Uncountable Ω

* if uncountable sample space were simple then Probability theory would be very easy

Ex: $\Omega = [0,1]$, for example we are throwing darts on this Line. it will Land somewhere in this Line. we want to model mathematically, the Intutive concept of dart Landing uniformly on this interval

in this case if we take $\mathcal{F} = 2^{\Omega}$

The set of all Possible subsets of $[0,1]$ interval, its a very huge collection of sets.

$\Omega = [0,1]$ uncountable

$\mathcal{F} = 2^{\Omega}$ very huge, it has

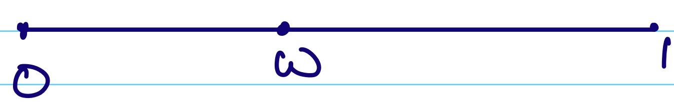
strictly bigger cardinality even the real number's

$$\left| 2^{[0,1]} \right| >> |\mathbb{R}|$$

$\Rightarrow$ That is too big σ-algebra to assign probabilites to.

$\Rightarrow$ This bring's us to the problem that we cannot assign probabilities to all sets of $[0,1]$ interval.

$\Rightarrow$ The elementary approach of simply assigning probabilities to singleton's definitely not gonna work.



$\Rightarrow$ if we were to assign $\mathbb{P}$ to each ω i.e $\mathbb{P}(\{\omega\})$ it cannot be anything positive. Because if it is positive, we would want to put same probability on different singleton i.e $\mathbb{P}(\{\omega_i\})$

$\Rightarrow$ But then we quickly find out that there is uncountable infinity ω's, and if any of them are +ve, the probability of interval will blow up.

$\Rightarrow$ The only thing we can do is $\mathbb{P}$ of singleton is 0

$\Rightarrow$ so, the way out of this is to stop worrying about singleton subsets, The idea is to directly assign $\mathbb{P}$ to sets we consider interesting subsets of Ω we found interesting.

* We want to put a Probability measure such as

① $\mathbb{P}((a,b)) = \mathbb{P}((a,b])$

$= \mathbb{P}([a,b))$

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② Translational invariance.

$$A \subseteq [0,1]$$

$$\mathbb{P}(A) = \mathbb{P}(A \oplus x)$$

$$A \oplus x = \{a+x \mid a \in A,\ a+x \leq 1\}$$

$$\cup \{a+x-1 \mid a \in A,\ a+x > 1\}$$

$$\Rightarrow \mathbb{P}\left(\left(\tfrac{1}{2}, 1\right)\right) = \mathbb{P}\left(\left(0, \tfrac{1}{4}\right) \cup \left(\tfrac{3}{4}, 1\right)\right)$$

$$= \mathbb{P}\left(\left(0, \tfrac{1}{2}\right)\right)$$

$$= \mathbb{P}\left(\left(\tfrac{1}{4}, \tfrac{3}{4}\right)\right)$$

Impossibility theorum

There does not exist a measure $\mu(A)$ defined on 2^{Ω}, i.e all subsets of $[0,1]$ satisfying ① and ② above.

Birth of Borel sets