Hierarchical Model's and Mixture Distribution's

Example (Binomial- Poisson hierarchy):-

① An insect Lay's a Large number of eggs each surviving with Prob P. on the averge, how many eggs will survive?

(i) Large number of egg's laid is a random variable

(ii) Survival of each egg is a random variable.

Therefore:

Y = number of egg's laid

X = number of survivor's

$$\Rightarrow \quad Y \sim \text{Poisson}(\lambda)$$

$$X \mid Y \sim \text{binomial}(Y, P)$$

$X|Y$ is $X|Y=y \sim \text{binomial}(y,p)$

The advantage of hierarchy is that complicated processes may be modelled by a seq of relatively simple models placed in hierarchy

Example 4.4.2

Therefore X = number of survival's

$$\Rightarrow \quad \mathbb{P}(X=x) = \sum_{y=0}^{\infty} \mathbb{P}(X=x, Y=y)$$

$$= \sum_{y=0}^{\infty} \mathbb{P}(X=x \mid Y=y)\, \mathbb{P}(Y=y)$$

$$= \sum_{y=0}^{\infty} \binom{y}{x} p^{x} (1-p)^{y-x} \cdot \frac{e^{-\lambda} \cdot \lambda^{y}}{y!}$$

$$= \sum_{y=0}^{\infty} \frac{\cancel{y!}}{(y-x)!\,x!} p^{x} \cdot (1-p)^{y-x} \cdot \frac{e^{-\lambda} \cdot \lambda^{y}}{\cancel{y!}}$$

for $y=0,1,\dots x$ this is 0

$$= \frac{1}{x!} \left(\frac{p}{1-p}\right)^x \cdot e^{-\lambda} \sum_{y=x}^{\infty} \frac{\left((1-p)\lambda\right)^y}{(y-x)!}$$

$$= \frac{1}{x!} p^x \cdot e^{-\lambda} \cdot \lambda^x \sum_{y=x}^{\infty} \frac{\left((1-p)\lambda\right)^{y-x}}{(y-x)!}$$

$$= \frac{1}{x!} p^x \cdot e^{-\lambda} \cdot \lambda^x \cdot e^{(1-p)\lambda}$$

$$\mathbb{P}(X=x) = \frac{(p\lambda)^x \cdot e^{-p\lambda}}{x!}$$

$$X \sim \text{Poisson}(p\lambda)$$

$$\Rightarrow \quad \mathbb{E}[X] = p\lambda$$

Theorem 4.4.3 :

if X and Y are any two r.v,

then $$\mathbb{E}[X] = \mathbb{E}\left[\mathbb{E}[X \mid Y]\right]$$

Proof:

$f(x,y)$ Joint pdf of X, Y

$$\mathbb{E}[X] = \int\int x f(x,y)\, dx\, dy$$

$$= \int\int x f(x|y) f(y)\, dx\, dy$$

$$= \int \left[\int x f(x|y)\, dx \right] f(y)\, dy$$

$$= \int \mathbb{E}[X|y]\, f(y)\, dy$$

$$= \mathbb{E}\left\{ \mathbb{E}[X|Y] \right\}$$

$$E_X[X] = \mathbb{E}_Y\left[\mathbb{E}_{X|Y}[X|Y] \right]$$

in the previous problem

$$E_X[X] = \underbrace{\mathbb{E}_Y\left[\underbrace{E_{X|Y}\,[X|Y]}_{\text{Binomial }(Y,p)} \right]}_{\text{Poisson }(\lambda)}$$

$$= \mathbb{E}_Y[YP] = \mathbb{E}_Y[Y] \cdot P$$
$$= \lambda P$$

$$\boxed{\Rightarrow \quad E_X[X] = \lambda P}$$

Mixture distribution = distribution arising from a heirarchical structure.

Definition: A random variable X is said to have a mixture distribution if the distribution of X depends on a quantity that also has a distribution.

Theorem 4.4.7 (conditional variance identity)

for any two r.v X and Y

$$\text{Var}(X) = \mathbb{E}[\text{Var}(X|Y)] + \text{Var}(E[X|Y])$$

Proof:

$$\mathrm{Var}(X) = \mathbb{E}_X\left[\left(X - E_X[X]\right)^2\right]$$

$$= \mathbb{E}_X\left[\left(X - E_{X|Y}[X|Y] + E_{X|Y}[X|Y] - E_X[X]\right)^2\right]$$

$$= \mathbb{E}_X\Big[\left(X - \mathbb{E}_{X|Y}[X|Y]\right)^2 + \left(E_{X|Y}(X|Y) - E_X(X)\right)^2$$

$$+ 2\left(X - E_{X|Y}(X|Y)\right)\left(E_{X|Y}(X|Y) - E_X(X)\right)\Big]$$

Last term

$=$

$$\Rightarrow \mathbb{E}\left[2\left(X - E_{X|Y}(X|Y)\right)\left(E_{X|Y}(X|Y) - E_X(X)\right)\right]$$

$$\Rightarrow 2\,\mathbb{E}\left[X - E_{X|Y}(X|Y)\right]\left[E\left[E_{X|Y}(X|Y)\right] - E_X(X)\right]$$

$$\Rightarrow 2\,\mathbb{E}\left[X - E_{X|Y}(X|Y)\right]\left[E[X] - E[X]\right]$$

$$\Rightarrow 0$$

$$\Rightarrow \mathbb{E}_X\left[\left(X-\mathbb{E}_{X|Y}[X|Y]\right)^2 + \left(E_{X|Y}(X|Y) - E_X(X)\right)^2\right]$$

$$\Rightarrow \mathbb{E}_X\left[\left\{X - \mathbb{E}_{X|Y}(X|Y)\right\}^2\right]$$

$$+ \; E_X\left[\left(E_{X|Y}(X|Y) - E_X(X)\right)^2\right]$$

$$\Rightarrow \mathbb{E}\left[\operatorname{var}(X|Y)\right] + \operatorname{var}\left(E(X|Y)\right)$$