Section 14 - Series

infinite series:

$$\sum_{n=m}^{\infty} a_n = a_m + a_{m+1} + \cdots + a_\infty$$

$$(S_n)_{n=m}^{\infty} \quad \text{where} \quad S_n = \sum_{k=m}^{n} a_k$$

The infinite series $\sum_{n=m}^{\infty} a_n$ is said to converge provided the sequence (S_n) of partial sum's converges to a real number S, in which case we define $\sum_{n=m}^{\infty} a_n = S$.

Thus $\sum_{n=m}^{\infty} a_n = S$ mean's

$$\lim S_n = S \quad \text{or} \quad \lim_{n \to \infty} \sum_{k=m}^{n} a_n = S.$$

* A series that does not converge is said to diverge. we say that $\sum_{n=m}^{\infty} a_n$

diverges to $+\infty$ and we write

$$\sum_{n=m}^{\infty} a_n = +\infty \implies \lim S_n = +\infty.$$

* The symbol $\sum_{n=m}^{\infty} a_n$ has no meaning unless the series converges, or diverges to $+\infty$ or $-\infty$.

* if the term's a_n of an infinite series $\sum a_n$ are all non-negative, then the seq^n (S_n) is monotone increasing

$\implies (S_n)$ either converge or diverge to $+\infty$

$\implies \sum |a_n|$ is a monotone increasing seq^n, $\implies$ converge or diverse to $+\infty$.

Geometric series

$$\sum_{k=0}^{\infty} ar^k \qquad r \neq 1$$

The partial sum $S_n = \sum_{k=0}^{n} ar^k$ is

$$(1-r) S_n = (1-r) \sum_{k=0}^{n} ar^k$$

$$= \sum_{k=0}^{n} ar^k - \sum_{k=0}^{n} ar^{k+1}$$

$$= \sum_{k=0}^{n} ar^k - \sum_{k=1}^{n+1} ar^k$$

$$= ar^0 - ar^{n+1}$$

$$\Rightarrow S_n = \frac{a(1-r^{n+1})}{1-r}$$

$$(S_n)_{n=0}^{\infty} = \left(\frac{a(1-r^{n+1})}{1-r} \right)$$

converges if $|r| < 1$

& $$\boxed{\lim_{n\to\infty} S_n = \frac{a}{1-r} \quad \text{if } |r|<1}$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges} \iff p>1}$$

14.3 Definition:

we say a series $\sum a_n$ satisfies the Cauchy Criterion if its seq (S_n) of partial sum's is a Cauchy seq^n.

$\forall \varepsilon>0$, $\exists N \in \mathbb{N}$ s.t $m,n > N$

$\implies$ $|S_n - S_m| < \varepsilon$

if we impose $n>m$

$\implies$ $\forall \varepsilon>0$, $\exists N$ s.t $n \geq m > N$

$|S_n - S_{m-1}| < \varepsilon$

$$\Rightarrow \left| \sum_{k=m}^{n} a_k \right| < \varepsilon$$

Def 14.4:-

A series converges $\Longleftrightarrow$ if it satisfies the Cauchy criterion.

14.5 corollary:

if a series $\sum a_n$ converges then $\lim a_n = 0$

Series $\sum a_n$ converges $\Longrightarrow$ $\lim a_n = 0$

$\cancel{\Longleftarrow}$

14.6 Comparison test:

Let $\sum a_n$ be a series where $a_n \geq 0 \ \forall n$

(i) if $\sum a_n$ converges and $|b_n| \leq a_n \ \forall n$, then $\sum b_n$ converges

(ii) if $\sum a_n = +\infty$ and $b_n \geq a_n \ \forall n$

then $\sum b_n = +\infty$.

Proof:

(i) $$\underbrace{\left|\sum_{k=m}^{n} b_k\right| \leq \sum_{k=m}^{n} |b_k|}_{\text{triangular inequality}} \leq \sum_{k=m}^{n} a_k$$

Since a_n converge's its satisfy Cauchy criterian.

$\Rightarrow$ $\sum b_n$ also satisfies the Cauchy criterian.

(ii) let $S_n = \sum_{k=m}^{n} a_k$ and $t_n = \sum_{k=m}^{n} b_k$

be the Partial Sum's of $\sum a_n$ and $\sum b_n$ respectively.

$b_n \geq a_n \quad \forall n$

$\Rightarrow \quad t_n \geq s_n \quad \forall n$

$\Rightarrow \quad 0 \leq s_n \leq t_n \quad \forall n$

$\Rightarrow \quad 0 \leq \lim s_n \leq \lim t_n$

$\Rightarrow \quad 0 \leq +\infty \leq \lim t_n$

$\Rightarrow \quad \lim t_n = +\infty.$

$\Rightarrow \quad \boxed{\sum b_n = +\infty}$

14.7 Corollary:

Absolutely Convergent series are Convergent.

Suppose $\sum b_n$ is absolutely Convergent. This mean's $\sum a_n$ converges where $a_n = |b_n| \ \forall n$. Then $|b_n| \leq a_n$ trivially, so $\sum b_n$ Converges by 14.6(i)

14.8 ratio test:

A series $\sum a_n$ of non-zero terms

(i) Converges absolutely if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$

(ii) diverges if $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$

(iii) o.w $\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$

and the test gives no information.

Root test:

Let $\sum a_n$ be a series and let $\alpha = \limsup |a_n|^{1/n}$.

Then series $\sum a_n$

(i) Converges absolutely if $\alpha < 1$

(ii) diverges if $\alpha > 1$

(iii) o.w $\alpha = 1$ and test gives no information.

Proof:

Suppose $\alpha < 1$, and select $\varepsilon > 0$ so that $\alpha + \varepsilon < 1$.

$$\alpha = \lim\sup |a_n|^{1/n}$$

$$\Rightarrow \alpha = \lim_{k \to \infty} \sup \{ |a_n|^{1/n} : n > k \}$$

$\exists N \in \mathbb{N}$, for $\varepsilon > 0$ $(\varepsilon + \alpha < 1)$ such that $\forall n > N$

(convergence of seq^n)

$$\left| \sup \{ |a_n|^{1/n} : n > N \} - \alpha \right| < \varepsilon$$

$$\Rightarrow -\varepsilon < \sup \{ |a_n|^{1/n} : n > N \} - \alpha < \varepsilon$$

$$\Rightarrow \alpha - \varepsilon < \sup \{ |a_n|^{1/n} : n > N \} < \varepsilon + \alpha$$

$$\Rightarrow |a_n|^{1/n} < \alpha + \varepsilon \quad \forall n > N, \text{ so}$$

$\Rightarrow \quad |a_n| < (\alpha+\varepsilon)^n \quad \forall n > N$, so

since $\alpha+\varepsilon < 1$

the geometric series $\sum_{n=N+1}^{\infty} (\alpha+\varepsilon)^n$

converges, from the comparison test

$\sum_{n=N+1}^{\infty} a_n$ also converges.

Example 3

$$\sum_{n=2}^{\infty} \left(-\frac{1}{3}\right)^n = \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \frac{1}{243} + \cdots$$

geometric seri with $q = -\frac{1}{3}$

$$|q| = \frac{1}{3} < 1$$

$$\Rightarrow \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

$$\Rightarrow \sum_{n=2}^{\infty} \left(\frac{-1}{3}\right)^n = \frac{1}{1+\frac{1}{3}} - 1 + \frac{1}{3}$$

$$= \frac{3}{4} - 1 + \frac{1}{3}$$

$$= \frac{3}{4} - \frac{2}{3} = \frac{9-8}{12} = \frac{1}{12}$$

<u>Ratio test:</u>

$$a_n = \left(\frac{-1}{3}\right)^n$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|-\left(\frac{1}{3}\right)^{n+1} \cdot 3^n\right|$$

$$= \frac{1}{3}$$

$$\limsup \left|\frac{a_{n+1}}{a_n}\right| = \limsup \frac{1}{3}$$

$$= \frac{1}{3} < 1$$

Hence $\sum_{n=2}^{\infty} \left(\frac{-1}{3}\right)^n$ converges absoutely.

Example 4 :

$$\sum \frac{n}{n^2+3}$$

$$a_n = \frac{n}{n^2+3} \qquad a_{n+1} = \frac{n+1}{n^2+2n+4}$$

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n} \cdot \frac{n^2+3}{n^2+2n+4}$$

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \frac{n^2+3}{n^2+2n+4} = 1$$

Ratio test gives no information.

* $$\frac{n}{n^2+3} \geq \frac{n}{n^2+3n^2} = \frac{1}{4n}$$

$$\Rightarrow \sum \frac{1}{n} \text{ diverges}$$

$$\Rightarrow \sum \frac{1}{4n} \text{ diverges.}$$

Ex5 :-

$$\sum \frac{n}{3^n}$$

$$a_n = \frac{n}{3^n}$$

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{3^{n+1}} \cdot \frac{3^n}{n} = \frac{n+1}{3n}$$

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{1+\frac{1}{n}}{3} = \frac{1}{3} < 1$$

Hence the series $\sum \frac{n}{3^n}$ converges by the ratio test.

Ex7:

$\sum a_n$ where $a_n = \left[\frac{2}{(-1)^n - 3}\right]^n$

Root test:

$$(a_n)^{1/n} = \frac{2}{(-1)^n - 3}$$

$$\limsup \left| \frac{2}{(-1)^n - 3} \right|$$

$$\left| \frac{2}{(-1)^n - 3} \right| = \begin{cases} 1 & \text{if } n \text{ is even} \\ \frac{1}{2} & \text{if } n \text{ is even} \end{cases}$$

$\Rightarrow \quad \limsup |a_n|^{1/n} = 1$

$\Rightarrow$ root test gives no information.

a_n do not converge to 0

$\Rightarrow \quad \sum a_n$ diverges.

Ex 8

$$\sum_{n=0}^{\infty} 2^{(-1)^n - n} = 2 + \frac{1}{4} + \frac{1}{4} + \frac{1}{16} + \frac{1}{8} + \frac{1}{64} + \cdots$$

$$a_n = \frac{1}{2^n - (-1)^n} \leq \frac{1}{2^n - 1} \quad \forall n$$