# Properties of Probability measure

* we defined $\sigma$-algebra in general, and we defined measures in general, we define Probability measure as a special case of measures.

* $\sigma$-algebra of events essentially consists of certain subsets which are closed under complimentation and countabe union and countable intersection. Eelements of $\sigma$-algebra are called events, and we defined measures on these events.

## Properties

① Suppose $A$ is a subset of $\Omega$, $A \in \mathcal{F}$.

Then $$\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$$

**Proof:** $A$ & $A^c$ are disjoint

$$\Rightarrow A \cap A^c = \phi$$

and $A \cup A^c = \Omega$

$$\Rightarrow \quad \mathbb{P}(A \cup A^c) = \mathbb{P}(\Omega)$$

$$\Rightarrow \quad \mathbb{P}(A) + \mathbb{P}(A^c) = 1$$

(disjoint) countable additivity property

$$\Rightarrow \quad \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$$

(2) This can be generalizes to finite Additivity

if $A_1\ A_2\ A_3 \dots A_n \in \mathcal{F}$ are disjoint

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i)$$

(3) Monotonicity: if $A \subseteq B$, $A \in \mathcal{F}$, $B \in \mathcal{F}$, then

$$\mathbb{P}(A) \leq \mathbb{P}(B)$$

Proof:

$$B = A \cup (B \setminus A)$$

Now $A$ & $B \setminus A$ are disjoint

$$\Rightarrow \mathbb{P}(B) = \mathbb{P}(A \cup (B \setminus A))$$

$$\Rightarrow \mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A)$$

Probability measure is non-negative

$$\Rightarrow \mathbb{P}(B \setminus A) \geq 0$$

$$\Rightarrow \mathbb{P}(B) = \mathbb{P}(A) + \underbrace{\mathbb{P}(B \setminus A)}_{\geq 0}$$

$$\Rightarrow \mathbb{P}(B) \geq \mathbb{P}(A)$$

Probability of set $B$ is atleast Probability of $A$

4 $A, B \in \mathcal{F}$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

**Proof:**

$$A \cup B = A \cup (A^c \cap B)$$

Both $A$ and $A^c \cap B$ are disjoint

$$\Rightarrow \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(A^c \cap B)$$

$$B = (A \cap B) \cup (A^c \cap B) \text{ (disjoint)}$$

$$\Rightarrow \mathbb{P}(B) = \mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B)$$

$$\Rightarrow \mathbb{P}(A^c \cap B) = \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

$$\Rightarrow \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

in general, $A_1, A_2, \ldots A_n \in \mathcal{F}$ then

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j)$$

$$+ \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) + \cdots +$$

$$+ (-1)^{n+1} \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right)$$

Inclusive Exclusive rules

* Can be Prooved using Induction or Indicator Random variable.

⑤ if $A_1, A_2, \ldots A_n, \ldots \in \mathcal{F}$ then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{m\to\infty} \mathbb{P}\left(\bigcup_{i=1}^{m} A_i\right)$$

Continuity of Probability measure.

Proof:

$$B_1 = A_1$$

$$B_2 = A_2 \setminus A_1 \qquad (A_2 - A_1)$$

$$B_3 = A_3 \setminus (A_1 \cup A_2)$$

$$\vdots$$

$$B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$$

$$\vdots$$

Now we can show

Claim 1: $B_i \cap B_j = \phi \quad \forall i \neq j$

(disjoint)

Claim 2: $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$

$$\Rightarrow \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$$

$$\Rightarrow \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} B_i\right)$$

All $B_i$'s are disjoint

$$= \sum_{i=1}^{\infty} \mathbb{P}(B_i)$$

This infinite summation (series) is a limit of finite summation

$$= \lim_{m \to \infty} \sum_{i=1}^{m} \mathbb{P}(B_i)$$

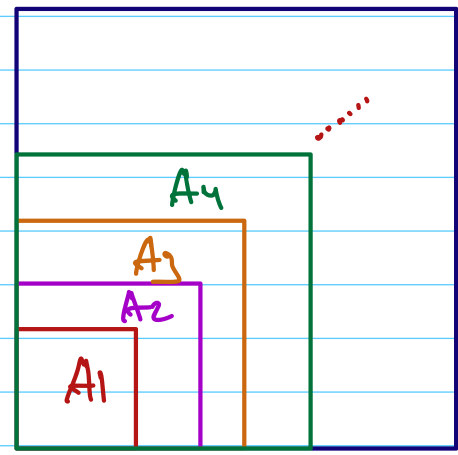
(finite additivity)

$$= \lim_{m \to \infty} \mathbb{P}\left(\bigcup_{i=1}^{m} B_i\right)$$

$$= \lim_{m \to \infty} \mathbb{P}\left(\bigcup_{i=1}^{m} A_i\right)$$

⑥ if $A_1, A_2, A_3 \ldots \in \mathcal{F}$ is a seq$^n$ of increasing nested events

i.e $A_i \subseteq A_{i+1}$ $\forall i \geq 1$ then




$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{m \to \infty} \mathbb{P}(A_m)$$

Proof:

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{m \to \infty} \mathbb{P}\left(\bigcup_{i=1}^{m} A_i\right)$$

Property 5

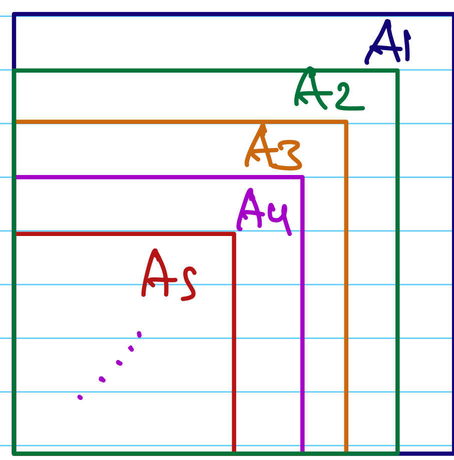
$\bigcup_{i=1}^{m} A_i = A_m$ because

$A_1 \subseteq A_2 \subseteq A_3 \cdots \subseteq A_m$

$$\Rightarrow \lim_{m \to \infty} \mathbb{P}\left( \bigcup_{i=1}^{m} A_i \right) = \lim_{m\to\infty} \mathbb{P}(A_m)$$

⑦ $A_1, A_2, A_3 \ldots \in \mathcal{F}$ nested decreasing events

$$A_1 \supseteq A_2 \supseteq A_3 \ldots$$




then

$$\mathbb{P}\left( \bigcap_{i=1}^{\infty} A_i \right) = \lim_{m\to\infty} \mathbb{P}(A_m)$$

## 8 Union Bound

Let $A_1, A_2, A_3, \ldots$ are events $\in \mathcal{F}$

then $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i)$

**Proof:**

$$B_1 = A_1$$

$$B_2 = A_2 \setminus A_1 \quad (A_2 - A_1)$$

$$B_3 = A_3 \setminus (A_1 \cup A_2)$$

$$\vdots$$

$$B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$$

$$\vdots$$

Now we can show

**Claim 1:** $B_i \cap B_j = \phi \quad \forall i \neq j$

(disjoint)

**Claim 2:** $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$

$$\Rightarrow \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} B_i\right)$$

all $B_i$'s are disjoint

$$= \sum_{i=1}^{\infty} \mathbb{P}(B_i)$$

Since $B_1 \subseteq A_1,\ B_2 \subseteq A_2 \ldots$

using property 2

$$\mathbb{P}(B_1) \leq \mathbb{P}(A_1)$$
$$\mathbb{P}(B_2) \leq \mathbb{P}(A_2)$$
$$\vdots$$

therefore using finite summation

$$\sum_{i=1}^{n} \mathbb{P}(B_i) \leq \sum_{i=1}^{n} \mathbb{P}(A_i)$$

then the Limit

$$\sum_{i=1}^{\infty} \mathbb{P}(B_i) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

$$\Rightarrow \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$