Borel sets And Lebesgue Measure

we are dealing with the Promblem, trying to define uniform measure on [0,1] interval.

* [0,1] is an uncountable sample space

Generated σ-algebra

Let $\mathcal{C}$ be an arbitary subsets of Ω, where $\mathcal{C}$ is our interesting subsets, subset's we want to keep in our smaller σ-algebra.

* in or $\Omega = [0,1]$ case we are interested in sub interval's of [0,1] such as (a,b).

* we want to generate σ-algebra that contain's all the element's of $\mathcal{C}$, $\mathcal{C}$ is not σ-algebra, its just collection of subsets, But we want to make a σ-algebra from it. we want to generate smallest (simplist) σ-alge-bra that contain's all the elements of $\mathcal{C}$

Theorm: There exists a unique σ-algebra, say $\sigma(\mathcal{C})$ which is the smallest σ-algebra containing all the elements of $\mathcal{C}$

$\Longrightarrow$ Mean's, if $\mathcal{H}$ is any σ-algebra that contain's $\mathcal{C}$, $\mathcal{C} \subseteq \mathcal{H}$, then $\sigma(\mathcal{C}) \subseteq \mathcal{H}$.

$\sigma(\mathcal{C})$ is called σ-algebra Generated by $\mathcal{C}$.

Proof:

Let's go and find all the σ-algebra's that contain $\mathcal{C}$.

* Let $\{\mathcal{F}_i , i \in I\}$ collection of all σ-algebra's that contain $\mathcal{C}$. This collection is definitely non-empty because 2^{Ω} is one such element of $\mathcal{F}$.

$$\sigma(\mathcal{C}) = \bigcap_{i \in I} \mathcal{F}_i$$

Intersection of σ-algebra's is a σ-algebra.

Definition:

(a) Consider $\Omega = (0,1]$. Let C_0 be the collection of all open subinterval's of $\Omega = (0,1]$. The $\sigma(C_0)$, the σ-algebra generated by C_0, is called the Borel σ-algebra. It is denoted by

$$\mathcal{B}((0,1]).$$

(b) An element of $\mathcal{B}((0,1])$ is called a Borel-measurable set, or simple Borel-set.

$\Rightarrow$ This Borel Sigma algebra $\mathcal{B}((0,1])$ turn's out, it is much smaller sigma algebra than 2^{Ω}

$$|\mathcal{B}((0,1])| << |2^{(0,1]}|$$

Lemma: Every singleton set $\{b\}$, $0 \leq b \leq 1$ is a Borel-set. i.e $\{b\} \in \mathcal{B}((0,1])$

Proof: How do we prove that some set is Borel-set, or an element in σ-algebra?

$\Rightarrow$ It should be a expressed as either complement's or countable union's or intersection's of the generating class (open intervals)

$$\{b\} = \bigcap_{n=1}^{\infty} \left[\left(b - 1/n, \; b + 1/n\right) \cap \Omega \right]$$

$$\Rightarrow \{b\} = \left[(b-1, b+1) \cap (0,1]\right] \cap \left[\left(b-\tfrac{1}{2}, b+\tfrac{1}{2}\right) \cap (0,1]\right]$$

$$\cap \left[\left(b-\tfrac{1}{3}, b+\tfrac{1}{3}\right) \cap (0,1]\right] \cap \left[\left(b-\tfrac{1}{4}, b+\tfrac{1}{4}\right) \cap (0,1]\right]$$

$\ldots$

therefore, we can write

$(a,b]$, (a,b), $[a,b]$, $[a,b)$ are Borel set's as all singleton's, $\{a\}, \{b\} \in \mathcal{B}$

Ex:
$$(a,b] = \bigcap_{n=1}^{\infty} \left(a, b+\tfrac{1}{n}\right) \cap \Omega$$

$$\Omega = (0,1]$$

measurable space $= (\Omega, \mathcal{B})$

Question is how to assign uniform probability measure on Borel sets?

* intuitively we know what measure we want for a open set (a,b) is $b-a$, the length of it. and it should be translation-aly invariant.

Let's start with simple collection of subsets.

$\Omega = (0,1]$

$\mathcal{F}_0 =$ collection of subsets of Ω which are finite union of disjoint intervals of the form $(a,b]$ plus the null set.

<u>Ex:</u> $(a_1, b_1] \cup (a_2, b_2] \cup \cdots \qquad \cup (a_n, b_n]$

$0 \le a_1 < b_1 \le a_2 < b_2 \le a_3 \cdots \quad < b_n$

<u>Lemma:</u> (i) $\mathcal{F}_0$ is an Algebra

<u>Explanation:</u> because $\phi \in \mathcal{F}_0$

$(0,1] \in \mathcal{F}_0$

(open, closed] interval

if $(a_1, b_1] \in \mathcal{F}_0$ then

$(0, a_1] \cup (b_1, 1] \in \mathcal{F}_0$. Both are (open, closed] set's

Lemma 2: $\mathcal{F}_0$ is not a sigma-algebra

$\mathcal{F}_0$ is an algebra, but not σ-algebra, only difference is countable union's are not (infinite) there.

Ex: $$A_n = \left(0, \frac{n}{n+1}\right] \quad n=1,2,\ldots$$

$$\Rightarrow \quad A_1 = \left(0, \frac{1}{2}\right]$$
$$A_2 = \left(0, \frac{2}{3}\right]$$
$$\vdots$$

All these $A_n \in \mathcal{F}_0$, for $n=1,2,\ldots$

Because they are of (open, closed] form.

* But if we take countable union

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \left(0, \frac{n}{n+1}\right]$$

$$= (0,1) \notin \mathcal{F}_0$$

Lemma 3:

$$\sigma(\mathcal{F}_0) = \mathcal{B}((0,1])$$

Borel sets

* the Borel σ-algebra is initially defined as σ-algebra generated by open interval's.
* we are saying, even if we generate σ-algebra by (open, closed] interval's it will still be the same Borel σ-algebra.

Proof:

$\sigma(\mathcal{F}_0) \subseteq \mathcal{B}$, it is enough to show $\mathcal{F}_0 \subseteq \mathcal{B}$

$$\{\phi, \Omega, (\text{open}, \text{closed}]\} = \mathcal{F}_0 \subseteq \mathcal{B}$$

2nd

$$\mathcal{B} \subseteq \sigma(\mathcal{F}_0)$$

we can write

$$(a,b) = \bigcup_{n=1}^{\infty} \left[\left(a, b-\frac{1}{n}\right] \cap \Omega \right]$$

$(a, b-\frac{1}{n}] \in \mathcal{F}_0$

(a,b) is countably infinite union of sets in $\mathcal{F}_0$

$\Rightarrow$ open intervals $\mathcal{C}_0 \subseteq \sigma(\mathcal{F}_0)$

$\Rightarrow$ $\sigma(\mathcal{C}_0) \subseteq \sigma(\mathcal{F}_0)$

$\Rightarrow$ $\mathcal{B} \subseteq \sigma(\mathcal{F}_0)$

so for every set $F \in \mathcal{F}_0$ of the form

$$F = (a_1, b_1] \cup (a_2, b_2] \cup (a_3, b_3] \\ \cdots \cup (a_n, b_n]$$

we define a function

$\mathbb{P}_0 : \mathcal{F}_0 \longrightarrow [0,1]$ s.t

$$P_0(\phi) = 0, \quad P_0(F) = \sum_{i=1}^{n} (b_i - a_i)$$

from this we can write

$$\mathbb{P}_0(\Omega) = \mathbb{P}_0((0,1]) = 1$$

if $(a_1, b_1] \cup (a_2, b_2] \cup (a_3, b_3]$
$\quad \ldots \quad \cup (a_n, b_n]$

are disjoint sets then

$$\mathbb{P}_0(F) = \sum_{i=1}^{n} \mathbb{P}_0((a_i, b_i])$$

$$= \sum_{i=1}^{n} (b_i - a_i)$$

(finite additivity)

Theorem: Caratheodory's Extension theorem

Let $\mathcal{F}_0$ be an algebra of subsets of Ω, and let $\mathcal{F} = \sigma(\mathcal{F}_0)$ be the σ-algebra that it generates. Suppose $\mathbb{P}_0 : \mathcal{F}_0 \to [0,1]$ that satisfies $\mathbb{P}_0(\Omega) = 1$, as well as countable additivity on $\mathcal{F}_0$.

Then $\mathbb{P}_0$ can be extended uniquely to a Probability measure on $(\Omega, \mathcal{F})$

that is, there exists a unique probability measure on $\mathbb{P}$ on $(\Omega, \mathcal{F})$ s.t

$$\mathbb{P}(A) = \mathbb{P}_0(A) \quad \forall A \in \mathcal{F}_0$$

This $\mathbb{P}$ is called the Lebesgue measure on $([0,1], \mathcal{B})$

Proof: Lebesgue measure on singleton.

$$\mathbb{P}(\{b\}) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} (b-\tfrac{1}{n}, b] \cap \Omega\right)$$

for $A_n = (b-\frac{1}{n}, b]$

$$\mathbb{P}(A_n) = \frac{1}{n}$$

Since A_n is a decreasing seq^n nested set's

$$\mathbb{P}(\{b\}) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right)$$

$$= \lim_{n\to\infty} \mathbb{P}(A_n)$$

$$= \lim_{n\to\infty} \frac{1}{n}$$

$$= 0$$

The Lebesgue measure on singleton is zero.

from this we can show

$$\mathbb{P}((a,b]) = \mathbb{P}((a,b)) = \mathbb{P}([a,b]) = \mathbb{P}([a,b))$$

Lebesgue measure or Probability of all rational number's in $(0,1]$?

$\mathbb{P}(\mathbb{Q} \cap (0,1]) = 0$, because this is a countable union of singleton's.

Borel sets on $\mathbb{R}$

(a) Let C is a collection of open interval's on $\mathbb{R}$. Then $\mathcal{B}(\mathbb{R}) = \sigma(C)$ is the Borel set on $\mathbb{R}$

(b) Let D be a collection of semi-infinite intervals $\{(-\infty, x] : x \in \mathbb{R}\}$, then

$$\sigma(D) = \mathcal{B}(\mathbb{R})$$

Lebesgue measure on $\mathbb{R}$

$(\mathbb{R}\ \mathcal{B}(\mathbb{R}), \lambda)$ is an infinite measure space, $\lambda \to$ Lebesgue measure.

with ① $\lambda(\mathbb{R}) = \infty$

② $\lambda(\phi) = 0$

③ The countable additivity property