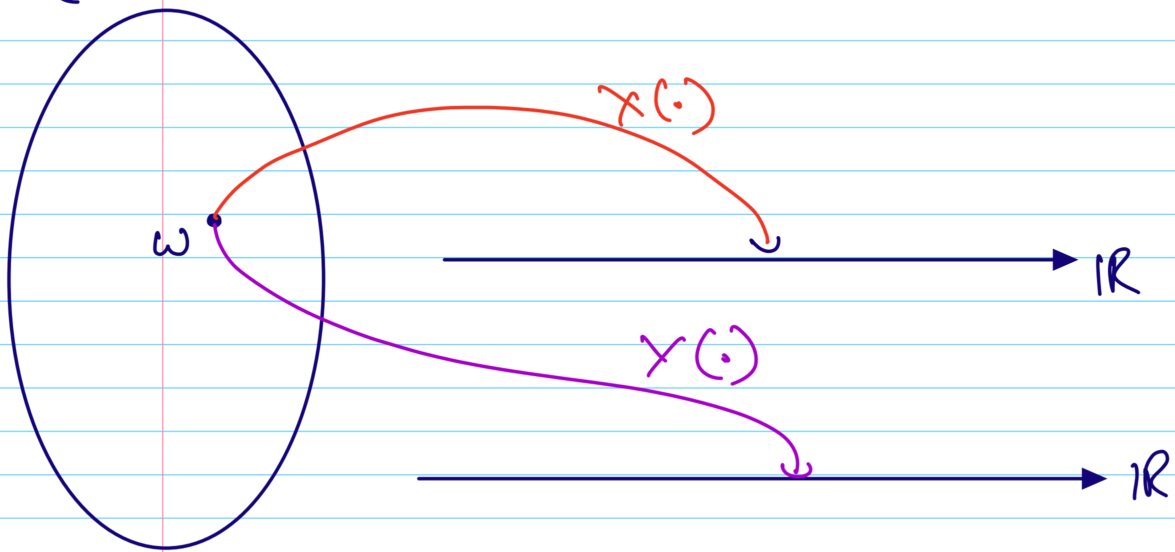
JOINT & MARGINAL Distribution

Definition 4.1.1 :

An n-dim random vector is a function from a sample space S into $\mathbb{R}^n$, n-dim Euclidean space

Multiple Random variables on $(\Omega, \mathcal{F}, \mathbb{P})$

$(\Omega, \mathcal{F}, \mathbb{P})$

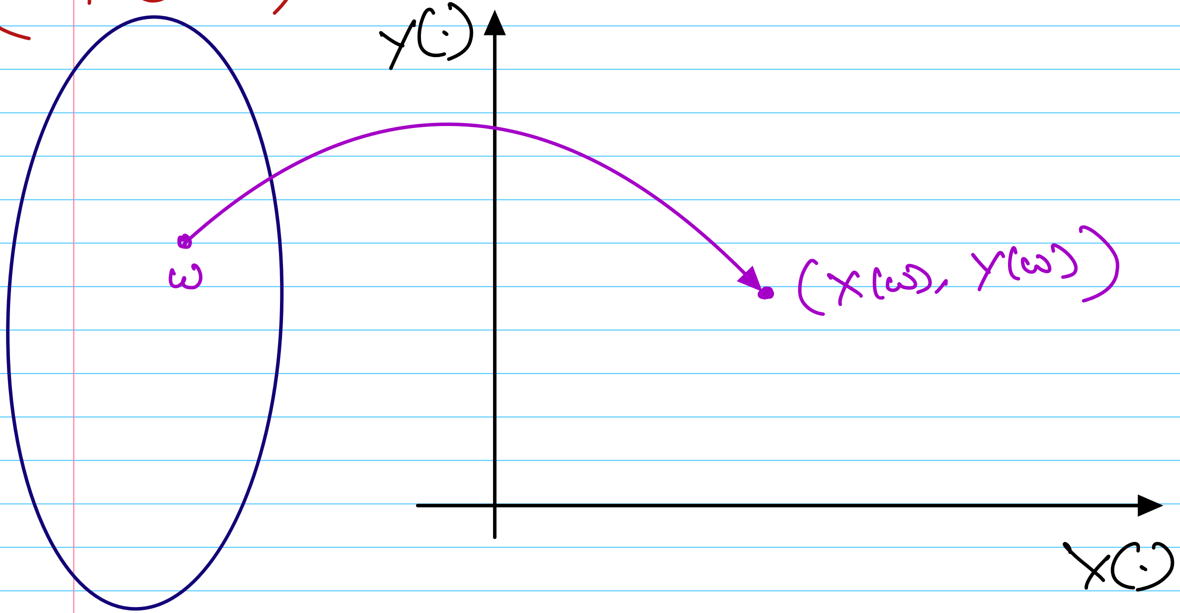


The Probability space is the same, the elementary outcome ω is the same, once the ω realizes $X(\cdot)$ captures Temperature (Example), $Y(\cdot)$ captures the Humidity (Example)

* The underlying Randomness feeding both X , Y are same.
* So in several r.v , the main issue is the capturing inter dependency b/w those r.v.
* The underlying randomness is same, if X(·) measuring # head's and Y(·) measuring # tails . we can clearly see the dependency b/w X & Y

for multiple random variable's

$(\Omega, \mathcal{F}, \mathbb{P})$



Example 4.1.2: consider the experiment of tossing two fair dice. $|\Omega| = 36$

$X =$ Sum of the two dice

$Y =$ | difference of two dice |

$$\mathbb{P}(X=5 \text{ and } Y=3)$$

$$= \mathbb{P}\left(\{(4,1),(1,4)\}\right) = \frac{2}{36} = \frac{1}{18}$$

$$\Rightarrow \mathbb{P}(X=5 \text{ and } Y=3) = \frac{1}{18}$$

we can write it as

$$\mathbb{P}(X=5, Y=3) = \frac{1}{18}$$ "," as AND

Definition 4.1.3: $f_{X,Y}(x,y)$

Let (X,Y) be a discrete bivariate random vector. The the Function $f(x,y)$ from $\mathbb{R}^2 \longrightarrow [0,1]$ defined by $f(x,y) = \mathbb{P}(X=x, Y=y)$ is called Joint PMF on (X,Y)

$$\mathbb{P}\left((X,Y)\in A\right) = \sum_{(x,y)\in A} f_{X,Y}(x,y)$$

where $A =$ event

Theorem 4.1.6 :- Let (X,Y) be a discrete bivariate random vector with joint pmf $f_{X,Y}(x,y)$. Then the marginal pmf's of X and Y,

$$f_X(x) = \mathbb{P}(X=x) \quad \text{and} \quad f_Y(y) = \mathbb{P}(Y=y)$$

$$f_X(x) = \sum_{y\in\mathbb{R}} f_{X,Y}(x,y) \quad \text{and}$$

$$f_Y(y) = \sum_{x\in\mathbb{R}} f_{X,Y}(x,y)$$

The marginal distribution's of X and Y, described by the marginal pmf's $f_X(x)$ and $f_Y(y)$, do not completely describe the joint distribution of X and Y. indeed, there are many different joint distribution's that have the same marginal distribution's.

The Joint pmf $f_{X,Y}(x,y)$ tells us additional information about the distribution of (X,Y) that is not found in the marginal distribution's.

Similarly for continuous case

$$\mathbb{P}((X,Y)\in A) = \iint_A f_{X,Y}(x,y)\,dx\,dy$$

$$\mathbb{E}(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\,dx\,dy$$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \qquad -\infty < x < \infty$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \qquad -\infty < y < \infty$$

Example 4.1.11

$$f(x,y) = \begin{cases} 6xy^2 & 0<x<1 \quad 0<y<1 \\ 0 & \text{o.w} \end{cases}$$

find $\mathbb{P}(X+Y \geq 1)$

Solution:

$$A = \{(x,y) : x+y \geq 1 ,\ 0<x<1,\ 0<y<1\}$$

$$= \{(x,y) : x \geq 1-y ,\ 0<x<1,\ 0<y<1\}$$

$$= \{(x,y) \mid 1-y<x<1 ,\ 0<y<1\}$$

$$\Rightarrow \mathbb{P}(X+Y \geq 1) = \iint_A f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_0^1 \int_{1-y}^1 6xy^2\,dx\,dy = \int_0^1 3x^2y^2\Big]_{1-y}^1 dy$$

$$= \int_0^1 \left(3y^2 - 3(1-y)^2y^2\right) dy$$

$$= \frac{9}{10}$$

Example 4.1.12

$$f(x,y) = \begin{cases} e^{-y} & , 0 < x < y < \infty \end{cases}$$

$$\Rightarrow f(x,y) = e^{-y} I_{\{(u,v):\, 0<u<v<\infty\}}(x,y)$$

The Joint Probability distribution of (X,Y) can be completely described with the Joint CDF.

$$F_{X,Y}(x,y) = \mathbb{P}(X \le x, Y \le y)$$

& for continuous case

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x,y)\, dy\, dx$$

$$\& \quad \frac{\partial^2 F(x,y)}{\partial x \, \partial y} = f(x,y)$$

4.2 Conditional distribution and Independence

Definition 4.2.1 :-

$$f(y|x) = \mathbb{P}(Y=y \mid X=x)$$

$$= \frac{\mathbb{P}(Y=y, X=x)}{\mathbb{P}(X=x)}$$

$$= \frac{f(x,y)}{f(x)}$$

where $\mathbb{P}(X=x) > 0$

$$\mathbb{E}[g(Y) \mid x] = \sum_{y} g(y) f(y|x) dx$$

Example 4.2.4

$$f(x,y) = e^{-y} \quad 0 < x < y < \infty$$

Soln

for $x > 0$, $f(x,y) = 0 \quad \forall\, y \leq x$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{x}^{\infty} e^{-y}\,dy = e^{-x}$$

<u>therefore:</u>

$$f(y|x) = \frac{f(x,y)}{f_x(x)}$$

$$\Rightarrow \quad f(y|x) = \begin{cases} \dfrac{e^{-y}}{e^{-x}} & \forall\, y > x \\ 0 & \forall\, y < x \end{cases}$$

Definition 4.2.5:- Let (X,Y) be a bivariate r.v with pdf or pmf $f(x,y)$ and marginal pdf's or pmf's as $f_X(x)$ $f_Y(y)$. Then X, Y are called independent r.v if $\forall x \in \mathbb{R}$, $\forall y \in \mathbb{R}$

$$f(x,y) = f_X(x)\, f_Y(y)$$

$$\Rightarrow f(y|x) = \frac{f(x,y)}{f_X(x)}$$

$$= \frac{f_X(x)\, f_Y(y)}{f_X(x)}$$

$$= f_Y(y)$$

$$\Rightarrow f(y|x) = f_Y(y)$$

Theorem 4.2.10 :- Let X and Y be independent random variables.

(a) for any $A \subset \mathbb{R}$, $B \subset \mathbb{R}$,

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\,\mathbb{P}(Y \in B);$$

that is events $\{X \in A\}$ and $\{Y \in B\}$ are independent events.

(b) Let $g(x)$ be a function of x and $h(y)$ be a function of y, then

$$\mathbb{E}[g(X)\,h(Y)] = E[g(X)] \cdot \mathbb{E}[h(Y)]$$

Theorem 4.2.12 :- if X, Y are independen r.v with MGF $M_X(t)$ $M_Y(t)$. Then the MGF of $Z = X + Y$ is

$$M_Z(t) = M_X(t)\,M_Y(t)$$

Proof:

$$M_Z(t) = \mathbb{E}\left[e^{tZ}\right]$$

$$= \mathbb{E}\left[e^{tX} \cdot e^{tY}\right]$$

$$= \mathbb{E}\left[e^{tX}\right] \mathbb{E}\left[e^{tY}\right]$$

$$= M_X(t) \; M_Y(t)$$