Exercises : section 9

(9.1) (a) $\lim \frac{n+1}{n} = 1$

Sol^n

$$\lim \frac{n+1}{n} = \lim \frac{1+\frac{1}{n}}{1}$$

$$= \frac{1 + \lim \frac{1}{n}}{1} = 1$$

(b)

$$\lim \frac{3n+7}{6n-5}$$

$$= \frac{3 + 7 \cdot \lim \frac{1}{n}}{6 - 5 \cdot \lim \frac{1}{n}} = \frac{3}{6} = \frac{1}{2}$$

(c) $\lim \frac{17n^5 + 73n^4 - 18n^2 + 3}{23n^5 + 13n^3} = \frac{17}{23}$

$$= \frac{17 + 73 \lim \frac{1}{n} - 18 \lim \frac{1}{n^3} + 3 \lim \frac{1}{n^5}}{23 + 13 \lim \frac{1}{n^2}}$$

$$= \frac{17 + 0 - 0 + 0}{23 + 0} = \frac{17}{23}$$

9.2 $\lim x_n = 3$, $\lim y_n = 7$ $y_n \neq 0 \ \forall n$

(a) $\lim (x_n + y_n)$

$$= \lim x_n + \lim y_n = 10$$

(b) $\lim \frac{3y_n - x_n}{y_n^2}$

$$= \frac{\lim \left(\frac{3}{y_n} - \frac{x_n}{y_n^2} \right)}{1}$$

$$= \frac{3 \cdot \lim \frac{1}{y_n} - \frac{\lim x_n}{\lim y_n^2}}{1}$$

$$= \frac{\frac{3}{7} - \frac{3}{49}}{1} = \frac{18}{49}$$

9.3 $\lim a_n = a$, $\lim b_n = b$ and $s_n = \dfrac{a_n^3 + 4a_n}{b_n^2 + 1}$

$$\lim s_n = \frac{\lim a_n^3 + 4 \lim a_n}{\lim b_n^2 + 1}$$

$$= \frac{a^3 + 4a}{b^2 + 1}$$

9.4 Let $s_1 = 1$, and $\forall n \geq 1$, let $s_{n+1} = \sqrt{s_n + 1}$

(a) List the first four term's of (s_n)

$\left(1, \sqrt{2}, \sqrt{\sqrt{2}+1}, \sqrt{\sqrt{\sqrt{2}+1}+1}\right.$

(b) $\lim s_n$?

as $n \to \infty$ $\quad s_n = s_{n+1} = s$

$\Rightarrow \quad s = \sqrt{s+1}$

$\Rightarrow \quad s^2 - s - 1 = 0$

$$\Rightarrow \quad S = \frac{1 \mp \sqrt{5}}{2}$$

S cannot be $-$ve

$\Rightarrow$ therefore

$$\lim_{n\to\infty} S_n = \frac{1+\sqrt{5}}{2}$$

9.5 Let $t_1 = 1$ and $t_{n+1} = \dfrac{t_n^2+2}{2t_n}$ for $n \geq 1$

What is $\lim t_n$?

Sol^n as t_n converges.

as $n \to \infty$ $\quad t_{n+1} = t_n = t$

$$t_{n+1} = t = \frac{t^2+2}{2t}$$

$$\Rightarrow 2t^2 = t^2 + 2$$

$$\Rightarrow t^2 = 2 \quad \Rightarrow \boxed{t = \sqrt{2}}$$

(9.6) let $x_1 = 1$ and $x_{n+1} = 3x_n^2$ for $n \geq 1$

(a) Show that if $a = \lim a_n$, then $a = \frac{1}{3}$ or 0

Soln as $n \to \infty$ $x_n = x_{n+1} = a$

$\Rightarrow$ $a = 3a^2$

$\Rightarrow$ $a(1-3a) = 0$

$\Rightarrow$ $a = 0$ or $a = \frac{1}{3}$

(b) Does $\lim x_n$ exists? Explain.

$(x_1 = 1, x_{n+1} = 3x_n^2 : \forall n \geq 1)$

$= (1, 3, 3^3, 3^7, 3^{15}, \ldots)$

The seq is diversinsins.

(c) The contradiction is we are ignoring x_1 (1st element)

9.8

(a) $\lim n^3$ (Not Exist)

(b) $\lim(-n^3) = -\lim n^3$ (Not Exist)

(c) $\lim(-n)^n =$ (Not Exist)

(d) $\lim(1.01)^n$ (Not Exist)

$\lim a^n = 0$ if $|a| < 1$

here $1.01 > 1$

(e) $\lim n^n$ (Not exist)

9.9

Suppose there exist's No such that

$s_n \leq t_n$ for all $n > N_0$

(a) Prove that if $\lim s_n = +\infty$, then $\lim t_n = +\infty$

Sol^n

we have $\lim_{n \to \infty} s_n = +\infty$

$\Rightarrow$ $\forall M > 0$, $\exists N_1 > N_0$, such that $\forall n > N_1$

we have $s_n > M$

we know $t_n \geq S_n \quad \forall n > N_0$

$\& \quad N_1 > N_0$

$\Rightarrow \quad t_n \geq S_n > M$

$t_n > M \quad \forall n > N_1 \quad \forall M$

Therefore $\boxed{\lim_{n \to \infty} t_n = +\infty}$

(b) Prove that if $\lim t_n = -\infty$, then $\lim S_n = -\infty$

sol^n

we have $\lim t_n = -\infty$

$\Rightarrow \quad \forall M < 0 \quad \exists N_1 > N_0$ such that $\forall n > N_1$ we have

$t_n < M$

$\& \quad S_n \leq t_n \quad \forall n > N_0$ and we know $N_1 > N_0$

$\Rightarrow \quad S_n \leq t_n < M$

$\Rightarrow \quad S_n < M \quad \forall n > N_1 \;\&\; \forall M < 0$

hence $\boxed{\lim_{n\to\infty} S_n = -\infty}$

(c) Prove that if $\lim S_n$ & $\lim t_n$ exist's then $\lim S_n \leq \lim t_n$

Sol^n

Suppose $\lim S_n = S$

$\lim t_n = t$

$\Rightarrow \forall \varepsilon > 0 \quad \exists N_1 \in \mathbb{N}$ such that $\forall n > N_1$

$|S_n - S| < \varepsilon \Rightarrow S - \varepsilon < S_n < S + \varepsilon$

$\Rightarrow \forall \varepsilon > 0 \quad \exists N_2 \in \mathbb{N}$ such that $\forall n > N_2$

$|t_n - t| < \varepsilon \Rightarrow t - \varepsilon < t_n < t + \varepsilon$

take $n > \max\{N_1, N_2, N_0\} \Rightarrow S_n \leq t_n$

$S - \varepsilon < S_n \leq t_n < t + \varepsilon$

$\Rightarrow \quad s-\varepsilon < t+\varepsilon \quad \forall \varepsilon > 0$

$\Rightarrow \quad s < t + 2\varepsilon \quad \forall \varepsilon > 0$

ε - is arbitrary, therefore take $\varepsilon \approx 0$

$$\Rightarrow \boxed{\begin{array}{l} s \leq t \\ \lim s_n \leq \lim t_n \end{array}}$$

9.10 (a) Show that if $\lim s_n = +\infty$ and $k > 0$, then $\lim (k s_n) = +\infty$

Sol^n

$\lim s_n = +\infty \Rightarrow$

$\forall M > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$

we have $s_n > M$

$\Rightarrow$ Give $k > 0 \Rightarrow k s_n > kM \quad \forall n > N$

$\Rightarrow \forall KM>0, \exists N \in \mathbb{N}$, s.t $\forall n>N$ we have

$$KS_n > KM \Rightarrow \lim KS_n = +\infty$$

9.10 (b) Show $\lim S_n = +\infty \Longleftrightarrow \lim(-S_n) = -\infty$

Sol^n ($\Longrightarrow$) let's take $\lim S_n = +\infty$, which mean's

$\forall M>0, \exists N \in \mathbb{N}$, s.t $\forall n>N$ we have

$$S_n > M$$

$$\Rightarrow -M < -S_n$$

$\Rightarrow \forall -M<0, \exists N \in \mathbb{N}$, s.t $\forall n>N$ we have

$$-M < -S_n$$

$$\Rightarrow \lim(-S_n) = -\infty.$$

($\Longleftarrow$) same proof bottom to top.

(c) Show that if $\lim s_n = +\infty$ and $k<0$ then $\lim (k s_n) = -\infty$

Sol^n $\lim s_n = -\infty \Rightarrow$

$\forall M>0, \exists N \in \mathbb{N}$, s.t $\forall n>N$ we have

$$s_n > M$$

& $k<0$ (negative value)

$$k s_n < kM < 0$$

$\Rightarrow \forall kM<0, \exists N \in \mathbb{N}$, s.t $\forall n>N$ we have

$$k s_n < kM < 0$$

$$\Rightarrow \quad \lim (k s_n) = -\infty$$

9.11 (a) Show that if $\lim s_n = +\infty$ and $\inf\{t_n : n \in \mathbb{N}\} > -\infty$, then $\lim (s_n + t_n) = +\infty$

Soln

$\lim s_n = +\infty \Rightarrow$

$\forall M > 0 \quad \exists N_1 \in \mathbb{N}$ such that $\forall n > N_1$ we have

$$s_n > M$$

$\inf\{t_n : n \in \mathbb{N}\} > -\infty$, $\Rightarrow$ for some $n = N_2$

$$t_{N_2} \geq t_n \quad \forall n \in \mathbb{N}$$

therefore $\forall M > 0 \quad \exists N_3 > \max\{N_1, N_2\}$

such the $\forall n > N_3$ we have

$$S_n < M$$

$$S_n + t_n < M + t_n$$

$$\Rightarrow \quad S_n + t_n \cdot M + t_n < \underbrace{M + t_{N_2}}_{\substack{M_1 \\ \text{Constant}}}$$

$$\Rightarrow \quad S_n + t_n < M_1$$

$$\Rightarrow \quad \boxed{\lim_{n \to \infty} S_n + t_n = +\infty}$$

(b) Show that if $\lim s_n = +\infty$ and $\lim t_n > -\infty$, then $\lim (s_n + t_n) = +\infty$

Sol^n $\lim t_n > -\infty$. assume $\lim t_n = t$

for $\varepsilon = 1$, $\exists N_1 \in \mathbb{N}$ such that $\forall n > N_1$ we have

$$|t_n - t| < 1$$

$$-1 < t_n - t < 1 \Rightarrow t - 1 < t_n < 1 + t$$

$\forall M > 0$ $\exists N_2 > N_1$ such that $\forall n > N_2$

we have

$$s_n > M - t + 1$$

Now we know $t_n > t - 1$ $\forall n > N_1$
$\Leftarrow N_2 > N_1$

$$\Rightarrow s_n + t_n > M - t + 1 + t - 1$$

$$\Rightarrow s_n + t_n > M \quad \forall M > 0, \forall n > N_2$$

$$\Rightarrow \boxed{\lim_{n \to \infty} (s_n + t_n) = +\infty}$$