Exercises: Section 8

8.1 (a) $\lim \frac{(-1)^n}{n} = 0$

Sol^n Let $\varepsilon > 0$, then we want to prove

for $n > N$

$$\left| \frac{(-1)^n}{n} \right| < \varepsilon$$

$$\Rightarrow \quad \frac{|(-1)^n|}{n} < \varepsilon$$

$$\Rightarrow \quad \frac{1}{n} < \varepsilon \quad \Rightarrow \quad \frac{1}{\varepsilon} < n$$

we can take $N = \frac{1}{\varepsilon}$

Formal Proof:

Let $\varepsilon > 0$, Let $N = \frac{1}{\varepsilon}$, Then $n > N$

$$\Rightarrow \quad n > \frac{1}{\varepsilon}$$

$$\Rightarrow \quad \frac{1}{n} < \varepsilon$$

$$\Rightarrow \frac{|(-1)^n|}{n} < \varepsilon$$

$$\Rightarrow \left| \frac{(-1)^n}{n} - 0 \right| < \varepsilon$$

therefore $\lim_{n \to \infty} \frac{(-1)^n}{n} = 0$

Q-1 b

$$\lim \frac{1}{n^{1/3}} = 0$$

Sol^n

Let $\varepsilon > 0$, we want to Prove for $n > N \in \mathbb{N}$

$$\left| \frac{1}{n^{1/3}} - 0 \right| < \varepsilon$$

$$\Rightarrow \frac{1}{n^{1/3}} < \varepsilon \Rightarrow \frac{1}{\varepsilon^3} < n$$

Therefore Take $N = \frac{1}{\varepsilon^3}$

Formal Proof:

let $\varepsilon > 0$, let $N = \frac{1}{\varepsilon^3}$, then for

$n > N$ i.e $n > \frac{1}{\varepsilon^3}$

$$\Rightarrow \frac{1}{n^{1/3}} < \varepsilon$$

$$= \left| \frac{1}{n^{1/3}} - 0 \right| < \varepsilon$$

Therefore $\lim \frac{1}{n^{1/3}} = 0$

8.1 C

Prove $\lim \frac{2n-1}{3n+2} = \frac{2}{3}$

__Proof:__ Let $\varepsilon > 0$, we want to prove for $n > N$

$$\left| \frac{2n-1}{3n+2} - \frac{2}{3} \right| < \varepsilon$$

$$\Rightarrow \left| \frac{6n-3-6n-6}{3(3n+2)} \right| < \varepsilon$$

$$\Rightarrow \left| \frac{-9}{3(3n+2)} \right| < \varepsilon$$

$$\Rightarrow \left|\frac{-3}{3n+2}\right| < \varepsilon$$

$$\Rightarrow \frac{3}{3n+2} < \varepsilon \Rightarrow \left(\frac{3}{\varepsilon}-2\right)\frac{1}{3} < n$$

$$\Rightarrow n > \frac{1}{\varepsilon} - \frac{2}{3}$$

Formal Proof:

Let $\varepsilon > 0$, and $N = \frac{1}{\varepsilon} - \frac{2}{3}$, $\forall n > N$

i.e $\quad n > \frac{1}{\varepsilon} - \frac{2}{3}$

$$\Rightarrow n + \frac{2}{3} > \frac{1}{\varepsilon}$$

$$\Rightarrow \frac{3}{3n+2} < \varepsilon$$

$$\Rightarrow \left|\frac{-9}{3(3n+2)}\right| < \varepsilon$$

$$\Rightarrow \left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| < \varepsilon$$

Hence Proved.

8.1d $\lim \frac{n+6}{n^2-6} = 0$

Sol^n Let $\varepsilon > 0$, we want to prove

$$\left| \frac{n+6}{n^2-6} - 0 \right| < \varepsilon$$

$$\Rightarrow \quad \left| \frac{n+6}{n^2-6} \right| < \varepsilon$$

It is difficult to solve for or isolate n.

we need not need to find least N

therefore: $\Rightarrow$ find upper bound to numerator
or
Lower bound to denominator.

i.e $n+6 \leq 7n$

& $n^2-6 \geq \frac{n^2}{2}$ for $n > 3$

therefore

$\forall n > 2$

$$\left|\frac{n+6}{n^2-6}\right| \leq \left|\frac{7n}{\frac{n^2}{2}}\right| < \varepsilon$$

$$\Rightarrow \quad \left|\frac{14}{n}\right| < \varepsilon$$

$\Rightarrow \quad n > \frac{14}{\varepsilon} \quad \Rightarrow$ take $N = \frac{14}{\varepsilon}$

Formal Proof:

let $\varepsilon > 0$, let $N = \max\left\{\frac{14}{\varepsilon}, 2\right\}$, the $\forall n > N$

$$n > \frac{14}{\varepsilon}$$

$$\Rightarrow \quad \left|\frac{14}{n}\right| < \varepsilon$$

$$\Rightarrow \quad \left|\frac{7n}{\frac{n^2}{2}}\right| < \varepsilon$$

$\Rightarrow \quad \left|\frac{n+6}{n^2-6}\right| < \varepsilon$ for $n > 2$

Hence proved

8.2

a) $a_n = \frac{n}{n^2+1}$

$$\lim_{n \to \infty} a_n = \frac{\frac{1}{n}}{1 + \frac{1}{n^2}} = 0$$

Discussion:

Let $\varepsilon > 0$, we want to prove

$$\left| \frac{n}{n^2+1} - 0 \right| < \varepsilon \qquad \forall n \in \mathbb{N}$$

$$\Rightarrow \quad \frac{n}{n^2+1} < \varepsilon$$

we cannot isolate n,

* we can use estimates

we need to find an upper Bound for the numerator and a lower Bound to denominator for denominator $n^2+1 > \frac{n^2}{2}$ $\forall n \in \mathbb{N}$

$$\Rightarrow \quad \frac{n}{n^2+1} \leq \frac{n}{\frac{n^2}{2}} < \varepsilon$$

$$\Rightarrow \quad \frac{2}{n} < \varepsilon$$

$\Rightarrow \quad n > \frac{2}{\varepsilon}$. This suggests we

put $N = \frac{2}{\varepsilon}$

Formal Proof:

Let $\varepsilon > 0$, Let $N = \frac{2}{\varepsilon}$, $\forall n > N$ i.e

$$n > \frac{2}{\varepsilon} \Rightarrow \frac{2}{n} < \varepsilon$$

$$\Rightarrow \quad \frac{n}{\frac{n^2}{2}} < \varepsilon$$

$$\Rightarrow \quad \frac{n}{n^2+1} \leq \frac{n}{\frac{n^2}{2}} < \varepsilon$$

$$\Rightarrow \quad \left| \frac{n}{n^2+1} - 0 \right| < \varepsilon$$

i.e
0 is the Limit.

8.2b

$$b_n = \frac{7n-19}{3n+7}$$

$$\lim_{n\to\infty} \frac{7n-19}{3n+7} = \lim_{n\to\infty} \frac{7-\frac{19}{n}}{3+\frac{7}{n}} = \frac{7}{3}$$

Discussion:

Let $\varepsilon > 0$, we want to prove

$$\left| \frac{7n-19}{3n+7} - \frac{7}{3} \right| < \varepsilon$$

$$\Rightarrow \left| \frac{\cancel{21n}-57-\cancel{21n}-49}{3(3n+7)} \right| < \varepsilon$$

$$\Rightarrow \frac{106}{3(3n+7)} < \varepsilon$$

$$\Rightarrow \frac{1}{3}\left(\frac{106}{3\varepsilon} - 7\right) < n$$

$$\Rightarrow \frac{106}{9\varepsilon} - \frac{7}{3} < n$$

take $N = \frac{106}{9\varepsilon} - \frac{7}{3}$

Formal Proof:

Let $\varepsilon > 0$, $N = \frac{106}{9\varepsilon} - \frac{7}{3}$, $\forall n > N$ i.e

$$n > \frac{106}{9\varepsilon} - \frac{7}{3}$$

$$\Rightarrow \left(n + \frac{7}{3}\right)\frac{9}{106} > \frac{1}{\varepsilon}$$

$$\Rightarrow \frac{3(3n+7)}{106} > \frac{1}{\varepsilon}$$

$$\Rightarrow \frac{106}{3(3n+7)} < \varepsilon$$

$$\Rightarrow \left| \frac{-106}{3(3n+7)} \right| < \varepsilon$$

$$\Rightarrow \left| \frac{7n-19}{3n+7} - \frac{7}{3} \right| < \varepsilon$$

Hence Proved. $\frac{7}{3}$ is the Limit.

8.2 c

$$c_n = \frac{4n+3}{7n-5}$$

Sol^n

$$\lim_{n\to\infty} \frac{4n+3}{7n-5} = \lim_{n\to\infty} \frac{4+\frac{3}{n}}{7-\frac{5}{n}} = \frac{4}{7}$$

Discussion:

Let $\varepsilon > 0$, we want to prove

$$\left| \frac{4n+3}{7n-5} - \frac{4}{7} \right| < \varepsilon$$

$$\Rightarrow \left| \frac{\cancel{28n}+21-\cancel{28n}+20}{7(7n-5)} \right| < \varepsilon$$

$$\Rightarrow \frac{41}{7(7n-5)} < \varepsilon$$

$$\Rightarrow \left(\frac{41}{7\varepsilon}+5\right)\frac{1}{7} < n$$

$$\Rightarrow \text{take } N = \left(\frac{41}{7\varepsilon}+5\right)\frac{1}{7}$$

Formal proof:

Let $\varepsilon > 0$, $N = \frac{1}{7}\left(\frac{41}{7\varepsilon} + 5\right)$, then

$\forall\, n > N$ i.e

$$n > \frac{1}{7}\left(\frac{41}{7\varepsilon} + 5\right)$$

$$\Rightarrow (7n-5)\frac{7}{41} > \frac{1}{\varepsilon}$$

$$\Rightarrow \varepsilon > \frac{41}{7(7n-5)}$$

$$\Rightarrow \varepsilon > \left|\frac{4n+3}{7n-5} - \frac{4}{7}\right|$$

Hence proved.

Q.2 d

$$d_n = \frac{2n+4}{5n+2}$$

$$\lim_{n\to\infty} \frac{2n+4}{5n+2} = \frac{2}{5}$$

Discussion: let $\varepsilon > 0$, we want to prove

$$\left|\frac{2n+4}{5n+2} - \frac{2}{5}\right| < \varepsilon$$

$$\Rightarrow \left|\frac{\cancel{10n}+20-\cancel{10n}-4}{(5n+2)(5)}\right| < \varepsilon$$

$$= \frac{16}{5(5n+2)} < \varepsilon$$

$$\Rightarrow \left(\frac{16}{5\varepsilon}-2\right)\frac{1}{5} < n$$

take $\quad N = \left(\frac{16}{5\varepsilon}-2\right)\frac{1}{5}$

Formal Proof:

let $\varepsilon > 0$, then for $\quad N = \frac{1}{5}\left(\frac{16}{5\varepsilon}-2\right)$

$\forall n > N$, i.e

$$n > \frac{1}{5}\left(\frac{16}{5\varepsilon}-2\right)$$

$$\Rightarrow \quad \frac{5n+2}{10} \cdot 5 > \frac{1}{\varepsilon}$$

$$\Rightarrow \quad \frac{10}{5}\left(\frac{1}{5n+2}\right) < \varepsilon$$

$$\Rightarrow \quad \left|\frac{2n+4}{5n+2} - \frac{2}{5}\right| < \varepsilon$$

Hence proved.

(e) $S_n = \frac{1}{n} \sin n$

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0$$

let $\varepsilon > 0$, we want to prove

$$\left|\frac{\sin n}{n} - 0\right| < \varepsilon$$

$$\Rightarrow \quad \left|\frac{\sin n}{n}\right| < \varepsilon$$

$$\Rightarrow \quad \frac{|\sin n|}{n} < \frac{1}{n} < \varepsilon$$

$$\Rightarrow \quad \frac{1}{\varepsilon} < n \Rightarrow \text{take } N = \frac{1}{\varepsilon}$$

Formal Proof:

let $\varepsilon > 0$, and let $N = \frac{1}{\varepsilon}$, then $\forall n > N$

i.e $\quad n > \frac{1}{\varepsilon}$

$$\Rightarrow \quad \frac{1}{n} < \varepsilon$$

$$\Rightarrow \quad \frac{|\sin n|}{n} < \frac{1}{n} < \varepsilon$$

$$\Rightarrow \quad \left| \frac{\sin n}{n} - 0 \right| < \varepsilon$$

8.3 Let (s_n) be a seq^n of non-negative real number's, and suppose

$\lim s_n = 0$. Prove that $\lim \sqrt{s_n} = 0$

This will complete the proof for Example 5.

Sol^n $\lim\limits_{n\to\infty} s_n = 0 \Leftarrow$ we know

$\Rightarrow \forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, such that

$\forall n > N$

$$|s_n| < \varepsilon$$

$$\Rightarrow s_n < \varepsilon$$

<u>we want to prove:</u>

$$|\sqrt{s_n}| < \varepsilon$$

$$\Rightarrow s_n < \varepsilon^2$$

we know from the convergence of S_n there exists a N for ε^2

s.t $|S_n| < \varepsilon^2 \quad \forall n > N$

$$\boxed{\text{Hence } \sqrt{S_n} \longrightarrow 0}$$

8.4 Let (t_n) be a bounded seq^n i.e $\exists M$ s.t $|t_n| \leq M \; \forall n$, and let (S_n) be a seq^n such that $\lim S_n = 0$. Prove $\lim(S_n t_n) = 0$

Sol^n

S_n is a convergence seq

t_n is bounded by M

$$|S_n t_n - 0|$$

$$\Rightarrow \quad |s_n t_n|$$

$$\Rightarrow \quad |s_n|\,|t_n|$$

$$\Rightarrow \quad |s_n|\,|t_n| \leq |s_n|\,M$$

Formal Proof:

let $\varepsilon > 0$, since $\lim_{n \to \infty} s_n = 0$,

is a convergent seq^n $\Rightarrow \exists N$, such that

$\forall n > N$,

$$|s_n| < \frac{\varepsilon}{M}$$

$$\Rightarrow \quad |s_n|\,M < \varepsilon$$

$$\Rightarrow \quad |s_n|\,|t_n| \leq |s_n|\,M < \varepsilon$$

$$\Rightarrow \quad |s_n|\,|t_n| < \varepsilon$$

$\Rightarrow \quad |s_n t_n| < \varepsilon$

Hence $(s_n t_n)$ is a convergent seq^n with limit 0.

8.5

(a) Consider three seq^n (a_n), (b_n) and (s_n) such that $a_n \le s_n \le b_n$ $\forall n$ and $\lim a_n = \lim b_n = s$.

Prove $\lim s_n = s$

This is called the "squeeze lemma"

Sol^n

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = s$$

$$a_n \le s_n \le b_n$$

$$\Rightarrow \quad \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = 0$$

$\Rightarrow$

$$\lim_{n\to\infty} (b_n - a_n) = 0$$

$$\Rightarrow \quad (b_n - a_n) \longrightarrow 0$$

we have

$$a_n \le S_n \le b_n$$

$$\Rightarrow \quad 0 \le S_n - a_n \le b_n - a_n$$

we know that $b_n - a_n$ is a convergent sequence

$\Rightarrow$ let $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t $\forall n > N$

$$|b_n - a_n| < \varepsilon$$

$(b_n - a_n)$ is a +ve number

$(S_n - a_n)$ is a +ve number

$\Rightarrow \quad |s_n - a_n| \leq |b_n - a_n| < \varepsilon$

$\Rightarrow \quad (s_n - a_n)$ is a convergent seq^n with limit 0

$$\Rightarrow \lim_{n\to\infty} (s_n - a_n) = 0$$

$$\boxed{\Rightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} a_n = s}$$

(b) Suppose (s_n) and (t_n) are seq^n such that $|s_n| \leq t_n \ \forall n$ and $\lim t_n = 0$. Prove $\lim s_n = 0$

Sol^n

$$|s_n| \leq t_n \quad \Rightarrow t_n \geq 0 \ \forall n \in \mathbb{N}$$

$$\Rightarrow \quad -t_n < s_n < t_n$$

(t_n) convergent seq^n

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t $\forall n > N$

$\Rightarrow \quad |t_n - 0| < \varepsilon$

$\Rightarrow \quad |t_n| < \varepsilon$

$\Rightarrow \quad |s_n| \leq |t_n| < \varepsilon$

$\Rightarrow \quad |s_n| < \varepsilon$

Hence s_n is convergent seq^n with Limit 0.

8.6 Let (s_n) be a seq^n in $\mathbb{R}$

(a) Prove $\lim s_n = 0 \iff \lim |s_n| = 0$

Sol^n $\quad s_n \longrightarrow 0 \quad$ (If Proof)

$\Rightarrow \forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, such that

$\forall n > N \qquad |S_n| < \varepsilon$

$\Rightarrow \qquad \big|\, |S_n| \,\big| < |\varepsilon|$

$\Rightarrow \qquad \big|\, |S_n| \,\big| < \varepsilon$

$\Rightarrow \qquad \big|\, |S_n| - 0 \,\big| < \varepsilon$

$\Rightarrow \lim\limits_{n\to\infty} |S_n| = 0$

(else proof)

need to prove if $\lim |S_n| = 0 \Rightarrow \lim S_n = 0$

as $|S_n| \to 0 \Rightarrow \forall \varepsilon_1 > 0, \exists N_1 \in \mathbb{N}$

such that

$\forall n > N_1 \qquad \big|\, |S_n| - 0 \,\big| < \varepsilon$

$\Rightarrow \quad |S_n| < \varepsilon$

$\Rightarrow \quad |S_n - 0| < \varepsilon$

Hence $\lim\limits_{n\to\infty} S_n = 0$

8.6

(b) Observe that if $s_n = (-1)^n$, then $\lim |s_n|$ exists, but $\lim s_n$ does not exist.

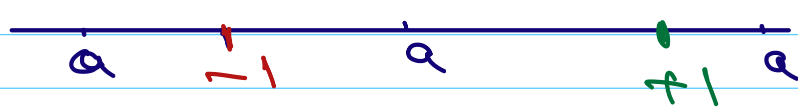
Sol^n

$$|s_n| = |(-1)^n| = 1$$

$$\Rightarrow \lim_{n\to\infty} |s_n| = 1$$

But $\lim_{n\to\infty} (-1)^n$ does not exists (not a convergent seq^n)

Let's assume $(-1)^n$ is a convergent seq^n with limit a.



Let $\varepsilon = 1$

$$\Rightarrow |s_n - a| < 1 \quad \text{assume } n = \text{even}$$

$$\Rightarrow |1 - a| < 1$$

then $\quad |s_{n+1} - a| < 1$

$$\Rightarrow \quad |-1-a| < 1$$

$$\Rightarrow \quad |1+a| < 1$$

$$\Rightarrow \quad |1-a| + |1-a| < 2$$

(violating triangle inequality)

8.7

a $\quad \cos\left(\frac{n\pi}{3}\right)$

$$\left(\cos\left(\frac{n\pi}{3}\right)\right)_{n\in\mathbb{N}} = \left(\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1\right)$$

The seq^n just repeats. this is a clearly non-convergent seq^n.

(8.9) Let (S_n) be a seq^n that converges

(a) Show that if $S_n \geq a$ for all but finitely many n, then $\lim S_n \geq a$

Sol^n (S_n) is a converging seq^n with L

$$\lim_{n \to \infty} S_n = L$$

$\forall \varepsilon > 0$ $\exists N_1 \in \mathbb{N}$ s.t $\forall n > N$

$$|S_n - L| < \varepsilon$$

& $\exists N_2 \in \mathbb{N}$ s.t $S_n \geq a$

$\forall n > N_2$

$\Rightarrow$ for $N = \max\{N_1, N_2\}$,

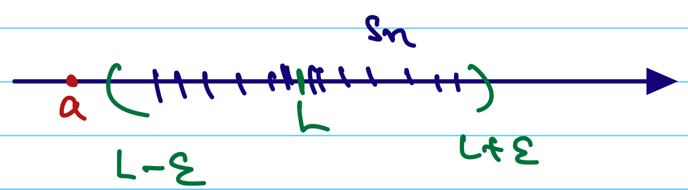
$\forall n > N$ we have

$$|S_n - L| < \varepsilon$$

$$S_n \geq a$$

$\Rightarrow$ $L-\varepsilon < S_n < L+\varepsilon$ & $a \leq S_n$

$\Rightarrow$



$S_n \geq 2$

$S_n > 1$