Lec02- Sequences and Limits

Sequences: A seq^n of real number's when we have a map from $\mathbb{N}$ to $\mathbb{R}$

a map $a: \mathbb{N} \longrightarrow \mathbb{R}$

or $a: \mathbb{N}_0 \longrightarrow \mathbb{R}$

Notation: $(a_1, a_2, \ldots)$ infinite List of number's

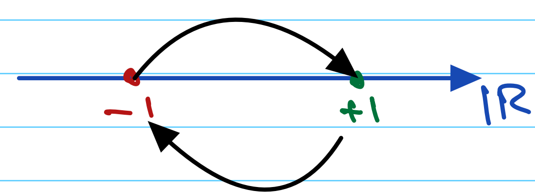
$$(a_n)_{n\in\mathbb{N}} \text{ or } (a_n)_{n=1}^{\infty}$$

$$\text{or } (a_n)$$

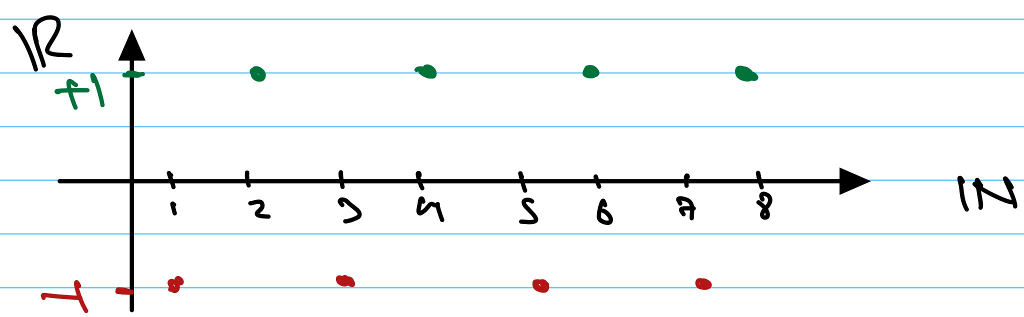
Examples: (0) $(a_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}}$

$= (-1, 1, -1, 1, -1, 1, \ldots\ldots)$

(i)



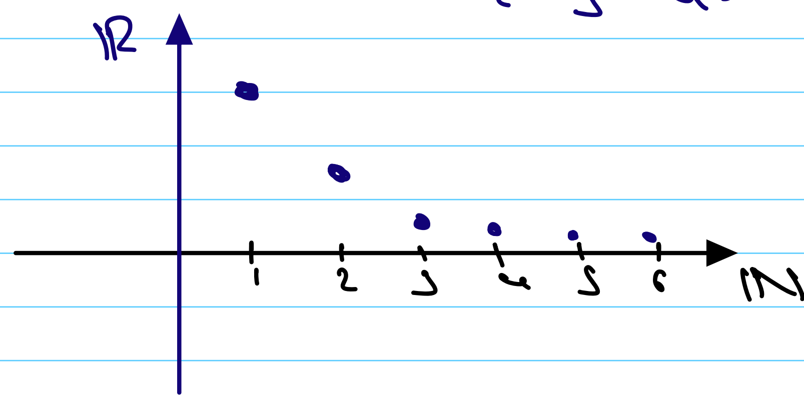
(ii)



we are interested in what happen's to the values of the seq^n when n goes to infinity.

(b) $(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$

$= \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right)$



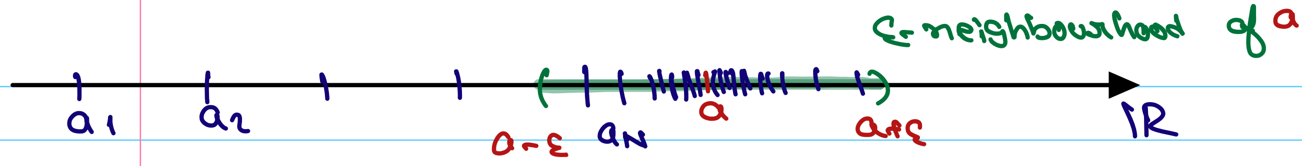
we will see $\lim_{n\to\infty} a_n = 0$

(c) $(a_n)_{n\in\mathbb{N}} = (2^n)_{n\in\mathbb{N}}$

$= (2, 4, 8, 16, 32, 64, \dots)$

Definition:

A sequence $(a_n)_{n\in\mathbb{N}}$ is called convergent to $a \in \mathbb{R}$, if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$, $\forall n \geq N$: $|a_n - a| < \varepsilon$



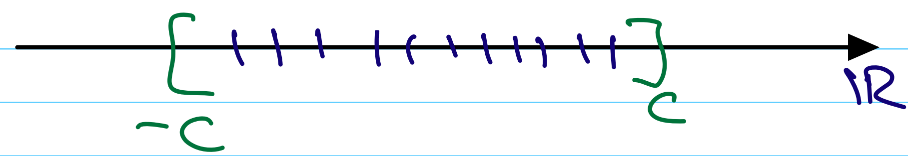
we need to get closer and closer to a with the seq^n member's. eventually all the seq^n member's have to lie ε- neighbourhood of a, only finitely many can lie outside.

No matter how small the ε is, there exist a $N \in \mathbb{N}$, (alway's works)

if there is no such $a \in \mathbb{R}$, we call the seq^n $(a_n)_{n \in \mathbb{N}}$ divergent.

Lec 03 – Bounded seq^n and Unique Limit

Definition: A seq^n $(a_n)_{n\in\mathbb{N}}$ is called Bounded if $\exists C \in \mathbb{N}$ $\forall n \in \mathbb{N}$
$|a_n| \leq C$



otherwise, the seq is called unbounded.

Important fact:

$(a_n)_{n\in\mathbb{N}}$ Convergent $\Longrightarrow$ $(a_n)_{n\in\mathbb{N}}$ Bounded

$(a_n)_{n\in\mathbb{N}}$ Convergent $\Longrightarrow$ There is only one Limit $a \in \mathbb{R}$

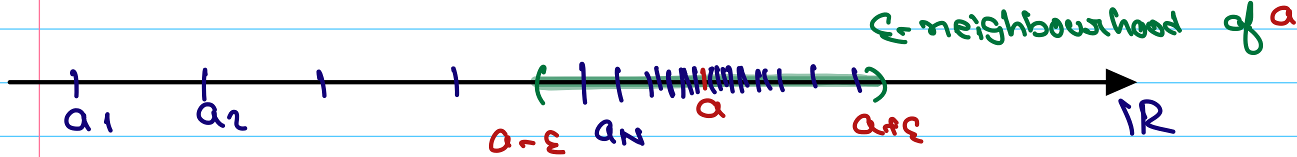
$$\lim_{n\to\infty} a_n = a$$

Lec04 - Theorem on Limits

$(a_n)_{n\in\mathbb{N}}$ Convergent to $a \in \mathbb{R}$:

$$\lim_{n\to\infty} a_n = a$$

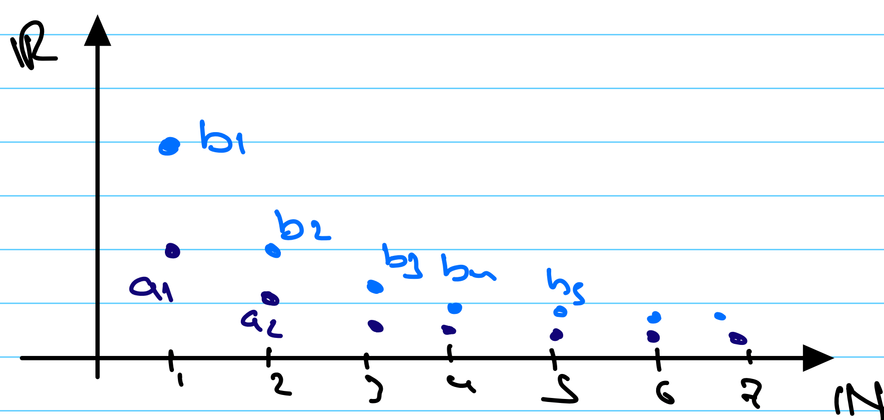
$$a_n \xrightarrow{n\to\infty} a$$



for all ε, the seqⁿ member's lie inside this ε-neighbourhood of a eventually.

Theorem's on Limits:

$(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ Convergent sequnes.



Then

(a) $$\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

we can Pull in the Limit when we have a sum, provided that these two seq^n and their limits actually exists.

(b) $$\lim_{n\to\infty} a_n \cdot b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

(c) $$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n \neq 0}$$

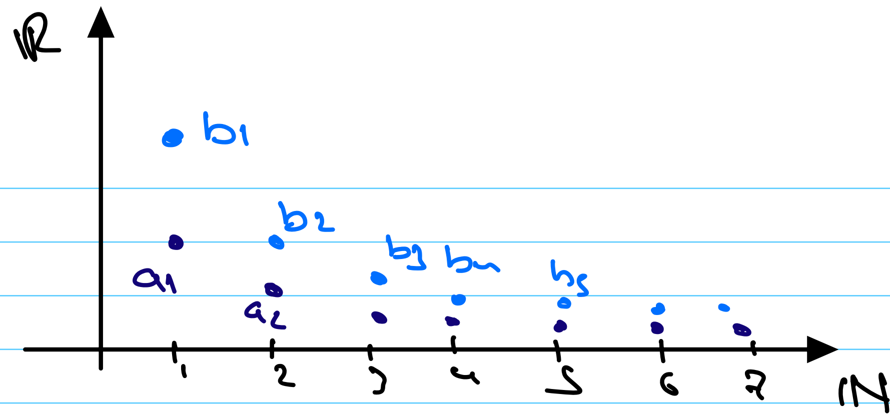
Lec05: Sandwich theorem

$(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ Convergent seq^n

then $$\lim_{n\to\infty}(a\cdot b_n) = a\cdot \lim_{n\to\infty} b_n$$

Properties:

(a) Monotonicity $a_n \leq b_n \ \forall n\in\mathbb{N}$



$$\Rightarrow \quad \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$

(b) Sandwich theorem

$$a_n \leq c_n \leq b_n \quad \forall n \in \mathbb{N} \quad \text{and}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

$$\Rightarrow (c_n)_{n\in\mathbb{N}} \text{ Convergent with}$$

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

Proof of (b):

$$(b_n - a_n) \xrightarrow{n\to\infty} \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = 0$$

$d_n := c_n - a_n$

$\Rightarrow \quad 0 \leq d_n \leq b_n - a_n$

let $\varepsilon > 0$. then $\exists N \in \mathbb{N}$, $\forall n \geq N$

$|b_n - a_n| < \varepsilon$

$\Rightarrow \quad |d_n - 0| < \varepsilon$

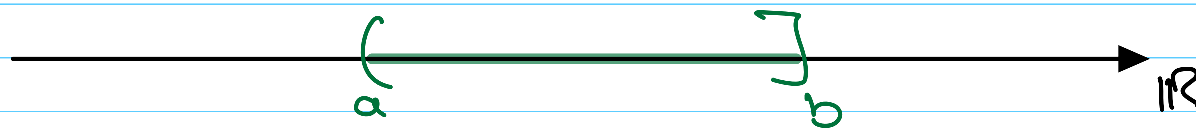
$\Rightarrow \quad d_n$ is convergent with limit 0.

$\Rightarrow \quad (c_n)_{n \in \mathbb{N}} = (d_n + a_n)_{n \in \mathbb{N}}$

is convergent with Limit a.

Lec 06 - Supremum & Infimum

Real Number's are ordered



interval $(a,b] := \{x \in \mathbb{R} \mid a < x \le b\}$

similarly $[a,b] := \{x \in \mathbb{R} \mid a \le x \le b\}$

$[a,\infty) := \{x \in \mathbb{R} \mid a \le x\}$

$(-\infty, b) := \{x \in \mathbb{R} \mid x < b\}$

these interval's are very nice subsets of real number's. (Not the only once)

Definition: For a subset $M \subseteq \mathbb{R}$: $b \in \mathbb{R}$ is called an upper bound for M if

$\forall x \in M: \quad x \leq b$

For $a \in \mathbb{R}$, is called a lower bound for M if $\forall x \in M: \quad x \geq a$

* if upper bound exists we call the set M is bounded from above

* if lower bound exists, the set M called bounded from below.

Both $\rightarrow$ set M is bounded.

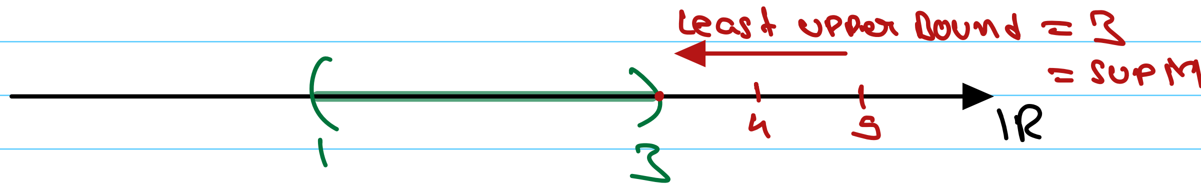
* if b is an upper bound for M and $b \in M$, then b is called a maximal element of M.

$b = \max M$

* if a is an lower bound for M and $a \in M$, then a is called a minimal element of M

$a = \min M$

Ex: $M = (1,3)$, $\max M$, $\min M$ do not exists



$\max M$, $\min M$ do not exists, because $1,3 \notin M$

Definition: for a subset $M \subseteq \mathbb{R}$; $s \in \mathbb{R}$ is called supremum of M if

* $\forall x \in M : x \leq s$ (upper bound for M)
* $\forall \varepsilon > 0, \exists \tilde{x} \in M : s - \varepsilon < \tilde{x}$
  ($s - \varepsilon$ is no upper bound for M)

Supremum = least upper bound.

$\sup M := s$ or $\sup M := \infty$ (if M is not bounded from above)

or $\sup \emptyset := -\infty$

Definition: for a subset $M \subseteq \mathbb{R}$; $\ell \in \mathbb{R}$ is called infimum of M if

* $\forall x \in M : x \geq \ell$ (lower bound for M)
* $\forall \varepsilon > 0, \exists \tilde{x} \in M : \ell + \varepsilon > \tilde{x}$
  ($\ell + \varepsilon$ is no lower bound for M)

Then write $\inf M := \ell$ or $\inf M = -\infty$

(if M is not bounded from below)

or $\inf \emptyset = +\infty$

Lec 07 : Cauchy seq^n & completeness

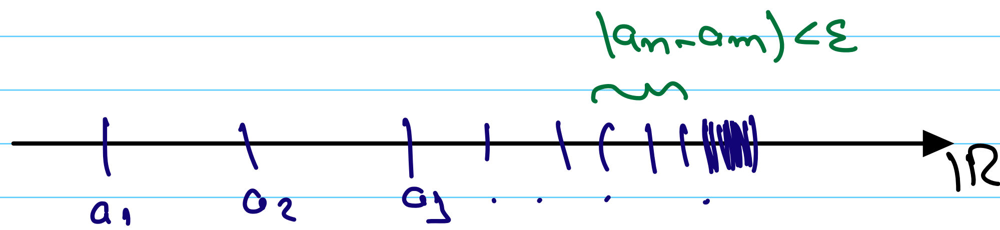
$(a_n)_{n \in \mathbb{N}}$ is convergent seq^n

which means there is a number a, s.t the seq member's here get arbitrarly close to this number a eventually.

$(a_n)_{n\in\mathbb{N}} \Rightarrow$ (there is a limit $a = \lim_{n\to\infty} a_n$)

The Problem with this definition is we need to know the value of Limit to show the convergence.

<u>Different idea:</u>



The seq^n member's gets closer and closer to each other.

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $\forall n, m \geq N$: we have

$$|a_n - a_m| < \varepsilon$$

(Cauchy sequence)

Definition: if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, such that $\forall n, m \geq N$ $|a_n - a_m| < \varepsilon$, then $(a_n)_{n\in\mathbb{N}}$ is called Cauchy seq^n

Important fact: For a seq^n of real number's

Cauchy seq^n $\Longleftrightarrow$ Convergent seq^n

Completeness Axiom: There are no holes in our complete real Number line.

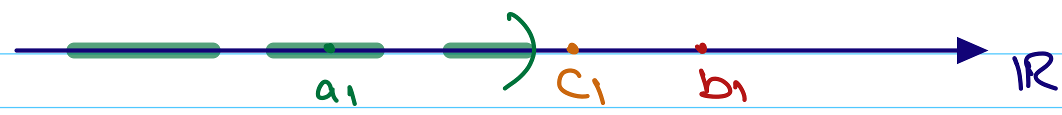
When we are working real number's, we don't have to distinguish Cauchy seq^n and convergent seq^n. They have different definition, But for real number's they mean the same thing.

Dedekind completeness:-

if $M \subseteq \mathbb{R}$ is bounded from above, then $\sup M \in \mathbb{R}$ (exists)

if $M \subseteq \mathbb{R}$ is bounded from below, then $\inf M \in \mathbb{R}$ (exists).

Proof:



$$c_1 := \frac{1}{2}(a_1+b_1)$$

Two cases:- (1) if c_1 is an upper bound for M:

$$b_2 := c_1$$
$$a_2 := a_1$$

(2) if c_2 is not an upper bound for M: $\exists x \in M : x > c_1$

$$b_2 := b_2$$
$$a_2 := x$$

$$c_n := \frac{1}{2}(a_n+b_n)$$

$$\vdots$$

Important fact:

if $(a_n)_{n\in\mathbb{N}}$ is monotonically decreasing

($a_{n+1} \le a_n \ \forall n$) and bounded from below

(the set $\{a_n\}_{n\in\mathbb{N}}$ has a lower bound) then $(a_n)_{n \in \mathbb{N}}$ is convergent.

Lec 08:

Important FACT: if $(a_n)_{n \in \mathbb{N}}$ is motonically increasing ($a_{n+1} \geq a_n \; \forall n$) and Bounded from above (The set $\{a_n\}_{n \in \mathbb{N}}$ has an upper Bound) then

$(a_n)_{n \in \mathbb{N}}$ is convergent.

(Monotone convergence criterion)

Ex:

$$(a_n)_{n \in \mathbb{N}} = \left(\left(1+\frac{1}{n}\right)^n\right)_{n \in \mathbb{N}} \text{ is convergent.}$$

Proof:

monotonicity:-

$$\frac{a_{n+1}}{a_n} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n}$$

$$= \left(1+\frac{1}{n}\right)\cdot\left(\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}}\right)^{n+1}$$

$$= \left(1+\frac{1}{n}\right)\left(\frac{n+2}{n+1}\cdot\frac{n}{n+1}\right)^{n+1}$$

$$= \left(1+\frac{1}{n}\right)\left(\frac{n^2+2n}{n^2+2n+1}\right)^{n+1}$$

$$= \left(1+\frac{1}{n}\right)\left(1-\frac{1}{(n+1)^2}\right)^{n+1}$$

Bernoulli inequality:

for $k \in \mathbb{N}$ and $x \geq -1$

$$(1+x)^k \geq 1+kx$$

$$\Rightarrow \quad \left(1+\frac{1}{n}\right)\left(1-\frac{1}{(n+1)^2}\right)^{n+1}$$

$$\geq \left(1+\frac{1}{n}\right)\left(1-(n+1)\left(\frac{1}{(n+1)^2}\right)\right)$$

$$\geq \left(1+\frac{1}{n}\right)\left(1-\frac{1}{n+1}\right)$$

$$\geq \left(\frac{n+1}{n}\right)\left(\frac{n}{n+1}\right) = 1$$

$$\Rightarrow \frac{a_{n+1}}{a_n} \geq 1 \quad \text{(monotone increasing)}$$

Now we need to show the seq'n is Bounded from above.

$$\#) \quad a_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k$$

$$= \left(\frac{1}{n}\right)^0 + \binom{n}{1}\left(\frac{1}{n}\right)^1 + \binom{n}{2}\left(\frac{1}{n}\right)^2 \cdots$$

$$= 1 + 1 + \sum_{k=2}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k$$

$$\binom{n}{k}\left(\frac{1}{n}\right)^k = \frac{n!}{(n-k)!\, k!} \cdot \frac{1}{n^k}$$

$$= \frac{n \cdot (n-1)(n-2) \cdots \cdot (n-k+1)}{n \cdot n \cdot n \cdot \cdots \cdot n} \cdot \frac{1}{k!}$$

$$\leq 1$$

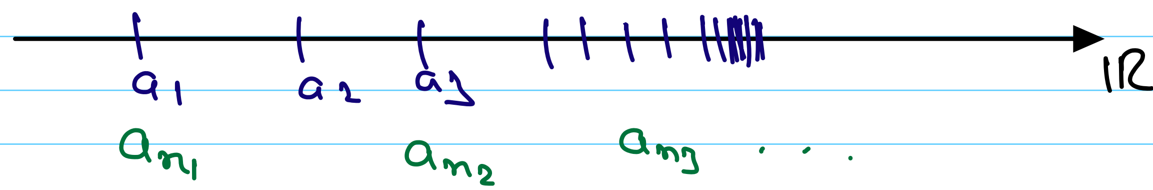
$$\Rightarrow \left(1+\frac{1}{n}\right)^n \leq 3$$

fact $\Longrightarrow$ The seq^n $(a_n)_{n\in\mathbb{N}}$ is convergent.

monotone convergence criterion

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$ Euler's Number

Lec 09 :- Subsequences and Limit Points



omit some member's of the seq^n.

Let $(n_k)_{k\in\mathbb{N}}$ be a seq^n of natural number's that is strictly monotonically increasing $(\forall k\in\mathbb{N} : n_{k+1} > n_k)$

then $(a_{n_k})_{k\in\mathbb{N}}$ is called an subsequence of $(a_n)_{n\in\mathbb{N}}$

Ex: $(a_n)_{n\in\mathbb{N}}$ given by $a_n = \frac{1}{n}$

$$(a_{n_k})_{k\in\mathbb{N}} = (a_{2k})_{k\in\mathbb{N}}$$

$$= (a_2, a_4, a_6, a_8, \ldots)$$

$$= \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\right)$$

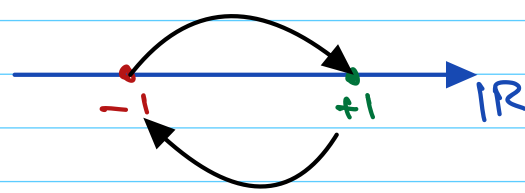
if we have a monotonic seq^n $(a_n)_{n\in\mathbb{N}}$ then the subseq^n $(a_{n_k})_{k\in\mathbb{N}}$ is also monotonic in the same way.

* we can omit lot of seq^n member's even infinitely many as long as infinitely many remain.

FACT: $(a_n)_{n\in\mathbb{N}}$ is convergent with $\lim\limits_{n\to\infty} a_n = a$, then

$\Rightarrow$ every subsequence $(a_{n_k})_{k\in\mathbb{N}}$ is convergent $\lim\limits_{k\to\infty} a_{n_k} = a$

Ex: $(a_n)_{n\in\mathbb{N}}$ given by $a_n = (-1)^n$



Subseq^n $(a_{n_k})_{k\in\mathbb{N}} = (a_{2k})_{k\in\mathbb{N}}$

$= (1, 1, 1, \ldots)$

$\lim\limits_{k\to\infty} a_{2k} = 1$ (limit 1)

$(a_{n_k})_{k\in\mathbb{N}} = (a_{2k+1})_{k\in\mathbb{N}}$

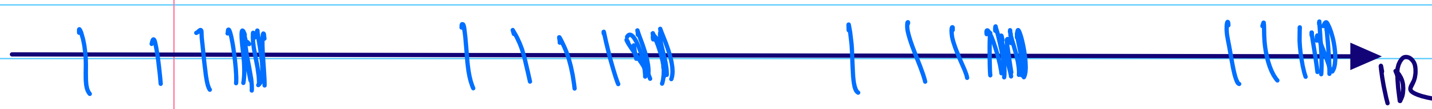
$\lim\limits_{k\to\infty} a_{2k+1} = -1$ (limit -1)

Def: $a \in \mathbb{R}$ is called an accomulation value of $(a_n)_{n\in\mathbb{N}}$ if there is a subseq^n $(a_{n_k})_{k\in\mathbb{N}}$ with $\lim\limits_{k\to\infty} a_{n_k} = a$

Accumulation value is a generalization of term Limit.

$\Rightarrow$ A convergent seq^n can only have one accumulation value = Limit

$\Rightarrow$ However for divergent seq^n could have different accumulation values



The Limit is getting closer to 4 different points. we don't have a Limit because the seq^n still Jumps around, But we get closer and closer to different accumulation values.

$\Rightarrow$ we could restrict ourselves to take seq^n memembors that only live in one convergent seq^n (Limit)

Accumulation value is just a point on the number line, where the seq^n accumulates

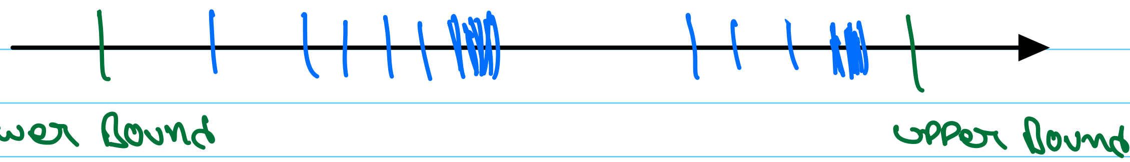
$a \in \mathbb{R}$ is an accumulation value of $(a_n)_{n\in\mathbb{N}}$

$\iff$ $\forall \varepsilon > 0$: The ε-neighbourhood of a contain's infinitely many seq^n member's of $(a_n)_{n\in\mathbb{N}}$.

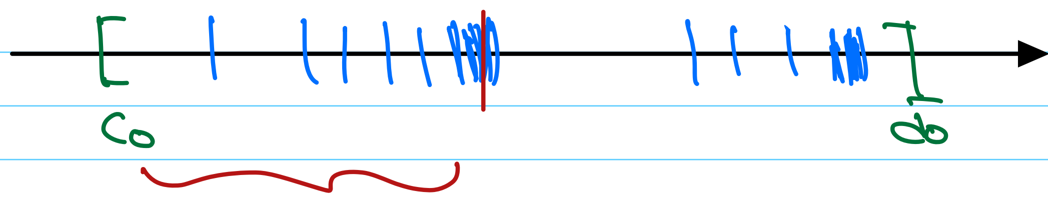
Lec 10: Bolzano-Weierstrass

$(a_n)_{n\in\mathbb{N}}$ bounded $\Longrightarrow$ $(a_n)_{n\in\mathbb{N}}$ has an accumulation value (has a convergent subseq^n)

having a accumulation value mean's the seq^n have a convergent subseq^n.



Proof:

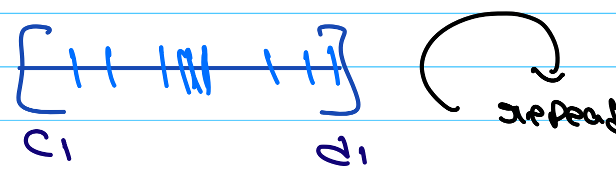


if infinitely many seq^n member's in it:

Choose left-hand interval

otherwise: Choose right hand interval.

New interval:



we set $[c_0, d_0] \supset [c_1, d_1] \supset [c_2, d_2] \dots$

and $d_1 - c_1 = \frac{1}{2}(d_0 - c_0)$

$d_2 - c_2 = \frac{1}{2}(d_1 - c_1)$

$d_3 - c_3 = \frac{1}{2}(d_2 - c_2)$

$\vdots$

$$d_n - c_n = \frac{1}{2^n}(d_0 - c_0) \xrightarrow{n\to\infty} 0$$

we know

$(c_n)_{n\in\mathbb{N}}$ monotonically increasing and bounded

$(d_n)_{n\in\mathbb{N}}$ monotonically decreasing and bounded

$\Rightarrow$ $(c_n)_{n\in\mathbb{N}}, (d_n)_{n\in\mathbb{N}}$ convergent

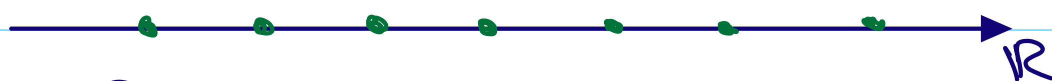
By Limit theorem

$$0 = \lim_{n\to\infty}(d_n - c_n)$$

$$\Rightarrow \lim_{n\to\infty} d_n = \lim_{n\to\infty} c_n$$

Lec 11: Limit superior and limit inferior

__Ex:__ $(a_n)_{n\in\mathbb{N}}$ given by $a_n = n$



$(a_n)_{n\in\mathbb{N}} = (n)_{n\in\mathbb{N}}$ is divergent to ∞

∞: symbol (Not as a number)

divergent to ∞ $:\Longleftrightarrow$ $\forall C>0\ \exists N\in\mathbb{N}$
$\forall n\geq N:\ a_n > C$

$\Leftarrow$ $\lim\limits_{n\to\infty} a_n = \infty$

divergent to $-\infty$ $:\Longleftrightarrow$ $\forall C<0,\ \exists N\in\mathbb{N}$
$\forall n\geq N:\ a_n < C$

$\Leftarrow$ $\lim\limits_{n\to\infty} a_n = -\infty$

$(a_n)_{n\in\mathbb{N}}$ has the improper accumulation value ∞ $:\Longleftrightarrow$ $(a_n)_{n\in\mathbb{N}}$ is not bounded from above

$(a_n)_{n\in\mathbb{N}}$ has the improper accumulation value $-\infty$

$:\Longleftrightarrow (a_n)_{n\in\mathbb{N}}$ is not bounded from below.

A given seq^n $(a_n)_{n\in\mathbb{N}}$ could have many accumulation values:

* we can talk about Largest and Smallest accumulation value, it could be a normal one or improper one.

Def: Let $(a_n)_{n\in\mathbb{N}}$ be a seq^n of real numbers. An element $a \in \mathbb{R} \cup \{-\infty, \infty\}$ is called

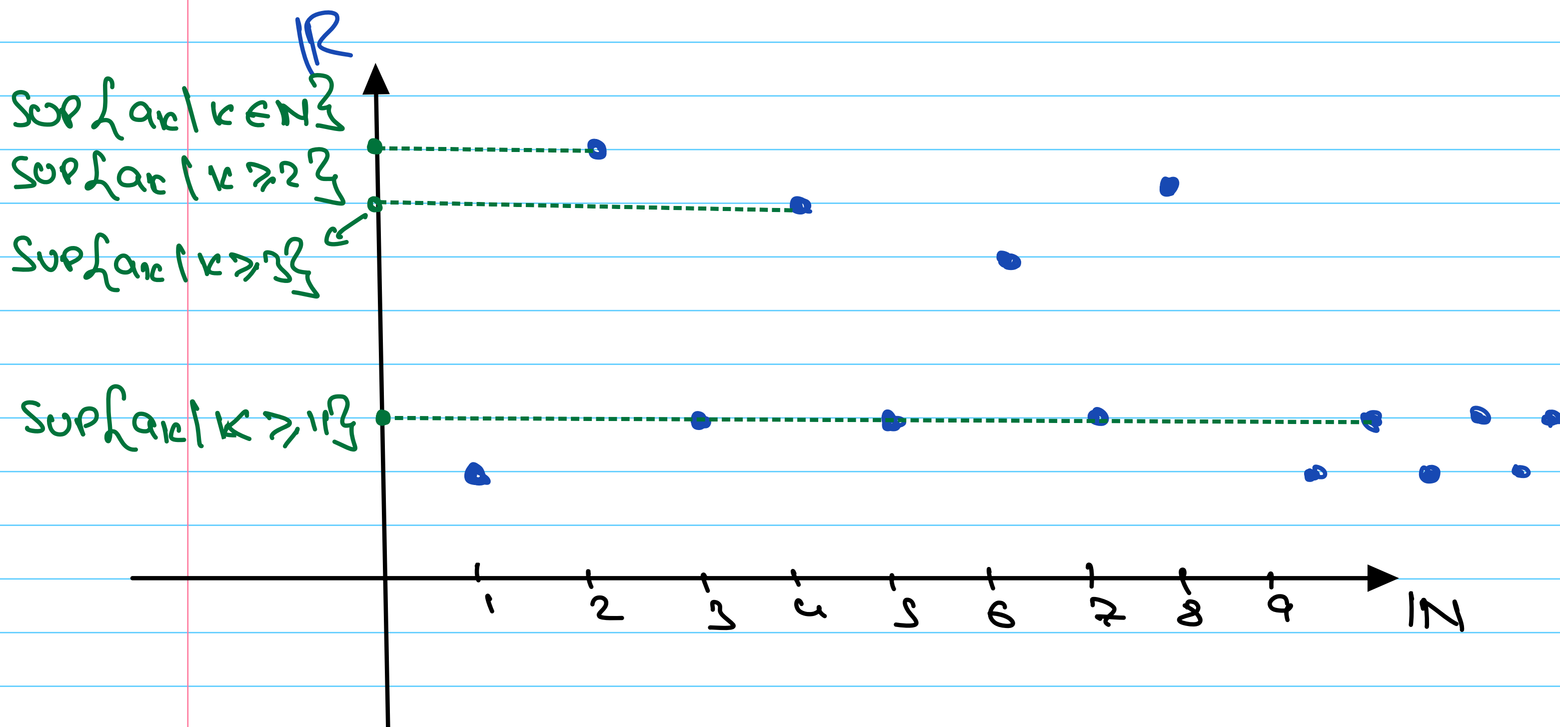
Limit superior of $(a_n)_{n\in\mathbb{N}}$

if a is the Largest (Improper) accumulation value of $(a_n)_{n\in\mathbb{N}}$

$$a := \limsup_{n\to\infty} a_n$$

- limit inferior of $(a_n)_{n\in\mathbb{N}}$ if a is the smallest (improper) accumulation value of $(a_n)_{n\in\mathbb{N}}$.

write $a = \liminf_{n\to\infty} a_n$



Limit superior describes what happen to with the supremum, when we cut of more and more at the begining of the seq

FACT:

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_n \mid k \geq n\}$$

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_n \mid k \geq n\}$$

Lec 12

Let $(a_n)_{n\in\mathbb{N}}$ be a seq^n of real numbers

$$\Rightarrow \limsup_{n\to\infty} a_n, \ \liminf_{n\to\infty} a_n \in \mathbb{R} \cup \{\pm\infty\}$$

$$= [-\infty, \infty]$$

Ex: $(a_n)_{n\in\mathbb{N}} = ((-1)^n \cdot n)_{n\in\mathbb{N}}$

$$= (-1, 2, -3, 4, -5, \ldots)$$

$$\limsup_{n\to\infty} a_n = \infty$$

$$\liminf_{n\to\infty} = -\infty$$

Properties:

(a) $(a_n)_{n\in\mathbb{N}}$ is convergent

$$\iff \limsup_{n\to\infty} = \liminf_{n\to\infty} \notin \{\pm\infty\}$$

(b) $(a_n)_{n\in\mathbb{N}}$ is divergent to ∞

$$\iff \left(\limsup_{n\to\infty} a_n\right) = \liminf_{n\to\infty} a_n = \infty$$

(c) $(a_n)_{n\in\mathbb{N}}$ is divergent to $-\infty$

$$\iff \limsup_{n\to\infty} a_n = \left(\liminf_{n\to\infty} a_n\right) = -\infty$$

(d) For $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, we have

$$\limsup_{n\to\infty}(a_n+b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

(only if right hand side is defined)

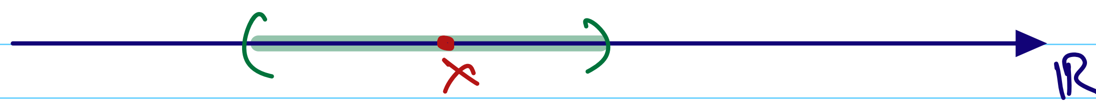
if $a_n, b_n \geq 0$

$$\limsup_{n\to\infty}(a_n \cdot b_n) \leq \limsup_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

(e) if $a_n, b_n \geq 0$

$$\liminf_{n\to\infty} (a_n \cdot b_n) \geq \liminf_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

Lec 13 - Open, closed, and compact sets



$(x-\varepsilon, x+\varepsilon)$ = ε-neighbour hood of x

$\forall \varepsilon > 0$: $(x-\varepsilon, x+\varepsilon) =: B_\varepsilon(x)$

ε-neighbour hood of x

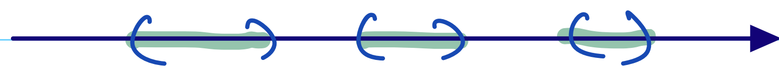
$M \subseteq \mathbb{R}$ is called a neighbourhood of x, if there is $\exists \varepsilon > 0$ s.t $M \supseteq B_\varepsilon(x)$

<u>Example:</u> $[-2,2]$ is a neighbourhood of $B_\varepsilon(0)$, $B_\varepsilon(1)$

$[-2,2]$ is not a neighbourhood of 2

Definition: $M \subseteq \mathbb{R}$ is called open (in $\mathbb{R}$) if, for all $x \in M$, M is a neighbourhood of x.

$\Rightarrow \forall x \in M, \exists \varepsilon > 0 : M \supseteq B_\varepsilon(x)$



$A \subseteq \mathbb{R}$ is closed (in $\mathbb{R}$) if $A^c := \mathbb{R} \setminus A$ is open

Open is Not the opposite of closed.

$\Rightarrow$ There could be neither open nor closed

$\Rightarrow$ There could be open & closed at the same time.

Ex: ① ϕ, $\mathbb{R}$ are Both open & closed

② $[-2, 2]$ is closed but not open

③ $(-2, 2)$ is open but not closed

④ $(-2, 2]$ is neither open nor closed

The subset of the real number's could be much more complicated than just an intervals.

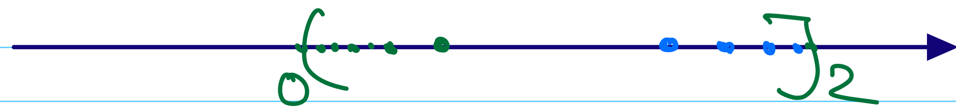
In order to deal with those sets, the next fact is very helpful.

FACT: It Gives us a criterion to check closeness with the help of seq^n.

$A \subseteq \mathbb{R}$ is closed $\Longleftrightarrow$ For all convergent seq^n $(a_n)_{n \in \mathbb{N}}$ with $a_n \in A$ $\forall n \in \mathbb{N}$ we have $\lim_{n \to \infty} a_n \in A$

It's not possible to leave the set with seq^n from Inside.

Ex: $A := (0, 2]$



Take $(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$

$\frac{1}{n} \in (0,2] \quad \forall n \in \mathbb{N}$

But $\lim\limits_{n\to\infty} a_n = 0 \notin A$

This set is not closed.

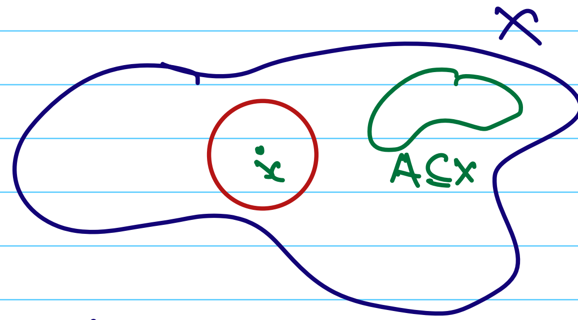
Definition:

$A \subseteq \mathbb{R}$ is called compact, if for all sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n \in A \ \forall n\in\mathbb{N}$, we find convergent subseq^n with a Limit Lies in A. $\Rightarrow (a_{n_k})_{n\in\mathbb{N}}$ with $\lim\limits_{k\to\infty} a_{n_k} \in A$

Functional Analysis lec 03

Open and closed sets

(X, d) : metric space



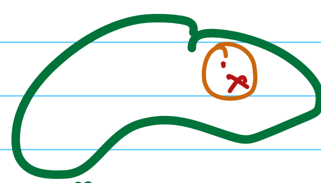
$B_\varepsilon(x)$ open ε-ball around x

$$B_\varepsilon(x) := \{ y \in X \mid d(x,y) < \varepsilon \}$$

(open ball of radius $\varepsilon > 0$, centered at x)

<u>Notion's:</u>

① <u>open sets:</u>

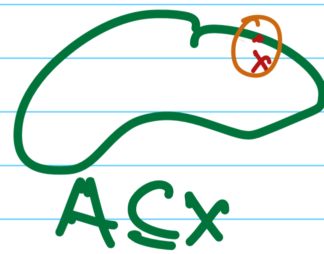


$A \subseteq X$

$A \subseteq X$ is called open, if for each $x \in A$, there is an open ball with $B_\varepsilon(x) \subseteq A$

i.e $\forall x \in A, \exists \varepsilon > 0$, such that $B_\varepsilon(x) \subseteq A$

(2) Boundary points:



$A \subseteq X$. $x \in X$ is called a Boundary Point for A if $\forall \varepsilon > 0$: $B_\varepsilon(x) \cap A \neq \emptyset$

$B_\varepsilon(x) \cap A^c \neq \emptyset$

$$\partial A := \{ x \in X \mid x \text{ is a boundary point for } A \}$$

Remember:

A open $\Longleftrightarrow$ $A \cap \partial A = \emptyset$

Closed set:

A closed $\Longleftrightarrow$ $A \cup \partial A = A$

A closed $\Longleftrightarrow$ $A^c := X \setminus A$ is open

④ Closure:

$\overline{A} := A \cup \partial A$ (always closed)

(smallest closed set that contains A)

Ex: $X := (1,3] \cup (4,\infty)$

$d(x,y) = |x-y|$

(X, d) metric space.

ⓐ $A := (1,3] \subseteq X$

Open: $\forall x \in A$, $\exists \varepsilon$ s.t $B_\varepsilon(x) \subseteq A \Rightarrow$ open

closed: $A^c = (4,\infty)$, $\forall x \in A^c$, $\exists \varepsilon$ s.t

$B_\varepsilon(x) \subseteq A^c \Rightarrow A^c$ is open

$\Rightarrow A$ is closed

ⓒ $C := (1,2]$

$\partial C = \{2\}$

Lec 14 Heine- Borel theorem

Compact set : (sequentially compact set)

we use sequence's to describe compactness.

* Any subset $A \subseteq \mathbb{R}$ is called compact set, if it forces any seq^n to cluster.

* so the seq^n with member's in A has at least one accumulation value, this accumulation value $\in A$

Any seq^n $(a_n)_{n \in \mathbb{N}} \subseteq A$ has a accumulation value $a \in A$.

Example: (a) ϕ is compact

(b) $\{5\}$ is compact

(c) $\mathbb{R}$

we need to look all the seq^n in real

number's $(a_n)_{n \in \mathbb{N}} = (n)_{n \in \mathbb{N}}$

$(n)_{n \in \mathbb{N}}$ divergent to $\infty \notin \mathbb{R}$

$\Rightarrow$ $\mathbb{R}$ is not compact

(d) $[c, d]$, $c \leq d$, compact set.

let $(a_n)_{n \in \mathbb{N}} \subseteq [c, d] \Rightarrow (a_n)_{n \in \mathbb{N}}$ Bounded

$(a_n)_{n \in \mathbb{N}}$ is Bounded $\Rightarrow$ apply Bolzano - weierstrass theorem

$\Downarrow$

Bounded seq^n have at least one accumulation value. $a \in \mathbb{R}$

$[c, d]$ is closed

$\Rightarrow$ we can't leave the set using seq^n

$\Rightarrow$ $a \in [c, d]$

$\Rightarrow$ $[c, d]$ compact set.

Heine-Borel theorem:

for $A \subseteq \mathbb{R}$, we have:

A is compact $\Longleftrightarrow$ A is bounded set and closed.

Lec 15: Series

infinite sum's are also called series.

Series: " infinite sum" , special seq^n.

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k$$

This is called a series, just adding infinitely many number.

Ex: seq^n $(a_k)_{k \in \mathbb{N}} = ((-1)^k)_{k \in \mathbb{N}}$

$$\sum_{k=1}^{\infty} a_k = (-1+1)+(-1+)\cdots\cdots = 0$$

$$= -1 + (1 + -1) + (1-1)\cdots\cdots = -1$$

we have 2 different possible results.

$\Rightarrow$ They don't act like an normal sum.

Definition: Let $(a_k)_{k \in \mathbb{N}}$ be a seq^n.

The seq^n $(S_n)_{n \in \mathbb{N}}$ given by

$$S_n := \sum_{k=1}^{n} a_k$$ is called a series.

The whole seq^n given by (S_n) is called a series.

if this seq^n (S_n) is convergent, then we have meaning for sum.

* if $(S_n)_{n \in \mathbb{N}}$ is convergent, we write

$$\sum_{k=1}^{\infty} a_k := \lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{\infty} a_k$$

Example:

$$\left(\sum_{k=1}^{n} (-1)^k \right)$$

$$= (-1, 0, -1, 0, \dots)$$

not convergent

$$\left(\sum_{k=1}^{n} 1^k \right)_{n \in \mathbb{N}} = (1, 2, 3, \dots)$$

divergent to ∞

Lec 16: Geometric series and Harmonic series

Series: series is just a spectical seq^n

$\sum_{k=1}^{\infty} a_k$ is the seq^n of partial sum's

$$S_n := \sum_{k=1}^{n} a_k$$

Example: Geometric series

$$\sum_{k=0}^{\infty} q^k \quad , \quad q \in \mathbb{R}$$

we show this series is convergent

$\Longleftrightarrow \quad |q| < 1$

Question: $S_n = \sum_{k=0}^{n} q^k = ?$

for $q \neq 1$ $\sum_{k=0}^{n} q^k = ?$

$$\Rightarrow \quad S_n(1-q) = (1-q)\sum_{k=0}^{n} q^k$$

$$= \sum_{k=0}^{n} q^k - \sum_{k=0}^{n} q^{k+1}$$

$$= \sum_{k=0}^{n} q^k - \sum_{k=1}^{n+1} q^k$$

$$= q^0 - q^{n+1}$$

$$= 1 - q^{n+1}$$

$$\Rightarrow \quad S_n \cdot (1-q) = 1 - q^{n+1}$$

$$\Rightarrow \quad S_n = \frac{1-q^{n+1}}{1-q}$$

we need to check in which cases (S_n) is a convergent seq^n

$$\Rightarrow (S_n)_{n\in\mathbb{N}} \text{ convergent} \Longleftrightarrow (q^n)_{n\in\mathbb{N}} \text{ is convergent}$$

$$\Longrightarrow \quad |q|<1$$

$$\lim_{n\to\infty} q^n = 0 \qquad |q|<1$$

$$\Rightarrow \quad \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-q^n}{1-q}$$

$$= \frac{1}{1-q}$$

$$\boxed{\Rightarrow \sum_{k=0}^{\infty} q^k = \lim_{n\to\infty} S_n = \frac{1}{1-q} \quad \text{if} \quad |q|<1}$$

geometric series

Ex: Harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty$$

(divergent to ∞)

Proof:

$$S_n = \sum_{k=1}^{n} \frac{1}{k}$$

(seq^n (S_n) is monotonically increasing

Now we need to prove the seq^n (S_n) is not bounded.

$$S_{2^m} = \sum_{k=1}^{2^m} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots + \frac{1}{2^m}$$

$$S_{2^m} = S_1 + (S_2 - S_1) + (S_4 - S_2) + \cdots + (S_{2^m} - S_{2^{m-1}})$$

$$\Rightarrow S_{2^m} = S_1 + (S_2 - S_1) + (S_4 - S_2) + (S_8 - S_4) + (S_{16} - S_8) + \cdots + (S_{2^m} - S_{2^{m-1}})$$

$$= S_1 + \sum_{j=1}^{m} \left(S_{2^j} - S_{2^{j-1}} \right)$$

$$S_{2^j} - S_{2^{j-1}} = \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{k} > \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{2^j}$$

$$\Rightarrow S_{2^j} - S_{2^{j-1}} > 2^{j-1} \cdot \frac{1}{2^j}$$

$$\Rightarrow S_{2^j} - S_{2^{j-1}} > \frac{1}{2}$$

$$\Rightarrow S_{2^m} = S_1 + \sum_{j=1}^{m} \left(S_{2^j} - S_{2^{j-1}} \right)$$

$$> S_1 + m \cdot \frac{1}{2}$$

$$\Rightarrow S_{2^m} > S_1 + m \cdot \frac{1}{2} \xrightarrow{m \to \infty} \infty$$

$$\Rightarrow S_n = \sum_{k=1}^{n} \frac{1}{k}$$ ist monotonisch.

increasing and unbounded from above

$$\boxed{\Rightarrow \quad \lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} \frac{1}{k} = +\infty}$$

Lec 17 : Cauchy criterion

* Criteria we have to check for convergence of series.

Series: Series is just a seq^n of partial sum's

$$S_n = \sum_{k=1}^{\infty} a_k \quad \text{seq}^n \text{ of partial sum's}$$

* This seq^n (S_n) is convergent we can denote as the limit

$$\sum_{k=1}^{\infty} a_k$$

Properties:

if $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent, then:

(a) $\sum_{k=1}^{\infty}(a_k + b_k)$ is also convergent.

$$= \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k = a + b$$

(b) $\sum_{k=1}^{\infty} \lambda a_k$ is also convergent

$$= \lambda \cdot \sum_{k=1}^{\infty} a_k$$

How can we check a given series is convergent?

① Cauchy Criterion:

it uses completeness axiom of $\mathbb{R}$

* for $\mathbb{R}$, being convergent seq^n $\Longleftrightarrow$ cauchy seq^n

$\Rightarrow$ for cauchy seq^n we don't have to know about the limit

$\Rightarrow$ therefore this criterion check for convergence, but does not tell us the limit.

$$\sum_{k=1}^{\infty} a_k \text{ is convergent}$$

$$\Longleftrightarrow \quad \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq m \geq N:$$

$$\left| \sum_{k=m}^{n} a_k \right| < \varepsilon$$

Proof:

$S_n := \sum_{k=1}^{n} a_k$, $(S_n)_{n\in\mathbb{N}}$ is convergent

$\underset{\text{Completeness}}{\Longleftrightarrow}$ $(S_n)_{n\in\mathbb{N}}$ is a cauchy seq^n

$\Longleftrightarrow$ $\forall \varepsilon > 0,\ \exists N \in \mathbb{N}\ \ \forall \tilde{n}, \tilde{m} \geq N:$

$$|S_{\tilde{n}} - S_{\tilde{m}}| < \varepsilon$$

$\Longleftrightarrow$ $\forall \varepsilon > 0,\ \exists N \in \mathbb{N}\ \ \forall n \geq m \geq N:$

$$|S_n - S_{m-1}| < \varepsilon$$

Ex: $\sum_{k=1}^{\infty} (-1)^k$ calculate

$$\left| \sum_{k=m}^{n} (-1)^k \right| \quad ?$$

$$\Rightarrow \left| \sum_{k=N}^{N+2} (-1)^k \right| = \left\{ \begin{matrix} |1 + (-1) + 1| \\ |-1 + 1 - 1| \end{matrix} \right\} = 1$$

$\Rightarrow$ Not convergent.

Important fact:

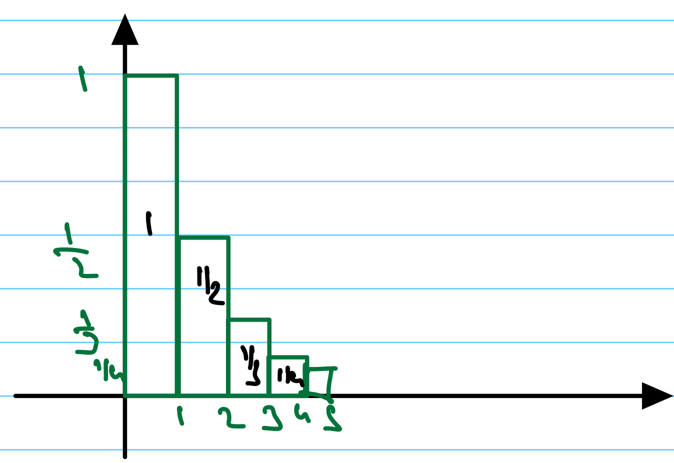
$\sum_{k=1}^{\infty} a_k$ is convergent $\Longrightarrow$ $(a_k)_{k \in \mathbb{N}}$ convergent with $\lim_{k \to \infty} a_k = 0$

Lec 18: Leibniz Criterion

This is the theorem that gives us the sufficient condition for a series to be convergent.

Harmonic series:

$$S_n = \sum_{k=1}^{n} \frac{1}{k}$$



The whole area $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$

$\Rightarrow$ The Harmonic series not converged. but divergent to ∞

Therefore having a seq^n inside the sum that goes to 0 is not sufficient for convergent of the series.

Cauchy Criterion:

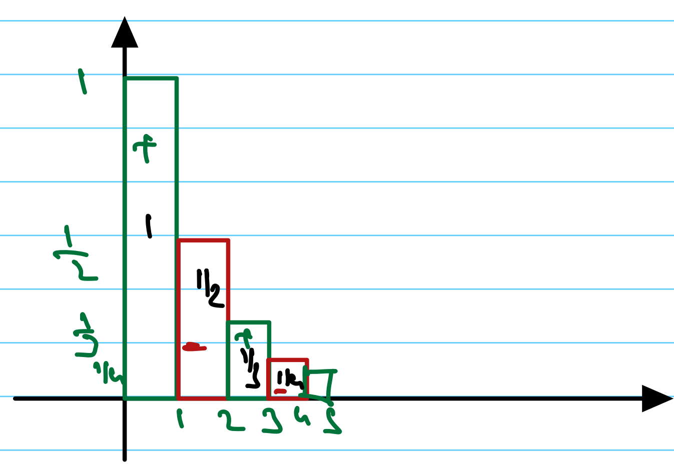
$$\sum_{k=1}^{\infty} a_k \text{ is convergent} \Longrightarrow \lim_{n\to\infty} a_k = 0$$

$$\overset{\times}{\Longleftarrow}$$

Ex:

$$\lim_{k\to\infty} \frac{1}{k} = 0 \quad \text{but} \quad \sum_{k=1}^{\infty} \frac{1}{k} \text{ is divergent to } \infty.$$

Leibniz Criterion:



$$S_n = \sum_{k=1}^{n} (-1)^{k+1} \cdot \frac{1}{k} \quad (\text{Convergent})$$

Theorem: (Alternating Series test, Leibniz Criterion, Leibniz test)

① Let $(a_k)_{k \in \mathbb{N}}$ be convergent with

$$\lim_{k \to \infty} a_k = 0$$

and monotonically decreasing

Then $\sum_{k=1}^{\infty} (-1)^{k} a_k$ is Convergent.

Proof:

$$S_n := \sum_{k=1}^{n} (-1)^k a_k$$

* Since all the a_k are non-negative

i.e $a_k \geq 0 \quad \forall k \in \mathbb{N}$

* and a_k's are monotonically decreasing

* $\lim_{k \to \infty} a_k = 0$

take a subseq^n (even)

$$S_{2l+2} - S_{2l} = \sum_{k=1}^{2l+2} (-1)^k a_k - \sum_{k=1}^{2l} (-1)^k a_k$$

$$= a_{2l+2} - a_{2l+1}$$

$\Rightarrow$ since the seq^n is monotonically decreasing

$$\boxed{S_{2l+2} - S_{2l} = a_{2l+2} - a_{2l+1} \leq 0}$$

$$S_{2l+3} - S_{2l+1} = \sum_{k=1}^{2l+3} (-1)^k a_k - \sum_{k=1}^{2l+1} (-1)^k a_k$$

$$= -a_{2l+3} + a_{2l+2}$$

$$\boxed{\Rightarrow \quad S_{2l+3} - S_{2l+1} = -a_{2l+3} + a_{2l+2} \geq 0}$$

The even subseq^n is monotonically decreasing

The odd subseq^n is monotonically increasing.

$$S_{2l+1} - S_{2l} = -a_{2l+1} \leq 0 \Rightarrow S_{2l+1} \leq S_{2l}$$

$S_{2l+1} \leq S_{2l}$ (this is usefue when we compare sub seq^n)

$(S_{2l+1})_{l \in \mathbb{N}}$ monotonically increasing sub seq^n

$\Rightarrow$ $S_1 \leq S_3 \leq S_5 \leq S_7 \ldots$

$(S_{2l})_{l \in \mathbb{N}}$ monotonically decreasing sub seq^n

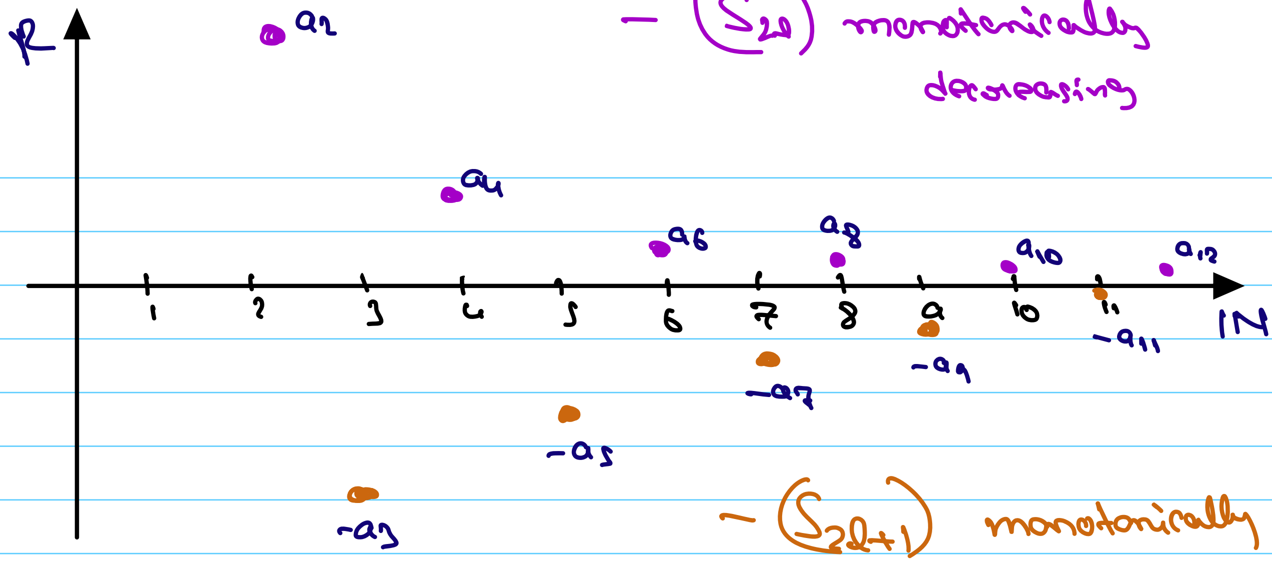
$S_2 \geq S_4 \geq S_6 \geq \ldots$

$\Rightarrow$ $S_3 \leq S_{2l+1} \leq S_{2l} \leq S_2$

(Bounded)

Both the subseq^n are monotonic & bounded

$\Rightarrow$ hence both the seq^n are converges



– (S_{2l}) monotonically decreasing

– (S_{2l+1}) monotonically increasing.

$$\lim_{l\to\infty}\left(S_{2l+1}-S_{2l}\right)=\lim_{l\to\infty}\left(-a_{2l+1}\right)=0$$

$$\Rightarrow\quad S:=\lim_{l\to\infty}S_{2l+1}=\lim_{l\to\infty}S_{2l}$$

$$\Rightarrow\quad \lim_{n\to\infty}S_n=S \qquad \text{(convergent)}$$

Lec 19 Comparison test:

* Comparison test is a Criterion used to decide if a Given series is convergent or not.

* we need a new definition about Convergence

$\sum_{k=1}^{\infty} a_k$ is called absolutely convergent

The series $\sum_{k=1}^{\infty} a_k$ is called absolutely convergent if the series given by the absolute value of a_k is simple convergent.

$\Rightarrow$ $\sum_{k=1}^{\infty} a_k$ is Convergent if $\sum_{k=1}^{\infty} |a_k|$ is convergent.

$\Rightarrow$ absoutely convergent

$\Longrightarrow$ convergent.

$\sum_{k=1}^{\infty} a_k$ is absolute convergent

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

for all $\forall n \geq m \geq N$:

$$\left| \sum_{k=m}^{n} |a_k| \right| < \varepsilon$$

(Cauchy criterion)

$$\Rightarrow \sum_{k=m}^{n} |a_k| < \varepsilon$$

$$\Rightarrow \overbrace{\left| \sum_{k=m}^{n} a_k \right| \leq \sum_{k=m}^{n} |a_k|}^{\text{triangular inequality}} < \varepsilon$$

$$\Rightarrow \left| \sum_{k=m}^{n} a_k \right| < \varepsilon$$

(Cauchy criterion for $\underbrace{\sum_{k=1}^{\infty} a_k}_{\text{original series}}$)

Example:

* $\sum_{k=1}^{\infty} \frac{1}{k}$ (Not convergent)

but * $a_k \geq 0 \ \forall k \in \mathbb{N}$

* monotonically decreasing

* $\lim_{n\to\infty} a_n = 0$

$\Rightarrow \sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ is Convergent using (Leibniz criterion)

But $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ is not absolutely convergent.

$\Rightarrow$ absolute convergent $\underset{\not\Leftarrow}{\Longrightarrow}$ convergent

$\Uparrow$

Stronger notion of convergence

This is exactly we need for comparison test.

Majorant Criterion:

$$\text{let } \sum_{k=1}^{\infty} a_k \text{ be a series}$$

if there is $n_0 \in \mathbb{N}$ and a convergent series $\sum_{k=1}^{\infty} b_k$
with $b_k \geq 0 \ \forall k$

$$|a_k| \leq b_k \quad \forall k \geq n_0$$

we have another seq^n b_k that is convergent.

$\Rightarrow$ then we can conclude the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

Proof:

apply cauchy criterion to $\sum_{k=1}^{\infty} b_k$

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq m > N$

we have
$$\left| \sum_{k=m}^{n} b_k \right| < \varepsilon$$

if we choose $N > n_0$ then

$$\sum_{k=m}^{n} |a_k| \leq \sum_{k=m}^{n} b_k = \left|\sum_{k=m}^{n} b_k\right| < \varepsilon$$

$$\Rightarrow \sum_{k=m}^{n} |a_k| < \varepsilon$$

$\Rightarrow \sum_{k=1}^{\infty} a_k$ is absolute convergent.

Minorant Criterion:

It works similarly but we don't check for convergence, but for divergence.

let $\sum_{k=1}^{\infty} a_k$ be a series with $a_k \geq 0$

if there is $n_0 \in \mathbb{N}$ and a divergent series

$$\sum_{k=1}^{\infty} b_k$$ with $b_k \geq 0$ and with

$a_k \geq b_k \quad \forall\, k \geq n_0$, then

$$\sum_{k=1}^{\infty} a_k$$ is divergent.

Lec 20: Ratio and Root test

Both Ratio and Root in the best case scenario they answer us if a given series is absoultely convergent.

$$\sum_{k=1}^{\infty} a_k$$ absolutely convergent?

if we have a majorant

There is a convergent majorant

$$\Downarrow$$

$$\sum_{k=1}^{\infty} a_k$$ is absoutely convergent.

we know that geometric series

$$\sum_{k=0}^{\infty} q^k \text{ convergent} \iff |q| < 1$$

<u>FACT:</u>

if there is $C, n_0 \in \mathbb{N}$ and $q \in \mathbb{R}$ with

$|q| < 1$ such that $|a_k| \leq C q^k \quad \forall k \geq n_0$

then $\sum_{k=1}^{\infty} a_k$ is absoulte convergent.

<u>Ratio test:</u> if there is $n_0 \in \mathbb{N}$ and

$q \in [0, 1)$ such that

$$\left| \frac{a_{k+1}}{a_k} \right| \leq q$$

Then $\sum_{k=1}^{\infty} a_k$ is absoulte convergent.

$\forall k \geq n_0$, then $\sum_{k=1}^{\infty} a_k$ is absoulte convergent.

Proof: $$|a_{k+1}| \leq q \cdot |a_k| \leq q \cdot q \cdot |a_{k-1}|$$

$$\Rightarrow |a_{k+1}| \leq q^{k+1-n_0} |a_{n_0}|$$

$$\Rightarrow |a_{k+1}| \leq q^{k+1} \cdot \frac{|a_{n_0}|}{q^{n_0}}$$

$$\Rightarrow |a_{k+1}| \leq q^{k+1} \cdot C$$

Root Test:

if there is $n_0 \in \mathbb{N}$ and $q \in (0,1)$ such that

$$\sqrt[k]{(|a_k|)} \leq q \qquad \forall k \geq n_0$$

then $\sum_{k=1}^{\infty} a_k$ is abs. convergent.

Proof: $$\sqrt[k]{|a_k|} \leq q$$

$$\Leftrightarrow |a_k| \leq q^k$$

$$\Rightarrow \text{convergent.}$$

Rember:

$$\limsup_{k\to\infty} \sqrt[k]{|a_k|} < 1$$

$$\Longrightarrow \sum_{k=1}^{\infty} a_k \text{ is abs. convergent.}$$

$$\limsup_{k\to\infty} \sqrt[k]{|a_k|} > 1 \Longrightarrow \sum_{k=1}^{\infty} a_k \text{ is divergent.}$$

Geometric series

$$\sum_{k=0}^{\infty} q^k \quad , \quad q \in \mathbb{R}$$

we show $\sum_{k=0}^{\infty} q^k$ convergent $\Longleftrightarrow |q|<1$

Question:

$$S_n = \sum_{k=0}^{n} q^k = ?$$

for $q \neq 1$: $S_n = \sum_{k=0}^{n} q^k$

$$\Rightarrow \quad (1-q) S_n = (1-q) \sum_{k=0}^{n} q^k$$

$$= \sum_{k=0}^{n} q^k - \sum_{k=0}^{n} q^{k+1}$$

$$= \sum_{k=0}^{n} q^k - \sum_{k=1}^{n+1} q^k$$

$$= q^0 - q^{n+1}$$

$$\Rightarrow \quad S_n = \frac{1-q^{n+1}}{1-q}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-q^{n+1}}{1-q}$$

$$= \frac{1}{1-q}\Big(1 - \underbrace{\lim_{n\to\infty} q^{n+1}}_{=0 \text{ if } |q|<1}\Big)$$

$$= \frac{1}{1-q} \quad \text{if} \quad |q|<1$$

Geometric Series

$$\boxed{\Rightarrow \sum_{k=0}^{\infty} q^k = \lim_{n\to\infty} \sum_{k=0}^{n} q^k = \lim_{n\to\infty} S_n = \frac{1}{1-q} \quad \text{if } |q|<1}$$

$(S_n)_{n\in\mathbb{N}}$ is convergent $\Longleftrightarrow$ $(q^n)_{n\in\mathbb{N}}$ convergent to 0

$\Longleftrightarrow |q|<1$

Harmonic Series

$$\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty$$

(divergent to ∞)

Proof:

$$S_n = \sum_{k=1}^{n} \frac{1}{k}$$

$(S_n)_{n\in\mathbb{N}}$ is monotonically increasing

$\Rightarrow$ either $\lim\limits_{n\to\infty} S_n \in \mathbb{R}$ (converges)

or $= +\infty$ (diverges to $+\infty$)

we need to show that $(S_n)_{n\in\mathbb{N}}$ is not bounded to show that S_n diverges to $+\infty$.

* Let's take a sub seq^n of (S_n)

$$S_{2^m} = \sum_{k=1}^{2^m} \frac{1}{k}$$

$$S_{2^m} = S_1 + (S_2 - S_1) + (S_4 - S_2) + (S_8 - S_4) + \cdots + (S_{2^m} - S_{2^{m-1}})$$

all all the term's cancel out each other and only left with S_{2^m}

$\Rightarrow$

$$S_{2^m} = S_1 + (S_2 - S_1) + (S_4 - S_2) + \cdots + (S_{2^m} - S_{2^{m-1}})$$

$$S_{2^m} = S_1 + \sum_{j=1}^{m} \left(S_{2^j} - S_{2^{j-1}}\right)$$

$$S_{2^j} - S_{2^{j-1}} = \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{k} > \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{2^j}$$

$$\Rightarrow \quad S_{2^j} - S_{2^{j-1}} > \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{2^j}$$

$$> 2^{j-1} \cdot \frac{1}{2^j} = \frac{1}{2}$$

$$\Rightarrow \quad S_{2j} - S_{2j-1} = \frac{1}{2}$$

$$\Rightarrow \quad S_{2m} = S_1 + \sum_{j=1}^{m} \left(S_{2j} - S_{2j-1} \right)$$

$$\Rightarrow \quad S_{2m} > S_1 + \frac{m}{2}$$

$$\Rightarrow \quad \lim_{m \to \infty} S_{2m} \geq \lim_{m \to \infty} S_1 + \frac{m}{2}$$

$$\Rightarrow \quad \lim_{m \to \infty} S_{2m} = +\infty$$

Hence the Harmonic series

$$(S_n)_{n \in \mathbb{N}} = \left(\sum_{k=1}^{n} \frac{1}{k} \right)_{n \in \mathbb{N}}$$

is divergent to $+\infty$

Harmonic series

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k} = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k} = +\infty}$$

Cauchy Criterion:

we say a series $\sum a_n$ satisfies the Cauchy Criterion, if its seq^n (S_n) the Partial sum's is a Cauchy Seq^n

$$\sum_{k=1}^{\infty} a_k \text{ is Convergent}$$

$$\Updownarrow$$

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t $\forall n, m \geq N$

$$|S_n - S_m| < \varepsilon$$

let's restric to $n \geq m$ such that

$$|S_n - S_{m-1}| < \varepsilon$$

$$\Rightarrow \left| \sum_{k=m}^{n} a_k \right| < \varepsilon$$

$\Rightarrow$ we can rewrite it as

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq m > N$

we have

$$\left| \sum_{k=m}^{n} a_k \right| < \varepsilon$$

A series converges $\Longleftrightarrow$ if it satisfy

Cauchy criterion.

Since the series converge it should

hold for $n = m$

$$\Rightarrow \quad |S_n - S_{m-1}| < \varepsilon$$

$$\Rightarrow \quad |S_n - S_{n-1}| < \varepsilon$$

$$\Rightarrow \quad |a_n| < \varepsilon \quad \left((a_n) \text{ converges to } 0 \right)$$

$\sum_{k=1}^{\infty} a_n$ converges $\Longrightarrow$ $(a_n)_{n \in \mathbb{N}}$ converges

$\cancel{\Longleftarrow}$ with $\lim_{n\to\infty} a_n = 0$

This is not equivalance

Leibniz Criterion:

Ex: Harmonic series: $S_n = \sum_{k=1}^{n} \frac{1}{k}$ diverges to $+\infty$

Theorem: (Alternating test, Leibniz Criterion)

① Let $(a_k)_{k \in \mathbb{N}}$ be convergent with $\lim_{n \to \infty} a_k = 0$

② $(a_k)_{k \in \mathbb{N}}$ is monotonically decreasing seq^n

$\Longrightarrow$ Then the alternating series

$$S_n := \sum_{k=1}^{\infty} (-1)^k a_k \text{ is convergent.}$$

Proof: $S_n = \sum_{k=1}^{n} (-1)^k a_k$

we know $(a_k)_n$ is monotonically decreasing

and the limit $= 0 \Rightarrow a_k \geq 0 \ \forall k \in \mathbb{N}$

let's take subseque of even and odd

i.e $(S_{2l})_{l \in \mathbb{N}}$ $(S_{2l+1})_{l \in \mathbb{N}}$

$$S_{2l+2} - S_{2l} = a_{2l+2} - a_{2l+1}$$

$(a_{2l+1} \geq a_{2l+2})$

$$S_{2l+2} - S_{2l} \leq 0$$

$$\Rightarrow S_{2l+2} \leq S_{2l}$$

$\Rightarrow (S_{2l+2})_{l \in \mathbb{N}}$ is monotonically decreasing seq

$$S_{2l+3} - S_{2l+1} = -a_{2l+3} + a_{2l+2}$$

$(a_{2l+2} \leq a_{2l+3})$

$$\geq 0$$

$$\Rightarrow S_{2l+3} \geq S_{2l+1}$$

$(S_{2l+1})_{l \in \mathbb{N}}$ is monotonically increasing seq^n

$$S_{2l+1} - S_{2l} = -a_{2l+1} \leq 0$$

$$\Rightarrow \quad S_{2l+1} \leq S_{2l}$$

(decreasing) (increasing)

$$\Rightarrow \quad S_3 \leq S_{2l+1} \leq S_{2l} \leq S_2$$

(bounded) Both subseq

$$\lim_{l\to\infty} \left(S_{2l+1} - S_{2l} \right) = 0$$

$$\Rightarrow \lim_{l\to\infty} \left(-a_{2l+1} \right) = \lim_{l\to\infty} -\frac{1}{k} = 0$$

$$\Rightarrow \lim_{l\to\infty} S_{2l+1} = \lim_{l\to\infty} S_{2l}$$

$$\Rightarrow \quad S := \lim_{l\to\infty} S_{2l+1} = \lim_{l\to\infty} S_{2l}$$

$$\Rightarrow \quad \lim_{n\to\infty} S_n = S \quad \text{(converged)}$$

Ex: $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}} = \sum_{k=1}^{\infty} (-1)^k \cdot a_k$

① $a_k = \frac{1}{\sqrt{k}}$ $\quad (a_k)_{k \in \mathbb{N}}$ monotonically decreasing

② $\lim_{n\to\infty} a_k = \lim_{n\to\infty} \frac{1}{\sqrt{k}} = 0$

③ $a_k \geq 0 \quad \forall k \in \mathbb{N}$

Let's apply Leibniz Criterion.

$\Rightarrow$ if the above 3 satisfies then the series $\sum_{k=1}^{\infty} (-1)^k a_k$ converges

$\left(\Rightarrow \sum_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k}} \text{ converges} \right)$

Comparison Test:

A new definition for convergence.

$\sum_{k=1}^{\infty} a_k$ is called absolutely convergent if $\sum_{k=1}^{\infty} |a_k|$ is convergent.

* for a series that has only positive a_k's i.e $a_k \geq 0 \ \forall k$, then this not a new definition.

However in general its a stronger convergence.

absoulty convergent $\Longrightarrow$ convergent
$\cancel{\Longleftarrow}$

$\Downarrow$

Stronger notion of convergence

Proof: $\sum_{k=1}^{\infty} |a_k|$ is convergent $\Longrightarrow$

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that

$\forall\, n \geq m > N$ we have

$$\left| \sum_{k=m}^{n} |a_k| \right| < \varepsilon$$

$$\Longrightarrow \quad \sum_{k=m}^{n} |a_k| < \varepsilon$$

$$\Longrightarrow \quad \underbrace{\left| \sum_{k=m}^{n} a_k \right| \leq \sum_{k=m}^{n} |a_k|}_{\text{triangle inequality}} < \varepsilon$$

$$\Longrightarrow \quad \left| \sum_{k=m}^{n} a_k \right| < \varepsilon$$

$\sum_{n=1}^{\infty} a_k$ is convergent.

Counter example:

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$$ is Convergent according to Leibniz Criterion.

However $$\sum_{k=1}^{\infty} \left| (-1)^{k} \frac{1}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k}$$ is not Convergent

i.e Convergence $\not\Longrightarrow$ absolute Convergence

Majorant Criterion:

Let $\sum_{k=1}^{\infty} a_k$ be a series.

if there is $n_0 \geq N$ and a Convergent series $\sum_{k=1}^{\infty} b_k$ with $b_k \geq 0 \;\; \forall k$, then

if $|a_k| \leq b_k \;\; \forall k \geq n_0$

then $\sum_{k=1}^{\infty} a_k$ is abs. Convergent.

<u>Proof:</u> Apply Cauchy criterion to

$$\sum_{k=1}^{\infty} b_k$$

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$\forall n \geq m > N$ we have

$$\left| \sum_{k=m}^{n} b_k \right| < \varepsilon$$

we know that

$$|a_k| \leq b_k \quad \forall k \geq n_0$$

$$\Rightarrow \quad \sum_{k=n_0}^{n} |a_k| \leq b_k$$

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists N \geq n_0$, such that

$\forall n \geq m > N$ we have

$$\left| \sum_{k=m}^{n} b_k \right| < \varepsilon$$

Since $b_k \geq 0$ $\forall k \in \mathbb{N}$

$$\sum_{k=m}^{n} b_k < \varepsilon$$

$$\Rightarrow \sum_{k=m}^{n} |a_k| \leq \sum_{k=m}^{n} b_k < \varepsilon$$

$$\Rightarrow \sum_{k=m}^{n} |a_k| < \varepsilon$$

$\Rightarrow$ $|a_k|$ is convergent

$\Rightarrow$ a_k is absoulte convergent.

Minorant Criterion:

let $\sum_{k=1}^{\infty} a_k$ be a series with $a_k \geq 0$

if there is $n_0 \in \mathbb{N}$ and a divergent series $\sum_{k=1}^{\infty} b_k$ with $b_k \geq 0$

and with $a_k \geq b_k \quad \forall k \geq n_0$,

then $\sum_{k=1}^{\infty} a_k$ is divergent.

Example:

$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ convergent or divergent?

Sol'

$\forall k > 0 \;\&\; \in \mathbb{N} \qquad \sqrt{k} < k$

$$\Rightarrow \frac{1}{\sqrt{k}} > \frac{1}{k} \quad \forall k > 0$$

we know the $\sum_{k=1}^{\infty} \frac{1}{k} = +\infty$ divergent

$\Rightarrow$ therefore $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ is divergent

$\sum_{k=1}^{\infty} a_k$ absolutely convergent?

There is a convergent majorant

$\Downarrow$

$\sum_{k=1}^{\infty} a_k$ is absolutely convergent

$\Downarrow$

$\sum_{k=1}^{\infty} a_k$ is convergent

$\Rightarrow$ we need to find a convergent majorant series to prove the convergence of $\sum_{k=1}^{\infty} a_k$

$\Rightarrow$ why don't we take

geometric series as Convergent majorant.

$\Rightarrow$ if there is a $n_0 \in \mathbb{N}$ and $q \in \mathbb{R}$ and $|q| < 1$ such that $|a_k| \leq C q^k$ $\forall k \geq n_0$ (convergent majorant)

$$\Downarrow$$

$\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

<u>RATIO test:</u> if there is $n_0 \in \mathbb{N}$ and $q \in [0,1)$ such that

$a_k \neq 0$ $$\left| \frac{a_{k+1}}{a_k} \right| \leq q \quad \forall k \geq n_0$$

Then $\sum_{k=1}^{\infty} a_k$ is abs convergent.

Proof:

$$|a_{k+1}| \leq q|a_k|$$

$$\Rightarrow |a_{k+1}| \leq q \cdot q\, |a_{k-1}|$$

$$\Rightarrow |a_{k+1}| \leq q^{k+1-n_0}\, |a_{n_0}|$$

$$\Rightarrow |a_{k+1}| \leq q^{k+1} \cdot \frac{|a_{n_0}|}{q^{n_0}}$$

$$\Rightarrow |a_{k+1}| \leq C \cdot q^{k+1}$$

(this satisfies the majorant criterion)

$\Rightarrow \sum_{k=1}^{\infty} a_k$ is absoulety convergent.

Ex: $\sum_{k=1}^{\infty} \frac{1}{k!}$ Convergent?

Sol^n

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{k!}{(k+1)!}\right| = \frac{1}{k+1} \leq \frac{1}{2} \quad \forall k \geq 1$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k!} \text{ is convergent.}$$