Section 11 : Subsequences

Suppose $(s_n)_{n\in\mathbb{N}}$ is a seq^n. A subseq^n of this seq^n is a seq^n of the form $(s_{n_k})_{k\in\mathbb{N}}$ where

$\forall$ k there is a positive integer n_k such that

$$n_1 < n_2 < \dots < n_k < n_{k+1} < \dots$$

Theorem 11.2 :

Let (s_n) be a seq^n

(i) if t is in $\mathbb{R}$, then there is a subseq^n of (s_n) converging to t $\iff$ if the set $\{n\in\mathbb{N} : |s_n - t| < \varepsilon\}$ is infinite $\forall$ $\varepsilon > 0$.

(ii) if the seq^n (s_n) is unbounded above, it has a subseq^n with limit $+\infty$

(iii) if the seq^n is unbounded below, then the seq^n has a subseq^n with limit $-\infty$.

Theorem 11.3:

if the seq^n (s_n) converges, then every subseq^n converges to the same limit

Proof:

Let $(s_{n_k})_{k\in\mathbb{N}}$ denote the subseq^n of $(s_n)_{n\in\mathbb{N}}$.

$$n_k \geq k \quad \forall k$$

* let $s = \lim s_n$ and let $\varepsilon > 0$.

$\exists N \in \mathbb{N}$ s.t $\forall n > N$

$$|s_n - s| < \varepsilon.$$

Now for $k > N \implies n_k > N$

$$|S_{n_k} - S| < \varepsilon$$

$$\boxed{\Rightarrow \quad \lim_{k \to \infty} S_{n_k} = S}$$

Theorem 11.4:

Every seq^n $(S_n)_{n \in \mathbb{N}}$ has a monotonic subseq^n

Proof:

Let's say n^{th} is dominant if it is greater than every term which follow it.

$$S_m < S_n \quad \forall \quad m > n.$$

CASE I: Suppose there are infinitely many dominant term's, then take (S_{n_k}) be a subseq^n consisting solely of dominant term's

$$\Rightarrow \quad S_{n_{k+1}} < S_{n_k} \quad \forall k$$

$\Rightarrow$ (S_{n_k}) is a monotonically decreasing seq^n.

CASE 2:

Suppose there are finitely many dominant term's.

then Select n_1 so that S_{n_1} is beyond all the dominant term's of the seq^n.

we can Produce monotonically increasing subseq^n form 1.

Theorem 11.5 : Bolzano Weierstrass theorem

Every bounded seq^n has a convergent seq^n.

Proof:

if (S_n) is a bounded seq^n, then it has a monotonic subseq^n by Theorem 11.4, which is bounded
$\Rightarrow$ convergent seq^n (Theorem 10.2)

Definition 11.6:

Let (S_n) be a seq^n in $\mathbb{R}$. A subsequential limit is any real number or symbol $+\infty$ or $-\infty$ that is the limit of some subseq^n of (S_n)

if $\lim_{n\to\infty} S_n = S \in \mathbb{R}$ then every subseq $\lim_{k \to \infty} S_{n_k} = S$.

Then the set of $\{S\}$ subsequential limits will be only $\{S\}$

The interesting case is when the original subseq^n does not have a limit

Ex: 5

$$S_n = n^2 (-1)^n$$

$$\lim_{n \to \infty} S_{2n} = \infty \quad \text{(even term's)}$$

$$\lim_{n \to \infty} S_{2n+1} = -\infty \quad \text{(odd term's)}$$

Subsequential limit's of $(S_n) = \{-\infty, +\infty\}$

Example 6:

$$a_n = \sin\left(\frac{n\pi}{3}\right)$$

The subsequential limits of a_n

$$\left\{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\right\}$$

$$(a_{3k})_{k\in\mathbb{N}} = 0$$

$$(a_{6k+1})_{k\in\mathbb{N}} = \frac{\sqrt{3}}{2}$$

$$(a_{6k+4})_{k\in\mathbb{N}} = -\frac{\sqrt{3}}{2}$$

Ex 7:

$$b_n = n\left[1+(-1)^n\right]$$

$$\lim_{k\to\infty} b_{2k+1} = 0$$

$$\lim_{k\to\infty} b_{2k} = +\infty$$

Theorem 11.7 :-

Let (S_n) be any seq^n. There exists a monotonic seq^n whose limit is $\lim\sup S_n$, $\exists$ a monotonic subseq^n whose limit is $\lim\inf S_n$

Theorem 8 :

Let (S_n) be any seq^n in $\mathbb{R}$, and let S denote the set of subseq^n limit's of (S_n)

(i) S is non-empty

(ii) $\sup S = \lim\sup S_n$

$\inf S = \lim\inf S_n$

iii) if $\lim S_n$ exists $\Longleftrightarrow$ set S has exactly one element which is $\lim S_n$