24. Uniform Convergence

24.1 Def:

Let (f_n) be a seq^n of real-valued function's defined on a set $\subseteq \mathbb{R}$. The seq (f_n) converges pointwise [i.e, at each point] to a function f defined on S if

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \forall x \in S.$$

we often write

(i) $\lim f_n = f$ pointwise [on S]

(ii) $f_n \longrightarrow f$ pointwise [on S]

EX 2

Let $f_n(x) = x^n$ for $x \in [0,1]$.

Then $f_n \longrightarrow f$ pointwise on $[0,1]$

where $f(x) = 0 \;\forall x \in [0,1)$ and $f(1) = 1$

$\forall \varepsilon > 0$ and $\forall x \in S$ $\exists N \in \mathbb{N}$ s.t

$\forall n > N$ we have $|f_n(x) - f(x)| < \varepsilon$.

$N(\varepsilon, x)$.

Def 24.2:

Let (f_n) be a seq of real-valued functions on a set $S \subseteq \mathbb{R}$. The seq (f_n) converges uniformly on S to a function f defined on S if

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, such that $\forall n > N$ we have

$$|f_n(x) - f(x)| < \varepsilon \quad \forall x \in S.$$

we write $\lim f_n = f$ uniformly on S

or $f_n \to f$ uniformly on S.

Ex4: let $f_n(x) = \frac{1}{n}\sin nx$ for $x \in \mathbb{R}$

then $f_n \to 0$ Pointwise on $\mathbb{R}$ as shown in Exercise 23.8.

<u>Uniform convergence:</u>

let $\varepsilon > 0$,

$$f_n(x) - f(x) = \frac{1}{n}\sin nx$$

$$\left|\frac{1}{n}\sin nx\right| \leq \frac{1}{n} < \varepsilon$$

$$\Rightarrow \quad n > \frac{1}{\varepsilon}$$

take $N = \frac{1}{\varepsilon}$, the $\forall n > \frac{1}{\varepsilon} = N$

we have $\left|\frac{1}{n}\sin nx - 0\right| < \varepsilon \quad \forall x \in \mathbb{R}$

$\Rightarrow$ $f_n(x) = \frac{1}{n}\sin nx$ converges uniformly

EXS: $f_n(x) = nx^n \quad \forall x \in (0,1)$

$f(x) = 0$

$f_n(x) - f(x) \quad \forall x \in [0,1)$

$\Rightarrow |nx^n - 0|$

$\Rightarrow |nx^n|$

if $f_n(x)$ uniformly converges to $f(x)$

then $\exists N \in \mathbb{N}$ for all $\varepsilon > 0$ such that

$|f_n(x) - f(x)| < \varepsilon \quad \forall n > N, \forall x \in [0,1)$

take $\varepsilon = 1$

$|nx^n| < 1 \quad \forall n > N, \forall x \in [0,1)$

this fails.

24.3 Theorem:

The uniform limit of continuous function's is continuous. More Precisely, let (f_n) be a seq of function's on a set $S \subseteq \mathbb{R}$; suppose $f_n \to f$ uniformly on S, and suppose $S = \mathrm{dom}(f)$.

if f_n is continuous at x_0 in S, then f is continuous at x_0.

$\Rightarrow$ SO if each f_n is continuous on S, then f is continuous on S.

Proof: Uniform convergence $\quad \varepsilon' = \frac{\varepsilon}{3}$

$\forall \varepsilon' > 0$, $\exists N \in \mathbb{N}$ such that

$\forall x \in S$, $\forall n > N$ we have

$$|f_n(x) - f(x)| < \varepsilon'$$

<u>Continuity of f_n:</u> for $x_0 \in S$

$\forall \varepsilon' > 0$, $\exists \delta > 0$ with $|x - x_0| < \delta$

$$\Rightarrow |f_n(x) - f_n(x_0)| < \varepsilon'$$

we want to prove continuity of $f(x)$ at x_0

$$|f(x) - f(x_0)|$$

$$= \left| f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0) \right|$$

$$\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$$

$$< \varepsilon' + \varepsilon' + \varepsilon' = \varepsilon$$

$\Rightarrow$ we have $\forall \varepsilon > 0$, for $x_0 \in S$, we have

$\exists N \in \mathbb{N}$ such that $\forall n > N$

$|x - x_0| < \delta$, $|f(x_0) - f(x)| < \varepsilon$

Hence $f(x)$ is continuous at x_0

24.4 Remark:

Uniform convergence can be reformulated as follow's.

A seq (f_n) of function's on a set $S \subseteq \mathbb{R}$ converges uniformly to a function f on $S \Longleftrightarrow$

$$\lim_{n \to \infty} \sup \{ |f(x) - f_n(x)| : x \in S \} = 0$$

Ex: $f_n(x) = \dfrac{x}{1+nx^2} \quad \forall x \in R$

Sol^n

$$\lim_{n\to\infty} f_n(x)$$

$$= \lim_{n\to\infty} \frac{x}{1+nx^2} \qquad \forall x \neq 0$$

$$= \lim_{n\to\infty} \frac{\frac{x}{n}}{\frac{1}{n}+x^2}$$

$$= 0$$

therefore $f_n(x) \longrightarrow 0$ Pointwise for $x \in R$

find max, min of $f_n(x)$

$$\Rightarrow \frac{d}{dx}\left(\frac{x}{1+nx^2}\right)$$

$$\Rightarrow \quad \frac{(1+nx^2) - x(2nx)}{(1+nx^2)^2} = 0$$

$$\Rightarrow \quad 1 + nx^2 - 2nx^2 = 0$$

$$\Rightarrow \quad nx^2 = 1$$

$$\Rightarrow \quad x = \pm \frac{1}{\sqrt{n}}$$

max $f_n(x)$ is at $x = \frac{1}{\sqrt{n}}$

min at $x = -\frac{1}{\sqrt{n}}$.

$$f_n\left(\pm \frac{1}{\sqrt{n}}\right) = \pm \frac{1}{2\sqrt{n}}$$

$$\sup\{|f_n(x)| : x \in S\} = \lim_{n \to \infty} \frac{1}{2\sqrt{n}}$$

$$= 0$$

There fore $f_n \to 0$ uniformly on $\mathbb{R}$.

Ex 8

$$f_n(x) = n^2 x^n (1-x) \qquad x \in [0,1]$$

Sol^n

$$\lim_{n\to\infty} f_n(x)$$

$$\Rightarrow \lim_{n\to\infty} n^2 x^n (1-x) \qquad \text{for } x \neq 1$$

$$\Rightarrow (1-x) \lim_{n\to\infty} x^n \cdot n^2$$

we need to find $\lim_{n\to\infty} x^n \cdot n^2$

find $\lim_{n\to\infty} \left| \frac{S_{n+1}}{S_n} \right|$

$$= \lim_{n\to\infty} \frac{x^{n+1} \cdot (n+1)^2}{x^n \cdot n^2}$$

$$\Rightarrow \lim_{n\to\infty} x \cdot \left(1+\frac{1}{n}\right)^2$$

$$\Rightarrow \quad x \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2$$

$$\Rightarrow \quad x \in [0,1) < 1$$

Hence $$\lim_{n\to\infty} n^2 x^n = 0$$

$$\Rightarrow \lim_{n\to\infty} f_n(x) = 0 \qquad \forall x \in [0,1)$$

$$\lim_{n\to\infty} f_n(1) = 1 \qquad x=1$$

$\Rightarrow \quad f_n \to 0$ Pointwise on $[0,1]$.

Let's find max & min of $f_n(x)$

$$\frac{d}{dx} f_n(x) = \frac{d}{dx} n^2 x^n (1-x)$$

$$= n^2 \frac{d}{dx} \left(x^n - x^{n+1}\right)$$

$$\Rightarrow n^2\left(n x^{n-1} - (n+1) x^{n} \right) = 0$$

$$\Rightarrow \boxed{\frac{n}{n+1} = x}$$

$$f_n\left(\frac{n}{n+1}\right) = n^2 \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right)$$

$$= \frac{n^2}{n+1} \left(\frac{n}{n+1}\right)^n$$

$$\lim_{n\to\infty} \sup\left\{ |f(x) - f_n(x)| : \forall x \in [0,1] \right\}$$

$$= \lim_{n\to\infty} \left(\frac{n^2}{n+1}\right) \cdot \left(\frac{n}{n+1}\right)^n$$

$$= \frac{1}{e} \lim_{n\to\infty} \left(\frac{n^2}{n+1}\right)$$

$$= \frac{1}{e} + \infty = +\infty$$

$\Rightarrow$ $f_n(x)$ does not converge uniformly