The set $\mathbb{N}$ of Natural Numbers

$\mathbb{N} = \{1, 2, 3, \ldots\}$ set of all positive integers.

⇒ Each positive integer n has a successor, namely $n+1$.

⇒ Thus the successor of 2 is 3, successor of 37 is 38.

Properties of $\mathbb{N}$: (Peano Axioms)
(Peano Postulates)

N1. $1 \in \mathbb{N}$

N2. if $n \in \mathbb{N}$, then $n+1 \in \mathbb{N}$ (successor)

N3. 1 is not the successor of any element in $\mathbb{N}$

N4. if n and $m \in \mathbb{N}$ have same successor, then $n = m$

N5. A subset of $\mathbb{N}$ which contain 1, and which contain's $n+1$, whenever it contains n, must equal to $\mathbb{N}$.

Example 1:-

Prove $1+2+\cdots+n = \frac{1}{2}n(n+1)$ for positive integers n.

Sol^n

$$P_n : 1+2+\cdots+n = \frac{n(n+1)}{n}$$

nth proposition.

① Base case:

$$n=1 \Rightarrow \quad 1 = \frac{1.2}{2} = 1 \quad \text{(True)}$$

② Induction step

Suppose P_n is true

$$\Rightarrow \quad 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

then $P_{n+1} = 1+2+\cdots+n+n+1$

$$= \frac{n(n+1)}{2} + n+1$$

$$= \frac{n(n+1) + 2(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

$\Rightarrow$ P_{n+1} is true if P_n is true.

By the Principle of mathematical induction P_n is true for all n.

Example 2

All number's of the form $5^n - 4n - 1$ are divisible by 16

Soln

Proposition P_n: "$5^n - 4n - 1$" is divisible by 16

① BASE CASE

$P_1 := 5^1 - 4 - 1 = 0$ is divisible by 16

② Induction case

Suppose P_n is true.

$\Rightarrow$ $5^n - 4n - 1$ is divisble by 16

$\Rightarrow$ $5^n - 4n - 1 = 16m$

then $P_{n+1} = 5^{n+1} - 4(n+1) - 1$

$= 5 \cdot 5^n - 4n - 1 - 4$

$$= 5\cdot(5^n - 4n - 1) + 16n$$

$$= 5\cdot 16m + 16n$$

$$= \text{divisible by } 16$$

$\Rightarrow$ by the Principle of mathematical induction

P_n: $5^n - 4n - 1$ is divisible by 16 is

true $\forall n \in \mathbb{N}$

Example 3 :-

Show $|\sin nx| \le n|\sin x|$ for all Positive

integer's n and all real number's x

Solution: our n^{th} Proposition is

P_n: " $|\sin nx| \le n|\sin x|$ $\forall x$

① BASE CASE

P_1: $|\sin x| \le |\sin x| \Rightarrow |\sin x| = |\sin x|$

True

② Induction step

Suppose P_n is true,

i.e $\quad |\sin nx| \leq n|\sin x| \quad \forall x \in \mathbb{R}$

then $\quad P_{n+1} \Rightarrow |\sin(n+1)x|$

$\Rightarrow |\sin(nx+x)|$

$\Rightarrow |\sin nx \cos x + \sin x \cos nx|$

$\Rightarrow |\sin(nx+x)| \leq |\sin nx||\cos x| + |\cos nx||\sin x|$

$|\cos x| \leq 1 \quad , \quad |\cos nx| \leq 1$

$\Rightarrow |\sin(n+1)x| \leq |\sin nx| + |\sin x|$

$\Rightarrow |\sin(n+1)x| \leq n|\sin nx| + |\sin x|$

$\Rightarrow |\sin(n+1)x| \leq (n+1)|\sin nx|$

P_{n+1} is true if P_n is true

$\Rightarrow$ According to Principles of mathematical induction, P_n is true $\forall n \in \mathbb{N}$