# Lec02 – Sequences and Limits

**Sequences:** A seq$^n$ of real number's when we have a map from $\mathbb{N}$ to $\mathbb{R}$

a map $a: \mathbb{N} \longrightarrow \mathbb{R}$

or $a: \mathbb{N}_0 \longrightarrow \mathbb{R}$

**Notation:** $(a_1, a_2, \ldots)$ infinite list of number's

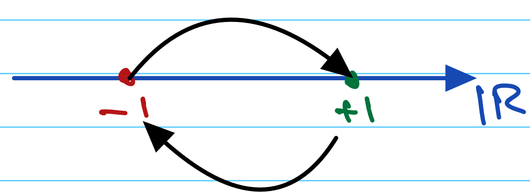
$$(a_n)_{n\in\mathbb{N}} \quad \text{or} \quad (a_n)_{n=1}^{\infty}$$

$$\text{or} \quad (a_n)$$

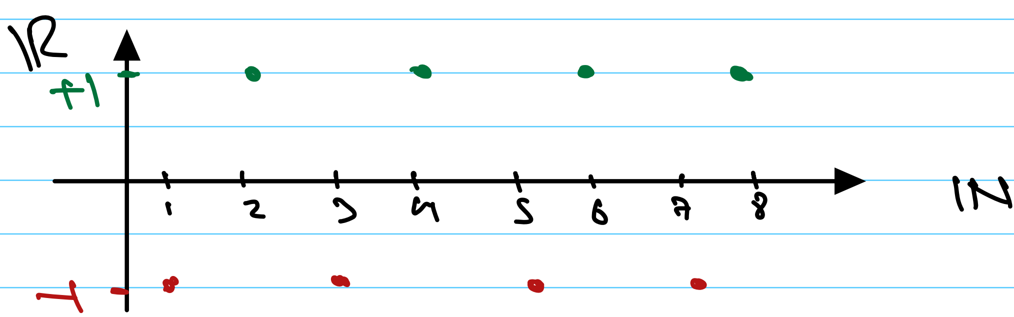
**Examples:** (0) $(a_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}}$

$$= (-1, 1, -1, 1, -1, 1, \ldots)$$

(i)



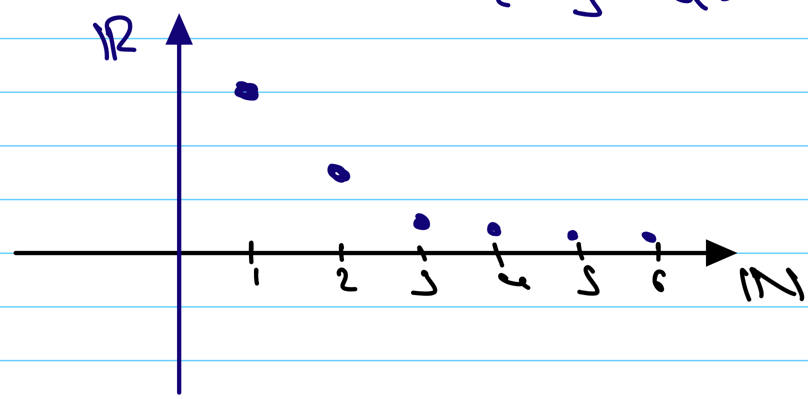

(ii)



we are interested in what happen's to the values of the seq$^n$ when $n$ goes to infinity.

(b) $(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$

$= \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$



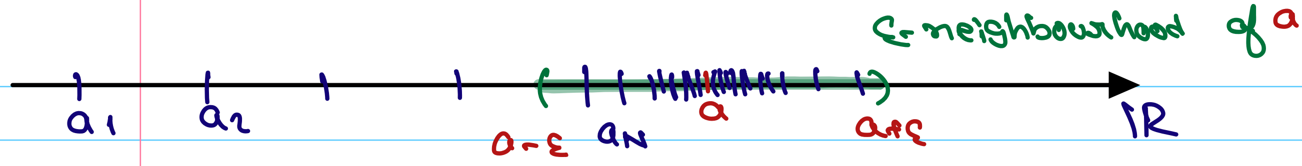

we will see $\lim_{n\to\infty} a_n = 0$

(c) $(a_n)_{n\in\mathbb{N}} = (2^n)_{n\in\mathbb{N}}$

$= (2, 4, 8, 16, 32, 64, \ldots)$

Definition:

A sequence $(a_n)_{n\in\mathbb{N}}$ is called convergent to $a \in \mathbb{R}$, if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}, \forall n \geq N$ : $|a_n - a| < \varepsilon$




we need to get closer and closer to a with the seq$^n$ members. eventually all the seq$^n$ members have to lie ε- neighbourhood of a, only finitely many can lie outside.

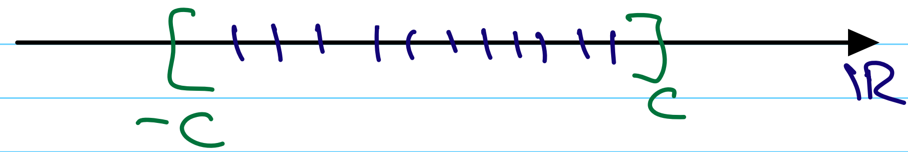
No matter how small the ε is, there exist a $N \in \mathbb{N}$, (always works)

if there is no such $a \in \mathbb{R}$, we call the seq$^n$ $(a_n)_{n \in \mathbb{N}}$ divergent.

# Lec 03 - Bounded seq$^n$ and Unique Limit

**Definition:** A seq$^n$ $(a_n)_{n\in\mathbb{N}}$ is called Bounded if $\exists C \in \mathbb{N}$ $\forall n \in \mathbb{N}$

$|a_n| \leq C$




otherwise, the seq is called unbounded.

**Important fact:**

$(a_n)_{n\in\mathbb{N}}$ Convergent $\Longrightarrow$ $(a_n)_{n\in\mathbb{N}}$ Bounded

$(a_n)_{n\in\mathbb{N}}$ Convergent $\Longrightarrow$ There is only one Limit $a \in \mathbb{R}$

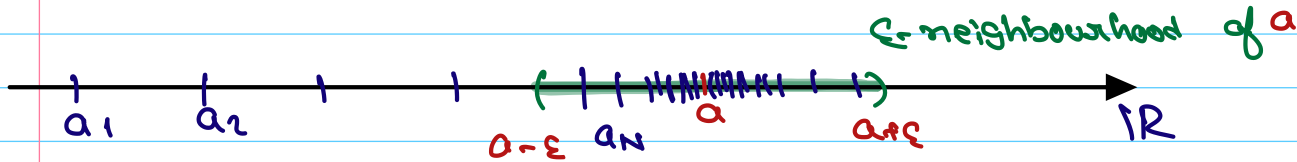
$$\lim_{n\to\infty} a_n = a$$

# Lec04 - Theorem on Limits

$(a_n)_{n\in\mathbb{N}}$ Convergent to $a \in \mathbb{R}$:

$$\lim_{n\to\infty} a_n = a$$

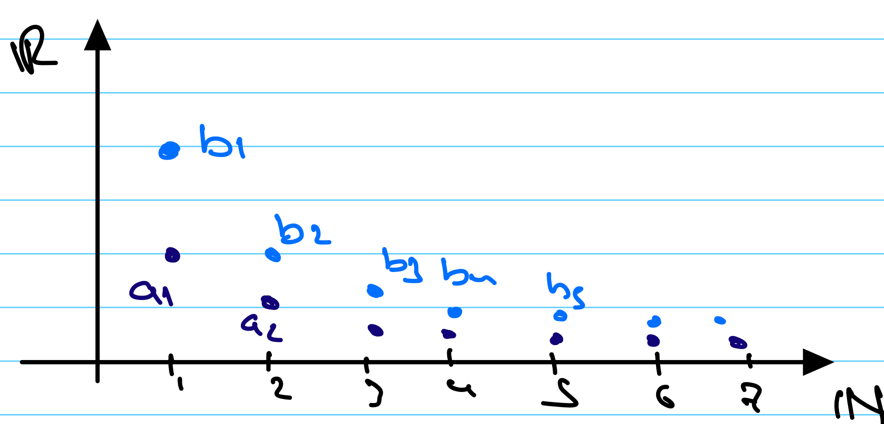
$$a_n \xrightarrow{n\to\infty} a$$




for all ε, the seqⁿ member's lie inside this ε-neighbourhood of a eventually.

## Theorem's on Limits:

$(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ Convergent sequnes.



Then

(a) $\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

we can pull in the limit when we have a sum, provided that these two seq^n and their limits actually exists.

(b) $\lim_{n\to\infty} a_n \cdot b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

(c) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n \neq 0}$

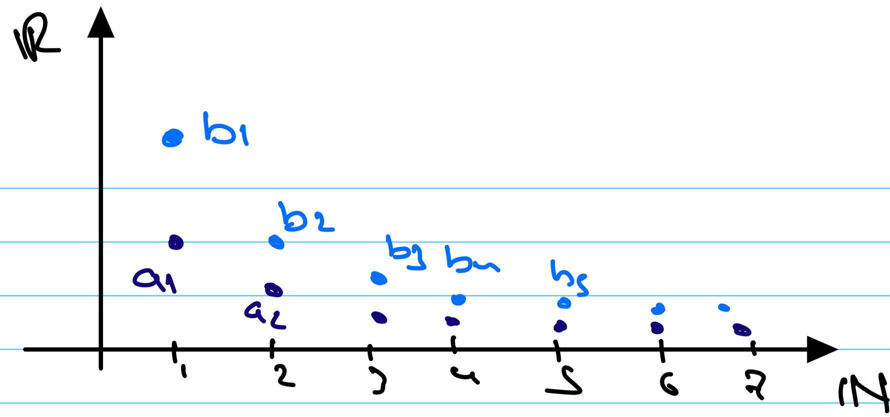
## Lec05: Sandwich theorem

$(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ convergent seq^n

then $\lim_{n\to\infty}(a \cdot b_n) = a \cdot \lim_{n\to\infty} b_n$

**Properties:**

(a) Monotonicity $a_n \leq b_n \ \forall n\in\mathbb{N}$




$$\Rightarrow \quad \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$

(b) Sandwich theorem

$$a_n \leq c_n \leq b_n \quad \forall n \in \mathbb{N} \quad \text{and}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

$\Rightarrow$ $(c_n)_{n\in\mathbb{N}}$ Convergent with

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

Proof of (b):

$$(b_n - a_n) \xrightarrow{n\to\infty} \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = 0$$

$d_n := c_n - a_n$

$\Rightarrow \quad 0 \leq d_n \leq b_n - a_n$

let $\varepsilon > 0$. then $\exists N \in \mathbb{N}$, $\forall n \geq N$

$|b_n - a_n| < \varepsilon$

$\Rightarrow \quad |d_n - 0| < \varepsilon$

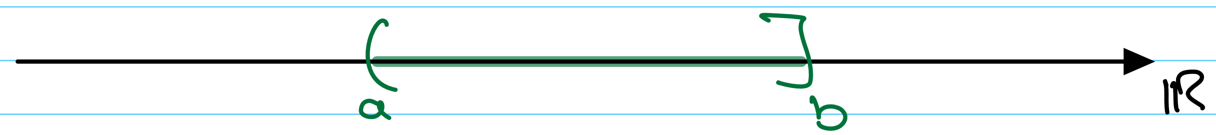
$\Rightarrow \quad d_n$ is convergent with limit $0$.

$\Rightarrow \quad (c_n)_{n\in\mathbb{N}} = (d_n + a_n)_{n\in\mathbb{N}}$

is convergent with limit $a$.

# Lec 06 - Supremum & Infimum

Real Number's are ordered



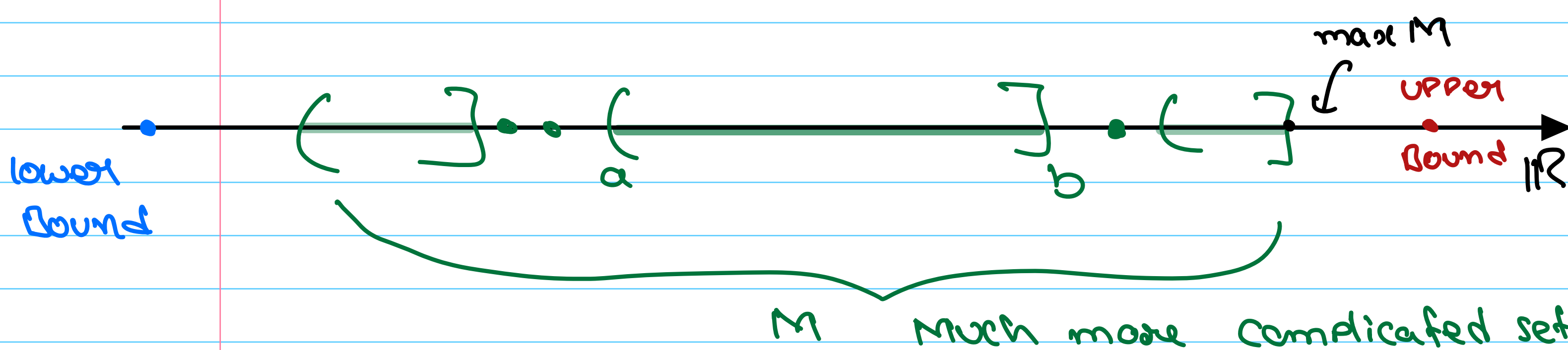

interval $(a,b] := \{x \in \mathbb{R} \mid a < x \leq b\}$

similarly $[a,b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$

$[a,\infty) := \{x \in \mathbb{R} \mid a \leq x\}$

$(-\infty, b) := \{x \in \mathbb{R} \mid x < b\}$

these interval's are very nice subsets of real number's. (Not the only once)



**Definition:** For a subset $M \subseteq \mathbb{R}$: $b \in \mathbb{R}$ is called an upper bound for $M$ if

$\forall x \in M: \quad x \leq b$

For $a \in \mathbb{R}$, is called a lower bound for $M$ if $\forall x \in M: \quad x \geq a$

* if upper bound exists we call the set $M$ is bounded from above

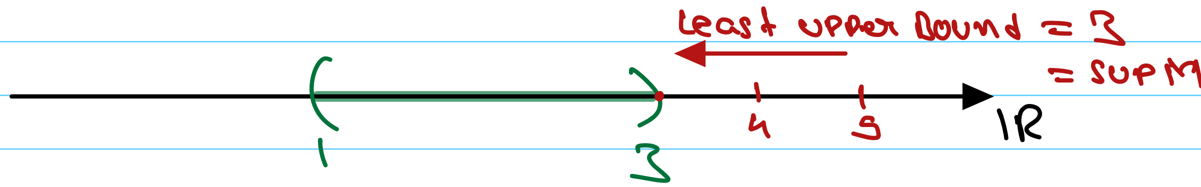
* if lower bound exists, the set $M$ called bounded from below.

Both $\rightarrow$ set $M$ is bounded.

* if $b$ is an upper bound for $M$ and $b \in M$, then $b$ is called a maximal element of $M$.
  $b = \max M$

* if $a$ is an lower bound for $M$ and $a \in M$, then $a$ is called a minimal element of $M$
  $a = \min M$

**Ex:** $M = (1,3)$ , max $M$, min $M$ do not exists




max $M$, min $M$ do not exists, because $1, 3 \notin M$

**Definition:** for a subset $M \subseteq \mathbb{R}$; $s \in \mathbb{R}$ is called supremum of $M$ if

* $\forall x \in M : x \leq s$ (upper bound for $M$)
* $\forall \varepsilon > 0, \exists \tilde{x} \in M : s - \varepsilon < \tilde{x}$

  ($s - \varepsilon$ is no upper bound for $M$)

Supremum = least upper bound.

$\sup M := s$ or $\sup M := \infty$

(if $M$ is not bounded from above)

or $\sup \emptyset := -\infty$

**Definition:** for a subset $M \subseteq \mathbb{R}$; $l \in \mathbb{R}$ is called infimum of $M$ if

* $\forall x \in M : x \geq l$ (lower bound for $M$)
* $\forall \varepsilon > 0, \exists \tilde{x} \in M : l + \varepsilon > \tilde{x}$ ($l + \varepsilon$ is no lower bound for $M$)

Then write $\inf M := l$ or $\inf M = -\infty$ (if $M$ is not bounded from below)

or $\inf \emptyset = +\infty$

## Lec 07 : Cauchy seq$^n$ & completeness

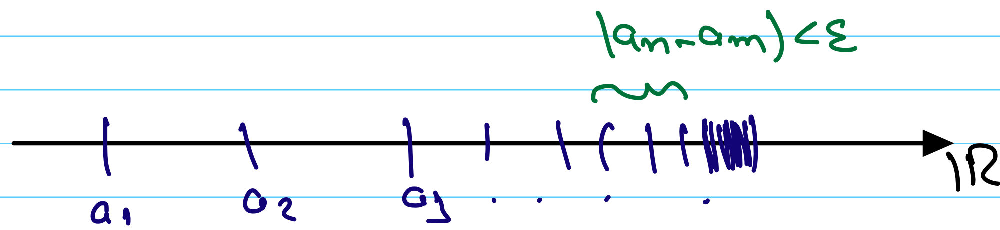
$(a_n)_{n \in \mathbb{N}}$ is convergent seq$^n$

which mean's there is a number $a$, s.t the seq member's here get arbitarly close to this number $a$ eventually.

$(a_n)_{n \in \mathbb{N}} \Rightarrow$ (there is a limit $a = \lim_{n \to \infty} a_n$)

The Problem with this definition is we need to know the value of Limit to show the convergence.

<u>Different idea:</u>




The seq$^n$ member's gets closer and closer to each other.

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $\forall n, m \geq N$ : we have

$$|a_n - a_m| < \varepsilon$$

(Cauchy sequence)

<u>Definition:</u> if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, such that $\forall n, m \geq N$ $|a_n - a_m| < \varepsilon$, then $(a_n)_{n \in \mathbb{N}}$ is called Cauchy seq$^n$

**Important fact:** For a seq$^n$ of real number's

$$\text{Cauchy seq}^n \iff \text{Convergent seq}^n$$

**Completeness Axiom:** There are no holes in our complete real Number line.

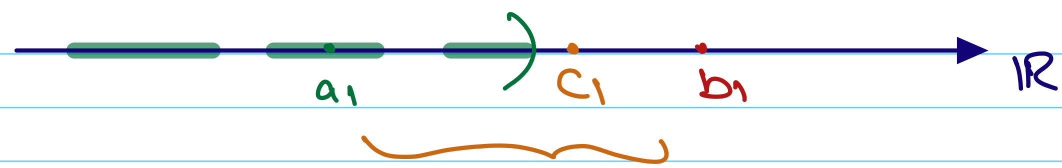
When we are working real number's, we don't have to distinguish Cauchy seq$^n$ and convergent seq$^n$. They have different definition, But for real number's they mean the same thing.

**Dedekind Completeness:-**

if $M \subseteq \mathbb{R}$ is bounded from above, then $\sup M \in \mathbb{R}$ (exists)

if $M \subseteq \mathbb{R}$ is bounded from below, then $\inf M \in \mathbb{R}$ (exists).

**Proof:**




$c_1 := \frac{1}{2}(a_1+b_1)$

Two cases: (1) if $c_1$ is an upper bound for $M$:

$$b_2 := c_1$$
$$a_2 := a_1$$

(2) if $c_2$ is not an upper bound for $M$: $\exists x \in M : x > c_1$

$$b_2 := b_2$$
$$a_2 := x$$

$$c_n := \frac{1}{2}(a_n + b_n)$$

$\vdots$

**Important fact:**

if $(a_n)_{n\in\mathbb{N}}$ is monotonically decreasing ($a_{n+1} \le a_n \ \forall n$) and bounded from below (the set $\{a_n\}_{n\in\mathbb{N}}$ has a lower bound) then $(a_n)_{n\in\mathbb{N}}$ is convergent.

## Lec 08:

**Important FACT:** if $(a_n)_{n\in\mathbb{N}}$ is monotonically increasing ($a_{n+1} \geq a_n \ \forall n$) and bounded from above (the set $\{a_n\}_{n\in\mathbb{N}}$ has an upper bound) then

$(a_n)_{n\in\mathbb{N}}$ is convergent.

(Monotone convergence criterion)

**Ex:** $(a_n)_{n\in\mathbb{N}} = \left(\left(1+\frac{1}{n}\right)^n\right)_{n\in\mathbb{N}}$ is convergent.

**Proof:**

monotonicity:-

$$\frac{a_{n+1}}{a_n} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n}$$

$$= \left(1+\frac{1}{n}\right)\cdot\left(\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}}\right)^{n+1}$$

$$= \left(1+\frac{1}{n}\right)\left(\frac{n+2}{n+1}\cdot\frac{n}{n+1}\right)^{n+1}$$

$$= \left(1+\frac{1}{n}\right)\left(\frac{n^2+2n}{n^2+2n+1}\right)^{n+1}$$

$$= \left(1+\frac{1}{n}\right)\left(1-\frac{1}{(n+1)^2}\right)^{n+1}$$

<u>Bernoulli inequality:</u>

for $k \in \mathbb{N}$ and $x \geq -1$

$$(1+x)^k \geq 1+kx$$

$$\Rightarrow \quad \left(1+\frac{1}{n}\right)\left(1-\frac{1}{(n+1)^2}\right)^{n+1}$$

$$\geq \left(1+\frac{1}{n}\right)\left(1-(n+1)\left(\frac{1}{(n+1)^2}\right)\right)$$

$$\geq \left(1+\frac{1}{n}\right)\left(1-\frac{1}{n+1}\right)$$

$$\geq \left(\frac{n+1}{n}\right)\left(\frac{n}{n+1}\right) = 1$$

$$\Rightarrow \frac{a_{n+1}}{a_n} \geq 1 \quad (\text{monotone increasing})$$

Now we need to show the seq$^n$ is Bounded from above.

$$\#) \quad a_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k$$

$$= \left(\frac{1}{n}\right)^0 + \binom{n}{1}\left(\frac{1}{n}\right)^1 + \binom{n}{2}\left(\frac{1}{n}\right)^2 \cdots$$

$$= 1+1+\sum_{k=2}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k$$

$$\binom{n}{k}\left(\frac{1}{n}\right)^k = \frac{n!}{(n-k)!\,k!}\cdot\frac{1}{n^k}$$

$$= \frac{n\cdot(n-1)(n-2)\cdots\cdot(n-k+1)}{n\cdot n\cdot n\cdot \cdots \cdot n}\cdot\frac{1}{k!}$$

$$\leq 1$$

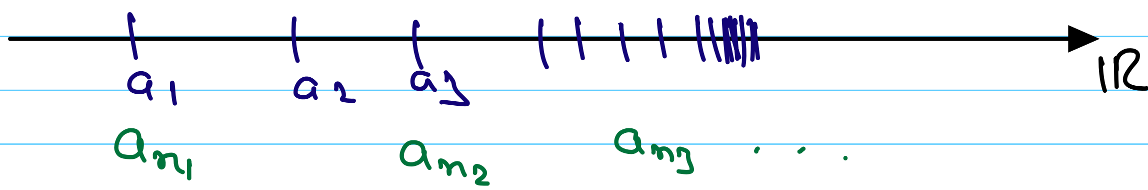
$$\Rightarrow \left(1+\frac{1}{n}\right)^n \leq 3$$

fact $\Longrightarrow$ The seq$^n$ $(a_n)_{n\in\mathbb{N}}$ is convergent.

monotone convergence criterion

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \quad \text{Euler's Number}$$

## Lec 09 :- Subsequences and Limit Points




Omit some member's of the seq$^n$.

Let $(n_k)_{k\in\mathbb{N}}$ be a seq$^n$ of natural number's that is strictly monotonically increasing $(\forall k\in\mathbb{N} : n_{k+1} > n_k)$

then $(a_{n_k})_{k\in\mathbb{N}}$ is called an subsequence of $(a_n)_{n\in\mathbb{N}}$

**Ex:** $(a_n)_{n\in\mathbb{N}}$ given by $a_n = \frac{1}{n}$

$$(a_{n_k})_{k\in\mathbb{N}} = (a_{2k})_{k\in\mathbb{N}}$$

$$= (a_2, a_4, a_6, a_8, \dots)$$

$$= \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots\right)$$

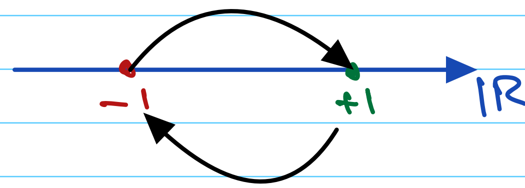
if we have a monotonic seq$^n$ $(a_n)_{n\in\mathbb{N}}$ then the subseq$^n$ $(a_{n_k})_{k\in\mathbb{N}}$ is also monotonic in the same way.

* we can omit lot of seq$^n$ members even infinitely many as long as infinitely many remain.

**FACT:** $(a_n)_{n\in\mathbb{N}}$ is convergent with $\lim_{n\to\infty} a_n = a$, then

$\Rightarrow$ every subsequence $(a_{n_k})_{k\in\mathbb{N}}$ is convergent $\lim\limits_{k\to\infty} a_{n_k} = a$

Ex: $(a_n)_{n\in\mathbb{N}}$ given by $a_n = (-1)^n$




Subseq$^n$ $(a_{n_k})_{k\in\mathbb{N}} = (a_{2k})_{k\in\mathbb{N}}$

$= (1, 1, 1, \ldots)$

$\lim\limits_{k\to\infty} a_{2k} = 1$ (limit 1)

$(a_{n_k})_{k\in\mathbb{N}} = (a_{2k+1})_{k\in\mathbb{N}}$

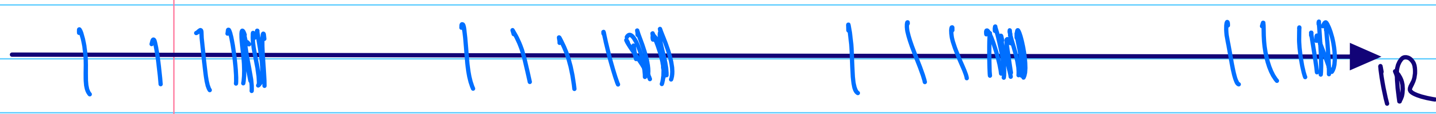
$\lim\limits_{k\to\infty} a_{2k+1} = -1$ (limit -1)

Def: $a \in \mathbb{R}$ is called an accumulation value of $(a_n)_{n\in\mathbb{N}}$ if there is a subseq$^n$ $(a_{n_k})_{k\in\mathbb{N}}$ with $\lim\limits_{k\to\infty} a_{n_k} = a$

Accumulation value is a generalization of term Limit.

=) A convergent seq^n can only have one accumulation value = Limit

=) However for divergent seq^n could have different accumulation values




The Limit is getting closer to 4 different points. we don't have a Limit because the seq^n still Jumps around, But we get closer and closer to different accumulation values.

=) we could restrict ourselves to take seq^n members that only live in one convergent seq^n (Limit)

Accumulation value is Just a Point on the number line, where the seq$^n$ accumulates

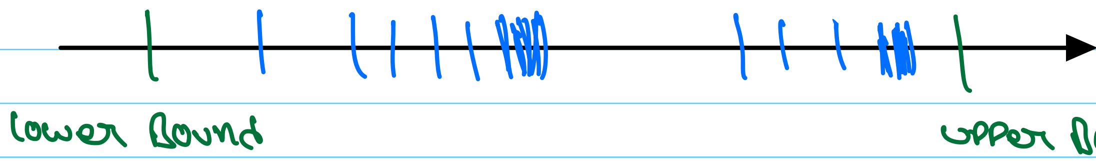
$a \in \mathbb{R}$ is an accumulation value of $(a_n)_{n \in \mathbb{N}}$

$\iff$ $\forall \varepsilon > 0$: The $\varepsilon$-neighbourhood of $a$ contain's infinitely many seq$^n$ member's of $(a_n)_{n \in \mathbb{N}}$.

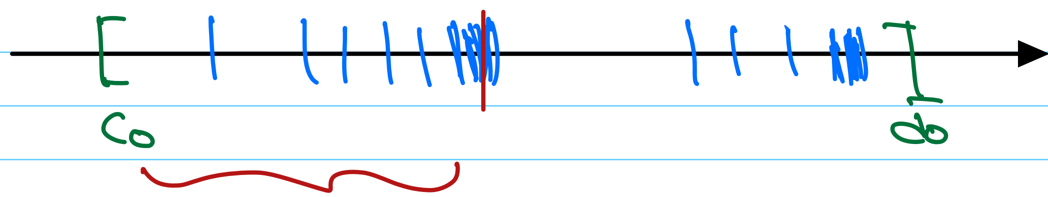
## Lec 10: Bolzano-Weierstrass

$(a_n)_{n \in \mathbb{N}}$ bounded $\implies$ $(a_n)_{n \in \mathbb{N}}$ has an accumulation value (has a convergent subseq$^n$)

having a accumulation value mean's the seq$^n$ have a convergent subseq$^n$.




**Proof:**

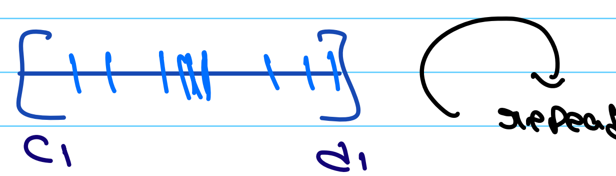



if infinitely many seq$^n$ member's in it:

Choose left-hand interval

otherwise: Choose right hand interval.

New interval:




we set $[c_0, d_0] \supset [c_1, d_1] \supset [c_2, d_2]$ ....

and $d_1 - c_1 = \frac{1}{2}(d_0 - c_0)$

$d_2 - c_2 = \frac{1}{2}(d_1 - c_1)$

$d_3 - c_3 = \frac{1}{2}(d_2 - c_2)$

$\vdots$

$$d_n - c_n = \frac{1}{2^n}(d_0 - c_0) \xrightarrow{n\to\infty} 0$$

we know

$(c_n)_{n\in\mathbb{N}}$ monotonically increasing and bounded

$(d_n)_{n\in\mathbb{N}}$ monotonically decreasing and bounded

$(c_n)_{n\in\mathbb{N}}, (d_n)_{n\in\mathbb{N}}$
Convergent

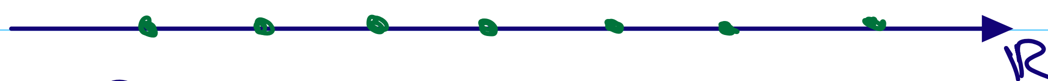
By Limit theorem

$$0 = \lim_{n\to\infty} (d_n - c_n)$$

$$\Rightarrow \lim_{n\to\infty} d_n = \lim_{n\to\infty} c_n$$

# Lec 11: Limit superior and limit inferior

**Ex:** $(a_n)_{n\in\mathbb{N}}$ given by $a_n = n$



$(a_n)_{n\in\mathbb{N}} = (n)_{n\in\mathbb{N}}$ is divergent to $\infty$

$\infty$: symbol (Not as a number)

divergent to $\infty$ $:\Longleftrightarrow$ $\forall C>0\ \exists N\in\mathbb{N}$
$\forall n\geq N:\ a_n > C$

$\Leftarrow$ $\lim\limits_{n\to\infty} a_n = \infty$

divergent to $-\infty$ $:\Longleftrightarrow$ $\forall C<0,\ \exists N\in\mathbb{N}$
$\forall n\geq N:\ a_n < C$

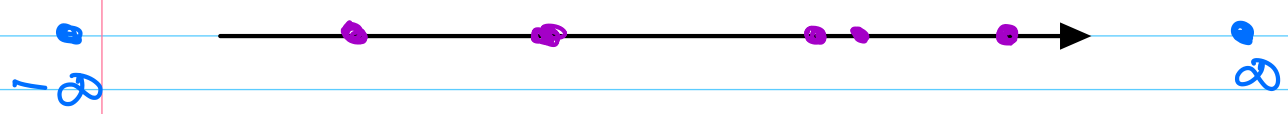
$\Leftarrow$ $\lim\limits_{n\to\infty} a_n = -\infty$

$(a_n)_{n\in\mathbb{N}}$ has the improper accumulation value $\infty$ $:\Longleftrightarrow$ $(a_n)_{n\in\mathbb{N}}$ is not bounded from above

$(a_n)_{n\in\mathbb{N}}$ has the improper accumulation value $-\infty$

$:\Longleftrightarrow$ $(a_n)_{n\in\mathbb{N}}$ is not bounded from below.

A given seq$^n$ $(a_n)_{n\in\mathbb{N}}$ could have many accumulation values:




* we can talk about Largest and Smallest accumulation value, it could be a normal one or improper one.

**Def:** Let $(a_n)_{n\in\mathbb{N}}$ be a seq$^n$ of real numbers. An element $a \in \mathbb{R} \cup \{-\infty, \infty\}$ is called

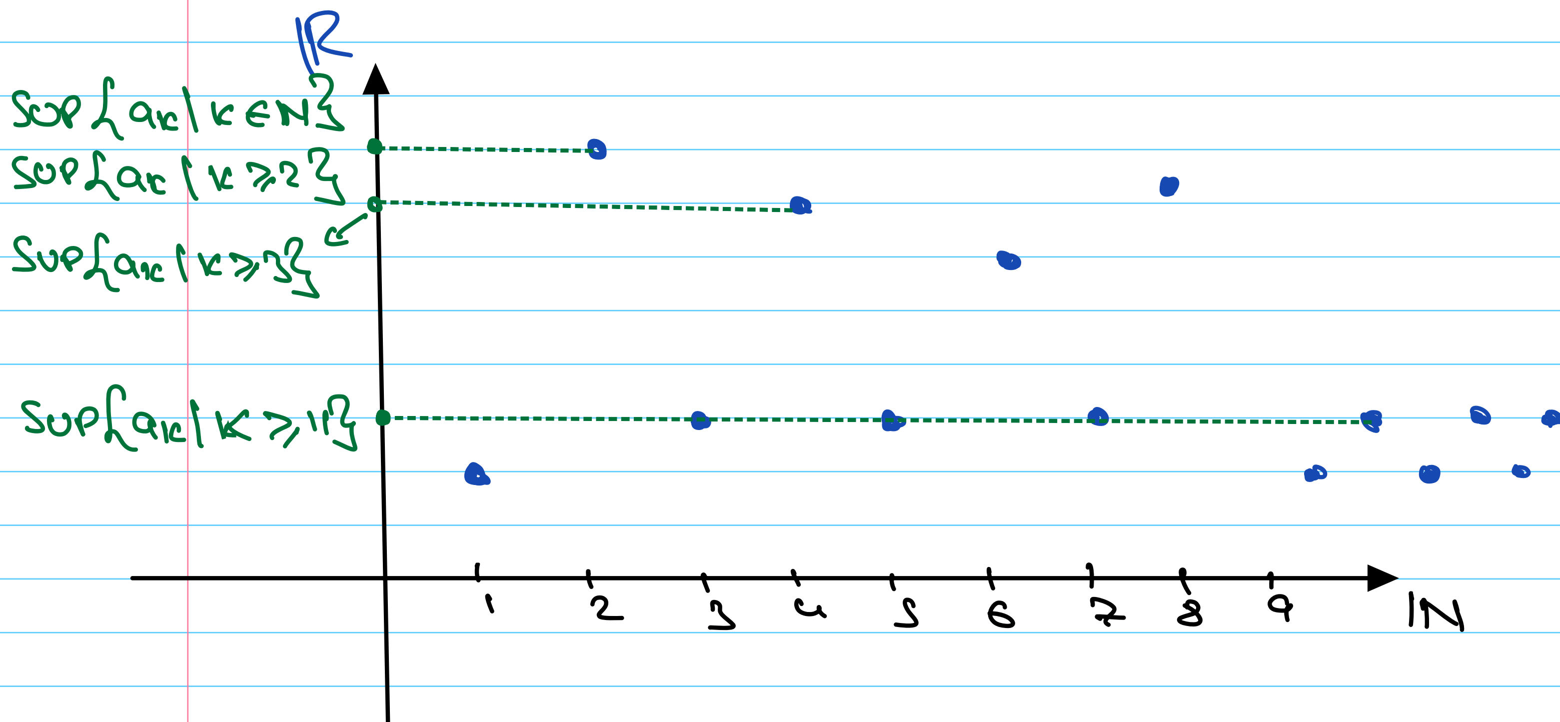
Limit superior of $(a_n)_{n\in\mathbb{N}}$

if $a$ is the Largest (Improper) accumulation value of $(a_n)_{n\in\mathbb{N}}$

$$a := \limsup_{n \to \infty} a_n$$

- limit inferior of $(a_n)_{n \in \mathbb{N}}$ if $a$ is the smallest (improper) accumulation value of $(a_n)_{n \in \mathbb{N}}$.

write $a = \liminf_{n \to \infty} a_n$




Limit superior describes what happen to with the supremum, when we cut of more and more at the begining of the seq

FACT:

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_n \mid k \geq n\}$$

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_n \mid k \geq n\}$$

## Lec 12

let $(a_n)_{n\in\mathbb{N}}$ be a seq$^n$ of real numbers

$$\Rightarrow \limsup_{n\to\infty} a_n,\ \liminf_{n\to\infty} a_n \in \mathbb{R} \cup \{\pm\infty\}$$
$$= [-\infty, \infty]$$

Ex: $(a_n)_{n\in\mathbb{N}} = ((-1)^n \cdot n)_{n\in\mathbb{N}}$

$$= (-1, 2, -3, 4, -5, \ldots)$$

$$\limsup_{n\to\infty} a_n = \infty$$

$$\liminf_{n\to\infty} = -\infty$$

**Properties:**

(a) $(a_n)_{n\in\mathbb{N}}$ is convergent

$$\iff \limsup_{n\to\infty} = \liminf_{n\to\infty} \notin \{\pm\infty\}$$

(b) $(a_n)_{n\in\mathbb{N}}$ is divergent to $\infty$

$$\iff \left(\limsup_{n\to\infty} a_n\right) = \liminf_{n\to\infty} a_n = \infty$$

(c) $(a_n)_{n\in\mathbb{N}}$ is divergent to $-\infty$

$$\iff \limsup_{n\to\infty} a_n = \left(\liminf_{n\to\infty} a_n\right) = -\infty$$

(d) For $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, we have

$$\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

(only if right hand side is defined)

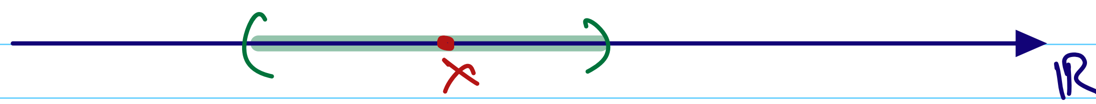
if $a_n, b_n \geq 0$

$$\limsup_{n\to\infty} (a_n \cdot b_n) \leq \limsup_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

(e) if $a_n, b_n \geq 0$

$$\liminf_{n\to\infty} (a_n \cdot b_n) \geq \liminf_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

# Lec 13 – Open, closed, and compact sets




$(x-\varepsilon, x+\varepsilon)$ = $\varepsilon$-neighbourhood of $x$

$\forall \varepsilon > 0$ : $(x-\varepsilon, x+\varepsilon) =: B_\varepsilon(x)$

$\varepsilon$-neighbourhood of $x$

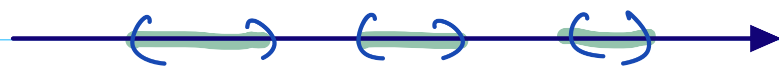
$M \subseteq \mathbb{R}$ is called a neighbourhood of $x$, if there is $\exists \varepsilon > 0$ s.t $M \supseteq B_\varepsilon(x)$

<u>Example:</u> $[-2,2]$ is a neighbourhood of $B_\varepsilon(0)$, $B_\varepsilon(1)$

$[-2,2]$ is not a neighbourhood of $2$

Definition: $M \subseteq \mathbb{R}$ is called open (in $\mathbb{R}$) if, for all $x \in M$, $M$ is a neighbourhood of $x$.

$\Rightarrow$ $\forall x \in M, \exists \varepsilon > 0 : M \supseteq B_\varepsilon(x)$



$A \subseteq \mathbb{R}$ is closed (in $\mathbb{R}$) if $A^c := \mathbb{R} \setminus A$ is open

Open is Not the opposite of closed.

$\Rightarrow$ There could be neither open nor closed

$\Rightarrow$ There could be open & closed at the same time.

Ex: (1) $\phi$, $\mathbb{R}$ are Both open & closed

(2) $[-2, 2]$ is closed but not open

(3) $(-2, 2)$ is open but not closed

(4) $(-2, 2]$ is neither open nor closed

The subset of the real number's could be much more complicated than Just an intervals.

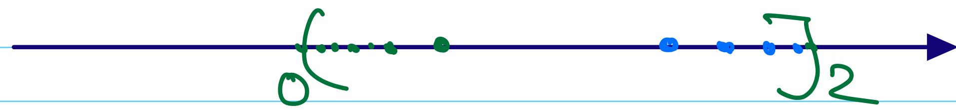
In order to deal with those sets, the next fact is very helpful.

FACT: It Gives us a Criterion to check Closeness with the help of seq$^n$.

$A \subseteq \mathbb{R}$ is closed $\Longleftrightarrow$ For all convergent seq$^n$ $(a_n)_{n \in \mathbb{N}}$ with $a_n \in A \;\forall n \in \mathbb{N}$ we have $\lim_{n \to \infty} a_n \in A$

It's not possible to leave the set with seq$^n$ from Inside.

<u>Ex:</u> $A := (0,2]$




Take $(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$

$\frac{1}{n} \in (0,2] \ \forall n\in\mathbb{N}$

But $\lim\limits_{n\to\infty} a_n = 0 \notin A$

This set is not closed.

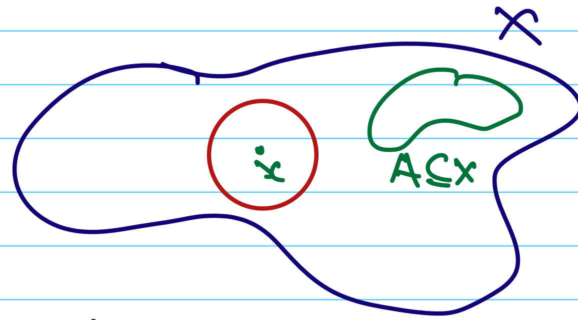
<u>Definition:</u>

$A \subseteq \mathbb{R}$ is called compact, if for all sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n \in A \ \forall n\in\mathbb{N}$, we find convergent subseq$^n$ with a limit lies in $A$. $\Rightarrow (a_{n_k})_{n\in\mathbb{N}}$ with $\lim\limits_{k\to\infty} a_{n_k} \in A$

Functional Analysis lec 03

Open and closed sets

$(X, d)$ : metric space



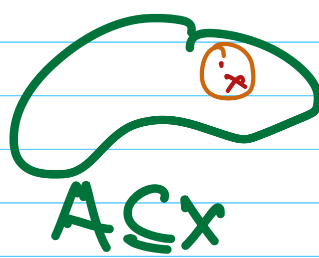

$B_\epsilon(x)$ open $\epsilon$-ball around $x$

$$B_\epsilon(x) := \{ y \in X \mid d(x,y) < \epsilon \}$$

(open ball of radius $\epsilon > 0$, centered at $x$)

Notion's:

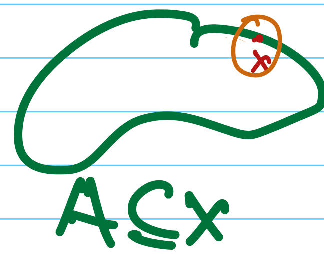
① Open sets:




$A \subseteq X$ is called open, if for each $x \in A$, there is an open ball with $B_\epsilon(x) \subseteq A$

i.e $\forall x \in A$, $\exists \epsilon > 0$, such that $B_\epsilon(x) \subseteq A$

## ② Boundary points:




$A \subseteq X$. $x \in X$ is called a Boundary Point for $A$ if $\forall \varepsilon > 0$: $B_\varepsilon(x) \cap A \neq \emptyset$

$B_\varepsilon(x) \cap A^c \neq \emptyset$

$$\partial A := \{ x \in X \mid x \text{ is a boundary point for } A \}$$

Remember:

$A$ open $\iff$ $A \cap \partial A = \emptyset$

Closed set:

$A$ closed $\iff$ $A \cup \partial A = A$

$A$ closed $\iff$ $A^c := X \setminus A$ is open

④ <u>Closure:</u>

$\overline{A} := A \cup \partial A$ (always closed)

(smallest closed set that contains A)

<u>Ex:</u> $X := (1,3] \cup (4,\infty)$

$d(x,y) = |x-y|$

$(X, d)$ metric space.

ⓐ $A := (1,3] \subseteq X$

Open: $\forall x \in A$, $\exists \varepsilon$ s.t $B_\varepsilon(x) \subseteq A \Rightarrow$ open

<u>closed:</u> $A^c = (4,\infty)$, $\forall x \in A^c$, $\exists \varepsilon$ s.t

$B_\varepsilon(x) \subseteq A^c \Rightarrow A^c$ is open

$\Rightarrow A$ is closed

ⓒ $C := (1,2]$

$\partial C = \{2\}$