Example 4.3.4 (Sum & Difference of Normal Distribution)

Let X, and Y be independent, standard normal r.v's. Consider the transformation $U = X+Y$, and $V = X-Y$.

Soln

$$X, Y \sim N(0,1)$$

$$\left.\begin{aligned} U &= X+Y \\ V &= X-Y \end{aligned}\right\} \Rightarrow \quad \begin{aligned} x &= \frac{U+V}{2} \\ y &= \frac{U-V}{2} \end{aligned}, \quad J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$|J| = \frac{1}{2}$$

$$f(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$

$$f_{U,V}(u,v) = \frac{1}{2\pi} \cdot e^{-\frac{1}{2}\left(\left(\frac{u+v}{2}\right)^2 + \left(\frac{u-v}{2}\right)^2\right)} \cdot \frac{1}{2}$$

$$f_{U,V}(u,v) = \frac{1}{2\pi} \cdot \frac{1}{2} \cdot e^{-\frac{1}{2}\left(\frac{u^2}{2} + \frac{v^2}{2}\right)}$$

$$\Rightarrow f_{U,V}(u,v) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2}} \cdot e^{-\frac{1}{2}\frac{u^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2}} \cdot e^{-\frac{1}{2}\frac{v^2}{2}}$$

$$\Rightarrow f_{U,V}(u,v) = f_U(u) \cdot f_V(u)$$

$$\Rightarrow \quad U \sim N(0, 2)$$

$$V \sim N(0, 2)$$

Therefore

if $X, Y \sim N(0,1)$ independent, then $X+Y \sim N(0,2)$, $X-Y \sim N(0,2)$ independent

FACT: The sum's and difference of independent normal r.v's are independent normal r.v's, i.e. true regardless of the mean's of X, and Y, as long as $\operatorname{Var} X = \operatorname{Var} Y$.

Theorem 4.2.12:- Let X and Y be independent r.v's with MGF $M_X(t)$, $M_Y(t)$. Then the moment generating function of the r.v $Z = X+Y$ is given by

$$M_Z(t) = M_X(t)\, M_Y(t)$$

Proof:

$$\begin{aligned}
M_Z(t) &= \mathbb{E}\left[e^{tZ}\right] \\
&= \mathbb{E}\left[e^{t(X+Y)}\right] \\
&= \mathbb{E}\left[e^{tX} \cdot e^{tY}\right] \qquad X, Y \text{ are independent} \\
&= \mathbb{E}\left[e^{tX}\right] \mathbb{E}\left[e^{tY}\right] \\
&= M_X(t)\, M_Y(t)
\end{aligned}$$

Example 4.2.13:- (Mgf of a sum of normal variables)

Let $X \sim N(\mu, \sigma^2)$ and $Y \sim N(\gamma, \tau^2)$ be independent normal r.v's. From Exercise 2.33 the mgf's of X and Y are

$$M_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

$$M_Y(t) = \exp\left(\gamma t + \frac{\tau^2 t^2}{2}\right)$$

Mgf of $Z = X + Y$

$$M_Z(t) = M_X(t)\, M_Y(t)$$

$$= \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \exp\left(\gamma t + \frac{\tau^2 t^2}{2}\right)$$

$$= \exp\left((\mu + \gamma) t + \frac{(\sigma^2 + \tau^2) t^2}{2}\right)$$

$$\boxed{\Rightarrow \quad Z \sim N(\mu + \gamma, \sigma^2 + \tau^2)}$$

Theorem 4.2.14:- Let $X \sim N(\mu, \sigma^2)$ and $Y \sim N(\gamma, \tau^2)$ be independent normal r.v. Then the r.v $Z = X + Y$ has a $n(\mu + \gamma, \sigma^2 + \tau^2)$ distribution.

Example 4.3.6 (Distribution of the ratio of normal variables)

$X \sim N(0,1)$, $Y \sim N(0,1)$ independent normal random variables. Consider a transformation $U = \frac{X}{Y}$ and $V = |Y|$

Sol^n $U = \frac{X}{Y}$, $V = |Y|$ not a one-to-one

transformation as for both $(1,-1)$, $(1,1)$ yields same $(U,V) = (1,1)$ set.

$A_1 = \{(x,y) : y < 0\}$ $A_2 = \{(x,y) : y > 0\}$
$A_0 = \{(x,y) : y = 0\}$

for A_1

$$\left.\begin{array}{l} U = \frac{X}{Y} \\ V = -Y \end{array}\right\} \Rightarrow \begin{array}{l} X = -UV \\ Y = -V \end{array} \qquad J = \begin{vmatrix} -v & -u \\ 0 & -1 \end{vmatrix}$$

$|J| = V$

$U = (-\infty, \infty)$ $V = (0, \infty)$

for A_2

$$U = \frac{X}{Y}, \quad V = Y \Bigg\} \Rightarrow \quad X = UV, \quad Y = V$$

$$J = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

$$|J| = v$$

$$f_{X,Y} = \frac{1}{2\pi} e^{-\left(\frac{x^2+y^2}{2}\right)}$$

$$f_{U,V}(u,v) = \frac{1}{2\pi} e^{-\left(\frac{u^2v^2+v^2}{2}\right)} \cdot v \cdot 2$$

$$\boxed{\Rightarrow f_{U,V}(u,v) = \frac{v}{\pi} \cdot e^{-\left(\frac{v^2(u^2+1)}{2}\right)} \qquad -\infty < u < \infty, \ 0 < v < \infty}$$

Marginal Pdf of u

$$f_U(u) = \int_{v=0}^{v=\infty} \frac{v}{\pi} e^{-\frac{(u^2+1)v^2}{2}} \, dv$$

$$\frac{(u^2+1)v^2}{2} = N$$

$$(u^2+1) v \, dv = dN$$

$$f_U(u) = \int_{v=0}^{v=\infty} \frac{1}{\pi(u^2+1)} e^{-v} \, dv$$

$$f_U(u) = \frac{1}{\pi(u^2+1)} \left[-e^{-v} \right]_{v=0}^{v=\infty}$$

$$\Rightarrow f_U(u) = \frac{1}{\pi(u^2+1)} \qquad -\infty < u < \infty$$

Therefore:

if $X, Y \sim N(0,1)$ independent,

then $\frac{X}{Y} \sim \text{Cauchy}(0)$

Ratio of two - independent standard normal random variables is a Cauchy random variable.

Marginal distribution of v

$$f_{U,V}(u,v) = \frac{v}{\pi} e^{-\frac{(u^2+1)v^2}{2}}$$

$$f_V(v) = \int_{u=-\infty}^{\infty} \frac{v}{\pi} e^{-\frac{1}{2}(u^2+1)v^2} \, du$$

$$= \frac{v}{\pi} \int_{-\infty}^{\infty} e^{-\frac{u^2 v^2}{2}} \cdot e^{-\frac{v^2}{2}} \, du$$

$$= \frac{v}{\pi} e^{-\frac{v^2}{2}} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2 \cdot \left(\frac{1}{v}\right)^2} u^2} \, du}_{N\left(0, \frac{1}{v}\right)}$$

$$= \frac{\cancel{v}}{\pi} \cdot e^{-\frac{v^2}{2}} \cdot \frac{\sqrt{2\pi}}{\cancel{v}}$$

$$\boxed{f_V(v) = \sqrt{\frac{2}{\pi}} \, e^{-\frac{v^2}{2}} \qquad 0 < v < \infty}$$

$v \sim$ half-normal distribution

if $X \sim N(0,1)$ then $|X| \sim$ half Normal

Theorem 4.6.7 :-

Let $X_1, X_2, .. X_n$ be mutually independent random variables with mgf's $M_{X_1}(t), M_{X_2}(t) .. , M_{X_n}(t)$. Let $Z = X_1 + X_2 + .. + X_n$. Then the mgf of Z is

$$M_Z(t) = M_{X_1}(t) \, M_{X_2}(t) .. M_{X_n}(t)$$

in particular, if $X_1, .. X_n$ all have the same distribution with mgf $M_X(t)$, then

$$M_Z(t) = \left[M_X(t)\right]^n$$

Corollary 4.6.9 : let $X_1, X_2, .. X_n$ be mutually independent random variables with mgf's) $M_{X_1}(t), M_{X_2}(t), \dots M_{X_n}(t)$. Let $a_1, a_2 .. a_n$ and $b_1, b_2, .. b_n$ be fixed constants. Let $Z = (a_1 X_1 + b) + \dots + (a_n X_n + b_n)$

Then the mgf of Z is

$$M_Z(t) = e^{t(\sum b_i)} M_{X_1}(a_1 t) \cdots M_{X_n}(a_n t)$$

Proof:

$$M_Z(t) = \mathbb{E}[e^{tZ}]$$

$$= \mathbb{E}\left[e^{t\sum a_i x_i + b_i}\right]$$

$$= \mathbb{E}\left[e^{t\sum b_i} \cdot e^{t\sum a_i x_i}\right]$$

$$= \mathbb{E}\left[e^{t\sum b_i}\right] \cdot \mathbb{E}\left[e^{t a_1 x_1}\right] \mathbb{E}\left[e^{t a_2 x_2}\right] \cdots \mathbb{E}\left[e^{t a_n x_n}\right]$$

$$= e^{t\sum b_i} \cdot M_{X_1}(a_1 t) \cdot M_{X_2}(a_2 t) \cdots M_{X_n}(a_n t)$$

Corollary 4.6.10: Let $X_1, X_2, \ldots X_n$ be mutually independent random variables with $X_i \sim N(\mu_i, \sigma_i^2)$. Let $a_1, a_2 \ldots a_n$ and $b_1, \ldots b_n$ be fixed constants.

Then

$$Z = \sum_{i=1}^{n} (a_i X_i + b_i) \sim N\left(\sum_{i=1}^{n} (a_i \mu_i + b_i), \sum_{i=1}^{n} a_i^2 \sigma_i^2\right)$$

Proof: from corollary 4.6.9

$$M_Z(t) = e^{t\sum b_i} \cdot e^{\mu_1 a_1 t + \frac{\sigma_1^2 a_1^2 t^2}{2}} \cdot \ldots \cdot e^{\mu_n a_n t + \frac{\sigma_n^2 a_n^2 t^2}{2}}$$

$$M_Z(t) = e^{t \cdot \sum a_i \mu_i + b_i + \frac{t^2}{2} \sum \sigma_i^2 a_i^2}$$

$$\Rightarrow \quad Z \sim N\left(\sum_{i=1}^{n} (a_i \mu_i + b_i), \sum_{i=1}^{n} a_i^2 \sigma_i^2\right)$$

Exercise 4.19 (a):

Let x_1 and x_2 be independent $n(0,1)$ random variables. Find the pdf of $\frac{(X_1 - X_2)^2}{2}$

soln

$X, Y \sim N(0,1)$ independent

Using corollary 4.6.10 $Z = a_1 X_1 + b_1 + a_2 X_2 + b_2$

then $Z \sim N\left(a_1 \mu_1 + a_2 \mu_2 + b_1 + b_2 , a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2\right)$

$$\Rightarrow \quad Z = \frac{X_1}{\sqrt{2}} - \frac{X_2}{\sqrt{2}}$$

$$\Rightarrow \quad Z \sim N\left(\frac{1}{\sqrt{2}} \cdot 0 - \frac{1}{\sqrt{2}} \cdot 0 , \left(\frac{1}{\sqrt{2}}\right)^2 \cdot 1 + \left(\frac{1}{\sqrt{2}}\right)^2 \cdot 1\right)$$

$$Z \sim N(0, 1)$$

$$\Rightarrow \quad \frac{X_1 - X_2}{\sqrt{2}} \sim N(0,1)$$

$$Y = Z^2 \Rightarrow \quad Z = -\sqrt{y} \qquad z \in (-\infty, 0)$$

$$= \sqrt{y} \qquad z \in (0, \infty)$$

$$\Rightarrow \quad f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \cdot \frac{1}{2\sqrt{y}} \cdot 2 \qquad 0 < y < \infty$$

$$\Rightarrow \quad f_Y(y) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{y}} \cdot e^{-\frac{y}{2}} \qquad 0<y<\infty$$

$Y \sim$ Chisquare (1)

$$\Rightarrow \quad \boxed{\begin{aligned} &\frac{X_1 - X_2}{\sqrt{2}} \sim N(0,1) \\ &\left(\frac{X_1 - X_2}{\sqrt{2}}\right)^2 \sim \chi_1^2 \end{aligned}}$$

Exercise 4.20 (a)

X_1 and X_2 are independent $n(0, \sigma^2)$

J.V

(a) Find the distribution of Y_1 and Y_2, where

$$Y_1 = X_1^2 + X_2^2 \qquad Y_2 = \frac{X_1}{\sqrt{Y_1}}$$

Sol^n

$$\left.\begin{aligned} Y_1 &= X_1^2 + X_2^2 \\ Y_2 &= \frac{X_1}{\sqrt{X_1^2 + X_2^2}} \end{aligned}\right\} \text{Not a one-to-one transformation}$$

$$\Rightarrow \quad A_0 = \{ (x_1, x_2) : x_2 = 0 \}$$

$$A_1 = \{ (x_1, x_2) : x_2 < 0 \}$$

$$A_2 = \{ (x_1, x_2) : x_2 > 0 \}$$

<u>for A_1</u>

$$\left.\begin{array}{l} Y_1 = X_1^2 + X_2^2 \\ Y_2 = \dfrac{X_1}{\sqrt{Y_1}} \end{array}\right\} \Rightarrow \begin{array}{l} X_1 = \sqrt{Y_1} \cdot Y_2 \\ X_2 = -\sqrt{y_1(1-y_2^2)} \end{array}$$

$$J = \begin{vmatrix} \frac{1}{2}\frac{Y_2}{\sqrt{y_1}} & \sqrt{y} \\ -\frac{1}{2}\frac{1-y_2^2}{\sqrt{y_1(1-y_2^2)}} & \frac{y_1 y_2}{\sqrt{y_1(1-y_2^2)}} \end{vmatrix}$$

$$|J| = \frac{1}{2}\frac{y_2^2}{\sqrt{1-y_2^2}} + \frac{1}{2}\frac{1-y_2^2}{\sqrt{1-y_2^2}}$$

$$\Rightarrow |J| = \frac{1}{2}\frac{1}{\sqrt{1-y_2^2}}$$

Similar for A_2

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(x_1^2+x_2^2)}$$

$$\Rightarrow f_{Y_1,Y_2}(y_1,y_2) = \cancel{2}\cdot\frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(y_1\cancel{y_2^2}+y_1-\cancel{y_1}y_2^2)} \cdot \frac{1}{\cancel{2}\sqrt{1-y_2^2}}$$

$$\boxed{f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\cdot y_1} \cdot \frac{1}{\sqrt{1-y_2^2}}}$$

(b) Show that Y_1 and Y_2 are independent and interpret this result geometrically.

<u>Sol^n</u>

$$f_{y_1,y_2}(y_1,y_2) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\cdot y_1} \cdot \frac{1}{\sqrt{1-y_2^2}}$$

$$f_{y_1}(y_1) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}y_1} \int_{y_2=-1}^{1} \frac{1}{\sqrt{1-y_2^2}}\, dy_2$$

$$\Rightarrow \quad f_{y_1}(y_1) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2} y_1} \cdot \pi$$

$$\Rightarrow \quad f_{y_1}(y_1) = \frac{1}{2\sigma^2} e^{-\frac{1}{2\sigma^2} y_1}$$

$$\boxed{Y_1 = X_1^2 + X_2^2 \sim \text{gamma}\left(\frac{2}{2}, 2\sigma^2\right)}$$

<u>Exercise 4.27:</u>

Let $X \sim N(\mu, \sigma^2)$ and let $Y \sim n(\gamma, \sigma^2)$. Suppose X and Y are independent. Define $U = X + Y$ and $V = X - Y$. Show that U, V are independent normal r.v. Find the distribution of each of them.

<u>Sol^n</u>

$$\left.\begin{aligned} U &= X + Y \\ V &= X - Y \end{aligned}\right\} \Rightarrow \quad \begin{aligned} X &= \frac{U+V}{2} \\ Y &= \frac{U-V}{2} \end{aligned} \qquad J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}$$

$$|J| = \frac{1}{2}$$

$$f_{X,Y}(x,y) = \frac{1}{2\pi}\sigma^2 e^{-\frac{1}{2}\left(\frac{(x-\mu)^2+(y-\gamma)^2}{\sigma^2}\right)}$$

$$f_{U,V}(u,v) = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}\left(x^2+\mu^2-2x\mu+y^2+\gamma^2-2y\gamma\right)} \cdot \frac{1}{2}$$

$$f_{U,V}(u,v) = \frac{1}{2\pi \cdot 2\sigma^2} e^{-\frac{1}{2\sigma^2}\left(\frac{U^2+V^2}{2}+\mu^2+\gamma^2-2\mu\left(\frac{U+V}{2}\right)-2\gamma\left(\frac{U-V}{2}\right)\right)}$$

$$= \frac{1}{2\pi \cdot 2\sigma^2} e^{-\frac{1}{2\sigma^2}\left(\frac{U^2}{2}+\frac{V^2}{2}+\mu^2+\gamma^2-2(\mu+\gamma)\frac{U}{2}-2(\mu-\gamma)\frac{V}{2}\right)}$$

$$= \frac{1}{2\pi(\sqrt{2}\sigma)^2} e^{-\frac{1}{2}\frac{1}{(\sqrt{2}\sigma)^2}\left(U^2+V^2+2(\mu^2+\gamma^2)-2(\mu+\gamma)U-2(\mu-\gamma)V\right)}$$

$$= \frac{1}{2\pi(\sqrt{2}\sigma)^2} e^{-\frac{1}{2}\frac{1}{(\sqrt{2}\sigma)^2}\left(U^2+V^2+(\mu+\gamma)^2+(\mu-\gamma)^2-2(\mu+\gamma)U-2(\mu-\gamma)V\right)}$$

$$= \frac{1}{2\pi(\sqrt{2}\sigma)^2} e^{-\frac{1}{2}\frac{1}{(\sqrt{2}\sigma)^2}\left((U-(\mu+\delta))^2 + (V-(\mu-\delta))^2\right)}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{2}\sigma} e^{-\frac{1}{2(\sqrt{2}\sigma)^2}(U-(\mu+\delta))^2}$$

$$\cdot \frac{1}{\sqrt{2\pi}\sqrt{2}\sigma} e^{-\frac{1}{2(\sqrt{2}\sigma)^2}(V-(\mu-\delta))^2}$$

$\Rightarrow$ $U = X + Y \sim N(\mu+\delta, 2\sigma^2)$

$V = X - Y \sim N(\mu-\delta, 2\sigma^2)$

and Both U, V are independent as we are able seperate the Joint distribution.

Exercise 4.28 Let X and Y be independent standard normal r.v's.

(a) Show that $\frac{X}{X+Y}$ has a Cauchy distribution.

Solution:

$$\left.\begin{aligned} U &= \frac{X}{X+Y} \\ V &= X+Y \end{aligned}\right\} \text{ one-to-one transformation}$$

$$\left.\begin{aligned} U &= g_1(X,Y) = \frac{X}{X+Y} \\ V &= g_2(X,Y) = X+Y \end{aligned}\right\} \Rightarrow \begin{aligned} X &= UV \\ Y &= V(1-U) \end{aligned} \qquad J = \begin{vmatrix} V & U \\ -V & 1-U \end{vmatrix}$$

$$J = V(1-U) + VU$$

$$J = V$$

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$

$$f_{U,V}(u,v) = \frac{1}{2\pi} e^{-\frac{1}{2}\left(u^2v^2 + v^2(1-u)^2\right)} \cdot |v|$$

$$f_{U,V}(u,v) = \frac{|v|}{2\pi} e^{-\frac{1}{2}\left(u^2v^2 + v^2 + u^2v^2 - 2uv^2\right)}$$

$$f_{U,V}(u,v) = \frac{|v|}{2\pi} e^{-\frac{1}{2}v^2\left(2u^2 + 1 - 2u\right)}$$

$$= \frac{|v|}{2\pi} e^{-\frac{v^2}{2}} \cdot e^{-\frac{1}{2}\cdot 2\left(u^2 + \frac{1}{2} - u\right)}$$

$$= \frac{|v|}{2\pi} e^{-\frac{v^2}{2}} \cdot e^{-\frac{1}{2}\cdot\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2}\left[\left(u - \frac{1}{2}\right)^2 + \frac{1}{4}\right]}$$

$$= \frac{|v|}{2\pi} e^{-\frac{v^2}{2}} \cdot e^{-\frac{1}{2}\cdot\frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2}\left[u - 1/2\right]^2} \cdot e^{-\frac{1}{4}}$$