08 A discussion about Proofs

Example 1

Prove $\lim \frac{1}{n^2} = 0$

Sol^n

Let $\varepsilon > 0$

we want $\left| \frac{1}{n^2} - 0 \right| < \varepsilon$

$$\Rightarrow \frac{1}{n^2} < \varepsilon$$

$$\Rightarrow \frac{1}{\sqrt{\varepsilon}} < n$$

$\Rightarrow$ for $n > \frac{1}{\sqrt{\varepsilon}}$, we have $\left|\frac{1}{n^2} - 0\right| < \varepsilon$

$$\Rightarrow N = \frac{1}{\sqrt{\varepsilon}}$$

Formal Proof:

Let $\varepsilon > 0$. Let $N = \frac{1}{\sqrt{\varepsilon}}$. Then $n > N$

$\Rightarrow \quad n > \frac{1}{\sqrt{\varepsilon}}$

$\Rightarrow \quad n^2 > \frac{1}{\varepsilon} \quad \Rightarrow \quad \varepsilon > \frac{1}{n^2}$

$\Rightarrow$ Thus for $n > N$ we have

$$\left| \frac{1}{n^2} - 0 \right| < \varepsilon$$

This proves $\lim \frac{1}{n^2} = 0$

Example 2

Prove $\lim \dfrac{3n+1}{7n-4} = \dfrac{3}{7}$

Sol^n

Let $\varepsilon > 0$,

we want $\left| \dfrac{3n+1}{7n-4} - \dfrac{3}{7} \right| < \varepsilon$

$$\Rightarrow \left| \frac{2\cancel{1n}+7-2\cancel{1n}+12}{7(7n-4)} \right| < \varepsilon$$

$$\Rightarrow \left| \frac{19}{7(7n-4)} \right| < \varepsilon$$

we know $7n-4 > 0$ becau $n \in \mathbb{N}$

$$\Rightarrow \frac{19}{7(7n-4)} < \varepsilon \Rightarrow \left(\frac{19}{7\varepsilon}+4\right)\frac{1}{7} < n$$

$$\Rightarrow n > \frac{19}{49\varepsilon} + \frac{4}{7}$$

Formal Proof:

Let $\varepsilon > 0$, and Let $N = \frac{19}{49\varepsilon} + \frac{4}{7}$.

Then for $n > N \Rightarrow n > \frac{19}{49\varepsilon} + \frac{4}{7}$

$$\Rightarrow \frac{19}{7(7n-4)} < \varepsilon$$

$$\Rightarrow \left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| < \varepsilon$$

$\Rightarrow$ This Proves $\lim \frac{3n+1}{7n-4} = \frac{3}{7}$

Example 3

Prove $\lim \frac{4n^3+3n}{n^3-6} = 4$

Solⁿ

let $\varepsilon > 0$

we want

$$\left| \frac{4n^3+3n}{n^3-6} - 4 \right| < \varepsilon$$

$$\Rightarrow \left| \frac{4n^3+3n-4n^3+24}{n^3-6} \right| < \varepsilon$$

$$\Rightarrow \left| \frac{3n+24}{n^3-6} \right| < \varepsilon$$

$$\Rightarrow \quad 3\left|\frac{n+6}{n^3-6}\right| < \varepsilon$$

Its very difficult to isolate n

we want $\left|\frac{3n+24}{n^3-6}\right| < \varepsilon$

lets Bound $\frac{3n+24}{n^3-6}$

$$3n+24 \leq 27n$$

& $n^3-6 \geq \frac{n^3}{2}$ $\forall n > 2$

$$\Rightarrow \quad \left|\frac{3n+24}{n^3-6}\right| \leq \frac{27n}{\frac{n^3}{2}} \quad \forall n > 2$$

So, we are checking

$$\frac{27n}{\frac{n^3}{2}} < \varepsilon$$

$$\Rightarrow \quad \frac{54}{n^2} < \varepsilon$$

$$\Rightarrow \quad n > \sqrt{\frac{54}{\varepsilon}} \quad , \quad n > 2$$

$$\Rightarrow \quad N = \max\left(\sqrt{\frac{54}{\varepsilon}}, 2\right)$$

Formal Proof:

Let $\varepsilon > 0$, and let $N = \max\left\{2, \sqrt{\frac{54}{\varepsilon}}\right\}$

Then $n > N \Rightarrow n > \sqrt{\frac{54}{\varepsilon}}$,

$$\Rightarrow \quad \frac{27n}{\frac{n^3}{2}} < \varepsilon$$

as $n > 0$, $\Rightarrow \quad \frac{n^3}{2} \leq n^3 - 6$

and $\quad 27n \geq 3n + 24$

$$\Rightarrow \quad \frac{3n+24}{n^3-6} \leq \frac{27n}{\frac{n^3}{2}} < \varepsilon$$

$$\Rightarrow \quad \frac{3n+24}{n^3-6} < \varepsilon$$

$$= \left| \frac{4n^3+3n}{n^3-6} - 4 \right| < \varepsilon$$

$$\Rightarrow \lim_{n \to \infty} \frac{4n^3+3n}{n^3-6} = 4.$$

Example 4

Show that seq^n $a_n = (-1)^n$ does not converge.

Sol^n assume $\lim_{n \to \infty} (-1)^n = a$

and obtain a contradiction.

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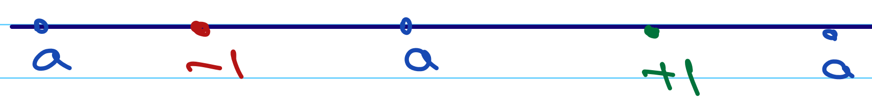


No matter what a is it will have distance atleast 1 from either 1 or -1

apply triangle inequality:

$$|1-a| + |-1-a| \geq |1-(-1)|$$

$$\Rightarrow \quad |1-a| + |1+a| \geq 2$$

To hold this inequality

$$|(-1)^n - a| < 1$$

$$\Rightarrow \quad |(-1)^n - a| < 1$$

$$\& \quad |(-1)^{n+1} - a| < 1$$

contradicting

$$\left.\begin{array}{l} \Rightarrow \quad |-1-a| < 1 \\ \Rightarrow \quad |1-a| < 1 \end{array}\right\} \Rightarrow |1-a| + |1+a| < 2$$

Formal Proof:

Assume $\lim (-1)^n = a \quad a \in \mathbb{R}$

Letting $\varepsilon = 1$ in the definition of the limit, we see that $\exists N \in \mathbb{N}$ s.t $n > N$

$$|(-1)^n - a| < 1$$

So by considering both $n = \text{odd}$, $n+1 = \text{even}$

$|1+a| + |1-a| < 2$ (fails triangle inequality)

Example 5

Let $\lim S_n = S$

then Prove $\lim \sqrt{S_n} = \sqrt{S}$

Sol^n we need to show $\forall \varepsilon > 0$,

$\exists N \in \mathbb{N}$ s.t $n > N$

$$|\sqrt{S_n} - \sqrt{S}| < \varepsilon$$

$$\sqrt{S_n} - \sqrt{S} = \frac{(\sqrt{S_n} - \sqrt{S})(\sqrt{S_n} + \sqrt{S})}{(\sqrt{S_n} + \sqrt{S})}$$

$$= \frac{S_n - S}{\sqrt{S_n} + \sqrt{S}}$$

$\Rightarrow$ if $S > 0$ then

$$\sqrt{S_n} - \sqrt{S} = \frac{S_n - S}{\sqrt{S_n} + \sqrt{S}} \leq \frac{S_n - S}{\sqrt{S}}$$

$\Rightarrow$ we know that

$$|S_n - S| < \varepsilon$$

$$\Rightarrow \left| \frac{S_n - S}{\sqrt{S}} \right| < \frac{\varepsilon}{\sqrt{S}}$$

$$\Rightarrow \left| \sqrt{S_n} - \sqrt{S} \right| \leq \left| \frac{S_n - S}{\sqrt{S}} \right| < \frac{\varepsilon}{\sqrt{S}}$$

there fore there Exis $N \in \mathbb{N}$ s.t this satistie.

Formal Proof:

Case 1: $S > 0$

Let $\varepsilon > 0$. Since $\lim S_n = S$, $\exists N \in \mathbb{N}$ s.t $\forall n > N$

$$|S_n - S| < \sqrt{S}\,\varepsilon$$

$$\Rightarrow \left| \frac{S_n - S}{\sqrt{S}} \right| < \varepsilon$$

$$\Rightarrow \left| \sqrt{S_n} - \sqrt{S} \right| \leq \left| \frac{S_n - S}{\sqrt{S}} \right| < \varepsilon$$

$\Rightarrow \quad \lim_{n\to\infty} \sqrt{s_n} = \sqrt{s}$

<u>CASE II</u> $\quad S = 0$

let $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t $n > N$

$$|s| < \varepsilon^2$$

$$\Rightarrow \quad -\varepsilon^2 < -s \leq |s| \leq s < \varepsilon^2$$

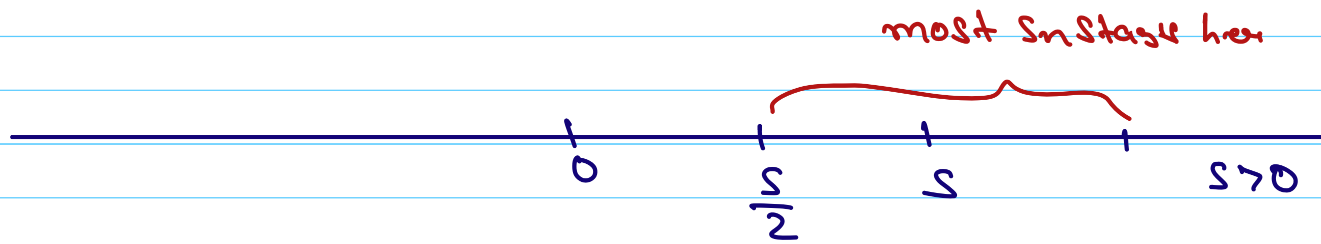
$$\Rightarrow \quad \sqrt{s} < \varepsilon$$

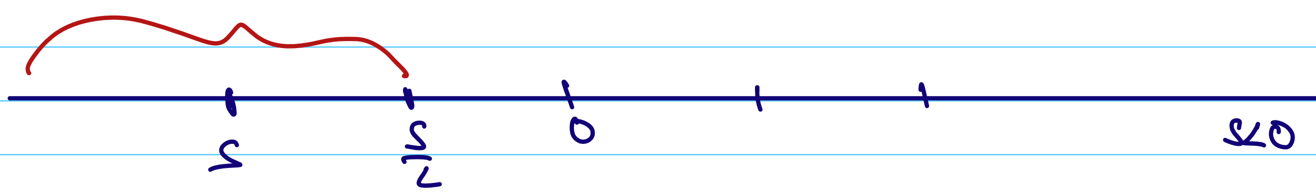
Example 6 :-

Let (S_n) be a convergent seq^n of real number's s.t $S_n \neq 0 \ \forall n \in \mathbb{N}$ and $\lim S_n = s \neq 0$.

Prove $\inf\{|S_n| : n \in \mathbb{N}\} > 0$

Sol^n





Proof: let $\varepsilon = \frac{1}{2}|s| > 0$

Since $\lim S_n = s$, then $\exists N \in \mathbb{N}$, s.t $\forall n > N$

$$|S_n - s| < \frac{|s|}{2}$$

$$|s| = |s - s_n + s_n| \leq |s - s_n| + |s_n|$$

$$\Rightarrow \quad |s| \leq |s - s_n| + |s_n|$$

$$\Rightarrow |s| \leq |s - s_n| + |s_n| \leq \frac{|s|}{2} + |s_n|$$

$$\boxed{\Rightarrow \quad |s_n| \geq \frac{|s|}{2}}$$

Therefore

$$\Rightarrow \quad m = \min\left\{ \frac{|s|}{2}, |s_1|, |s_2|, \ldots, |s_N| \right\}$$

hence $m > 0$, $|s_n| \geq m \quad \forall n \in \mathbb{N}$