Exercises : Section 10

10.1

(a) $\frac{1}{n}$ decreasing, bounded

(b) $\frac{(-1)^n}{n^2}$ bounded

(c) n^5 increasing

(d) $\sin\left(\frac{n\pi}{7}\right)$ Bounded

(e) $(-2)^n$ neither

(f) $\frac{n}{3^n}$ decreasing, Bounded

10.2 Prove theorem 10.2 for Bounded decreasing seq^n

Let (S_n) be a bounded decreasing seq^n

$$S = \{S_n : n \in \mathbb{N}\}$$

let $v = \inf S \in \mathbb{R}$ (because (S_n) is Bounded)

we want to prove $\lim S_n = v$

Let $\varepsilon > 0$. Since $v+\varepsilon$ is not an lower Bound any more of set S, $\exists N$ s.t

$\forall n > N$ $\quad S_N < v+\varepsilon$

$S_n \leq S_N < v+\varepsilon$

$\Rightarrow \quad S_n < v+\varepsilon$

$\Rightarrow \quad S_n - v < \varepsilon$

& $\quad v \leq S_n \quad \forall n \in \mathbb{N}$ $\Big\}\ 0 \leq S_n - v$

$\Rightarrow \quad |S_n - v| < \varepsilon$

Hence S_n converges to v

$\Rightarrow \quad \boxed{\lim S_n = v}$

10.4 Prove theorem 10.4 (ii)

Sol^n if (S_n) is an unbounded decreasing seq^n, then $\lim S_n = -\infty$

Proof:

(S_n) unbounded decreasing seq^n

let $M < 0$.

The set $S = \{S_n : n \in \mathbb{N}\}$ is bounded from above by s_1, and unbounded below.

$\exists N \in \mathbb{N}$ where $S_N < M$

$\Rightarrow$ clear $\forall n > N$ $\quad S_n \leq S_N < M$

$\Rightarrow \forall n > N \quad S_n < M$

therefore $\boxed{\lim_{n \to \infty} S_N = -\infty}$

10.6 (a) let (S_n) be a seq^n s.t

$$|S_{n+1} - S_n| < \frac{1}{2^n} \quad \forall n \in \mathbb{N}$$

Prove (S_n) is a Cauchy seq^n and hence a convergent seq^n.

Sol^n

Discussion:

let $\varepsilon > 0$ then we want to prove

$\exists N$ s.t $\forall n, m > N$

$$|S_n - S_m| < \frac{1}{2^n} < \varepsilon$$

$$|S_{n+1} - S_n| < \frac{1}{2^n} < \varepsilon$$

$$\Rightarrow \frac{1}{2^n} < \varepsilon$$

$$\Rightarrow \log \frac{1}{2^n} < \log \varepsilon$$

$$\Rightarrow -n \log 2 < \log \varepsilon$$

$$\Rightarrow n \log 2 > -\log \varepsilon$$

$$\Rightarrow \quad n > -\frac{\log \varepsilon}{\log 2}$$

take $N = -\frac{\log \varepsilon}{\log 2}$

10.8

Let (S_n) be an increasing seq of +ve number's and define $\sigma_n = \frac{1}{n}(S_1 + S_2 + \cdots + S_n)$. Prove (σ_n) is an increasing seq^n.

Sol^n

(S_n) is increasing seq^n

$$\Rightarrow \quad S_{n+1} \geq S_n$$

$$\sigma_{n+1} = \frac{1}{n+1}(S_1 + S_2 + \cdots + S_n + S_{n+1})$$

$$\sigma_{n+1} = \frac{S_1 + S_2 + \cdots + S_n}{n+1} + \frac{S_{n+1}}{n+1}$$

$$\sigma_{n+1} = \frac{n}{n+1}\sigma_n + \frac{S_{n+1}}{n+1}$$

$$\Rightarrow \quad \sigma_{n+1} = \frac{n\sigma_n + S_{n+1}}{n+1}$$

as $S_{n+1} \geq \sigma_n$ (because σ_n is average)

$$\Rightarrow \quad \sigma_{n+1} = \frac{n\sigma_n + S_{n+1}}{n+1} \geq \frac{(n+1)\sigma_n}{\sigma_n}$$

$$\Rightarrow \quad \frac{\sigma_{n+1}}{\sigma_n} \geq 1 \quad \text{(Increasing) seq}^n$$

10.9 Let $S_1 = 1$, and $S_{n+1} = \left(\frac{n}{n+1}\right) S_n^2$ $\forall\, n \geq 1$

(a) Find S_2, S_3 S_4

$S_2 = \frac{1}{2} \cdot 1^2 = \frac{1}{2}$ $\quad S_3 = \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6}$

$S_4 = \frac{3}{4} \cdot \frac{1}{6^2} = \frac{1}{48}$

(b) show $\lim s_n$ exists.

$$s_{n+1} = \left(\frac{n}{n+1}\right) s_n^2$$

$$\Rightarrow \frac{s_{n+1}}{s_n} = \frac{n}{n+1} s_n$$

to show $\frac{s_{n+1}}{s_n}$ is a decreasing seq

we need to show

$$\frac{s_{n+1}}{s_n} \leq 1 \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \left(\frac{n}{n+1}\right) s_n \leq 1$$

$$\Rightarrow s_n \leq \frac{n+1}{n}$$

Proof By Induction:

Base: $S_1 = 1 < \frac{2}{1}$ (True)

Inductive step

Suppose $S_n \leq \frac{n+1}{n}$

then we need to show

$$S_{n+1} \leq \frac{n+2}{n+1}$$

$$\Rightarrow \quad S_{n+1} = \frac{n}{n+1} \cdot S_n^2$$

$$\leq \left(\frac{n}{n+1}\right)\left(\frac{n+1}{n}\right)^2$$

$$\boxed{S_{n+1} \leq \frac{n+1}{n}}$$

Hence $\frac{S_{n+1}}{S_n} \leq 1$

Hence the (S_n) is decreasing and

Bounded $\Rightarrow$ (S_n) is converges to $\in \mathbb{R}$

(10.9) c Prove $\lim S_n = 0$

Clearly $S_n \geq 0$ (+ve)

$$S_{n+1} = \left(\frac{n}{n+1}\right) S_n^2$$

* $\sup\{S_n : n \in \mathbb{N}\} = 1$

* The seq^n $(S_n)_{n\in\mathbb{N}}$ is monotonically decreasing and bounded below by zero

$\Rightarrow$ The limit of the seq^n is the infimum of set (S_n)

$$\Rightarrow \lim_{n\to\infty} S_n = \inf\{S_n : n \geq 0\}$$

$$= 0$$

10.10 Let $S_1 = 1$ and $S_{n+1} = \frac{1}{3}(S_n + 1)$ $\forall n \geq 1$

a) Find S_2, S_3, and S_4.

Sol^n

$S_1 = 1$

$S_2 = \frac{2}{3}$

$S_3 = \frac{5}{9}$

$S_4 = \frac{14}{27}$

b) Use Induction to show $S_n > \frac{1}{2}$ $\forall n$

i) <u>Base case:</u>

P_n: " $S_n > \frac{1}{2}$ $\forall n \geq 1$ "

$P_1 \Rightarrow S_1 = 1$ (True)

<u>Induction step:</u>

Suppose P_n is true

i.e $S_n > \frac{1}{2}$

we need to show P_{n+1} is true

i.e $\quad S_{n+1} > \frac{1}{2}$

$\Rightarrow \quad S_n > \frac{1}{2}$

$S_{n+1} = \frac{1}{3}(S_n + 1)$

$\Rightarrow \quad S_n > \frac{1}{2}$

$\Rightarrow \quad S_n + 1 > \frac{3}{2}$

$\Rightarrow \quad \frac{1}{3}(S_n + 1) > \frac{1}{2}$

$\Rightarrow \quad S_{n+1} > \frac{1}{2} \quad$ (True)

$$\boxed{\text{Hence} \quad S_n > \frac{1}{2} \quad \forall n \geq 1}$$

(c) Show that (S_n) is a decreasing seq^n

$S_{n+1} = \frac{1}{3}(S_n + 1)$

$S_{n+1} - S_n = \frac{1}{3} - \frac{2}{3} S_n$

we have determine $S_n > \frac{1}{2}$ $\forall n \geq 1$

$\Rightarrow$ $2S_n > 1$

$\Rightarrow$ $\frac{2S_n}{3} > \frac{1}{3}$

$\Rightarrow$ $-\frac{2S_n}{3} < -\frac{1}{3}$

$\Rightarrow$ $\frac{1}{3} - \frac{2S_n}{3} < 0$

$\Rightarrow$ $S_{n+1} - S_n < 0$ $\forall n \geq 1$

(Hence (S_n) is a decreasing seq^n)

(d) Show $\lim S_n$ exists and find $\lim S_n$

$(S_n)_{n \in \mathbb{N}}$ is monotonically decreasing and Bounded, Hence limit S_n exists.

as $n \to \infty$ $\lim s_n = \lim s_{n+1} = L$

$\Rightarrow$ $L = \frac{1}{3}(L+1)$

$\Rightarrow$ $L = \frac{1}{2}$

Hence $\lim_{n \to \infty} s_n = \frac{1}{2}$

(10.11) Let $t_1 = 1$ and $t_{n+1} = \left[1 - \frac{1}{4n^2}\right] t_n$ $\forall\, n \geq 1$

(a) Show $\lim t_n$ exists.

$$t_{n+1} = \left(1 - \frac{1}{4n^2}\right) t_n$$

$$\Rightarrow \frac{t_{n+1}}{t_n} = 1 - \frac{1}{4n^2}$$

$n \geq 1$

$\Rightarrow$ $n^2 \geq 1$ & $4n^2 \geq 4$

$$\Rightarrow \quad \frac{1}{4n^2} \leq \frac{1}{4}$$

$$\Rightarrow \quad 1 - \frac{1}{4n^2} \geq \frac{3}{4}$$

and $\quad \lim_{n \to \infty} 1 - \frac{1}{4n^2} = 1$

hence $\quad \frac{3}{4} \leq 1 - \frac{1}{4n^2} < 1$

Hence

$$\frac{3}{4} \leq \frac{t_{n+1}}{t_n} = 1 - \frac{1}{4n^2} < 1$$

$\Rightarrow$ $(t_n)_{n \in \mathbb{N}}$ is a monotonically decreasing seq

and always's +ve ie $t_n > 0$

Hence (t_n) exists and real.

(b) what do you think $\lim t_n$ is

$$t_{n+1} = \left(1 - \frac{1}{4n^2}\right) \cdot t_n$$

$$t_n = \prod_{k=1}^{n-1} \left(1 - \frac{1}{4k^2}\right)$$

$$\ln t_n = \sum_{k=1}^{n-1} \ln\left(1 - \frac{1}{4k^2}\right)$$

$$\ln(1-x) \approx -x$$

$$\Rightarrow \ln\left(1 - \frac{1}{4k^2}\right) \approx -\frac{1}{4k^2}$$

$$\Rightarrow \ln t_n \approx -1 \cdot \sum_{k=1}^{n-1} \frac{1}{4k^2}$$

$$\Rightarrow \ln t_n \approx -\frac{1}{4} \sum_{k=1}^{n-1} \frac{1}{k^2}$$

$$\lim_{n\to\infty} \ln t_n \approx -\frac{1}{4} \underbrace{\sum_{k=1}^{\infty} \frac{1}{k^2}}_{\frac{\pi^2}{6}}$$

$$\Rightarrow \ln t \approx \frac{-\pi^2}{24}$$

$$\Rightarrow t \approx e^{\frac{-\pi^2}{24}}$$

10.12 Let $t_1 = 1$ and $t_{n+1} = \left(1 - \frac{1}{(n+1)^2}\right) \cdot t_n$

$\forall n \geq 1$

(a) Show that $\lim t_n$ exists?

$$n \geq 1 \Rightarrow (n+1)^2 \geq 4$$

$$\Rightarrow \frac{1}{(n+1)^2} \leq \frac{1}{4}$$

$$\Rightarrow 1 - \frac{1}{(n+1)^2} \geq \frac{3}{4}$$

$$\lim_{n\to\infty} 1-\frac{1}{(n+1)^2} = 1$$

$$\Rightarrow \frac{3}{4} \le \frac{t_{n+1}}{t_n} < 1$$

Hence t_{n+1} is monotonically decreasing and bounded below by 0 i.e $t_n > 0 \ \forall n \geq 1$

Hence t_n limit converges & exits.

(b)

What do you think t_n is?

$$t_n = \prod_{k=1}^{n-1} \left[1-\frac{1}{(n+1)^2}\right]$$

$$\ln t_n = \sum_{k=1}^{n-1} \ln\left(1-\frac{1}{(n+1)^2}\right]$$

$$\ln t_n \approx \sum_{k=1}^{n-1} -\frac{1}{(n+1)^2}$$

$$\lim_{n\to\infty} \ln t_n \approx -1 \cdot \sum_{k=1}^{\infty} \frac{1}{(n+1)^2}$$

$$\approx -1 \left[\sum_{k=2}^{\infty} \frac{1}{k^2} \right]$$

$$\approx -1 \left[\sum_{k=1}^{\infty} \frac{1}{k^2} - 1 \right]$$

$$\approx -1 \left[\frac{\pi^2}{6} - 1 \right]$$

$$\approx 1 - \frac{\pi^2}{6}$$

$$\boxed{t = e^{\left(1 - \frac{\pi^2}{6}\right)}}$$