Functional Analysis

Lec 01:

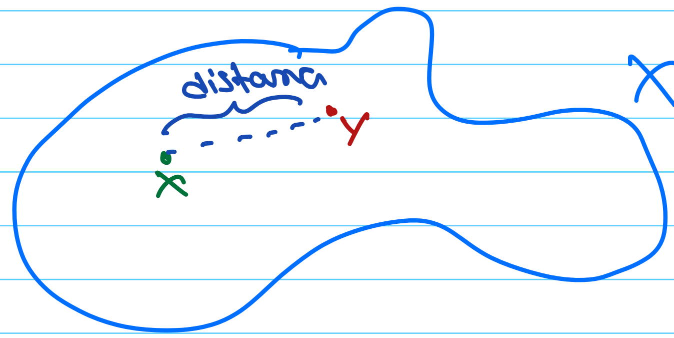
linear algebra dim = ∞	Real and complex Analysis

in rough sense if you look at infinite dimentional space in the Linear algebra sense, and if we want to apply some analysis for this infinite dimentional spaces then we are in the realm functional analysis

Spaces consists of Function's or of seq^n and also linear maps b/w such spaces

Mectric SPACE:

X set is a collection of points



we want to give set X more structure

a metric $d : X \times X \longrightarrow [0, \infty)$

① $d(x,y) = 0 \iff x = y$

② $d(x,y) = d(y,x)$ (symmetry)

③ $d(x,y) \le d(x,z) + d(z,y)$

(triangle inequality)

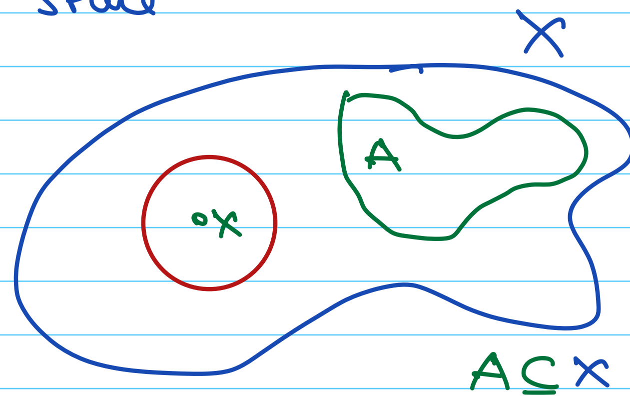
Lec 02

X set $+$ $d : X \times X \to [0,\infty)$ metric

$=$ metric space (X,d)

Lec03:

(X, d) metric space



Open ε-Ball around x:

$$B_\varepsilon(x) := \{ y \in X \mid d(x,y) < \varepsilon \}$$

(Open ball of radius $\varepsilon > 0$ centered at x)

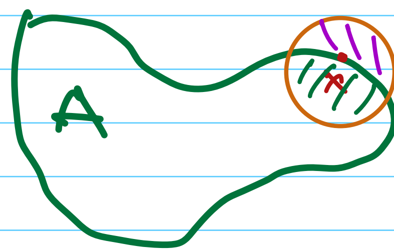
Notion's:

① Open sets:

$A \subseteq X$, descriptively openness should mean that if we are inside set A, we should never see the Boundary of set if we fix $x \in A$, there should be enough points in all direction's around this point that also belongs to set A

Def: $A \subseteq X$ is called open if $\forall x \in A$ $\exists \varepsilon > 0$, s.t the open Ball around x $B_\varepsilon(x) \subseteq A$

② <u>Boundary points:</u> $A \subseteq X$



$x \in X$ is called a Boundary Point for A if $\forall \varepsilon > 0$: $B_\varepsilon(x) \cap A \neq \phi$ and

$$B_\varepsilon(x) \cap A^c \neq \phi$$

where $[A^c := X \setminus A]$

- The Boundary Point can be inside or ouside set A
- The notion Boundary Point can makes only sense with respect to given subset A

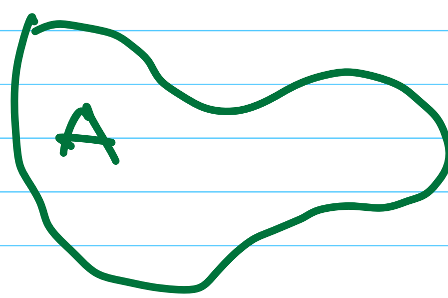
$\partial A :=$ boundary points of set A

Remember: A open $\iff A \cap \partial A = \phi$

(3) Closed set: A subset $A \subseteq X$ is closed if $A^c := X \setminus A$ is open.

Remember A closed $\iff A \cup \partial A = A$

(4) Closure:



$A \cup \partial A$ is called closure of A

$\overline{A} := A \cup \partial A$ (always closed)
smallest closed set that contain's A.

Example: $X := (1,3] \cup (4,\infty)$

$d(x,y) := |x-y|$

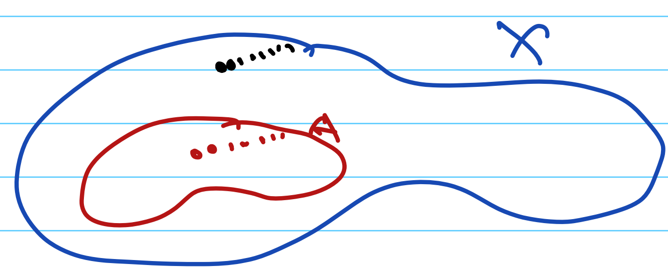
(a) $A := (1,3] \subseteq X$

A is open set.

A is closed set. because $A^c = (4,\infty)$ is open.

Lec04

(X,d) metric space



* one can use seq^n to describe the properties of a metric space.

Proposition:

$A \subseteq X$ is closed

$\Longleftrightarrow$ we can't leave the set from inside by just using seq^n.

$\Longleftrightarrow$ for every convergent seq^n $(a_n)_{n\in\mathbb{N}}$ where $a_n \in A$ $\forall n \in \mathbb{N}$ $(a_n)_{n\in\mathbb{N}} \subseteq A$ one has

$$\lim_{n\to\infty} a_n \in A$$

Proof: $(\Longleftarrow)$: Show it by contradiction.

Assume A is not closed $\Rightarrow$ A^c is not open.

$\Rightarrow$ $A^c := X \setminus A$ is not open

$\Rightarrow$ There is an $\tilde{x} \in A^c$ with $B_\varepsilon(\tilde{x}) \cap A \neq \emptyset$ $\forall \varepsilon > 0$

$\Rightarrow$ There is a seq $(a_n)_{n\in\mathbb{N}}$ with

$a_n \in B_{1/n}(\tilde{x}) \cap A$

$\Rightarrow \lim_{n \to \infty} a_n = \tilde{x} \notin A$

Lec05

Def: let (X,d) be a metric space.

A seq^n $(x_n)_{n \in \mathbb{N}} \subseteq X$ is called Cauchy seq^n if $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ $\forall n,m > N$;

$$d(x_m, x_n) < \varepsilon$$

(X,d) is called complete if all Cauchy seq^n converge.