17: Continuous Function's

Continuous & uniformly continuous function's.

ⓐ $dom(f)$ is called domain of f

$dom(f) \subseteq \mathbb{R}$ when f is a Real valued function.

Definition 17.1

Let f be a real-valued function whose domain is a subset of $\mathbb{R}$.

$$dom(f) \subseteq \mathbb{R}.$$

The function f is continuous at x_0 in $dom(f)$ if, for every seq (x_n) in $dom(f)$ converging to x_0, we have $\lim_n f(x_n) = f(x_0)$

Our definition say's that the values of $f(x)$ are close to $f(x_0)$ when the value x are close to x_0.

Theorem 17.2:

Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. Then f is continuous at x_0 in $\mathrm{dom}(f)$ $\iff$

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $x \in \mathrm{dom}(f)$ and $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$ — (1)

Proof:

Consider a seq^n $(x_n) \in \mathrm{dom}(f)$ s.t $\lim x_n = x_0$.

We need to prove $\lim_{n \to \infty} f(x_n) = f(x_0)$.

Let $\varepsilon > 0$. By (1) $\exists \delta > 0$ such that

$x \in \mathrm{dom}(f)$ and $|x - x_0| < \delta$

$\Rightarrow \quad |f(x) - f(x_0)| < \varepsilon$

Since $\lim x_n = x_0$, $\exists N \in \mathbb{N}$ s.t $\forall n > N$

$$|x_n - x_0| < \delta$$

for $\varepsilon > 0$

$\Rightarrow \quad \forall n > N \quad |f(x_n) - f(x_0)| < \varepsilon$

$$\Rightarrow \quad \lim_{n \to \infty} f(x_n) = f(x_0)$$

Now assume f is continuous at x_0, but ① fails.

Then $\exists \varepsilon > 0$, such that

" $x \in dom(f)$ and $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$ "

fails for each $\delta > 0$.

$\Rightarrow$ " $x \in dom(f)$ and $|x - x_0| < \frac{1}{n} \Rightarrow |f(x) - f(x_0)| < \varepsilon$

fails for $n \in \mathbb{N}$.

So $\forall n \in \mathbb{N}$ $\exists x_n \in dom(f)$ such that

$|x_n - x_0| < \frac{1}{n}$ and yet $|f(x_0) - f(x_n)| \geq \varepsilon$.

$\Rightarrow$ we have $\lim x_n = x_0$, but we cannot have

$$\lim_{n\to\infty} f(x_n) = f(x_0)$$

$\Rightarrow$ This shows f is not continuous at x_0

Contradiction

Example:

$f(x) = 2x^2+1$ $x \in \mathbb{R}$. Prove f is continuous on $\mathbb{R}$ by

ⓐ Using the definion.

Sol: Suppose $(x_n)_{n\in\mathbb{N}}$ is a seq^n with

$$\lim_{n\to\infty} x_n = x_0$$

$\Rightarrow$

$$\lim f(x_n) = \lim 2x_n^2+1$$

$$= 2(\lim x_n)^2+1$$

$$= 2x_0^2+1$$

$$= f(x_0)$$

$\Rightarrow$ f is continuous at each $x_0 \in \mathbb{R}$

(b) Using the ε-δ theorem.

Sol: Let $x_0 \in \mathbb{R}$, let $\varepsilon > 0$. We want to show $|f(x) - f(x_0)| < \varepsilon$ provided $|x - x_0|$ is significantly small.

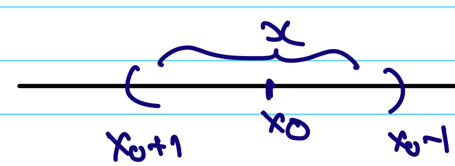
$$|f(x) - f(x_0)| = |2x^2 + 1 - (2x_0^2 + 1)|$$

$$= 2|x^2 - x_0^2|$$

$$= 2|(x - x_0)(x + x_0)|$$

$$= 2|x - x_0||x + x_0|$$

if $|x - x_0| < 1$



then $x_0 - 1 < x < x_0 + 1$

$\Rightarrow$ $|x| < |x_0| + 1$

$\Rightarrow$ $|x + x_0| \leq |x| + |x_0| < 2|x_0| + 1$

$$\Rightarrow |f(x) - f(x_0)| = 2|x - x_0| \cdot |x + x_0|$$

$$< 2|x - x_0| (2|x_0| + 1)$$

provided $|x - x_0| < 1$

to make $2|x - x_0| (2|x_0| + 1) < \varepsilon$

$$\Rightarrow |x - x_0| < \frac{\varepsilon}{2(2|x_0| + 1)}$$

$$\& \quad |x - x_0| < 1$$

$\Rightarrow$ take $\delta = \min\left\{1, \frac{\varepsilon}{2(2|x_0|+1)}\right\}$

$\Rightarrow$ we show $|x - x_0| < \delta$

$$\Rightarrow |f(x) - f(x_0)| < \varepsilon$$

Hence $f(x)$ is continuous.

Ex2:- $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ for $x \neq 0$

and $f(0) = 0$.

Prove f is continuous at $x = 0$

<u>Sol</u>

Solⁿ let $\varepsilon > 0$, clearly

$$|f(x) - f(0)| = |f(x)|$$

$$\Rightarrow |f(x)| = \left| x^2 \sin\left(\frac{1}{x}\right)\right| \leq x^2 < \varepsilon$$

$\forall x$

$$|x| < \sqrt{\varepsilon}$$

$$|x - 0| < \sqrt{\varepsilon}$$

take $\delta = \sqrt{\varepsilon}$

$\Rightarrow$ for $|f(x) - f(0)| < \varepsilon$

we have $|x - 0| < \sqrt{\varepsilon}$

Hence $f(x)$ is continous at $x=0$

Ex3 let $f(x) = \frac{1}{x} \sin\left(\frac{1}{x^2}\right)$ for $x \neq 0$

and $f(0) = 0$. Show f is discontinuous

i.e not continuous at $x=0$

Sol^n $f(x) = \frac{1}{x}\sin\left(\frac{1}{x^2}\right)$ let $\varepsilon > 0$

$$|f(x) - f(x_0)| = |f(x)|$$

$$\Rightarrow |f(x)| = \left|\frac{1}{x}\sin\left(\frac{1}{x^2}\right)\right| \leq \left|\frac{1}{x}\right| < \varepsilon$$

$$\Rightarrow \frac{1}{\varepsilon} < |x|$$

$$\Rightarrow \frac{1}{\varepsilon} < |x - 0|$$

we cannot find such δ such that

$$|x - 0| < \delta$$

Hence $f(x)$ is discontinuous.

Theorem 17.3:

Let f be a real-valued function with $dom(f) \subseteq \mathbb{R}$. if f is continuous at x_0 in $dom(f)$, then $|f|$ and kf, $k \in \mathbb{R}$ are continuous at x_0

Proof:

let $(x_n)_{n \in \mathbb{N}} \in dom(f)$

with $\lim_{n \to \infty} x_n = x_0$

Since f is continuous at x_0,

we have $\lim_{n \to \infty} f(x_n) = f(x_0)$

$$\Rightarrow \lim_{n \to \infty} k f(x_n) = k \lim_{n \to \infty} f(x_n)$$

$$= k f(x_0)$$

$\Rightarrow$ $kf(x)$ is continuous.

<u>Proof $|f|$</u>

we need to prove

$$\lim_{n\to\infty} |f(x_n)| = |f(x_0)|$$

$$\Big| \, |f(x_n)| - |f(x_0)| \, \Big|$$

$$\leq |f(x_n) - f(x_0)|$$

$\Rightarrow$ let $\varepsilon > 0$. since $\lim f(x_n) = f(x_0)$

$\exists N$ s.t $n > N \Rightarrow |f(x_n) - f(x_0)| < \varepsilon$

$$\Rightarrow \Big| |f(x_n)| - |f(x_0)| \Big| \leq |f(x_n) - f(x_0)| < \varepsilon$$

$$\Rightarrow \Big| \, |f(x_n)| - |f(x_0)| \, \Big| < \varepsilon$$

$\forall n > N$

$$\Rightarrow \lim |f(x_n)| = |f(x_0)|$$

1. $(f+g)(x) = f(x) + g(x)$

2. $fg(x) = f(x)\,g(x)$

3. $f/g\,(x) = \dfrac{f(x)}{g(x)}$

4. $g\circ f(x) = g\big(f(x)\big)$

5. $\max(f,g)(x) = \max\{f(x), g(x)\}$

6. $\min(f,g)(x) = \min\{f(x), g(x)\}$

domain of $f+g$, fg, $\max(f,g)$, $\min(f,g)$

is $\operatorname{dom}(f) \cap \operatorname{dom}(g)$

domain of $f/g(x)$ is

$\operatorname{dom}(f) \cap \{x \in \operatorname{dom}(g) : g(x) \neq 0\}$

domain of $g\circ f\,(x)$

is $\{x \in \operatorname{dom}(f) : f(x) \in \operatorname{dom}(g)\}$

These new function's are continuous if f, g are continuous

Theorem 17.4:

Let f and g are real-valued function that are continuous at x_0 in $\mathbb{R}$. Then

1. $f+g$ is continuous at x_0
2. fg is continuous at x_0
3. $\frac{f}{g}$ is continuous at x_0 if $g(x_0) \neq 0$

Proof:

$$x_0 \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$$

$(x_n)_{n \in \mathbb{N}} \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ with

$\lim\limits_{n \to \infty} x_n = x_0$, we have

$\lim f(x_n) = f(x_0)$, $\lim g(x_n) = g(x_0)$

$$\lim (f+g)(x_n) = \lim f(x_n) + \lim g(x_n)$$
$$= f(x_0) + g(x_0)$$
$$= (f+g)(x_0)$$

$\Rightarrow$ $f+g$ is continuous

Simillary

$$\lim_{n\to\infty} (fg)(x_n) = \lim_{n\to\infty} f(x_n)\, g(x_n)$$
$$= \lim f(x_n) \lim g(x_n)$$
$$= f(x_0)\, g(x_0)$$
$$= (fg)(x_0)$$

$\Rightarrow$ (fg) is continuous

Simillarly for $\frac{f}{g}$

x_0, $(x_n)_{n\in\mathbb{N}} \in \operatorname{dom} f \cap \{x \in \operatorname{dom}(g) : g(x) \neq 0\}$

S.t $\lim x_n = x_0$

$$\lim \left(\frac{f}{g}\right) x_n = \lim \frac{f(x)}{g(x)} = \frac{f(x_0)}{g(x_0)} = \left(\frac{f}{g}\right)(x_0)$$

Theorem 17.5:

f is continuous at x_0, g is continuous at $f(x_0)$ then the $g \circ f$ is continuous at x_0

Proof:

$x_0 \in dom(f)$ and $f(x_0) \in dom(g)$

$(x_n)_{n \in \mathbb{N}} \in \{x \in dom(f) : f(x) \in dom(g)\}$

s.t $\lim x_n = x_0$

we know $\lim_{n \to \infty} f(x_n) = f(x_0)$ &

$\lim_{n \to \infty} g(f(x_n)) = g(f(x_0))$

$$\lim g \circ f(x_n) = \lim g(f(x_n))$$
$$= g \circ f(x_0)$$

$\Rightarrow$ $g \circ f$ is continuous at x_0

Ex5 if f, g are continuous at $x_0 \in \mathbb{R}$

Prove $\max(f, g)$ is continuous at $x_0 \in \mathbb{R}$

Sol^n

$$\max(f, g) = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$$