25. More on Uniform Convergence

We can interchange integral's and uniform limits.

25.1 Discussion:

(a) if g and h are integrable on $[a,b]$ and if $g(x) \leq h(x)$ $\forall x \in [a,b]$ then
$$\int_a^b g(x)\,dx \leq \int_a^b h(x)\,dx.$$

(b) if g is integrable on $[a,b]$, then
$$\left| \int_a^b g(x)\,dx \right| \leq \int_a^b |g(x)|\,dx$$

Theorem 25.2 :-

Let (f_n) be seq^n of continuous functions on $[a,b]$, and suppose

$f_n \to f$ uniformly on $[a,b]$. then

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

Proof: f is continuous

$\Rightarrow$ $f_n - f$ is integrable on $[a,b]$

Let $\varepsilon > 0$. $\exists N \in \mathbb{N}$ such that $\forall x \in [a,b]$

$\forall n > N$ $\quad |f_n(x) - f(x)| < \dfrac{\varepsilon}{b-a}$

$$\Rightarrow \left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right|$$

$$= \left| \int_a^b [f_n(x) - f(x)]\,dx \right.$$

$$\leq \int_a^b |f_n(x) - f(x)|\, dx$$

$$\leq \int_a^b \frac{\varepsilon}{b-a}\, dx = \varepsilon$$

$$\Rightarrow \left| \int_a^b f_n(x)\, dx - \int_a^b f(x)\, dx \right| < \varepsilon$$

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$\left| \int_a^b f_n(x)\, dx - \int_a^b f(x)\, dx \right| \leq \varepsilon$$

$\forall n > N$

$$\boxed{\Rightarrow \lim_{n \to \infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dx}$$

"Uniformly Cauchy"

25.3 Def:

A seq (f_n) of function's defined on set $S \subseteq \mathbb{R}$ is uniformly Cauchy on S if

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| < \varepsilon \quad \forall x \in S$$
$$\forall m, n > N$$

Theorem 25.4 :-

Let (f_n) be a seq^n of function's defined and uniformly Cauchy on a set $S \subseteq \mathbb{R}$. Then $\exists$ a function f on S such that $f_n \longrightarrow f$ uniformly on S.

Proof: we need to find f

* $\forall x_0 \in S$ The seq $(f_n(x_0))_{n\in\mathbb{N}}$ is a cauchy seq^n of $\mathbb{R}$

* $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$\forall m, n > N$, $\forall x \in S$ $|f_n(x) - f_m(x)| < \varepsilon$

$\Rightarrow$ for x_0 $|f_n(x_0) - f_m(x_0)| < \varepsilon$

$\Rightarrow (f_n(x_0))_{n\in\mathbb{N}}$ is cauchy.

define $f(x) = \lim_{n\to\infty} f_n(x)$ (Pointwise)

(found)

we need to prove $f_n \to f$ (uniform) on S

Let $\varepsilon > 0$, Then $\exists N$ such that

$\forall\, m, n > N$ $\quad |f_n(x) - f_m(x)| < \frac{\varepsilon}{2} \quad \forall x \in S$

$\Rightarrow \quad f_n(x) \in \left(f_m(x) - \frac{\varepsilon}{2},\ f_m(x) + \frac{\varepsilon}{2}\right)$

$\vdots$