

## LEC 02 - Sequences and Limits

Sequence: A seq<sup>n</sup> of real number's when we have a map from  $\mathbb{N}$  to  $\mathbb{R}$

a map  $a: \mathbb{N} \rightarrow \mathbb{R}$

or  $a: \mathbb{N}_0 \rightarrow \mathbb{R}$

Notation:  $(a_1, a_2, \dots)$  infinite List of number's

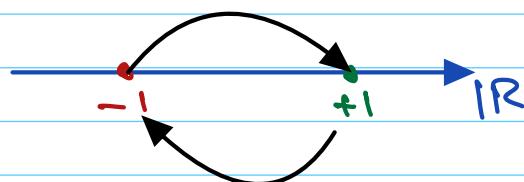
$(a_n)_{n \in \mathbb{N}}$  or  $(a_n)_{n=1}^\infty$

or  $(a_n)$

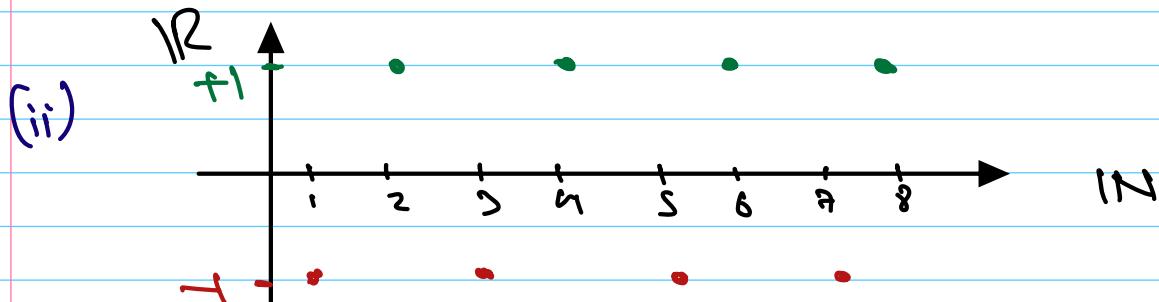
Example: (i)  $(a_n)_{n \in \mathbb{N}} = (-1)^n$

$$= (-1, 1, -1, 1, -1, 1, \dots)$$

(i)



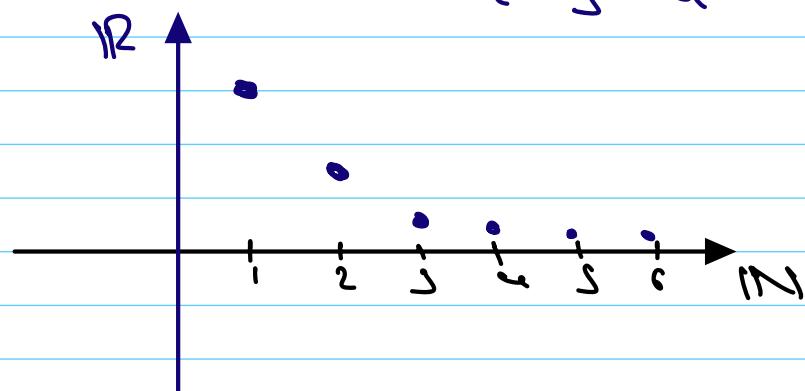
(ii)



we are interested in what happens to the values of the seqn when  $n$  goes to infinity.

$$(b) (a_n)_{n \in \mathbb{N}} = \left(\frac{1}{n}\right)_{n \in \mathbb{N}}$$

$$= \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right)$$



we will see  $\lim_{n \rightarrow \infty} a_n = 0$

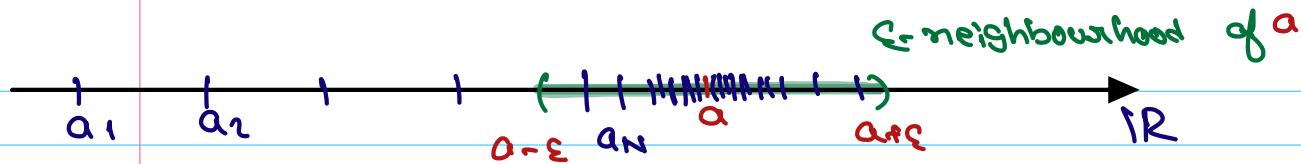
$$(c) (a_n)_{n \in \mathbb{N}} = (2^n)_{n \in \mathbb{N}}$$

$$= (2, 4, 8, 16, 32, 64, \dots)$$

### Definition:

A sequence  $(a_n)_{n \in \mathbb{N}}$  is called convergent to a  $a \in \mathbb{R}$ , if  $\forall \varepsilon > 0 \exists N \in \mathbb{N}, \forall n \geq N$

$$\therefore |a_n - a| < \varepsilon$$



we need to get closer and closer to  $a$  with the seq<sup>n</sup> member's. Eventually all the seq<sup>n</sup> member's have to lie  $\Sigma$ -neighbourhood of  $a$ , only finitely many can lie outside.

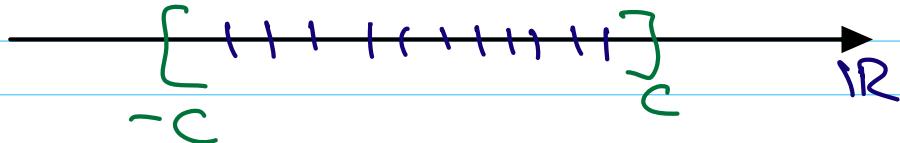
No matter how small the  $\Sigma$  is, there exist a  $N \in \mathbb{N}$ , (always works)

if there is no such  $a \in \mathbb{R}$ , we call the seq<sup>n</sup>  $(a_n)_{n \in \mathbb{N}}$  divergent.

## Lec 03 - Bounded seq<sup>n</sup> and Unique Limit

Definition: A seq<sup>n</sup>  $(a_n)_{n \in \mathbb{N}}$  is called Bounded if  $\exists C \in \mathbb{N} \ \forall n \in \mathbb{N}$

$$|a_n| \leq C$$



Otherwise, the seq is called unbounded.

Important fact:

$$(a_n)_{n \in \mathbb{N}} \text{ convergent} \Rightarrow (a_n)_{n \in \mathbb{N}} \text{ bounded}$$

$(a_n)_{n \in \mathbb{N}}$  convergent  $\Rightarrow$  There is only one limit  $a \in \mathbb{R}$

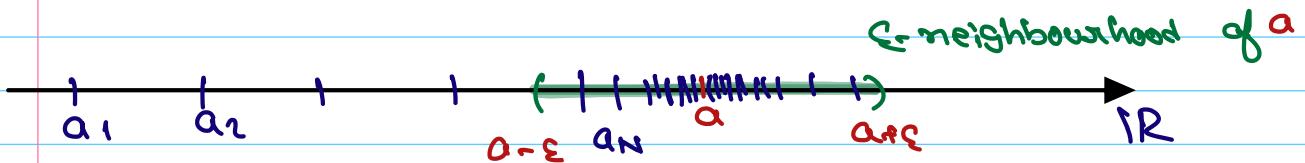
$$\lim_{n \rightarrow \infty} a_n = a$$

## Lec04 - Theorem on Limits

$(a_n)_{n \in \mathbb{N}}$  convergent to  $a \in \mathbb{R}$ :

$$\lim_{n \rightarrow \infty} a_n = a$$

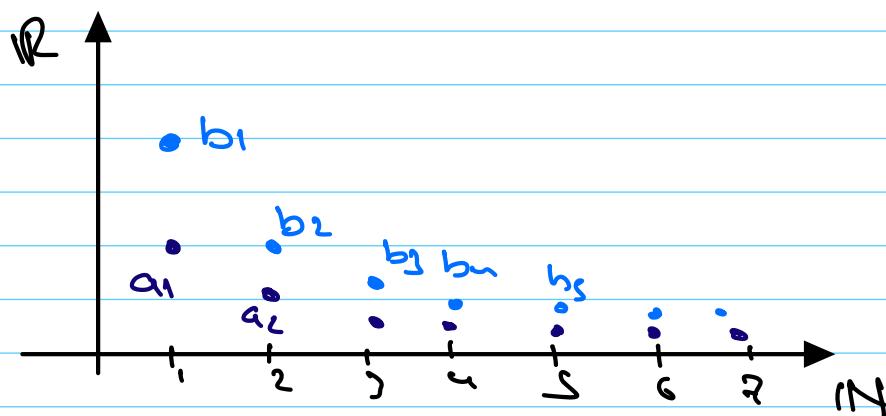
$$a_n \xrightarrow{n \rightarrow \infty} a$$



for all  $\epsilon$ , the seq<sup>n</sup> members lie inside this  
 $\epsilon$ -neighbourhood of  $a$  eventually.

## Theorem's on Limits:

$(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$  convergent sequences.



Then

$$(a) \lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$$

We can pull in the limit when we have a sum, provided that these two seq<sup>n</sup> and their limits actually exists.

(b)

$$\lim_{n \rightarrow \infty} a_n \cdot b_n = \lim_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$$

(c)

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n \neq 0}$$

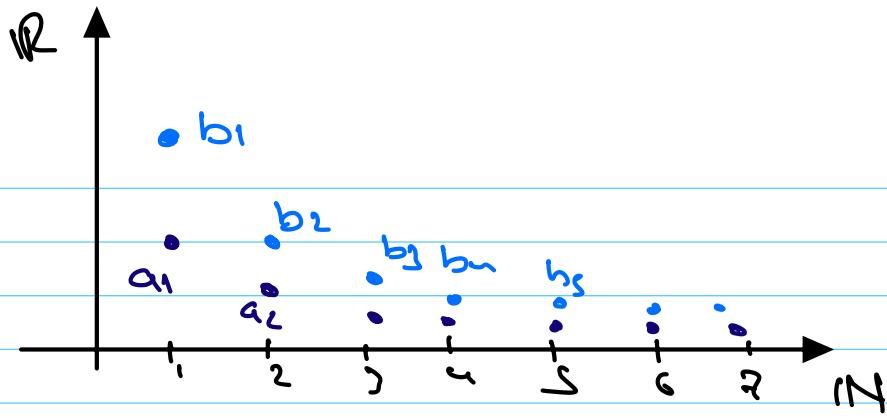
### Lec 05: Sandwich theorem

$(a_n)_{n \in \mathbb{N}}$ ,  $(b_n)_{n \in \mathbb{N}}$  convergent seq<sup>n</sup>

then  $\lim_{n \rightarrow \infty} (a_n \cdot b_n) = a \cdot \lim_{n \rightarrow \infty} b_n$

Properties:

(a) Monotonicity  $a_n \leq b_n \forall n \in \mathbb{N}$



$$\Rightarrow \lim_{n \rightarrow \infty} a_n \leq \lim_{n \rightarrow \infty} b_n$$

(b) Sandwich theorem

$$a_n \leq c_n \leq b_n \quad \forall n \in \mathbb{N} \text{ and}$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$$

$\Rightarrow (c_n)_{n \in \mathbb{N}}$  converges with

$$\lim_{n \rightarrow \infty} c_n = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$$

Proof of (b):

$$(b_n - a_n) \xrightarrow{n \rightarrow \infty} \lim_{n \rightarrow \infty} b_n - \lim_{n \rightarrow \infty} a_n = 0$$

$$d_n := c_n - a_n$$

$$\Rightarrow 0 \leq d_n \leq b_n - a_n$$

Let  $\varepsilon > 0$ . Then  $\exists N \in \mathbb{N}$ ,  $\forall n \geq N$

$$|b_n - a_n| < \varepsilon$$

$$\Rightarrow |d_n - 0| < \varepsilon$$

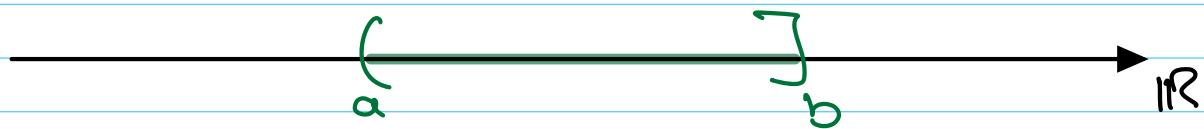
$\Rightarrow d_n$  is convergent with limit 0.

$$\Rightarrow (c_n)_{n \in \mathbb{N}} = (d_n + a_n)_{n \in \mathbb{N}}$$

is convergent with limit a.

## Lec 06 - Supremum & Infimum

Real Number's are ordered



interval

$$(a, b] := \{x \in \mathbb{R} \mid a < x \leq b\}$$

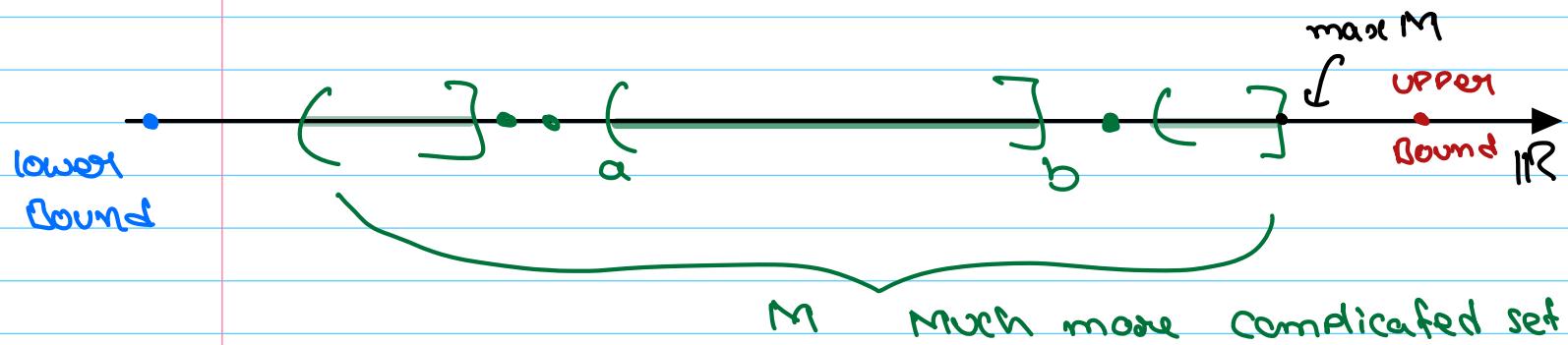
similarly

$$[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

$$[a, \infty) := \{x \in \mathbb{R} \mid a \leq x\}$$

$$(-\infty, b) := \{x \in \mathbb{R} \mid x < b\}$$

these intervals are very nice subsets of real numbers. (Not the only ones)



Definition: For a subset  $M \subseteq \mathbb{R}$ :  $b \in \mathbb{R}$  is

called an upper bound for  $M$  if

$$\forall x \in M : x \leq b$$

For  $a \in \mathbb{R}$ , is called a lower bound for  $M$

$$\text{if } \forall x \in M : x \geq a$$

\* if upper bound exists we call the set  $M$

is bounded from above

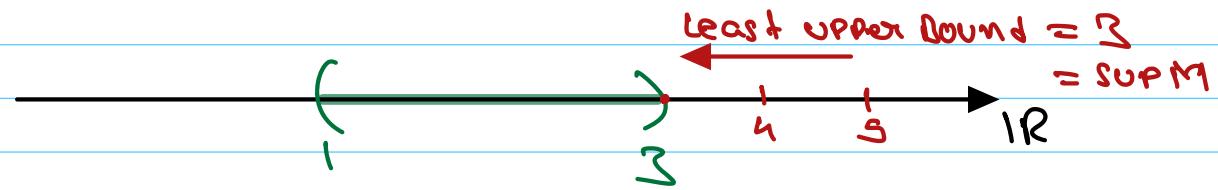
\* if lower bound exists, the set  $M$  called  
bounded from below.

Both  $\rightarrow$  set  $M$  is bounded.

\* if  $b$  is an upper bound for  $M$  and  $b \in M$ ,  
then  $b$  is called a maximal element of  $M$ .  
 $b = \max M$

\* if  $a$  is an lower bound for  $M$  and  $a \in M$ ,  
then  $a$  is called a minimal element of  $M$ .  
 $a = \min M$

Ex:  $M = (1, 3)$ ,  $\max M$ ,  $\min M$  do not exist



$\max M$ ,  $\min M$  do not exist, because  $1, 3 \notin M$

Definition: for a subset  $M \subseteq \mathbb{R}$ ;  $s \in \mathbb{R}$  is

called supremum of  $M$  if

\*  $\forall x \in M : x \leq s$  (upper bound for  $M$ )

\*  $\forall \varepsilon > 0, \exists x \in M : s - \varepsilon < x$

( $s - \varepsilon$  is no upper bound for  $M$ )

Supremum = least upper bound.

$\text{SUP } M := s$  or  $\text{SUP } M := \infty$

(if  $M$  is not bounded from above)

or  $\text{SUP } \emptyset := -\infty$

Definition: for a subset  $M \subseteq \mathbb{R}$ ;  $l \in \mathbb{R}$  is

called infimum of  $M$  if

\*  $\forall x \in M : x \geq l$  (lower bound for  $M$ )

\*  $\forall \varepsilon > 0, \exists \tilde{x} \in M : l + \varepsilon > \tilde{x}$

( $l + \varepsilon$  is no lower bound for  $M$ )

Then  $\inf M = l$  or  $\inf M = -\infty$

(if  $M$  is not bounded from below)

or  $\inf \emptyset = +\infty$

## Lec 07: Cauchy seq<sup>n</sup> & completeness

$(a_n)_{n \in \mathbb{N}}$  is convergent seq<sup>n</sup>

which mean's there is a number

$a$ , s.t. the seq member's here get

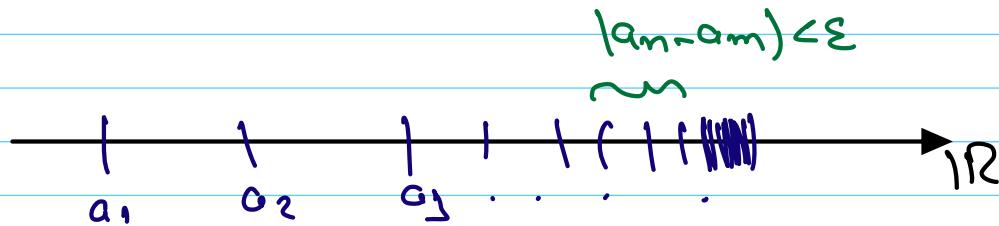
arbitrarily close to this number  $a$

eventually

$(a_n)_{n \in \mathbb{N}} \Rightarrow$  (there is a limit  $a = \lim_{n \rightarrow \infty} a_n$ )

The Problem with this definition is  
we need to know the value of limit to  
show the convergence.

Different idea:



The seq'n member's gets closer and closer to  
each other.

$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n, m > N$  : we have

$$|a_n - a_m| < \varepsilon$$

(Cauchy sequence)

Definition: if  $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ , such that  $\forall n, m > N$

$|a_n - a_m| < \varepsilon$ , then  $(a_n)_{n \in \mathbb{N}}$  is called  
Cauchy seq'n

Important fact: For a seq<sup>n</sup> of real numbers's

Cauchy seq<sup>n</sup>  $\iff$  Convergent seq<sup>n</sup>

Completeness Axiom: There are no holes  
in our Complete Real Number line.

When we are working real number's, we  
don't have to distinguish Cauchy seq<sup>n</sup> and  
convergent seq<sup>n</sup>. They have different  
definition, But for real number's they  
mean the same thing.

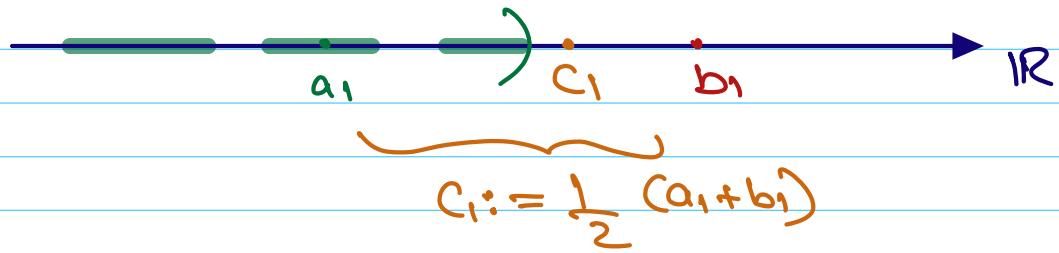
Dedekind Completeness:

if  $M \subseteq \mathbb{R}$  is bounded  
from above, then  $\sup M \in \mathbb{R}$  (exists)

if  $M \subseteq \mathbb{R}$  is bounded from below,

then  $\inf M \in \mathbb{R}$  (exists).

Proof:



Two Cases: (1) if  $c_1$  is an upper bound

$$\text{for } M: b_2 := c_1$$

$$a_2 := a_1$$

(2) if  $c_1$  is not an upper

bound for  $M$ :  $\exists x \in M : x > c_1$

$$b_2 := b_2$$

$$a_2 := x$$

$$c_n := \frac{1}{2}(a_n + b_n)$$

⋮

Important fact:

if  $(a_n)_{n \in \mathbb{N}}$  is monotonically decreasing

$(a_{n+1} \leq a_n \forall n)$  and bounded from below

(the set  $\{a_n\}_{n \in \mathbb{N}}$  has a lower bound)

then  $(a_n)_{n \in \mathbb{N}}$  is convergent.

### Lec 08:

Important FACT: if  $(a_n)_{n \in \mathbb{N}}$  is monotonically increasing ( $a_{n+1} > a_n \forall n$ ) and bounded from above (the set  $\{a_n\}_{n \in \mathbb{N}}$  has an upper bound) then

$(a_n)_{n \in \mathbb{N}}$  is convergent.

(Monotone Convergence Criterion)

Ex:  $(a_n)_{n \in \mathbb{N}} = \left( \left(1 + \frac{1}{n}\right)^n \right)_{n \in \mathbb{N}}$  is convergent.

Proof:

monotonicity:-

$$\frac{a_{n+1}}{a_n} = \frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^n}$$

$$= \left(1 + \frac{1}{n}\right) \cdot \left(\frac{1 + \frac{1}{n+1}}{1 + \frac{1}{n}}\right)^{n+1}$$

$$= \left(1 + \frac{1}{n}\right) \left(\frac{n+2}{n+1} \cdot \frac{n}{n+1}\right)^{n+1}$$

$$= \left(1 + \frac{1}{n}\right) \left(\frac{n^2 + 2n}{n^2 + 2n + 1}\right)^{n+1}$$

$$= \left(1 + \frac{1}{n}\right) \left(1 - \frac{1}{(n+1)^2}\right)^{n+1}$$

Bernoulli inequality:

for  $k \in \mathbb{N}$  and  $x \geq -1$

$$(1+x)^k \geq 1+kx$$

$$\Rightarrow \left(1 + \frac{1}{n}\right) \left(1 - \frac{1}{(n+1)^2}\right)^{n+1}$$

$$\geq \left(1 + \frac{1}{n}\right) \left(1 - (n+1)\left(\frac{1}{(n+1)^2}\right)\right)$$

$$\geq \left(1 + \frac{1}{n}\right) \left(1 - \frac{1}{n+1}\right)$$

$$\geq \left(\frac{n+1}{n}\right) \left(\frac{n}{n+1}\right) = 1$$

$$\Rightarrow \frac{a_{n+1}}{a_n} > 1 \quad (\text{monotone increasing})$$

Now we need to show the seq'n  
is bounded from above.

$$\Leftrightarrow a_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^n \binom{n}{k} \left(\frac{1}{n}\right)^k$$

$$= \left(\frac{1}{n}\right)^0 + \binom{n}{1} \left(\frac{1}{n}\right)^1 + \binom{n}{2} \left(\frac{1}{n}\right)^2 \dots$$

$$= 1 + 1 + \sum_{k=2}^n \binom{n}{k} \left(\frac{1}{n}\right)^k$$

$$\binom{n}{k} \left(\frac{1}{n}\right)^k = \frac{n!}{(n-k)! k!} \cdot \frac{1}{n^k}$$

$$= \frac{n \cdot (n-1) \cdot (n-2) \dots \cdot (n-k+1)}{n \cdot n \cdot n \cdot \dots \cdot n} \cdot \frac{1}{k!}$$

$$\leq 1$$

$$\Rightarrow \left(1 + \frac{1}{n}\right)^n \leq 3$$

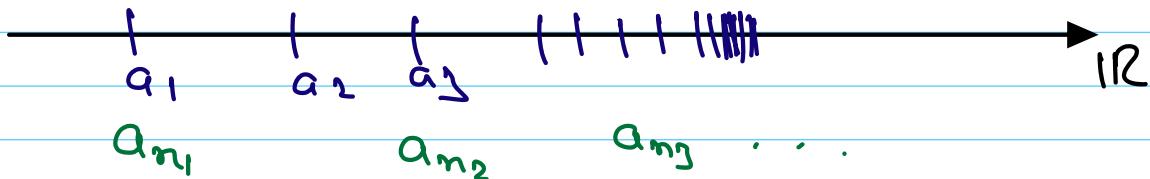
fact

The seq<sup>n</sup>  $(a_n)_{n \in \mathbb{N}}$  is convergent.

monotone  
convergence  
criterion

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \quad \text{Euler's Number}$$

## Lec 09 :- Subsequences and Limit Points



Omit some member's of the seq<sup>n</sup>.

Let  $(n_k)_{k \in \mathbb{N}}$  be a seq<sup>n</sup> of natural numbers that is strictly monotonically increasing ( $\forall k \in \mathbb{N} : n_{k+1} > n_k$ )

then  $(a_{n_k})_{k \in \mathbb{N}}$  is called an subsequence of  $(a_n)_{n \in \mathbb{N}}$

Eg:  $(a_n)_{n \in \mathbb{N}}$  given by  $a_n = \frac{1}{n}$

$$(a_{n_k})_{k \in \mathbb{N}} = (a_{2^k})_{k \in \mathbb{N}}$$

$$= (a_2, a_4, a_6, a_8, \dots)$$

$$= (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots)$$

if we have a monotonic seq<sup>n</sup>  $(a_n)_{n \in \mathbb{N}}$

then the subseq<sup>n</sup>  $(a_{n_k})_{k \in \mathbb{N}}$  is also

monotonic in the same way.

If we can omit lot of seq<sup>n</sup> member's even infinitely many as long as infinitely many remain

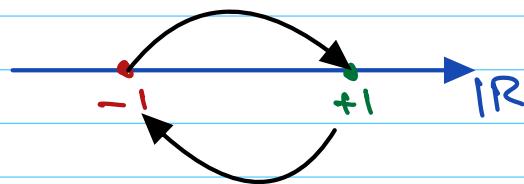
FACT:  $(a_n)_{n \in \mathbb{N}}$  is convergent with

$\lim_{n \rightarrow \infty} a_n = a$ , then

$\Rightarrow$  Every Subsequence  $(a_{n_k})_{k \in \mathbb{N}}$

is convergent  $\lim_{k \rightarrow \infty} a_{n_k} = a$

Ex:  $(a_n)_{n \in \mathbb{N}}$  given by  $a_n = (-1)^n$



Subseqn  $(a_{n_k})_{k \in \mathbb{N}} = (a_{2k})_{k \in \mathbb{N}}$

$$= (1, 1, 1, \dots)$$

$\lim_{k \rightarrow \infty} a_{2k} = 1$  (limit 1)

$(a_{n_k})_{k \in \mathbb{N}} = (a_{2k+1})_{k \in \mathbb{N}}$

$\lim_{k \rightarrow \infty} a_{2k+1} = -1$  (limit -1)

Def:  $a \in \mathbb{R}$  is called an accumulation

value of  $(a_n)_{n \in \mathbb{N}}$  if there is a

subseqn  $(a_{n_k})_{k \in \mathbb{N}}$  with  $\lim_{k \rightarrow \infty} a_{n_k} = a$

Accumulation value is a generalization  
of term limit.

=> A convergent seq<sup>n</sup> can only have  
one accumulation value = Limit

=> However for divergent seq<sup>n</sup> could  
have different accumulation values



The limit is getting closer to 4 different  
points. we don't have a limit because  
the seq<sup>n</sup> still jumps around, But  
we get closer and closer to different  
accumulation values.

=> we could restrict ourselves to take  
seq<sup>n</sup> members that only live  
in one convergent seq<sup>n</sup> (limit)

Accumulation value is just a point on  
the number line, where the seq<sup>n</sup>  
accumulates

$a \in \mathbb{R}$  is an accumulation value of

$(a_n)_{n \in \mathbb{N}}$

$\iff \forall \varepsilon > 0$ : The  $\varepsilon$ -neighbourhood of  $a$   
contains infinitely many seq<sup>n</sup> members  
of  $(a_n)_{n \in \mathbb{N}}$ .

## Lec 10: Bolzano-Weierstrass

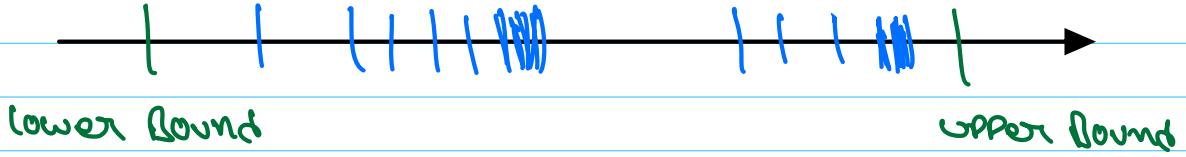
$(a_n)_{n \in \mathbb{N}}$  bounded  $\Rightarrow (a_n)_{n \in \mathbb{N}}$  has an

accumulation value

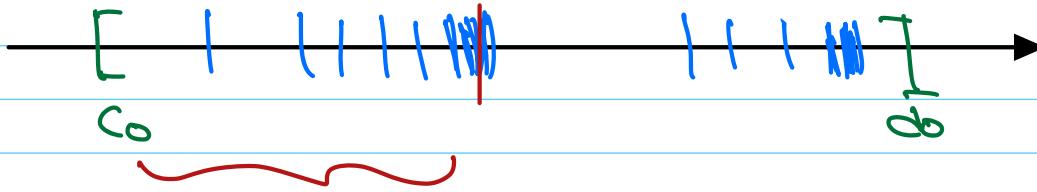
(has a convergent subseq<sup>n</sup>)

having a accumulation value means

the seq<sup>n</sup> have a convergent subseq<sup>n</sup>.



Proof:



if infinitely many seq'n member's win it:

Choose left-hand interval

otherwise: choose - right hand interval.

New interval:



we set  $[c_0, d_0] \supset [c_1, d_1] \supset [c_2, d_2]$

.....

$$\text{and } d_1 - c_1 = \frac{1}{2} (d_0 - c_0)$$

$$d_2 - c_2 = \frac{1}{2} (d_1 - c_1)$$

$$d_3 - c_3 = \frac{1}{2} (d_2 - c_2)$$

⋮

$$d_n - c_n = \frac{1}{2^n} (d_0 - c_0) \xrightarrow{n \rightarrow \infty} 0$$

we know

$(c_n)_{n \in \mathbb{N}}$  monotonically increasing and bounded

$(d_n)_{n \in \mathbb{N}}$  monotonically decreasing and bounded

$(c_n)_{n \in \mathbb{N}}, (d_n)_{n \in \mathbb{N}}$

Convergent

By Limit theorem

$$0 = \lim_{n \rightarrow \infty} (d_n - c_n)$$

$$\Rightarrow \lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} c_n$$

## Lec 11: Limit Superior and Limit Inferior

Ex:  $(a_n)_{n \in \mathbb{N}}$  given by  $a_n = n$



$(a_n)_{n \in \mathbb{N}} = (n)_{n \in \mathbb{N}}$  is divergent to  $\infty$

$\infty$ : symbol (not as a number)

divergent to  $\infty$  :  $\Leftrightarrow \forall C > 0 \exists N \in \mathbb{N}$

$\lim_{n \rightarrow \infty} a_n = \infty$

$\forall n \geq N : a_n > C$

divergent to  $-\infty$  :  $\Leftrightarrow \forall C < 0, \exists N \in \mathbb{N}$

$\lim_{n \rightarrow \infty} a_n = -\infty$

$\forall n \geq N : a_n < C$

$(a_n)_{n \in \mathbb{N}}$  has the improper accumulation

value  $\infty$  :  $\Leftrightarrow (a_n)_{n \in \mathbb{N}}$  is not

bounded from above

$(a_n)_{n \in \mathbb{N}}$  has the improper accumulation value  $\infty$

$\Leftrightarrow (a_n)_{n \in \mathbb{N}}$  is not bounded

from below.

A given seq<sup>n</sup>  $(a_n)_{n \in \mathbb{N}}$  could have many accumulation values:



\* we can talk about largest and smallest accumulation value, it could be a normal one or improper one.

Def: Let  $(a_n)_{n \in \mathbb{N}}$  be a seq<sup>n</sup> of real numbers. An element  $a \in \mathbb{R} \cup \{-\infty, \infty\}$  is called

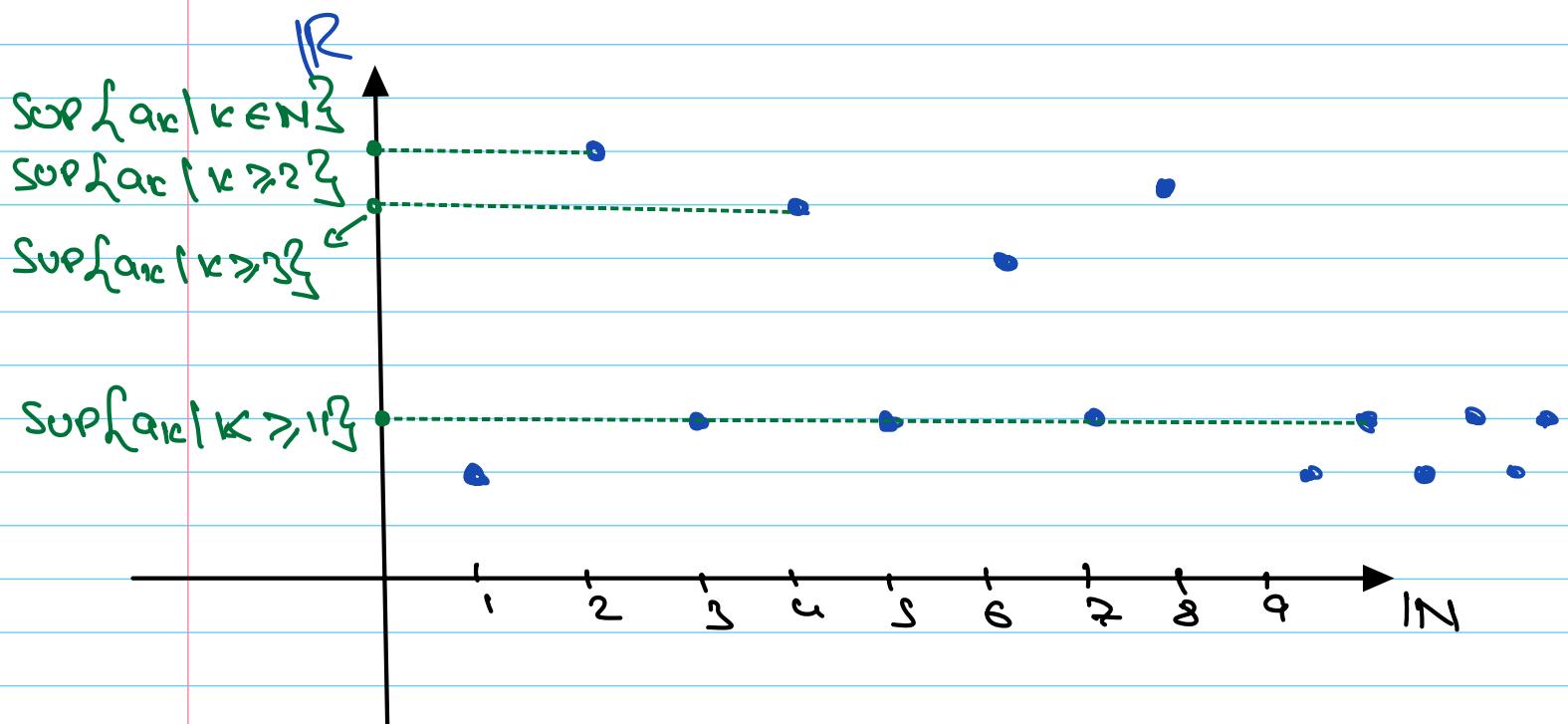
limit superior of  $(a_n)_{n \in \mathbb{N}}$

if  $a$  is the largest (improper) accumulation value of  $(a_n)_{n \in \mathbb{N}}$

$$a := \limsup_{n \rightarrow \infty} a_n$$

• limit inferior of  $(a_n)_{n \in \mathbb{N}}$  if  
 $a$  is the smallest (improper)  
 accumulation value of  $(a_n)_{n \in \mathbb{N}}$ .

Write  $a = \liminf_{n \rightarrow \infty} a_n$



Limit Superior describes what happen to  
 with the supremum, when we cut off more and  
 more at the beginning of the seq

FACT:

$$\limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sup \{a_n \mid k > n\}$$

$$\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \inf \{a_n \mid k > n\}$$

## LEC 12

let  $(a_n)_{n \in \mathbb{N}}$  be a seqn of real numbers

$$\Rightarrow \limsup_{n \rightarrow \infty} a_n, \quad \liminf_{n \rightarrow \infty} a_n \in \mathbb{R} \cup \{\pm\infty\}$$
$$= [-\infty, \infty]$$

Ex:

$$(a_n)_{n \in \mathbb{N}} = ((-1)^n \cdot n)_{n \in \mathbb{N}}$$

$$= (-1, 2, -3, 4, -5, \dots)$$

$$\limsup_{n \rightarrow \infty} a_n = \infty$$

$$\liminf_{n \rightarrow \infty} = -\infty$$

### Properties:

(a)  $(a_n)_{n \in \mathbb{N}}$  is convergent

$$\iff \limsup_{n \rightarrow \infty} = \liminf_{n \rightarrow \infty} \notin \{-\infty\}$$

(b)  $(a_n)_{n \in \mathbb{N}}$  is divergent to  $\infty$

$$\iff (\limsup_{n \rightarrow \infty} a_n) = \liminf_{n \rightarrow \infty} a_n = \infty$$

(c)  $(a_n)_{n \in \mathbb{N}}$  is divergent to  $-\infty$

$$\iff \limsup_{n \rightarrow \infty} a_n = (\liminf_{n \rightarrow \infty} a_n) = -\infty$$

(d) For  $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$ , we have

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$$

if  $a_n, b_n > 0$

(only if right hand side is defined)

$$\limsup_{n \rightarrow \infty} (a_n \cdot b_n) \leq \limsup_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$$

(e)

if  $a_n, b_n > 0$ 

$$\liminf_{n \rightarrow \infty} (a_n, b_n) \geq \liminf_{n \rightarrow \infty} a_n \cdot \lim_{n \rightarrow \infty} b_n$$

## Lec 13 - Open, closed, and Compact sets



$(x - \varepsilon, x + \varepsilon)$  =  $\varepsilon$ -neighborhood

of  $x$

$\forall \varepsilon > 0 : (x - \varepsilon, x + \varepsilon) =: B_\varepsilon(x)$

$\varepsilon$ -neighborhood of  $x$

$M \subseteq \mathbb{R}$  is called a neighbourhood of  $x$ , if there is  $\exists \varepsilon > 0$  s.t  $M \supset B_\varepsilon(x)$

Example:  $[-2, 2]$  is a neighbourhood of  $B_\varepsilon(0), B_\varepsilon(1)$

$[-2, 2]$  is not a neighbourhood of 2

definition:

$M \subseteq \mathbb{R}$  is called open (in  $\mathbb{R}$ )

if, for all  $x \in M$ ,  $M$  is a neighbourhood  
of  $x$ .

$$\Rightarrow \forall x \in M, \exists \varepsilon > 0 : M \supseteq B_\varepsilon(x)$$



$A \subseteq \mathbb{R}$  is closed (in  $\mathbb{R}$ ) if  $A^c := \mathbb{R} \setminus A$   
is open

Open is Not the opposite of  
closed.

- $\Rightarrow$  There could be neither open nor closed
- $\Rightarrow$  There could be open & closed at the same time.

Ex: ①  $\emptyset, \mathbb{R}$  are Both open & closed

②  $[-2, 2]$  is closed but not open

③  $(-2, 2)$  is open but not closed

④  $[-2, 2]$  is neither open nor closed

The subset of the real numbers could be much more complicated than just an interval.

In order to deal with those sets, the next fact is very helpful.

FACT: It gives us a criterion to check closeness with the help of seq<sup>n</sup>.

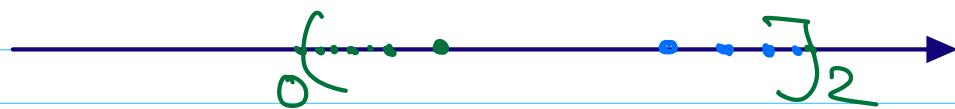
$A \subseteq \mathbb{R}$  is closed  $\iff$  For all convergent seq<sup>n</sup>

$(a_n)_{n \in \mathbb{N}}$  with  $a_n \in A \forall n \in \mathbb{N}$

we have  $\lim_{n \rightarrow \infty} a_n \in A$

It's not possible to leave the set with seqn from Inside.

Ex:  $A := (0, 2]$



$$\text{Take } (a_n)_{n \in \mathbb{N}} = \left(\frac{1}{n}\right)_{n \in \mathbb{N}}$$

$$\frac{1}{n} \in (0, 2] \quad \forall n \in \mathbb{N}$$

But  $\lim_{n \rightarrow \infty} a_n = 0 \notin A$

This set is not closed.

Definition:

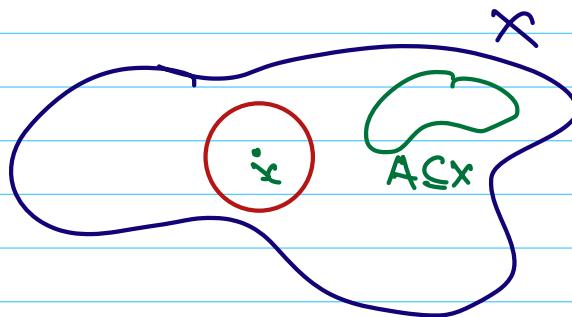
$A \subseteq \mathbb{R}$  is called compact, if for all sequences  $(a_n)_{n \in \mathbb{N}}$  with  $a_n \in A \quad \forall n \in \mathbb{N}$ , we find convergent subseqn with a limit lies

in  $A$ .  $\Rightarrow (a_{n_k})_{n \in \mathbb{N}}$  with  $\lim_{k \rightarrow \infty} a_{n_k} \in A$

# Functional Analysis lec 03

## Open and Closed sets

$(X, d)$  : metric space



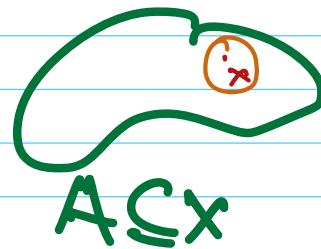
$B_\epsilon(x)$  open  $\epsilon$ -Ball around  $x$

$$B_\epsilon(x) := \{ y \in X \mid d(x, y) < \epsilon \}$$

(open ball of radius  $\epsilon > 0$ , centered at  $x$ )

Notion's:

① Open sets:



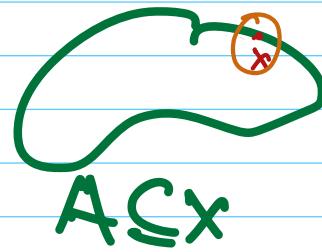
$A \subseteq X$  is called open, if for each

$x \in A$ , there is an open ball with  $B_\epsilon(x) \subseteq A$

i.e.  $\forall x \in A, \exists \epsilon > 0$ , such that

$$B_\epsilon(x) \subseteq A$$

② Boundary points:



$A \subseteq X$ .  $x \in X$  is called a Boundary Point

for  $A$  if  $\forall \varepsilon > 0$ :  $B_\varepsilon(x) \cap A \neq \emptyset$

$$B_\varepsilon(x) \cap A^c \neq \emptyset$$

$$\partial A := \left\{ x \in X \mid \begin{array}{l} x \text{ is a boundary} \\ \text{point for } A \end{array} \right\}$$

Remember:

$$A \text{ open} \iff A \cap \partial A = \emptyset$$

Closed set:

$$A \text{ closed} \iff A \cup \partial A = A$$

$$A \text{ closed} \iff A^c := X \setminus A \text{ is open}$$

④

Closure:

$$\overline{A} := A \cup \partial A \quad (\text{Cantor's closed})$$

(smallest closed set  
that contains A)

Ex:

$$X := (1, 3] \cup (4, \infty)$$

$$d(x, y) = |x - y|$$

$(X, d)$  metric space.

⑤

$$A := (1, 3] \subseteq X$$

Open:  $\forall x \in A, \exists \epsilon \text{ s.t } B_\epsilon(x) \subseteq A \Rightarrow$  open

Closed:

$$A^c = (4, \infty), \forall x \in A^c, \exists \epsilon \text{ s.t}$$

$$B_\epsilon(x) \subseteq A^c \Rightarrow A^c \text{ is open}$$

$\Rightarrow A$  is closed

⑥

$$C := [1, 2]$$

$$\partial C = \{2\}$$

## Lec 14 Heine-Borel theorem

Compact set : (sequentially compact set)

We use sequence's to describe compactness.

- \* Any subset  $A \subseteq \mathbb{R}$  is called compact set , if it forces any seqn to cluster.
- \* So the seqn with members in A has at atleast one accumulation value , this accumulation value  $\in A$

Any seqn  $(a_n)_{n \in \mathbb{N}} \subseteq A$  has a accumulation value  $a \in A$ .

Example: (a)  $\emptyset$  is compact

(b)  $\{s\}$  is compact

(c)  $\mathbb{R}$

We need to look all the seqn in detail

number's  $(a_n)_{n \in \mathbb{N}} = (n)_{n \in \mathbb{N}}$

$(n)_{n \in \mathbb{N}}$  divergent to  $\infty \notin \mathbb{R}$

$\Rightarrow \mathbb{R}$  is not compact

(d)  $[c, d]$ ,  $c \leq d$ , compact set.

let  $(a_n)_{n \in \mathbb{N}} \subseteq [c, d] \Rightarrow (a_n)_{n \in \mathbb{N}}$  bounded

$(a_n)_{n \in \mathbb{N}}$  is bounded  $\Rightarrow$  apply Bolzano

- weierstrass theorem

||

Bounded seq<sup>n</sup> have at least one accumulation value.  $a \in \mathbb{R}$

$[c, d]$  is closed

$\Rightarrow$  we can't leave the set

using seq<sup>n</sup>

$\Rightarrow a \in [c, d]$

$\Rightarrow [c, d]$  compact set.

## Heine-Borel theorem:

for  $A \subseteq \mathbb{R}$ , we have :

$A$  is compact  $\iff$   $A$  is bounded  
set  
and closed.

## Lec 13: Series

infinite sum's are also called series.

Series: "infinite sum", special seq<sup>n</sup>.

$$a_1 + a_2 + \dots + a_n + \dots = \sum_{k=1}^{\infty} a_k$$

This is called a series, just adding infinitely many numbers.

Ex: seq<sup>n</sup>  $(a_k)_{k \in \mathbb{N}} = ((-1)^k)_{k \in \mathbb{N}}$

$$\begin{aligned}\sum_{k=1}^{\infty} a_k &= (-1 + 1) + (-1 + 1) + \dots = 0 \\ &= -1 + (1 + -1) + (1 - 1) + \dots = -1\end{aligned}$$

we have 2 different possible results.

$\Rightarrow$  They don't act like a normal sum.

Definition: Let  $(a_k)_{k \in \mathbb{N}}$  be a seq<sup>n</sup>.

The seq<sup>n</sup>  $(S_n)_{n \in \mathbb{N}}$  given by

$$S_n := \sum_{k=1}^n a_k \quad \text{is called a series.}$$

The whole seq<sup>n</sup> given by  $(S_n)$  is called a series.

If this seq<sup>n</sup>  $(S_n)$  is convergent, then we have meaning for sum.

\* If  $(S_n)_{n \in \mathbb{N}}$  is convergent, we write

$$\sum_{k=1}^{\infty} a_k := \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k$$

Example:

$$\left( \sum_{k=1}^n (-1)^k \right)$$

$$= (-1, 0, -1, 0, \dots)$$

not convergent

$$\left( \sum_{k=1}^n k \right)_{n \in \mathbb{N}} = (1, 2, 3, \dots)$$

divergent to  $\infty$

## LPC 16: Geometric Series

and Harmonic Series

Series: Series is just a special seq<sup>n</sup>

$\sum_{k=1}^{\infty} a_k$  is the seq<sup>n</sup> of partial sum's

$$S_n := \sum_{k=1}^n a_k$$

Example :- Geometric Series

$$\sum_{k=0}^{\infty} q^k, \quad q \in \mathbb{R}$$

we show this series is convergent

$$\iff |q| < 1$$

Question:  $S_n = \sum_{k=0}^n q^k = ?$

for  $q \neq 1$   $\sum_{k=0}^n q^k = ?$

$$\Rightarrow S_n (1-q) = (1-q) \sum_{k=0}^n q^k$$

$$= \sum_{k=0}^n q^k - \sum_{k=0}^n q^{k+1}$$

$$= \sum_{k=0}^n q^k - \sum_{k=1}^{n+1} q^k$$

$$= q^0 - q^{n+1}$$

$$= 1 - q^{n+1}$$

$$\Rightarrow S_n \cdot (1-q) = 1 - q^{n+1}$$

$$\Rightarrow S_n = \frac{1-q^{n+1}}{1-q}$$

we need to check in which cases  $(S_n)$  is a convergent seqn

$\Rightarrow (S_n)_{n \in \mathbb{N}}$  convergent  $\Leftrightarrow (q^n)_{n \in \mathbb{N}}$  is convergent

$$\Rightarrow |q| < 1$$

$$\lim_{n \rightarrow \infty} q^n = 0 \quad |q| < 1$$

$$\Rightarrow \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1-q^n}{1-q}$$

$$= \frac{1}{1-q}$$

$$\Rightarrow \sum_{k=0}^{\infty} q^k = \lim_{n \rightarrow \infty} S_n = \frac{1}{1-q} \text{ if } |q| < 1$$

geometric series

Ex: Harmonic Series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots = \infty$$

(Divergent to  $\infty$ )

Proof:

$$S_n = \sum_{k=1}^n \frac{1}{k}$$

(seq<sup>n</sup> ( $S_n$ ) is

monotonically increasing

Now we need to prove the seq<sup>n</sup>

( $S_n$ ) is not bounded.

$$S_{2m} = \sum_{k=1}^{2^m} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{2^m}$$

$$\begin{aligned} S_{2m} &= S_1 + (S_2 - S_1) + (S_4 - S_2) \\ &\quad + \dots + (S_{2^m} - S_{2^{m-1}}) \end{aligned}$$

$$\begin{aligned} \Rightarrow S_{2m} &= S_1 + (S_2 - S_1) + (S_4 - S_2) + (S_8 - S_4) \\ &\quad + (S_{16} - S_8) + \dots + (S_{2^m} - S_{2^{m-1}}) \end{aligned}$$

$$= S_1 + \sum_{j=1}^m (S_{2^j} - S_{2^{j-1}})$$

$$S_{2^j} - S_{2^{j-1}} = \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{k} > \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{2^j}$$

$$\Rightarrow S_{2^j} - S_{2^{j-1}} > 2^{j-1} \cdot \frac{1}{2^j}$$

$$\Rightarrow S_{2^m} - S_{2^{m-1}} > \frac{1}{2}$$

$$\Rightarrow S_{2^m} = S_1 + \sum_{j=1}^m (S_{2^j} - S_{2^{j-1}})$$

$$> S_1 + m \cdot \frac{1}{2}$$

$$\Rightarrow S_{2^m} > S_1 + m \cdot \frac{1}{2} \xrightarrow{m \rightarrow \infty} \infty$$

$$\Rightarrow S_n = \sum_{k=1}^n \frac{1}{k} \text{ is monotonically increasing.}$$

increasing and unbounded from above

$$\Rightarrow \lim_{n \rightarrow \infty} S_n = \sum_{k=1}^{\infty} \frac{1}{k} = +\infty$$

### Lec 17 :- Cauchy Criterion

\* Criteria we have to check for convergence of series.

Series: Series is just a seq<sup>n</sup> of partial sum's

$$S_n = \sum_{k=1}^{\infty} a_k \text{ seq}^n \text{ of partial sum's}$$

\* This seq<sup>n</sup> ( $S_n$ ) is convergent

we can denote as the limit

$$\sum_{k=1}^{\infty} a_k$$

Properties:

if  $\sum_{k=1}^{\infty} a_k$  and  $\sum_{k=1}^{\infty} b_k$  are

Convergent, then:

$$(a) \sum_{k=1}^{\infty} (a_k + b_k) \text{ is also convergent.}$$

$$= \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k = a + b$$

$$(b) \sum_{k=1}^{\infty} \lambda a_k \text{ is also convergent}$$
$$= \lambda \cdot \sum_{k=1}^{\infty} a_k$$

How can we check a given series  
is convergent?

①

Cauchy Criterion:

it uses completeness

Axiom of  $\mathbb{R}$

\* for  $\mathbb{R}$ , being convergent seq<sup>n</sup>

$\iff$  Cauchy seq<sup>n</sup>

$\Rightarrow$  for Cauchy seq<sup>n</sup> we don't have to know about the limit

$\Rightarrow$  therefore this criterion checks for convergence, but does not tell us the limit.

$$\sum_{k=1}^{\infty} a_k \text{ is convergent}$$

$\iff \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n > m \geq N :$

$$\left| \sum_{k=m}^n a_k \right| < \varepsilon$$

Proof:

$$s_n := \sum_{k=1}^n a_k, (s_n)_{n \in \mathbb{N}} \text{ is convergent}$$

$\iff (s_n)_{n \in \mathbb{N}}$  is a Cauchy seq<sup>n</sup>  
Completeness

$\Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N} \quad \forall n, m \geq N:$

$$|S_n - S_m| < \epsilon$$

$\Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N} \quad \forall n, m \geq N:$

$$|S_n - S_{m-1}| < \epsilon$$

Ex:

$$\sum_{k=1}^{\infty} (-1)^k \quad \text{calculat-}$$

$$\left| \sum_{k=m}^n (-1)^k \right| ?$$

$$\Rightarrow \left( \sum_{k=n}^{n+2} (-1)^k \right) = \begin{cases} |1 + (-1) + 1| \\ |-1 + 1 - 1| \end{cases} = 1$$

$\Rightarrow$  Not convergent.

Important fact:

$\sum_{k=1}^{\infty} a_k$  is convergent  $\Rightarrow (a_{1k})_{k \in \mathbb{N}}$  convergent with

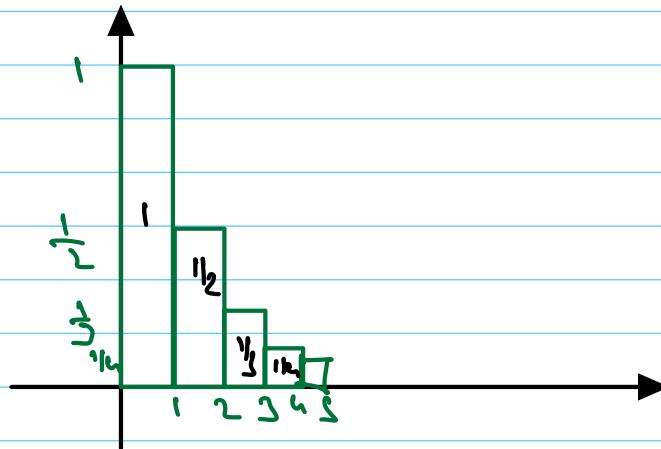
$$\lim_{k \rightarrow \infty} a_{1k} = 0$$

## Lec 18: Leibniz Criterion

This is the theorem that gives us the sufficient condition for a series to be convergent.

Harmonic Series:

$$S_n = \sum_{k=1}^n \frac{1}{k}$$



The whole area  $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$

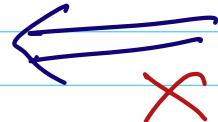
$\Rightarrow$  The Harmonic series not converges.

but divergent to  $\infty$

Therefore having a seq<sup>n</sup> inside the sum  
that goes to 0 is not sufficient for  
convergent of the series.

### Cauchy Criterion:

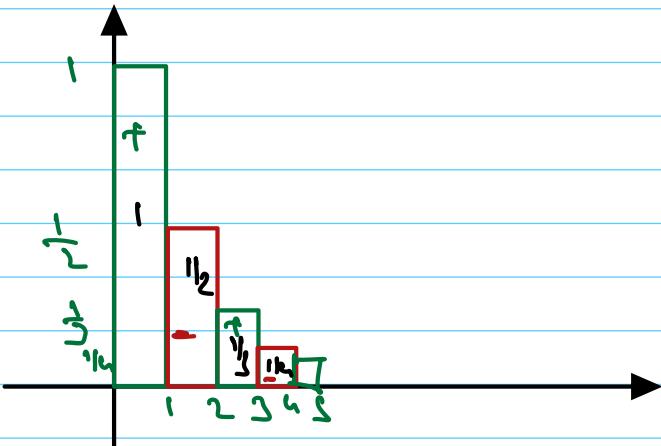
$$\sum_{k=1}^{\infty} a_k \text{ is convergent} \implies \lim_{n \rightarrow \infty} a_n = 0$$



Ex:

$$\lim_{k \rightarrow \infty} \frac{1}{k} = 0 \quad \text{but} \quad \sum_{k=1}^{\infty} \frac{1}{k} \text{ is divergent to } \infty.$$

### Leibniz Criterion:



$$S_n = \sum_{k=1}^n (-1)^{k+1} \cdot \frac{1}{k} \quad (\text{Convergent})$$

Theorem: (Alternating Series test, Leibniz Criterion, Leibniz test)

① Let  $(a_k)_{k \in \mathbb{N}}$  be convergent with

$$\lim_{k \rightarrow \infty} a_k = 0$$

and monotonically decreasing

Then  $\sum_{k=1}^{\infty} (-1)^k a_k$  is convergent.

Proof:

$$S_n := \sum_{k=1}^n (-1)^k a_k$$

\* Since all the  $a_k$ 's are non-negative

$$\text{i.e. } a_k \geq 0 \quad \forall k \in \mathbb{N}$$

\* and  $a_k$ 's are monotonically decreasing

$$\lim_{k \rightarrow \infty} a_k = 0$$

take a SUBSEQ<sup>n</sup> (even)

$$S_{2l+2} - S_{2l} = \sum_{k=1}^{2l+2} (-1)^k a_{1k} - \sum_{k=1}^{2l} (-1)^k a_{1k}$$
$$= a_{2l+2} - a_{2l+1}$$

$\Rightarrow$  Since the seq<sup>n</sup> is monotonically decreasing

$$S_{2l+2} - S_{2l} = a_{2l+2} - a_{2l+1} \leq 0$$

$$S_{2l+3} - S_{2l+1} = \sum_{k=1}^{2l+3} (-1)^k a_k - \sum_{k=1}^{2l+1} (-1)^k a_k$$
$$= -a_{2l+3} + a_{2l+2}$$

$$\Rightarrow S_{2l+3} - S_{2l+1} = -a_{2l+3} + a_{2l+2} \geq 0$$

The even SUB SEQ<sup>n</sup> is monotonically decreasing

The odd SUB SEQ<sup>n</sup> is monotonically increasing.

$$S_{2l+1} - S_{2l} = -a_{2l+1} \leq 0 \Rightarrow S_{2l+1} \leq S_{2l}$$

$S_{2l+1} \leq S_{2l}$  (this is useful when we compare subseq<sup>n</sup>)

$$(S_{2l+1})_{l \in \mathbb{N}}$$

monotonically increasing  
sub seq<sup>n</sup>

$\Rightarrow$

$$S_1 \leq S_3 \leq S_5 \leq S_7 \dots$$

$$(S_{2l})_{l \in \mathbb{N}}$$

monotonically decreasing  
sub seq<sup>n</sup>

$$S_2 \geq S_4 \geq S_6 \geq \dots$$

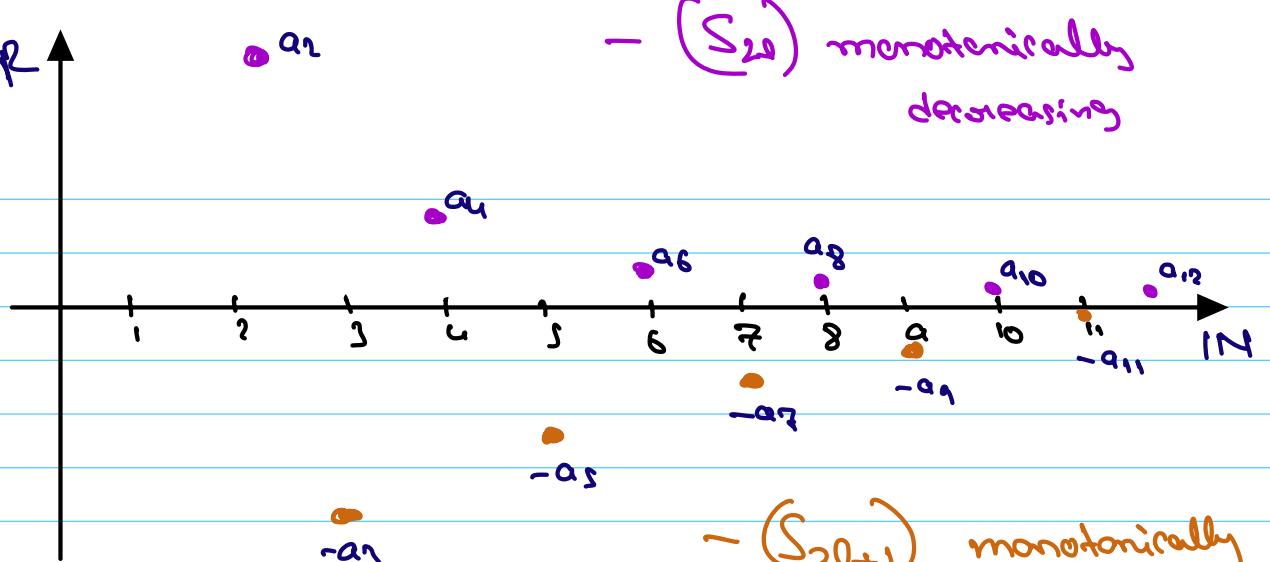
$\Rightarrow$

$$S_2 \leq S_{2l+1} \leq S_{2l} \leq S_2$$

(Bounded)

Both the subseq<sup>n</sup> are monotonic &  
Bounded

$\Rightarrow$  hence both the seq<sup>n</sup> are  
Converges



$$\lim_{l \rightarrow \infty} (S_{2l+1} - S_{2l}) = \lim_{l \rightarrow \infty} (-a_{2l+1}) = 0$$

$$\Rightarrow S := \lim_{l \rightarrow \infty} S_{2l+1} = \lim_{l \rightarrow \infty} S_{2l}$$

$$\Rightarrow \lim_{n \rightarrow \infty} S_n = S \quad (\text{convergent})$$

## Lec 19 Comparison test:

- \* Comparison test is a criterion used to decide if a given series is convergent or not.
- \* we need a new definition about convergence

$\sum_{k=1}^{\infty} a_k$  is called absolutely convergent

The series  $\sum_{k=1}^{\infty} |a_k|$  is called absolutely

Convergent if the series given by the absolute value of  $|a_k|$  is simple convergent.

$\Rightarrow \sum_{k=1}^{\infty} a_k$  is convergent if  $\sum_{k=1}^{\infty} |a_k|$  is convergent.

$\Rightarrow$  absolutely convergent

$\implies$  convergent.

$\sum_{k=1}^{\infty} a_{ik}$  is absolute convergent

$\Rightarrow \forall \varepsilon > 0, \exists N \in \mathbb{N}$  such that

for all  $n \geq m \geq N$ :

$$\left| \sum_{k=m}^n |a_{ik}| \right| < \varepsilon$$

(Cauchy criterion)

$$\Rightarrow \sum_{k=m}^n |a_{ik}| < \varepsilon$$

triangular inequality

$$\Rightarrow \left| \sum_{k=m}^n a_{ik} \right| \leq \sum_{k=m}^n |a_{ik}| < \varepsilon$$

$$\Rightarrow \left| \sum_{k=m}^n a_{ik} \right| < \varepsilon$$

(Cauchy criterion for  $\sum_{k=1}^{\infty} a_{ik}$ )

Original series

Example:

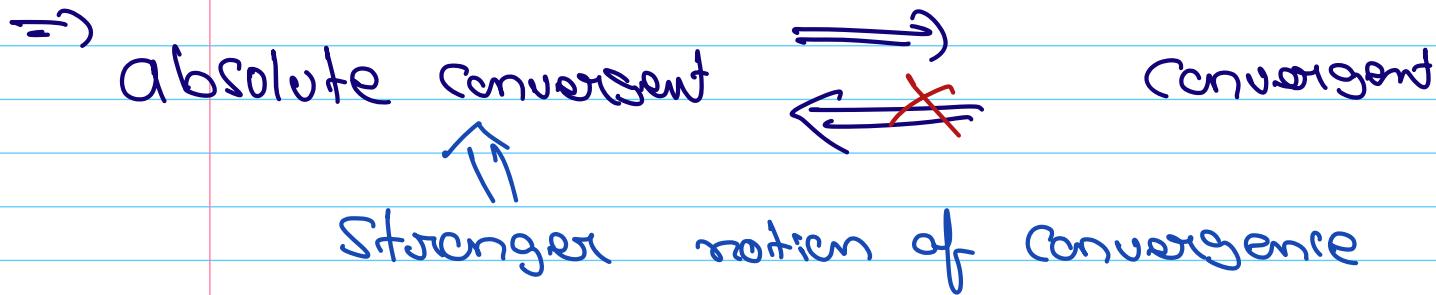
\*  $\sum_{k=1}^{\infty} \frac{1}{k}$  (Not convergent)

but  $a_k \geq 0 \quad \forall k \in \mathbb{N}$

- \* monotonically decreasing
- \*  $\lim_{n \rightarrow \infty} a_n = 0$

$\Rightarrow \sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$  is convergent using  
(Leibniz Kriterien)

But  $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$  is not absolutely convergent.



This is exactly we need for comparison test.

## Majorant Criterion:

Let  $\sum_{k=1}^{\infty} a_k$  be a series

if there is no  $n \in \mathbb{N}$  and a convergent series  $\sum_{k=n}^{\infty} b_k$   
with  $b_k > 0 \quad \forall k$

$$|a_k| \leq b_k \quad \forall k \geq n$$

we have another seq<sup>n</sup>  $b_k$  that is  
convergent.

$\Rightarrow$  then we can conclude the series

$\sum_{k=1}^{\infty} a_k$  is absolutely convergent.

## Proof:

apply Cauchy criterion to  $\sum_{k=m}^{\infty} b_k$

$\forall \epsilon > 0, \exists N \in \mathbb{N}$  such that  $\forall n > m > N$

we have

$$\left| \sum_{k=m}^n b_k \right| < \epsilon$$

if we choose  $N > n_0$  then

$$\sum_{k=m}^N |a_k| \leq \sum_{k=m}^N b_k = \left( \sum_{k=m}^N b_k \right) < \varepsilon$$

$$\Rightarrow \sum_{k=m}^N |a_k| < \varepsilon$$

$\Rightarrow \sum_{k=1}^{\infty} a_k$  is absolute convergent.

### Minorant Criterion:

It works similarly but we don't check for convergence, but for divergence.

let  $\sum_{k=1}^{\infty} a_k$  be a series with  $a_k > 0$

if there is  $n_0 \in \mathbb{N}$  and a divergent series

$\sum_{k=1}^{\infty} b_k$  with  $b_k \geq 0$  and with

$a_k \geq b_k \quad \forall k > n_0 \rightarrow$  then

$\sum_{k=1}^{\infty} a_k$  is divergent.

## Lec 20: Ratio and Root test

Both Ratio and Root in the best case

scenario they answer us if a given series is absolutely convergent.

$\sum_{k=1}^{\infty} a_k$  absolutely convergent?

if we have a majorant

There is a convergent majorant



$\sum_{k=1}^{\infty} a_k$  is absolutely convergent.

we know that geometric series

$$\sum_{k=0}^{\infty} q^k \text{ convergent} \iff |q| < 1$$

FACT:

if there is  $c, n_0 \in \mathbb{N}$  and  $q \in \mathbb{R}$  with

$|q| < 1$  such that  $|a_k| \leq c q^k \forall k \geq n_0$

then  $\sum_{k=n_0}^{\infty} a_k$  is absolute convergent.

Ratio test: if there is  $n_0 \in \mathbb{N}$  and  $q \in [0, 1)$  such that

$$\left| \frac{a_{k+1}}{a_k} \right| \leq q$$

Then  $\sum_{k=n_0}^{\infty} a_k$  is absolute convergent.

$\forall k \geq n_0$ , then  $\sum_{k=n_0}^{\infty} a_k$  is absolute convergent.

Proof:  $|a_{k+1}| \leq q \cdot |a_k| \leq q \cdot q \cdot |a_{k+1}|$

$$\Rightarrow |a_{k+1}| \leq q^{k+1-n_0} |a_{n_0}|$$

$$\Rightarrow |a_{k+1}| \leq q^{k+1} \cdot \frac{|a_{n_0}|}{q^{n_0}}$$

$$\Rightarrow |a_{k+1}| \leq q^{k+1} \cdot c$$

### Root Test:

if there is  $n_0 \in \mathbb{N}$  and  $q \in (0, 1)$

such that

$$\sqrt[k]{(|a_k|)} \leq q \quad \forall k > n_0$$

then  $\sum_{k=1}^{\infty} a_k$  is abs. convergent.

### Proof:

$$\sqrt[k]{|a_k|} \leq q$$

$$\Leftrightarrow |a_k| \leq q^k$$

$\Rightarrow$  Converges.

Remember:  $\limsup_{k \rightarrow \infty} \sqrt[k]{|a_k|} < 1$

$\Rightarrow \sum_{k=1}^{\infty} |a_k|$  is abs-convergent.

$\limsup_{k \rightarrow \infty} \sqrt[k]{|a_k|} > 1 \Rightarrow \sum_{k=1}^{\infty} |a_k|$  is divergent.

## Geometric Series

$$\sum_{k=0}^{\infty} q^k, \quad q \in \mathbb{R}$$

we show  $\sum_{k=0}^{\infty} q^k$  convergent  $\Leftrightarrow |q| < 1$

Question:

$$S_n = \sum_{k=0}^n q^k = ?$$

for  $q \neq 1$ :  $S_n = \sum_{k=0}^n q^k$

$$\Rightarrow (1-q) S_n = (1-q) \sum_{k=0}^n q^k$$

$$= \sum_{k=0}^n q^k - \sum_{k=0}^n q^{k+1}$$

$$= \sum_{k=0}^n q^k - \sum_{k=1}^{n+1} q^k$$

$$= q^0 - q^{n+1}$$

$$\Rightarrow S_n = \frac{1-q^{n+1}}{1-q}$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{1-q^{n+1}}{1-q}$$

$$= \frac{1}{1-q} \left( 1 - \lim_{n \rightarrow \infty} q^{n+1} \right) \leq 0$$

if  $|q| < 1$

$$= \frac{1}{1-q} \quad \text{if } |q| < 1$$

### geometric series

$$\Rightarrow \sum_{k=0}^{\infty} q^k = \lim_{n \rightarrow \infty} \sum_{k=0}^n q^k = \lim_{n \rightarrow \infty} S_n = \frac{1}{1-q}$$

if  $|q| < 1$

$(S_n)_{n \in \mathbb{N}}$  in convergent  $\Leftrightarrow (q^n)_{n \in \mathbb{N}}$  convergent

$\rightarrow 0$

$\Leftrightarrow |q| < 1$

## Harmonic Series

$$\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \infty$$

(divergent to  $\infty$ )

Proof.

$$S_n = \sum_{k=1}^n \frac{1}{k}$$

$(S_n)_{n \in \mathbb{N}}$  is monotonically increasing

$\Rightarrow$  either  $\lim_{n \rightarrow \infty} S_n \in \mathbb{R}$  (converges)

or  $= +\infty$  (diverges to  $+\infty$ )

we need to show that  $(S_n)_{n \in \mathbb{N}}$

is not bounded to show that  $S_n$  diverges to  $+\infty$ .

\* Let's take a sub seq<sup>n</sup> of  $(S_n)$

$$S_{2^m} = \sum_{k=1}^{2^m} \frac{1}{k}$$

$$S_{2m} = S_1 + (S_2 - S_1) + (S_4 - S_2) \\ + (S_8 - S_7) + \dots + (S_{2m} - S_{2m-1})$$

all all the terms cancel out each other and only left with  $S_{2m}$

$\Rightarrow$

$$S_{2m} = S_1 + (S_2 - S_1) + (S_4 - S_2) + \dots + (S_{2m} - S_{2m-1})$$

$$S_{2m} = S_1 + \sum_{j=1}^{2^0} (S_{2^j} - S_{2^{j-1}})$$

$$S_{2^j} - S_{2^{j-1}} = \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{k} > \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{2^j}$$

$$\Rightarrow S_{2^j} - S_{2^{j-1}} > \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{2^j}$$

$$> 2^{j-1} \cdot \frac{1}{2^j} = \frac{1}{2}$$

$$\Rightarrow S_{2^j} - S_{2^{j-1}} = \frac{1}{2}$$

$$\Rightarrow S_{2^m} = S_1 + \sum_{j=1}^m (S_{2^j} - S_{2^{j-1}})$$

$$\Rightarrow S_{2^m} > S_1 + \frac{m}{2}$$

$$\Rightarrow \lim_{m \rightarrow \infty} S_{2^m} > \lim_{m \rightarrow \infty} S_1 + \frac{m}{2}$$

$$\Rightarrow \lim_{m \rightarrow \infty} S_{2^m} = +\infty$$

Hence the Harmonic series

$$(S_n)_{n \in \mathbb{N}} = \left( \sum_{k=1}^n \frac{1}{k} \right)_{n \in \mathbb{N}}$$

is divergent to  $+\infty$  Harmonic series

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k} = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k} = +\infty}$$

## Cauchy Criterion:

We say a series  $\sum a_k$  satisfies the Cauchy criterion if its seq<sup>n</sup> ( $s_n$ ) the partial sum's is a Cauchy seq<sup>n</sup>

$\sum_{k=1}^{\infty} a_k$  is convergent

$\forall \epsilon > 0, \exists N \in \mathbb{N}$  s.t  $\forall n, m > N$

$$|s_n - s_m| < \epsilon$$

Let's restrict to  $n > m$  such that

$$|s_n - s_{m+1}| < \epsilon$$

$$\Rightarrow \left| \sum_{k=m+1}^n a_k \right| < \epsilon$$

$\Rightarrow$  we can rewrite it as

$\forall \epsilon > 0, \exists N \in \mathbb{N}$  such that  $\forall n \geq m > N$

we have

$$\left| \sum_{k=m}^n a_k \right| < \epsilon$$

A series converges  $\iff$  if it satisfy

Cauchy criterion.

Since the series converge it should hold for  $n = m$

$$\Rightarrow |S_n - S_{m-1}| < \epsilon$$

$$\Rightarrow |S_n - S_{n-1}| < \epsilon$$

$$\Rightarrow |a_n| < \epsilon \quad ((a_n) \text{ converges to } 0)$$

$\sum_{k=1}^{\infty} a_n$  converges

$\Rightarrow$

$(a_n)_{n \in \mathbb{N}}$  converges

$\cancel{\Leftarrow}$

with  $\lim_{n \rightarrow \infty} a_n = 0$

This is not equivalence

## Leibniz Criterion:

Ex: Harmonic series:  $S_n = \sum_{k=1}^n \frac{1}{k}$  diverges to  $+\infty$

Theorem: (Alternating test, Leibniz Criterion)

① Let  $(a_k)_{k \in \mathbb{N}}$  be convergent with  $\lim_{n \rightarrow \infty} a_k = 0$

②  $(a_k)_{k \in \mathbb{N}}$  is monotonically decreasing seq<sup>n</sup>

⇒ Then the alternating series

$$S_n := \sum_{k=1}^{\infty} (-1)^k a_k \text{ is convergent.}$$

Proof:  $S_n = \sum_{k=1}^n (-1)^k a_k$

we know  $(a_k)_{k \in \mathbb{N}}$  is monotonically decreasing

and the limit  $\Rightarrow a_k > 0 \quad \forall k \in \mathbb{N}$

let's take subsequence of even and odds

i.e  $(S_{2l})_{l \in \mathbb{N}}$

$(S_{2l+1})_{l \in \mathbb{N}}$

$$S_{2l+2} - S_{2l} = a_{2l+2} - a_{2l+1}$$

$$(a_{2l+1} > a_{2l+2})$$

$$S_{2l+2} - S_{2l} \leq 0$$

$$\Rightarrow S_{2l+2} \leq S_{2l}$$

$\Rightarrow (S_{2l+2})_{l \in \mathbb{N}}$  is monotonically  
decreasing seq

$$S_{2l+3} - S_{2l+1} = -a_{2l+3} + a_{2l+2}$$

$$(a_{2l+2} \leq a_{2l+3})$$

$$> 0$$

$$\Rightarrow S_{2l+3} > S_{2l+1}$$

$(S_{2l+1})_{l \in \mathbb{N}}$  is monotonically increasing  
seq,

$$S_{2l+1} - S_{2l} = -a_{2l+1} \leq 0$$

$$\Rightarrow S_{2l+1} \leq S_{2l}$$

(decreasing)                          (increasing)

$$\Rightarrow S_3 \leq S_{2l+1} \leq S_{2l} \leq S_2$$

(Bounded) Both subseq<sup>n</sup>

$$\lim_{l \rightarrow \infty} (S_{2l+1} - S_{2l}) = 0$$

$$\Rightarrow \lim_{l \rightarrow \infty} (-a_{2l+1}) = \lim_{l \rightarrow \infty} -\frac{1}{k} = 0$$

$$\Rightarrow \lim_{l \rightarrow \infty} S_{2l+1} = \lim_{l \rightarrow \infty} S_{2l}$$

$$\Rightarrow S := \lim_{l \rightarrow \infty} S_{2l+1} = \lim_{l \rightarrow \infty} S_{2l}$$

$$\Rightarrow \lim_{n \rightarrow \infty} S_n = S \quad (\text{convergent})$$

Ex:

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}} = \sum_{k=1}^{\infty} (-1)^k \cdot a_k$$

①

$$a_k = \frac{1}{\sqrt{k}}$$

$$(a_k)_{k \in \mathbb{N}}$$

monotonically  
decreasing

②

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$$

③

$$a_k > 0 \quad \forall k \in \mathbb{N}$$

Let's apply Leibniz Criterion.

$\Rightarrow$  if the above 3 satisfies then the

series  $\sum_{k=1}^{\infty} (-1)^k a_k$  converges

$\left( \Rightarrow \sum_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k}} \text{ converges} \right)$

## Comparison Test:

A new definition for convergence.

$\sum_{k=1}^{\infty} a_k$  is called absolutely convergent  
if  $\sum_{k=1}^{\infty} |a_k|$  is convergent.

\* for a series that has only positive  $a_k$ 's ie  $a_k \geq 0 \forall k$ , then this  
not a new definition.

However in general its a stronger  
convergence.

Absolutely Convergent  $\Rightarrow$  Convergent  
 $\Downarrow$   ~~$\Leftarrow$~~

Stronger notion of convergence

Proof:

$$\sum_{k=1}^{\infty} |a_k| \text{ is convergent} \implies$$

$\forall \epsilon > 0, \exists N \in \mathbb{N}$  such that

$\forall n > m > N$  we have

$$\left| \sum_{k=m}^n |a_k| \right| < \epsilon$$

$$\Rightarrow \sum_{k=m}^n |a_k| < \epsilon$$

$$\Rightarrow \left| \sum_{k=m}^n a_k \right| \leq \sum_{k=m}^n |a_k| < \epsilon$$

triangle inequality

$$\Rightarrow \left| \sum_{k=m}^n a_k \right| < \epsilon$$

$$\sum_{k=1}^{\infty} a_k \text{ is convergent}$$

Counter example:

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$$

is convergent  
according to

Leibniz Criterion.

However

$$\sum_{k=1}^{\infty} |(-1)^k \frac{1}{k}| = \sum_{k=1}^{\infty} \frac{1}{k}$$

not convergent

i.e. Convergence  $\cancel{\Rightarrow}$  absolute convergence

Majorant Criterion:

Let  $\sum_{k=1}^{\infty} a_k$  be a series.

If there is  $n_0 > N$  and a convergent series

$\sum_{k=1}^{\infty} b_k$  with  $b_k \geq 0 \quad \forall k$ , then

If  $|a_{k,c}| \leq b_k \quad \forall k > n_0$

then  $\sum_{k=1}^{\infty} a_k$  is abs. convergent.

Proof: Apply Cauchy Criterion to

$$\sum_{k=1}^{\infty} b_k$$

$\forall \epsilon > 0$ ,  $\exists N \in \mathbb{N}$ . such that

$\forall n \geq m \geq N$  we have

$$\left| \sum_{k=m}^n b_k \right| < \epsilon$$

we know that

$$(a_k) \leq b_k \quad \forall k \geq n_0$$

$$\Rightarrow \sum_{k=n_0}^n |a_k| \leq b_k$$

$\Rightarrow \forall \varepsilon > 0, \exists N > n_0$ , such that

$\forall n \geq m > N$  we have

$$\left| \sum_{k=m}^n b_k \right| < \varepsilon$$

Since  $b_k \geq 0 \quad \forall k \in \mathbb{N}$

$$\sum_{k=m}^n b_k \leq \varepsilon$$

$$\Rightarrow \sum_{k=m}^n |a_k| \leq \sum_{k=m}^n b_k < \varepsilon$$

$$\Rightarrow \sum_{k=m}^n |a_k| < \varepsilon$$

$\Rightarrow |a_k| \text{ is convergent}$

$\Rightarrow a_k \text{ is absolute convergent.}$

## Mинорант критерий:

Let  $\sum_{k=1}^{\infty} a_k$  be a series with  $a_k > 0$

If there is  $n_0 \in \mathbb{N}$  and a divergent series  $\sum_{k=n_0}^{\infty} b_k$  with  $b_k > 0$

and with  $a_k \geq b_k \quad \forall k \geq n_0$ ,

then  $\sum_{k=1}^{\infty} a_k$  is divergent.

Example:

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

Convergent or divergent?

Sol:

$$\forall k > 0 \quad \exists n \in \mathbb{N}$$

$$\sqrt{k} < n$$

$$\Rightarrow \frac{1}{\sqrt{k}} > \frac{1}{n} \quad \forall k > 0$$

we know tho  $\sum_{k=1}^{\infty} \frac{1}{k} = +\infty$  divergent

$\Rightarrow$  therefore  $\sum_{k=1}^{\infty} \frac{1}{5^k}$  is divergent

$\sum_{k=1}^{\infty} a_k$  absolutely Convergent?

There is a convergent majorant



$\sum_{k=1}^{\infty} a_k$  is absolutely Convergent



$\sum_{k=1}^{\infty} a_{1k}$  is Convergent

$\Rightarrow$  we need to find a convergent majorant

series to prove the convergence of

$\sum_{k=1}^{\infty} a_{1k}$

$\Rightarrow$  why don't we take

geometric series or Convergent majorant.

$\Rightarrow$  if there is a  $n_0 \in \mathbb{N}$  and  $q \in \mathbb{R}$  and  $|q| < 1$  such that  $|a_k| \leq c q^k$

$\forall k \geq n_0$

(convergent majorant)



$\sum_{k=1}^{\infty} a_k$  is absolutely convergent.

RATIO test: if there is  $n_0 \in \mathbb{N}$  and  $q_r \in [0, 1)$  such that

$$a_k \neq 0 \quad \left| \frac{a_{k+1}}{a_k} \right| \leq q_r \quad \forall k \geq n_0$$

Then  $\sum_{k=1}^{\infty} a_k$  is abs convergent.

Proof:

$$|a_{k+1}| \leq q |a_k|$$

$$\Rightarrow |a_{k+1}| \leq q \cdot q \cdot |a_{k-1}|$$

$$\Rightarrow |a_{k+1}| \leq q^{k+1-n_0} |a_{n_0}|$$

$$\Rightarrow |a_{k+1}| \leq q^{k+1} \cdot \frac{|a_{n_0}|}{q^{n_0}}$$

$$\Rightarrow |a_{k+1}| \leq c \cdot q^{k+1}$$

(this satisfies the majorant criterion)

$$\Rightarrow \sum_{k=1}^{\infty} a_{k+1} \text{ is absolutely convergent.}$$

Ex:

$$\sum_{k=1}^{\infty} \frac{1}{k!} \quad \text{Convergent?}$$

Sol:

$$\left| \frac{a_{k+1}}{a_k} \right| = \left| \frac{k!}{(k+1)!} \right| = \frac{1}{k+1} \leq \frac{1}{2} \quad \forall k \geq 1$$

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k!}$  is convergent.

## Lec 22 Cauchy Product

Cauchy Product for series

$$\sum_{k=0}^{\infty} a_k, \quad \sum_{k=0}^{\infty} b_k \quad \xrightarrow{\text{How to multiply}} \quad \sum_{k=0}^{\infty} c_k$$

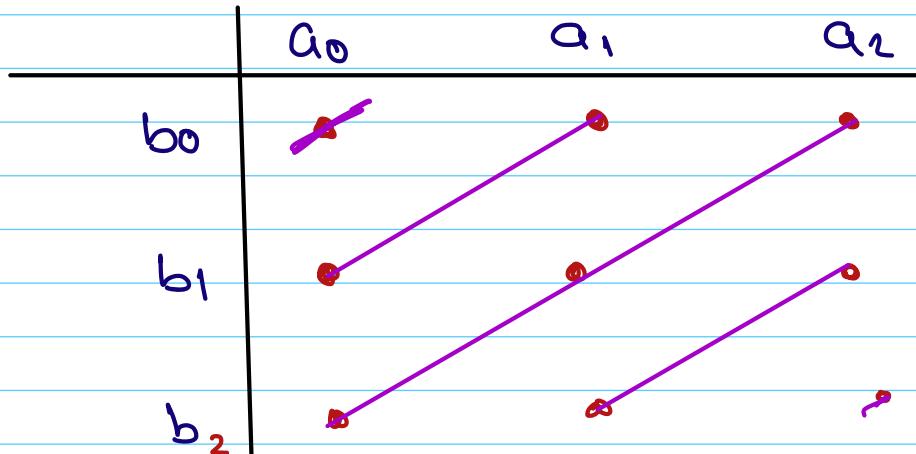
four finite sum's:

$$(a_0 + a_1 + a_2) (b_0 + b_1 + b_2)$$

$$\Rightarrow a_0 b_0 + a_1 b_0 + a_2 b_0 +$$

$$a_0 b_1 + a_1 b_1 + a_2 b_1 +$$

$$a_0 b_2 + a_1 b_2 + a_2 b_2$$



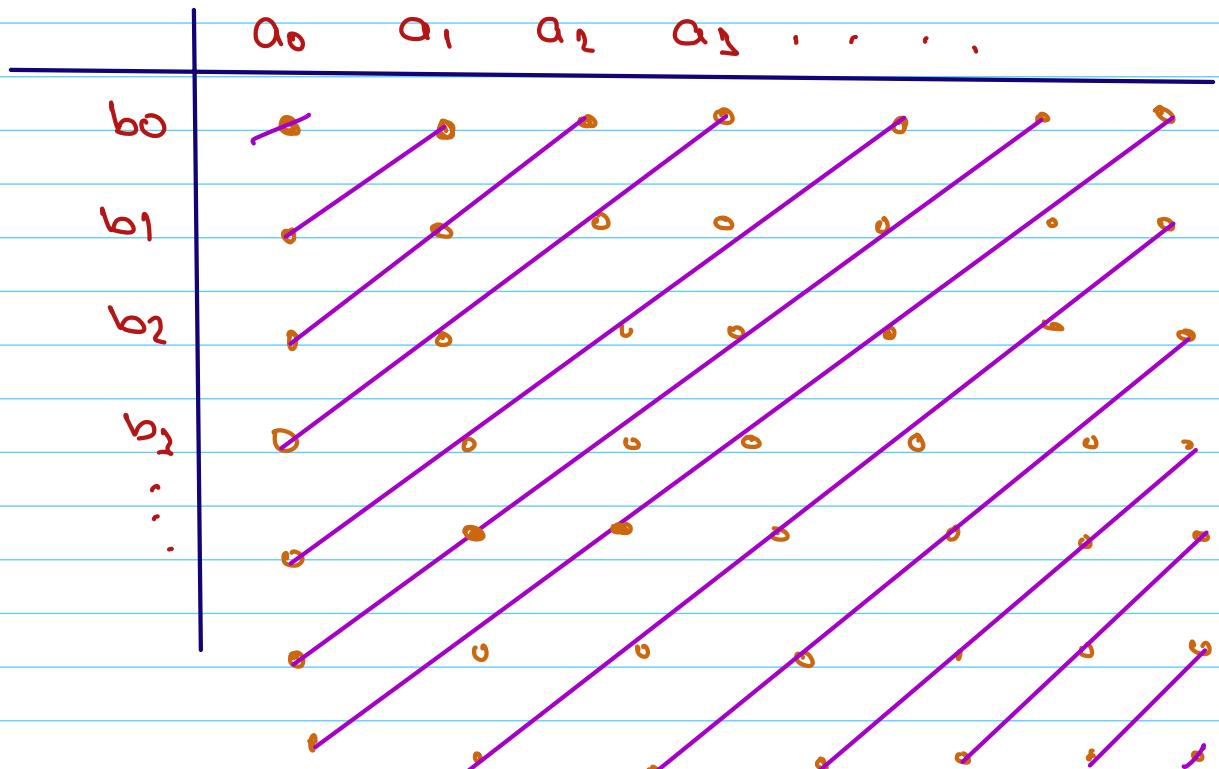
$$= a_0 b_0 + (a_1 b_0 + a_0 b_1)$$

$$+ (a_2 b_0 + a_1 b_1 + a_0 b_2)$$

$$+ (a_2 b_1 + a_1 b_2) + (a_2 b_2)$$

Cauchy Product : for two series

$$\sum_{k=0}^{\infty} a_k c, \quad \sum_{k=0}^{\infty} b_k c$$



for two series  $\sum_{n=0}^{\infty} a_n$ ,  $\sum_{n=0}^{\infty} b_n$  Cauchy Product

$\sum_{k=0}^{\infty} c_k$  with  $c_k = \sum_{l=0}^k a_l b_{k-l}$  is called

the Cauchy Product

Can we tell anything about convergence  
of this series?

Theorem:

when we have two convergent  
series  $\sum_{k=0}^{\infty} a_k$ ,  $\sum_{k=0}^{\infty} b_k$  and at least one

of them is absolutely convergent, then

the series  $\sum_{k=0}^{\infty} c_k$ ,  $c_k = \sum_{l=0}^k a_l b_{k-l}$

is also absolutely convergent.

$$\sum_{k=0}^{\infty} c_k = \left( \sum_{k=0}^{\infty} a_k \right) \left( \sum_{k=0}^{\infty} b_k \right)$$

formal theorem:

if  $\sum_{k=0}^{\infty} a_k$  is absolutely convergent

and  $\sum_{k=0}^{\infty} b_k$  is convergent, then Cauchy

Product  $\sum_{k=0}^{\infty} c_k$  is absolutely convergent.

and

$$\sum_{k=0}^{\infty} c_{ik} = \left( \sum_{k=0}^{\infty} a_k \right) \left( \sum_{k=0}^{\infty} b_{ik} \right)$$

Ex:

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{for } x \in \mathbb{R}$$

(abs. convergent by the ratio test)

$$e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

apply Cauchy Product to  $e^x, e^y$

$$e^x \cdot e^y = \sum_{k=0}^{\infty} \frac{x^k}{k!} \cdot \sum_{k=0}^{\infty} \frac{y^k}{k!}$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^k \frac{x^l}{l!} \cdot \frac{y^{k-l}}{(k-l)!}$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^k \frac{1}{k!} \frac{k!}{l! (k-l)!} x^l \cdot y^{k-l}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{l=0}^k \binom{k}{l} x^l \cdot y^{k-l}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \cdot (x+y)^k$$

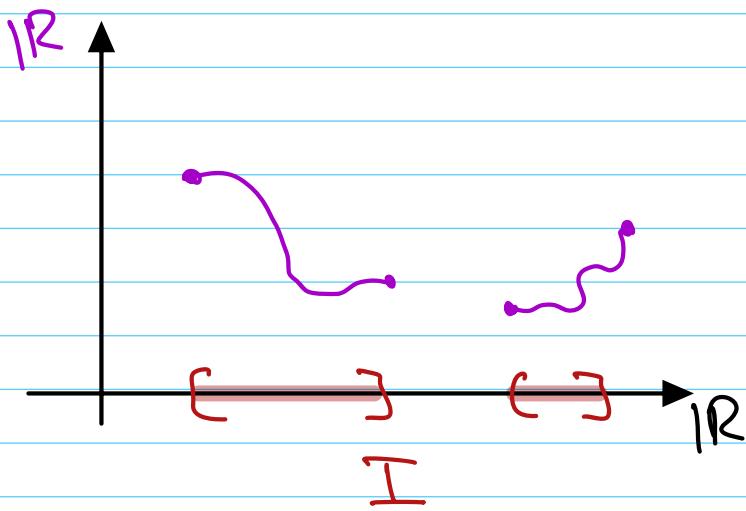
$$= \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!} = e^{x+y}$$

$$\Rightarrow e^x \cdot e^y = e^{x+y}$$

## Lec 23: Seq<sup>n</sup> of Function's

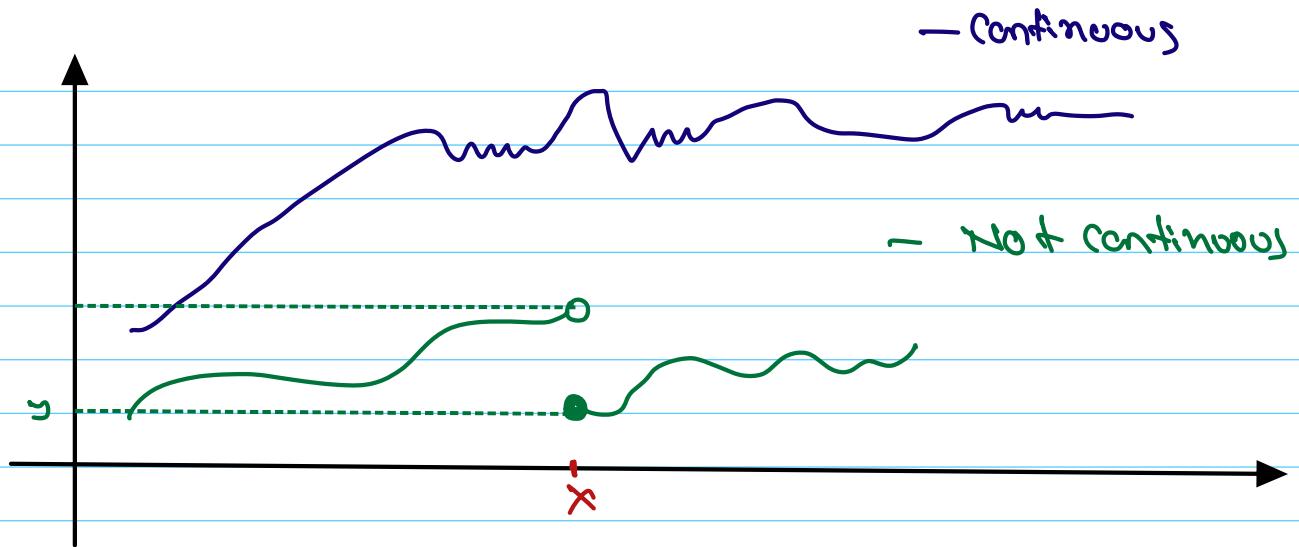
Function:  $f: I \rightarrow \mathbb{R}$

$(I \subseteq \mathbb{R})$



Later: Continuous function  $f: \mathbb{R} \rightarrow \mathbb{R}$

\* Continuous function is a function whose graph has no jumps, we don't need any smoothness, But for the function defined on whole number line, we have a connected graph.



Idea: small errors on  $x$ -axis  $\rightsquigarrow$  small errors on  $y$ -axis

### Bounded function:

$f: I \rightarrow \mathbb{R}$  is called a

Bounded function if, range of the function is bounded i.e

$$\{f(x) \mid x \in I\} = \text{Range}(f)$$

$$= f(I)$$

is a bounded

set in  $\mathbb{R}$

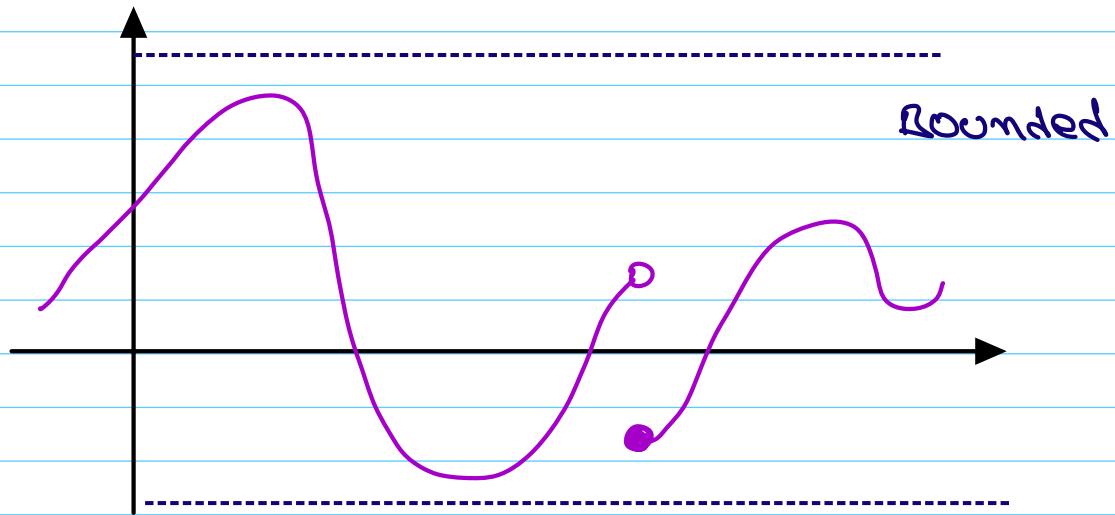


$$\sup_{x \in I} |f(x)| < \infty$$

Therefore, equivalently we can say supremum  
and inf of the set Range is finite

or equivalently we could say, the  
sup of abs value is finite

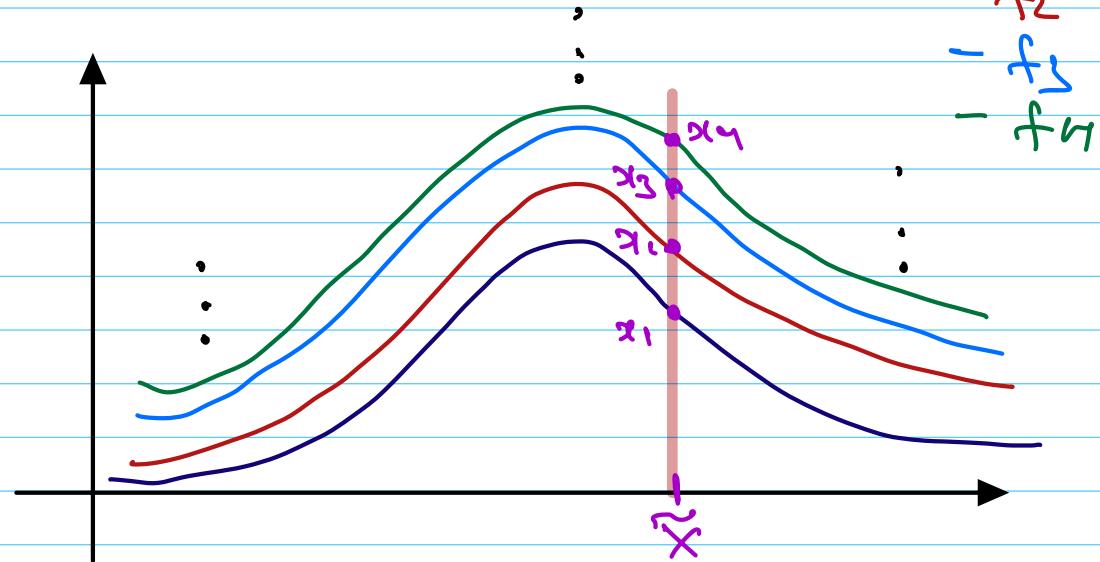
$$\Leftrightarrow \sup_{x \in I} |f(x)| < \infty$$



This is a straight forward generalization  
of bounded seq<sup>n</sup>.

combine both concepts.

Sequence of function's :



Sequence of function's :

$$(f_1, f_2, f_3, f_4, \dots)$$

with sequence members

$$f_1 : I \rightarrow \mathbb{R}$$

$$f_2 : I \rightarrow \mathbb{R}$$

$$f_3 : I \rightarrow \mathbb{R}$$

$$f_4 : I \rightarrow \mathbb{R}$$

All the function's in seq<sup>n</sup> have  
Same domain I

for a fixed  $\tilde{x} \in I$ , we get an

ordinary seq of real numbers

$$(f_1(\tilde{x}), f_2(\tilde{x}), f_3(\tilde{x}), \dots, f_n(\tilde{x}), \dots)$$

$\Rightarrow$  we could ask whether this seq is  
bounded? or is it monotonically increasing?  
or is it even converging?

$\Rightarrow$  However we can ask more. all of these  
questions depend on  $\tilde{x}$ .

$\Rightarrow$  so we immediately see we have  
infinitely many seq<sup>n</sup> here

$\Rightarrow$  infinitely many questions about  
convergence

$\Rightarrow$  All these questions translate to  
the question this seq of functions  
 $(f_1, f_2, f_3, \dots)$  in some sense

## Convergence.

=> In order to answer this, we need to define what is the distance b/w two function's.

### Lec 24: Pointwise Convergence

Seq of functions:  $(f_1, f_2, f_3, f_4, \dots)$

$f_n: I \rightarrow \mathbb{R}$

#### Pointwise Convergence:

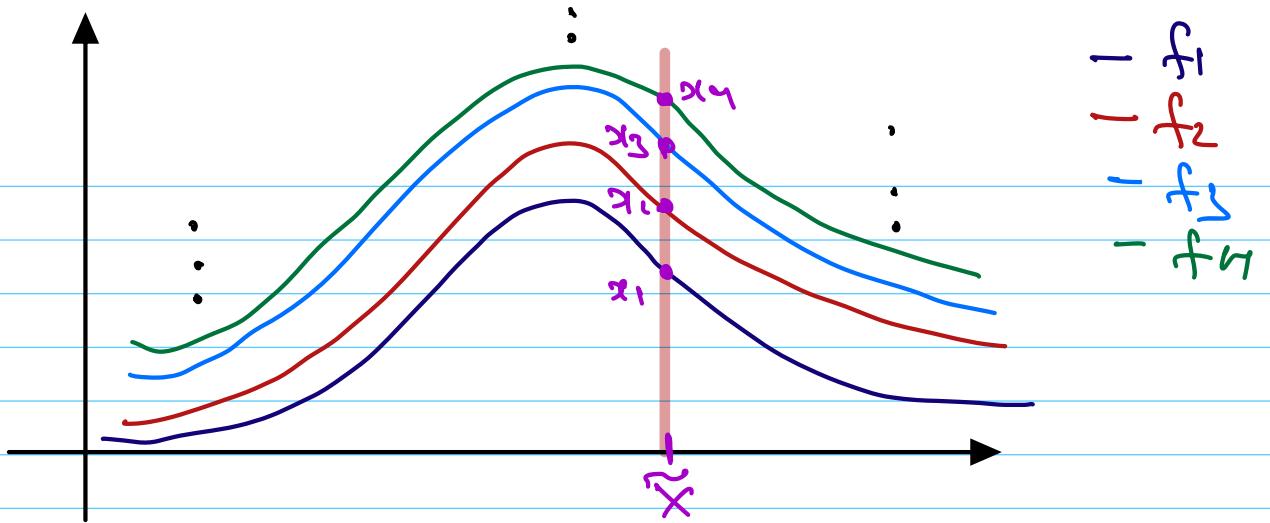
$(f_1, f_2, f_3, f_4, \dots)$  is

Pointwisely convergent to a function

$f: I \rightarrow \mathbb{R}$  if for all  $\tilde{x} \in I$ :

$(f_1(\tilde{x}), f_2(\tilde{x}), f_3(\tilde{x}), \dots)$

is convergent to  $f(\tilde{x})$

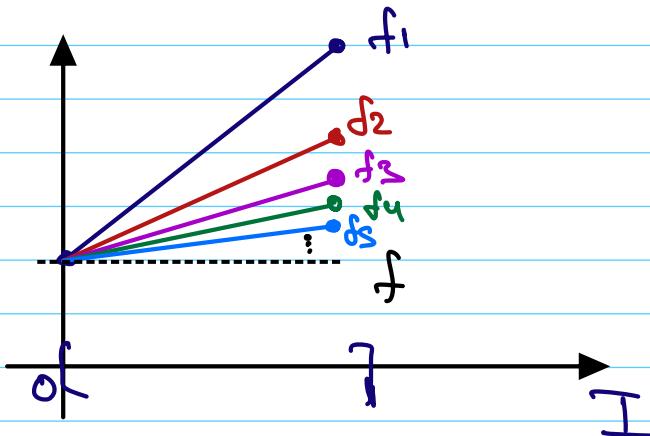


$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N: \quad |f_n(x) - f(x)| < \varepsilon$

Example:

$$f_n: [0, 1] \rightarrow \mathbb{R}$$

$$f_n(x) = \frac{1}{n}x + 1$$



for  $\tilde{x} \in [0,1] : f_n(\tilde{x}) = \frac{1}{n} \tilde{x} + 1$

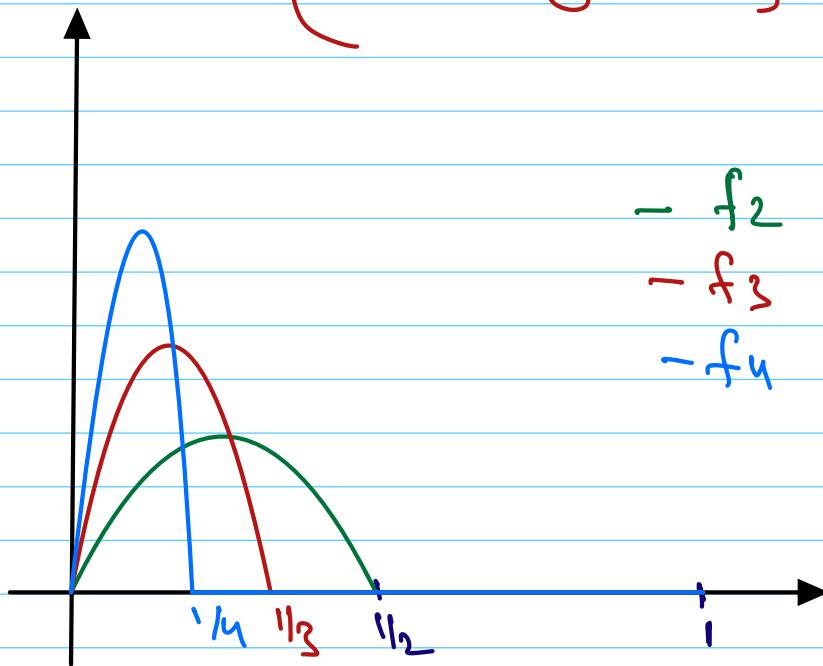
$$f_n(\tilde{x}) = \frac{1}{n} \tilde{x} + 1 \xrightarrow{n \rightarrow \infty} 1$$

$\Rightarrow$  (Pointwise) limit function

$$f: [0,1] \rightarrow \mathbb{R}, f(x) =$$

Example 2  $f_n: [0,1] \rightarrow \mathbb{R}$ ,

$$f_n(x) = \begin{cases} n^2(1-nx) & , x \in [0, \frac{1}{n}] \\ 0 & , x \in (\frac{1}{n}, 1] \end{cases}$$



DO we have Pointwise Continuous?

for  $x=0$  :  $f_n(x)=0 \quad \forall n \in \mathbb{N}$

for  $x>0$  :  $f_n(x) = 0 \quad \text{for all } n > \frac{1}{x}$

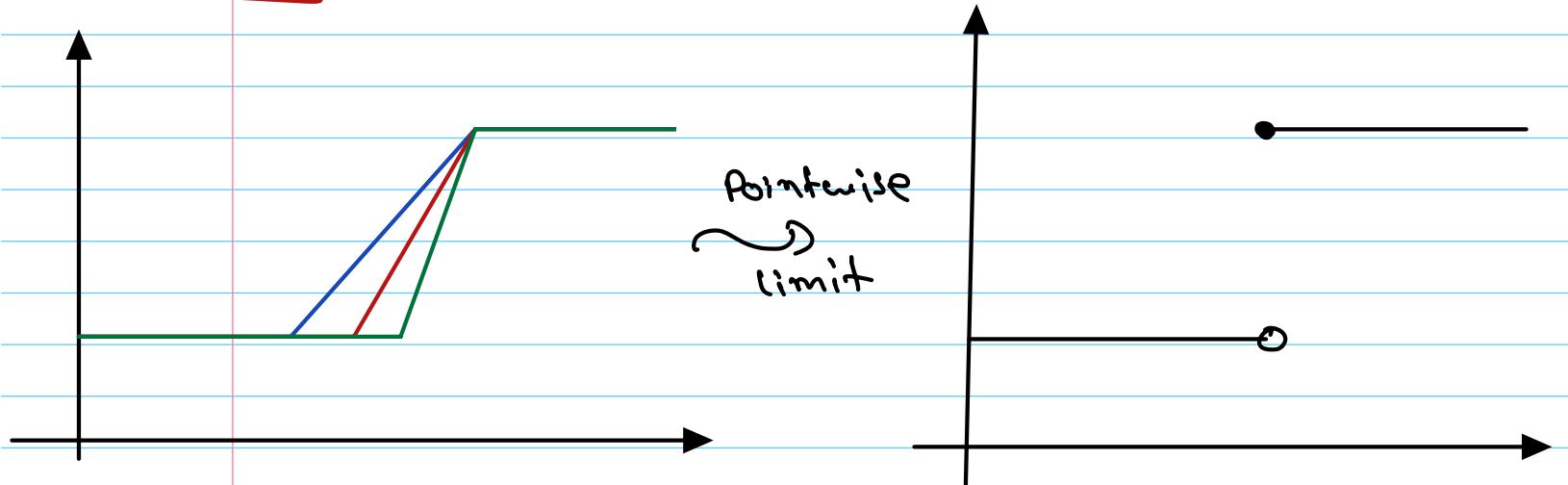
$\Rightarrow$  (Pointwise) limit function

$f: [0,1] \rightarrow \mathbb{R}, f(x) = 0$

Pointwise convergence is not strong enough

for our need's

Example:



\* we get a function with a Jump, although we stored the whole seq<sup>n</sup> without Jumps

That's indeed a good reason why we want a stronger convergence property.

## Lec 28: Uniform Convergence

---

$(f_1, f_2, f_3, f_4, \dots)$  is pointwise convergent

to  $f: I \rightarrow \mathbb{R}$  if

$$\forall x \in I, \forall \epsilon > 0, \exists N \in \mathbb{N} : |f_n(x) - f(x)| < \epsilon$$

\* with this formulation with the quantifiers we are allowed to choose different  $N \in \mathbb{N}$  depending on  $x$ .

\* This is the only thing we change to get the uniform convergence.

Def:  $(f_1, f_2, f_3, f_4, \dots)$  is

uniformly convergent to  $f: I \rightarrow \mathbb{R}$

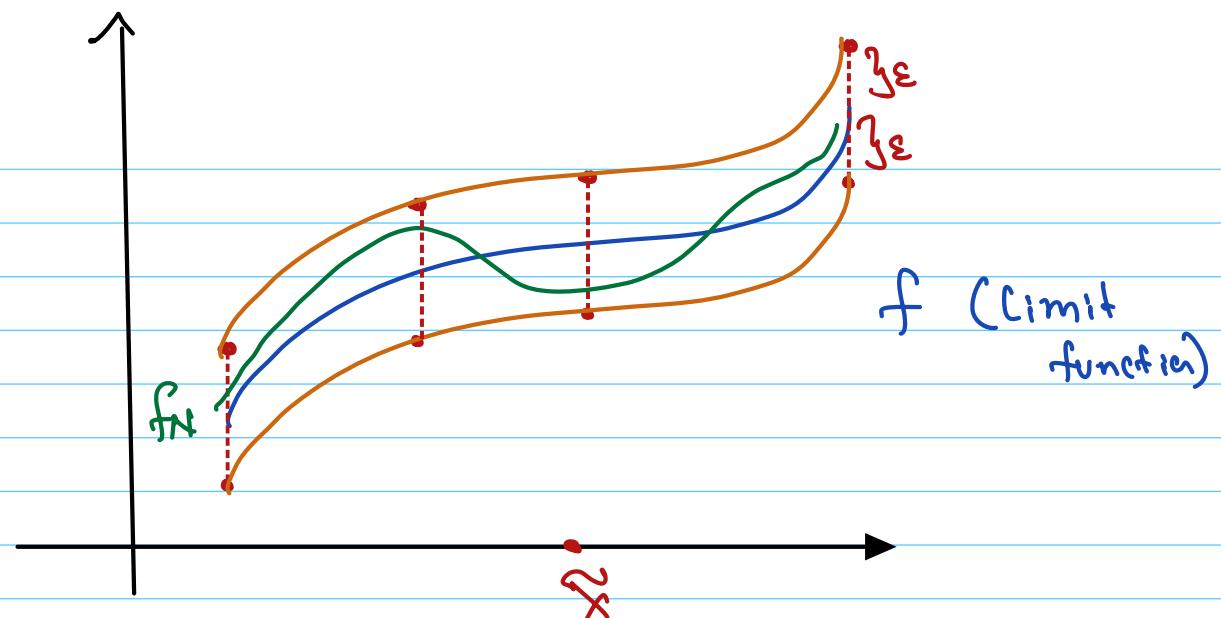
if  $\forall \epsilon > 0$ ,  $\exists N \in \mathbb{N}$  such that  $\forall n > N$

$$\forall \tilde{x} \in I : |f_n(\tilde{x}) - f(\tilde{x})| < \epsilon$$

$\Rightarrow$  Now the  $N \in \mathbb{N}$  works uniformly  
for all points  $\tilde{x} \in I$

$\Rightarrow$  that's different from before, because  
in Pointwise Convergence, we have  
 $N \in \mathbb{N}$  depending on  $\tilde{x}$ , so,  
there could be different  $N$  in  
Pointwise Convergence  
 $N(\tilde{x}) \in \mathbb{N}$

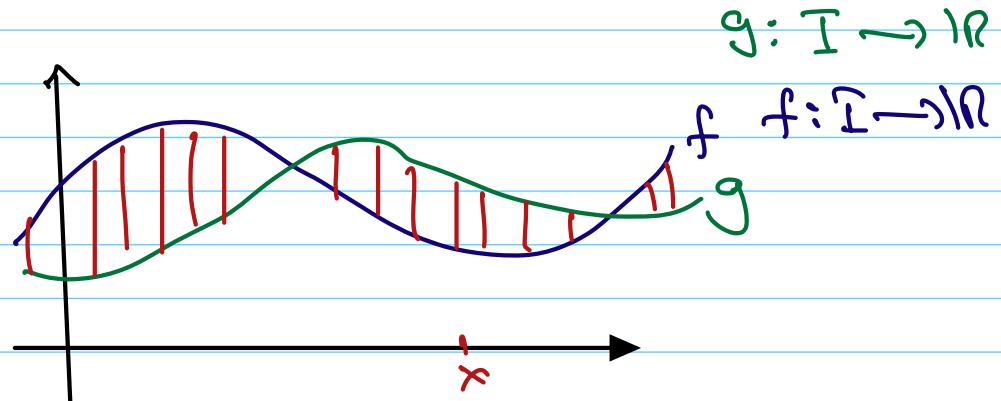
However for the uniform convergence  
 $N \in \mathbb{N}$  has to work for all  $\tilde{x} \in I$   
at the same time.



$\forall n > N \quad f_n(x) \text{ lies b/w}$   
the epsilon tube.

This also tells us its possible to  
measure distance b/w two functions

### Distance for function's



$$\|f - g\|_{\infty} = \sup_{x \in I} |f(x) - g(x)|$$

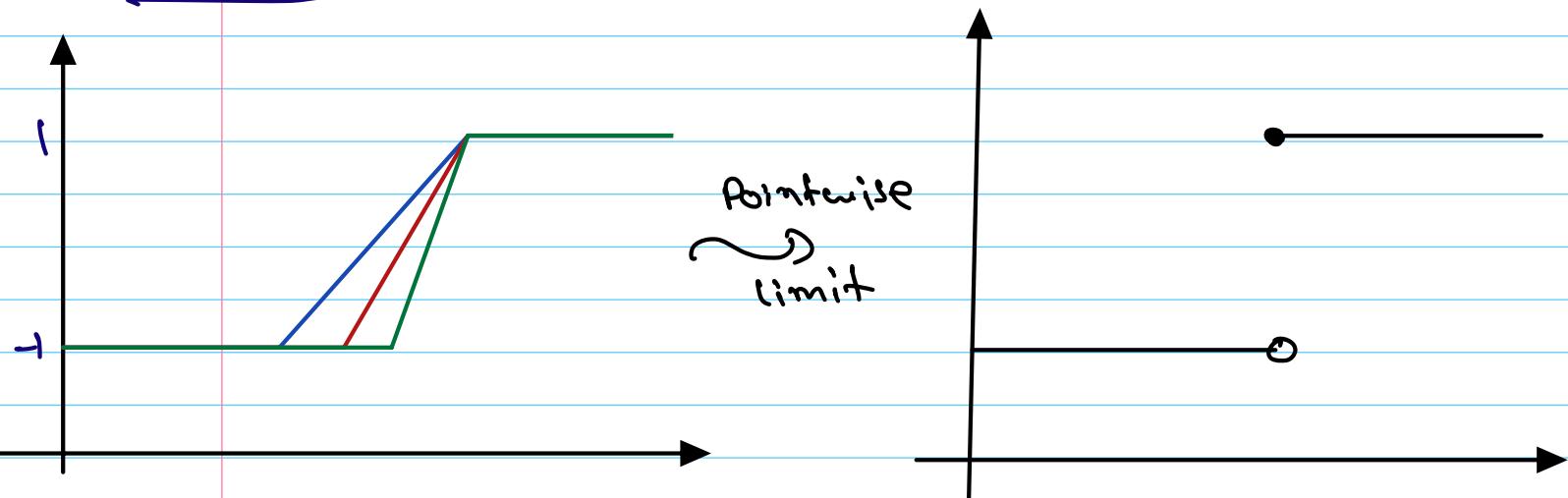
Supremum norm of  $f - g$

$$\|f - g\|_{\infty} = \sup_{x \in I} |f(x) - g(x)|$$

Uniform Convergence means:

$$\|f_n - f\|_{\infty} \xrightarrow{n \rightarrow \infty} 0$$

Example:



$$\|f_n - f\|_{\infty} \geq 1 \quad \forall n$$

Result:

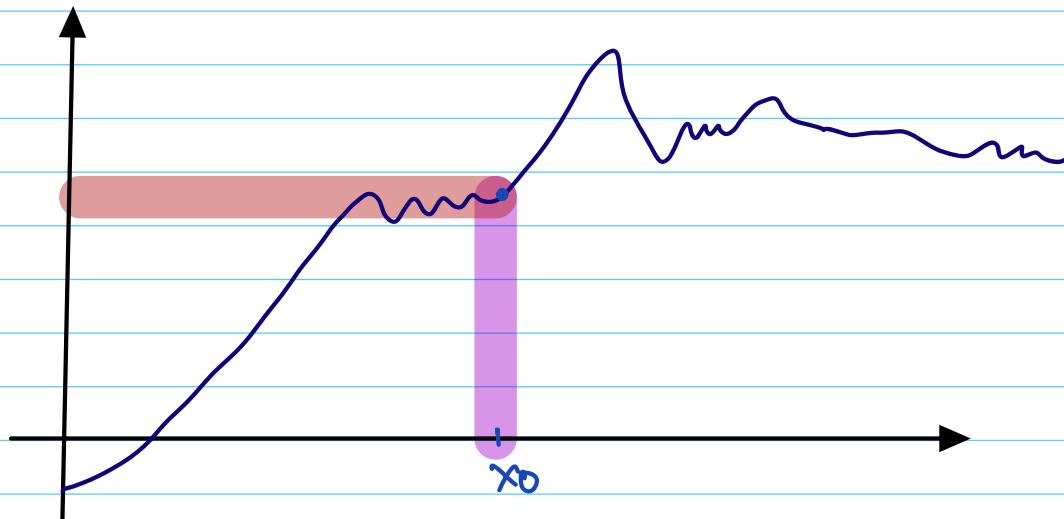
Pointwise convergence  $\not\Rightarrow$  Uniform convergence

Uniform Convergence is stronger than  
Pointwise Convergence.

- \* Uniform Convergence is very useful,  
as it conserves nice properties of  
functions (Ex, continuity)

## Lec 26 : Limits of Functions

The Property Continuous will be local  
Property.



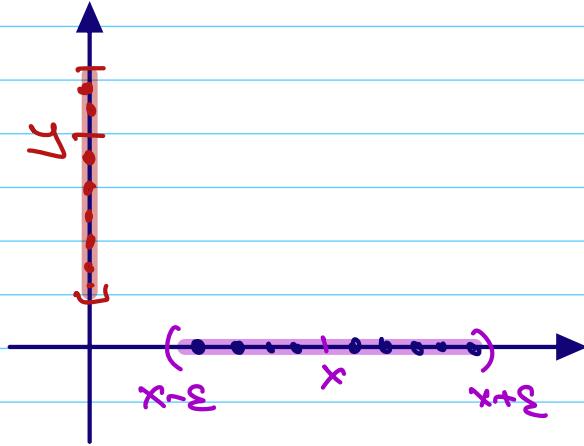
for a given function, we will  
define continuous for point  $x_0$ .

what it mean's  $f$  is continuous at  
point  $x_0$ ?

\* we want to know what happen's  
to the values of function when  $x_0$   
wobbles a little bit around.

$\Rightarrow$  Small error in  $x$  translates to small  
error's in  $y \Rightarrow$  this is what we  
call continuous at  $x_0$

Definition:



we have a input seq<sup>n</sup> around  $x_0$   
 Converge to  $x_0$ , and an output seq<sup>n</sup>  
 there in  $y$ -axis converge to  $y_0 = f(x_0)$

$\Rightarrow$  To really exclude Jum's around  $x_0$   
 we have to look at all possible seq<sup>n</sup>  
 here.

Def: Let  $f: I \rightarrow \mathbb{R}$ ,  $x_0 \in I$ , if there in  $c \in \mathbb{R}$

all seq<sup>n</sup>  $(x_n)_{n \in \mathbb{N}}$  with  $\lim_{n \rightarrow \infty} x_n = x_0$

we have  $(f(x_n))_{n \in \mathbb{N}}$  with also convergent with  
 $\lim_{n \rightarrow \infty} f(x_n) = c$

then we write

$$\boxed{\lim_{x \rightarrow x_0} f(x) = c}$$

This is the definition of limit when we  
 have a function  $f(\cdot)$ .

Definition:  $f: I \rightarrow \mathbb{R}$ ,  $x_0 \in I$  ( $x_0$  is fixed),  $c \in \mathbb{R}$  (fixed)

we have a seq<sup>n</sup>  $(x_n)_{n \in \mathbb{N}}$  that is convergent to  $x_0$

all seq<sup>n</sup>  $(x_n)_{n \in \mathbb{N}}$  with  $\lim_{n \rightarrow \infty} x_n = x_0$

and then we have output seq  $(f(x_n))_{n \in \mathbb{N}}$   
the we have

$(x_n)_{n \in \mathbb{N}}$  with  $\lim_{n \rightarrow \infty} x_n = x_0$

$(f(x_n))_{n \in \mathbb{N}}$  is also convergent with  $\lim_{n \rightarrow \infty} f(x_n) = c$

the limit  $c$  does not depend upon chosen

seq  $(x_n)_{n \in \mathbb{N}} \Rightarrow$  we get same number  $c$

for all seq  $(x_n)_{n \in \mathbb{N}} \rightarrow x_0$ . So we

fix  $c \in \mathbb{R}$

then we write

$$\boxed{\lim_{x \rightarrow x_0} f(x) = c}$$

$\lim_{x \rightarrow x_0} f(x)$  describes what happens  
with value  $f(x)$  when  $x$  gets closer and  
closer to  $x_0$ . This mean's that seq<sup>n</sup>  
 $(x_n)_{n \in \mathbb{N}}$  come from outside and converges  
to  $x_0$ , therefore the seq  $(x_n)$  does  
not have  $x_0$  as member.

It defined by using the limit notion  
that we had for seq<sup>n</sup>.

$$\lim_{x \rightarrow x_0} f(x) = c \quad (\text{It describes what happens with the values of } f(x) \text{ when } x \text{ gets closer & closer to } x_0)$$

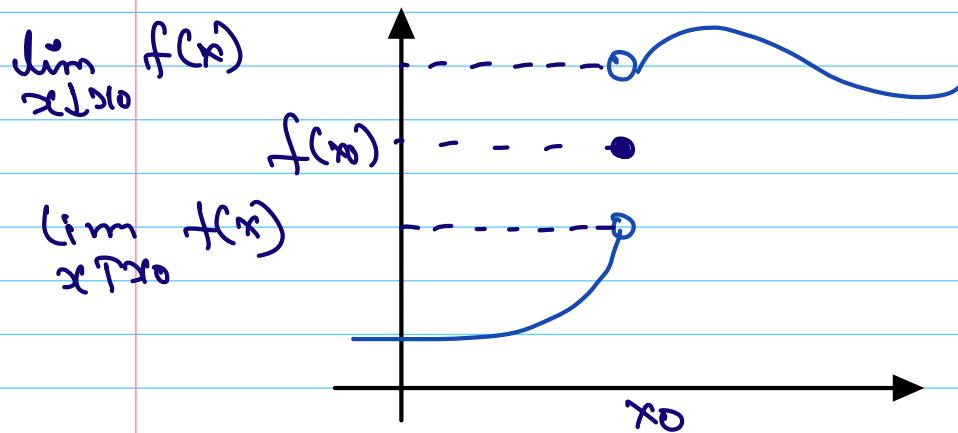
happens with the values of  $f(x)$  when  
 $x$  gets closer & closer to  $x_0$

and

$$\lim_{x \uparrow x_0} f(x) = c$$

if  $x_n < x_0 \ \forall n \in \mathbb{N}$

$$\lim_{x \downarrow x_0} f(x) = c \quad \text{if } x_n > x_0 \ \forall n \in \mathbb{N}$$



Ex:

(a)  $f(x) = \begin{cases} 0 & , x \neq 0 \\ 1 & , x = 0 \end{cases}$

$$\lim_{x \rightarrow 0} f(x) = 0 \neq 1 = f(0)$$

(Jump)

(b)  $f(x) = a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x^1$

$\rightarrow \infty$

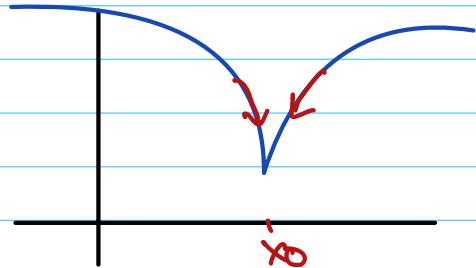
## Lec 27 : Continuity.

Definition:

Let  $f: I \rightarrow \mathbb{R}$  be a function with  $I \subseteq \mathbb{R}$ .

$f$  is called continuous at  $x_0 \in I$  if

$$\lim_{x \rightarrow x_0} f(x) = f(x_0)$$



This definition is a local property  $\Rightarrow$  we are fixing point  $x_0$  and zoom in around  $x_0$  and look what happens.

\* ALSO in definition of limits we have two direction's , right limit , left limit.

$$\lim_{x \rightarrow x_0} f(x) = f(x_0) = \lim_{x \uparrow x_0} f(x) = \lim_{x \downarrow x_0} f(x)$$

Definition: Let  $f: I \rightarrow \mathbb{R}$  be a function with  $I \subseteq \mathbb{R}$ .  $f$  is called continuous if  $f$  is continuous at  $x_0 \forall x_0 \in I$ .

To remember:

Continuity implies

$$\lim_{n \rightarrow \infty} f(x_n) = f\left(\lim_{n \rightarrow \infty} x_n\right)$$

(if  $\lim_{n \rightarrow \infty} x_n \in I$ )

Example: (a)  $f: I \rightarrow \mathbb{R}$  constant function

## Lec 28 : Epsilon-Delta:

Continuity:  $f$  is called continuous at  $x_0 \in I$

$$\text{if } \lim_{x \rightarrow x_0} f(x) = f(x_0)$$

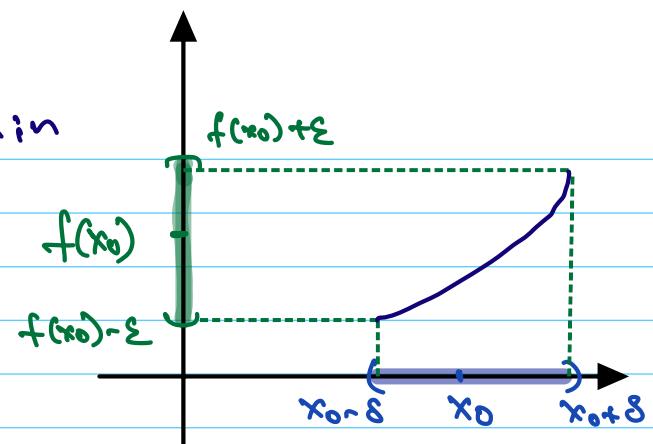
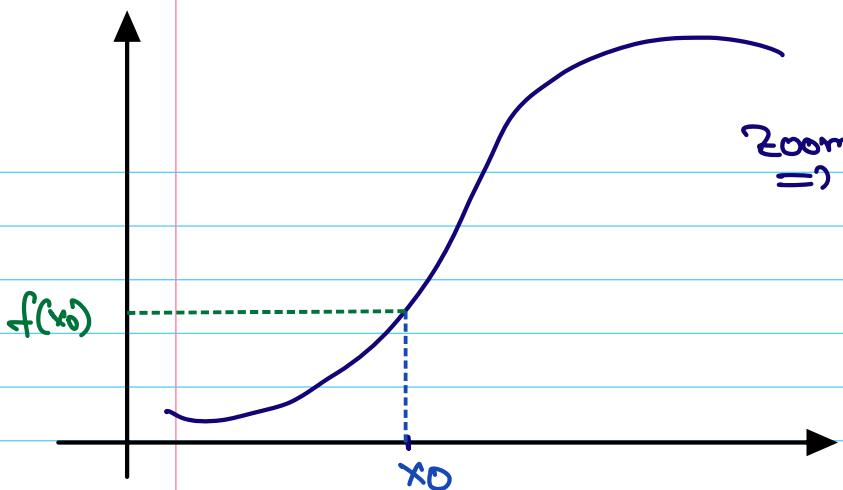
\* Here in this definition we have to work with seq<sup>n</sup>, simple because in the def of this limit, we have to put in seq<sup>n</sup>'s. Therefore if we want to avoid the seq<sup>n</sup>, The  $\epsilon-\delta$  criterion is very useful.

Theorem: Let  $f: I \rightarrow \mathbb{R}$  be a function with  $I \subseteq \mathbb{R}$ . For  $x_0 \in I$ , we have

$f$  is continuous at  $x_0 \in I$

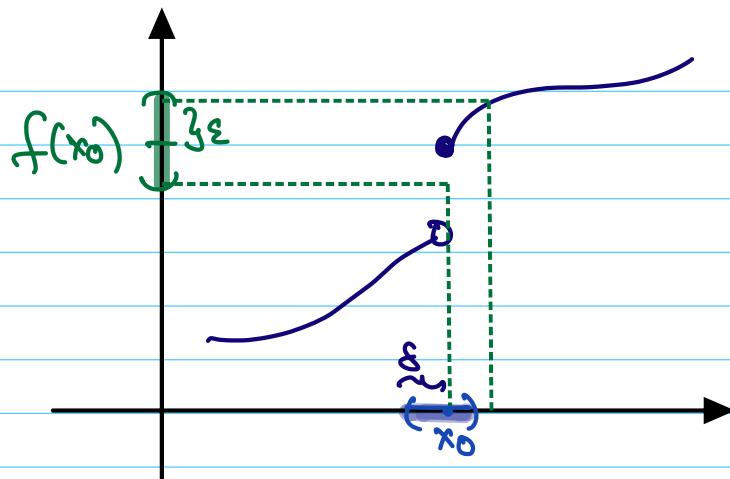
$\Leftrightarrow \forall \epsilon > 0, \exists \delta > 0$ , such that  $\forall x \in I$ :

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$$



- \* All the points that have distance less than  $\delta$  from  $x_0$  ( $|x-x_0|<\delta$ ) on the x-axis give us values on y-axis that full fill this  $|f(x)-f(x_0)|<\epsilon$  inequality.
- \* The close points on x-axis are sent to close points on y-axis. This is exactly what we want from continuity.
- \* This should work no matter how small we choose  $\epsilon$ . This won't work when we have jump in graph.

Example:



No matter how small we choose  $\delta$   
we choose for any fixed  $\epsilon$ , we would  
always hit points here on the left.