09 - Limit theorem's for Sequences

* First we need to Prove Convergent seq^n are Bounded.

A seq^n (S_n) of real number's is said to be bounded if the set $\{S_n : n \in \mathbb{N}\}$ is a bounded set.

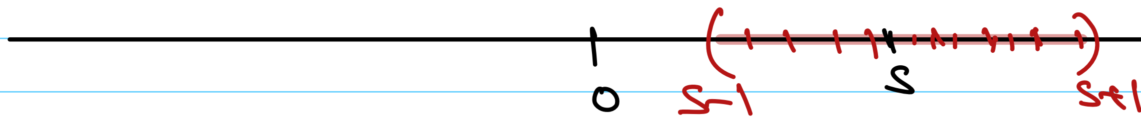
i.e there exists a constant M s.t

$$|S_n| < M \quad \forall n \in \mathbb{N}$$

Theorem 9.1 :-

let (S_n) be a convergent seq^n, and let $s = \lim S_n$. Applying Def 7.1 with $\varepsilon = 1$, we obtain N in $\mathbb{N}$ so that

$n > N$ implies $|s_n - s| < 1$



$\forall n > N$ lies in this set

only finitley many are left outside which ar $(s_1, s_2, \ldots s_n)$

apply toriangle inequality

$$|s_n - s + s| \leq |s_n - s| + |s|$$

$$\Rightarrow |s_n| \leq |s_n - s| + |s| < 1 + |s|$$

$$\Rightarrow |s_n| < 1 + |s|$$

Define

$$M = \max\{|s|+1, |s_1|, |s_2|, |s_3|, \ldots, |s_N|\}$$

$\Rightarrow \quad |S_n| \leq M \quad \forall n \in \mathbb{N}.$

$\Rightarrow \quad S_n$ is a bounded seq^n.

Theorem 9.2 :-

if the sequence (S_n) converges to S and k is in $\mathbb{R}$; then the seq^n (kS_n) converges to kS,

$$\boxed{\lim kS_n = k \lim S_n = kS.}$$

Proof:

we assume $k \neq 0$, we need to show

$$|kS_n - kS| < \varepsilon$$

$\Rightarrow \quad |S_n - S| < \dfrac{\varepsilon}{|k|}$ according to convergence of (S_n) there exist some $N \in \mathbb{N}$ s.t $\forall n > N$ $|S_n - S| < \dfrac{\varepsilon}{|k|}$

Theorem 9.3:-

if S_n converges to S, t_n converges to t, then $(S_n + t_n)$ converges to $S+t$, That is

$$\lim_{n\to\infty} (S_n + t_n) = \lim S_n + \lim t_n.$$

Proof:

Let $\varepsilon > 0$, we need to show

$|S_n + t_n - S - t| < \varepsilon$ for large n.

$$|S_n - S + t_n - t| \le |S_n - S| + |t_n - t|$$

Since S_n converges $\exists\, N_1$ such that

$\forall n > N_1 \quad |S_n - S| < \frac{\varepsilon}{2}$

Simillar $\exists N_2$ such that

$\forall n \in \mathbb{N}$ $\quad |t_n - t| < \frac{\varepsilon}{2}$

let $N = \max\{N_1, N_2\}$

$$|s_n + t_n - s - t| \leq |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence Proved.

Theorem 9.4:-

if $\lim s_n = s$, $\lim t_n = t$, then

$\lim s_n \cdot t_n = st$.

Discussion:

$$|s_n t_n - st| = |s_n t_n - s_n t + s_n t - st|$$

$$\leq |s_n t_n - s_n t| + |s_n t - st|$$

$$\leq |s_n|\,|t_n - t| + |t|\,|s_n - s|$$

Therefore for large n

$$|t_n - t| < \varepsilon$$

$$|s_n - s| < \varepsilon$$

$|t|$ is constant

$|s_n|$ is bounded

$\Rightarrow |s_n| \leq M$

$\Rightarrow$ $|s_n t_n - st|$ is small

Proof:

let $\varepsilon > 0$, By theorem 9.1 $\exists M > 0$

s.t $|s_n| \leq M \ \forall n \in \mathbb{N}$.

Since $\lim t_n = t$ $\exists N_1$ such that

$$n > N_1 \Rightarrow |t_n - t| < \frac{\varepsilon}{2M}$$

also, since $\lim s_n = s$, $\exists N_2$ such that

$$n > N_2 \Rightarrow |s_n - s| < \frac{\varepsilon}{2(|t|+1)}$$

then for $n > N$

$$|s_n t_n - st| \le |s_n|\,|t_n - t| + |t| \cdot |s_n - s|$$

$$\le M \cdot \frac{\varepsilon}{2M} + |t| \cdot \frac{\varepsilon}{2(|t|+1)}$$

$$\le \frac{\varepsilon}{2} + \frac{|t|}{(|t|+1)} \cdot \frac{\varepsilon}{2}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$< \varepsilon$$

Hence it is convergent.

Lemma 9.5:-

if (s_n) converges to s, if $s_n \neq 0$, $\forall n \in \mathbb{N}$, then $\left(\frac{1}{s_n}\right) \longrightarrow \frac{1}{s}$

Disussion:

$$\left|\frac{1}{s_n} - \frac{1}{s}\right|$$

$$= \left| \frac{s - s_n}{s_n s} \right| < \frac{|s - s_n|}{m|s|}$$

Theorem 9.6:-

$\lim s_n = s$, $\lim t_n = t$, if $s \neq 0$ and $s_n \neq 0$ $\forall n \in \mathbb{N}$, then

$$\lim_{n \to \infty} \frac{t_n}{s_n} = \frac{t}{s}$$

Theorem 9.7 (Basic Examples)

(a) $\lim_{n \to \infty} \frac{1}{n^p} = 0$ for $p > 0$

(b) $\lim_{n \to \infty} a^n = 0$ if $|a| > 1$

(c) $\lim_{n \to \infty} n^{1/n} = 1$

(d) $\lim_{n \to \infty} (a^{1/n}) = 1$ $\forall a > 0$

Def 9.8:-

for a seq (S_n), we write $\lim S_n = +\infty$

Provided $\forall M > 0$, $\exists N$ s.t $n > N$

$$S_n > M$$

we say sequence diverges to $+\infty$

Similarly we write $\lim S_n = -\infty$

Provided $\forall M < 0$ $\exists N$ such that

$\forall n > N$ $\quad S_n < M$

Sequence diverges to $-\infty$

Theorem 9.9:-

let $\lim S_n = +\infty$, and $\lim t_n > 0$

[$\lim t_n$ can be finite or $+\infty$]

Then $\lim\limits_{t \to \infty} S_n t_n = +\infty$