LECTURE 14

The Poisson process

- **Readings:** Start Section 6.2.

Lecture outline

- Review of Bernoulli process
- Definition of Poisson process
- Distribution of number of arrivals
- Distribution of interarrival times
- Other properties of the Poisson process

Bernoulli review

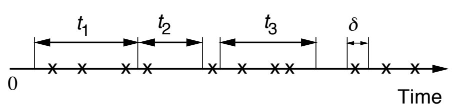
- Discrete time; success probability p
- Number of arrivals in n time slots: binomial pmf
- Interarrival times: geometric pmf
- Time to k arrivals: Pascal pmf
- Memorylessness

$Y_k = T_1 + T_2 + T_3 + \cdots + T_k$

$\mathbb{P}(Y_k = t) = \binom{t-1}{k-1} p^{k-1} (1-p)^{t-k} \cdot p$

(Pascal distribution)

Definition of the Poisson process

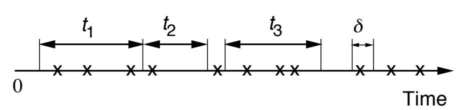


- **Time homogeneity:**
  $P(k, \tau)$ = Prob. of k arrivals in interval of duration τ
- Numbers of arrivals in disjoint time intervals are **independent**
- **Small interval probabilities:**
  For VERY small δ:

$$P(k,\delta) \approx \begin{cases} 1 - \lambda\delta, & \text{if } k = 0; \\ \lambda\delta, & \text{if } k = 1; \\ 0, & \text{if } k > 1. \end{cases}$$

  – λ: "arrival rate"

PMF of Number of Arrivals N



- Finely discretize $[0, t]$: approximately Bernoulli
- N_t (of discrete approximation): binomial
- Taking $\delta \to 0$ (or $n \to \infty$) gives:

$$P(k,\tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!}, \qquad k = 0, 1, \ldots$$

- $\mathbf{E}[N_t] = \lambda t, \qquad \text{var}(N_t) = \lambda t$

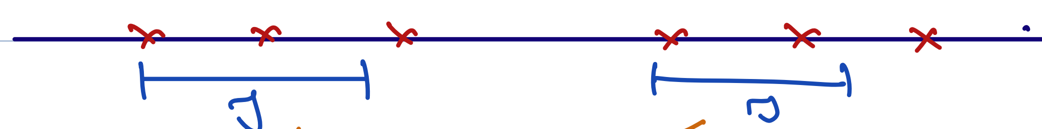
Possion distribution:

* think of an interval of some given length
$\Rightarrow$ during the interval of that length, there is going to be random number of arrivals
* this random number of arrival has a Prob dist, is denoted by $P(k, T)$

$\mathbb{P}(k, J) =$ Prob of k arrivals in interval of duration J, J is fixed

$$\Rightarrow \sum_{k} \mathbb{P}(k, J) = 1$$

Time homogeneity: The Prob distribution of number of arrival's only depend's on length of the interval, But not the exact location of the interval on time axis.

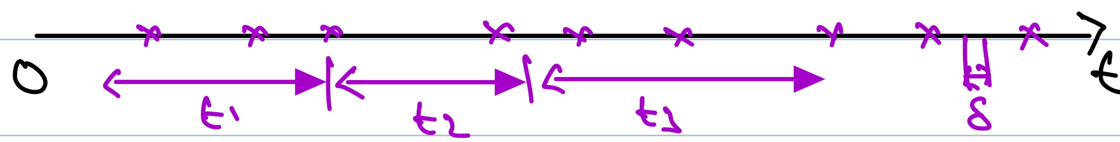


Both interval's are of same length. number of arrival's have same Prob in Both these interval's

Statistical behaviour of number of arrival's in Both the interval's are Same

$\Rightarrow$ disjoint time interval's are statistically independent.

Small interval Probabilities



* if we look at the time interval of length δ, δ (small time interval), there is a Probability that we get exactly one arrival is $\lambda\delta$, where λ = intensity of arrival process.

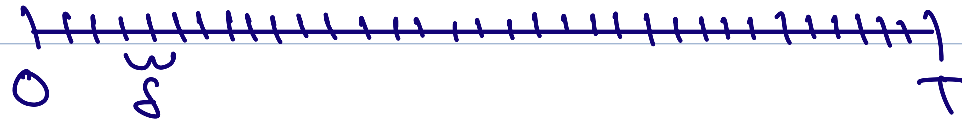
$$\mathbb{P}(k, \delta) = \begin{cases} 1-\lambda\delta & \text{if } k=0 \\ \lambda\delta & \text{if } k=1 \\ 0 & \text{if } k>1 \end{cases}$$

$$\lim_{\delta \to 0} \frac{\mathbb{P}(1,\delta)}{\delta} = \lambda \quad \text{(arrival rate)}$$

$$\mathbb{E}[\#\text{ arrival's in } \delta \text{ interval}] = 1\cdot(\lambda\delta) + 0\cdot(1-\lambda\delta)$$
$$= \lambda\delta$$

$$\lambda = \frac{\mathbb{E}[\#\text{ arrival's in } \delta \text{ interval}]}{\delta}$$ = Expected number arrival's in unit time.

$\Rightarrow$ <u>Poisson distribution:</u> number of arrival's in a given amount of time t (Bis length)



Split it into δ time interval's

$\Rightarrow$ number of time slots $= \dfrac{T}{\delta} = n$

$\Rightarrow$ $\mathbb{P}(k, \delta)$ = this is a Bernoulli distributio with $\mathbb{P}(\text{success}) = \lambda\delta = p$

$\mathbb{P}(\text{fail}) = 1 - \lambda\delta.$

$\Rightarrow$ Total number of success in n slots $= \dfrac{T}{\delta}$ (Binomial)

$$\mathbb{P}(k \text{ arrival's}) = \binom{n}{k} (\lambda\delta)^k (1-\lambda\delta)^{n-k}$$

$$= \binom{n}{k} \left(\frac{\lambda T}{n}\right) \left(1 - \frac{\lambda T}{n}\right)$$

for Possion Process $\delta \to 0$, $\Rightarrow n \to \infty$

$$\mathbb{P}(k, \tau) = \frac{(\lambda\tau)^k e^{-\lambda\tau}}{k!} \qquad k = 0, 1, \ldots$$

Example

- You get email according to a Poisson process at a rate of $\lambda = 5$ messages per hour. You check your email every thirty minutes.

- Prob(no new messages) =

- Prob(one new message) =

Interarrival Times

- Y_k time of kth arrival

- **Erlang** distribution:

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \qquad y \geq 0$$

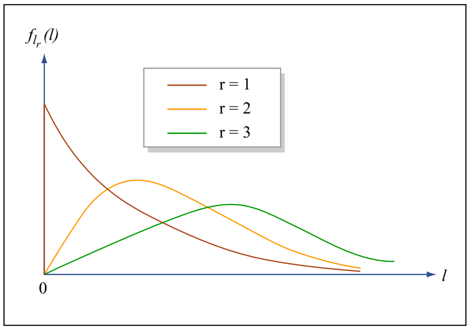


Image by MIT OpenCourseWare.

- Time of first arrival ($k = 1$): **exponential**: $f_{Y_1}(y) = \lambda e^{-\lambda y}, \quad y \geq 0$
  - Memoryless property: The time to the next arrival is independent of the past

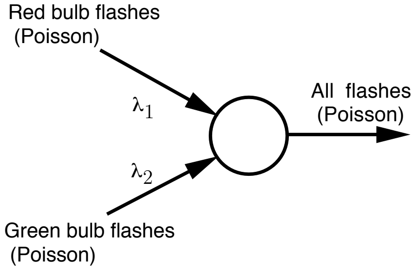
Bernoulli/Poisson Relation

	POISSON	**BERNOULLI**
Times of Arrival	Continuous	Discrete
Arrival Rate	λ/unit time	p/per trial
PMF of # of Arrivals	Poisson	Binomial
Interarrival Time Distr.	Exponential	Geometric
Time to k-th arrival	Erlang	Pascal

Merging Poisson Processes

- Sum of independent Poisson **random variables** is Poisson

- Merging of independent Poisson **processes** is Poisson



  - What is the probability that the next arrival comes from the first process?

MIT OpenCourseWare
http://ocw.mit.edu

6.041 / 6.431 Probabilistic Systems Analysis and Applied Probability
Fall 2010

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.