18 Properties of Continuous Function

A real valued function f is said to be bounded if $\{f(x) : x \in \operatorname{dom} f\}$ is a bounded set, i.e., if there exists a real number M such that $|f(x)| \le M$ $\forall x \in \operatorname{dom}(f)$

Theorem 18.1:

Let f be a continuous real valued function on a closed interval $[a,b]$. Then f is a bounded function. Moreover, f assumes its maximum and minimum values on $[a,b]$; that is, $\exists x_0, y_0 \in [a,b]$ such that $f(x_0) \le f(x) \le f(y_0)$ $\forall x \in [a,b]$

Proof:

Assume f is not bounded on $[a,b]$

$\Rightarrow$ $\forall n \in \mathbb{N}$, $\exists x_n \in [a,b]$ s.t

$$|f(x_n)| > n$$

This gives us a seq^n $(x_n)_{n\in\mathbb{N}} \in [a,b]$

where the value's of $|f(x_n)|$ grow arbitarly large as n increases.

$$\lim_{n\to\infty} |f(x_n)| = +\infty$$

Using Bolzano Weierstrass theorem:

(Every bounded seq^n has a convergent sub seq^n) (x_n) has a sub seq^n $(x_{n_k})_{k\in\mathbb{N}}$ that converges to some real number x_0

$\Rightarrow$ $(x_n)_{n\in\mathbb{N}} \in [a,b]$

$\Rightarrow$ according to Bolzano Weierstrass theorem

$\exists\ (x_{n_k})_{k\in\mathbb{N}}$ a subseq^n of $(x_n)_{n\in\mathbb{N}}$

Such that $\lim\limits_{k\to\infty} (x_{n_k})_{k\in\mathbb{N}} = x_0$

where $x_0 \in [a,b]$

* Since f is continuous at x_0,

we have

$$\lim_{k\to\infty} f(x_{n_k}) = f\left(\lim_{n\to\infty} x_{n_k}\right)$$

$$= f(x_0)$$

$$\Rightarrow \left.\begin{array}{l} \lim\limits_{k\to\infty} f(x_{n_k}) = f(x_0) \\ \lim\limits_{n\to\infty} |f(x_n)| = +\infty \end{array}\right\} \text{(contradiction)}$$

$\Rightarrow$ f is Bounded.

$M = \sup \{ f(x) : x \in [a,b] \}$

M is finite.

$\Rightarrow \forall n \in \mathbb{N}$, $\exists\, y_n \in [a,b]$ s.t

$$M - \frac{1}{n} < f(y_n) < M.$$

$$\Rightarrow \quad \lim_{n \to \infty} f(y_n) = M.$$

$\Rightarrow$ By Bolzano weierstrass theorem,

$\exists$ subseq^n (y_{n_k}) of (y_n)

converging to limit $y_0 \in [a,b]$.

Since f is continuous at y_0.

$$\Rightarrow \quad f(y_0) = \lim_{k \to \infty} f(y_{n_k})$$

Since (y_{n_k}) is subseq^n of (y_n)

Hence $\lim_{k \to \infty} f(y_{n_k}) = \lim_{n \to \infty} f(y_n)$

$= M$

$\Rightarrow f(y_0) = M$

$\Rightarrow$ $-f$ has maximum at $x_0 \in [a,b]$

$\Rightarrow$ f has minimum at x_0 &

f has maximum at y_0

$\Rightarrow$ $x_0, y_0 \in [a,b]$

Theorem 18.2: Intermediate Value Theorem

if f is continuous real-valued function on an interval I, then f has the intermediate value property on I:

whenever $a, b \in I$, $a < b$ and y lies b/w $f(a)$, $f(b)$ [ie $f(a) < y < f(b)$] or [$f(b) < y < f(a)$] $\exists$ at least one $x \in [a, b]$ s.t $f(x) = y$.

Proof:

we assume $f(a) < y < f(b)$

let $S = \{x \in [a, b] : f(x) < y\}$

$a \in S \Rightarrow S$ is non-empty set

$x_0 = \sup S \in [a, b]$

18.3 Corollary:

if f is continuous on a interval I, then the set $f(I) = \{ f(x) : x \in I \}$ is also an interval or a single point.