10- Monotone seq^n and Cauchy

Definition 10.1:-

A seq^n (s_n) of real number's is called an increasing seq^n if $s_n \leq s_{n+1}$ $\forall n \in \mathbb{N}$, and (s_n) is called a decreasing seq^n if $s_n \geq s_{n+1}$ $\forall n$.

Note that if (s_n) is increasing, then $s_n \leq s_m$ whenever $n < m$.

A seq^n that is increasing or decreasing will be called a monotone seq^n or a monotonic seq^n.

Theorem 10.2:-

All Bounded monotone seq^n Converge.

Proof: Let (s_n) be a Bounded increasing seq^n.

$\Rightarrow$ $S = \{ s_n : n \in \mathbb{N} \}$ set

then $u = \sup S$.

Since S is bounded $\Rightarrow u \in \mathbb{R}$

we need to show $\lim s_n = u$.

Let $\varepsilon > 0$. Since $u - \varepsilon$ is not a upper bound for S. $\exists N$ such that $s_N > u - \varepsilon$.

$\Rightarrow$ $s_n \geq s_N > u - \varepsilon \quad \forall n > N$

& $s_n < u$ (as u is $\sup S$)

$$\Rightarrow \left. \begin{array}{l} s_n < u \\ s_n > u - \varepsilon \end{array} \right\} \Rightarrow |s_n - u| < \varepsilon$$

$\Rightarrow$ This show's $\lim s_n = u$

Theorem 10.4

(i) if (S_n) is an unbounded increasing seq^n; then $\lim S_n = +\infty$

(ii) if (S_n) is an unbounded decreasing seq^n, then $\lim S_n = -\infty$.

Proof:

(i) S_n be an unbounded increasing seq^n

Let $M > 0$, since the set $\{S_n : n \in \mathbb{N}\}$ is unbounded and it is bounded below by S_1, and unbounded above.

Hence $\exists N \in \mathbb{N}$ such that $S_N > M$.

$$S_N > M \Rightarrow \forall n > N \quad S_n > M$$

(increasing seq)

$$\Rightarrow \lim S_n = +\infty$$

10.5 Corollary:

if (s_n) is a monotone seq^n, then the seq either converges, diverges to $+\infty$ or $-\infty$. Thus $\lim s_n$ is always meaningful for monotone seq^n.

Proof:

if (s_n) monotone + bounded

$\Rightarrow \lim s_n \in \mathbb{R}$

if s_n monotone + unbounded

$\lim s_n \in \{-\infty, \infty\}$

either way $\lim s_n$ exists

Let (S_n) be a bounded seq^n

$\Rightarrow$ It may or may not converge
(yet to prove: if (S_n) is bounded, $\exists$ a sub-seq^n that converges)

$\Rightarrow$ The limiting behaviour of (S_n) depends only on sets of the form $\{S_n : n > N\}$

$\Rightarrow$ if $\lim S_n$ exists, clearly it lies in the interval $[u_N, v_N]$, where

$$u_N = \inf\{S_n : n > N\}$$
$$v_N = \sup\{S_n : n > N\}$$

as N increases u_N increases
v_n decreases.

$\Rightarrow$ $u_1 \leq u_2 \leq u_3 \leq u_4 \leq u_5 \ldots$

$\ldots v_5 \leq v_4 \leq v_3 \leq v_2 \leq v_1$

$\Rightarrow \quad u = \lim_{N\to\infty} U_N \ , \ V = \lim_{n\to\infty} V_N$

(Exists: Bounded, monotone seq^n converges)

and $U \le V$ since $U_N \le V_N \ \forall N \in \mathbb{N}$

$\Rightarrow (S_n)$ limit exist

$U_N \le \lim S_N \le V_N \quad \forall N \in \mathbb{N}$

$\Rightarrow \quad u \le \lim S_N \le V_N$

Definition 10.6 :-

Let (S_n) be a seq^n in $\mathbb{R}$, we define

$$\lim\sup S_n = \lim_{N\to\infty} \sup\{S_n : n > N\}$$

and

$$\lim\inf S_n = \lim_{N\to\infty} \inf\{S_n : n > N\}$$

Theorem 10.7

Let (s_n) be a seq^n in $\mathbb{R}$

(i) if $\lim s_n$ is defined [as a real number, $+\infty$ or $-\infty$], then

$$\liminf s_n = \lim s_n = \limsup s_n$$

(ii) if $\liminf s_n = \limsup s_n$, then

$\lim s_n$ is defined and $\lim s_n = \liminf s_n$

$= \limsup s_n$.

Proof:

$$u_N = \inf\{s_n : n > N\}$$

$$v_N = \sup\{s_n : n > N\}$$

$$u = \lim_{N\to\infty} u_N, \qquad v = \lim_{N\to\infty} v_N$$

(i) Suppose $\lim_{n\to\infty} s_n = +\infty$. Let M

be a positive number. Then $\exists N \in \mathbb{N}$

s.t $\quad \forall n > N \quad s_n > M$

Then $\quad u_N = \inf\{s_n : n > N\} \geq M$

Therefore $\quad \lim u_N = +\infty$

$\Rightarrow \quad \lim\inf s_n = +\infty$

|||ly $\quad \lim\sup s_n = +\infty$

Definition 10.8:

A seq^n (S_n) of real number's is called a Cauchy seq^n if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, such that $\forall m, n > N$

$$|S_n - S_m| < \varepsilon$$

10.9 Lemma:

Convergent seq^n are Cauchy seq^n

Proof.

$$|S_n - S_m| = |S_n - S + S - S_m|$$
$$\leq |S_n - S| + |S_m - S|$$

To be Presise, let $\varepsilon > 0$, $\exists N$ s.t

$\forall n > N$ $\quad |S_n - S| < \frac{\varepsilon}{2}$

& $\quad |S_m - S| < \frac{\varepsilon}{2}$

$\Rightarrow \forall m,n > N \quad |S_n - S| + |S_m - S| < \varepsilon$

$\Rightarrow \quad |S_n - S_m| \leq |S_n - S| + |S_m - S| < \varepsilon$

$\Rightarrow \quad |S_n - S_m| < \varepsilon$

$\Rightarrow$ Cauchy seq^n

Lemma 10.10

Cauchy seq^n are bounded.

Proof:

let $\varepsilon = 1$ $\exists N \in \mathbb{N}$, s.t $\forall m,n > N$

$|S_n - S_m| < 1 \Rightarrow |S_n - S_{N+1}| < 1$

apply triangular inequality to

$S_n - S_{N+1}, \quad S_{N+1}$

$$\Rightarrow |S_n - S_{N+1} + S_{N+1}| \leq |S_n - S_{N+1}| + |S_{N+1}|$$

$$\Rightarrow |S_n| \leq |S_n - S_{N+1}| + |S_{N+1}| \leq 1 + |S_{N+1}|$$

$$\Rightarrow |S_n| \leq 1 + |S_{N+1}| \quad \forall n > N$$

take

$$M = \max\{1 + |S_{N+1}|, |S_1|, |S_2|, \ldots, |S_N|\}$$

Theorem 10.11

A seq^n is a convergent seq^n $\Longleftrightarrow$ Cauchy seq^n

Proof:

(i) Convergent seq^n $\Rightarrow$ Cauchy seq^n

(ii) Cauchy seq^n $\Rightarrow$ Convergent seq^n

Important Properties

① All Bounded monotone seq^n converge

② if (S_n) is an unBounded increasing seq^n, then $\lim_{n\to\infty} S_n = +\infty$

if (S_n) is an unbounded decreasing seq^n, then $\lim_{n\to\infty} S_n = -\infty$

③ if (S_n) is monotone, then the seq^n either converges, or diverges to $+\infty$ or diverges to $-\infty$

④

$$\lim_{n\to\infty} \sup S_n = \lim_{N\to\infty} \sup\{S_n : n > N\}$$

$$\lim_{n\to\infty} \inf S_n = \lim_{N\to\infty} \inf\{S_n : n > N\}$$

⑤ Cauchy seq^n are Bounded

Cauchy seq^n $\Longleftrightarrow$ Convergent seq^n