# 2 Signals and Systems: Part I

[solution's at the end]

## Recommended Problems

### P2.1

Let $x(t) = \cos(\omega_x(t + \tau_x) + \theta_x)$.

**(a)** Determine the frequency in hertz and the period of $x(t)$ for each of the following three cases:

| | $\omega_x$ | $\tau_x$ | $\theta_x$ |
|---|---|---|---|
| (i) | $\pi/3$ | 0 | $2\pi$ |
| (ii) | $3\pi/4$ | 1/2 | $\pi/4$ |
| (iii) | 3/4 | 1/2 | 1/4 |

**(b)** With $x(t) = \cos(\omega_x(t + \tau_x) + \theta_x)$ and $y(t) = \sin(\omega_y(t + \tau_y) + \theta_y)$, determine for which of the following combinations $x(t)$ and $y(t)$ are identically equal for all $t$.

| | $\omega_x$ | $\tau_x$ | $\theta_x$ | $\omega_y$ | $\tau_y$ | $\theta_y$ |
|---|---|---|---|---|---|---|
| (i) | $\pi/3$ | 0 | $2\pi$ | $\pi/3$ | 1 | $-\pi/3$ |
| (ii) | $3\pi/4$ | 1/2 | $\pi/4$ | $11\pi/4$ | 1 | $3\pi/8$ |
| (iii) | 3/4 | 1/2 | 1/4 | 3/4 | 1 | 3/8 |

### P2.2

Let $x[n] = \cos(\Omega_x(n + P_x) + \theta_x)$.

**(a)** Determine the period of $x[n]$ for each of the following three cases:

| | $\Omega_x$ | $P_x$ | $\theta_x$ |
|---|---|---|---|
| (i) | $\pi/3$ | 0 | $2\pi$ |
| (ii) | $3\pi/4$ | 2 | $\pi/4$ |
| (iii) | 3/4 | 1 | 1/4 |

**(b)** With $x[n] = \cos(\Omega_x(n + P_x) + \theta_x)$ and $y[n] = \cos(\Omega_y(n + P_y) + \theta_y)$, determine for which of the following combinations $x[n]$ and $y[n]$ are identically equal for all $n$.

| | $\Omega_x$ | $P_x$ | $\theta_x$ | $\Omega_y$ | $P_y$ | $\theta_y$ |
|---|---|---|---|---|---|---|
| (i) | $\pi/3$ | 0 | $2\pi$ | $8\pi/3$ | 0 | 0 |
| (ii) | $3\pi/4$ | 2 | $\pi/4$ | $3\pi/4$ | 1 | $-\pi$ |
| (iii) | 3/4 | 1 | 1/4 | 3/4 | 0 | 1 |

### P2.3

**(a)** A discrete-time signal $x[n]$ is shown in Figure P2.3.

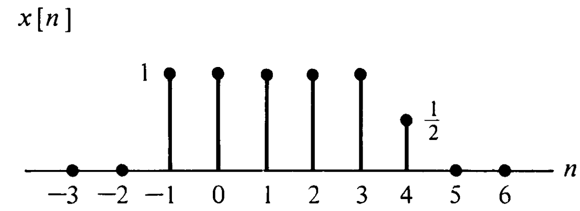



**Figure P2.3**

Sketch and carefully label each of the following signals:

(i) $x[n-2]$

(ii) $x[4-n]$

(iii) $x[2n]$

**(b)** What difficulty arises when we try to define a signal as $x[n/2]$?

## P2.4

For each of the following signals, determine whether it is even, odd, or neither.

**(a)**

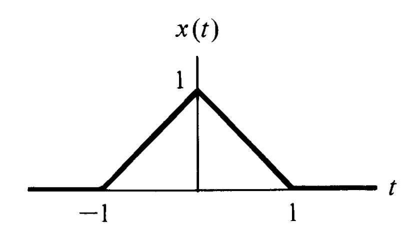



**Figure P2.4-1**

**(b)**

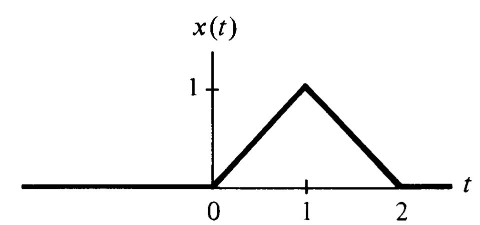



**Figure P2.4-2**

**(c)**

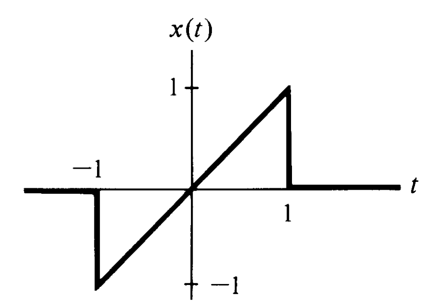



**Figure P2.4-3**

**(d)**

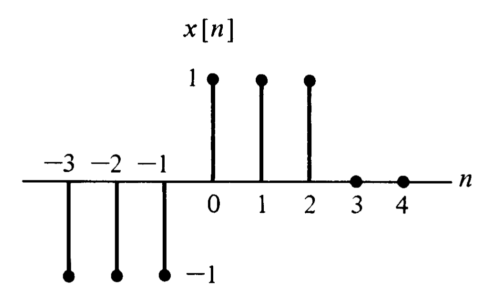



**Figure P2.4-4**

(e)

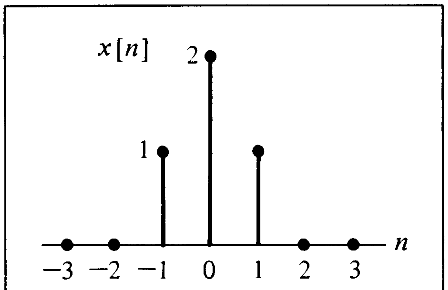


**Figure P2.4-5**

(f)

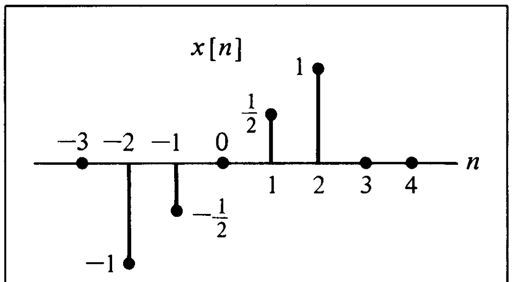


**Figure P2.4-6**

## P2.5

Consider the signal $y[n]$ in Figure P2.5.

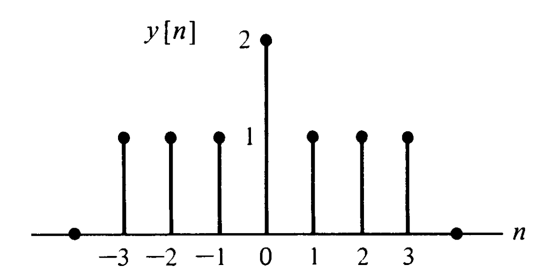


**Figure P2.5**

**(a)** Find the signal $x[n]$ such that $Ev\{x[n]\} = y[n]$ for $n \geq 0$, and $Od\{x[n]\} = y[n]$ for $n < 0$.

**(b)** Suppose that $Ev\{w[n]\} = y[n]$ for all $n$. Also assume that $w[n] = 0$ for $n < 0$. Find $w[n]$.

## P2.6

**(a)** Sketch $x[n] = \alpha^n$ for a typical $\alpha$ in the range $-1 < \alpha < 0$.

**(b)** Assume that $\alpha = -e^{-1}$ and define $y(t)$ as $y(t) = e^{\beta t}$. Find a complex number $\beta$ such that $y(t)$, when evaluated at $t$ equal to an integer $n$, is described by $(-e^{-1})^n$.

**(c)** For $y(t)$ found in part (b), find an expression for $Re\{y(t)\}$ and $Im\{y(t)\}$. Plot $Re\{y(t)\}$ and $Im\{y(t)\}$ for $t$ equal to an integer.

## P2.7

Let $x(t) = \sqrt{2}(1 + j)e^{j\pi/4}e^{(-1+j2\pi)t}$. Sketch and label the following:

**(a)** $Re\{x(t)\}$

**(b)** $Im\{x(t)\}$

**(c)** $x(t + 2) + x^*(t + 2)$

## P2.8

Evaluate the following sums:

**(a)** $\sum_{n=0}^{5} 2\left(\frac{3}{a}\right)^n$

**(b)** $\sum_{n=2}^{6} b^n$

**(c)** $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{2n}$

*Hint:* Convert each sum to the form

$$C \sum_{n=0}^{N-1} \alpha^n = S_N \qquad \text{or} \qquad C \sum_{n=0}^{\infty} \alpha^n = S_\infty$$

and use the formulas

$$S_N = C\left(\frac{1-\alpha^N}{1-\alpha}\right), \qquad S_\infty = \frac{C}{1-\alpha} \qquad \text{for } |\alpha| < 1$$

## P2.9

**(a)** Let $x(t)$ and $y(t)$ be periodic signals with fundamental periods $T_1$ and $T_2$, respectively. Under what conditions is the sum $x(t) + y(t)$ periodic, and what is the fundamental period of this signal if it is periodic?

**(b)** Let $x[n]$ and $y[n]$ be periodic signals with fundamental periods $N_1$ and $N_2$, respectively. Under what conditions is the sum $x[n] + y[n]$ periodic, and what is the fundamental period of this signal if it is periodic?

**(c)** Consider the signals

$$x(t) = \cos\frac{2\pi t}{3} + 2\sin\frac{16\pi t}{3},$$
$$y(t) = \sin \pi t$$

Show that $z(t) = x(t)y(t)$ is periodic, and write $z(t)$ as a linear combination of harmonically related complex exponentials. That is, find a number $T$ and complex numbers $c_k$ such that

$$z(t) = \sum_k c_k e^{jk(2\pi/T)t}$$

## P2.10

In this problem we explore several of the properties of even and odd signals.

**(a)** Show that if $x[n]$ is an odd signal, then

$$\sum_{n=-\infty}^{+\infty} x[n] = 0$$

**(b)** Show that if $x_1[n]$ is an odd signal and $x_2[n]$ is an even signal, then $x_1[n]x_2[n]$ is an odd signal.

**(c)** Let $x[n]$ be an arbitrary signal with even and odd parts denoted by

$$x_e[n] = Ev\{x[n]\}, \qquad x_o[n] = Od\{x[n]\}$$

Show that

$$\sum_{n=-\infty}^{+\infty} x^2[n] = \sum_{n=-\infty}^{+\infty} x_e^2[n] + \sum_{n=-\infty}^{+\infty} x_o^2[n]$$

**(d)** Although parts (a)–(c) have been stated in terms of discrete-time signals, the analogous properties are also valid in continuous time. To demonstrate this, show that

$$\int_{-\infty}^{+\infty} x^2(t)\,dt = \int_{-\infty}^{+\infty} x_e^2(t)\,dt + \int_{-\infty}^{+\infty} x_o^2(t)\,dt,$$

where $x_e(t)$ and $x_o(t)$ are, respectively, the even and odd parts of $x(t)$.

## P2.11

Let $x(t)$ be the continuous-time complex exponential signal $x(t) = e^{j\omega_0 t}$ with fundamental frequency $\omega_0$ and fundamental period $T_0 = 2\pi/\omega_0$. Consider the discrete-time signal obtained by taking equally spaced samples of $x(t)$. That is, $x[n] = x(nT) = e^{j\omega_0 nT}$.

**(a)** Show that $x[n]$ is periodic if and only if $T/T_0$ is a rational number, that is, if and only if some multiple of the sampling interval *exactly equals* a multiple of the period $x(t)$.

**(b)** Suppose that $x[n]$ is periodic, that is, that

$$\frac{T}{T_0} = \frac{p}{q}, \qquad \text{(P2.11-1)}$$

where $p$ and $q$ are integers. What are the fundamental period and fundamental frequency of $x[n]$? Express the fundamental frequency as a fraction of $\omega_0 T$.

**(c)** Again assuming that $T/T_0$ satisfies eq. (P2.11-1), determine precisely how many periods of $x(t)$ are needed to obtain the samples that form a single period of $x[n]$.

P2.1

$$x(t) = \cos(\omega_x(t+T_x)+\theta_x)$$

a) i

$\omega_x = \frac{\pi}{3}$, $T_x = 0$ $\theta_x = 2\pi$

$$x(t) = \cos\left(\frac{\pi}{3}t + 2\pi\right)$$

$$\Rightarrow \quad x(t) = \cos\frac{\pi}{3}t$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi/3} = 6 \text{ sec}$$

$$f = \frac{1}{T} = \frac{1}{6} \text{ Hz}$$

ii

$\omega = \frac{3\pi}{4}$ $T_x = 1/2$ $\theta_x = \frac{\pi}{4}$

$$T = \frac{2\pi}{\frac{3\pi}{4}} = \frac{8}{3} \text{ sec} \qquad f = \frac{3}{8} \text{ Hz}$$

iii

$\omega_x = \frac{3}{4}$

$$T = \frac{2\pi}{3/4} = \frac{8\pi}{3} \text{ sec}$$

$$f = \frac{3}{8\pi} \text{ Hz}$$

2.1 b

$$x(t) = \cos(\omega_x(t+T_x)+\theta_x)$$

$$y(t) = \sin(\omega_x(t+T_y)+\theta_y)$$

(i) $x(t) = \cos\left(\frac{\pi}{3}t + 2\pi\right)$

$y(x) = \sin\left(\frac{\pi}{3}t + \frac{\pi}{3} - \frac{\pi}{3}\right)$ (Not equal)

(ii) $x(t) = \cos\left(\frac{3\pi}{4}t + \frac{5\pi}{8}\right)$

$y(t) = \cos\left(\frac{11\pi}{4}t + \phi\right)$ } diff freq (not equal)

(iii) $x(t) = \cos\left(\frac{3}{4}t + \frac{5}{8}\right)$

$y(t) = \sin\left(\frac{3}{4}t + \frac{9}{8}\right)$ (Not equal)

P2.2 (a) $x[n] = \cos\left(\Omega_x(n + P_x) + \theta_x\right)$

(i) $x[n] = \cos\left(\frac{\pi}{3}n + 2\pi\right) = \cos\left(\frac{\pi}{3}n\right)$

$\frac{\pi}{3}N = 2\pi m$ for an interger $N, m$

$\Rightarrow N = 6,\ m = 1$ ($\Rightarrow$ Period $= 6$)

(ii) $x[n] = \cos\left(\frac{3\pi}{4}n + \frac{3\pi}{2} + \frac{\pi}{4}\right)$

$= \cos\left(\frac{3\pi}{4}n + \frac{7\pi}{4}\right)$

$$\frac{3\pi}{4} N = 2\pi m$$

$$N = \frac{8}{3} m \Rightarrow N = 8, \quad m = 3$$

Period $= 8$

(iii)

$$x[n] = \cos\left(\frac{3}{4} n + 2\right)$$

$$\frac{3}{4} N = 2\pi m$$

$$N = \frac{8\pi}{3} m \quad \text{(Not Periodic)}$$

P2.2 b

$$x[n] = \cos(\Omega_x(n + P_x) + \theta_x)$$

$$y[n] = \cos(\Omega_y(n + P_y) + \theta_y)$$

(i) $\omega_x = \frac{\pi}{3}$ $\quad \omega_y = \frac{8\pi}{3}$ $\Rightarrow$ Not equal

(ii) $x(n) = \cos\left(\frac{3\pi}{4} n + \frac{7\pi}{4}\right)$

$$y(n) = \cos\left(\frac{3\pi}{4} n - \frac{\pi}{4}\right) = \cos\left(\frac{3\pi}{4} n + \frac{7\pi}{4}\right)$$

same $\forall n$

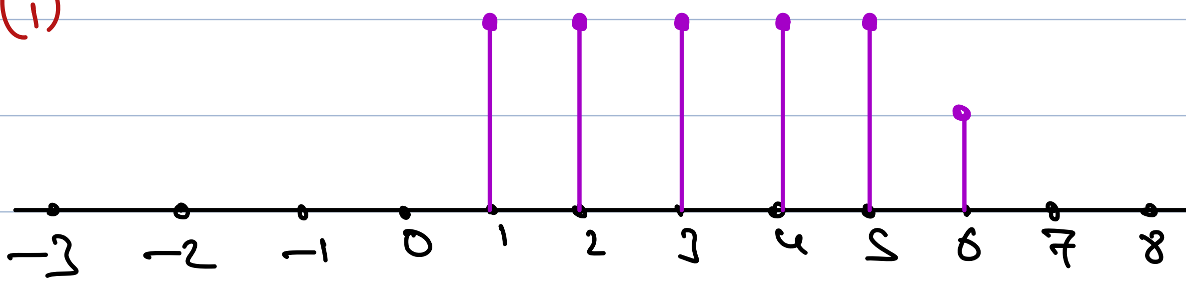
(iii) $x[n] = \cos\left(\frac{3}{4}n + 1\right)$

$y(n) = \cos\left(\frac{3}{4}n + 1\right)$

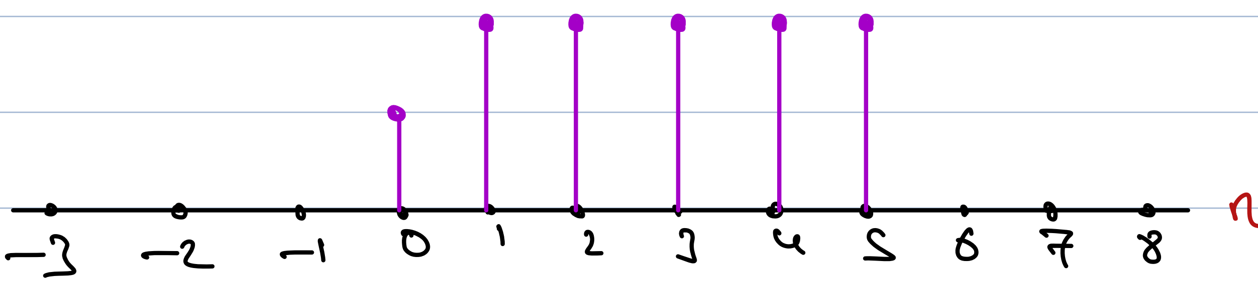
$x[n] = y[n]$
$\forall n$

**P2.3a**

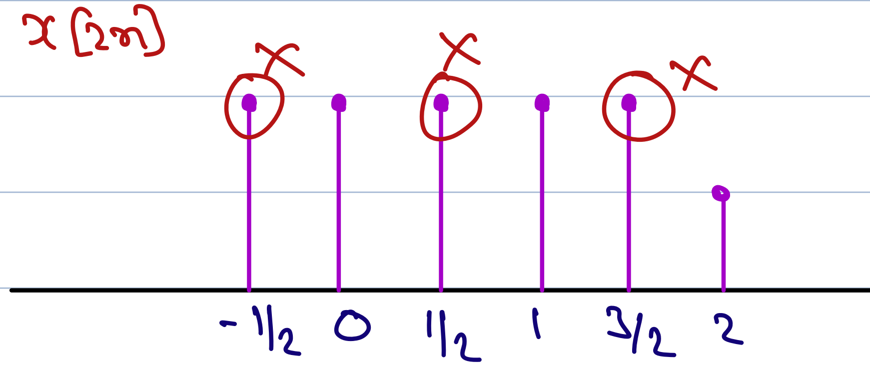
(i)




(ii) $x[4-n]$




(iii) $x[2n]$



2.4

ⓐ even

ⓑ neither

ⓒ odd

ⓓ neither

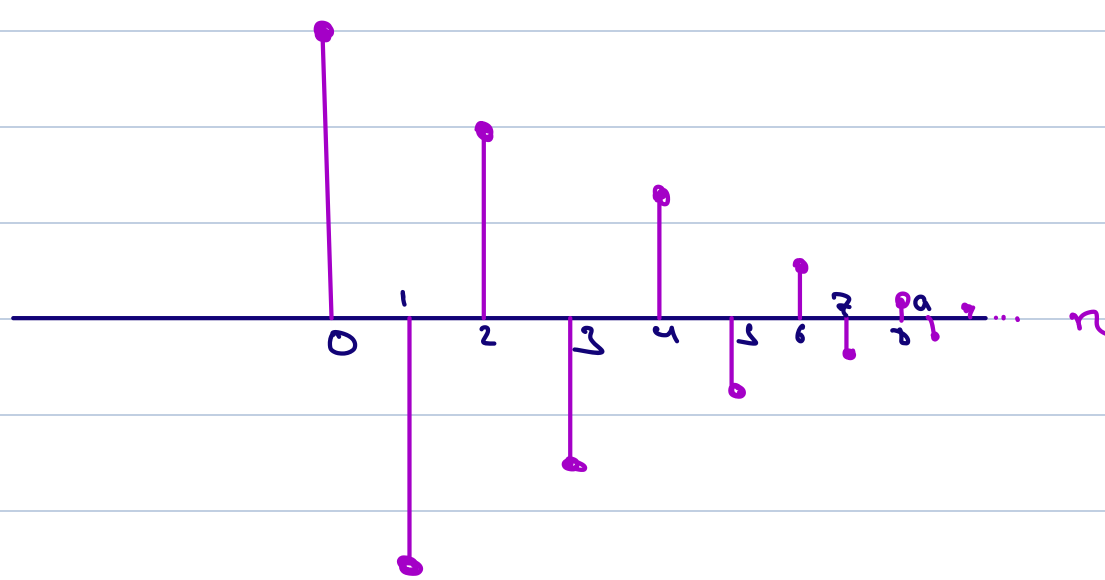
ⓔ even

ⓕ odd

P2.6

ⓐ $x[n] = \alpha^n \qquad -1 < \alpha < 0$

Shrinking as $n$ increases and alternating for even & odd



2.7

$$x(t) = \sqrt{2}(1+j)\, e^{j\pi/4}\, e^{(-1+j2\pi)t}$$

$$\Rightarrow \quad x(t) = \sqrt{2} \cdot \sqrt{2}\, e^{\frac{\pi}{4}j}\, e^{j\frac{\pi}{4}}\, e^{-t}\, e^{2\pi j t}$$

$$x(t) = 2e^{-t}\, e^{j\left(\frac{\pi}{4} + 2\pi t\right)}$$

$$x(t) = 2e^{-t}\left[\cos\left(2\pi t + \frac{\pi}{4}\right) + i\sin\left(2\pi t + \frac{\pi}{4}\right)\right]$$

(a) $$\mathrm{Re}\{x(t)\} = 2e^{-t}\cos\left(2\pi t + \frac{\pi}{4}\right)$$

(b) $$\mathrm{Im}\{x(t)\} = 2e^{-t}\sin\left(2\pi t + \frac{\pi}{4}\right)$$

(c) $$x(t+2) + x^*(t+2)$$

$$= 2\,\mathrm{Re}\{x(t+2)\}$$

$$= 4e^{-t}\cos\left(2\pi(t+2) + \frac{\pi}{4}\right)$$