Identities

These identities are useful in establishing theorem's.

Theorem 3.6.4:

Let $X_{\alpha,\beta} \sim \text{gamma}(\alpha, \beta)$ is r.v with $\alpha > 1$. Then for any constants a and b.

$$\mathbb{P}(a < X_{\alpha,\beta} < b) = \beta\left[f(a|\alpha,\beta) - f(b|\alpha,\beta)\right]$$

$$+ P(a < X_{\alpha-1,\beta} < b)$$

Proof:

$$\mathbb{P}(a < X_{\alpha,b} < b) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_a^b x^{\alpha-1} e^{-x/\beta}\, dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^{\alpha}}\left[x^{\alpha-1} \cdot e^{-x/\beta} - \beta \Big]_a^b + \int_a^b (\alpha-1) x^{\alpha-2} e^{-x/\beta} \beta\, dx \right]$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha}\left[a^{\alpha-1}\cdot e^{-a/\beta}\beta - b^{\alpha-1}\cdot e^{-b/\beta}\cdot\beta + (\alpha-1)\beta\int_a^b x^{\alpha-2}\cdot e^{-x/\beta}dx\right]$$

$$= \frac{\beta}{\Gamma(\alpha)\beta^\alpha}\left[a^{\alpha-1}\cdot e^{-a/\beta} - b^{\alpha-1}\cdot e^{-b/\beta}\right] + \mathbb{P}(a < X_{\alpha-1,\beta} < b)$$

$$= \beta\left[f(a\mid\alpha,\beta) - f(b\mid\alpha,\beta)\right] + \mathbb{P}(a < X_{\alpha-1,\beta} < b)$$

Lemma 3.6.5 (Stein's Lemma)

① $X \sim N(\theta, \sigma^2)$

② $g(\cdot)$ be differentiable function.

satisfying $\mathbb{E}\,|g'(x)| < \infty$

Then $\mathbb{E}[g(x)(X-\theta)] = \sigma^2\,\mathbb{E}[g'(x)]$

Proof:

$$\mathbb{E}[g(x)(x-\theta)]$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} g(x)(x-\theta) e^{-\frac{(x-\theta)^2}{2\sigma^2}} dx$$

$$u = g(x) \qquad v = (x-\theta) \cdot e^{-\frac{(x-\theta)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \left[g(x) \int (x-\theta) e^{-\frac{(x-\theta)^2}{2\sigma^2}} \Big]_{-\infty}^{\infty} - \int g'(x) \int (x-\theta) e^{-\frac{(x-\theta)^2}{2\sigma^2}} \right.$$

$$\int (x-\theta) \cdot e^{-\frac{(x-\theta)^2}{2\sigma^2}} dx$$

$$\frac{(x-\theta)^2}{2\sigma^2} = u$$

$$\frac{(x-\theta)}{\sigma^2} dx = du$$

$$\Rightarrow \quad \sigma^2 \int e^{-u} du = -\sigma^2 e^{-\frac{(x-\theta)^2}{2\sigma^2}}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}\sigma}\left[-\sigma^2 g(x) e^{-\frac{(x-\theta)^2}{2\sigma^2}}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} g'(x)\,\sigma^2 e^{-\frac{(x-\theta)^2}{2\sigma^2}}\,dx$$

$=0$

$$\Rightarrow \sigma^2 \cdot \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} g'(x)\cdot e^{-\frac{(x-\theta)^2}{2\sigma^2}}\,dx$$

$$\Rightarrow \sigma^2 \, \mathbb{E}[g'(x)]$$

Example 3.6.6 :-

$$X \sim N(\theta, \sigma^2)$$

$$\mathbb{E}[X^3] = \mathbb{E}[X^2(X-\theta+\theta)]$$

$$= \mathbb{E}[X^2(X-\theta)] + \theta\,\mathbb{E}[X^2]$$

$$= \sigma^2\,\mathbb{E}[2X] + \theta(\sigma^2+\theta^2)$$

$$= 2\sigma^2\theta + \sigma^2\theta + \theta^3$$

$$E[X^3] = 3\sigma^2\theta + \theta^3$$

Theorem 3.6.7

$\chi_p^2 \sim$ chi squared r.v with p dof

for any function $h(x)$

$$\mathbb{E}\left[h(\chi_p^2)\right] = p\,\mathbb{E}\left[\frac{h(\chi_{p+2}^2)}{\chi_{p+2}^2}\right]$$

Proof:

$$f(x) = \frac{1}{\Gamma(p/2)\,2^{p/2}} x^{p/2-1} \cdot e^{-x/2} dx$$

$$\mathbb{E}\left[h(\chi_p^2)\right] = \frac{1}{\Gamma(p/2)\,2^{p/2}} \int_0^\infty h(x)\, x^{p/2-1} \cdot e^{-x/2} dx$$

$$= \frac{1}{\Gamma(p/2)\,2^{p/2}} \int_0^\infty \frac{h(x)}{x}\, x^{\frac{p+2}{2}-1} \cdot e^{-x/2} dx$$

$$\Rightarrow \Gamma\left(\frac{p}{2}\right) \cdot 2^{p/2} = \frac{\Gamma(p+2/2)\, 2^{\frac{p+2}{2}}}{p}$$

$$\Rightarrow \frac{p}{\Gamma\left(\frac{p+2}{2}\right) 2^{\frac{p+2}{2}}} \int_0^\infty \frac{h(x)}{x} \, x^{\frac{p+2}{2}-1} e^{-x/2} \, dx$$

$$\mathbb{E}\left[\chi_p^2\right] = p \, \mathbb{E}\left[\frac{h\left(\chi^2_{p+2}\right)}{\chi^2_{p+2}}\right]$$