Exercises

(1.1) Prove $1^2+2^2+\dots+n^2 = \frac{1}{6}n(n+1)(2n+1)$
$\forall n \in \mathbb{N}$

Sol^n P_n: " $1^2+2^2+\dots+n^2 = \frac{1}{6}n(n+1)(2n+1)$

(1) Base case:

$P_1 = 1^2 = \frac{1}{6}\cdot 1\cdot 2\cdot 3 = 1$

P_1 is true

(2) Induction step:

Suppose P_n is true.

$\Rightarrow 1^2+2^2+\dots+n^2 = \frac{n}{6}(n+1)(2n+1)$

then

$P_{n+1} \Rightarrow 1^2+2^2+\dots+n^2+(n+1)^2$

$\Rightarrow \frac{n}{6}(n+1)(2n+1)+(n+1)^2$

$\Rightarrow \frac{(n+1)}{6}\left(n(2n+1)+6n+6\right)$

$\Rightarrow \left(\frac{n+1}{6}\right)\left(2n^2+7n+6\right)$

$$\Rightarrow \quad \frac{(n+1)}{6}\cdot(n+2)(2(n+1)+1)$$

$\Rightarrow$ P_{n+1} is true

Therefore according to Principle of mathematical induction, we can conclude P_n is true $\forall n \in \mathbb{N}$

(1.2) Prove $3+11+\cdots+(8n-5) = 4n^2-n \quad \forall n\in\mathbb{N}$

Sol^n

(1) Base step

$$P_1 = 3 = 4\cdot 1^2 - 1 = 3 \quad (\text{true})$$

(2) Induction step:

Suppose P_n is true

$$\Rightarrow \quad 3+11+\cdots+(8n-5) = 4n^2-n$$

$$P_{n+1}: \; 3+11+\cdots+(8n-5) + 8(n+1)-5$$

$$\Rightarrow \quad 4n^2-n + 8n+3$$

$$\Rightarrow \quad 4n^2+7n+3 = 4n^2+8n+4-(n+1)$$

$$\Rightarrow \quad 4(n+1)^2 - (n+1)$$

$\Rightarrow$ P_{n+1} is true

(1.3) Prove $1^3+2^3+\cdots+n^3 = (1+2+\cdots+n)^2$

$\forall n \in \mathbb{N}$

Sol^n

Our Proposition

P_n : " $1^3+2^3+\cdots+n^3 = (1+2+\cdots+n)^2$ "

(1) Base case:

$P_1 = 1^3 = 1^2$ (True)

(2) Induction step:

Suppose P_n is true.

$$P_{n+1} = 1^3+2^3+\cdots+n^3+(n+1)^3$$

$$= (1+2+\cdots+n)^2 + (n+1)^3$$

$$= (1+2+\cdots+n+n+1-n+1)^2 + (n+1)^3$$

$$= (1+2+\cdots+n+n+1)^2 - (n+1)^2 - 2(1+2+\cdots+n)(n+1) + (n+1)^3$$

$$= P_{n+1} - (n+1)^2 - \cancel{2} \cdot \frac{n(n+1)}{\cancel{2}} \cdot (n+1) + (n+1)^3$$

$$= P_{n+1} - (n+1)^2 - n(n+1)^2 + (n+1)^3$$

$$= P_{n+1} - (n+1)^2 \cancel{(1+n)} + \cancel{(n+1)^3}$$

$$= P_{n+1}$$

$\Rightarrow$ P_{n+1} is true if P_n is true

1.4a Guess a formula for $1+3+ + \cdots + (2n-1)$ by evaluating the sum for $n=1,2,3,$ and 4 [For $n=1$, the sum is simply 1)

Sol^n $P_n : 1+3+ \cdots + (2n-1)$

$P_1 = 1$

$P_2 = 1+3 = 4$

$P_3 = 1+3+5 = 9$

$P_4 = 1+3+5+7 = 16$ $\Rightarrow$ $\boxed{P_n = n^2}$

1.4 b Prove your formula using mathematical induction.

Sol^n

$$P_n : \text{"} 1+3+\cdots+(2n-1) = n^2 \text{"}$$

① <u>BASE CASE:</u>

$P_1 = 1 = 1^2$ (True)

② <u>Induction step</u>

Suppose P_n = true

$$P_{n+1} = 1+3+\cdots+(2n-1)+2(n+1)-1$$

$$= n^2+2n+1$$

$$= (n+1)^2$$

$\Rightarrow$ P_{n+1} is true if P_n is true.

1.5 Prove $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n} = 2-\frac{1}{2^n}$ $\forall n \in \mathbb{N}$

Sol^n P_n : " $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n} = 2-\frac{1}{2^n}$ "

① Base case:

$$P_1 = 1 + \frac{1}{2^1} = 2 - \frac{1}{2^1} \quad \text{(true)}$$

② Inductive step:

Suppose P_n is true

$$1 + \frac{1}{2^1} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$$

$$P_{n+1}: \; 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}}$$

$$= 2 - \frac{1}{2^n} + \frac{1}{2^{n+1}}$$

$$= 2 - \frac{1}{2^{n+1}}\left[2 - 1\right]$$

$$= 2 - \frac{1}{2^{n+1}}$$

P_{n+1} is true, if P_n is true.

Hence, $1 + \frac{1}{2} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n} \quad \forall n \in \mathbb{N}$

(1.6) Prove $11^n - 4^n$ is divisible by 7 $n \in \mathbb{N}$

Sol^n P_n : " $11^n - 4^n$ is divisible by 7 "

(1) Base case

P_1 : $11 - 4 = 7$ (true)

(2) Induction step:

if P_n is true.

$\Rightarrow$ $11^n - 4^n = 7m$

then P_{n+1} : $11^{n+1} - 4^{n+1}$

$= 11 \cdot 11^n - 4 \cdot 4^n$

$= 7 \cdot 11^n + 4(11^n - 4^n)$

$= \underbrace{7 \cdot 11^n + 4 \cdot 7m}_{\text{divisible by } 7}$

$\Rightarrow$ P_{n+1} is true if P_n is true

$\Rightarrow$ $11^n - 4^n$ divisible by 7

1.7 Prove $7^n - 6n - 1$ is divisible by 36

Soln P_n: "$7^n - 6n - 1$ is divisible by 36"

① BASE CASE:

$P_1 = 7^1 - 6 - 1 = 0$ (true)

② Inductive Step:

P_n is true. $\Rightarrow 7^n - 6n - 1 = 36m$

$\Rightarrow P_{n+1}: \ 7^{n+1} - 6(n+1) - 1$

$\Rightarrow 7 \cdot 7^n - 6n - 6 - 1$

$\Rightarrow 7(7^n - 6n - 1) + 36n$

$\Rightarrow \underbrace{7 \cdot 36m + 36n}_{\text{divisible by } 36}$

$\Rightarrow$ P_{n+1} is true if P_n is true

$\Rightarrow$ $7^n - 6n - 1$ is divisible by 36

(1.8) The Principle of mathematical induction can be extended as follow's.

A List $P_m, P_{m+1}, \ldots$ of Proposition's is true Provided (i) P_m is true,

(ii) P_{n+1} is true whenever P_n is true and $n \geq m$

(a) Prove $n^2 > n+1$ $\forall$ integers $n \geq 2$

Sol^n P_n : "$n^2 > n+1$"

(1) <u>BASE CASE</u>:

P_2 : $2^2 > 2+1$

$\Rightarrow$ $4 > 3$ (true)

(2) <u>Inductive step</u>

Suppose P_n is true for $n > 2$

$\Rightarrow$ $n^2 > n+1$

then P_{n+1} : $(n+1)^2$

$\Rightarrow$ $n^2 + 2n + 1$

$\geq n+1 + 2n + 1$

$\geq 3n+2$

$\geq n+2$

$\Rightarrow$ P_{n+1} is true whenever P_n is true.

(b) Prove $n! > n^2$ for all integer's $n \geq 4$

Soln ① Base case:

$4! > 4^2$

$\Rightarrow 24 > 16$ (true)

② Inductive step:

P_n is true when $n > 4$

$\Rightarrow$ P_{n+1} : $(n+1)!$

$(n+1)! = (n+1) \cdot n!$

$= (n+1) n^2$

$= n^3 + n^2$

$(n+1)^2$
n^2+2n+1

for $k > 4$ we know $k^3 > 2k+1$

$\Rightarrow (n+1)! = n^3 + n^2 > 2k+1+n^2$

$> (n+1)^2$

$\Rightarrow \quad P_{n+1}$ is true.

1.9a Decide for which integer's the inequality $2^n > n^2$ is true.

Sol^n

$P_n : 2^n > n^2$

$P_1 : 2^1 > 1^2$ (True)

$P_2 : 2^2 > 2^2$ (False)

$P_3 : 2^3 > 3^2$ (False)

$P_4 : 2^4 > 4^2$ (False)

$P_5 : 2^5 > 5^2$ (true)

$P_6 : 2^6 > 6^2$ (true)

$P_7 : 2^7 > 7^2$ (true)

$\Rightarrow$ this is true for $n \geq 7$

(10.9b) Prove your Claim in (a) by Induction.

Sol^n P_n : " $2^n > n^2$ "

$2^7 \quad \frac{16 \times 8}{102}$

① <u>Base case:</u>

P_7 : $2^7 > 7^2$

$\Rightarrow$ $102 > 49$ (true)

② <u>Inductive step:</u>

P_n is true $\Rightarrow$ $2^n > n^2$

then P_{n+1} : $2^{n+1} = 2 \cdot 2^n$

$\geq 2 \cdot n^2$

$\geq (n+1)^2 + n^2 - 2n - 1$

$\Rightarrow$ $n^2 - 2n - 1 > 0 \quad \forall n \geq 2$

$\Rightarrow$ therefor P_{n+1} is true if P_n true $\forall$ $n \geq 2$

(1.10) Prove $(2n+1)+(2n+3)+\cdots+(4n-1)$
$=3n^2 \quad \forall n \in \mathbb{N}$

Sol^n

(1) BASE CASE:-

$P_1 : 2\cdot 1+1=3=3\cdot 1^2$ (true)

(2) Inductive step:

P_n is true.

i.e $(2n+1)+(2n+3)+\cdots+(2n+2n-1)$
$=3n^2$

$P_{n+1} : (2(n+1)+1)+(2(n+1)+3)+\cdots+(2(n+1)+2(n+1)-1)$

$\Rightarrow (2n+1)+2+(2n+3)+2+\cdots+2(n+1)+2(n+1)-3$
$+2(n+1)+2(n+1)-1$

$\Rightarrow (2n+1)+2+(2n+3)+2+\cdots+(2n+2n-1)+2$
$+2(n+1)+2(n+1)-1$

$\Rightarrow (2n+1)+(2n+3)+\dots+(2n+2n-1)+2n$

$+\ 2(n+1)+2(n+1)-1)$

$\Rightarrow \quad 3n^2+2n+4n+3$

$\Rightarrow \quad 3n^2+6n+3$

$\Rightarrow \quad 3(n+1)^2 \quad$ (true)

1.11 For each $n \in \mathbb{N}$, Let P_n denote the assertion "n^2+5n+1" is an even integer

(a) Prove P_{n+1} is true whenever P_n is true

Sol^n P_n is true (suppose)

$\Rightarrow P_n : \ n^2+5n+1 = 2m$

$P_{n+1} : \ (n+1)^2 + 5(n+1)+1$

$= \ n^2+2n+1+5n+5+1$

$$= n^2+5n+1+2n+6$$

$$= 2m + 2(n+3)$$

(even integer)

hence P_{n+1} is true.

(b) For which n is P_n actually true?
What is the moral of this exercise.

soln

$$n^2+5n+1$$

P_1: $1^2+5\cdot1+1 = 7$ (odd)

P_2: $2^2+5\cdot2+1 = 11$ (odd)

P_3: $3^2+5\cdot3+1 = 25$ (odd)

Moral: need to check BASE CASE always.

1.12 For $n \in \mathbb{N}$, let $n!$ denote the