Sec 12: Lim sup's and lim inf's

let (s_n) be any seq of real numbers, and let S be the set of subsequential limits of (s_n)

then

$$\limsup_{n\to\infty} s_n = \lim_{N\to\infty} \sup\{s_n : n > N\}$$
$$= \sup S$$

$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} \inf\{s_n : n > N\}$$
$$= \inf S$$

Theorem 12.1

if (s_n) converges to any positive real number s and (t_n) is any seq^n then

$$\limsup s_n t_n = s \cdot \limsup t_n$$

where $s \cdot +\infty = +\infty \quad \forall s > 0$

$s \cdot (-\infty) = -\infty \quad \forall s > 0$

Proof:

$$\limsup s_n t_n \geq s \cdot \limsup t_n$$

let $\beta = \limsup t_n$

CASE 1 :- β = finite

$\Rightarrow \exists\, (t_{n_k})$ of (t_n) s.t

$$\lim_{k \to \infty} t_{n_k} = \beta$$

& we have $\lim_{k \to \infty} s_{n_k} = s$

we know

$$\lim_{k \to \infty} s_{n_k} t_{n_k} = s \cdot \beta$$

$\Rightarrow$ $(s_{n_k} \cdot t_{n_k})_k$ is a subseq^n of $(s_n \cdot t_n)_n$ converging to $s\beta$.

$$\Rightarrow \quad \underbrace{\lim_{k\to\infty} (s_{n_k} \cdot t_{n_k})}_{} \leq \underbrace{\limsup_{n\to\infty} (s_n \cdot t_n)}_{\text{largest possible limit of seq}^n \text{ of } (s_n t_n)}$$

CASE 2: $\beta = \limsup_{k\to\infty} t_{n_k} = +\infty$

$\Rightarrow$ $\exists\ (t_{n_k})$ of t_n s.t. $\lim_{k\to\infty} t_{n_k} = +\infty$

Since $\lim_{k\to\infty} s_{n_k} = s > 0$.

$$\Rightarrow \lim_{k\to\infty} s_{n_k} t_{n_k} = +\infty$$

$$\& \quad \limsup_{n\to\infty} (s_n \cdot t_n) = +\infty$$

Theorem 12.2

Let (s_n) be any seq^n of nonzero real number's. Then we have

$$\liminf \left|\frac{s_{n+1}}{s_n}\right| \le \liminf |s_n|^{1/n} \le \limsup |s_n|^{1/n}$$
$$\le \limsup \left|\frac{s_{n+1}}{s_n}\right|$$

Proof:

$$\limsup |s_n|^{1/n} \le \limsup \left|\frac{s_{n+1}}{s_n}\right|$$

Let $\alpha = \limsup |s_n|^{1/n}$

and $L = \limsup \left|\frac{s_{n+1}}{s_n}\right|$

we need to prove $\alpha \le L$

Assume $L < +\infty$.

we can show $\alpha \le L_1$ for any $L_1 > L$

$$L = \limsup \left| \frac{s_{n+1}}{s_n} \right|$$

$$= \lim_{N \to \infty} \sup \left\{ \left| \frac{s_{n+1}}{s_n} \right| : n > N \right\} < L_1$$

$$\Rightarrow \left| \frac{s_{n+1}}{s_n} \right| < L_1 \quad \text{for} \quad n \geq N$$

for $n > N$ we can write

$$|s_n| = \left| \frac{s_n}{s_{n-1}} \right| \left| \frac{s_{n-1}}{s_{n-2}} \right| \cdots \left| \frac{s_{N+1}}{s_N} \right| \cdot |s_N|$$

$$\Rightarrow |s_n| < L_1^{n-N} |s_N| \quad \forall n > N$$

$$a = \frac{|s_N|}{L_1^N}$$

$$\Rightarrow |s_n| < L_1^n . a$$

$$\Rightarrow |s_n|^{1/n} < L_1 a^{1/n} \quad \forall n > N$$

$\Rightarrow$ since $\lim a^{1/n} = 1$

$$\Rightarrow \quad \limsup |s_n|^{1/n} \leq L_1$$

$$\Rightarrow \quad \alpha \leq L_1 \quad \forall L_1 > L$$

12.3 Corollary:

if $\lim \left|\frac{s_{n+1}}{s_n}\right|$ exists, then $\lim |s_n|^{1/n}$ exists. and Both are equal