Probabilities to all subsets of Ω.

⟹ when Ω is countable (finite or countable infinite) we can afford to take $\mathcal{F} = 2^{\Omega}$, and can assign probabilities to all subsets of Ω, i.e to all singelton's.

⟹ But when Ω is uncountable, it turns out that, assigning probabilities consistantly to all subsets of Real line is not always possible.

⟹ for an uncountable Ω, we have to design σ-algebra $\mathcal{F}$, that is smaller than 2^{Ω}

$$|\mathcal{F}| < |2^{\Omega}|$$