DESCRIPTION

This document provides solution's to 4.15 to 4.31 from chapter 4 of statistical inference Book by Casella and Berger, Focusing on Bivariate transformation's.

A few Problem's are yet to be solved, but I will update the document soon.

4.15 Let $X \sim Poisson(\theta)$, $Y \sim Poisson(\lambda)$, independent. $X+Y \sim Poisson(\theta+\lambda)$. Show that distribution $X \mid X+Y$ is Binomial with success Prob $\frac{\theta}{\theta+\lambda}$. what is the distribution of $Y \mid X+Y$?

<u>solution:</u>

in example 4.3.1

$U = X+Y$ $\quad V = Y$ and Joint

Pmf is

$$f_{U,V}(u,v) = \frac{\theta^{u-v} e^{-\theta}}{(u-v)!} \cdot \frac{\lambda^{v} e^{-\lambda}}{v!}$$

$v = 0, 1, 2, \ldots$

$u = v, v+1, v+2, \ldots$

$$\mathbb{P}(Y \mid X+Y) = \mathbb{P}(V \mid U)$$

$$= \frac{f_{U,V}(u,v)}{f_U(u)}$$

$$f_{V|U}(v|u) = \frac{\theta^{u-v} \cancel{e^{-\theta}} \lambda^{v} \cancel{e^{-\lambda}}}{(u-v)!\ v!} \cdot \frac{u!}{\cancel{e^{-(\theta+\lambda)}} \cdot (\theta+\lambda)^{u}}$$

for a fixed u

$v = 0, 1, \ldots u$

$$= \frac{u!}{(u-v)!\,v!} \cdot \theta^{u-v} \cdot \frac{\lambda^{v}}{(\theta+\lambda)^{u}}$$

$$= \binom{u}{v} \left(\frac{\lambda}{\theta+\lambda}\right)^{v} \cdot \left(\frac{\theta}{\theta+\lambda}\right)^{u-v}$$

$v = 0, 1, 2, \ldots u$

$$\Rightarrow f_{V|U}(v|u) = \binom{u}{v} \left(\frac{\lambda}{\theta+\lambda}\right)^{v} \cdot \left(\frac{\theta}{\theta+\lambda}\right)^{u-v}$$

$$\Rightarrow V|U \sim \text{Binomial}\left(U, \frac{\lambda}{\theta+\lambda}\right)$$

$$\Rightarrow X|X+Y \sim \text{Binomial}\left(X+Y, \frac{\lambda}{\theta+\lambda}\right)$$

using symmetricity

$$X \mid X+Y \sim \text{Binomial}\left(X+Y, \frac{\theta}{\theta+\lambda}\right)$$

4.16

$$\left.\begin{array}{l} X \sim \text{geometric}(p) \\ Y \sim \text{geometric}(p) \end{array}\right\} \text{independent}$$

a Show that U and V are independent where U, V are defined by

$$U = \min(X, Y) \quad \text{, and} \quad V = X - Y$$

Solution:

To show independence find Joint distribution, and show that we can split.

$$U = \begin{cases} X & \text{if } Y \geq X \\ Y & \text{if } X > Y \end{cases}$$

$$V = X - Y$$

$$U = 1, 2, 3, \ldots$$

$$V = \mathbb{Z}$$

$$f_{X,Y}(x,y) = (1-p)^{x-1} \cdot p \cdot (1-p)^{y-1} \cdot p$$

$$f_{X,Y}(x,y) = (1-p)^{x+y-2} \cdot p^2$$

$$A_1 = \{(u,v) \mid v<0\}$$

$$A_2 = \{(u,v) \mid v>0\}$$

<u>for $v<0$</u>

$$U = g_1(X,Y) = X \qquad\qquad X = U = h_1(U,V)$$

$$\Rightarrow$$

$$V = g_2(X,Y) = X-Y \qquad\qquad Y = U-V = h_2(U,V)$$

$$J_1 = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

<u>for $v>0$</u>

$$U = g_1(X,Y) = Y \quad \Rightarrow \quad X = U+V = h_1(U,V)$$

$$V = g_2(X,Y) = X-Y \qquad\qquad Y = U = h_2(U,V)$$

$$J_2 = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1$$

for $V<0$

$$f_{U,V}(u,v) = (1-p)^{2u-v-2} \cdot p^2$$ (we can seperate u,v)

for $V>0$

$$f_{U,V}(u,v) = (1-p)^{2u+v-2} \cdot p^2$$ (we can seperate u,v)

for $V=0 \Rightarrow X=Y$

$$f_{U,V}(u,0) = (1-p)^{u-2} \cdot p^2$$

(we can seperate u,v)

Therefore:

$$f_{U,V}(u,v) = (1-p)^{2u+|v|-2} \cdot p^2$$

$$= (1-p)^{2u-2} \cdot (1-p)^{|v|} \cdot p^2$$

$$= g(u) \cdot h(v)$$

$\Rightarrow$ U, V are independent

(c) Joint Pdf of X and $X+Y$

$$U = X \qquad \Rightarrow \qquad X = u$$
$$V = X+Y \qquad\qquad Y = V-U$$

$$\Rightarrow J = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

$$\boxed{f_{U,V}(u,v) = (1-p)^{v-2} \cdot p^2}$$

4.17 $X \sim \text{exponential}(1)$

Y = integer part of $X+1$

$\Rightarrow Y = i+1 \iff i \leq X < i+1,\ i = 0,1,2,\ldots$

Solution.

$$y = 1,2,3,\ldots$$

$$f_X(x) = e^{-x} \qquad x \in (0,\infty)$$

$$\mathbb{P}(Y=y) = \int_{y-1}^{y} f_X(x)\,dx$$

$$= \int_{y-1}^{y} e^{-x}\,dx = \left. -e^{-x} \right]_{y-1}^{y}$$

$$= e^{-y+1} - e^{-y}$$

$$\Rightarrow \mathbb{P}(Y=y) = e^{-y}(e-1)$$

$$\Rightarrow \mathbb{P}(Y=y) = e^{-y} \cdot e \left(1-\frac{1}{e}\right)$$

$$= e^{-(y-1)} \cdot \left(1-\frac{1}{e}\right)$$

$$\mathbb{P}(Y=y) = \left(\frac{1}{e}\right)^{y-1} \cdot \left(1-\frac{1}{e}\right)$$

$$\Rightarrow \mathbb{P}(Y=y) = \left[1-\left(1-\frac{1}{e}\right)\right]^{y-1} \cdot \left(1-\frac{1}{e}\right)$$

$$\boxed{Y \sim \text{geometric}\left(1-\frac{1}{e}\right)}$$

(b) Find the conditional distribution of $X-Y$ given $Y \geq 5$

Solution: $\mathbb{P}(X-4 \mid Y \geq 5)$

$$Y \geq 5 \iff X \geq 4$$

$$\Rightarrow \quad X-4 \geq 0$$

$$\Rightarrow \quad X-4 \mid X \geq 4 \sim \text{geometric}(1)$$

Memorylessness property.

4.18 Given that $g(x) \geq 0$ has the property that

$$\int_0^\infty g(x)\,dx = 1$$

Show that

$$f(x,y) = \frac{2g\left(\sqrt{x^2+y^2}\right)}{\pi\sqrt{x^2+y^2}}, \quad x,y>0$$

is a pdf.

Solution:

$$f(x,y) = \frac{2g\left(\sqrt{x^2+y^2}\right)}{\pi\sqrt{x^2+y^2}} > 0 \quad \forall x,y>0$$

$$\int_0^\infty \int_0^\infty \frac{2\, g\left(\sqrt{x^2+y^2}\right)}{\pi \sqrt{x^2+y^2}} \quad , x, y > 0$$

Change to polar co-ordinates.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$dx\, dy = r\, dr\, d\theta$$

$$\int_0^\infty \int_0^{\frac{\pi}{2}} \frac{2\, g(r)}{\pi \cancel{r}} \cdot \cancel{r}\, dr\, d\theta$$

$$= \int_0^\infty \frac{\cancel{2}}{\cancel{\pi}}\, g(r) \cdot \frac{\cancel{\pi}}{\cancel{2}}\, dr$$

$$= \int_0^\infty g(r)\, dr = 1$$

Hence its a Pdf

4.19 (a) $X_1 \sim N(0,1)$, $X_2 \sim N(0,1)$ independent.

find the pdf of $\frac{(X_1 - X_2)^2}{2}$

Soln

$X_1 \sim N(0,1)$

$X_2 \sim N(0,1)$

$\Rightarrow X_1 - X_2 \sim (0-0, 2)$

$\Rightarrow \frac{X_1 - X_2}{\sqrt{2}} \sim \left(0, \frac{2}{(\sqrt{2})^2}\right)$

$\Rightarrow \frac{X_1 - X_2}{\sqrt{2}} \sim N(0,1)$

if $\frac{X_1 - X_2}{\sqrt{2}} \sim N(0,1)$ then

$\left(\frac{X_1 - X_2}{\sqrt{2}}\right)^2 \sim$ chi square distributio

$$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{y}} e^{-y/2} \quad 0 < y < \infty}$$

4.19b if X_i, $i = 1, 2$, are independent gamma$(\alpha_i, 1)$ r.v's, find the marginal distribution of

$$\frac{X_1}{X_1+X_2} \quad \text{and} \quad \frac{X_2}{X_1+X_2}$$

Solution:

$$u = \frac{X_1}{X_1+X_2}$$, find a V

such that its a one-to-one transformation

$$V = X_1 + X_2$$

$$\Rightarrow \left.\begin{aligned} u &= \frac{X_1}{X_1+X_2} \\ V &= X_1 + X_2 \end{aligned}\right\} \Rightarrow \begin{aligned} X_1 &= uV \\ X_2 &= V(1-u) \end{aligned}$$

$$J = \begin{vmatrix} V & u \\ -V & 1-u \end{vmatrix}$$

$$J = V(1-u) + Vu$$
$$J = V$$

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{\Gamma(\alpha_1)} \cdot x_1^{\alpha_1-1} \cdot e^{-x_1}$$

$$\frac{1}{\Gamma(\alpha_2)} \cdot x_2^{\alpha_2-1} \cdot e^{-x_2}$$

$$f_{U,V}(u,v) = \frac{1}{\Gamma(\alpha_1)} (uv)^{\alpha_1-1} \cdot e^{-uv} \cdot \frac{1}{\Gamma(\alpha_2)} \cdot (v(1-u))^{\alpha_2-1}$$

$$\cdot e^{-v(1-u)} \cdot v$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \cdot u^{\alpha_1-1} \cdot (1-u)^{\alpha_2-1} \cdot v^{\alpha_1-1} \cdot v^{\alpha_2-1} \cdot v$$

$$\cdot e^{-uv-v+uv}$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \cdot u^{\alpha_1-1} (1-u)^{\alpha_2-1} \cdot v^{\alpha_1+\alpha_2-1}$$

$$\cdot e^{-v}$$

$$= \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \cdot u^{\alpha_1-1} \cdot (1-u)^{\alpha_2-1} \cdot \frac{1}{\Gamma(\alpha_1+\alpha_2)} \cdot v^{\alpha_1+\alpha_2-1} \cdot e^{-v}$$

$\Rightarrow$ $U \sim Beta(\alpha_1, \alpha_2)$

$V \sim gamma(\alpha_1 + \alpha_2, 1)$

therefore $\dfrac{X_1}{X_1 + X_2} \sim Beta(\alpha_1, \alpha_2)$

using symmetric

$$\frac{X_2}{X_1 + X_2} \sim Beta(\alpha_2, \alpha_1)$$

4.20 X_1, X_2 are independent $n(0, \sigma^2)$

a) Find the distribution of Y_1 and Y_2 where

$$Y_1 = X_1^2 + X_2^2$$

$$Y_2 = \frac{X_1}{\sqrt{Y_1}}$$

Solution: The transformation is not one-to-one

$$A_0 = \{(x_1, x_2) \mid -\infty < x_1 < \infty,\ x_2 = 0\}$$

$$A_1 = \{(x_1, x_2) \mid -\infty < x_1 < \infty,\ x_2 < 0\}$$

$$A_2 = \{(x_1, x_2) \mid -\infty < x_1 < \infty,\ x_2 > 0\}$$

$$B = ((y_1, y_2) \mid 0 < y_1 < \infty,\ -\infty < y_2 < \infty)$$

$$\left.\begin{aligned} y_1 &= g_{11}(x_1, x_2) = x_1^2 + x_2^2 \\ y_2 &= g_2(x_1, x_2) = \frac{x_1}{\sqrt{y_1}} \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 &= \sqrt{y_1} \cdot y_2 \\ x_2 &= -\sqrt{y_1(1-y_2^2)} \end{aligned}$$

$$J = \begin{vmatrix} \frac{\partial h_1(y_1, y_2)}{\partial y_1} & \frac{\partial h_1(y_1, y_2)}{\partial y_2} \\ \frac{\partial h_2(y_1, y_2)}{\partial y_2} & \frac{\partial h_2(y_1, y_2)}{\partial y_2} \end{vmatrix}$$

$$J = \begin{vmatrix} \dfrac{y_2}{2\sqrt{y_1}} & \sqrt{y_1} \\ \dfrac{-(1-y_2^2)}{2\sqrt{y_1(1-y_2^2)}} & \dfrac{y_1 y_2}{\sqrt{y_1(1-y_2^2)}} \end{vmatrix}$$

$$J = \frac{y_2^2}{2\sqrt{1-y_2^2}} + \frac{1-y_2^2}{2\sqrt{1-y_2^2}}$$

$$\Rightarrow \boxed{J = \frac{1}{2\sqrt{1-y_2^2}}}$$

$$\left.\begin{aligned} g_{12}(x_1, x_2) &= x_1^2 + x_2^2 \\ g_{22}(x_1, x_2) &= \frac{x_1}{\sqrt{y_1}} \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 &= \sqrt{y_1} \cdot y_2 \\ x_2 &= \sqrt{y_1(1-y_2^2)} \end{aligned}$$

$$J = \frac{1}{2\sqrt{1-y_2^2}}$$

$$f_{x_1, x_2}(x_1, x_2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x_1^2}{2\sigma^2}} \cdot e^{-\frac{x_2^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}}$$

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}\frac{(x_1^2+x_2^2)}{\sigma^2}} \cdot |J| \cdot 2$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}\frac{y_1}{\sigma^2}} \cdot \frac{1}{2\sqrt{1-y_2^2}}$$

$$\boxed{f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{2\pi\sigma^2} \frac{e^{-\frac{y_1}{2\sigma^2}}}{\sqrt{1-y_2^2}} \quad \begin{array}{l} 0<y_1<\infty \\ -\infty<y_2<\infty \end{array}}$$

4.20 b Show that Y_1 and Y_2 are independent, and interpret this result geometrically.

Solution:- The Joint Pdf can be written as

$$f_{Y_1,Y_2}(y_1,y_2) = f_{Y_1}(y_1)\, f_{Y_2}(y_2)$$

Therefore Y_1, Y_2 are independent

$$Y_1 = X_1^2+X_2^2 = r^2 \quad (\text{distance})$$

$$Y_2 = \frac{X_1}{\sqrt{X_1^2+X_2^2}} = \cos\theta \quad (\text{cosine angle})$$

4.21 A point is generated at random in the plane according to the following polar scheme. A radius R is chosen, where the distribution of R^2 is χ^2 with 2 DOF. independently an angle is choosen, where $\Theta \sim$ uniform$(0,2\pi)$. Find the joint distribution of $X = R\cos\Theta$, $Y = R\sin\Theta$

Solution:

$$Y \sim \text{Chi-square } (P \text{ dof})$$

$$f_Y(y) = \frac{1}{2} e^{-\frac{y}{2}} \qquad y \geq 0$$

$$r = \sqrt{y} \Rightarrow y = r^2$$

$$\frac{dr^2}{dr} = 2r$$

$$f_r(r) = \frac{1}{2} e^{-\frac{r^2}{2}} \cdot 2r$$

$$f_r(r) = r e^{-\frac{r^2}{2}} \qquad r > 0$$

$\Theta \sim \text{unif}(0, 2\pi)$

Joint pdf

$$f_{R,\Theta}(r, \theta) = \frac{r}{2\pi} e^{-\frac{r^2}{2}} \qquad \begin{matrix} 0 < r < \infty \\ 0 < \theta < 2\pi \end{matrix}$$

$$\begin{aligned} u &= r\cos\theta \\ v &= r\sin\theta \end{aligned} \quad \Rightarrow \quad \begin{aligned} r &= \sqrt{u^2+v^2} \\ \theta &= \tan^{-1}\left(\frac{v}{u}\right) \end{aligned}$$

$$\begin{matrix} -\infty < v < +\infty \\ -\infty < u < \infty \end{matrix} \Rightarrow J = \begin{vmatrix} \frac{u}{\sqrt{u^2+v^2}} & \frac{v}{\sqrt{u^2+v^2}} \\ \frac{-v}{u^2+v^2} & \frac{u}{u^2+v^2} \end{vmatrix}$$

$$|J| = \frac{u^2}{(u^2+v^2)^{3/2}} + \frac{v^2}{(u^2+v^2)^{3/2}}$$

$$= \frac{1}{\sqrt{u^2+v^2}}$$

$$f_{U,V}(u,v) = \frac{\sqrt{u^2+v^2}}{2\pi} e^{-\frac{(u^2+v^2)}{2}} \cdot \frac{1}{\sqrt{u^2+v^2}}$$

$$\Rightarrow \boxed{f_{U,V}(u,v) = \frac{1}{2\pi} e^{-\frac{(u^2+v^2)}{2}}, \quad \begin{array}{l} -\infty < u < \infty \\ -\infty < v < \infty \end{array}}$$

$$f_{U,V}(u,v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}$$

4.22 Let (X,Y) be a bivariate random vector with joint pdf $f(x,y)$. Let $U = aX+b$ and $V = cY+d$, where a,b,c,d are fixed constants. with $a>0$ and $c>0$. Show that the joint pdf of U,V is

$$f_{U,V}(u,v) = \frac{1}{ac} f\left(\frac{u-b}{a}, \frac{v-d}{c}\right)$$

Solution:

$$\begin{array}{l} U = aX+b \\ V = cY+d \end{array} \Rightarrow \begin{array}{l} x = \frac{u-b}{a} \\ y = \frac{v-d}{c} \end{array}$$

$$J = \begin{vmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{c} \end{vmatrix} = \frac{1}{ac}$$

$$f_{U,V}(u,v) = f_{X,Y}\left(h_1(u,v), h_2(u,v)\right) \cdot |J|$$

$$= \frac{1}{ac} f\left(\frac{u-b}{a}, \frac{v-d}{c}\right)$$

Here Proved.

4.23 for X and Y as in example 4.3.3 find the distribution of XY by making the transformation's given in (a) and (b) and integrating out V

$$X \sim beta(\alpha, \beta)$$
$$Y \sim beta(\alpha+\beta, \gamma)$$

(a) $U = XY, \quad V = Y$

solution:

$$A = \{(x,y) \mid 0<x<1, \; 0<y<1\}$$

$$U = XY \qquad \Rightarrow \qquad X = \frac{U}{V} \qquad \Rightarrow \qquad J = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix}$$
$$V = Y \qquad\qquad\quad Y = V$$

$$\Rightarrow \quad J = 1$$

$$B = \{ (u,v) \mid 0 < u < v < 1 \}$$

$$f_{X,Y}(x,y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$$

$$\times \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)} y^{\alpha+\beta-1} (1-y)^{\gamma-1}$$

$$f_{U,V}(u,v) = f_{X,Y}(h_1(u,v), h_2(u,v)) \cdot |J|$$

$$= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \left(\frac{u}{v}\right)^{\alpha-1} \cdot \left(1 - \frac{u}{v}\right)^{\beta-1} \cdot v^{\alpha+\beta-1} \cdot (1-v)^{\gamma-1} \cdot \frac{1}{v}$$

$$0 < u < v < 1$$

$$f_U(u) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1} \int_u^1 (v-u)^{\beta-1} (1-v)^{\gamma-1} \, dv$$

solving this

gte $U \sim \text{beta}(\alpha, \beta+\gamma)$

4.24 Let X, Y are independent random variables

$X \sim \text{gamma}(r,1)$

$Y \sim \text{gamma}(s,1)$.

Show that $Z_1 = X + Y$

$Z_2 = \dfrac{X}{X+Y}$ are independent.

and find the distribution of each

Solution:

$$f_X(x) = \frac{1}{\Gamma(r)} x^{r-1} e^{-x}$$

$$f_Y(y) = \frac{1}{\Gamma(s)} y^{s-1} e^{-y}$$

$$f_{X,Y}(x,y) = \frac{1}{\Gamma(r)\Gamma(s)} x^{r-1} e^{-x} \cdot y^{s-1} \cdot e^{-y}$$

$$\left.\begin{aligned} Z_1 &= X+Y \\ Z_2 &= \frac{X}{X+Y} \end{aligned}\right\} \Rightarrow \begin{aligned} X &= Z_1 Z_2 \\ Y &= Z_1(1-Z_2) \end{aligned} \qquad J = Z_1$$

$$A = \{(x,y) \mid x>0,\ y>0\}$$

$$B = \{(z_1,z_2) \mid z_1>0,\ 0<z_2<1\}$$

$$f_{Z_1Z_2}(z_1,z_2) = \frac{1}{\Gamma(\gamma)\Gamma(\delta)} (z_1z_2)^{\gamma-1} \cdot e^{-z_1z_2} \cdot (z_1(1-z_2))^{\delta-1} \cdot e^{-z_1(1-z_2)} \cdot z_1$$

$$= \frac{1}{\Gamma(\gamma)\Gamma(\delta)} z_1^{\gamma+\delta-1} \cdot e^{-z_1} \cdot z_2^{\gamma-1} \cdot (1-z_2)^{\delta-1}$$

$$= \frac{1}{\Gamma(\gamma+\delta)} z_1^{\gamma+\delta-1} \cdot e^{-z_1} \cdot \frac{\Gamma(\gamma+\delta)}{\Gamma(\gamma)\Gamma(\delta)} \cdot z_2^{\gamma-1} \cdot (1-z_2)^{\delta-1}$$

$$\Rightarrow \boxed{\begin{array}{l} Z_1 \sim \text{gamma}(\gamma+\delta, 1) \\ Z_2 \sim \text{beta}(\gamma,\delta) \end{array}}$$

$$f_{Z_1}(z_1) = \frac{1}{\Gamma(\gamma+\delta)} z_1^{\gamma+\delta-1} \cdot e^{-z_1}$$

$$f_{Z_2}(z_2) = \frac{\Gamma(\gamma+\delta)}{\Gamma(\gamma)\Gamma(\delta)} z_2^{\gamma-1} (1-z_2)^{\delta-1}$$

4.26 $X \sim \text{exponential}(\lambda)$, $Y \sim \text{exponential}(\mu)$ } independent.

It is impossible to obtain direct observation's of X and Y. instead, we observe the r.v Z and W, where

$$Z = \min\{X, Y\}$$

$$W = \begin{cases} 1 & \text{if } Z = X \\ 0 & \text{if } Z = Y \end{cases}$$

(a) find Joint distribution of Z and W

Solution:

$$Z = \min\{X, Y\}, \quad W = \begin{cases} 1 & \text{if } Z = X \Rightarrow X < Y \\ 0 & \text{if } Z = Y \Rightarrow Y < X \end{cases}$$

Case 1 if $X < Y$

$\Rightarrow$ $Z = X$ and $W = 1$

$\Rightarrow$ $X = Z$, $W = 1$

4.27

$X \sim N(\mu, \sigma^2)$
$Y \sim N(\gamma, \sigma^2)$ } independent.

Define $U = X + Y$ and $V = X - Y$

Show that U, V are independent Normal variables. Find the distribution of each

Solution:

$$f_{X,Y}(x,y) = \frac{1}{2\pi \cdot \sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \cdot e^{-\frac{(y-\gamma)^2}{2\sigma^2}}$$

$$\begin{matrix} U = X + Y \\ V = X - Y \end{matrix} \implies \begin{matrix} X = \frac{U+V}{2} \\ Y = \frac{U-V}{2} \end{matrix} \qquad J = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$|J| = \frac{1}{2}$$

$$f_{U,V}(u,v) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left(x^2+y^2+\mu^2+\gamma^2-2x\mu-2y\gamma\right)}$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left(\frac{x^2+y^2}{2}+\frac{x^2+y^2}{2}-2xy+2xy+\mu^2+\gamma^2-2x\mu-2y\gamma\right)}$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left(u^2 + v^2 + \mu^2 + \gamma^2 - 2x\mu - 2y\gamma\right)}$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left(u^2 + v^2 + \mu^2 + \gamma^2 - 2\frac{(u+v)}{2}\mu - 2\frac{(u-v)}{2}\gamma\right)}$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left(u^2 + v^2 + \mu^2 + \gamma^2 - u(\mu+\gamma) - v(\mu-\gamma)\right)}$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left(u^2 - 2u\left(\frac{\mu+\gamma}{2}\right) + \left(\frac{\mu+\gamma}{2}\right)^2 + v^2 - 2v\left(\frac{\mu-\gamma}{2}\right) + \left(\frac{\mu-\gamma}{2}\right)^2\right)}$$

$$= \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left[\left(u - \frac{\mu+\gamma}{2}\right)^2 + \left(v - \frac{\mu-\gamma}{2}\right)^2\right]}$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}\left(u - \left(\frac{\mu+\gamma}{2}\right)\right)^2} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}\left(v - \left(\frac{\mu-\gamma}{2}\right)\right)^2}$$

We can seperate U, V, so they are independent

$$\boxed{U \sim N\left(\frac{\mu+\gamma}{2}, \sigma\right), \quad V \sim N\left(\frac{\mu-\gamma}{2}, \sigma\right)}$$

4.28 Let X and Y be independent Standard Normal variables.

(a) Show that $\frac{X}{X+Y}$ has a Cauchy distribution.

Solution:

$$X \sim N(0,1) \qquad Y \sim N(0,1)$$

$$\left.\begin{aligned} U &= \frac{X}{X+Y} \\ V &= X+Y \end{aligned}\right\} \text{ one-to-one transformation.}$$

$$\left.\begin{aligned} U &= g_1(X,Y) = \frac{X}{X+Y} \\ V &= g_2(X,Y) = X+Y \end{aligned}\right\} \Rightarrow \begin{aligned} X &= UV \\ Y &= V(1-U) \end{aligned} \qquad J = \begin{vmatrix} V & U \\ -V & 1-U \end{vmatrix}$$

$$J = V(1-U) + VU$$

$$J = V$$

$$f_{U,V}(u,v) = \frac{1}{2\pi} e^{-\frac{1}{2}\left(u^2v^2 + v^2(1-u)^2\right)} \cdot v$$

$$f_{U,V}(u,v) = \frac{1}{2\pi} e^{-\frac{1}{2}v^2(u^2+1+u^2-2u)} \cdot |v|$$

$$f_{U,V}(u,v) = \frac{1}{2\pi} e^{-\frac{1}{2}v^2(2u^2-2u+1)} \cdot |v|$$

$$f_U(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |v| \cdot e^{-\frac{1}{2}v^2(2u^2-2u+1)} \, dv$$

$$f_U(u) = \frac{1}{2\pi} \int_{-\infty}^{0} -v \cdot e^{-\frac{1}{2}v^2(2u^2-2u+1)} \, dv$$

$$+ \frac{1}{2\pi} \int_{0}^{\infty} v \cdot e^{-\frac{1}{2}v^2(2u^2-2u+1)}$$

$$\frac{v^2}{2} = y \Rightarrow \quad v\,dv = dy$$

$$f_U(u) = \frac{1}{2\pi} \left[\int -e^{-y(2u^2-2u+1)} \, dy + \int e^{-y(2u^2-2u+1)} \, dy \right.$$

$$= \frac{1}{2\pi}\left[\left[\frac{e^{-\frac{v^2}{2}(2v^2-2v+1)}}{2v^2-2v+1}\right]_{-\infty}^{0} + \left[\frac{-e^{-\frac{v^2}{2}(2v^2-2v+1)}}{2v^2-2v+1}\right]_{0}^{\infty}\right]$$

$$= \frac{1}{2\pi}\left(\frac{1}{2v^2-2v+1} + \frac{1}{2v^2-2v+1}\right)$$

$$= \frac{1}{\pi}\cdot\frac{1}{2v^2-2v+1} =$$