19- Uniform Continuity

let f be a real-valued function whose domain is a subset of $\mathbb{R}$.

The theorem 17.2 tell us that f is continuous on a set $S \subseteq dom(f)$

$\iff$

$\forall x_0 \in S$, and $\forall \varepsilon > 0$, $\exists \delta > 0$

such that $x \in dom(f)$ and

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$$

The choice of $\delta > 0$ depends on $\varepsilon > 0$, on point $x_0 \in S$

$\Rightarrow$ i.e $\delta(x_0, \varepsilon)$

Ex1 $f(x) = \frac{1}{x^2}, \ x \in (0, \infty)$

let $x_0 > 0, \ \varepsilon > 0$

$$f(x) - f(x_0) = \frac{1}{x^2} - \frac{1}{x_0^2}$$

$$= \frac{x_0^2 - x^2}{x^2 x_0^2}$$

$$= \frac{(x_0 - x)(x_0 + x)}{x^2 x_0^2}$$

If $|x - x_0| < \frac{x_0}{2}$

$\frac{x_0}{2} < |x| < x_0 + \frac{x_0}{2}$

$\Rightarrow \frac{x_0}{2} < |x| < \frac{3x_0}{2}$

$\Rightarrow x^2 > \frac{x_0}{2} \Rightarrow \frac{1}{x^2} < \frac{1}{\left(\frac{x_0}{2}\right)^2}$

& $|x + x_0| \le |x| + |x_0|$

$\le \frac{3x_0}{2} + x_0$

$|x + x_0| \le \frac{5}{2} x_0$

$$\Rightarrow |f(x) - f(x_0)| = \frac{|x - x_0|\,|x + x_0|}{x^2 x_0^2}$$

$$\leq \frac{|x - x_0|\,\frac{5x_0}{2}}{\frac{x_0^2}{4} \cdot x_0^2}$$

$$\leq \frac{|x - x_0|\,10}{x_0^3} < \varepsilon$$

$$\Rightarrow |x - x_0| < \varepsilon \cdot \frac{x_0^3}{10}$$

$\Rightarrow$ for $\forall \varepsilon > 0$, $\forall x_0 > 0$, we have

$\delta = \min\left\{ \varepsilon \cdot \frac{x_0^3}{10}, \frac{x_0}{2} \right\}$ such that

$$|x - x_0| < \frac{\varepsilon x_0^3}{10} \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

Hence $f(x) = \frac{1}{x^2}$ is continuous.

$$\delta(x_0, \varepsilon) = \min\left\{ \frac{x_0}{2}, \frac{\varepsilon x_0^2}{3} \right\}$$

$\Rightarrow$ even if ε is fixed, δ gets small when x_0 gets small.

$\Rightarrow$ δ depends on Both ε, x_0.

Def 19.1

Let f be a real-valued function defined on a set $S \subseteq \mathbb{R}$. Then f is uniformly continuous on S if

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $x, y \in S$ and $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

It makes no sense to speak of a function being uniformly continuous at each point.

Ex 2

$f(x) = \frac{1}{x^2}$ is uniformly continuous on any set of the form $[a, \infty)$ where $a > 0$. Here a is fixed.

Sol^n let $\varepsilon > 0$ we need to $\exists \delta > 0$ s.t

$\forall x, y \in [a, \infty)$ $\quad |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

$$f(x) - f(y) = \frac{1}{x^2} - \frac{1}{y^2}$$

$$= \frac{y^2 - x^2}{x^2 y^2}$$

$$= \frac{(y - x)(y + x)}{x^2 y^2}$$

$$\frac{y + x}{x^2 y^2} = \frac{1}{y x^2} + \frac{1}{x y^2}$$

$$x \geq a \, ; \, y \geq a \Rightarrow \frac{1}{x} \leq \frac{1}{a} \, , \, \frac{1}{y} \leq \frac{1}{a}$$

$$\Rightarrow \quad x^2 \geq a^2 \, ; \quad y^2 \geq a^2$$

$$\Rightarrow \quad \frac{1}{x^2} \leq \frac{1}{a^2} \, ; \quad \frac{1}{y^2} \leq \frac{1}{a^2}$$

$$\Rightarrow \quad \frac{1}{x^2 y} \leq \frac{1}{a^3} \qquad \frac{1}{y^2 x} \leq \frac{1}{a^3}$$

$$\Rightarrow \quad \frac{1}{x^2 y} + \frac{1}{y^2 x} \leq \frac{2}{a^3}$$

$$\Rightarrow \quad |f(x) - f(y)| = \frac{|y-x| \cdot (y+x)}{x^2 y^2}$$

$$\leq \frac{|y-x| \cdot 2}{a^3} < \varepsilon$$

$$\Rightarrow \quad |y-x| < \frac{a^3 \varepsilon}{2}$$

take $\delta = \frac{a^3 \varepsilon}{2}$

$\forall \varepsilon > 0, \ \exists \delta = \frac{a^3 \varepsilon}{2}$, such that $\forall x, y > a$

$$|x-y| < \frac{a^3 \varepsilon}{2} \Rightarrow |f(x) - f(y)| < \varepsilon$$

$\Rightarrow \quad f(x) = \frac{1}{x^2}$ is uniform continuous on $[a, \infty)$

Exam 3 $f(x) = \frac{1}{x^2}$ is not uniformly continuous on the set $(0, \infty)$ or even on $(0,1)$

Ex 4 : $f(x) = x^2$ is uniformly conversup on $[-7,7]$?

Sol^n let $\varepsilon > 0$.

$$f(x) - f(y) = x^2 - y^2$$
$$= (x-y)(x+y)$$

$$-7 \le x \le 7$$
$$-7 \le y \le 7$$

$$\Rightarrow -14 \le x+y \le 14 \Rightarrow |x+y| \le 14$$

$$\Rightarrow |f(x) - f(y)| = |x-y| \, |x+y|$$
$$\le 14 |x-y| < \varepsilon$$

take $\delta = \frac{\varepsilon}{14}$

$\forall \varepsilon > 0$, $\exists \delta = \frac{\varepsilon}{14}$ such that $x, y \in [-7, 7]$

$|x-y| < \frac{\varepsilon}{14} \Rightarrow |f(x) - f(y)| < \varepsilon$

Hence $f(x) = x^2$ is uniformly continuous on $[-7, 7]$

19.2 Theorem:

if f is continuous on a closed interval $[a, b]$; then f is uniformly continuous on $[a, b]$.

Proof: Assume f is not uniformly continuous on $[a, b]$.

$\Rightarrow \exists \varepsilon > 0$ s.t for each $\delta > 0$,

" $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ "

fail's

$\Rightarrow$ $\forall \delta > 0$, $x, y \in [a,b]$

$|x-y| < \delta$ but $|f(x)-f(y)| \geq \varepsilon$

$\Rightarrow$ $x_n, y_n \in [a,b)$, $\forall n \in \mathbb{N}$ such that

$|x_n - y_n| < \frac{1}{n}$ and yet

$|f(x_n) - f(y_n)| \geq \varepsilon.$

By Bolzano Weierstrass theorem
(Every Bounded seq^n has a convergent subseq^n)

There a subseq^n (x_{n_k}) of (x_n)
converges.

$$x_0 = \lim_{k \to \infty} x_{n_k}$$

$$x_0 \in [a,b].$$

$$x_0 = \lim_{k \to \infty} y_{n_k}$$

Since f is Continuous at x_0

$$f(x_0) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f(y_{n_k})$$

$$\Rightarrow \lim_{n\to\infty} f(x_{n_k} - y_{n_k}) = 0$$

$\Rightarrow$ Since $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon \ \forall k$

(contradiction)

$\Rightarrow$ f is uniformly continuous on $[a,b]$

one of the important application's of uniform continuity concer's the integrability of continuous function's on closed intervals.

Proof: A real valued function continuous, non-negative f on $[0,1]$

$\forall n \in \mathbb{N}$ and $i = 0, 1, 2, \ldots n-1$

let

$$M_{i,n} = \sup\left\{ f(x) : x \in \left[\frac{i}{n}, \frac{i+1}{n}\right] \right\}$$

$$m_{i,n} = \inf\left\{ f(x) : x \in \left[\frac{i}{n}, \frac{i+1}{n}\right] \right\}$$

$$U_n = \frac{1}{n} \sum_{i=0}^{n-1} M_{i,n}$$

$$L_n = \frac{1}{n} \sum_{i=0}^{n-1} m_{i,n}$$

$$\lim_{n \to \infty} (U_n - L_n) = 0$$

$$\Rightarrow \int_0^1 f(x)\,dx = \lim_{n \to \infty} U_n$$

$$= \lim_{n \to \infty} L_n$$

$$0 \leq U_n - L_n$$

$$\Rightarrow \quad 0 \le \frac{1}{n} \sum_{i=0}^{n-1} \left(M_{i,n} - m_{i,n} \right) \qquad \forall n.$$

Let $\varepsilon > 0$, by theorem 19.2, f is uniformly continuous on $[0,1]$.

$\Rightarrow \exists\, \delta > 0$, s.t $x, y \in [0,1]$

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \varepsilon$$

19.4 Theorem:

if f is uniformly continuous on a set S and (s_n) is a Cauchy seq^n in S, then $(f(s_n))$ is a Cauchy seq^n.

Proof:

Let $(s_n)_{n\in\mathbb{N}}$ is a Cauchy seq^n in S. let $\varepsilon>0$

* since f is uniforly continuous in S $\exists\,\delta>0$ such that

$\forall x,y\in S$ $$|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$$

* (s_n) is a Cauchy seq^n $\Rightarrow$ for $\varepsilon>0$

$\exists N\in\mathbb{N}$ s.t $\forall n,m>N$ we have

$$|s_n-s_m|<\delta$$

$\Rightarrow$ from f uniform Continuity

$$|f(x_n) - f(x_m)| < \varepsilon$$

$\forall\, n, m > N$,

$\Rightarrow \left(f(s_n)\right)_{n \in \mathbb{N}}$ is a Cauchy seq^n.

Ex: 6

Show $f(x) = \frac{1}{x^2}$ is not uniform continuous in $(0,1)$

Sol^n

Let $s_n = \frac{1}{n}$ $\forall n \in \mathbb{N}$

$\Rightarrow \left(\frac{1}{n}\right)_{n \in \mathbb{N}}$ Cauchy seq^n in $(0,1)$

$f(s_n) = n^2$ should be Cauchy seq^n if f is uniform continuous.

But $\Rightarrow \left(f(s_n) = n^2\right)_{n \in \mathbb{N}}$ is not a Cauchy seq^n.

$\Rightarrow$ f is not uniform continuous on $(0,1)$

Ex 7: $f(x) = x\sin\left(\frac{1}{x}\right)$ for $x \in \left(0, \frac{\pi}{4}\right]$

$$\tilde{f}(x) = \begin{cases} x\sin\frac{1}{x} \\ 0 \end{cases}$$ is an

extension of $f(x)$

Theorem 19.5:

A real-valued function f on (a,b) is uniformly continuous on (a,b) $\Longleftrightarrow$ it can be extended to an continuous $\tilde{f}$ on $[a,b]$.

Proof:

define

$(s_n)_{n\in\mathbb{N}}$ is a converging seq^n in (a,b)

$\lim_{n\to\infty} s_n = a$

then define $\tilde{f}(a) = \lim_{n\to\infty} f(a)$

Ex: 9

$$h(x) = \frac{\sin x}{x} \quad \text{for } x \neq 0$$

The function $\tilde{h} = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$