04 - The Completeness Axiom

in this section we give you the completeness axiom of $\mathbb{R}$. This is to ensure us $\mathbb{R}$ has no "gaps".

4.1 Definition:

let S be an nonempty subset of $\mathbb{R}$.

(a) if S contain's a largest element s_0 [$s_0 \in S$, $s \leq s_0$ $\forall s \in S$], then we call s_0 the maximum of S and write

$s_0 = \max S$

(b) if S contain's a smallest element, then we call the smallest element the minimum of S and write it as $\min S$.

① Every finite non-empty subset of $\mathbb{R}$ has a maximum and a minimum.

$$\max\{1,2,3,4,5\} = 5 \quad , \min = 1$$

ⓑ Real number's a, b , where $a < b$, The following notation will be used throughout

$[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ closed interval

$[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$ half-open (or)

$(a,b) = \{x \in \mathbb{R} : a < x < b\}$ open interval

$(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$ semi-open interval

$$\max[a,b] = b \quad ; \quad \min[a,b] = a$$

The set (a,b) has no max & min as $a, b \notin (a,b)$

ⓒ The set $\{x \in \mathbb{Q} : 0 \leq x \leq \sqrt{2}\}$ has a minimum $= 0$, But no maximum.

$\sqrt{2} \notin \mathbb{Q}$

There are infinitely many rational number's close to $\sqrt{2}$ $\Rightarrow$ maximum does not exists.

(e) $\left\{ n^{(-1)^n} : n \in \mathbb{N} \right\} = \left\{ 1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \frac{1}{7} \dots \right\}$

This set has no maximum & minimum.

Definition 4.2 :-

Let S be a non-empty subset of $\mathbb{R}$

(a) if a real number M satisfies $s \leq M$ for $\forall s \in S$, then M is called an upper bound of S and the set S is said to be bounded above

(b) if a real number m satisfy $m \leq s$ $\forall s \in S$, then m is called a lower bound and the set S is said to be bounded below.

(c) The set S is said to be bounded if it is bounded above and bounded below.

Thus S is bounded if $\exists\ m, M \in \mathbb{R}$, s.t

$$S \subseteq [m, M]$$

4.3 Definition:

Let S be a non-empty subset of $\mathbb{R}$.

(a) if S is bounded above and S has a least upper bound, then we will call it the supremum of S and denote it by $\sup S$

(b) if S is bounded below, and S has a greatest lower bound, then we will call it the infimum of S and denote it by $\inf S$.

4.4 Completeness Axiom:

Every non-empty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. In other words, $\sup S$ exists and is a real number.

Ex: Completeness axiom for $\mathbb{Q}$ (won't satisfy)

$$A = \{ r \in \mathbb{Q} : 0 \le r < \sqrt{2} \}$$

set A is bounded from above. Ex $\frac{3}{2}$ is upper bound.

$\Rightarrow$ But there is no least upper bound to the set A that is rational number.

Thus the completeness Axiom does not hold for $\mathbb{Q}$.

4.5 Corollary:

Every nonempty subset S of $\mathbb{R}$ that is bounded below has a greatest lower bound $\inf S$.

4.6 Archimedean Property

if $a>0$, and $b>0$, then for som +ve integer n, we have $na>b$.

+ even if a is quite small, b is quite big Some integer multiple of a exceeds b.