Section 13 : Some Topological Concepts in Metric Spaces

Def 13.1:-

Let S be a set, and suppose d is a function defined for all the pairs (x,y) of elements from S satisfying

D1. $d(x,x)=0 \quad \forall x \in S$ and $d(x,y)>0 \quad \forall x,y \in S, \ x \neq y$

D2: $d(x,y) = d(y,x) \quad \forall x,y \in S$

D3: $d(x,z) \leq d(x,y) + d(y,z) \quad \forall x,y,z \in S$

$(S,d) \to$ metric space

Ex2

$$S = \mathbb{R}^K$$

$x = (x_1, x_2, \cdots x_K)$ where

$x_j \in \mathbb{R} \quad \forall j = 1, 2, \cdots K$

$$d(x,y) = \left(\sum_{j=1}^{K} (x_j - y_j)^2 \right)^{1/2}$$

$$= \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_K-y_K)^2}$$

Proof: that d is a metric

Def 13.2:-

A seq^n (s_n) in a metric space (S,d) converges to s in S if $\lim_{n\to\infty} d(s_n, s) = 0$.

A seq^n (s_n) in S is a Cauchy seq^n if $\forall \varepsilon > 0$, $\exists N$ s.t

$$m, n > N \implies d(s_m, s_n) < \varepsilon$$

(S, d) is a complete metric space

if every Cauchy seq^n in S converges to some element in S.

Lemma 13.3

A seq^n $(x^{(n)})$ in $\mathbb{R}^k$ converges

$\Longleftrightarrow$ $\forall j = 1, 2, \ldots k$, the seq^n $(x_j^{(n)})$ converges in $\mathbb{R}$.

A seq^n $(x^{(n)})$ in $\mathbb{R}^k$ is a Cauchy seq^n $\Longleftrightarrow$ $(x_j^{(n)})$ is a Cauchy seq^n in $\mathbb{R}$ $\forall j = 1, 2, \ldots k$.

Proof:

$$|x_j - y_j| \leq d(x, y) \leq \sqrt{k} \max\{|x_j - y_j| : j = 1, 2, \ldots k\}$$

for a fixed j, let $\varepsilon > 0$, $\exists N \in \mathbb{N}$

such that $\forall m, n > N$

$$d(x^{(m)}, x^{(n)}) < \varepsilon$$

$$\Rightarrow \quad |x_j^{(m)} - x_j^{(n)}| < \varepsilon$$

$\Rightarrow$ $\left(x_j^{(n)}\right)$ is a Cauchy seq^n

$\Rightarrow$ $\forall j = 1, 2, \ldots K$, each seq^n

$\left(x_j^{(n)}\right)$ is a Cauchy seq^n in $\mathbb{R}$.

Therefore for $\varepsilon > 0$, $\exists N_j$ such that

$m, n > N_j \quad |x_j^{(m)} - x_j^{(n)}| < \frac{\varepsilon}{\sqrt{K}}$

if $N = \max\{N_1, N_2, \ldots N_K\}$

$$\max\{|x_j^{(m)} - x_j^{(n)}|\} < \frac{\varepsilon}{\sqrt{k}}$$

$$\Rightarrow \sqrt{k}\max\{x_j^{(m)} - x_j^{(n)} : j = 1, 2 \cdots k\} < \varepsilon$$

$$\Rightarrow d(x^{(m)}, x^{(n)}) < \varepsilon$$

$\Rightarrow (x^{(n)})$ is a Cauchy seq^n in $\mathbb{R}^k$.

Theorem 13.4:

Euclidean k-space $\mathbb{R}^k$ is complete.

Theorem 13.5 (Bolzano-Weierstrass theorem)

Every bounded seq^n in $\mathbb{R}^k$ has a convergent sub-seq^n.

Definition 13.6:

Let (S, d) be a metric space. Let E be a subset of S. An element $s_0 \in E$ is interior of E if for some $r > 0$ we have

$$\{s \in S : d(s, s_0) < r\} \subseteq E$$

E° = set of points in E that are interior to E.

if E is open then $E^\circ = E$

13.7 Discussion:

(1) S is open in S

(2) The empty set ϕ is open

(3) The union of any collection of open sets is open.

(4) The intersection of finitely many

open sets is again an open set.

Def 13.8:

(S,d) metric space

$E \subseteq S$ is closed

$\iff E^c = S \setminus E$ is open.

Closure: $\bar{E}$ = intersection of all closed sets containing E.

Boundary $\partial E = \bar{E} \setminus E^\circ$

Theorem 13.10:

Let (F_n) be a decreasing seq^n [i.e., $F_1 \supseteq F_2 \supseteq \cdots$] of closed bounded non-empty sets in $\mathbb{R}^k$.

Then $F = \bigcap_{n=1}^{\infty} F_n$ is also closed bounded and non-empty.

Heine-Borel theorem:

A subset E of $\mathbb{R}^k$ is compact

$\Longleftrightarrow$ closed and bounded