Lec02- Sequences and Limits

Sequences: A seq^n of real number's when we have a map from $\mathbb{N}$ to $\mathbb{R}$

a map $a: \mathbb{N} \longrightarrow \mathbb{R}$

or $a: \mathbb{N}_0 \longrightarrow \mathbb{R}$

Notation: $(a_1, a_2, \ldots)$ infinite List of number's

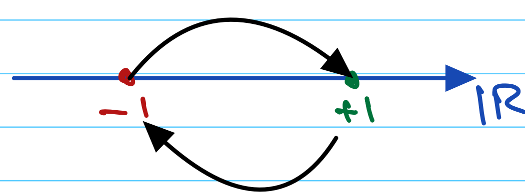
$(a_n)_{n\in\mathbb{N}}$ or $(a_n)_{n=1}^{\infty}$

or (a_n)

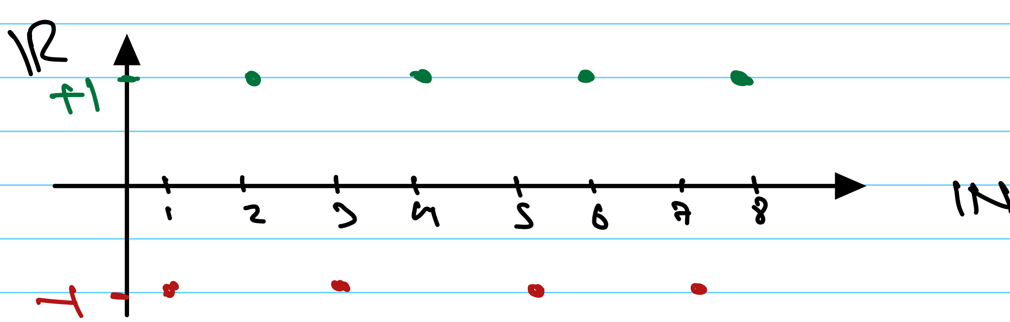
Examples: (0) $(a_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}}$

$= (-1, 1, -1, 1, -1, 1, \ldots)$

(i)



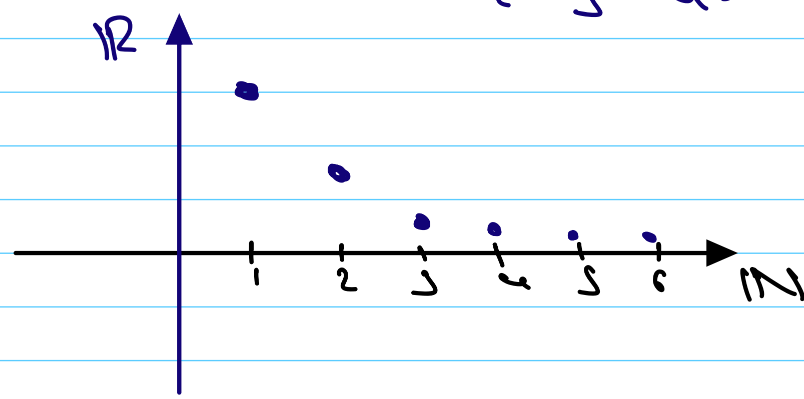
(ii)



we are interested in what happen's to the values of the seq^n when n goes to infinity.

(b) $(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$

$= \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$



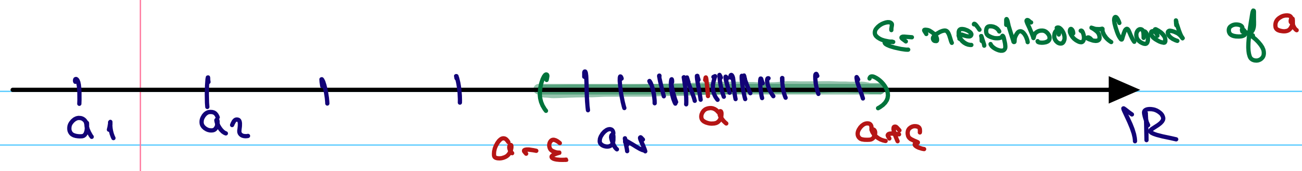
we will see $\lim_{n\to\infty} a_n = 0$

(c) $(a_n)_{n\in\mathbb{N}} = (2^n)_{n\in\mathbb{N}}$

$= (2, 4, 8, 16, 32, 64, \ldots)$

Definition:

A sequence $(a_n)_{n\in\mathbb{N}}$ is called convergent to $a \in \mathbb{R}$, if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$, $\forall n \geq N$: $|a_n - a| < \varepsilon$



we need to get closer and closer to a with the seq^n member's. eventually all the seq^n member's have to lie ε- neighbourhood of a, only finitely many can lie outside.

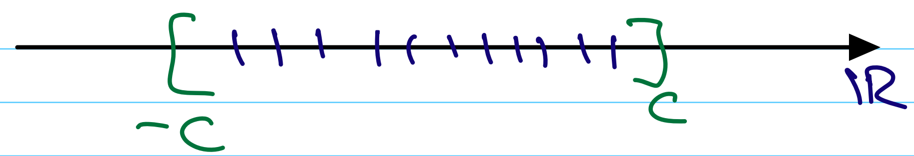
No matter how small the ε is, there exist a $N \in \mathbb{N}$, (alway's works)

if there is no such $a \in \mathbb{R}$, we call the seq^n $(a_n)_{n \in \mathbb{N}}$ divergent.

Lec 03 – Bounded seq^n and Unique Limit

Definition: A seq^n $(a_n)_{n\in\mathbb{N}}$ is called Bounded if $\exists C \in \mathbb{N}$ $\forall n \in \mathbb{N}$

$$|a_n| \leq C$$



otherwise, the seq is called unbounded.

Important fact:

$(a_n)_{n\in\mathbb{N}}$ Convergent $\Longrightarrow$ $(a_n)_{n\in\mathbb{N}}$ Bounded

$(a_n)_{n\in\mathbb{N}}$ Convergent $\Longrightarrow$ There is only one Limit $a \in \mathbb{R}$

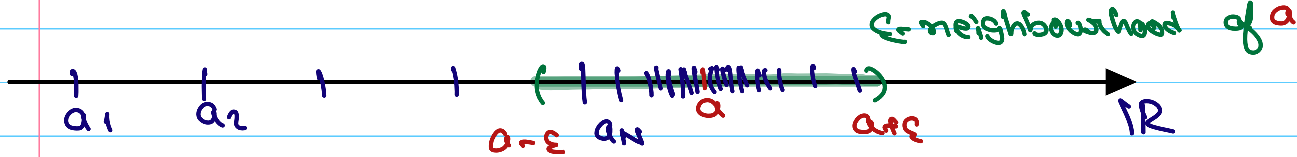
$$\lim_{n\to\infty} a_n = a$$

Lec04 - Theorem on Limits

$(a_n)_{n\in\mathbb{N}}$ Convergent to $a \in \mathbb{R}$:

$$\lim_{n\to\infty} a_n = a$$

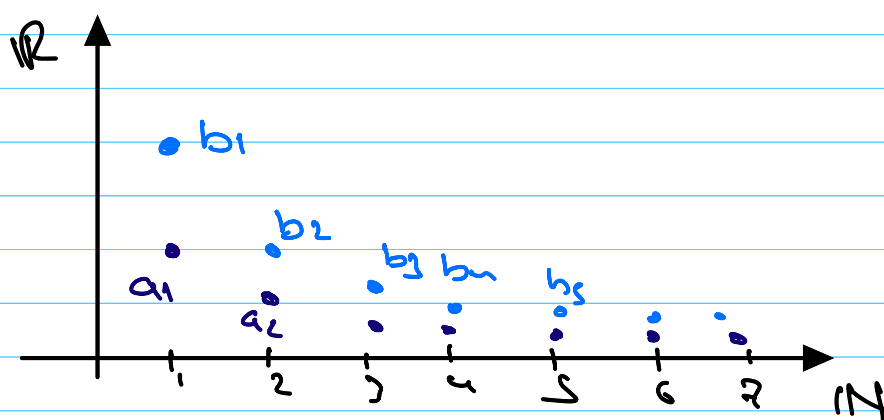
$$a_n \xrightarrow{n\to\infty} a$$



for all ε, the seqⁿ member's lie inside this ε-neighbourhood of a eventually.

Theorem's on Limits:

$(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ Convergent sequnes.



Then

(a) $$\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

we can pull in the Limit when we have a sum, provided that these two seq^n and their limits actually exists.

(b) $$\lim_{n\to\infty} a_n \cdot b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

(c) $$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n \neq 0}$$

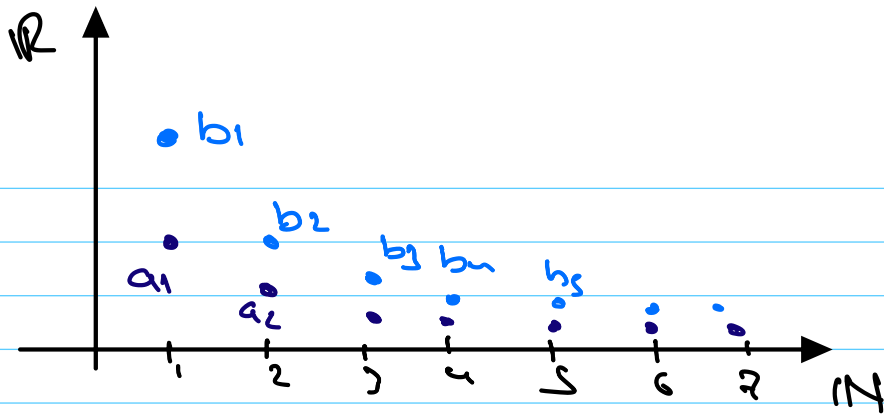
Lec05: Sandwich theorem

$(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ convergent seq^n

then $$\lim_{n\to\infty}(a \cdot b_n) = a \cdot \lim_{n\to\infty} b_n$$

Properties:

(a) Monotonicity $a_n \leq b_n \ \forall n \in \mathbb{N}$



$$\Rightarrow \quad \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$

(b) Sandwich theorem

$$a_n \leq c_n \leq b_n \quad \forall n \in \mathbb{N} \quad \text{and}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

$\Rightarrow$ $(c_n)_{n\in\mathbb{N}}$ convergent with

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

Proof of (b):

$$(b_n - a_n) \xrightarrow{n\to\infty} \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = 0$$

$d_n := c_n - a_n$

$\Rightarrow \quad 0 \leq d_n \leq b_n - a_n$

let $\varepsilon > 0$. then $\exists N \in \mathbb{N}$, $\forall n \geq N$

$|b_n - a_n| < \varepsilon$

$\Rightarrow \quad |d_n - 0| < \varepsilon$

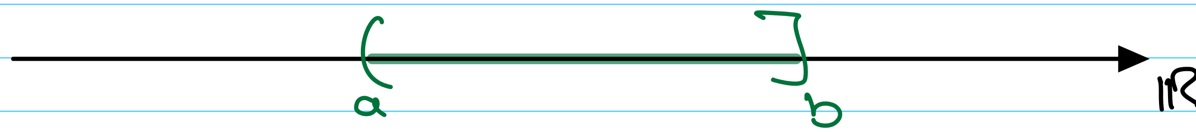
$\Rightarrow$ d_n is convergent with limit 0.

$\Rightarrow \quad (c_n)_{n \in \mathbb{N}} = (d_n + a_n)_{n \in \mathbb{N}}$

is convergent with limit a.

Lec 06 - Supremum & Infimum

Real Number's are ordered



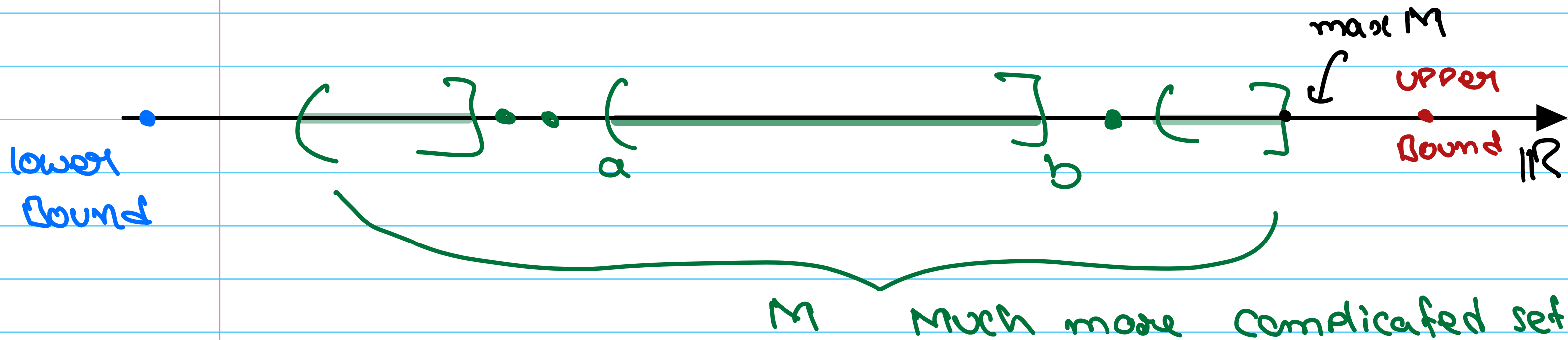
interval $(a,b] := \{x \in \mathbb{R} \mid a < x \le b\}$

similarly $[a,b] := \{x \in \mathbb{R} \mid a \le x \le b\}$

$[a,\infty) := \{x \in \mathbb{R} \mid a \le x\}$

$(-\infty, b) := \{x \in \mathbb{R} \mid x < b\}$

these interval's are very nice subsets of real number's. (Not the only ones)



Definition: For a subset $M \subseteq \mathbb{R}$: $b \in \mathbb{R}$ is called an upper bound for M if

$\forall x \in M : x \leq b$

For $a \in \mathbb{R}$, is called a lower bound for M if $\forall x \in M : x \geq a$

* if upper bound exists we call the set M is bounded from above

* if lower bound exists, the set M called bounded from below.

Both → set M is bounded.

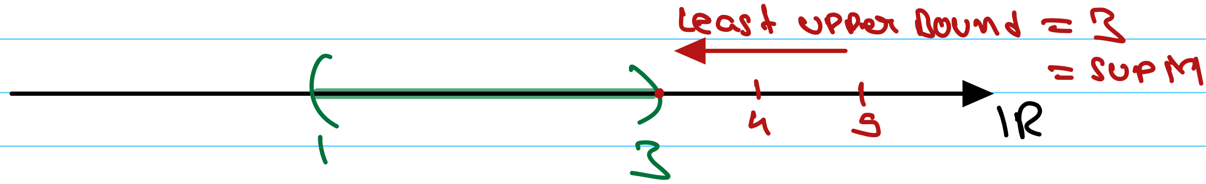
* if b is an upper bound for M and $b \in M$, then b is called a maximal element of M.

  $b = \max M$

* if a is an lower bound for M and $a \in M$, then a is called a minimal element of M

  $a = \min M$

Ex: $M = (1,3)$, $\max M$, $\min M$ do not exists



$\max M$, $\min M$ do not exists, because $1,3 \notin M$

Definition: for a subset $M \subseteq \mathbb{R}$; $S \in \mathbb{R}$ is called supremum of M if

* $\forall x \in M : x \leq S$ (upper bound for M)
* $\forall \varepsilon > 0, \exists \tilde{x} \in M : S - \varepsilon < \tilde{x}$ ($S - \varepsilon$ is no upper bound for M)

Supremum = least upper bound.

$\sup M := S$ or $\sup M := \infty$ (if M is not bounded from above)

or $\sup \emptyset := -\infty$

Definition: for a subset $M \subseteq \mathbb{R}$; $l \in \mathbb{R}$ is called infimum of M if

* $\forall x \in M : x \geq l$ (lower bound for M)
* $\forall \varepsilon > 0, \exists \tilde{x} \in M : l + \varepsilon > \tilde{x}$ ($l + \varepsilon$ is no lower bound for M)

Then write $\inf M := l$ or $\inf M = -\infty$ (if M is not bounded from below)

or $\inf \emptyset = +\infty$

Lec 07 : Cauchy seq^n & completeness

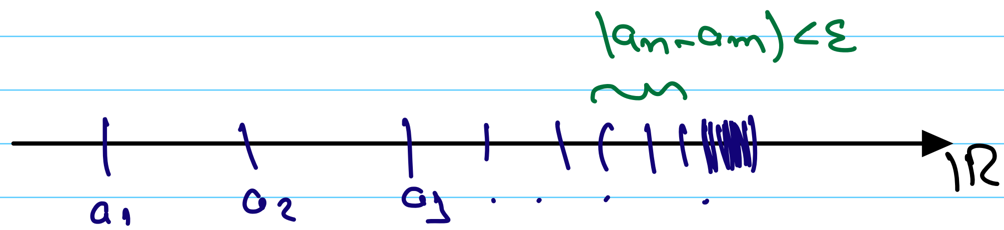
$(a_n)_{n \in \mathbb{N}}$ is convergent seq^n

which means there is a number a, s.t the seq member's here get arbitrarly close to this number a eventually.

$(a_n)_{n\in\mathbb{N}} \Rightarrow$ (there is a limit $a = \lim_{n\to\infty} a_n$)

The Problem with this definition is we need to know the value of limit to show the convergence.

<u>Different idea:</u>



The seqn member's gets closer and closer to each other.

$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n,m \geq N$: we have

$$|a_n - a_m| < \varepsilon$$

(Cauchy sequence)

<u>Definition:</u> if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$, such that $\forall n,m \geq N$ $|a_n - a_m| < \varepsilon$, then $(a_n)_{n\in\mathbb{N}}$ is called Cauchy seqn

Important fact: For a seq^n of real number's

Cauchy seq^n $\Longleftrightarrow$ Convergent seq^n

Completeness Axiom: There are no holes in our complete real Number line.

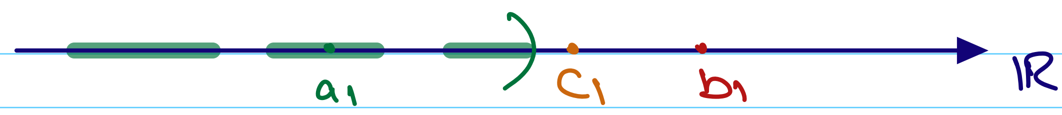
When we are working real number's, we don't have to distinguish Cauchy seq^n and Convergent seq^n. They have different definition, But for real number's they mean the same thing.

Dedekind Completeness:-

if $M \subseteq \mathbb{R}$ is bounded from above, then $\sup M \in \mathbb{R}$ (exists)

if $M \subseteq \mathbb{R}$ is bounded from below, then $\inf M \in \mathbb{R}$ (exists).

Proof:



$c_1 := \frac{1}{2}(a_1 + b_1)$

Two cases: (1) if c_1 is an upper bound for M: $b_2 := c_1$

$a_2 := a_1$

(2) if c_2 is not an upper bound for M: $\exists x \in M : x > c_1$

$b_2 := b_2$

$a_2 := x$

$c_n := \frac{1}{2}(a_n + b_n)$

$\vdots$

Important fact:

if $(a_n)_{n\in\mathbb{N}}$ is monotonically decreasing

$(a_{n+1} \leq a_n \ \forall n)$ and bounded from below

(the set $\{a_n\}_{n\in\mathbb{N}}$ has a lower bound) then $(a_n)_{n\in\mathbb{N}}$ is convergent.

Lec 08:

Important FACT: if $(a_n)_{n\in\mathbb{N}}$ is monotonically increasing ($a_{n+1} \geq a_n \ \forall n$) and bounded from above (the set $\{a_n\}_{n\in\mathbb{N}}$ has an upper bound) then

$(a_n)_{n\in\mathbb{N}}$ is convergent.

(Monotone convergence criterion)

Ex: $(a_n)_{n\in\mathbb{N}} = \left(\left(1+\frac{1}{n}\right)^n\right)_{n\in\mathbb{N}}$ is convergent.

Proof:

monotonicity:-

$$\frac{a_{n+1}}{a_n} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n}$$

$$= \left(1+\frac{1}{n}\right) \cdot \left(\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}} \right)^{n+1}$$

$$= \left(1+\frac{1}{n}\right) \left(\frac{n+2}{n+1} \cdot \frac{n}{n+1} \right)^{n+1}$$

$$= \left(1+\frac{1}{n}\right) \left(\frac{n^2+2n}{n^2+2n+1} \right)^{n+1}$$

$$= \left(1+\frac{1}{n}\right) \left(1 - \frac{1}{(n+1)^2} \right)^{n+1}$$

<u>Bernoulli inequality:</u>

for $k \in \mathbb{N}$ and $x \geq -1$

$$(1+x)^k \geq 1 + kx$$

$$\Rightarrow \quad \left(1+\frac{1}{n}\right) \left(1 - \frac{1}{(n+1)^2} \right)^{n+1}$$

$$\geq \left(1+\frac{1}{n}\right) \left(1 - (n+1)\left(\frac{1}{(n+1)^2}\right) \right)$$

$$\geq \left(1+\frac{1}{n}\right) \left(1 - \frac{1}{n+1} \right)$$

$$\geq \left(\frac{n+1}{n}\right)\left(\frac{n}{n+1}\right) = 1$$

$$\Rightarrow \frac{a_{n+1}}{a_n} \geq 1 \quad (\text{monotone increasing})$$

Now we need to show the seq^n is Bounded from above.

$$\Rightarrow \quad a_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k$$

$$= \left(\frac{1}{n}\right)^0 + \binom{n}{1}\left(\frac{1}{n}\right)^1 + \binom{n}{2}\left(\frac{1}{n}\right)^2 \cdots$$

$$= 1 + 1 + \sum_{k=2}^{n} \binom{n}{k}\left(\frac{1}{n}\right)^k$$

$$\binom{n}{k}\left(\frac{1}{n}\right)^k = \frac{n!}{(n-k)!\,k!} \cdot \frac{1}{n^k}$$

$$= \frac{n\cdot(n-1)(n-2)\cdots\cdot(n-k+1)}{n\cdot n\cdot n\cdot \quad \cdot n} \cdot \frac{1}{k!}$$

$$\leq 1$$

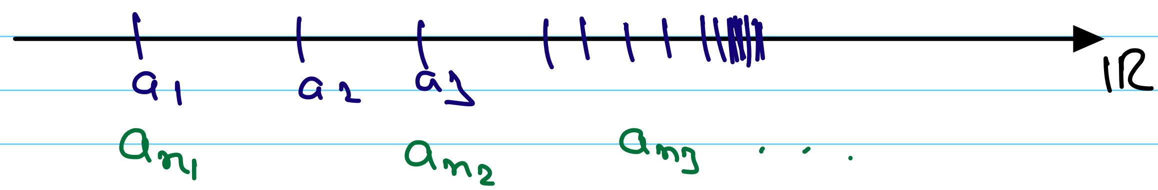
$$\Rightarrow \left(1+\frac{1}{n}\right)^n \leq 3$$

fact $\Longrightarrow$ The seq^n $(a_n)_{n\in\mathbb{N}}$ is convergent.

monotone convergence criterion

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \quad \text{Euler's Number}$$

Lec 09 :- Subsequences and Limit Points



Omit some member's of the seq^n.

Let $(n_k)_{k\in\mathbb{N}}$ be a seq^n of natural number's that is strictly monotonically increasing $(\forall k\in\mathbb{N} : n_{k+1} > n_k)$

then $(a_{n_k})_{k\in\mathbb{N}}$ is called an subsequence of $(a_n)_{n\in\mathbb{N}}$

Ex: $(a_n)_{n\in\mathbb{N}}$ given by $a_n = \frac{1}{n}$

$$(a_{n_k})_{k\in\mathbb{N}} = (a_{2k})_{k\in\mathbb{N}}$$

$$= (a_2, a_4, a_6, a_8 \ldots)$$

$$= \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\right)$$

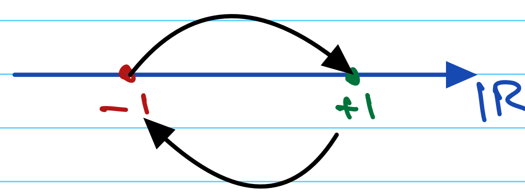
if we have a monotonic seq^n $(a_n)_{n\in\mathbb{N}}$ then the subseq^n $(a_{n_k})_{k\in\mathbb{N}}$ is also monotonic in the same way.

* we can omit lot of seq^n member's even infinitely many as long as infinitely many remain.

FACT: $(a_n)_{n\in\mathbb{N}}$ is convergent with $\lim\limits_{n\to\infty} a_n = a$, then

$\Rightarrow$ every subsequence $(a_{n_k})_{k\in\mathbb{N}}$ is convergent $\lim_{k\to\infty} a_{n_k} = a$

Ex: $(a_n)_{n\in\mathbb{N}}$ given by $a_n = (-1)^n$



Subseq^n $(a_{n_k})_{k\in\mathbb{N}} = (a_{2k})_{k\in\mathbb{N}}$

$= (1, 1, 1, \ldots)$

$\lim_{k\to\infty} a_{2k} = 1$ (limit 1)

$(a_{n_k})_{k\in\mathbb{N}} = (a_{2k+1})_{k\in\mathbb{N}}$

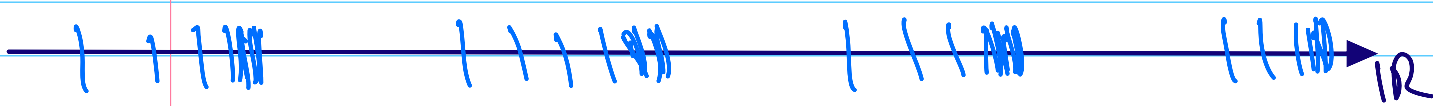
$\lim_{k\to\infty} a_{2k+1} = -1$ (limit -1)

Def: $a \in \mathbb{R}$ is called an accumulation value of $(a_n)_{n\in\mathbb{N}}$ if there is a subseq^n $(a_{n_k})_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} a_{n_k} = a$

Accumulation value is a generalization of term Limit.

$\Rightarrow$ A convergent seq^n can only have one accumulation value = Limit

$\Rightarrow$ However for divergent seq^n could have different accumulation values



The Limit is getting closer to 4 different points. we don't have a Limit because the seq^n still Jumps around, But we get closer and closer to different accumulation values.

$\Rightarrow$ we could restrict ourselves to take seq^n members that only live in one convergent seq^n (Limit)

Accumulation value is Just a Point on the number line, where the seq^n accumulates

$a \in \mathbb{R}$ is an accumulation value of $(a_n)_{n \in \mathbb{N}}$

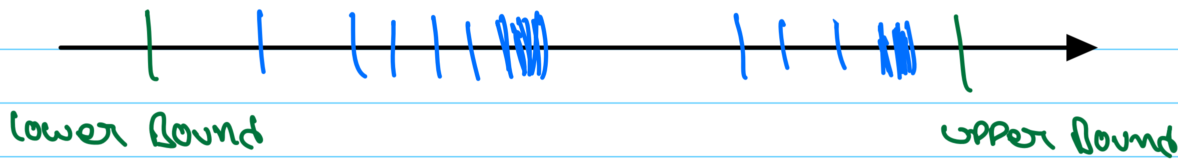
$\Longleftrightarrow$ $\forall \varepsilon > 0$: The ε-neighbourhood of a contain's infinitely many seq^n member's of $(a_n)_{n \in \mathbb{N}}$.

Lec 10: Bolzano-Weierstrass

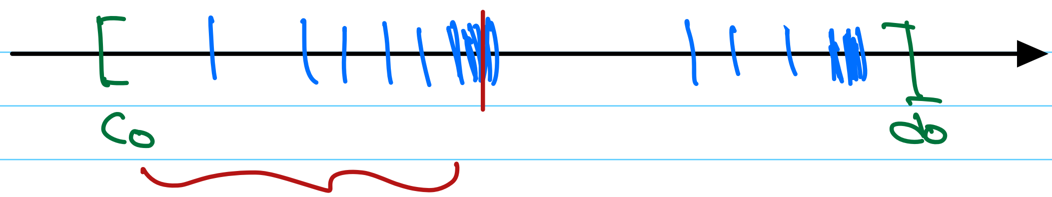
$(a_n)_{n \in \mathbb{N}}$ bounded $\Longrightarrow$ $(a_n)_{n \in \mathbb{N}}$ has an accumulation value

(has a convergent subseq^n)

having a accumulation value mean's the seq^n have a convergent subseq^n.



Proof:

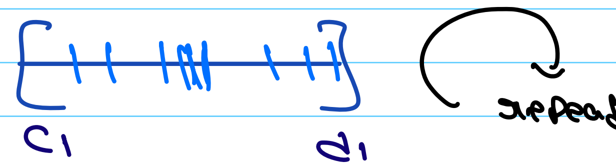


if infinitely many seq^n member's in it:

Choose left-hand interval

otherwise: Choose right hand interval.

New interval:



we set $[c_0, d_0] \supset [c_1, d_1] \supset [c_2, d_2] \ldots$

and $d_1 - c_1 = \frac{1}{2}(d_0 - c_0)$

$d_2 - c_2 = \frac{1}{2}(d_1 - c_1)$

$d_3 - c_3 = \frac{1}{2}(d_2 - c_2)$

$\vdots$

$$d_n - c_n = \frac{1}{2^n}(d_0 - c_0) \xrightarrow{n\to\infty} 0$$

we know

$(c_n)_{n\in\mathbb{N}}$ monotonically increasing and bounded

$(d_n)_{n\in\mathbb{N}}$ monotonically decreasing and bounded

$\Rightarrow$ $(c_n)_{n\in\mathbb{N}}, (d_n)_{n\in\mathbb{N}}$ convergent

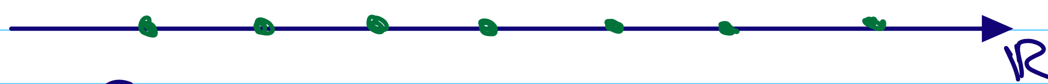
By Limit theorem

$$0 = \lim_{n\to\infty}(d_n - c_n)$$

$$\Rightarrow \lim_{n\to\infty} d_n = \lim_{n\to\infty} c_n$$

Lec 11: Limit superior and limit inferior

Ex: $(a_n)_{n\in\mathbb{N}}$ given by $a_n = n$



$(a_n)_{n\in\mathbb{N}} = (n)_{n\in\mathbb{N}}$ is divergent to ∞

∞: symbol (Not as a number)

divergent to ∞ $:\Longleftrightarrow$ $\forall C>0\ \exists N\in\mathbb{N}$
$\forall n\geq N: \ a_n > C$

$\Leftarrow$ $\lim\limits_{n\to\infty} a_n = \infty$

divergent to $-\infty$ $:\Longleftrightarrow$ $\forall C<0,\ \exists N\in\mathbb{N}$
$\forall n\geq N: \ a_n < C$

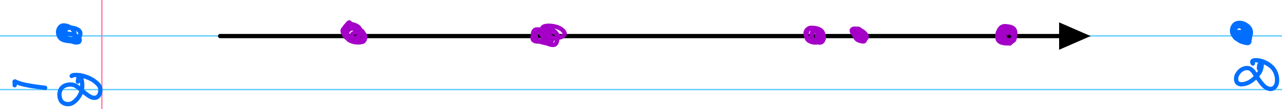
$\Leftarrow$ $\lim\limits_{n\to\infty} a_n = -\infty$

$(a_n)_{n\in\mathbb{N}}$ has the improper accumulation value ∞ $:\Longleftrightarrow$ $(a_n)_{n\in\mathbb{N}}$ is not bounded from above

$(a_n)_{n\in\mathbb{N}}$ has the improper accumulation value $-\infty$

$:\Longleftrightarrow$ $(a_n)_{n\in\mathbb{N}}$ is not bounded from below.

A given seq^n $(a_n)_{n\in\mathbb{N}}$ could have many accumulation values:



* we can talk about Largest and smallest accumulation value, it could be a normal one or improper one.

Def: Let $(a_n)_{n\in\mathbb{N}}$ be a seq^n of real numbers. An element $a \in \mathbb{R}\cup\{-\infty,\infty\}$ is called

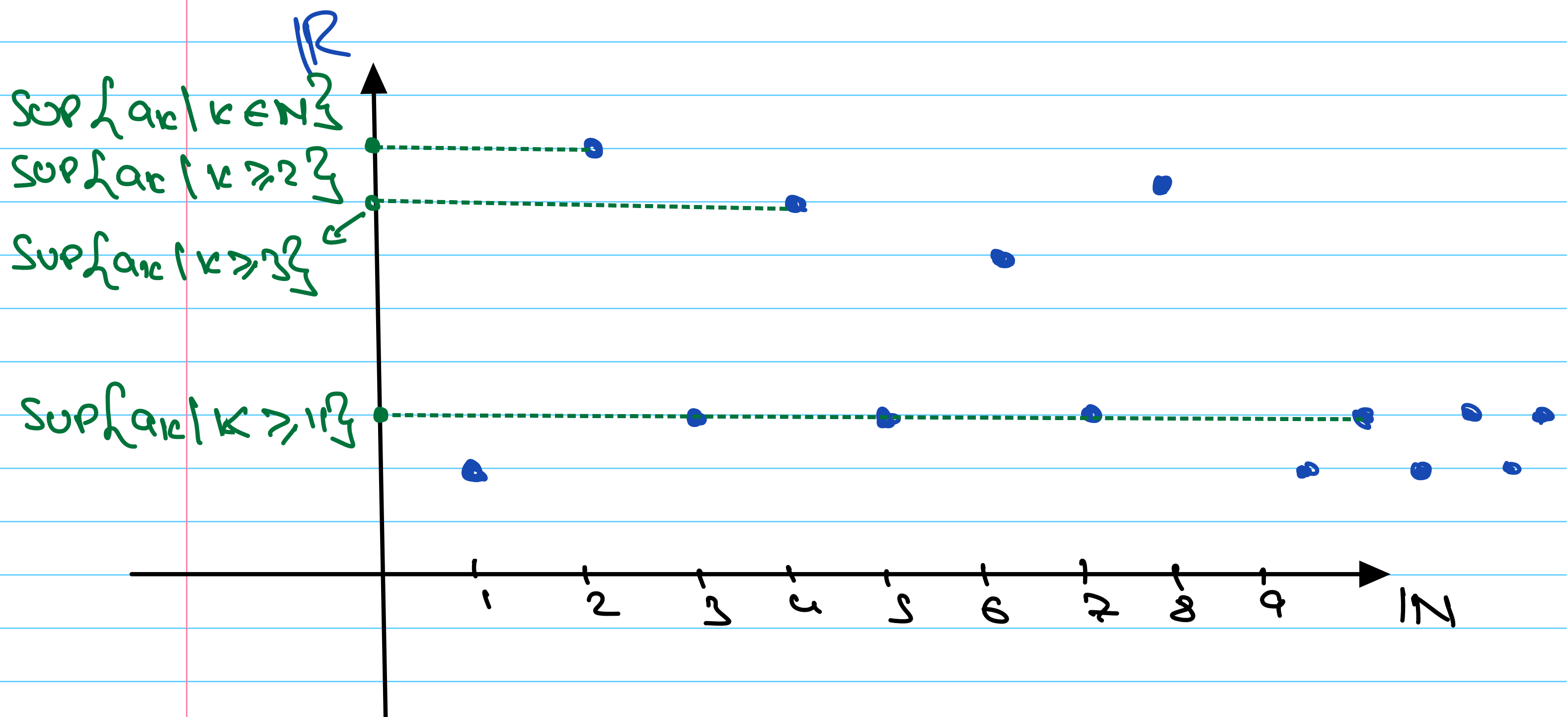
limit superior of $(a_n)_{n\in\mathbb{N}}$

if a is the Largest (Improper) accumulation value of $(a_n)_{n\in\mathbb{N}}$

$$a := \limsup_{n\to\infty} a_n$$

- limit inferior of $(a_n)_{n\in\mathbb{N}}$ if a is the smallest (improper) accumulation value of $(a_n)_{n\in\mathbb{N}}$.

write $a = \liminf_{n\to\infty} a_n$



Limit superior describes what happen to with the supremum, when we cut of more and more at the begining of the seq

FACT:

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_n \mid k \geq n\}$$

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_n \mid k \geq n\}$$

Lec 12

Let $(a_n)_{n\in\mathbb{N}}$ be a seq^n of real numbers

$$\Rightarrow \limsup_{n\to\infty} a_n, \ \liminf_{n\to\infty} a_n \in \mathbb{R} \cup \{\pm\infty\}$$
$$= [-\infty, \infty]$$

Ex: $(a_n)_{n\in\mathbb{N}} = ((-1)^n \cdot n)_{n\in\mathbb{N}}$

$$= (-1, 2, -3, 4, -5, \ldots)$$

$$\limsup_{n\to\infty} a_n = \infty$$

$$\liminf_{n\to\infty} = -\infty$$

Properties:

(a) $(a_n)_{n\in\mathbb{N}}$ is convergent

$$\iff \limsup_{n\to\infty} = \liminf_{n\to\infty} \notin \{\pm\infty\}$$

(b) $(a_n)_{n\in\mathbb{N}}$ is divergent to ∞

$$\iff \left(\limsup_{n\to\infty} a_n\right) = \liminf_{n\to\infty} a_n = \infty$$

(c) $(a_n)_{n\in\mathbb{N}}$ is divergent to $-\infty$

$$\iff \limsup_{n\to\infty} a_n = \left(\liminf_{n\to\infty} a_n\right) = -\infty$$

(d) For $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, we have

$$\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

(only if right hand side is defined)

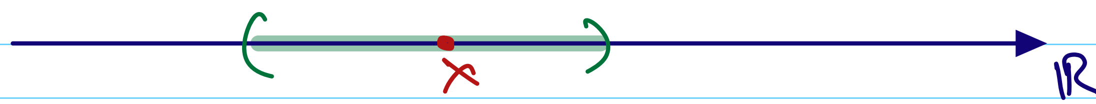
if $a_n, b_n \geq 0$

$$\limsup_{n\to\infty} (a_n \cdot b_n) \leq \limsup_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

(e) if $a_n, b_n \geq 0$

$$\liminf_{n\to\infty} (a_n \cdot b_n) \geq \liminf_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

Lec 13 - Open, closed, and compact sets



$(x-\varepsilon, x+\varepsilon)$ = ε-neighbour hood of x

$\forall \varepsilon > 0$: $(x-\varepsilon, x+\varepsilon) =: B_\varepsilon(x)$

ε-neighbour hood of x

$M \subseteq \mathbb{R}$ is called a neighbourhood of x, if there is $\exists \varepsilon > 0$ s.t $M \supseteq B_\varepsilon(x)$

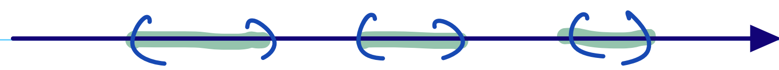
Example: $[-2, 2]$ is a neighbourhood of $B_\varepsilon(0)$, $B_\varepsilon(1)$

$[-2, 2]$ is not a neighbourhood of 2

Definition: $M \subseteq \mathbb{R}$ is called open (in $\mathbb{R}$)

if, for all $x \in M$, M is a neighborhood of x.

$\Rightarrow \quad \forall x \in M, \ \exists \varepsilon > 0 : M \supseteq B_\varepsilon(x)$



$A \subseteq \mathbb{R}$ is closed (in $\mathbb{R}$) if $A^c := \mathbb{R} \setminus A$ is open

Open is Not the opposite of closed.

$\Rightarrow$ There could be neither open nor closed

$\Rightarrow$ There could be open & closed at the same time.

Ex: ① ϕ, $\mathbb{R}$ are Both open & closed

② $[-2, 2]$ is closed but not open

③ $(-2, 2)$ is open but not closed

④ $(-2, 2]$ is neither open nor closed

The subset of the real number's could be much more complicated than just an intervals.

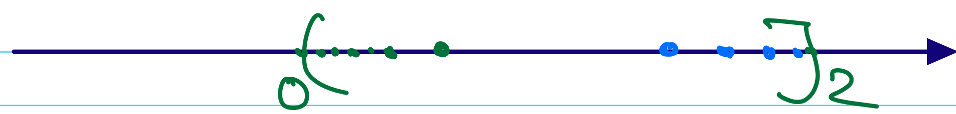
In order to deal with those sets, the next fact is very helpful.

FACT: It gives us a criterion to check closeness with the help of seq^n.

$A \subseteq \mathbb{R}$ is closed $\iff$ For all convergent seq^n $(a_n)_{n\in\mathbb{N}}$ with $a_n \in A$ $\forall n \in \mathbb{N}$ we have $\lim_{n\to\infty} a_n \in A$

It's not possible to leave the set with seq^n from inside.

Ex: $A := (0,2]$



Take $(a_n)_{n\in\mathbb{N}} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}}$

$\frac{1}{n} \in (0,2] \ \forall n\in\mathbb{N}$

But $\lim\limits_{n\to\infty} a_n = 0 \notin A$

This set is not closed.

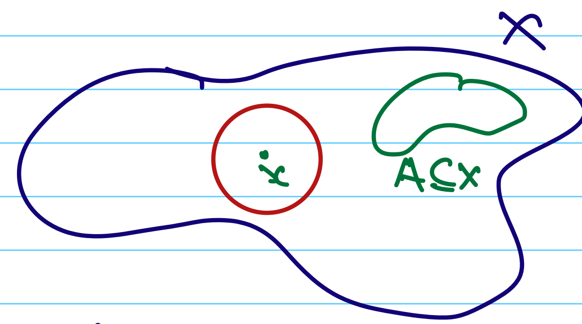
Definition:

$A \subseteq \mathbb{R}$ is called compact, if for all sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n \in A \ \forall n\in\mathbb{N}$, we find convergent subseq^n with a limit lies in A. $\Rightarrow (a_{n_k})_{n\in\mathbb{N}}$ with $\lim\limits_{k\to\infty} a_{n_k} \in A$

Functional Analysis lec 03

Open and closed sets

(X, d) : metric space



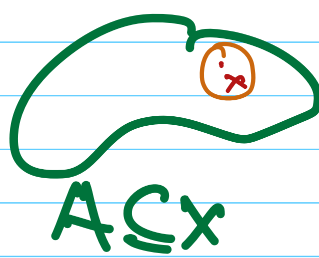
$B_\varepsilon(x)$ open ε-ball around x

$$B_\varepsilon(x) := \{ y \in X \mid d(x,y) < \varepsilon \}$$

(open ball of radius $\varepsilon > 0$, centered at x)

Notion's:

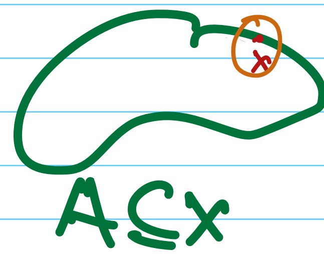
① **Open sets:**



$A \subseteq X$ is called open, if for each $x \in A$, there is an open ball with $B_\varepsilon(x) \subseteq A$

i.e $\forall x \in A$, $\exists \varepsilon > 0$, such that $B_\varepsilon(x) \subseteq A$

② **Boundary points:**



$A \subseteq X$. $x \in X$ is called a Boundary Point for A if $\forall \varepsilon > 0$: $B_\varepsilon(x) \cap A \neq \emptyset$

$B_\varepsilon(x) \cap A^c \neq \emptyset$

$$\partial A := \{ x \in X \mid x \text{ is a boundary point for } A \}$$

Remember:

A open $\iff$ $A \cap \partial A = \emptyset$

Closed set:

A closed $\iff$ $A \cup \partial A = A$

A closed $\iff$ $A^c := X \setminus A$ is open

(4) Closure:

$\overline{A} := A \cup \partial A$ (always closed)

(smallest closed set that contain's A)

Ex: $X := (1,3] \cup (4,\infty)$

$d(x,y) = |x-y|$

(X, d) metric space.

(a) $A := (1,3] \subseteq X$

Open: $\forall x \in A$, $\exists \varepsilon$ s.t $B_\varepsilon(x) \subseteq A \Rightarrow$ open

closed: $A^c = (4,\infty)$, $\forall x \in A^c$, $\exists \varepsilon$ s.t $B_\varepsilon(x) \subseteq A^c \Rightarrow A^c$ is open $\Rightarrow A$ is closed

(c) $C := (1,2]$

$\partial C = \{2\}$

Lec 14 Heine-Borel theorem

Compact set : (sequentially compact set)

we use sequence's to describe compactness.

* Any subset $A \subseteq \mathbb{R}$ is called compact set, if it forces any seq^n to cluster.

* so the seq^n with member's in A has at atleast one accumulation value, this accumulation value $\in A$

Any seq^n $(a_n)_{n \in \mathbb{N}} \subseteq A$ has a accumulation value $a \in A$.

Example: (a) ϕ is compact

(b) $\{5\}$ is compact

(c) $\mathbb{R}$

we need to look all the seq^n in real

number's $\quad (a_n)_{n\in\mathbb{N}} = (n)_{n\in\mathbb{N}}$

$(n)_{n\in\mathbb{N}}$ divergent to $\infty \notin \mathbb{R}$

$\Rightarrow$ $\mathbb{R}$ is not compact

(d) $\quad [c, d] \quad , \; c \leq d \, ,$ compact set.

let $(a_n)_{n\in\mathbb{N}} \subseteq [c,d] \Rightarrow (a_n)_{n\in\mathbb{N}}$ Bounded

$(a_n)_{n\in\mathbb{N}}$ is Bounded $\Rightarrow$ apply Bolzano - weierstrass theorem

$\Downarrow$

Bounded seq^n have at least one accumulation value. $a \in \mathbb{R}$

$[c,d]$ is closed

$\Rightarrow$ we can't leave the set using seq^n

$\Rightarrow$ $a \in [c,d]$

$\Rightarrow$ $[c,d]$ compact set.

Heine-Borel theorem:

for $A \subseteq \mathbb{R}$, we have:

A is compact $\Longleftrightarrow$ A is bounded set and closed.

Lec 15: Series

infinite sum's are also called series.

Series: " infinite sum" , special seq^n.

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{k=1}^{\infty} a_k$$

This is called a series, just adding infinitely many numbers.

Ex: seq^n $(a_k)_{k \in \mathbb{N}} = ((-1)^k)_{k \in \mathbb{N}}$

$$\sum_{k=1}^{\infty} a_k = (-1+1)+(-1+1) \cdots \quad = 0$$

$$= -1 + (1 + -1) + (1-1) \cdots \quad = -1$$

we have 2 different possible results.

$\Rightarrow$ They don't act like an normal sum.

Definition: Let $(a_k)_{k \in \mathbb{N}}$ be a seq^n.

The seq^n $(S_n)_{n \in \mathbb{N}}$ given by

$$S_n := \sum_{k=1}^{n} a_k$$ is called a series.

The whole seq^n given by (S_n) is called a series.

if this seq^n (S_n) is convergent, then we have meaning for sum.

* if $(S_n)_{n \in \mathbb{N}}$ is convergent, we write

$$\sum_{k=1}^{\infty} a_k := \lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{\infty} a_k$$

Example:

$$\left(\sum_{k=1}^{n} (-1)^k \right)$$

$$= (-1, 0, -1, 0, \ldots)$$

not convergent

$$\left(\sum_{k=1}^{n} 1^k \right)_{n \in \mathbb{N}} = (1, 2, 3, \dots)$$

divergent to ∞

Lec 16: Geometric series and Harmonic series

Series: series is just a spectial seq^n

$\sum_{k=1}^{\infty} a_k$ is the seq^n of partial sum's

$$S_n := \sum_{k=1}^{n} a_k$$

Example: Geometric series

$$\sum_{k=0}^{\infty} q^k \quad , \quad q \in \mathbb{R}$$

we show this series is convergent

$\iff \quad |q| < 1$

Question: $S_n = \sum_{k=0}^{n} q^k = ?$

for $q \neq 1$ $\sum_{k=0}^{n} q^k = ?$

$$\Rightarrow \quad S_n (1-q) = (1-q) \sum_{k=0}^{n} q^k$$

$$= \sum_{k=0}^{n} q^k - \sum_{k=0}^{n} q^{k+1}$$

$$= \sum_{k=0}^{n} q^k - \sum_{k=1}^{n+1} q^k$$

$$= q^0 - q^{n+1}$$

$$= 1 - q^{n+1}$$

$$\Rightarrow \quad S_n \cdot (1-q) = 1 - q^{n+1}$$

$$\Rightarrow \quad S_n = \frac{1-q^{n+1}}{1-q}$$

we need to check in which cases (S_n) is a convergent seq^n

$\Rightarrow (S_n)_{n\in\mathbb{N}}$ convergent $\Leftrightarrow (q^n)_{n\in\mathbb{N}}$ is convergent

$\Rightarrow \quad |q| < 1$

$$\lim_{n\to\infty} q^n = 0 \qquad |q| < 1$$

$$\Rightarrow \quad \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-q^n}{1-q}$$

$$= \frac{1}{1-q}$$

$$\boxed{\Rightarrow \sum_{k=0}^{\infty} q^k = \lim_{n\to\infty} S_n = \frac{1}{1-q} \text{ if } |q|<1}$$

geometric series

Ex: Harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty$$

(divergent to ∞)

Proof:

$$S_n = \sum_{k=1}^{n} \frac{1}{k}$$

(seq^n (S_n) is monotonically increasing

Now we need to prove the seq^n (S_n) is not bounded.

$$S_{2^m} = \sum_{k=1}^{2^m} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots + \frac{1}{2^m}$$

$$S_{2^m} = S_1 + (S_2 - S_1) + (S_4 - S_2) + \cdots + (S_{2^m} - S_{2^{m-1}})$$

$$\Rightarrow S_{2^m} = S_1 + (S_2 - S_1) + (S_4 - S_2) + (S_8 - S_4) + (S_{16} - S_8) + \cdots + (S_{2^m} - S_{2^{m-1}})$$

$$= S_1 + \sum_{j=1}^{m} \left(S_{2^j} - S_{2^{j-1}} \right)$$

$$S_{2^j} - S_{2^{j-1}} = \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{k} > \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{2^j}$$

$$\Rightarrow S_{2^j} - S_{2^{j-1}} > 2^{j-1} \cdot \frac{1}{2^j}$$

$$\Rightarrow S_{2^j} - S_{2^{j-1}} > \frac{1}{2}$$

$$\Rightarrow S_{2^m} = S_1 + \sum_{j=1}^{m} \left(S_{2^j} - S_{2^{j-1}} \right)$$

$$> S_1 + m \cdot \frac{1}{2}$$

$$\Rightarrow S_{2^m} > S_1 + m \cdot \frac{1}{2} \xrightarrow{m \to \infty} \infty$$

$\Rightarrow S_n = \sum_{k=1}^{n} \frac{1}{k}$ ist monotonically.

increasing and unbounded from above

$$\boxed{\Rightarrow \quad \lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} \frac{1}{k} = +\infty}$$

Lec 17 : Cauchy criterion

* Criteria we have to check for convergence of series.

Series: Series is just a seq^n of partial sum's

$$S_n = \sum_{k=1}^{\infty} a_k \quad \text{seq}^n \text{ of partial sum's}$$

* This seq^n (S_n) is convergent we can denote as the limit

$$\sum_{k=1}^{\infty} a_k$$

Properties:

if $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent, then:

(a) $\sum_{k=1}^{\infty} (a_k + b_k)$ is also convergent.

$$= \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k = a + b$$

(b) $\sum_{k=1}^{\infty} \lambda a_k$ is also convergent

$$= \lambda \cdot \sum_{k=1}^{\infty} a_k$$

How can we check a given series is convergent?

1 **Cauchy Criterion:**

it uses completeness axiom of $\mathbb{R}$

* for $\mathbb{R}$, being convergent seq^n

$\Longleftrightarrow$ cauchy seq^n

$\Rightarrow$ for Cauchy seq^n we don't have to know about the limit

$\Rightarrow$ therefore this criterion check for Convergence, but does not tell us the limit.

$$\sum_{k=1}^{\infty} a_k \text{ is convergent}$$

$$\Longleftrightarrow \quad \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq m \geq N:$$

$$\left| \sum_{k=m}^{n} a_k \right| < \varepsilon$$

Proof: $S_n := \sum_{k=1}^{n} a_k$, $(S_n)_{n\in\mathbb{N}}$ is convergent

$\underset{\text{Completeness}}{\Longleftrightarrow}$ $(S_n)_{n\in\mathbb{N}}$ is a Cauchy seq^n

$\Longleftrightarrow$ $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ $\forall \tilde{n}, \tilde{m} \geq N$:

$$|S_{\tilde{n}} - S_{\tilde{m}}| < \varepsilon$$

$\Longleftrightarrow$ $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ $\forall n \geq m \geq N$:

$$|S_n - S_{m-1}| < \varepsilon$$

Ex: $\sum_{k=1}^{\infty} (-1)^k$ calculate

$$\left| \sum_{k=m}^{n} (-1)^k \right| \quad ?$$

$$\Rightarrow \left| \sum_{k=N}^{N+2} (-1)^k \right| = \begin{cases} |1 + (-1) + 1| \\ |-1 + 1 - 1| \end{cases} = 1$$

$\Rightarrow$ Not convergent.

Important fact:

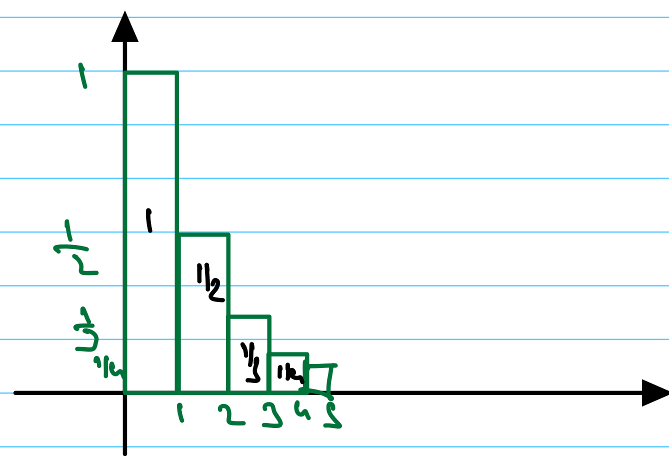
$\sum_{k=1}^{\infty} a_k$ is convergent $\Longrightarrow$ $(a_k)_{k \in \mathbb{N}}$ convergent with $\lim_{k \to \infty} a_k = 0$

Lec 18: Leibniz Criterion

This is the theorem that gives us the sufficient condition for a series to be convergent.

Harmonic series:

$$S_n = \sum_{k=1}^{n} \frac{1}{k}$$



The whole area $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$

$\Rightarrow$ The Harmonic series not converged. but divergent to ∞

Therefore having a seq^n inside the sum that goes to 0 is not sufficient for convergent of the series.

Cauchy Criterion:

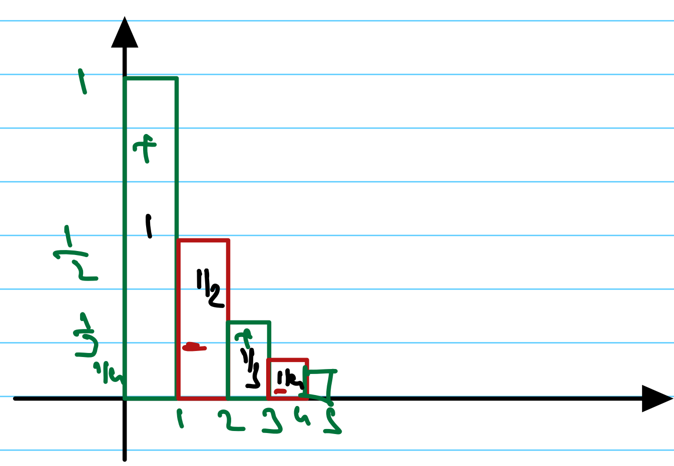
$$\sum_{k=1}^{\infty} a_k \text{ is convergent} \implies \lim_{n\to\infty} a_k = 0$$

$$\overset{\times}{\Longleftarrow}$$

Ex:

$\lim_{k\to\infty} \frac{1}{k} = 0$ but $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent to ∞.

Leibniz Criterion:



$$S_n = \sum_{k=1}^{n} (-1)^{k+1} \cdot \frac{1}{k} \quad (\text{Convergent})$$

Theorem: (Alternating Series test, Leibniz Criterion, Leibniz test)

① Let $(a_k)_{k \in \mathbb{N}}$ be convergent with

$$\lim_{k \to \infty} a_k = 0$$

and monotonically decreasing

Then $\sum_{k=1}^{\infty} (-1)^{\infty} a_k$ is Convergent.

Proof:

$$S_n := \sum_{k=1}^{n} (-1)^k a_k$$

* Since all the a_k are non-negative

i.e $a_k \geq 0 \quad \forall k \in \mathbb{N}$

* and a_k's are monotonically decreasing

* $\lim_{k \to \infty} a_k = 0$

take a subseq^n (even)

$$S_{2l+2} - S_{2l} = \sum_{k=1}^{2l+2} (-1)^k a_k - \sum_{k=1}^{2l} (-1)^k a_k$$

$$= a_{2l+2} - a_{2l+1}$$

$\Rightarrow$ since the seq^n is monotonically decreasing

$$\boxed{S_{2l+2} - S_{2l} = a_{2l+2} - a_{2l+1} \leq 0}$$

$$S_{2l+3} - S_{2l+1} = \sum_{k=1}^{2l+3} (-1)^k a_k - \sum_{k=1}^{2l+1} (-1)^k a_k$$

$$= -a_{2l+3} + a_{2l+2}$$

$$\boxed{\Rightarrow \quad S_{2l+3} - S_{2l+1} = -a_{2l+3} + a_{2l+2} \geq 0}$$

The even subseq^n is monotonically decreasing

The odd subseq^n is monotonically increasing.

$$S_{2l+1} - S_{2l} = -a_{2l+1} \leq 0 \Rightarrow S_{2l+1} \leq S_{2l}$$

$S_{2l+1} \leq S_{2l}$ (this is usefue when we compare sub seq^n)

$(S_{2l+1})_{l \in \mathbb{N}}$ monotonically increasing sub seq^n

$\Rightarrow$ $S_1 \leq S_3 \leq S_5 \leq S_7 \ldots$

$(S_{2l})_{l \in \mathbb{N}}$ monotonically decreasing sub seq^n

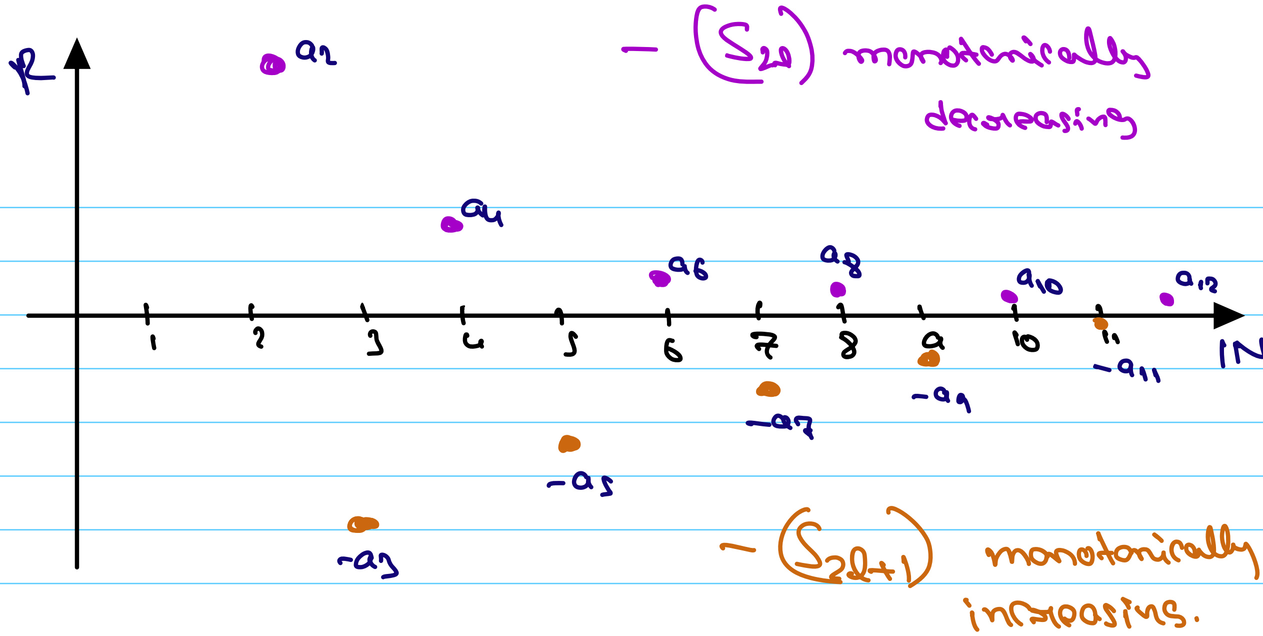
$S_2 \geq S_4 \geq S_6 \geq \ldots$

$\Rightarrow$ $S_3 \leq S_{2l+1} \leq S_{2l} \leq S_2$

(Bounded)

Both the subseq^n are monotonic & bounded

$\Rightarrow$ hence both the seq^n are converges



$$\lim_{l\to\infty}\left(S_{2l+1}-S_{2l}\right)=\lim_{l\to\infty}\left(-a_{2l+1}\right)=0$$

$$\Rightarrow\quad S:=\lim_{l\to\infty} S_{2l+1}=\lim_{l\to\infty} S_{2l}$$

$$\Rightarrow\quad \lim_{n\to\infty} S_n=S \qquad \text{(convergent)}$$

Lec 19 Comparison test:

* Comparison test is a Criterion used to decide if a given series is convergent or not.

* we need a new definition about convergence

$\sum_{k=1}^{\infty} a_k$ is called **absolutely convergent**

The series $\sum_{k=1}^{\infty} a_k$ is called absolutely convergent if the series given by the absolute value of a_k is simple convergent.

$\Rightarrow$ $\sum_{k=1}^{\infty} a_k$ is convergent if $\sum_{k=1}^{\infty} |a_k|$ is convergent.

$\Rightarrow$ absoutely convergent

$\Longrightarrow$ convergent.

$\sum_{k=1}^{\infty} a_k$ is absolute convergent

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

for all $\forall n \geq m \geq N$:

$$\left| \sum_{k=m}^{n} |a_k| \right| < \varepsilon$$

(Cauchy criterion)

$$\Rightarrow \sum_{k=m}^{n} |a_k| < \varepsilon$$

$$\Rightarrow \overbrace{\left| \sum_{k=m}^{n} a_k \right| \leq \sum_{k=m}^{n} |a_k|}^{\text{triangular inequality}} < \varepsilon$$

$$\Rightarrow \left| \sum_{k=m}^{n} a_k \right| < \varepsilon$$

(Cauchy criterion for $\underbrace{\sum_{k=1}^{\infty} a_k}_{\text{original series}}$)

Example:

* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ (Not convergent)

but * $a_k \geq 0 \quad \forall k \in \mathbb{N}$

* monotonically decreasing

* $\lim_{n\to\infty} a_n = 0$

$\Longrightarrow$ $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ is convergent using (Leibniz criterion)

But $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ is not absoutely convergent.

$\Rightarrow$ absolute convergent $\Longrightarrow$ convergent

absolute convergent $\not\Longleftarrow$ convergent

$\Uparrow$

Stronger notion of convergence

This is exactly we need for comparison test.

Majorant Criterion:

let $\sum_{k=1}^{\infty} a_k$ be a series

if there is $n_0 \in \mathbb{N}$ and a convergent series $\sum_{k=1}^{\infty} b_k$ with $b_k \geq 0 \ \forall k$

$$|a_k| \leq b_k \quad \forall k \geq n_0$$

we have another seq^n b_k that is convergent.

$\Rightarrow$ then we can conclude the series $\sum_{k=1}^{\infty} a_k$ is absoutely convergent.

Proof: apply cauchy criterion to $\sum_{k=1}^{\infty} b_k$

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq m > N$

we have
$$\left| \sum_{k=m}^{n} b_k \right| < \varepsilon$$

if we choose $N > n_0$ then

$$\sum_{k=m}^{n} |a_k| \leq \sum_{k=m}^{n} b_k = \left|\sum_{k=m}^{n} b_k\right| < \varepsilon$$

$$\Rightarrow \sum_{k=m}^{n} |a_k| < \varepsilon$$

$\Rightarrow \sum_{k=1}^{\infty} a_k$ is absolute convergent.

Minorant Criterion:

It works similary but we don't check for convergence, but for divergence.

let $\sum_{k=1}^{\infty} a_k$ be a series with $a_k \geq 0$

if there is $n_0 \in \mathbb{N}$ and a divergent series

$$\sum_{k=1}^{\infty} b_k \quad \text{with} \quad b_k \geq 0 \quad \text{and with}$$

$$a_k \geq b_k \quad \forall\, k \geq n_0 \quad , \text{then}$$

$$\sum_{k=1}^{\infty} a_k \quad \text{is} \quad \text{divergent.}$$

Lec 20: Ratio and Root test

Both Ratio and Root in the best case scenario they answer us if a given series is absolutely convergent.

$$\sum_{k=1}^{\infty} a_k \quad \text{absolutely convergent?}$$

if we have a majorant

There is a convergent majorant

$$\Downarrow$$

$$\sum_{k=1}^{\infty} a_k \quad \text{is} \quad \text{absolutely convergent.}$$

we know that geometric series

$$\sum_{k=0}^{\infty} q^k \text{ convergent} \iff |q| < 1$$

<u>FACT</u>:

if there is $C, n_0 \in \mathbb{N}$ and $q \in \mathbb{R}$ with $|q| < 1$ such that $|a_k| \leq C q^k \quad \forall k \geq n_0$

then $\sum_{k=1}^{\infty} a_k$ is absoulte convergent.

<u>Ratio test:</u> if there is $n_0 \in \mathbb{N}$ and $q \in [0, 1)$ such that

$$\left| \frac{a_{k+1}}{a_k} \right| \leq q$$

Then $\sum_{k=1}^{\infty} a_k$ is absoulte convergent.

$\forall k \geq n_0$, then $\sum_{k=1}^{\infty} a_k$ is absoulte convergent.

Proof: $$|a_{k+1}| \le q \cdot |a_k| \le q \cdot q \cdot |a_{k-1}|$$

$$\Rightarrow |a_{k+1}| \le q^{k+1-n_0} |a_{n_0}|$$

$$\Rightarrow |a_{k+1}| \le q^{k+1} \cdot \frac{|a_{n_0}|}{q^{n_0}}$$

$$\Rightarrow |a_{k+1}| \le q^{k+1} \cdot C$$

Root Test:

if there is $n_0 \in \mathbb{N}$ and $q \in (0,1)$
such that

$$\sqrt[k]{(|a_k|)} \le q \qquad \forall k \ge n_0$$

then $\sum_{k=1}^{\infty} a_k$ is abs. convergent.

Proof:

$$\sqrt[k]{|a_k|} \le q$$

$$\Longleftrightarrow |a_k| \le q^k$$

$$\Rightarrow \text{convergent.}$$

Rember:

$$\limsup_{k\to\infty} \sqrt[k]{|a_k|} < 1$$

$$\Longrightarrow \sum_{k=1}^{\infty} a_k \text{ is abs. convergent.}$$

$$\limsup_{k\to\infty} \sqrt[k]{|a_k|} > 1 \Longrightarrow \sum_{k=1}^{\infty} a_k \text{ is divergent.}$$

Geometric series

$$\sum_{k=0}^{\infty} q^k \quad , \quad q \in \mathbb{R}$$

we show $\sum_{k=0}^{\infty} q^k$ convergent $\Leftrightarrow |q|<1$

Question:

$$S_n = \sum_{k=0}^{n} q^k = ?$$

for $q \neq 1$: $\quad S_n = \sum_{k=0}^{n} q^k$

$$\Rightarrow \quad (1-q) S_n = (1-q) \sum_{k=0}^{n} q^k$$

$$= \sum_{k=0}^{n} q^k - \sum_{k=0}^{n} q^{k+1}$$

$$= \sum_{k=0}^{n} q^k - \sum_{k=1}^{n+1} q^k$$

$$= q^0 - q^{n+1}$$

$$\Rightarrow \quad S_n = \frac{1-q^{n+1}}{1-q}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-q^{n+1}}{1-q}$$

$$= \frac{1}{1-q}\left(1 - \underbrace{\lim_{n\to\infty} q^{n+1}}_{=0 \text{ if } |q|<1}\right)$$

$$= \frac{1}{1-q} \quad \text{if} \quad |q|<1$$

Geometric Series

$$\boxed{\Rightarrow \sum_{k=0}^{\infty} q^k = \lim_{n\to\infty} \sum_{k=0}^{n} q^k = \lim_{n\to\infty} S_n = \frac{1}{1-q} \quad \text{if } |q|<1}$$

$(S_n)_{n\in\mathbb{N}}$ is convergent $\Longleftrightarrow$ $(q^n)_{n\in\mathbb{N}}$ convergent to 0

$\Longleftrightarrow |q|<1$

Harmonic Series

$$\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty$$

(divergent to ∞)

Proof:

$$S_n = \sum_{k=1}^{n} \frac{1}{k}$$

$(S_n)_{n\in\mathbb{N}}$ is monotonically increasing

$\Rightarrow$ either $\lim_{n\to\infty} S_n \in \mathbb{R}$ (converges)

or $= +\infty$ (diverges to $+\infty$)

we need to show that $(S_n)_{n\in\mathbb{N}}$ is not bounded to show that S_n diverges to $+\infty$.

* Let's take a sub seq^n of (S_n)

$$S_{2^m} = \sum_{k=1}^{2^m} \frac{1}{k}$$

$$S_{2^m} = S_1 + (S_2 - S_1) + (S_4 - S_2) + (S_8 - S_4) + \cdots + (S_{2^m} - S_{2^{m-1}})$$

all all the term's cancel out each other and only left with S_{2^m}

$\Rightarrow$

$$S_{2^m} = S_1 + (S_2 - S_1) + (S_4 - S_2) + \cdots + (S_{2^m} - S_{2^{m-1}})$$

$$S_{2^m} = S_1 + \sum_{j=1}^{m} (S_{2^j} - S_{2^{j-1}})$$

$$S_{2^j} - S_{2^{j-1}} = \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{k} > \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{2^j}$$

$$\Rightarrow \quad S_{2^j} - S_{2^{j-1}} > \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{2^j}$$

$$> 2^{j-1} \cdot \frac{1}{2^j} = \frac{1}{2}$$

$$\Rightarrow \quad S_{2j} - S_{2j-1} = \frac{1}{2}$$

$$\Rightarrow \quad S_{2m} = S_1 + \sum_{j=1}^{m} \left(S_{2j} - S_{2j-1} \right)$$

$$\Rightarrow \quad S_{2m} > S_1 + \frac{m}{2}$$

$$\Rightarrow \quad \lim_{m \to \infty} S_{2m} \geq \lim_{m \to \infty} S_1 + \frac{m}{2}$$

$$\Rightarrow \quad \lim_{m \to \infty} S_{2m} = +\infty$$

Hence the Harmonic series

$$\left(S_n \right)_{n \in \mathbb{N}} = \left(\sum_{k=1}^{n} \frac{1}{k} \right)_{n \in \mathbb{N}}$$

is divergent to $+\infty$

Harmonic series

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k} = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k} = +\infty}$$

Cauchy Criterion:

we say a series $\sum a_n$ satisfies the Cauchy Criterion, if its seq^n (S_n) the Partial sum's is a Cauchy Seq^n

$$\sum_{k=1}^{\infty} a_k \quad \text{is} \quad \text{Convergent}$$

$$\Updownarrow$$

$\forall \varepsilon > 0, \ \exists N \in \mathbb{N}$ s.t $\forall n, m \geq N$

$$|S_n - S_m| < \varepsilon$$

let's restric to $n \geq m$ such that

$$|S_n - S_{m-1}| < \varepsilon$$

$$\Rightarrow \left| \sum_{k=m}^{n} a_k \right| < \varepsilon$$

$\Rightarrow$ we can rewrite it as

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq m > N$

we have

$$\left| \sum_{k=m}^{n} a_k \right| < \varepsilon$$

A series converges $\Longleftrightarrow$ if it satisfy

Cauchy criterion.

Since the series converge it should

hold for $n = m$

$$\Rightarrow \quad |S_n - S_{m-1}| < \varepsilon$$

$$\Rightarrow \quad |S_n - S_{n-1}| < \varepsilon$$

$$\Rightarrow \quad |a_n| < \varepsilon \quad ((a_n) \text{ converges to } 0)$$

$\sum_{k=1}^{\infty} a_n$ converges $\Longrightarrow$ $(a_n)_{n \in \mathbb{N}}$ converges

$\cancel{\Longleftarrow}$ with $\lim_{n \to \infty} a_n = 0$

This is not equivalance

Leibniz Criterion:

Ex: Harmonic series: $S_n = \sum_{k=1}^{n} \frac{1}{k}$ diverges to $+\infty$

Theorem: (Alternating test, Leibniz Criterion)

① Let $(a_k)_{k\in\mathbb{N}}$ be convergent with $\lim_{n\to\infty} a_k = 0$

② $(a_k)_{k\in\mathbb{N}}$ is monotonically decreasing seq^n

$\Longrightarrow$ Then the alternating series

$$S_n := \sum_{k=1}^{\infty} (-1)^k a_k \text{ is convergent.}$$

Proof: $S_n = \sum_{k=1}^{n} (-1)^k a_k$

we know $(a_k)_n$ is monotonically decreasing and the limit $=0$ $\Rightarrow$ $a_k \geq 0$ $\forall k\in\mathbb{N}$

let's take subseque of even and odd

i.e $(S_{2l})_{l\in\mathbb{N}}$ $(S_{2l+1})_{l\in\mathbb{N}}$

$S_{2l+2} - S_{2l} = a_{2l+2} - a_{2l+1}$
$(a_{2l+1} \geq a_{2l+2})$

$S_{2l+2} - S_{2l} \leq 0$

$\Rightarrow S_{2l+2} \leq S_{2l}$

$\Rightarrow (S_{2l+2})_{l\in\mathbb{N}}$ is monotonically decreasing seq

$S_{2l+3} - S_{2l+1} = -a_{2l+3} + a_{2l+2}$
$(a_{2l+2} \leq a_{2l+3})$

≥ 0

$\Rightarrow S_{2l+3} \geq S_{2l+1}$

$(S_{2l+1})_{l\in\mathbb{N}}$ is monotonically increasing seq^n

$$S_{2l+1} - S_{2l} = -a_{2l+1} \leq 0$$

$$\Rightarrow \quad S_{2l+1} \leq S_{2l}$$

(decreasing) (increasing)

$$\Rightarrow \quad S_3 \leq S_{2l+1} \leq S_{2l} \leq S_2$$

(bounded) both subseq

$$\lim_{l\to\infty} (S_{2l+1} - S_{2l}) = 0$$

$$\Rightarrow \lim_{l\to\infty} (-a_{2l+1}) = \lim_{l\to\infty} -\frac{1}{k} = 0$$

$$\Rightarrow \lim_{l\to\infty} S_{2l+1} = \lim_{l\to\infty} S_{2l}$$

$$\Rightarrow \quad S := \lim_{l\to\infty} S_{2l+1} = \lim_{l\to\infty} S_{2l}$$

$$\Rightarrow \quad \lim_{n\to\infty} S_n = S \quad \text{(converged)}$$

Ex: $$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}} = \sum_{k=1}^{\infty} (-1)^k \cdot a_k$$

1. $a_k = \frac{1}{\sqrt{k}}$ $\quad (a_k)_{k \in \mathbb{N}}$ monotonically decreasing

2. $\lim\limits_{n \to \infty} a_k = \lim\limits_{n \to \infty} \frac{1}{\sqrt{k}} = 0$

3. $a_k \geq 0 \quad \forall k \in \mathbb{N}$

Let's apply Leibniz Criterion.

$\Rightarrow$ if the above 3 satisfies then the series $\sum_{k=1}^{\infty} (-1)^k a_k$ converges

$\left(\Rightarrow \sum_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k}} \text{ converges} \right)$

Comparison Test:

A new definition for Convergence.

$\sum_{k=1}^{\infty} a_k$ is called absolutely convergent

if $\sum_{k=1}^{\infty} |a_k|$ is convergent.

* for a series that has only positive a_k's i.e $a_k \geq 0 \ \forall k$, then this not a new definition.

However in general its a stronger convergence.

absoulty convergent $\Longrightarrow$ convergent
$\;\;\;\;\not\Longleftarrow$

$\Downarrow$

Stronger notion of convergence

Proof: $\sum_{k=1}^{\infty} |a_k|$ is convergent $\Longrightarrow$

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that

$\forall\, n \geq m > N$ we have

$$\left| \sum_{k=m}^{n} |a_k| \right| < \varepsilon$$

$$\Longrightarrow \quad \sum_{k=m}^{n} |a_k| < \varepsilon$$

$$\Longrightarrow \quad \underbrace{\left| \sum_{k=m}^{n} a_k \right| \leq \sum_{k=m}^{n} |a_k|}_{\text{triangle inequality}} < \varepsilon$$

$$\Longrightarrow \quad \left| \sum_{k=m}^{n} a_k \right| < \varepsilon$$

$\sum_{n=1}^{\infty} a_k$ is convergent.

Counter example:

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$$ is convergent according to Leibniz criterion.

However $$\sum_{k=1}^{\infty} \left| (-1)^{k} \frac{1}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k}$$ is not convergent

i.e Convergence $\not\Longrightarrow$ absolute convergence

Majorant Criterion:

Let $\sum_{k=1}^{\infty} a_k$ be a series.

if there is $n_0 \geq N$ and a convergent series $\sum_{k=1}^{\infty} b_k$ with $b_k \geq 0$ $\forall k$, then

if $|a_k| \leq b_k$ $\forall k \geq n_0$

then $\sum_{k=1}^{\infty} a_k$ is abs. convergent.

Proof: Apply Cauchy criterion to

$$\sum_{k=1}^{\infty} b_k$$

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that

$\forall n \geq m > N$ we have

$$\left| \sum_{k=m}^{n} b_k \right| < \varepsilon$$

we know that

$$|a_k| \leq b_k \quad \forall k \geq n_0$$

$$\Rightarrow \quad \sum_{k=n_0}^{n} |a_k| \leq b_k$$

$\Rightarrow \forall \varepsilon > 0, \exists N \geq n_0,$ such that

$\forall n \geq m > N$ we have

$$\left| \sum_{k=m}^{n} b_k \right| < \varepsilon$$

Since $b_k \geq 0 \quad \forall k \in \mathbb{N}$

$$\sum_{k=m}^{n} b_k < \varepsilon$$

$$\Rightarrow \sum_{k=m}^{n} |a_k| \leq \sum_{k=m}^{n} b_k < \varepsilon$$

$$\Rightarrow \sum_{k=m}^{n} |a_k| < \varepsilon$$

$\Rightarrow |a_k|$ is convergent

$\Rightarrow a_k$ is absoulte convergent.

Minorant Criterion:

let $\sum_{k=1}^{\infty} a_k$ be a series with $a_k \geq 0$

if there is $n_0 \in \mathbb{N}$ and a divergent series $\sum_{k=1}^{\infty} b_k$ with $b_k \geq 0$

and with $a_k \geq b_k \quad \forall k \geq n_0$,

then $\sum_{k=1}^{\infty} a_k$ is divergent.

Example:

$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ convergent or divergent?

Sol:

$\forall k > 0 \ \& \in \mathbb{N} \qquad \sqrt{k} < k$

$$\Rightarrow \frac{1}{\sqrt{k}} > \frac{1}{k} \quad \forall k > 0$$

we know the $\sum_{k=1}^{\infty} \frac{1}{k} = +\infty$ divergent

$\Rightarrow$ therefore $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ is divergent

$\sum_{k=1}^{\infty} a_k$ absolutely Convergent?

There is a convergent majorant

$\Downarrow$

$\sum_{k=1}^{\infty} a_k$ is absolutely Convergent

$\Downarrow$

$\sum_{k=1}^{\infty} a_k$ is Convergent

$\Rightarrow$ we need to find a Convergent majorant series to prove the convergence of $\sum_{k=1}^{\infty} a_k$

$\Rightarrow$ why don't we take

geometric series as convergent majorant.

$\Rightarrow$ if there is a $n_0 \in \mathbb{N}$ and $q \in \mathbb{R}$ and $|q| < 1$ such that $|a_k| \leq C q^k$ $\forall k \geq n_0$ (convergent majorant)

$\Downarrow$

$\sum_{k=1}^{\infty} a_k$ is absolutely convergent.

<u>RATIO test:</u> if there is $n_0 \in \mathbb{N}$ and $q \in [0,1)$ such that

$a_k \neq 0$ $$\left| \frac{a_{k+1}}{a_k} \right| \leq q \qquad \forall k \geq n_0$$

Then $\sum_{k=1}^{\infty} a_k$ is abs convergent.

Proof:

$$|a_{k+1}| \leq q|a_k|$$

$$\Rightarrow |a_{k+1}| \leq q \cdot q \, |a_{k-1}|$$

$$\Rightarrow |a_{k+1}| \leq q^{k+1-n_0} \, |a_{n_0}|$$

$$\Rightarrow |a_{k+1}| \leq q^{k+1} \cdot \frac{|a_{n_0}|}{q^{n_0}}$$

$$\Rightarrow |a_{k+1}| \leq C \cdot q^{k+1}$$

(this satisfies the majorant criterion)

$\Rightarrow \sum_{k=1}^{\infty} a_k$ is absolutely convergent.

Ex: $\sum_{k=1}^{\infty} \frac{1}{k!}$ Convergent?

Sol^n

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{k!}{(k+1)!}\right| = \frac{1}{k+1} \leq \frac{1}{2} \quad \forall k \geq 1$$

$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k!}$ is convergent.

Lec 22 Cauchy Product

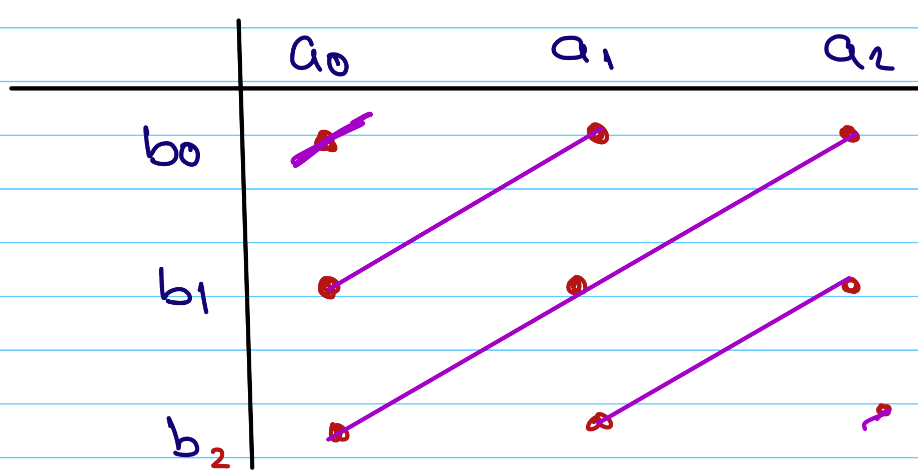
Cauchy Product for series

$$\sum_{k=0}^{\infty} a_k , \quad \sum_{k=0}^{\infty} b_k \quad \xrightarrow{\text{How to multiply}} \quad \sum_{k=0}^{\infty} c_k$$

for finite sum's :-

$$(a_0 + a_1 + a_2)(b_0 + b_1 + b_2)$$

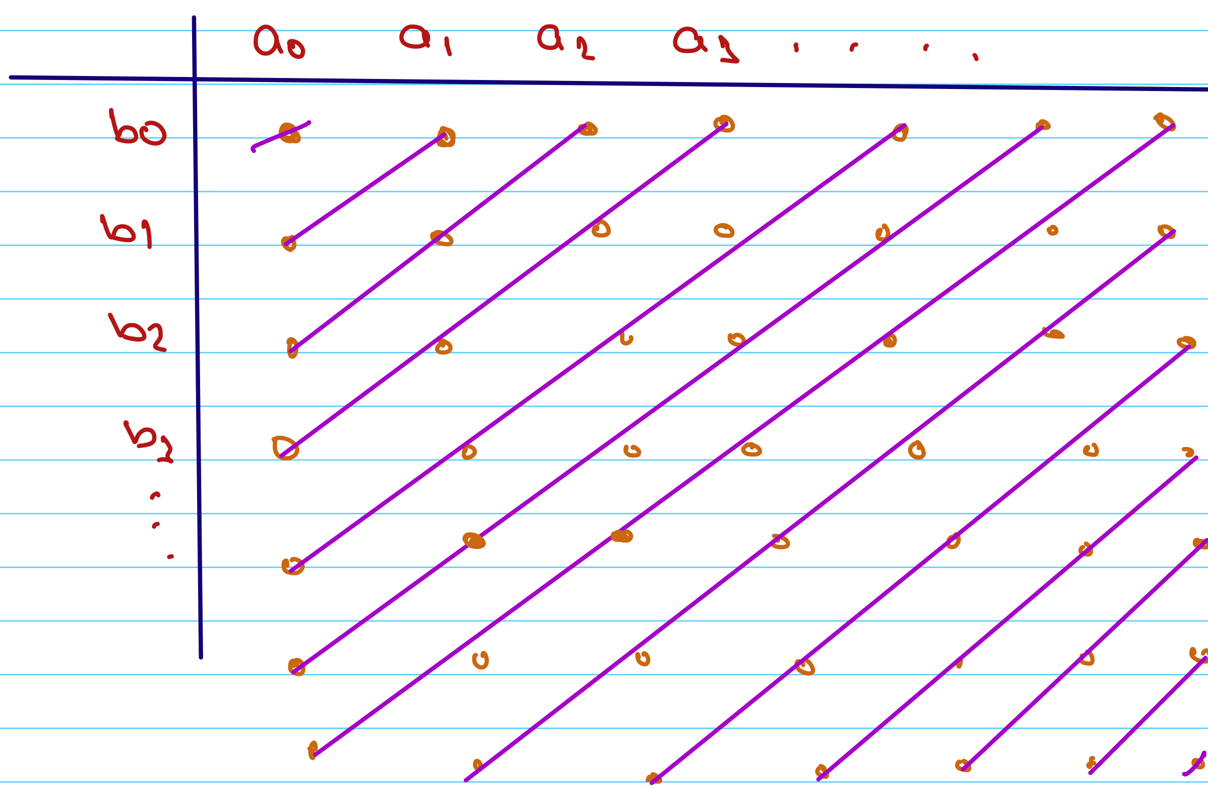
$$\Rightarrow \quad a_0 b_0 + a_1 b_0 + a_2 b_0 +$$
$$a_0 b_1 + a_1 b_1 + a_2 b_1 +$$
$$a_0 b_2 + a_1 b_2 + a_2 b_2$$



$$= a_0b_0 + (a_1b_0 + a_0b_1)$$
$$+ (a_2b_0 + a_1b_1 + a_0b_2)$$
$$+ (a_2b_1 + a_1b_2) + (a_2b_2)$$

<u>Cauchy Product</u>: for two series

$$\sum_{k=0}^{\infty} a_k, \quad \sum_{k=0}^{\infty} b_k$$



for two serie $\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n$ Cauchy Product

$\sum_{k=0}^{\infty} c_k$ with $c_k = \sum_{l=0}^{k} a_l b_{k-l}$ is called

the Cauchy Product

Can we tell anything about convergence of this series?

Theorem:

when we have two convergent series $\sum_{k=0}^{\infty} a_k, \sum_{k=0}^{\infty} b_k$ and atleast one of them is absoultely convergent, then the Series $\sum_{k=0}^{\infty} c_k$, $c_k = \sum_{l=0}^{k} a_l b_{k-l}$ is also absoolutely convergent.

$$\sum_{k=0}^{\infty} c_k = \left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right)$$

formal theorem:

if $\sum_{k=0}^{\infty} a_k$ is absoultely convergent and $\sum_{k=0}^{\infty} b_k$ is convergent, then Cauchy Product $\sum_{k=0}^{\infty} c_k$ is absoutely convergent.

and $$\sum_{k=0}^{\infty} c_k = \left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right)$$

Ex!

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{for } x \in \mathbb{R}$$

(abs. convergent by the ratio test)

$$e^x := \sum_{k=0}^{\infty} \frac{x^k}{x!}$$

apply Cauchy Product to e^x, e^y

$$e^x \cdot e^y = \sum_{k=0}^{\infty} \frac{x^k}{k!} \cdot \sum_{k=0}^{\infty} \frac{y^k}{k!}$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{x^l}{l!} \cdot \frac{y^{k-l}}{(k-l)!}$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{1}{k!} \frac{k!}{l!\,(k-l)!} x^l \cdot y^{k-l}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{l=0}^{k} \binom{k}{l} x^l \cdot y^{k-l}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \cdot (x+y)^k$$

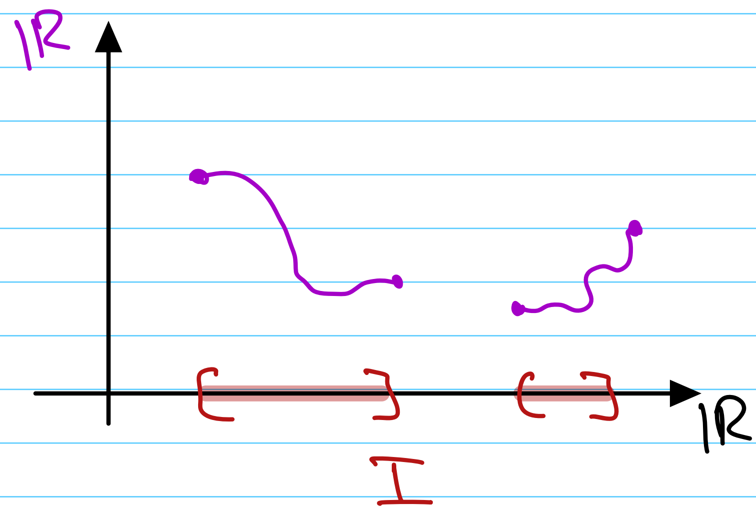
$$= \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!} = e^{x+y}$$

$$\Rightarrow \quad e^x \cdot e^y = e^{x+y}$$

Lec 23 : Seq^n of Function's

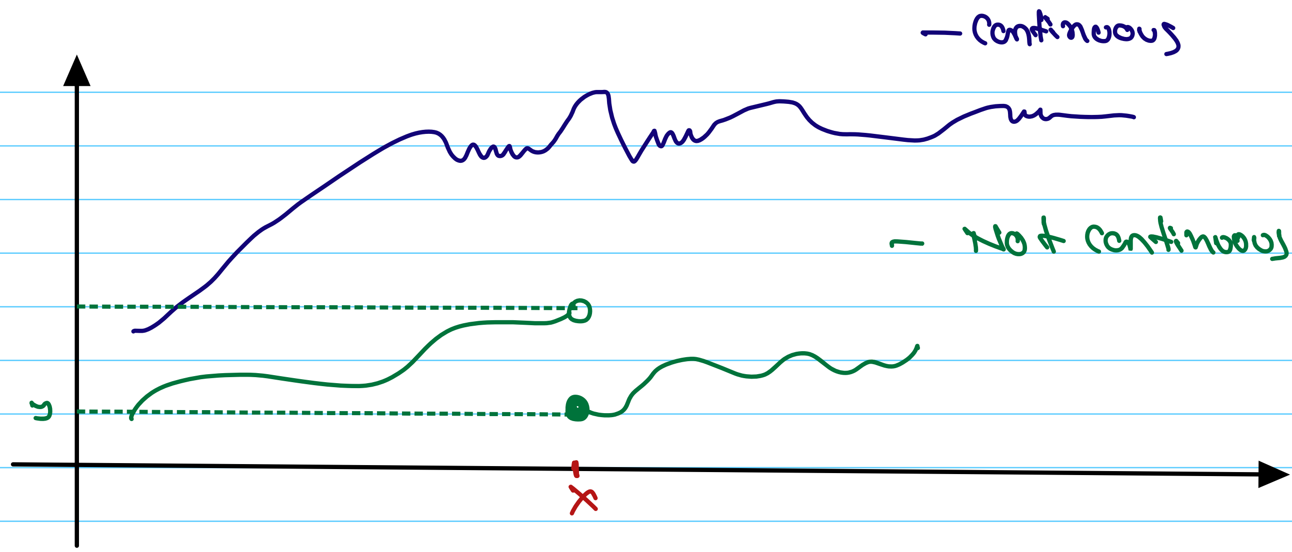
Function :- $f: I \to \mathbb{R}$

$(I \subseteq \mathbb{R})$



Later: Continuous function $f: \mathbb{R} \to \mathbb{R}$

* Continuous function is a function whose graph has no Jumps, we don't need any smoothness, But for the function defined on whole number line, we have a Connected graph.



Idea: small errors on x-axis $\rightsquigarrow$ small error's on y-axis

Bounded function:

$f : I \longrightarrow \mathbb{R}$ is called a

Bounded function if, range of the function is Bounded i.e

$$\{f(x) \mid x \in I\} = \text{Range}(f)$$
$$= f(I)$$

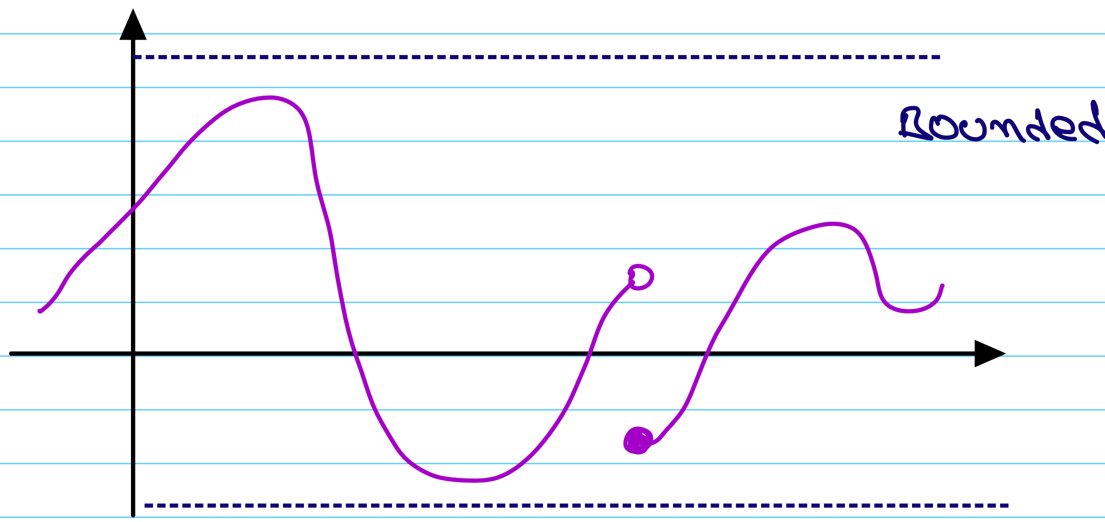
is a bounded set in $\mathbb{R}$

$$\Longleftrightarrow \sup_{x \in I} |f(x)| < \infty$$

Therefore, equivalently we can say supremum and inf of the set Range is finite

or equivalenty we could say, the sup of abs value is finite

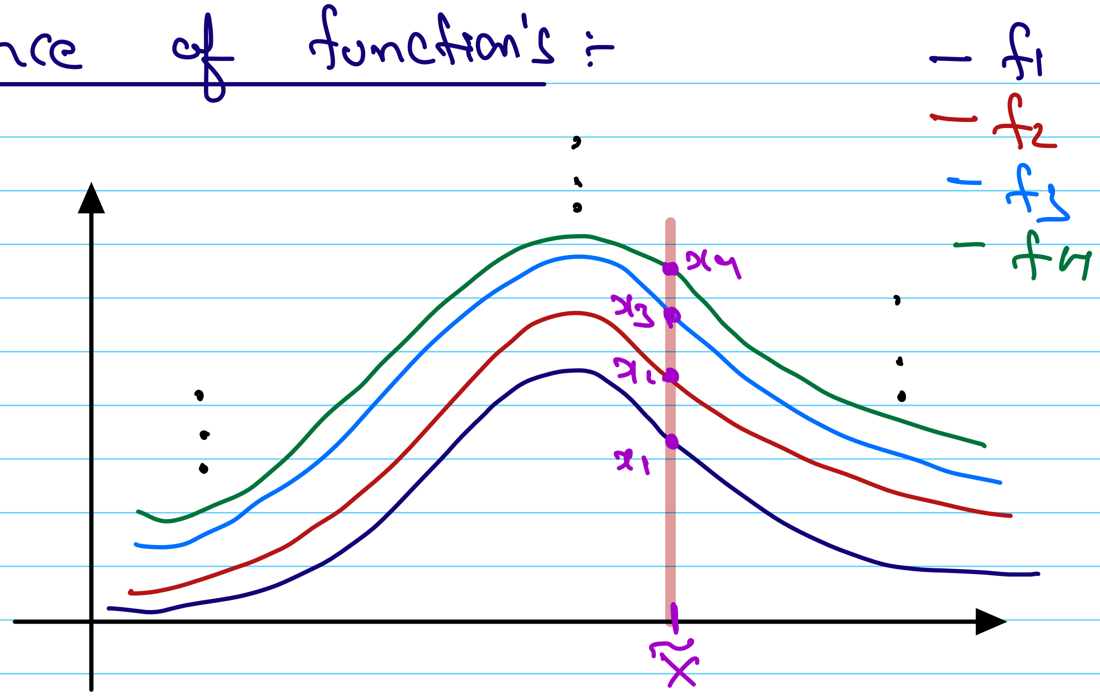
$$\Longleftrightarrow \sup_{x \in I} |f(x)| < \infty$$



This is a straight forward generalization of bounded seq^n.

combine both concepts.

Sequence of function's :-



Sequence of function's :

$$(f_1, f_2, f_3, f_4, \dots)$$

with sequence member's

$$f_1 : I \longrightarrow \mathbb{R}$$

$$f_2 : I \longrightarrow \mathbb{R}$$

$$f_3 : I \longrightarrow \mathbb{R}$$

$$f_4 : I \longrightarrow \mathbb{R}$$

All the function's in seq^n have same domain I

for a fixed $\tilde{x} \in I$, we get an ordinary seq of real number's

$$\left(f_1(\tilde{x}), f_2(\tilde{x}), f_3(\tilde{x}) \dots f_n(\tilde{x}) \dots \right)$$

$\Rightarrow$ we could ask whether this seq^n is bounded? or is it monotonically increasing? or is it even converging?

$\Rightarrow$ However we can ask more. all of these question depend on $\tilde{x}$.

$\Rightarrow$ so we immediatly see we have infinitely many seq^n here

$\Rightarrow$ infinitely many Question's about convergence

$\Rightarrow$ All these Question's transulate to the Question this seq^n of function's $(f_1, f_2, f_3, \dots \quad)$ in some sense

Convergence.

$\Rightarrow$ in order to answer this, we need to define what is the distance b/w two function's.

Lec 24: Pointwise Convergence

Seq of functions: $(f_1, f_2, f_3, f_4, \dots)$

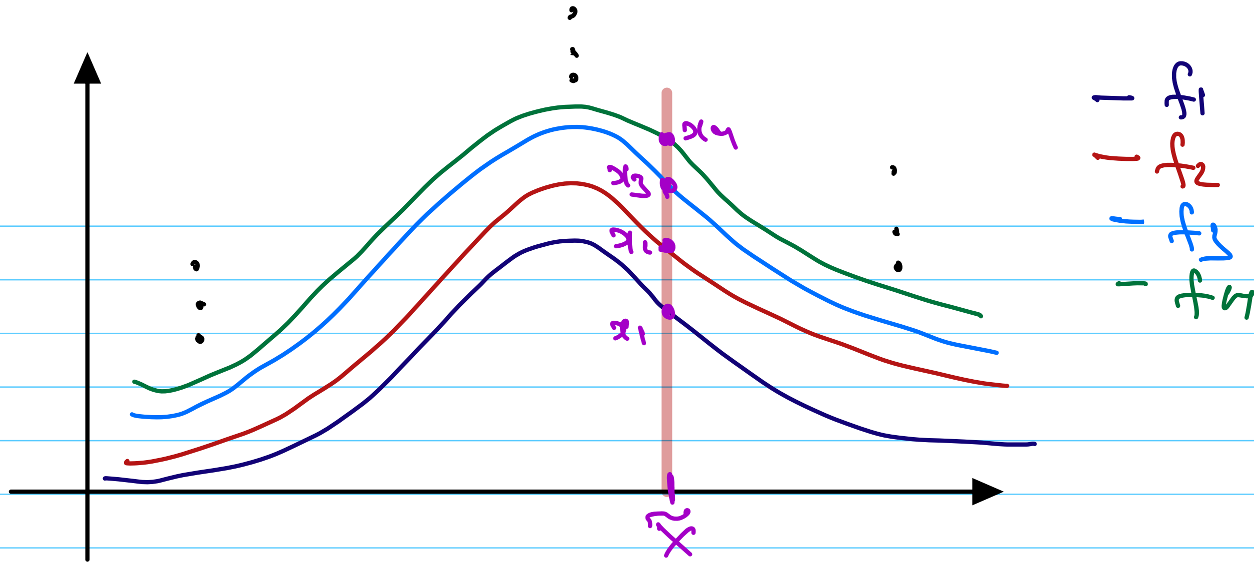
$$f_n : I \longrightarrow \mathbb{R}$$

Pointwise convergence:

$(f_1, f_2, f_3, f_4, \dots)$ is pointwisely convergent to a function $f : I \longrightarrow \mathbb{R}$ if for all $\tilde{x} \in I$:

$(f_1(\tilde{x}), f_2(\tilde{x}), f_3(\tilde{x}), \dots)$

is convergent to $f(\tilde{x})$

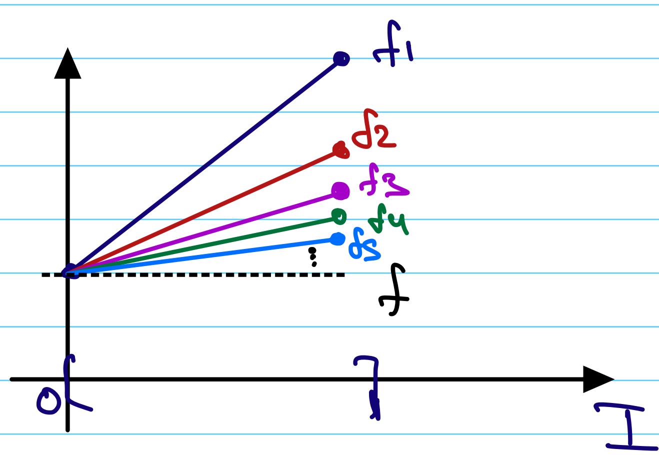


$$\left[\ \forall \tilde{x} \in I,\ \forall \varepsilon > 0\ \ \exists N \in \mathbb{N},\ \forall n \geq N: \quad |f_n(\tilde{x}) - f(\tilde{x})| < \varepsilon \ \right]$$

Example:

$$f_n : [0,1] \longrightarrow \mathbb{R}$$

$$f_n(x) = \frac{1}{n}x + 1$$



for $\tilde{x} \in [0,1]$: $f_n(\tilde{x}) = \frac{1}{n}\tilde{x} + 1$

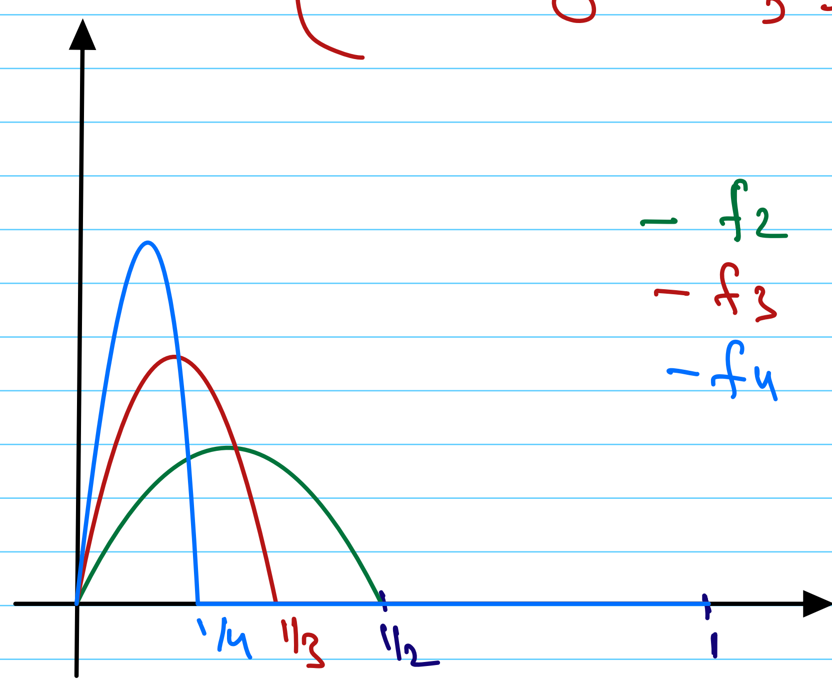
$$f_n(\tilde{x}) = \frac{1}{n}\tilde{x} + 1 \xrightarrow{n \to \infty} 1$$

$\Longrightarrow$ (Pointwise) limit function

$$f : [0,1] \longrightarrow \mathbb{R}, \quad f(x) = 1$$

Example 2 $\quad f_n : [0,1] \longrightarrow \mathbb{R}$,

$$f_n(x) = \begin{cases} n^2(1-nx) & , x \in [0, \frac{1}{n}) \\ 0 & , x \in (\frac{1}{n}, 1] \end{cases}$$



Do we have Pointwise continuous?

for $x=0$: $f_n(x)=0 \quad \forall n \in \mathbb{N}$

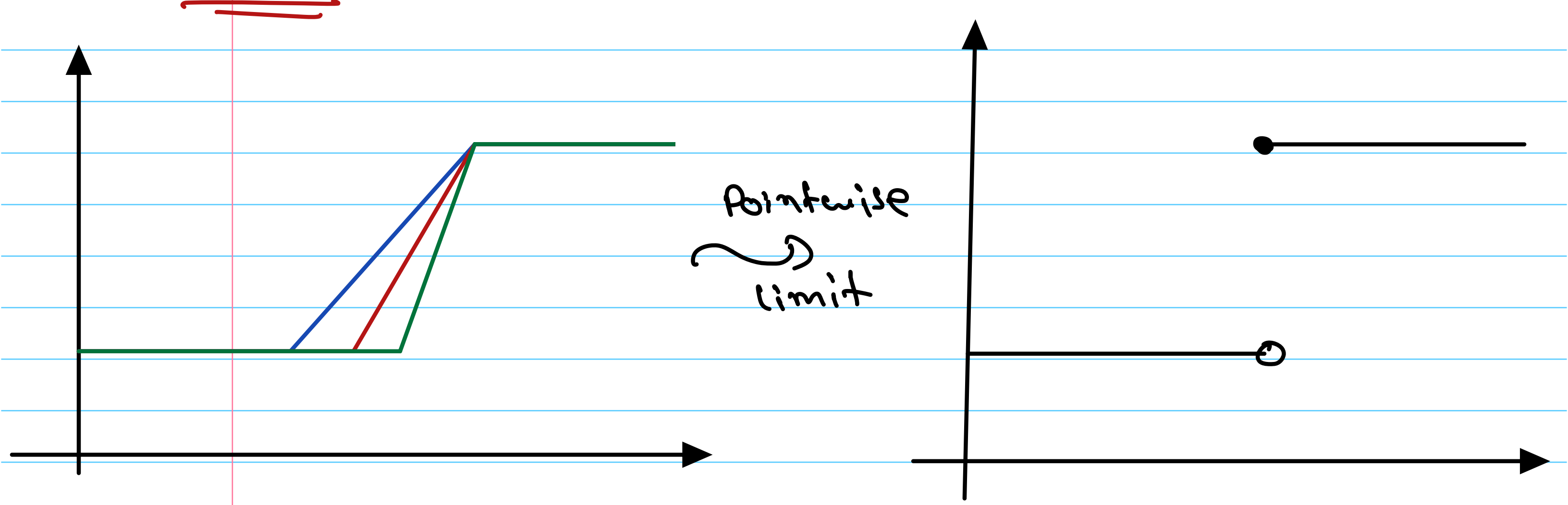
for $x>0$: $f_n(x) = 0$ for all $n > \frac{1}{x}$

$\Rightarrow$ (Pointwise) Limit function

$f:[0,1] \longrightarrow \mathbb{R} \quad , f(x)=0$

Poitwise convergence is not strong enough for our need's

Example:



* we get a function with a Jump, although we stored the whole seq^n without Jump's

That's indeed a good reason why we want a stricter convergence property.

Lec 25 : Uniform Convergence

$(f_1, f_2, f_3, f_4, \dots)$ is pointwise convergent to $f : I \longrightarrow \mathbb{R}$ if

$$\forall x \in I, \quad \forall \varepsilon > 0, \quad \exists N \in \mathbb{N} : |f_n(x) - f(x)| < \varepsilon$$

* with this formolation with the quantifier's we are allowed to choose different $N \in \mathbb{N}$ dependending on x.

* This is the only thing we change to get the uniform convergence.

Def: $(f_1, f_2, f_3, f_4, \ldots)$ is uniformly convergent to $f: I \to \mathbb{R}$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$

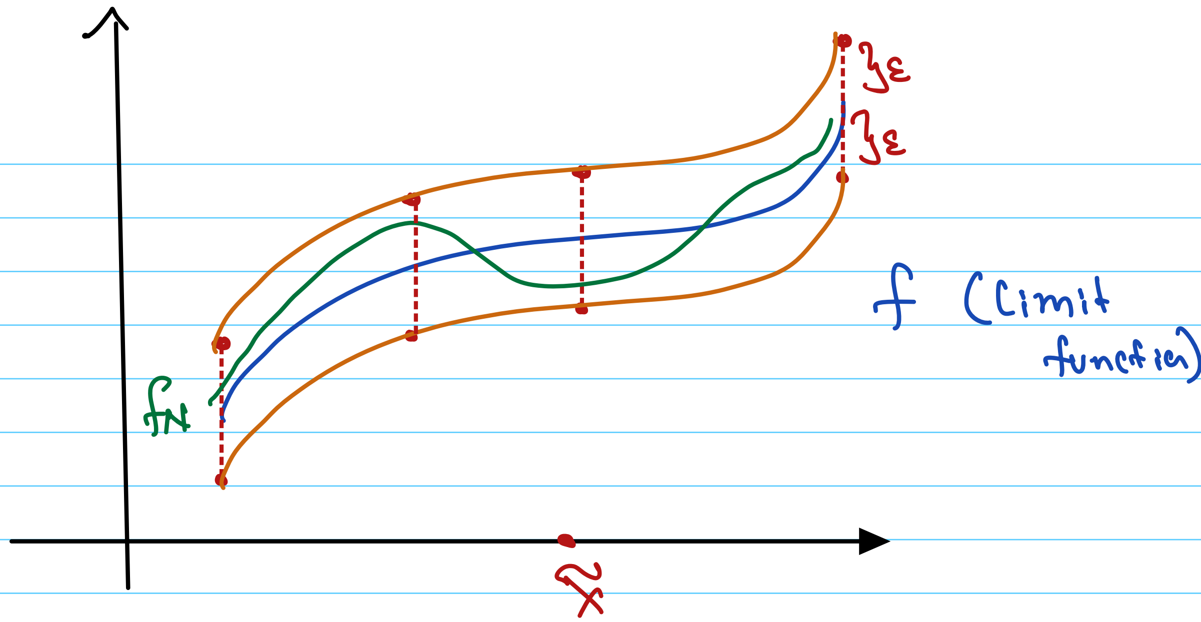
$$\forall \tilde{x} \in I : |f_n(\tilde{x}) - f(\tilde{x})| < \varepsilon$$

$\Rightarrow$ Now the $N \in \mathbb{N}$ works uniformly for all points $\tilde{x} \in I$

$\Rightarrow$ that's different from before, because in Pointwise convergence, we have $N \in \mathbb{N}$ depending on $\tilde{x}$, so, there could be different N in Pointwise convergence

$N(\tilde{x}) \in \mathbb{N}$

However for the uniform convergence $N \in \mathbb{N}$ has to work for all $\tilde{x} \in I$ at the same time.



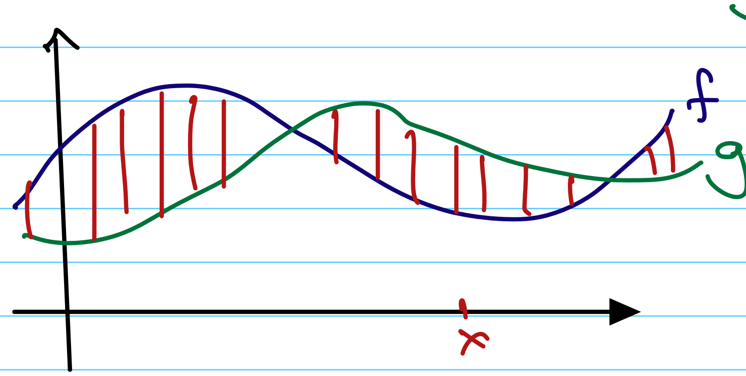
$\forall n > N$ $f_n(x)$ lies b/w the epsilon tube.

This also tells us its possible to measure distance b/w two function's

Distance for function's

$g: I \to \mathbb{R}$

$f: I \to \mathbb{R}$



$$\|f-g\|_\infty = \sup_{x \in I} |f(x) - g(x)|$$

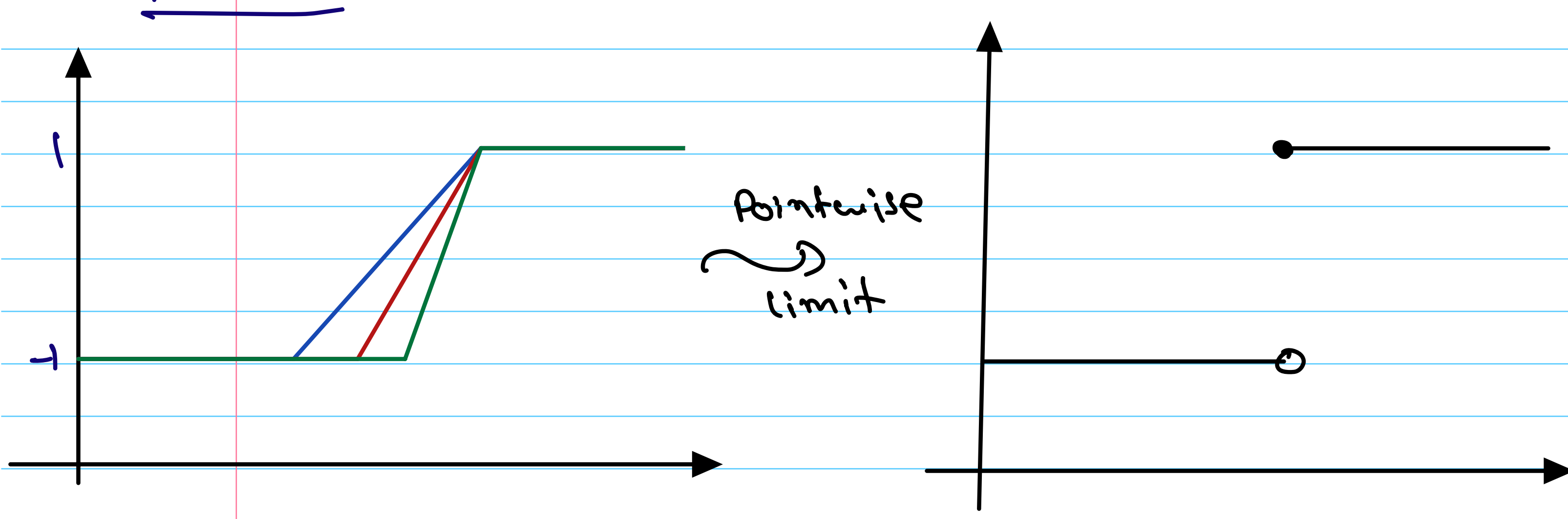
Supremum norm of $f-g$

$$\|f-g\|_\infty = \sup_{x \in I} |f(x) - g(x)|$$

Uniform Convergence mean's:

$$\|f_n - f\|_\infty \xrightarrow[n \to \infty]{} 0$$

Example:



$$\|f_n - f\|_\infty \geq 1 \quad \forall n$$

Result:

Pointwise convergence $\not\Longrightarrow$ Uniform convergence

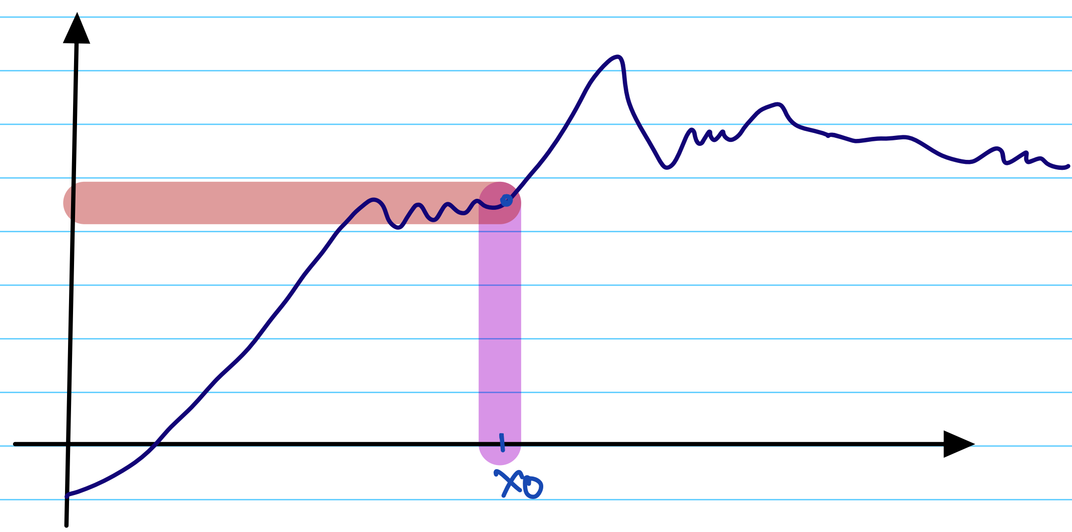
$\Longleftarrow$

Uniform Convergence is Stronger than pointwise Convergence.

* Uniform Convergence is very useful, as it conserve nice properties of function's (Ex, continuity)

Lec 26 : Limits of Function's

The Property Continuous will be local Property.



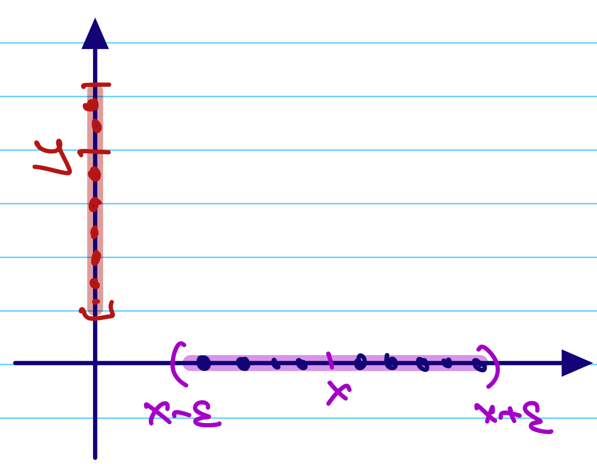
for a given function, we will define continuous for point x_0.

what it mean's f is continuous at point x_0?

* we want to know what happen's to the value of function when x_0 wobbles a little bit around.

$\Rightarrow$ Small error in x translate to small error's in y $\Rightarrow$ this is what we call continuous at x_0

Definition:



we have a input seq^n around x_0 converges to x_0, and an out put seq^n there in y-axis converges to $y_0 = f(x_0)$

$\Rightarrow$ To really exclude Jum's around x_0 we have to look at all possible seq^n here.

<u>Def</u>: Let $f: I \longrightarrow \mathbb{R}$, $x_0 \in I$, if there is $C \in \mathbb{R}$

all seq^n $(x_n)_{n \in \mathbb{N}}$ with $\lim_{n \to \infty} x_n = x_0$

we have $\left(f(x_n) \right)_{n \in \mathbb{N}}$ with also converges with

$$\lim_{n \to \infty} f(x_n) = C$$

then we write $\boxed{\lim_{x \to x_0} f(x) = C}$

This is the definition of Limit when we have a function $f(\cdot)$.

Definition: $f: I \to \mathbb{R}$, $x_0 \in I$ (x_0 is fixed), $c \in \mathbb{R}$ (fixed)

we have a seq^n $(x_n)_{n\in\mathbb{N}}$ that is convergent to x_0

all seq^n $(x_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} x_n = x_0$

and then we have out put seq $(f(x_n))_{n\in\mathbb{N}}$

the we have

$(x_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} x_n = x_0$

$(f(x_n))_{n\in\mathbb{N}}$ is also convergent with $\lim_{n\to\infty} f(x_n) = c$

the limit c does not depend upon choosen seq $(x_n)_{n\in\mathbb{N}}$ $\Rightarrow$ we get same number c for all seq $(x_n)_{n\in\mathbb{N}} \to x_0$. so we fix $c \in \mathbb{R}$

then we write

$$\boxed{\lim_{x\to x_0} f(x) = c}$$

$\lim_{x \to x_0} f(x)$ describes what happen's with value $f(x)$ when x gets closer and closer to x_0. This mean's that seq^n $(x_n)_{n \in \mathbb{N}}$ comes from outside and converges to x_0, therefore the seq (x_n) does not have x_0 as member.

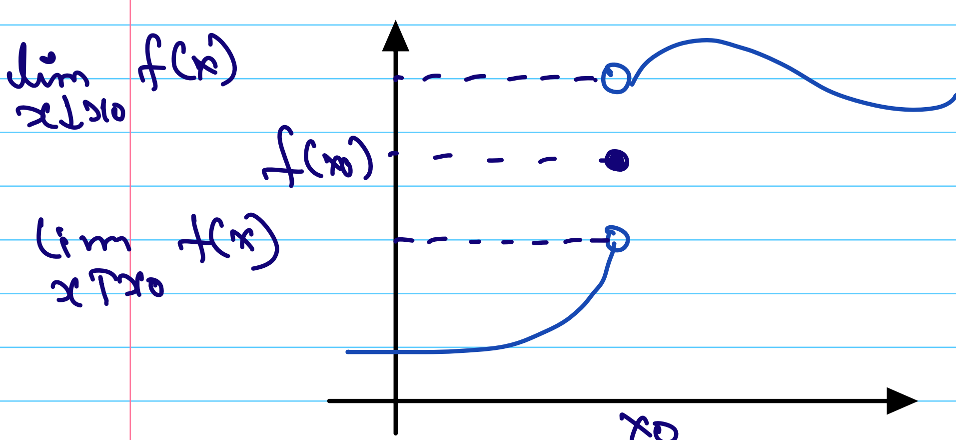
It defined by using the limit notion that we had for seq^n.

$\lim_{x \to x_0} f(x) = c$ (It describes what happen's with the value's of $f(x)$ when x gets closer & closer to x_0

and $\lim_{x \uparrow x_0} f(x) = c$

if $x_n < x_0 \ \forall n \in \mathbb{N}$

$\lim_{x \downarrow x_0} f(x) = c$ if $x_n > x_0 \ \forall n \in \mathbb{N}$



Ex:

(a) $f(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$

$$\lim_{x \to 0} f(x) = 0 \neq 1 = f(0)$$

(Jump)

(b) $f(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x^1 + a_0$

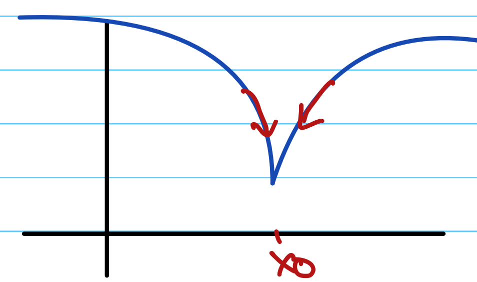
Lec 27 : Continuity.

Definition:

Let $f: I \longrightarrow \mathbb{R}$ be a function with $I \subseteq \mathbb{R}$.

f is called continuous at $x_0 \in I$ if

$$\lim_{x \to x_0} f(x) = f(x_0)$$



This definition is a local property $\Rightarrow$ we are fixing point x_0 and zoom in around x_0 and look what happen's.

* Also in definition of limit's we have two direction's, right limit, Left limit.

$$\lim_{x \to x_0} f(x) = f(x_0) = \lim_{x \uparrow x_0} f(x) = \lim_{x \downarrow x_0} f(x_0)$$

Definition: let $f : I \longrightarrow \mathbb{R}$ be a function with $I \subseteq \mathbb{R}$ f is called continuous if f is continuous at x_0 $\forall x_0 \in I$.

To Remember:

Continuity implies

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right)$$

$$\left(\text{if } \lim_{n\to\infty} x_n \in I\right)$$

Example: (a) $f : I \to \mathbb{R}$ Constant function

Lec 28 : Epsilon- Delta:

Continuity: f is called continuous at $x_0 \in I$

if $\lim_{x \to x_0} f(x) = f(x_0)$

* Here in this definicn we have to work with seqⁿ, simple because in the def of this Limit, we have to put in seqⁿ's. Therefore if we want to avoid the seqⁿ, The ε-δ Criterion is very useful.

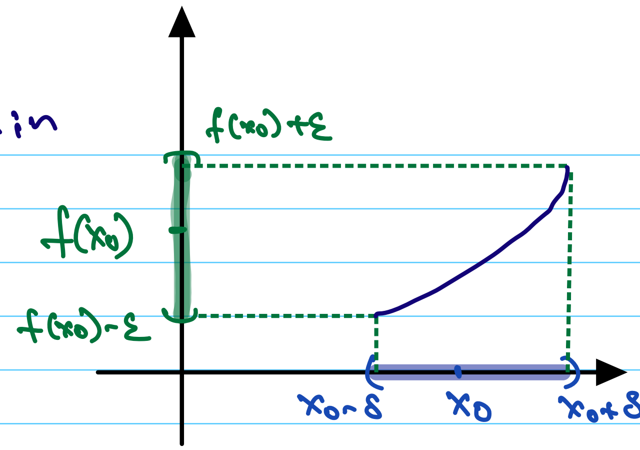
Theorem: Let $f: I \longrightarrow \mathbb{R}$ be a function with $I \subseteq \mathbb{R}$. For $x_0 \in I$, we have

f is continuous at $x_0 \in I$

$\Longleftrightarrow$ $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $\forall x \in I$:

$$|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon$$

Zoom in $\Rightarrow$

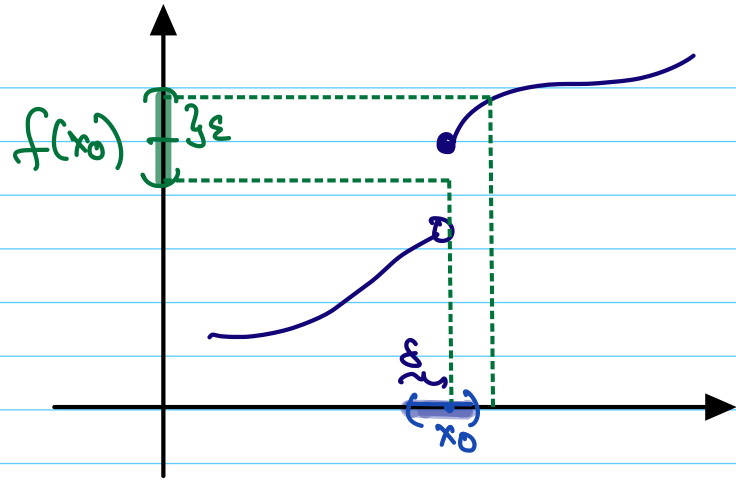


* All the points that have distance less than δ from x_0 ($|x-x_0|<\delta$) on the x-axis gives us values on Y-axis that fullfill this $|f(x)-f(x_0)|<\varepsilon$ inequality.

* The close points on x-axis are sent to close points on Y-axis. This is exactly what we want from continuity.

* This should work no matter how small we choose ε. This won't work when we have jump in graph.

Example:



No matter how small we choose δ we choose for any fixed ε, we would always hit points here on the left.

Proof:

($\Longrightarrow$) Assume $\exists \varepsilon > 0, \forall \delta > 0, \exists x \in I$:

$$|x - x_0| < \delta \quad \wedge \quad |f(x) - f(x_0)| \geq \varepsilon$$

This works for all $\delta > 0$, let's take $\delta = \frac{1}{n}$, $n \in \mathbb{N}$

$\Rightarrow$ for all $n \in \mathbb{N}$, we find $x_n \in I \setminus \{x_0\}$ with $|x_n - x_0| < \frac{1}{n}$ and $|f(x_n) - f(x_0)| \geq \varepsilon$

$\Rightarrow$ f is not continuos at $x_0 \in I$

($\Leftarrow$) we have ε-δ criterion and now we want to show $\forall$ seq^n that converge to x_0 the Images also converge to $f(x_0)$

Choose a seq^n $(x_n)_{n \in \mathbb{N}} \subseteq I \setminus \{x_0\}$ with limit x_0. let $\varepsilon > 0$. Take $\delta > 0$

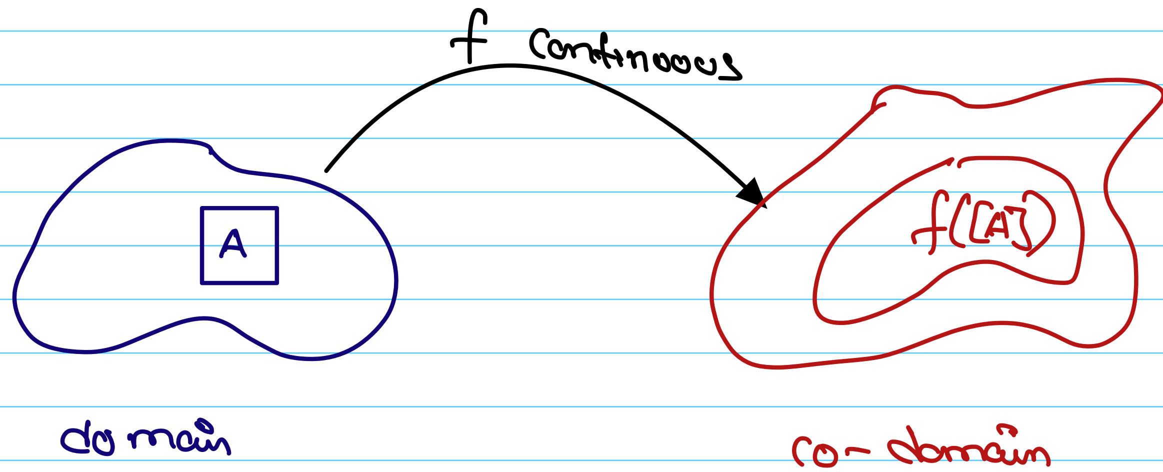
there is $N \in \mathbb{N}$ such that $\forall n \geq N$ we have $|x_n - x_0| < \delta$

$\Rightarrow$ from Assumption we have

$$|f(x_n) - f(x_0)| < \varepsilon$$

$\Rightarrow$ seq^n $(f(x_n))_{n \in \mathbb{N}}$ is convergent to $f(x_0)$

Lec 30 :-



Set A is compact $\xrightarrow{f}$ compact

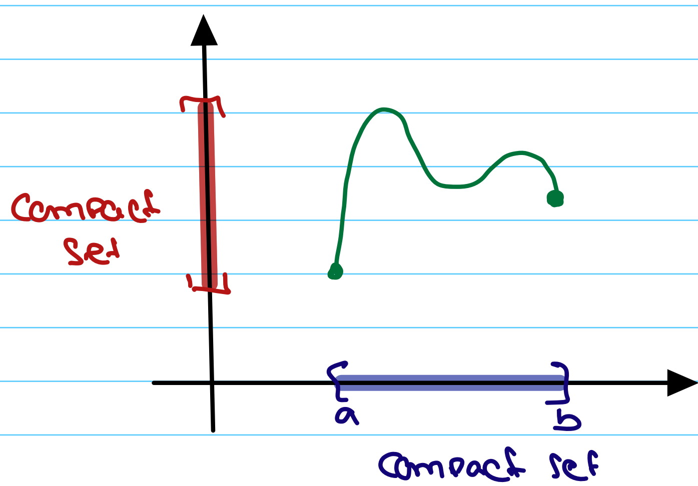
Theorem: $f : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ compact,

$f : I \to \mathbb{R}$ continuous.

Then $f[I] \subseteq \mathbb{R}$ is compact

(= bounded + closed)

Heine-Borel



and there are x^+ and $x^- \in I$ with

$$f(x^+) = \sup \{ f(x) \mid x \in I \}$$

$$f(x^-) = \inf \{ f(x) \mid x \in I \}$$

x^-, x^+ dont need to lie on Boundary, are need to be Unique

<u>Proof:</u>

<u>Compact set:</u>

if we choose any seq^n with elements from this set, the convergent $subseq^n$ where the limil lies also in set.

$\Rightarrow$ In short every seq^n has a convergent $subseq^n$.

Let $(y_n)_{n \in \mathbb{N}} \subseteq f[I]$ be a seq^n

$\forall\, y_n$ there is $x_n \in I$ with $f(x_n) = y_n$

$\Rightarrow$ A new seq form the element's (x_n)

$\Rightarrow$ New sequence $(x_n)_{n\in\mathbb{N}} \subseteq I$

$\overset{I \text{ compact}}{\Longrightarrow}$ There is a subseq^n $(x_{n_k})_{k\in\mathbb{N}}$ that is convergent. :

$$x := \lim_{k\to\infty} x_{n_k} \in I$$

$$y_{n_k} = f(x_{n_k})$$

Since f is continuous, x_{n_k} is convergent seq to x

$$\lim_{k\to\infty} y_{n_k} = \lim_{n\to\infty} f(x_{n_k})$$

$$= f\left(\lim_{k\to\infty} x_{n_k}\right)$$

$$= f(x) =: y$$

so $(y_{n_k})_{k\in\mathbb{N}}$ is convergent with limit

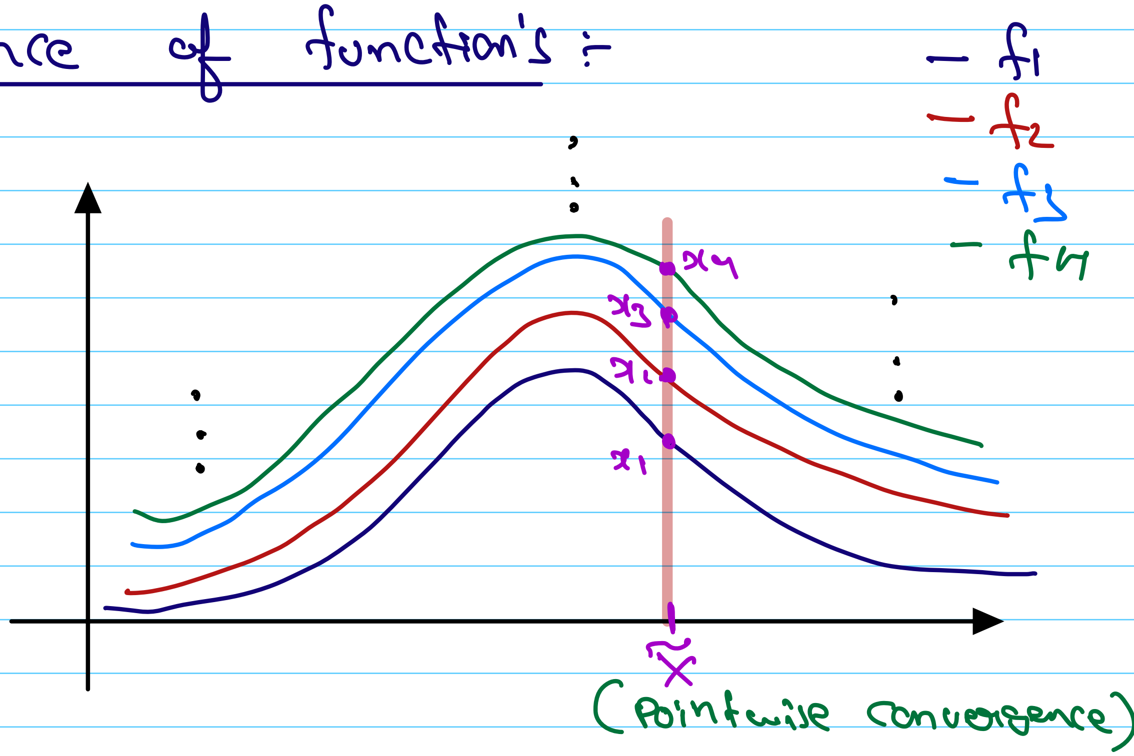
$y \in f[I] \implies f[I]$ is compact.

* Images of compact set are also compact if we consider continuous function.

* An Important Implication from this is that a continuous function defined on a compact set always attain's its maximum, minimum.

Lec 31 : Uniform Convergence of Continuous function's are continuous.

Sequence of function's :-



Sequence of function's :

$$(f_1, f_2, f_3, f_4, \cdots)$$

with sequence member's

$$f_1 : I \longrightarrow \mathbb{R}$$

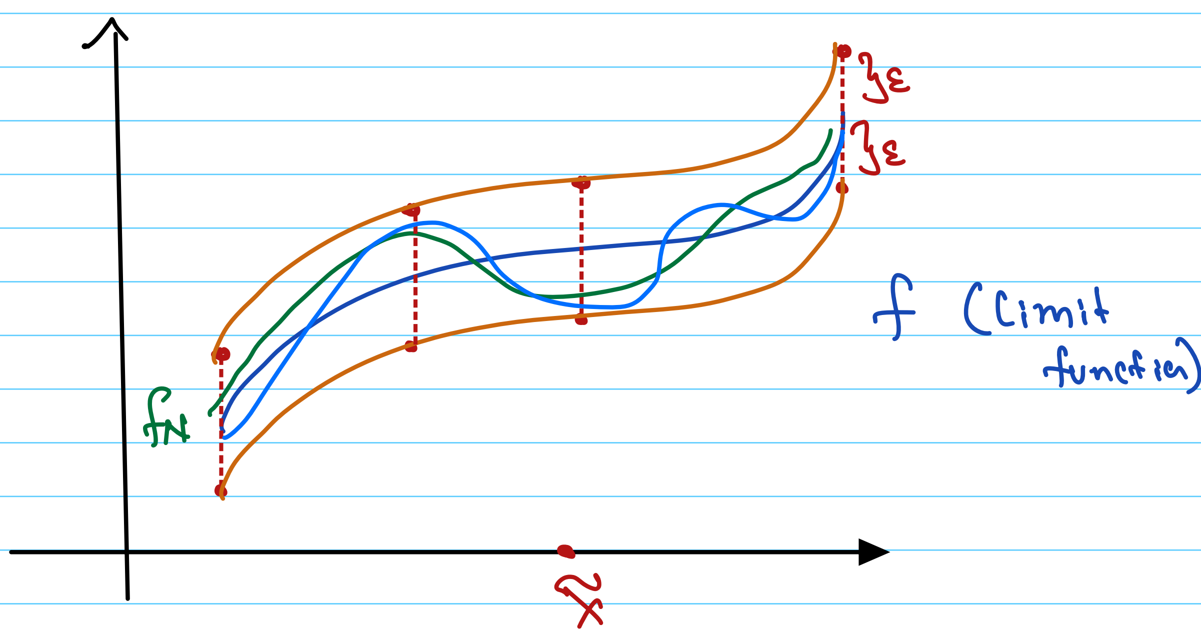
$$f_2 : I \longrightarrow \mathbb{R}$$

$$f_3 : I \longrightarrow \mathbb{R}$$

All the function's in seq^n have same domain I

* in Pointwise convergence we look at each $\tilde{x}$ of dom (f) seperately, and then at this point we see what happen's to the value $f_n(\tilde{x})$ as $n \to \infty$.

* in uniform convergence we look at all points at the same time.



Theorem: $I \subseteq \mathbb{R}$ is domain $f_n : I \to \mathbb{R}$ is a seq^n of continuous function's (for all $n \in \mathbb{N}$)

and $(f_n)_{n\in\mathbb{N}}$ uniformely converges to $f : I \to \mathbb{R}$.

Then: f is also continuous.

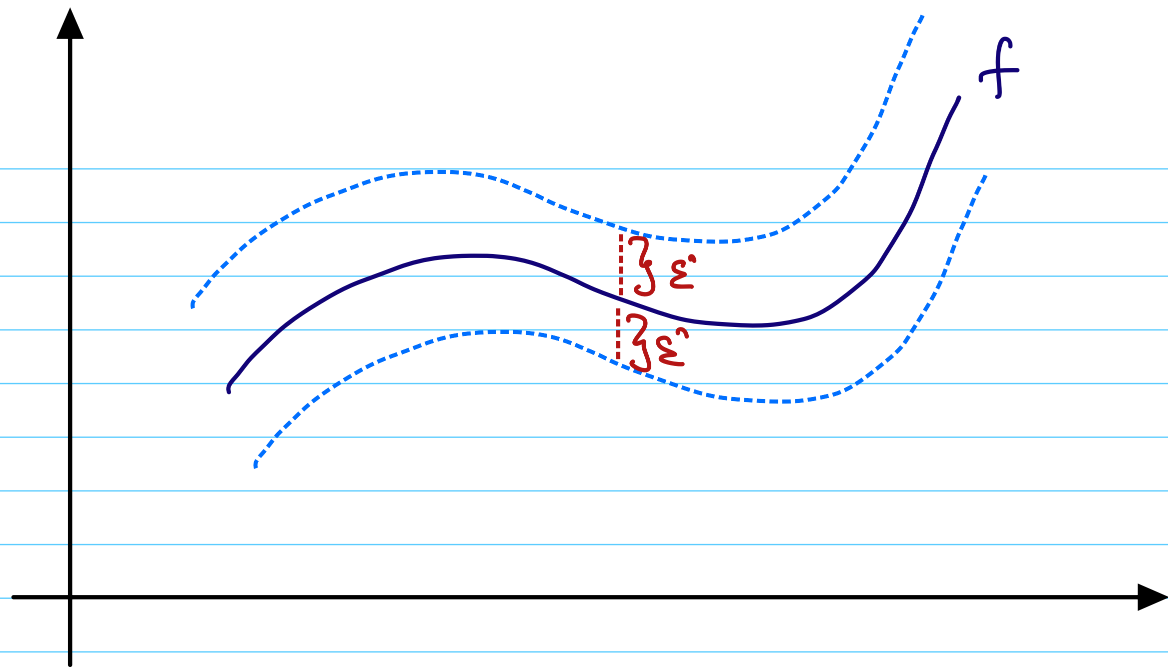
Proof:

we can use two thing's

① we can use the ε- tobe from the uniform convergence

② we can use the $\varepsilon-\delta$ criterion form the continuity.

Let $\varepsilon > 0$. let $x_0 \in I$. Set: $\varepsilon' := \frac{\varepsilon}{3}$



Uniform Convergence : $\forall \varepsilon' > 0 \ \exists N \in \mathbb{N}$, and $\forall n \geq N$ $\forall \tilde{x} \in I$: $|f_n(\tilde{x}) - f(\tilde{x})| < \varepsilon'$

Continuity of f_N : we find $\delta > 0$ with

$$|x - x_0| < \delta \implies |f_N(x) - f_N(x_0)| < \varepsilon'$$

Hence $|f(x) - f(x_0)|$

$$= |f(x) - f_N(x) + f_N(x) - f_N(x_0) + f_N(x_0) - f(x_0)|$$

$$\leq \underbrace{|f(x) - f_N(x)|}_{< \varepsilon'} + \underbrace{|f_N(x) - f_N(x_0)|}_{< \varepsilon'} + \underbrace{|f_N(x_0) - f(x_0)|}_{< \varepsilon'}$$

$$\Rightarrow \quad |f(x) - f(x_0)| < 3 \cdot \varepsilon'$$

$$\Rightarrow \quad |f(x) - f(x_0)| < \varepsilon'$$

Conclusion: we find $\delta > 0$ with: $\forall x \in I$:

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

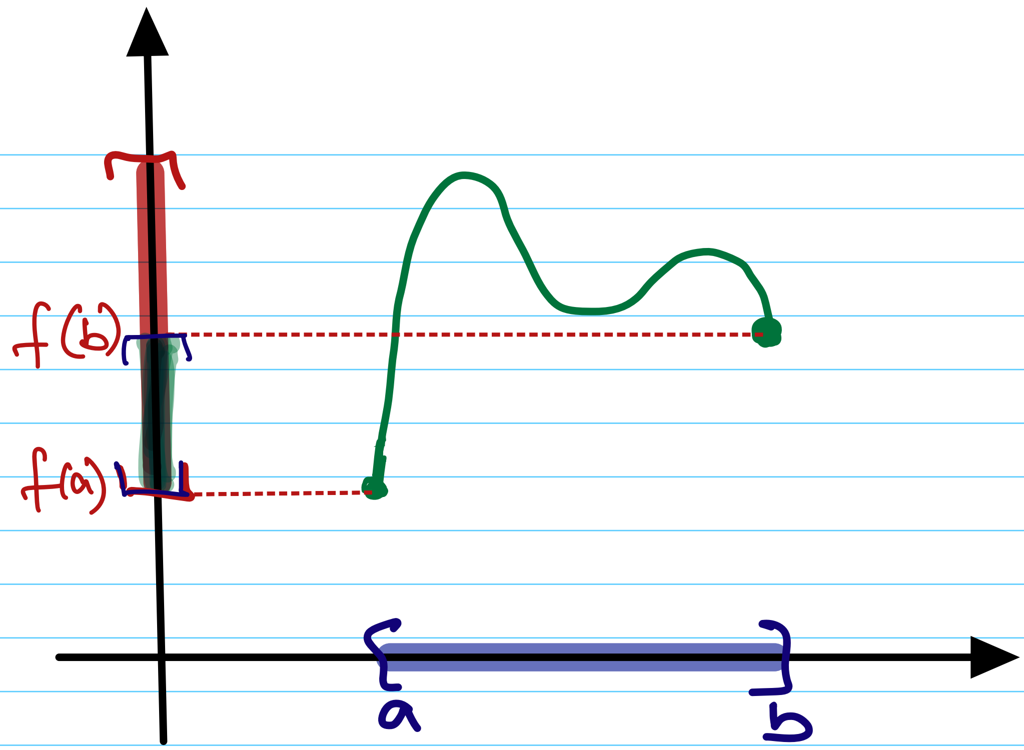
Hence $f(x)$ is continuous at x_0

$\xRightarrow{x_0 \text{ is arbitrary}}$ f is continuous.

Lec 32:- Intermediate value theorem.

$f : I \longrightarrow \mathbb{R}$ continuous.

$I = [a, b] \subseteq \mathbb{R}$



The intermediate value theorem:

Intermediate value theorem:

Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous

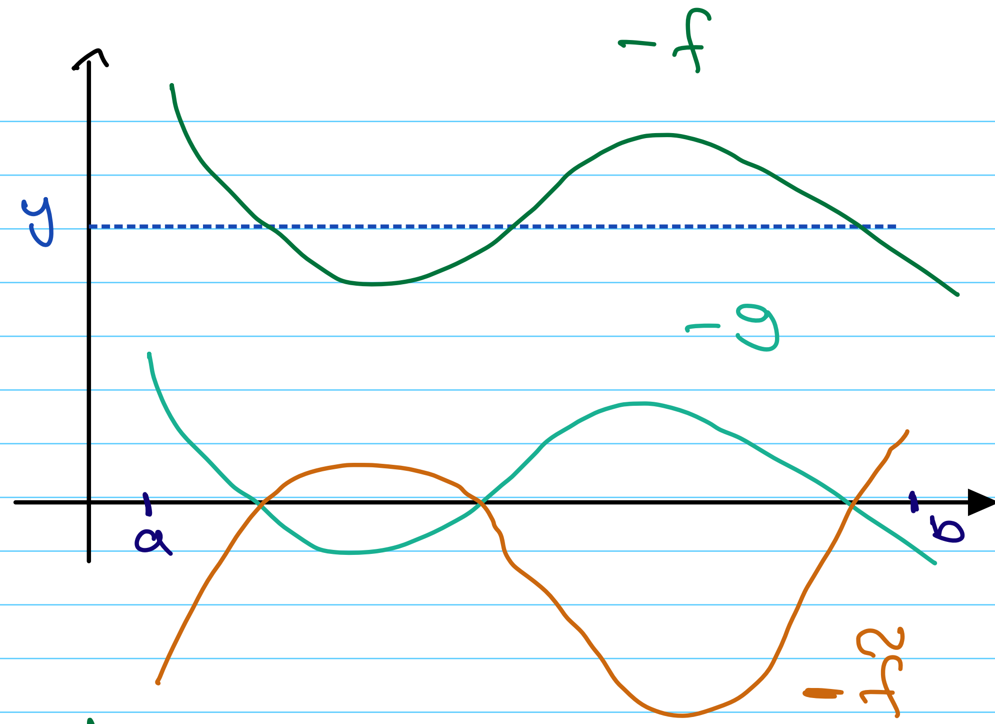
and $y \in [f(a), f(b)]$ or $y \in [f(b), f(a)]$

Then: there is $\tilde{x} \in [a,b]$ with $f(\tilde{x}) = y$

Corollary:

$f\left[[a,b]\right]$ is also an interval.

Proof:



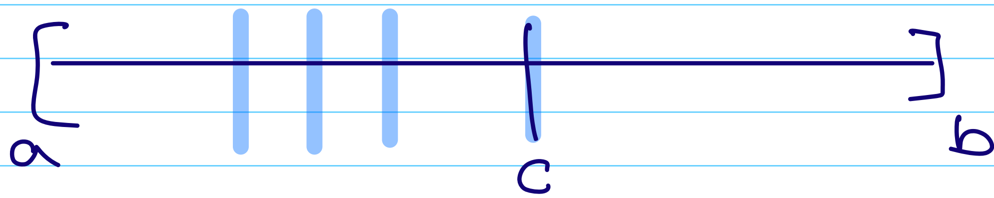
New function:

$$g := f - y$$

$$\tilde{f} := \begin{cases} -g & \text{if } g(a) > 0 \\ g & \text{if } g(a) \leq 0 \end{cases}$$

Then $\tilde{f}$ is continuous: $\tilde{y} := 0$ and

$\tilde{f}(a) \leq 0$, $\tilde{f}(b) \geq 0$



$c = \frac{a+b}{2}$ check $\tilde{f}(c)$:

if $\tilde{f}(c) > 0$: $b_1 := c,\ a_1 := a$

if $\tilde{f}(c) \leq 0$: $a_1 := c,\ b_1 := b$

repeat

At the end we got two seq^n out

$(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and

$$(b_n - a_n) \xrightarrow[n\to\infty]{} 0$$

$$\Longrightarrow \quad \tilde{x} := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \in [a,b]$$

we know $\tilde{f}(a_n) \leq 0$, $\tilde{f}(b_n) \geq 0$

$$\Downarrow$$

$$\tilde{f}\left(\lim_{n\to\infty} a_n\right) \leq 0 \qquad \tilde{f}(\tilde{x}) \leq 0$$

$$\Longrightarrow$$

$$\tilde{f}\left(\lim_{n\to\infty} b_n\right) \geq 0 \qquad \tilde{f}(\tilde{x}) \geq 0$$

Hence $\tilde{f}(\tilde{x}) = 0$

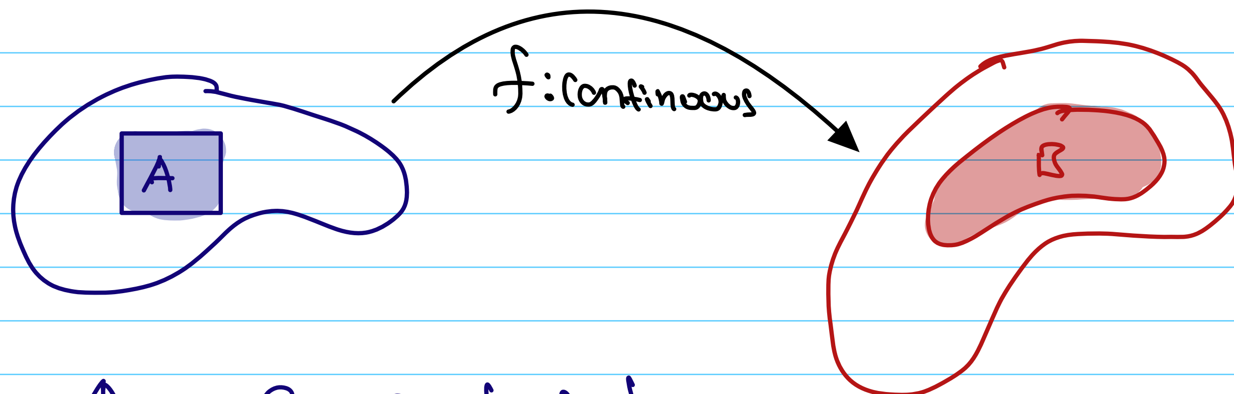
$\Rightarrow \quad g(\tilde{x}) = 0$

$\Rightarrow \quad f(\tilde{x}) - y = 0$

$\Rightarrow \quad f(\tilde{x}) = y$

Lec 30:

An important property continuous function have. we will show that continuous function conserve the property compact.



A = compact set

f: continuous function

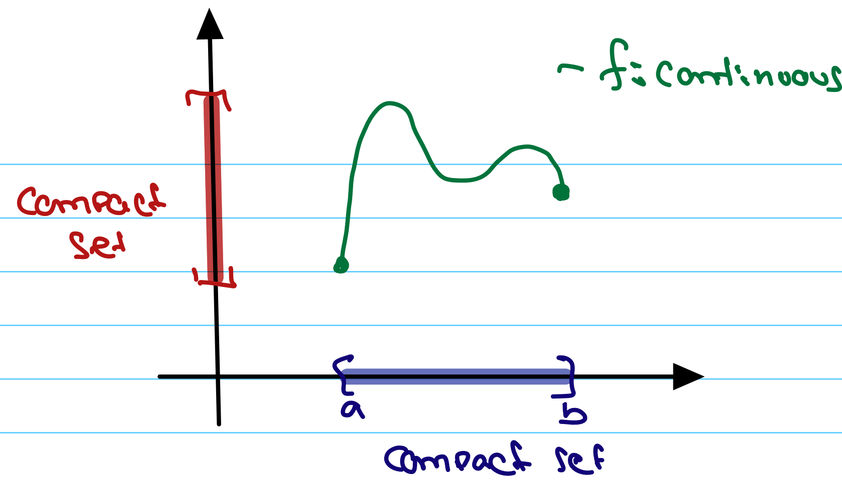
then the image of A

$$f[A] = B \text{ is compact set.}$$

Theorem: $I \subseteq \mathbb{R}$ compact, $f: I \to \mathbb{R}$ continuous.

Then $f[I] \subseteq \mathbb{R}$ is compact

(= bounded + closed)
Heine-Borel theorem)

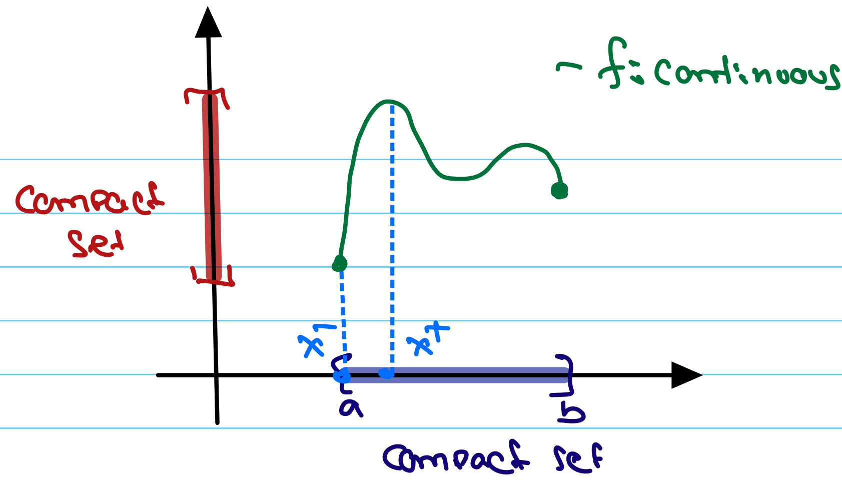


for all subset's of the real number's, we find the sup & inf of set $f[I]$, However we already know that the Image is bounded and closed $\Rightarrow$ Hence sup & inf are max and min. (they are elements of set)

and there is $x^+, x^- \in I$ with

$$f(x^+) = \sup\{f(x) \mid x \in I\}$$

$$f(x^-) = \inf\{f(x) \mid x \in I\}$$



x^-, and x^+ don't need to lie on boundary of I, and don't need to be unique.

Proof:

def: Compact for a set mean's :
if we choose any seq^n with elements from this set, there is a convergent sub seq^n, with limit also lies in that set.

Every seq^n have a convergent subseq^n.

Let $(y_n)_{n\in\mathbb{N}} \subseteq f[I]$ be a seq^n with $f(x_n) = y_n$

$\Rightarrow$ New seq^n $(x_n)_{n\in\mathbb{N}} \subseteq I$

I is compact and $(x_n)_{n\in\mathbb{N}} \subseteq I$

$\Rightarrow$ There is a subseq^n $(x_{n_k})_{k\in\mathbb{N}}$ that is convergent

$$\lim_{k\to\infty} x_{n_k} := x \in I$$

$\Rightarrow$ we can transulathe that to y-axis using the function f.

$$\Rightarrow \quad y_{n_k} = f(x_{n_k})$$

Since f is continuous

$$\lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(x_{n_k})$$

$$= f\left(\lim_{k\to\infty} x_{n_k}\right)$$

$$= f(x) := y$$

So our subseq^n (y_{n_k}) is convergent with $y \in f[I]$

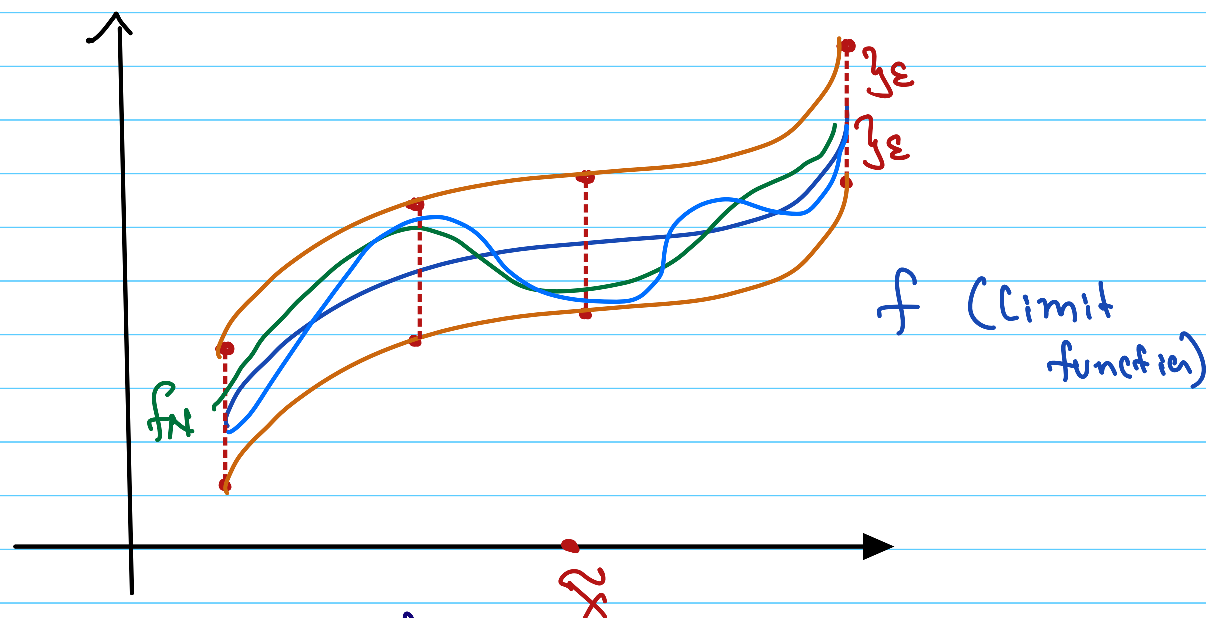
$\Rightarrow$ $f[I]$ is compact.

Lec 31: Uniform Limits of Continuous functions are continuous

f_n continuous

$\|f_n - f\|_\infty \xrightarrow{n\to\infty} 0$

$\Rightarrow$ f is continuous



uniform convergence.

Theorem: $I \subseteq \mathbb{R}$, $f_n : I \to \mathbb{R}$ continuous (for all $n \in \mathbb{N}$), and $(f_n)_{n\in\mathbb{N}}$ uniformly converges to $f : I \to \mathbb{R}$.

Then f is also continuous.

Proof:

Let $\varepsilon > 0$. Let $x_0 \in I$. Set $\varepsilon' := \frac{\varepsilon}{3}$

Uniform convergence:

$\forall \varepsilon' > 0$, $\exists N \in \mathbb{N}$ such that $\forall \tilde{x} \in I$,

$\forall n > N$ $\quad |f_n(\tilde{x}) - f(\tilde{x})| < \varepsilon'$

Continuity of f_N: (Apply ε- δ Criterion)

we find $\delta > 0$ with

$$|x - x_0| < \delta \implies |f_N(x) - f_N(x_0)| < \varepsilon'$$

Hence:

$$|f(x) - f(x_0)|$$

$$= |f(x) - f_N(x) + f_N(x) - f_N(x_0) + f_N(x_0) - f(x_0)|$$

$$\leq \underbrace{|f(x) - f_N(x)|}_{\substack{\text{Limit of seq}^n \\ \text{of function's } < \varepsilon'}} + \underbrace{|f_N(x) - f_N(x_0)|}_{\substack{\text{Continuity of } f_N \\ < \varepsilon'}} + \underbrace{|f_N(x_0) - f(x_0)|}_{\substack{\text{Limit of seq}^n \text{ of} \\ \text{function } < \varepsilon'}}$$

$$|f(x) - f(x_0)| < \varepsilon$$

$\Rightarrow$ for $\forall \delta > 0$ with $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \varepsilon$

$\Rightarrow$ f is Continuous.

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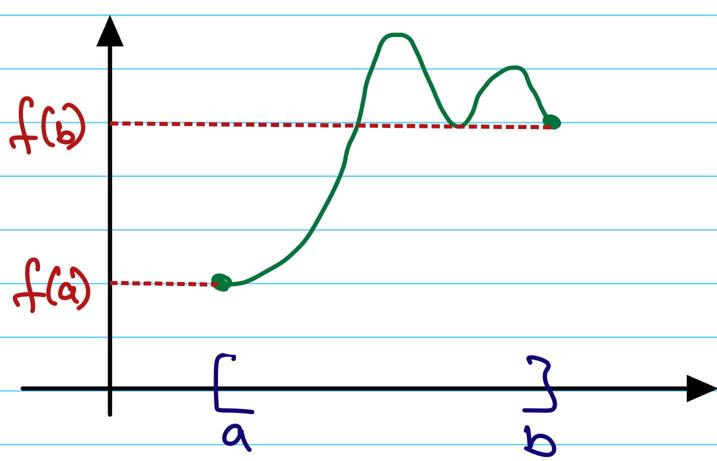
$f: I \longrightarrow \mathbb{R}$ Continuous

$\Downarrow$

$[a,b]$



Therefoor, the intermediate value theorem gives a meaning to the sentence that the graph of continuous function can be drawn from one stroke.

<u>Intermediate value theorem:</u>

let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous

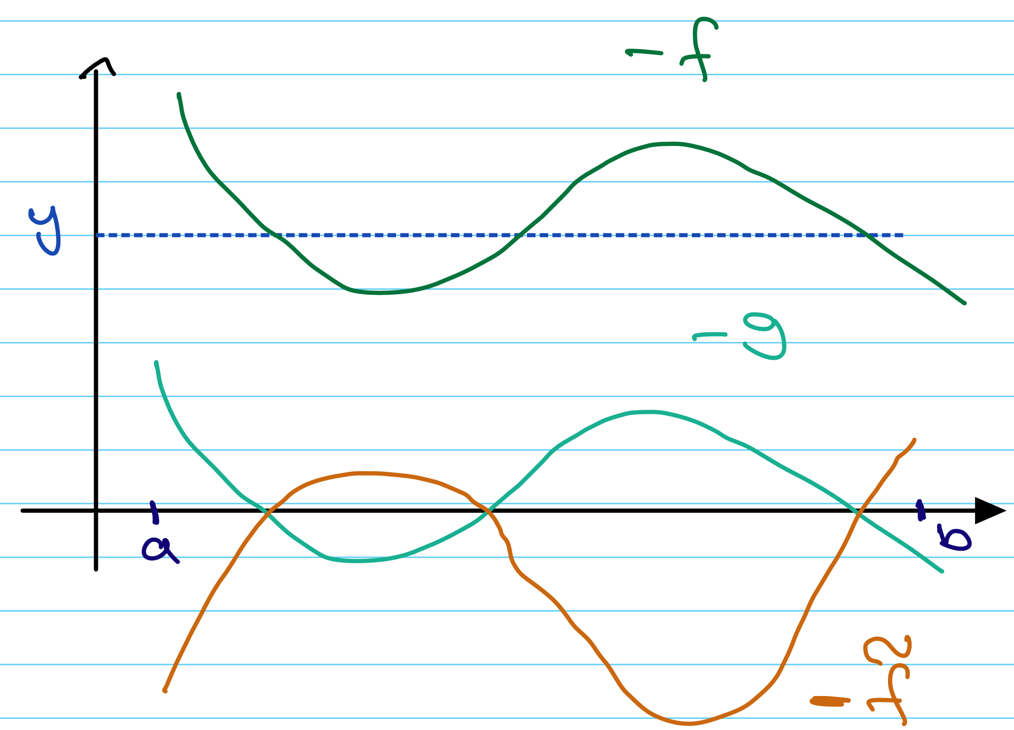
and $y \in [f(a), f(b)]$ or

$y \in [f(b), f(a)]$

Then there is $\tilde{x} \in [a,b]$ with $f(\tilde{x}) = y$

<u>Corollary:</u> $f[a,b]$ is also an interval

<u>Proof</u>



New function:

$$g := f - y$$

$$\tilde{f} := \begin{cases} -g & \text{if } g(a) > 0 \\ g & \text{if } g(a) \leq 0 \end{cases}$$