Exercises

3.1a Which of the properties A1–A4, M1–M4, DL, O1–O5 fail for $\mathbb{N}$?

Sol^n

① A4: $\forall a \in \mathbb{N}\ \exists -a$ s.t $a+(-a)=0$
(FAIL)

② M4: $\forall a \neq 0\ \exists a^{-1}$ s.t $a \cdot a^{-1} = 1$
(FAIL)

③ A3: $a+0=a$, $0 \notin \mathbb{N}$

b Which of these properties fail for $\mathbb{Z}$?

Sol^n

M4: $\forall a \neq 0\ \exists a^{-1} \in \mathbb{Z}$ s.t $a \cdot a^{-1} = 1$
(FAIL)

3.2a The commutative law A2 was used in the proof of (ii) in theorem 3.1. Where?

Sol^n (ii) $a \cdot 0 = 0 \quad \forall a$

Commutative law $a+b = b+c$, $ab = bc$

3.3 Prove (iv) and (v) of theorem 3.1

Sol^n (iv) $(-a)(-b) = ab \quad \forall a, b$

$$-a + a = 0$$

$$\Rightarrow (-a+a)(-b) = 0$$

$$\Rightarrow (-a)(-b) + a(-b) = 0 \quad — (1)$$

& $$-a+a=0$$

$$\Rightarrow (-a+a)b=0$$

$$\Rightarrow (-a)b + ab = 0 \quad — (2)$$

$$\boxed{\Rightarrow (-a)(-b) = ab}$$

(v) $ac = bc$ and $c \neq 0 \Rightarrow a = b$

$$\Rightarrow ac - bc = 0$$

$\Rightarrow (a-b)c = 0$ here $c \neq 0$

$\Rightarrow a-b = 0$

$\Rightarrow \boxed{a = b}$

3.4 Prove (v) & (vii) of theorem 3.2

Sol^n (v) $0 < 1$

if $0 < a$

$\Rightarrow a \cdot 1 = a$ & $a \cdot 0 = 0$

$\Rightarrow$ as we know $0 < a$

$\Rightarrow a \cdot 0 < a \cdot 1$

$\Rightarrow \boxed{0 < 1}$

if $a < 0$

$\Rightarrow a \cdot 1 < a \cdot 0$ (as $a < 0$)

$\Rightarrow \boxed{1 > 0}$

(vii) if $0 < a < b$ then $0 < b^{-1} < a^{-1}$

$\forall a, b, c \in \mathbb{R}$

3.5 (a) Show $|b| \leq a \iff -a \leq b \leq a$

Sol^n if Proof:- $|b| \leq a$

$-|b| \leq b \leq |b|$ $\Bigg\} \Rightarrow b \leq a$

if $|b| \leq a \Rightarrow -a \leq |b|$

$\Rightarrow -a \leq |b| \leq b$ $\Bigg\} \Rightarrow -a \leq b$

$\Rightarrow -a \leq b \leq a$

Only if Proof:

$-a \leq b \leq a$ we know $-a \leq |a| \leq a$

$-|b| \leq b \leq |b| \Rightarrow$ if b is +ve $b = |b|$

if b is -ve $b = -|b|$ $\Bigg\}$ tight bounds

therefore $\Rightarrow b$

(b) Show that $||a| - |b|| \leq |a-b|$

$\forall a, b \in \mathbb{R}$

Sol[n]

we have triangle inequality

$|x+y| \leq |x| + |y| \quad \forall x, y \in \mathbb{R}$

substitute $x = x-y$

$y = y$

$\Rightarrow \quad |x-y+y| \leq |x-y| + |y|$

$\Rightarrow \quad |x| \leq |x-y| + |y|$

$\Rightarrow \quad |x| - |y| \leq |x-y| \quad — (1)$

from Symmetry $|x-y| = |y-x|$ we can write

$|y| - |x| \leq |x-y| \quad — (2)$

$$\left.\begin{aligned} |x| - |y| &\leq |x-y| \\ -(|x| - |y|) &\leq |x-y| \end{aligned}\right\} \Rightarrow \boxed{\big||x| - |y|\big| \leq |x-y|}$$

3.6 a Prove $|a+b+c| \leq |a|+|b|+|c| \quad \forall a,b,c \in \mathbb{R}.$

Sol^n

$$|a+b| \leq |a|+|b|$$

$$|b+c| \leq |b|+|c|$$

$$|a+b+c| \leq |a+b| + |c| \leq |a|+|b|+|c|$$

$$\Rightarrow \quad |a+b+c| \leq |a|+|b|+|c|$$

b Use induction to prove

$$|a_1+a_2+\dots+a_n| \leq |a_1|+|a_2|+\dots+|a_n|$$

for n numbers $a_1, a_2, \dots a_n$

Sol^n

Base case:

P_2: $\quad |a_1+a_2| \leq |a_1|+|a_2|$

(triangle inequality)

Inductive step:

Suppose P_n is true

$$\Rightarrow \quad |a_1+a_2+\dots+a_n| \leq |a_1|+|a_2|+\dots|a_n|$$

then

P_{n+1}:

$$|a_1 + a_2 + \cdots + a_n + a_{n+1}|$$
$$\leq |a_1 + a_2 + \cdots + a_n| + |a_{n+1}|$$
$$\leq |a_1| + |a_2| + \cdots + |a_n| + |a_{n+1}|$$

$\Rightarrow$ P_{n+1} is true.

3.7a show $|b| < a \iff -a < b < a$

Sol^n **if Proof**:

$\Longrightarrow$

$|b| < a$

$-|b| \leq b \leq |b|$ (we know)

if b is $+ve$ then $b = |b|$

if b is $-ve$ then $b = -|b|$

Given $|b| < a$

$\Rightarrow \quad -|b| \leq b \leq |b| < a$

$\Rightarrow \quad \boxed{b < a}$

if $\quad |b| < a \quad$ then $\quad -a < -|b|$

$\Rightarrow \quad -a < -|b| \leq b$

$\Rightarrow \quad \boxed{-a < b}$

therefore $\quad \boxed{-a < b < a}$

$\Leftarrow$ (Proof)

Given $\quad -a < b < a$