Exercise : Sec 19 : Uniform Continuity.

(19.1)

(a) $f(x) = x^{17}\sin x - e^{x}\cos 3x$

on $[0,\pi]$

$f(x)$ is continuous on a closed interval $[0,\pi]$ $\Rightarrow$ f is uniformly continuous on $[a,b]$

(b) $f(x) = x^3$ on $[0,1]$

uniformly continuous $\Rightarrow$ Theorem 19.2

(c) $f(x) = x^3$ on $(0,1)$

if $f(x)$ is uniformly continuous on $[0,1]$ $\Longrightarrow$ $f(x)$ is uniformly continuous on $(0,1)$

(d) $f(x) = x^3$ on $\mathbb{R}$

Not uniformly continuous.

(e) $f(x) = \frac{1}{x^3}$ on $(0,1]$

Not uniformly continuous on $(0,1]$

take $(s_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}} \in (0,1]$

this is a Cauchy seq^n on dom(f)

if f is uniform continuous then

$f(s_n)$ should be Cauchy seq^n

$\Rightarrow \lim\limits_{n\to\infty} \frac{1}{n} = 0 = x_0$

$\Rightarrow \lim\limits_{n\to\infty} f\left(\frac{1}{n}\right) = \lim\limits_{n\to\infty} n^3 = +\infty \neq f(0)$

Hence f is not uniformly continuous

(f) $f(x) = \sin \frac{1}{x^2}$ on $(0,1]$

Sol^n we can use theorem 19.5

if we can exetent $f(x)$ on interval $[0,1]$ and show its continuous then $f(x)$ is uniformly continuous on $(0,1]$

let's take the Cauchy seq^n $(x_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

the $$\tilde{f}(0) = \lim_{n\to\infty} f(x_n)$$

$$= \underbrace{\lim_{n\to\infty} \sin n^2}_{\text{do not converse}}$$

$\Rightarrow$ $\sin\left(\frac{1}{x^2}\right)$ is not uniformly continuous on $(0,1]$

(9) $f(x) = x^2 \sin \frac{1}{x}$ on $(0,1]$

Sol^n use theorem 19.5 $x_n = \frac{1}{n}$

$$\tilde{f}(0) = \lim_{n\to\infty} f(x_n)$$

$$= \lim_{n\to\infty} f\left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \sin(n)$$

$$-\lim_{n\to\infty} \frac{1}{n^2} \le \tilde{f}(0) = \lim_{n\to\infty} \frac{1}{n^2} \sin(n) \le + \lim_{n\to\infty} \frac{1}{n^2}$$

$$\Rightarrow \quad 0 \le \tilde{f}(0) \le 0$$

$$\Rightarrow \quad \tilde{f}(0) = 0$$

Hence we can extend $f(x)$ to

$$\tilde{f}(x) = \begin{cases} \sin \frac{1}{x^2} & \text{on } (0,1] \\ 0 & \text{at } x = 0 \end{cases}$$

$\Rightarrow$ $f(x) = x^2 \sin \frac{1}{x}$ is uniform continuous on $(0,1]$

Apply $\varepsilon - \delta$ Criterion.

19.2

(a) $f(x) = 3x+11$ on $\mathbb{R}$

let $\varepsilon > 0$

$$f(x) - f(y) = 3(x-y)$$

$$\Rightarrow |f(x) - f(y)| = 3|x-y| < \varepsilon$$

take $\delta = \frac{\varepsilon}{3}$

there fore

$\forall \varepsilon > 0, \exists \delta = \frac{\varepsilon}{3} > 0$ s.t $\forall x, y \in \mathbb{R}$

$$|x-y| < \frac{\varepsilon}{2} \Longrightarrow |f(x) - f(y)| < \varepsilon$$

Hence uniform continous

(b) $f(x) = x^2$ on $[0,3]$

let $\varepsilon > 0$,

$f(x) - f(y) = x^2 - y^2$

$= (x-y)(x+y)$

$x, y \in [0, 3]$

$\Rightarrow 0 \leq x \leq 3$

$\Rightarrow 0 \leq y \leq 3$

$\Rightarrow 0 \leq x+y \leq 6$

$\Rightarrow |f(x) - f(y)| = |x-y|\,|x+y|$

$\leq |x-y|\, 6 < \varepsilon$

$\Rightarrow |x-y| < \frac{\varepsilon}{6}$

take $\delta = \frac{\varepsilon}{6}$

Hence $\forall \varepsilon > 0$, $\exists \delta = \frac{\varepsilon}{6}$ such that

$\forall x, y \in [0, 3]$ $|x-y| < \frac{\varepsilon}{6} \Rightarrow |f(x) - f(y)| < \varepsilon$

Hence $f(x)$ is uniform continuous on $[0, 3]$

(c) $f(x) = \frac{1}{x}$ on $[1/2, \infty)$

let $\varepsilon > 0$

$$f(x) - f(y) = \frac{1}{x} - \frac{1}{y}$$

$$= \frac{y-x}{xy}$$

$x \geq \frac{1}{2}$, $y \geq \frac{1}{2}$

$\Rightarrow \frac{1}{x} \leq 2$, $\frac{1}{y} \leq 2$

$\Rightarrow \frac{1}{xy} \leq 4$

$\Rightarrow |f(x) - f(y)| = \frac{|y-x|}{xy} < \frac{|y-x|}{4}$

$\Rightarrow |f(x) - f(y)| < \frac{|y-x|}{4} < \varepsilon$

take $\delta = 4\varepsilon$

Therefore,

$\forall \varepsilon > 0$, $\exists\, \delta = 4\varepsilon$, such that $\forall x, y \in [\frac{1}{2}, \infty)$

we have

$|x-y| < 4\varepsilon \Rightarrow |f(x) - f(y)| < \varepsilon$

$\Rightarrow f(x)$ is uniformly continuous on $[1/2, \infty)$

19.3 (a) $f(x) = \frac{x}{x+1}$ on $[0,2]$

$$f(x) - f(y) = \frac{x}{x+1} - \frac{y}{y+1} = \frac{xy + x - xy - y}{(x+1)(y+1)}$$

$$= \frac{x-y}{(x+1)(y+1)}$$

$x, y \in [0,2] \Rightarrow 0 \le x \le 2$

$\Rightarrow 0 \le y \le 2$

$\Rightarrow$ $1 \le x+1 \le 3$, $1 \le y+1 \le 3$ $\Rightarrow 1 \le (x+1)(y+1) \le 9$

$$\Rightarrow \frac{1}{9} \le \frac{1}{(x+1)(y+1)} \le 1$$

$$\Rightarrow |f(x) - f(y)| = \frac{|x-y|}{(x+1)(y+1)} \le |x-y| < \varepsilon$$

take $\delta = 1 \Rightarrow$

therefore $\forall \varepsilon > 0, \exists \delta = \varepsilon$ s.t $\forall x, y \in [0,2]$

we have

$|x-y| < \varepsilon \Rightarrow |f(x) - f(y)| < \varepsilon$

Hence

$f(x) = \frac{x}{x+1}$ is uniformly continuous

on $[0,2]$

(b) $f(x) = \frac{5x}{2x-1}$ on $[1, \infty)$

$$f(x) - f(y) = \frac{5x}{2x-1} - \frac{5y}{2y-1}$$

$$= \frac{10xy - 5x - 10xy + 5y}{(2x-1)(2y-1)} = \frac{5(y-x)}{(2x-1)(2y-1)}$$

$x, y \geq 1 \Rightarrow 2x-1,\ 2y-1 \geq 1 \Rightarrow (2x-1)(2y-1) \geq 1$

$$\Rightarrow \frac{1}{(2x-1)(2y-1)} \leq 1$$

$$\Rightarrow |f(x) - f(y)| = \frac{5|y-x|}{|(2x-1)(2y-1)|} \leq \frac{5|y-x|}{1}$$

take $\delta = \frac{\varepsilon}{5}$

Therefore $\forall \varepsilon > 0$, $\exists \delta = \frac{\varepsilon}{5}$, such that

$\forall x, y \in [1, \infty)$ we have

$$|x-y| < \frac{\varepsilon}{5} \Rightarrow |f(x) - f(y)| < \varepsilon$$

Hence $f(x) = \frac{5x}{2x-1}$ is uniformly continuous on $[1, \infty)$

19.4 (a) Prove that if f is uniformly continuous on a bounded set S, then f is a bounded function on S.

Sol^n

Assume f is not bounded on set S.

Then to each $n \in \mathbb{N}$ there corresponding an $x_n \in S$ such that

$$|f(x_n)| > n$$

$$\Rightarrow \lim_{n \to \infty} |f(x_n)| = +\infty.$$

Using Bolzano-weierstrass theorem 11.5 (every bounded seq has a convergent subseq^n)

$$\Rightarrow (x_n)_{n \in \mathbb{N}} \in S \quad \text{(bounded)}$$

Therefore $\exists$ a subseq^n $(x_{n_k})_{k \in \mathbb{N}} \in S$

that converges to $x_0 \in S$

$$\Rightarrow \lim_{k \to \infty} x_{n_k} = x_0 \in S$$

Since f is uniformly continuous on a set S, and $(x_{n_k})_{k \in \mathbb{N}}$ is a convergent cauchy seq^n.

then $f(x_{n_k})$ is a cauchy seq^n with $\lim_{k \to \infty} f(x_{n_k}) = f(x_0)$

But we have $\lim_{n \to \infty} |f(x_{n_k})| = +\infty$

(contradiction)

Hence f is Bounded

(19.4 b) use (a) to prove that $\frac{1}{x^2}$ is not uniformly continuous on $(0,1)$

let $x_n = \frac{1}{n}$ be Cauchy seq^n on $(0,1)$

$\Rightarrow (x_n)_{n \in \mathbb{N}} \in (0,1)$

if f is uniformly continuous then

$f(x_n)$ should be Cauchy seq^n

$\Rightarrow f(x_n) = n^2$

$(n^2)_{n \in \mathbb{N}}$ is a divergent

seq^n.

$\Rightarrow$ f is not uniformly continuous.

19.5 b $\tan x$ on $[0, \frac{\pi}{2})$

$\tan x$ is not bounded on

$[0, \frac{\pi}{4}] \Rightarrow$ according to (19.4a)

$\tan x$ is not uniformly continous

a $\tan x$ on $[0, \frac{\pi}{4}]$

$\tan x$ is bounded on $[0, \frac{\pi}{4}]$

$\Rightarrow$ uniformly continuous on $[0, \frac{\pi}{4}]$

c $\frac{1}{x} \sin^2 x$ on $(0, \pi]$

$\sin^2 x \longrightarrow$ continuous on $(0, \pi]$

$\frac{1}{x} \longrightarrow$ continuous on $(0, \pi]$

$\Rightarrow f(x) = \frac{1}{x} \cdot \sin^2 x$ is continuous on $(0, \pi]$

Using theorem 19.5

find $\tilde{f}(0)$

$\Rightarrow$ let's take a cauchy seq

$$(x_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$$

with limit $\lim_{n\to\infty} \frac{1}{n} = 0$

then

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} n \cdot \sin^2\left(\tfrac{1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \cdot \frac{\sin^2\left(\frac{1}{n}\right)}{(1/n)^2}$$

$$= \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{\sin^2\left(\frac{1}{n}\right)}{(1/n)^2}$$

$$= 0$$

$\Rightarrow$ $\tilde{f}(x)$ is continous

$\Rightarrow$ $f(x)$ is uniformy continuous.

(d)

$$f(x) = \frac{1}{x-3} \quad \text{on} \quad (0,3)$$

let's take an cauchy seq^n $\in (0,3)$

$$x_n = \left(3-\frac{1}{n}\right)_{n\in\mathbb{N}} \in (0,3)$$

$$\lim_{n\to\infty}\left(3-\frac{1}{n}\right) = 3$$

$$\tilde{f}(3) = \lim_{n\to\infty} f(x_n)$$

$$= \lim_{n\to 3} \frac{1}{3-\frac{1}{n}-3} = -\lim_{n\to\infty} n$$

$$= -\infty$$

Hence $f(x) = \frac{1}{x-3}$ is not uniformly continuous on $(0,3)$

(e) $\frac{1}{x-3}$ on $(3,\infty)$

Same reason as (d)

Not uniform continuous