Exercise : sec 17 Continuous function

17.1 let $f(x) = \sqrt{4-x}$ for $x \leq 4$

and $g(x) = x^2$ $\forall x \in \mathbb{R}$

(a) domains of $f+g$, fg, $f \circ g$, and $g \circ f$

Ans:

domain of, $f+g$, fg is

$$\text{dom}(f) \cap \text{dom}(g)$$

$$= \{x : x \leq 4\} \cap \mathbb{R}$$

$$= \{x : x \leq 4\}$$

domain of $f \circ g$ is

$$\{x \in \text{dom}(g) : g(x) \in \text{dom}(f)\}$$

$$= \{x \in \mathbb{R} : x^2 \in (-\infty, 4]\}$$

$$= \{ x \in \mathbb{R} : x \in [-2, 2] \}$$

domain of $g \circ f$.

$$= \{ x \in dom(f) : f(x) \in dom(g) \}$$

$$= \{ x \in (-\infty, 4] : \sqrt{4-x} \in \mathbb{R} \}$$

$$= \{ x \in (-\infty, 4] \}$$

(b)

$$f \circ g(0) = f(g(0))$$
$$= f(0)$$
$$= 2$$

$$g \circ f(0) = g(f(0))$$
$$= 4$$

$$f \circ g(1) = f(g(1))$$
$$= f(1) = \sqrt{3}$$

$$gof(1) = g(f(1))$$
$$= 3$$

$$fog(2) = f(g(2))$$
$$= 0$$

$$gof(2) = g(f(2))$$
$$= 2$$

(c) Are the function's fog & gof equal? No

(d)

$$fog(3) = f(g(3))$$
$$= f(9)$$ Not meaningful

$$gof(3) = g(f(3))$$
$$= g(\sqrt{1})$$
$$= 1$$ Meaningful

17. 2

a, b

let $f(x) = 4$ for $x \geq 0$, $f(x) = 0$ for $x < 0$, and $g(x) = x^2$, $x \in \mathbb{R}$

$\Rightarrow$ $dom(f) = dom(g) = \mathbb{R}$

(a) Determine the following function's $f+g$, fg, $f \circ g$, $g \circ f$, Be sure to specify their domain's

Sol^n

$$f+g = \begin{cases} x^2 & x < 0 \\ 4+x^2 & x \geq 0 \end{cases}$$

discontinuous

$dom(f+g) = \mathbb{R}$

$$fg(x) = \begin{cases} 0 & x < 0 \\ 4x^2 & x \geq 0 \end{cases}$$

Continuous

$dom(fg) = \mathbb{R}$

$f \circ g$

$\text{dom}(f \circ g) =$

$\{x \in \text{dom}(g) : g(x) \in \text{dom}(f)\}$

$= \{x \in \mathbb{R} : x^2 \in \mathbb{R}\}$

$= \{x \in \mathbb{R}\}$

$f \circ g = f(g(x)) = \begin{cases} 4 & x \in \mathbb{R} \end{cases}$

$f \circ g$ is continous

$g \circ f$

$\text{dom}(g \circ f)$

$= \{x \in \text{dom}(f) : f(x) \in \text{dom}(g)\}$

$= \{x \in \mathbb{R} : \{4, 0\} \in \mathbb{R}\}$

$\Rightarrow \{x \in \mathbb{R}\}$

$$g \circ f = g(f(x)) = \begin{cases} 16 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

discontinuous

$\underline{\underline{f}}$:- discontinuous

$\underline{\underline{g}}$:- continuous

17.3 Accept the faith that the following function's are continuous on their domain's . $\sin x$, $\cos x$, e^x, 2^x

$\log_e x \quad \forall x > 0$

x^p for $x > 0$

(a) $\log_e(1 + \cos^4 x)$

$\cos x$ is continuous : given

$\Rightarrow \cos^4 x$ is continuous : give x^p continuous i.e $\cos^4 x = (\cos x)^4 = f \circ g$

$1 + \cos^4 x$ is continuous

$\Rightarrow \log_e(1+\cos^4 x)$ is continuous.

(b) $(\sin^2 x + \cos^6 x)^{\pi}$

$\Rightarrow e^{\log_e (\sin^2 x + \cos^6 x)^{\pi}} = e^{\pi \log_e (\sin^2 x + \cos^6 x)}$

$\cos x$, $\sin x$ continuous

$\Rightarrow \cos^6 x$, $\sin x^2$ continuous

$\Rightarrow x^p$

$\Rightarrow \cos^6 x + \sin^2 x$ continuous

$\Rightarrow f+g$

$\Rightarrow \log_e(\cos^6 x + \sin^2 x)$ continuous

$\Rightarrow \log_e x$

$\Rightarrow \pi \log_e (\cos^6 x + \sin^2 x)$

$\Rightarrow k < f$

$\Rightarrow e^{\pi \log_e (\cos^6 x + \sin^2 x)}$

$\Rightarrow e^x$

(c) 2^{x^2}

$\Rightarrow x^2$ is continuous

& 2^{x^2} is continuous $\because 2^x$ is continuous

(d) $8^x = 2^{3x} = (2^x)^3$

(e) $\tan x$ for $x \neq$ odd multiple of $\frac{\pi}{2}$

$\tan x = \frac{\sin x}{\cos x}$

$\Rightarrow$ continuous except for $\cos x \neq 0$

$\Rightarrow$ Continuous except for odd multiple of $\frac{\pi}{2}$

(f) (g) (h) $x \sin\left(\frac{1}{x}\right)$ for $x \neq 0$

x is continous

$\Rightarrow \frac{1}{x}$ is continuous if x is continuous
for $x \neq 0$

$\Rightarrow \sin\left(\frac{1}{x}\right)$ is continuous for $x \neq 0$
if $\frac{1}{x}$ is continuous
for $x \neq 0$

then $x \sin\left(\frac{1}{x}\right)$, $x^2 \sin\left(\frac{1}{x}\right)$, $\frac{1}{x} \sin\left(\frac{1}{x^2}\right)$

is continuous for $x \neq 0$

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(17.4) Prove the function $\sqrt{x}$ is continuous on its domain $[0,\infty)$.

Sol^n let $x_0 \in [0,\infty)$ and $\varepsilon > 0$,

we want show $|\sqrt{x} - \sqrt{x_0}| < \varepsilon$
Provided $|x - x_0| < \delta$.

$$\sqrt{x} - \sqrt{x_0} = \sqrt{x} - \sqrt{x_0} \cdot \frac{(\sqrt{x} + \sqrt{x_0})}{\sqrt{x} + \sqrt{x_0}}$$

$$= \frac{x - x_0}{\sqrt{x} + \sqrt{x_0}}$$

for $x_0 > 0$ we have

$$\sqrt{x} - \sqrt{x_0} = \frac{x - x_0}{\sqrt{x} + \sqrt{x_0}} < \frac{x - x_0}{\sqrt{x_0}}$$

$$\Rightarrow |\sqrt{x} - \sqrt{x_0}| < \frac{|x - x_0|}{\sqrt{x_0}} < \varepsilon$$

then $|x - x_0| < \varepsilon \sqrt{x_0}$

then take $\delta = \varepsilon\sqrt{x_0}$

This show's for $x_0 > 0$, then

$$|x - x_0| < \delta \Rightarrow |\sqrt{x} - \sqrt{x_0}| < \varepsilon$$

as desired

$\Rightarrow$ $\sqrt{x}$ is continuous for $x > 0$

for $x_0 = 0 \Rightarrow \sqrt{x_0} = 0$

Let's take a $(x_n)_{n \in \mathbb{N}} \in [0, \infty)$

where $\lim\limits_{n \to \infty} x_n = 0$

then $\lim\limits_{n \to \infty} \sqrt{x_n} = \sqrt{\lim\limits_{n \to \infty} x_n}$

$= 0$

$\Rightarrow \lim\limits_{n \to \infty} f(x_n) = f(x_0)$

Hence $\sqrt{x}$ is continuous at $x = 0$

17.5 (a) Prove that if $m \in \mathbb{N}$, then the function $f(x) = x^m$ is continuous on $\mathbb{R}$

Sol^n let's assume a seq^n $(x_n)_{n \in \mathbb{N}}$

s.t $\lim x_n = x_0$

then $\lim\limits_{n \to \infty} x_n^m = \left(\lim\limits_{n \to \infty} x_n\right)^m$

$= x_0^m$

$\Rightarrow \lim\limits_{n \to \infty} f(x_n) = f(x_0)$

$\Rightarrow f(x) = x^m$ is continuous.

(b) Prove that every polynomial function $P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

Sol^n let $\lim\limits_{n \to \infty} x_n = x_0$

$$\lim_{n\to\infty} P(x_n) = \lim_{n\to\infty} (a_0 + a_1 x_n + \cdots + a_n x^n)$$

$$= a_0 + a_1 x_0 + \cdots + a_n x_0^n$$

$$= P(x_0)$$

Hence every polynomial is continuous

17.6 A rational function is a function f of the form $\frac{p}{q}$ where p and q are polynomial functions. The domain of f is $\{x \in \mathbb{R} : q(x) \neq 0\}$. Prove that every rational is continuous.

Sol^n

$$f(x) = \frac{P(x)}{q(x)}$$

we proved in (17.5 b) that every polynomail is continuous

$\Rightarrow$ And we Proved that if f, g are continuous function's

then $\dfrac{f}{g}$ is continuous function's

with $\operatorname{dom}\left(\dfrac{f}{g}\right) = \operatorname{dom}(f) \cap \{x \in \operatorname{dom}(g) : g(x) \neq 0\}$

Hence $P(x), q(x)$ are Polynomial's which are continuous

$\Rightarrow \dfrac{P(x)}{q(x)}$ is continuous over the domain

$$\operatorname{dom}(P) \cap \{x \in \operatorname{dom}(q) : q(x) \neq 0\}$$

$$\Rightarrow \mathbb{R} \cap \{x \in \mathbb{R} : q(x) \neq 0\}$$

$$\Rightarrow \{x \in \mathbb{R} : q(x) \neq 0\}$$

17.7 b Prove $|x|$ is continous function on $\mathbb{R}$

$f(x) = x$ is continuous

we proved if $f(x)$ is continuous then $|f(x)|$ is continuous, Hence $|x|$ is continuous.

Formal Proof:

let $x_0 \in \mathbb{R}$ s.t $\lim_{n \to \infty} x_n = x_0$

$$\big|\, |x| - |x_0| \,\big| \leq |x - x_0| < \varepsilon$$

for $|x - x_0| < \delta = \varepsilon$ we have

$$\big|\, |x| - |x_0| \,\big| < \varepsilon$$

$\Rightarrow$ $|x|$ is continuous.

17.8 Let f and g be real-valued functions.

(a) Show that

$$\min(f,g) = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$$

Sol^n

$$\min(f,g) = \begin{cases} f(x) & \text{if } f(x) \le g(x) \\ g(x) & \text{if } f(x) > g(x) \end{cases}$$

$$|f(x) - g(x)| = \begin{cases} g(x) - f(x) & \text{if } f(x) \le g(x) \\ f(x) - g(x) & \text{if } f(x) > g(x) \end{cases}$$

Case 1 $\quad f(x) \le g(x)$

then $\quad \min(f,g) = f(x)$

$$\frac{1}{2}(f+g) - \frac{1}{2}|f-g|$$

$$= \frac{1}{2}\left(f(x)+g(x)\right) - \frac{1}{2}\left(g(x) - f(x)\right)$$

$$= f(x)$$

CASE 2

$$g(x) < f(x)$$

then $\min(f(x), g(x)) = g(x)$

$$\frac{1}{2}(f(x)+g(x)) - \frac{1}{2}|f(x)-g(x)|$$

$$\Rightarrow \frac{f(x)}{2} + \frac{g(x)}{2} - \frac{f(x)}{2} + \frac{g(x)}{2}$$

$$\Rightarrow g(x)$$

Hence Proved.

(b) Show that $\min(f, g)$

$$= -\max(-f, -g)$$

$$\min(f, g) = \begin{cases} f(x) & \text{if } f(x) \leq g(x) \\ g(x) & \text{if } g(x) < f(x) \end{cases}$$

$$\max(-f,-g) = \begin{cases} -f & \text{if } -f(x) \geq -g(x) \\ -g & \text{if } -g(x) > f(x) \end{cases}$$

$$= \begin{cases} -f & \text{if } f(x) \leq g(x) \\ -g & \text{if } g(x) < f(x) \end{cases}$$

$$-\max(f,g) = \begin{cases} f & \text{if } f(x) \leq g(x) \\ g & \text{if } g(x) < f(x) \end{cases}$$

$$= \min(f,g)$$

Hence proved.

(c) Prove if f, g are continuous at $x_0 \in \mathbb{R}$ then $\min(f,g)$ is continuous at x_0.

Sol^n $f(x)$, $g(x)$ is continous at x_0

from continous theorem

$\frac{1}{2}\left(f(x)+g(x)\right)$ is continous

$f(x)-g(x)$ is continuous

$\Rightarrow$ $\frac{1}{2}|f(x)-g(x)|$ is continous

$\Rightarrow$ $\frac{1}{2}(f+g) - \frac{1}{2}|f-g|$ is continuous

$\Rightarrow$ $\min(f,g)$ is continuous.

17.9

(a) $f(x) = x^2$, $x_0 = 2$

Sol^n

Let $x_0 = 2$ & $(x_n)_{n\in\mathbb{N}}$ is a seq^n with $\lim_{n\to\infty} x_n = 2$

we want to show that

$|x^2 - x_0^2| < \varepsilon$ Provided

$|x - x_0|$ is very small $< \delta$

$\Rightarrow |x^2 - x_0^2|$

$\Rightarrow |(x - x_0)(x + x_0)|$

$\Rightarrow |x - x_0| \, |x + x_0|$

if lets take $|x - x_0| < 1$

then $|x| < |x_0| + 1$

then $|x + x_0| \leq |x| + |x_0| < 2|x_0| + 1$

$\Rightarrow |x^2 - x_0^2| < |x - x_0| \left(2|x_0| + 1 \right) < \varepsilon$

$\Rightarrow |x - x_0| < \dfrac{\varepsilon}{2|x_0| + 1}$

take $\delta = \min\left\{1, \dfrac{\varepsilon}{2|x_0|+1}\right\}$

$\Rightarrow$ if $|x - x_0| < \delta$, then

$$|x^2 - x_0^2| < \varepsilon$$

Hence $f(x) = x^2$ is continuous

(b) $f(x) = \sqrt{x}$, $x_0 = 2$

Sol^n $$|\sqrt{x} - \sqrt{2}| = |\sqrt{x} - \sqrt{2}| \frac{|\sqrt{x} + \sqrt{2}|}{\sqrt{x} + \sqrt{2}}$$

$$= \frac{|x-2|}{\sqrt{x}+\sqrt{2}} < \frac{|x-2|}{\sqrt{2}}$$

$$\Rightarrow |\sqrt{x} - \sqrt{2}| < \frac{|x-2|}{\sqrt{2}} < \varepsilon$$

lets take $\delta = \varepsilon\sqrt{2}$

$\Rightarrow$ if $|\sqrt{x}-\sqrt{2}| < \varepsilon$ then

$|x-2| < \varepsilon\sqrt{2}$

Hence $f(x)=\sqrt{x}$ is continuous at $x_0=2$

(c) $f(x)= x\sin\left(\frac{1}{x}\right)$ for $x\neq 0$

and $f(0)=0$, $x_0=0$

Soln

$$|f(x)-f(0)|$$

$$= |f(x)-0| = |f(x)|$$

$$\left| x\sin\left(\frac{1}{x}\right)\right| < |x| < \varepsilon$$

then take $\delta=\varepsilon$

$|x-0| < \delta = \varepsilon$

Hence for $|x-0| < \delta = \varepsilon$

the $|x \sin(\frac{1}{x}) - 0| < \delta = \varepsilon$

Hence continuous.

(d) $g(x) = x^3$, x_0 arbitrary.

Sol^n

$$|x^3 - x_0^3| = |x - x_0| \, |x^2 + x_0 x + x_0^2|$$

if $|x - x_0| < 1 \Rightarrow |x| < |x_0| + 1$

then $|x^2 + x_0 x + x_0^2|$

$$< |x|^2 + |x_0||x| + x_0^2$$

$$\Rightarrow |x^2 + x_0 x + x_0^2| < x_0^2 + 1 + 2|x_0| + x_0^2 + |x_0| + x_0^2$$

$$\Rightarrow |x^2 + x_0 x + x_0^2| < 3x_0^2 + 3|x_0| + 1$$

Assuming $|x^3 - x_0^3| = |x - x_0|\,|x^2 + x_0 x + x_0^2|$

$$< |x - x_0|\left(3x_0^2 + 3|x_0| + 1\right)$$
$$< \varepsilon$$

$$\Rightarrow |x - x_0| < \frac{\varepsilon}{\left(3x_0^2 + 3|x_0| + 1\right)}$$

take

$$\delta = \min\left\{1, \frac{\varepsilon}{3x_0^2 + 3|x_0| + 1}\right\}$$

Here if $|x^3 - x_0^3| < \varepsilon$ then

$$|x - x_0| < \delta$$

Hence $f(x) = x^3$ is continuous.

17.10 $f(x) = 1$ for $x > 0$

and $f(x) = 0$ for $x \leq 0$, $x_0 = 0$

Sol^n $f(x) = 0$ at $x = 0 \Rightarrow f(0) = 0$

take $\varepsilon = \frac{1}{2}$

$$\Rightarrow |f(x) - f(0)| < \frac{1}{2}$$

$$\Rightarrow |f(x)| < \frac{1}{2}$$

we need to find a δ that satisfies this condition

$|x - 0| < \delta$ such that $|f(x)| < \frac{1}{2}$

for $x > 0$ $|f(x)| = 1 > \frac{1}{2}$

(Not satisfying)

No matter how small δ is we cannot satisfy $|f(x)| < \frac{1}{2} \Rightarrow$ discontinuous.

(b) $g(x) = \sin\left(\frac{1}{x}\right)$ for $x \neq 0$

and $g(0)=0$, $x_0 = 0$

Sol^n

$$|g(x) - g(0)|$$

$$= \left|\sin\left(\frac{1}{x}\right)\right|$$

let's find a seq^n $(x_n)_{n \in \mathbb{N}}$ which is converging to 0 but $\sin\left(\frac{1}{x_n}\right)$ not converging to 0.

$$\sin\left(\frac{1}{x}\right) = 1 \implies \frac{1}{x} = \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$$

$$\implies \frac{1}{x} = 2n\pi + \frac{\pi}{2}$$

$$\implies \frac{1}{x} = \frac{4n\pi + \pi}{2} = \frac{\pi}{2}(4n+1)$$

$$\Rightarrow \quad x = \frac{2}{\pi(4n+1)}$$

Let's a seq^n $(x_n)_{n \in \mathbb{N}_0} = \left(\frac{2}{\pi(4n+1)} \right)_{n \in \mathbb{N}_0}$

$$\lim_{n \to \infty} \frac{2}{\pi(4n+1)} = \lim_{n \to \infty} \frac{\frac{2}{n}}{\pi\left(4+\frac{1}{n}\right)}$$

$$= 0$$

Then $\lim_{n \to \infty} \sin\left(\frac{1}{x_n}\right)$

$$\Rightarrow \lim_{n \to \infty} \sin\left(\frac{\pi(4n+1)}{2}\right)$$

$$\Rightarrow \lim_{n \to \infty} \sin\left(2n\pi + \frac{\pi}{2}\right)$$

$$\Rightarrow \lim_{n \to \infty} 1 = 1$$

Then $\lim_{n \to \infty} x_n = 0$ but $\lim_{n \to \infty} f(x_n) = 1$

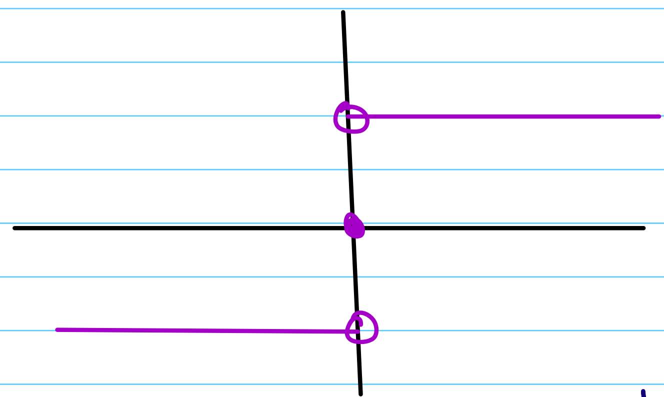
$\Rightarrow \sin\left(\frac{1}{x}\right)$ is discontinuous at $x = 0$

17.10 (c)

$$\operatorname{sgn}(x) = -1 \quad \forall x<0$$
$$= 1 \quad \text{for } x>0$$
$$= 0 \quad \text{for } x=0$$

$$\Rightarrow \operatorname{sgn}(x) = \frac{x}{|x|} \quad \text{for } x \neq 0$$

Sol^n



let's take $\varepsilon = \frac{1}{2}$

$$|\operatorname{sgn}(x) - \operatorname{sgn}(0)|$$

$$= |\operatorname{sgn}(x)| < \frac{1}{2}$$

for $x>0$

$$|\operatorname{sgn}(x)| = 1 > \frac{1}{2} \quad \text{(contradiction)}$$

for $x<1$

$$|\operatorname{sgn}(x)| = |-1| = 1 > \frac{1}{2} \quad \text{(contradiction)}$$

Hence we cannot find a δ for $\epsilon = \frac{1}{2}$

$\Rightarrow$ discontinuous.

17.11 Let f be a real-valued function with $dom(f) \subseteq \mathbb{R}$. Prove f is continuous at $x_0 \iff$ for every monotonic seq (x_n) in $dom(f)$ converging to x_0 we have $\lim f(x_n) = f(x_0)$

Soln

if a function $f(x)$ is continuous at point x_0

$\Updownarrow$

every seq $(x_n)_{n \in \mathbb{N}}$ with $\lim x_n = x_0$ should have $\lim f(x_n) = f(x_0)$

from theorem 11.4 we know

Every seq (s_n) has a monotonotic subseq^n

$\Rightarrow$ $(x_n)_{n \in \mathbb{N}}$ is a seq^n with

$$\lim_{n \to \infty} x_n = x_0$$

$\Rightarrow$ $\exists$ a subseq^n $(x_{n_k})_{k \in \mathbb{N}}$ monotone seq^n such that $\lim_{k \to \infty} x_{n_k} = x_0$

$\Rightarrow$ for the function $f(x)$ to be convergent at x_0, it should satisfy

$$\lim_{k \to \infty} f(x_{n_k}) = f(x_0)$$

(Hence Proved)