Exercise : 24 : Uniform Convergence

24.1 let $f_n(x) = \dfrac{1+2\cos^2 nx}{\sqrt{n}}$

Prove carefully that (f_n) converges uniformly to 0 on $\mathbb{R}$

Sol^n

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1+2\cos^2 nx}{\sqrt{n}}$$

$$= \lim \frac{1}{\sqrt{n}} + 2\lim \frac{\cos^2 nx}{\sqrt{n}}$$

$$= 0 + 0$$

$\Rightarrow f_n \longrightarrow 0$ Pointwise on $\mathbb{R}$

Uniform:

maximum of $f_n(x)$

$$\frac{d}{dx}\left(\frac{1+2\cos^2 nx}{\sqrt{n}}\right)=0$$

$$\Rightarrow \frac{1}{\sqrt{n}}\left(0+2\frac{d}{dx}\cos^2 nx\right)=0$$

$$\Rightarrow \frac{d}{dx}\cos^2 nx = 2\cos nx \cdot -\sin nx \cdot n = 0$$

$$\Rightarrow 2\sin nx \cos nx = 0$$

$$\Rightarrow \sin 2nx = 0$$

$$2nx = m\pi \quad \forall m \in \mathbb{N}_0$$

$$x = \frac{m\pi}{2n}$$

take $m=0$ for maximum.

$$\Rightarrow x=0$$

$$f_n(0)=\frac{1+2\cos^2 0}{\sqrt{n}}=\frac{3}{\sqrt{n}}$$

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} |f_n(x) - f(x)|$$

$$= \lim_{n\to\infty} \frac{3}{\sqrt{n}} = 0$$

Hence

$f_n \longrightarrow 0$ uniformly on $\mathbb{R}$

24.2 for $x \in [0,\infty)$, let $f_n(x) = \frac{x}{n}$

(a) find $f(x) = \lim f_n(x)$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{n}$$

$$= x \lim_{n\to\infty} \frac{1}{n} = 0$$

$f(x) = 0$

$\Rightarrow$ $f_n(x) \longrightarrow 0$ Pointwise on $[0,\infty)$

(b) Determine whether $f_n \to f$ uniformly on $[0,1]$

Sol^n for $x \in [0,1]$

$f_n(x)$ has maximum at $x=1$

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

$$\Rightarrow \lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| = 0$$

$\Rightarrow f_n(x) \longrightarrow 0$ uniformly on $[0,1]$

(c) Determine whether $f_n \to f$ uniformly on $[0,\infty)$

$$\sup_{x\in[0,\infty)} |f_n(x) - f(x)| = +\infty$$

$\Rightarrow \lim_{n\to\infty} +\infty = +\infty$

$\Rightarrow$ f_n do not converge uniformly on $[0, \infty)$

24.3 $\quad f_n(x) = \dfrac{1}{1+x^n}$

(a) $\lim_{n\to\infty} \dfrac{1}{1+x^n}$

for $|x|>1$ $\quad \lim_{n\to\infty} x^n = +\infty$ or $-\infty$

$\Rightarrow \lim_{n\to\infty} \dfrac{1}{1+x^n} = 0$

for $x=1$

$\lim_{n\to\infty} \dfrac{1}{2} = \dfrac{1}{2}$

for $|x|<1$

$$\lim_{n\to\infty} x^n = 0$$

$$\lim_{n\to\infty} \frac{1}{1+x^n} = 1$$

for $x=-1$

$\lim_{n\to\infty} \frac{1}{1+x^n}$ is not defined

(b) Determine whether $f_n \to f$ uniformly on $[0,1]$

$$\sup_{x\in[0,1]} \left| f_n(x) - f(x) \right|$$

$$\Rightarrow \sup_{x\in[0,1]} \left| \frac{1}{1+x^n} - 1 \right|$$

$$\frac{d}{dx}\left(1-\frac{1}{1+x^n}\right)=0$$

$$-1\cdot\frac{-1\cdot nx^n}{(1+x^n)^2}=0$$

$$\Rightarrow \frac{nx^n}{(1+x^n)^2}>0$$

$\Rightarrow$ maximum at $x=1$

$$f_n(1)=\frac{1}{2}$$

$$\Rightarrow \lim_{n\to\infty}\sup_{x\in[0,1]}|f_n(x)-f(x)|$$

$$=\lim_{n\to\infty}\frac{1}{2}$$

$$=\frac{1}{2}$$

$\Rightarrow$ $f_n(x)$ does not converge uniformly on $[0,1]$

(c) Determine whether $f_n \to f$ uniformly $[0, \infty)$

Ans:

f_n is continuous function on $(0, \infty)$

f is discontinuous at $x = 1$

$\Rightarrow$ f does not converge uniformly

(24.4) Repeat for $f_n(x) = \frac{x^n}{1+x^n}$

(a) for $|x| < 1$

$$\lim_{n \to \infty} x^n = 0$$

$$\Rightarrow \lim_{n \to \infty} \frac{x^n}{1+x^n} = 0$$

for $x = 1$

$$\lim_{n \to \infty} \frac{x^n}{1+x^n} = \frac{1}{2}$$

for $x > 1$

$$\lim_{n\to\infty} x^n = +\infty$$

$$\lim_{n\to\infty} \frac{x^n}{1+x^n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{x^n}}$$

$$= 1$$

$$\Rightarrow \quad f(x) = \begin{cases} 0 & \forall\ x \in [0,1) \\ \frac{1}{2} & x = 1 \\ 1 & x > 1 \end{cases}$$

b, c

f is discontinuous even though f_n's are continuous

$\Rightarrow$ f does not converge uniformly

24.5 Repeat for $f_n(x) = \frac{x^n}{n + x^n}$

for $0 < x < 1$

$$\lim_{n \to \infty} x^n = 0$$

$$\lim_{n \to \infty} \frac{1}{\frac{n}{x^n} + 1} = 0$$

for $x = 1$

$$\lim_{n \to \infty} \frac{1}{n+1} = 0$$

for $x > 1$

$$\lim_{n \to \infty} \frac{1}{\frac{n}{x^n} + 1} = 1$$

$$f(x) = \begin{cases} 0 & x \in [0,1] \\ 1 & x > 1 \end{cases}$$

(b) Determine whether $f_n \to f$ uniformly on $[0,1]$

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| = \sup_{x \in [0,1]} |f_n(x)|$$

$\Rightarrow$ again its a monotonic increasing function on $[0,1]$

$\Rightarrow$ max at $x = 1$

$\Rightarrow$ $f_n(1) = \dfrac{1}{n+1}$

$\Rightarrow$ $\lim_{n \to \infty} \dfrac{1}{n+1} = 0$

$\Rightarrow$ $f_n(x) \longrightarrow f$ uniformly over $[0,1]$

(c) for $x \in [0, \infty)$

f is discontinoous

$\Rightarrow$ f do not converge uniformly

(24.6) Let $f_n(x) = \left(x - \frac{1}{n}\right)^n$ for $x \in [0,1]$

(a) Pointwise ?

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \left(x - \frac{1}{n}\right)^2$$

$$= \lim_{n \to \infty} \left(x^2 + \frac{1}{n^2} - \frac{2x}{n}\right)$$

$$= x^2$$

$\lim_{n \to \infty} f_n(x) = x^2$ Yes, it

conver's pointwise

(b) Does (f_n) converge uniformly on $[0,1]$

$$\sup_{x\in[0,1]} |f_n(x) - f(x)|$$

$$= \sup_{x\in[0,1]} \left| \left(x-\frac{1}{n}\right)^2 - x^2 \right|$$

$$\Rightarrow \left| x^2 + \frac{1}{n^2} - \frac{2x}{n} - x^2 \right|$$

$$\Rightarrow \left| \frac{1}{n^2} - \frac{2x}{n} \right| = \frac{2x}{n} - \frac{1}{n^2}$$

$$\frac{d}{dx}\left(\frac{2x}{n} - \frac{1}{n^2}\right) = \frac{2}{n} \quad \text{(Linear)}$$

Maximum at $x=1$ $\quad f_n(1) = \left(1-\frac{1}{n}\right)^2$

$$|f_n(1) - f(1)| = \left| \left(1-\frac{1}{n}\right)^2 - 1 \right|$$

$$= \frac{2}{n} - \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{2}{n} - \frac{1}{n^2} = 0$$

$f_n(x) = \left(x - \frac{1}{n}\right)^2$ uniformly converges on $x \in [0,1]$

24.7 $f_n(x) = x - x^n \qquad x \in [0,1]$

a) $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x - x^n$

$\Rightarrow x - \lim_{n\to\infty} x^n = x$ for $x \in [0,1)$

$x-1$ for $x=1$

$$f(x) = \begin{cases} x & \forall x \in [0,1) \\ x-1 & \text{for } x=1 \end{cases}$$

b) f_n does not converge uniformly because f is discontinuous

24.8 $f_n(x) = \sum_{k=0}^{n} x^k = 1 + x^2 + x^3 + \cdots + x^n$

$\lim_{n\to\infty} f_n(x) = \sum_{k=0} x^\infty = \frac{1}{1-x}$ for $|x|<1$

for $x = 1$ $f(1) = \infty$

$\Rightarrow$ $f(x) = \begin{cases} \dfrac{1}{1-x} & |x| < 1 \\ \infty & x = 1 \end{cases}$

(b) on $x \in [0,1]$

$$\sup_{x \in [0,1]} \left| f_n(x) - f(x) \right|$$

24.9 $f_n(x) = nx^n(1-x)$ for $x \in [0,1]$

(a) find $f(x) = \lim f_n(x)$

$$\lim f_n(x) = \lim nx^n(1-x)$$

$$= \lim nx^n - \lim n.x^{n+1}$$

$$= \lim \frac{n}{(1/x)^n} - \lim \frac{n}{\left(\frac{1}{x}\right)^{n+1}}$$

$\left(\frac{1}{x}\right)^n$ grow's expoentially

$$\Rightarrow \lim_{n\to\infty} nx^n(1-x) = 0$$

$\forall [0,1]$

(b) Does $f_n \to f$ uniformly on $[0,1]$?

$$\sup_{x\in[0,1]} |f_n(x) - f(x)|$$

$$= \sup_{x\in[0,1]} |nx^n(1-x)|$$

$$\frac{d}{dx} nx^n(1-x) = n\left[nx^{n-1} - (n+1)x^n\right] = 0$$

$$\Rightarrow nx^{n-1} = (n+1)x^n$$

$$\Rightarrow x = \frac{n}{n+1}$$

$$f_n\left(\frac{n}{n+1}\right) = n \cdot \frac{n^n}{(n+1)^n}\left(1 - \frac{n}{n+1}\right)$$

$$= \frac{n^{n+1}}{(n+1)^n}\left(\frac{1}{n+1}\right)$$

$$= \left(\frac{n}{n+1}\right)^{n+1}$$

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n+1} = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^{n+1}$$

$$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n+1}} = \frac{1}{e}$$

Do not converge uniformly

24.10 (a) Prove that if $f_n \to f$ uniformly on a set S, and if $g_n \to g$ uniformly on set S, then

$$f_n + g_n \longrightarrow f+g \text{ uniformly}$$

on set S.

Sol^n

$$f_n \longrightarrow f \quad \text{uniformly on set } S$$

$$g_n \longrightarrow g \quad \text{uniformly on set } S$$

$$\Rightarrow \lim_{n\to\infty} \sup_{x\in S} |f_n(x) - f(x)| = 0$$

$$\lim_{n\to\infty} \sup_{x\in S} |g_n(x) - g(x)| = 0$$

then

$$\lim_{n\to\infty} \sup_{x\in S} |f_n(x) - f + g_n(x) - g|$$

$$0 \leq \lim_{n\to\infty} \sup_{x\in S} |f_n(x) - f| = 0$$

$$+ \lim_{n\to\infty} \sup_{x\in S} |g_n(x) - g| = 0$$

$\Rightarrow$ $f_n + g_n \longrightarrow f+g$ uniformly on S