Lec02 – Sequences and Limits

Sequences: A seq^n of real number's when we have a map from $\mathbb{N}$ to $\mathbb{R}$

a map $a : \mathbb{N} \longrightarrow \mathbb{R}$

or $a : \mathbb{N}_0 \longrightarrow \mathbb{R}$

Notation: $(a_1, a_2, \ldots)$ infinite List of number's

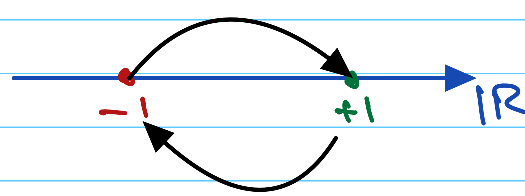
$$(a_n)_{n\in\mathbb{N}} \quad \text{or} \quad (a_n)_{n=1}^{\infty}$$

$$\text{or} \quad (a_n)$$

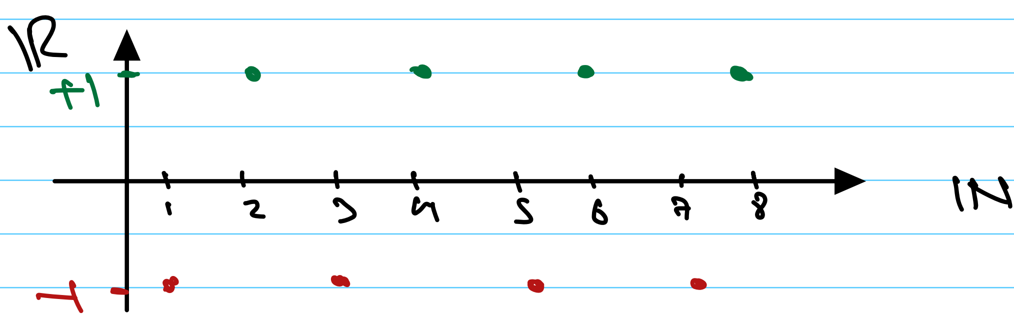
Examples: (0) $(a_n)_{n\in\mathbb{N}} = \left((-1)^n\right)_{n\in\mathbb{N}}$

$$= (-1, 1, -1, 1, -1, 1, \ldots)$$

(i)



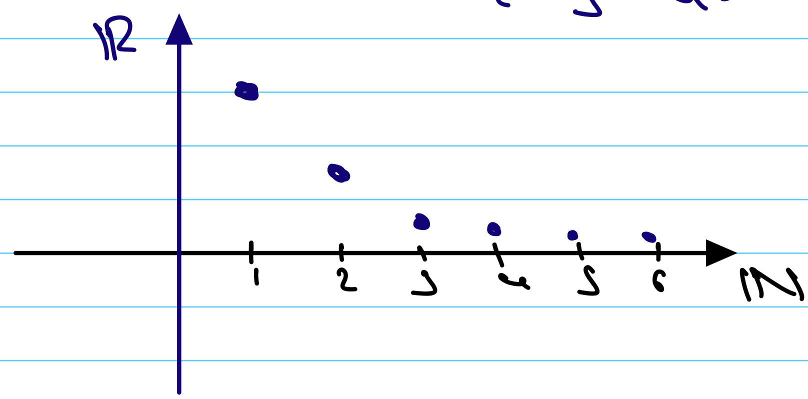
(ii)



we are interested in what happen's to the values of the seq^n when n goes to infinity.

(b) $(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$

$= \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right)$



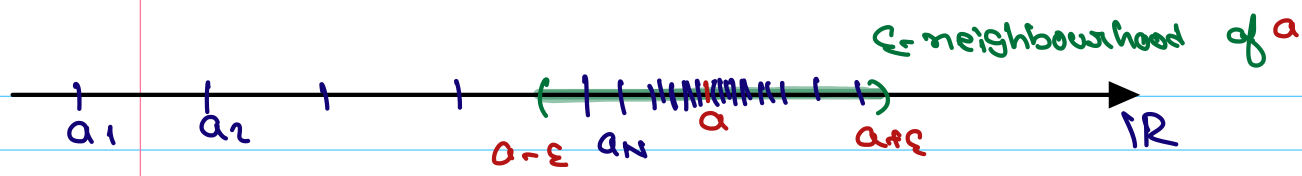
we will see $\lim_{n\to\infty} a_n = 0$

(c) $(a_n)_{n\in\mathbb{N}} = (2^n)_{n\in\mathbb{N}}$

$= (2, 4, 8, 16, 32, 64, \dots)$

Definition:

A sequence $(a_n)_{n\in\mathbb{N}}$ is called convergent to $a \in \mathbb{R}$, if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$, $\forall n \geq N$: $|a_n - a| < \varepsilon$



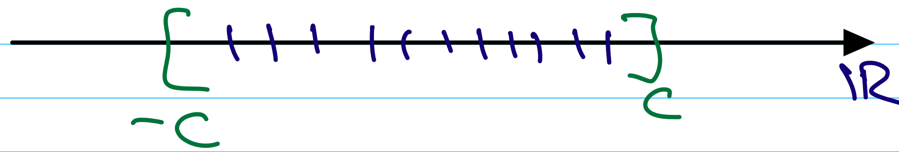
we need to get closer and closer to a with the seq^n member's. eventually all the seq^n member's have to lie ε- neighbourhood of a, only finitely many can lie outside.

No matter how small the ε is, there exist a $N \in \mathbb{N}$, (alway's works)

if there is no such $a \in \mathbb{R}$, we call the seq^n $(a_n)_{n\in\mathbb{N}}$ divergent.

Lec 03 - Bounded seq^n and Unique Limit

Definition: A seq^n $(a_n)_{n\in\mathbb{N}}$ is called Bounded if $\exists C \in \mathbb{N}$ $\forall n \in \mathbb{N}$
$|a_n| \le C$



otherwise, the seq is called unbounded.

Important fact:

$(a_n)_{n\in\mathbb{N}}$ Convergent $\Longrightarrow$ $(a_n)_{n\in\mathbb{N}}$ Bounded

$(a_n)_{n\in\mathbb{N}}$ Convergent $\Longrightarrow$ There is only one limit $a \in \mathbb{R}$

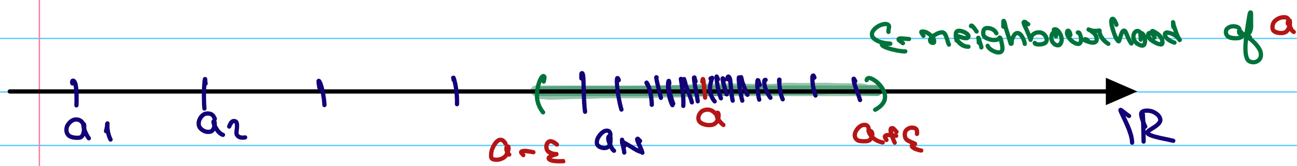
$$\lim_{n\to\infty} a_n = a$$

Lec04 - Theorem on Limits

$(a_n)_{n\in\mathbb{N}}$ convergent to $a \in \mathbb{R}$:

$$\lim_{n\to\infty} a_n = a$$

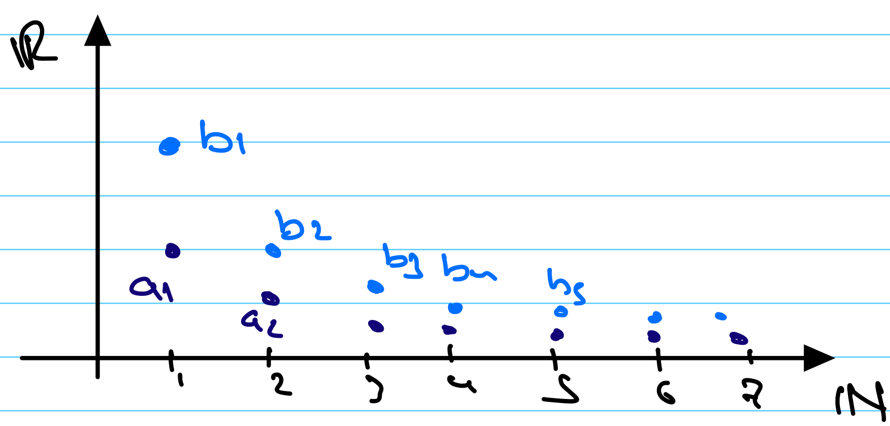
$$a_n \xrightarrow{n\to\infty} a$$



for all ε, the seq^n member's lie inside this ε-neighbourhood of a eventually.

Theorem's on Limits:

$(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ convergent sequnes.



Then

(a) $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

we can Pull in the Limit when we have a sum, provided that these two seq^n and their limits actually exists.

(b) $\lim_{n\to\infty} a_n \cdot b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

(c) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n \neq 0}$

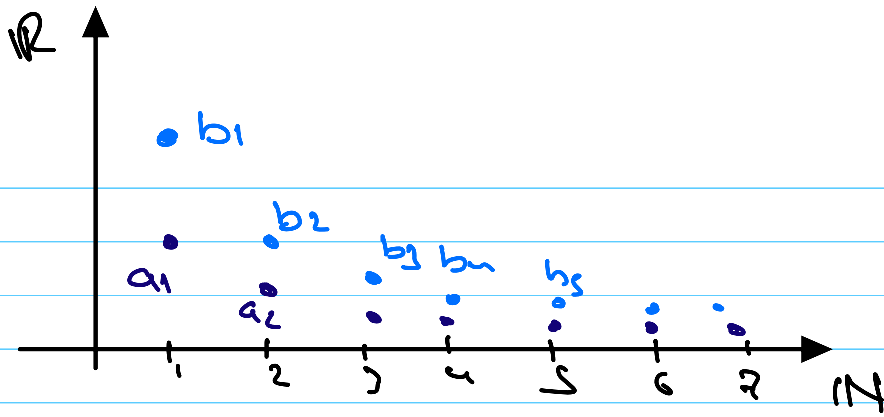
Lec05: Sandwich theorem

$(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ Convergent seq^n

then $\lim_{n\to\infty} (a \cdot b_n) = a \cdot \lim_{n\to\infty} b_n$

Properties:

(a) Monotonicity $a_n \leq b_n \ \forall n \in \mathbb{N}$



$$\Longrightarrow \quad \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$

(b) Sandwich theorem

$$a_n \leq c_n \leq b_n \quad \forall n \in \mathbb{N} \quad \text{and}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

$$\Longrightarrow (c_n)_{n\in\mathbb{N}} \text{ Convergent with}$$

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

Proof of (b):

$$(b_n - a_n) \xrightarrow{n\to\infty} \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = 0$$

$d_n := c_n - a_n$

$\Rightarrow \quad 0 \leq d_n \leq b_n - a_n$

let $\varepsilon > 0$. then $\exists N \in \mathbb{N}$, $\forall n \geq N$

$|b_n - a_n| < \varepsilon$

$\Rightarrow \quad |d_n - 0| < \varepsilon$

$\Rightarrow$ d_n is convergent with limit 0.

$\Rightarrow \quad (c_n)_{n \in \mathbb{N}} = (d_n + a_n)_{n \in \mathbb{N}}$

is convergent with Limit a.