09 – Limit theorem's for seq^n

A seq^n $(S_n) \in \mathbb{R}$ is said to be bounded seq^n if the set $\{S_n : n \in N\}$ is a bounded set. i.e $\exists M$ constant such that $|S_n| \leq M \quad \forall n \in \mathbb{N}$

Theorem 9.1:-

Convergent seq^n are Bounded

Proof:

S_n is a convergent seq^n

Therefore for $\varepsilon = 1$, $\exists N \in \mathbb{N}$ such that for all $n > N$

$$|S_n - S| < 1$$

* apply triangle inequality to

$S_n - S,\ S$

$$|S_n - S + S| \leq |S_n - S| + |S|$$

$$\Rightarrow \quad |S_n| \leq |S_n - S| + |S|$$

$$\Rightarrow \quad |S_n| \leq |S_n - S| + |S| \leq 1 + |S|$$

$$\Rightarrow \quad |S_n| \leq 1 + |S|$$

Lets Define

$$M = \max\{|S_1|, |S_2|, |S_3|, \dots |S_{N-1}|, |S_N|, 1+|S|\}$$

Because for all $n > N$ $|S_n| < |S| + 1$

Hence all convergent seq^n are Bounded.

Theorem 9.2 :-

if the sequence (S_n) converges to S and $k \in \mathbb{R}$, then the seq^n (kS_n) converges to KS. That is $\lim (KS_n) = K \cdot \lim S_n$

Proof:

$K \neq 0$

let $\varepsilon > 0$, we want to show

$|KS_n - KS| < \varepsilon$ for large n

Since $S_n \to S$, $\exists N \in \mathbb{N}$, such that

$\forall n > N$ $\quad |S_n - S| < \dfrac{\varepsilon}{|K|}$

$\Rightarrow$

$|KS_n - KS| < \varepsilon$

Therefore (KS_n) converges to KS

$\Rightarrow \lim KS_n = K \cdot \lim S_n$

Theorem 9.3

if s_n converges to s, and t_n converges to t, then

$$\lim(s_n+t_n) = \lim s_n + \lim t_n$$

Proof: Let $\varepsilon>0$, $\exists N_1 \in \mathbb{N}$ such that $\forall n > N_1$

$$|s_n - s| < \frac{\varepsilon}{2}$$

Similarly for the same $\varepsilon>0$, $\exists N_2 \in \mathbb{N}$ such that $\forall n > N_2$

$$|t_n - t| < \frac{\varepsilon}{2}$$

Now take $N = \max\{N_1, N_2\}$

$\Rightarrow \forall n > N$

$$|s_n + t_n - s - t|$$

$$|s_n - s + t_n - t| \le |s_n - s| + |t_n - t|$$

$$\Rightarrow \quad |s_n - s + t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\Rightarrow \quad |s_n - s + t_n - t| < \frac{\varepsilon}{2}$$

$$\Rightarrow \quad \boxed{\lim (s_n + t_n) = s + t}$$

Theorem 9.4

if (s_n) converges to s and t_n converges to t, then $(s_n t_n)$ converges to st.

$$\lim s_n t_n = \lim s_n \cdot \lim t_n$$

Proof:

$$|s_n t_n - st| = |s_n t_n - s_n t + s_n t - st|$$

$$= |s_n (t_n - t) + (s_n - s) t$$

$$|s_n t_n - st| \leq |s_n(t_n - t)| + |(s_n - s)t|$$

$$\leq |s_n|\,|t_n - t| + |s_n - s|\,|t|$$

(i) $|t_n - t|$ is very small for large n

(ii) $|t|$ is Bounded i.e $t \in \mathbb{R}$ Constant.

(iii) $|s_n - s|$ is very small for large n.

(iv) from theorem 9.1, $|s_n|$ is Bounded ie $\exists M \in \mathbb{R}$, $|s_n| \leq M$

therefore

$|s_n t_n - st|$ is Bounded.

<u>formal Proof:</u>

let $\varepsilon > 0$, $\exists M > 0$ s.t $|s_n| \leq M \; \forall n$,

$\exists N_1 \in \mathbb{N}$, s.t $|t_n - t| < \dfrac{\varepsilon}{2M}$

$\exists N_2 \in \mathbb{N}$, s.t $\forall n > N_2$

$$|s_n - s| < \frac{\varepsilon}{2(|t|+1)}$$

$$\Rightarrow |s_n t_n - st| \leq |s_n||t_n - t| + |s_n - s||t|$$

$$\leq \cancel{M} \cdot \frac{\varepsilon}{2\cancel{M}} + \frac{\varepsilon}{2(|t|+1)} \cdot |t|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \cdot \frac{|t|}{|t|+1} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon$$

i.e

$|s_n t_n - st| < \varepsilon$ for all $n > \max\{N_1, N_2\}$

Therefore

$$\lim_{n \to \infty} s_n t_n = \lim_{n \to \infty} s_n \cdot \lim_{n \to \infty} t_n.$$

Lemma 9.5:-

if (s_n) converges to s, if $s_n \neq 0$, $\forall n \in \mathbb{N}$, and if $s \neq 0$, then

$$\lim \frac{1}{s_n} = \frac{1}{s}$$

Proof: we want to prove

$$\left| \frac{1}{s_n} - \frac{1}{s} \right| = \frac{|s - s_n|}{|s_n \cdot s|}$$

s_n is a convergent seq^n $\Rightarrow$ Bounded

and there exists $m = \inf\{|s_n| : n \in \mathbb{N}\} > 0$

$$\Rightarrow \left| \frac{1}{s_n} - \frac{1}{s} \right| = \frac{|s_n - s|}{|s_n| \cdot |s|} \leq \frac{|s_n - s|}{m \cdot |s|}$$

Formal Proof:

Let $\varepsilon > 0$, $\exists N \in \mathbb{N}$, such that

$\forall n > N$ $\quad |S_n - S| \leq \varepsilon \cdot m|S|$

$$\Rightarrow \frac{|S_n - S|}{m|S|} < \varepsilon$$

$$\Rightarrow \left| \frac{1}{S_n} - \frac{1}{S} \right| \leq \frac{|S_n - S|}{m|S|} < \varepsilon$$

$$\Rightarrow \left| \frac{1}{S_n} - \frac{1}{S} \right| < \varepsilon$$

Hence $\left(\frac{1}{S_n}\right)$ is convergent seq^n with limit $\frac{1}{S}$

Theorem 9.6 :-

Suppose (s_n) converges to s, (t_n) converges to t. if $s \neq 0$ and $s_n \neq 0 \ \forall n$,

then (t_n / s_n) converges to t/s

Proof: $\lim \dfrac{t_n}{s_n} = \lim \dfrac{1}{s_n} \cdot \lim t_n$ (Theorem 9.4)

Exist (Lemma 9.5)

$$= \frac{1}{s} \cdot t = \frac{t}{s}$$

Theorem 9.7

(a) $\lim\limits_{n \to \infty} \dfrac{1}{n^p} = 0$ for $p > 0$

(b) $\lim\limits_{n \to \infty} a^n = 0$ if $|a| < 1$

(c) $\lim n^{1/n} = 1$

(d) $\lim\limits_{n \to \infty} a^{1/n} = 1$ for $a > 0$

Proof:

(a) Let $\varepsilon > 0$, and $N = \frac{1}{\varepsilon^{1/p}}$, then

$\forall n > N$ i.e

$$n > \frac{1}{\varepsilon^{1/p}}$$

$$\Rightarrow \frac{1}{n} < \varepsilon^{1/p}$$

$$\Rightarrow \frac{1}{n^p} < \varepsilon$$

$$\Rightarrow \left|\frac{1}{n^p} - 0\right| < \varepsilon$$ $\Rightarrow$ convergent to 0

(b) $|a| < 1 \Rightarrow |a| = \frac{1}{1+b}$ where $b > 0$

$(1+b)^n = 1 + nb + \cdots$ (binomial theorem) $(b > 0)$

$$\Rightarrow (1+b)^n \geq 1 + nb > nb$$

therefore

$$|a^n - 0| = |a^n| = \frac{1}{(1+b)^n}$$

$$\Rightarrow \quad |a^n - a| < \frac{1}{nb}$$

Now consider $\varepsilon > 0$, let $N = \frac{1}{\varepsilon b}$, then $n > N$

$$n > \frac{1}{\varepsilon b}$$

$$\Rightarrow \quad \frac{1}{nb} < \varepsilon$$

$$\Rightarrow \quad |a^n - 0| < \frac{1}{nb} < \varepsilon$$

$$\Rightarrow \quad |a^n - 0| < \varepsilon \quad (\text{convergent to } 0)$$

Definition 9.8 :-

for a seq (s_n), we write $\lim s_n = +\infty$ provided for each $M > 0$ there is a number N such that $n > N \Rightarrow s_n > M$.

The seqn diverges to $+\infty$.

Similarly, we write $\lim S_n = -\infty$, Provided for each $M < 0$, there is a Number N such that $n > N$ $S_n < M$.

limit exists Provided S_n converges or diverges to $+\infty$, or diverges to $-\infty$.

Example 4

$\lim n^2 = +\infty$

$\lim (-n) = -\infty$

$\lim 2^n = +\infty$

$\lim (\sqrt{n+7}) = \infty$

Limit exists

$\lim (-1)^n n \implies$ Limit does not exits.

Examples: $\lim(\sqrt{n}+7) = +\infty$

for an arbitrary $M > 0$, $\exists N \in \mathbb{N}$

s.t $\forall n > N$ $\quad \sqrt{n}+7 > M$

$\Rightarrow \quad \sqrt{n} > M-7$

$\Rightarrow \quad n > (M-7)^2$

Formal Proof:

Let $M > 0$ and $N = (M-7)^2$. Then

$\forall n > N$

$\Rightarrow \quad n > (M-7)^2$

$\Rightarrow \quad \sqrt{n}+7 > M$

$\Rightarrow \quad \lim(\sqrt{n}+7) = +\infty$

Example 6

$$\lim \frac{n^2+3}{n+1} = +\infty$$

Discussion:

an arbitrary M, $\exists N \in \mathbb{N}$

$$\forall n > N \qquad \frac{n^2+3}{n+1} > M$$

$$\Rightarrow \quad \frac{n^2+3}{n+1} \geqslant \frac{n^2}{2n} > M$$

$$\Rightarrow \quad \frac{n^2}{2n} > M$$

$$\Rightarrow \quad \frac{n}{2} > M \qquad N = 2M$$

formal Proof:

let $M > 0$, then $N = 2M$, then $n > N$

implies

$$\frac{n}{2} > M$$

$$\Rightarrow \quad \frac{n^2+3}{n+1} \geqslant \frac{n^2}{2n} > M$$

$$\Rightarrow \quad \frac{n^2+3}{n+1} > M$$

$$\Rightarrow \quad \lim \frac{n^2+3}{n+1} = +\infty$$

Theorem 9.9:

Let (s_n) and (t_n) be seq^n such that

$\lim s_n = +\infty$ and $\lim t_n > 0$ $(+\infty, \text{or } \mathbb{R})$

Then $\lim s_n t_n = +\infty$.

9.10 Theorem:

for a seq^n (s_n) of $\mathbb{R}^+$, we have

$\lim s_n = +\infty \iff \lim \frac{1}{s_n} = 0$

Properties:

① Convergent seq^n are bounded

② $\lim_{n\to\infty} k s_n = k \cdot \lim_{n\to\infty} s_n$

③ $\lim_{n\to\infty} (s_n + t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n$

④ $\lim_{n\to\infty} s_n \cdot t_n = \lim_{n\to\infty} s_n \cdot \lim_{n\to\infty} t_n$

(5) if $\lim_{n\to\infty} s_n = s \neq 0$, $\Rightarrow \lim \frac{1}{s_n} = \frac{1}{s}$

(6) $\lim t_n = t$, $\lim s_n = s \neq 0$

then $\lim \frac{t_n}{s_n} = \frac{t}{s}$