07 - Limits of sequences.

$(a_n)_{n \in \mathbb{N}}$ $\quad$ Seq^n $\quad a : \mathbb{N} \to \mathbb{R}$

It is important to distingush b/w a seq^n and its set of values; since the validity of many results in this Book depends on whether we are working with a seq^n or a set.

$(\ )$ - to signify a Sequence

$\{\ \}$ - to signify a set.

a seq $(a_n)_{n \in \mathbb{Z}^+} = ((-1)^n)_{n \in \mathbb{Z}^+}$

$= (1, -1, 1, -1, 1, -1, \ldots)$

The set associated with the seq^n

set $\{(-1)^n : n = 0, 1, 2, \ldots\} = \{-1, 1\}$

② $\left(\cos\left(\frac{n\pi}{3}\right)\right)_{n\in\mathbb{N}} = \left(\frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}, 1, \ldots\right)$

The set of values is $\left\{\cos\left(\frac{n\pi}{3}\right) : n\in\mathbb{N}\right\}$

$$= \left\{\frac{1}{2}, -\frac{1}{2}, -1, 1\right\}$$

The "limit" of a sequence (s_n) is a real number that the values s_n are close to for large value's of n.

Definition 7.1

A seq^n (s_n) of real number's is said to converge to the real number s Provided that

for each $\varepsilon > 0$, there exists a Number N s.t $n > N$ implie $|s_n - s| < \varepsilon$.

$$\lim_{n\to\infty} s_n = s \quad \text{or} \quad s_n \longrightarrow s.$$

$S =$ Limit of the sequence (S_n)

Example 2

$$S_n = \frac{3n+1}{7n-4}$$

$$\Rightarrow \quad S_n = \frac{3+\frac{1}{n}}{7-\frac{4}{n}}$$

$\frac{1}{n}, \frac{4}{n}$ are very small

$$\Rightarrow \quad S = \frac{3}{7}$$

Proof $\quad \forall \varepsilon > 0, \exists N \in \mathbb{N}$, such that

$\forall n > N \quad |S_n - S| < \varepsilon$

$$n > 0 \Rightarrow \left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| < 1$$

$$n > 4 \Rightarrow \left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| < 0.1$$

$$n > 39 \Rightarrow \left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| < 0.01$$

$$n > 308 \Rightarrow \left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| < 0.001$$

$$n > 387755 \Rightarrow \left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| < 0.000001$$

Limits are unique. $\Rightarrow$ if $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} s_n = t$, then $s = t$.

Proof $\varepsilon > 0 \quad \exists N_1 \in \mathbb{N}$

s.t. $n > N_1 \Rightarrow |s_n - s| < \frac{\varepsilon}{2}$

$n > N_2 \Rightarrow |s_n - t| < \frac{\varepsilon}{2}$

for $n > \max\{N_1, N_2\}$

$$|s-t| = |(s - s_n) + (s_n - t)| \le |s - s_n| + |s_n - t|$$

$$\Rightarrow |s-t| \leq |s-s_n| + |s_n-t|$$

$$\Rightarrow |s-t| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This shows $|s-t| < \varepsilon \quad \forall \varepsilon > 0$.

$$\Rightarrow |s-t| = 0 \Rightarrow s = t$$