Lec05: Properties of LTI system's

* for LTI system's if we know the response of system to single impulse at $t=0$, or in fact at any time, then we can determine from that its response to an arbitrary input through an convolution.

* The class of LTI system's is very rich class, there are lot of system's that have that property, and in addition, there are lot's of very interesting thing's we can do with LTI system's.

Today we will focus on, convolution as an algebraic operation. then we see that it has many algebraic properties, that in turn have important implication for LTI

* we will discuss what the characterization of LTI system's through convolution
⟹ in term's of the relationship with various other system properties to the Impulse response

Convolution sum:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\,h[n-k] = x[n] * h[n]$$

Convolution - integral

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau = x(t) * h(t)$$

Convolution as algebraic operation has number of properties

① **Commutative:**

$$x * h = h * x$$

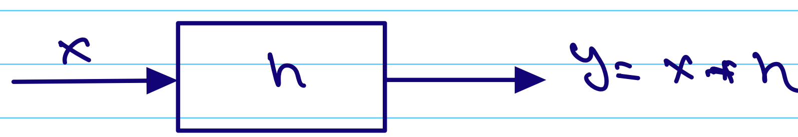
Order doesn't effect output result.

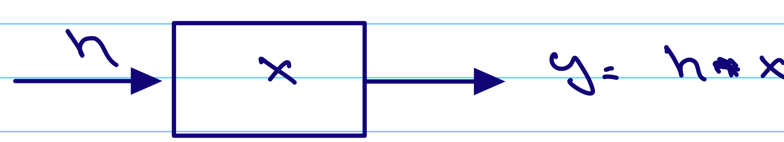
$$x[n] * h[n] = h[n] * x[n]$$

$$x(t) * h(t) = h(t) * x(t)$$

drop the independent variable

$$x * h = h * x$$





<u>LTI system:</u> output the same if input & output response interchanged

2 Associative:

$$x * \{h_1 * h_2\} = \{x * h_1\} * h_2$$

LTI system can be cascaded in any order

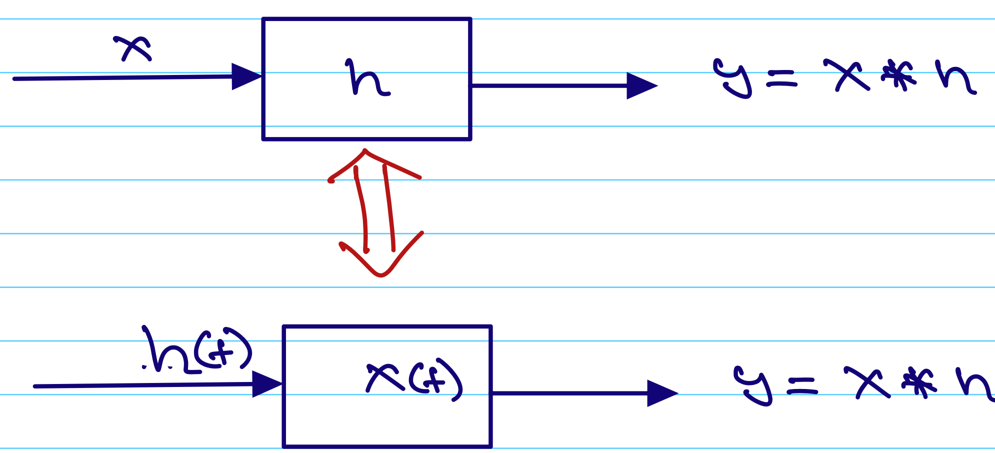
3 Distributive:

$$x * \{h_1 + h_2\} = x * h_1 + x * h_2$$

Commutative Property:

$h(t) \longrightarrow$ impulse response

$x(t) \longrightarrow$ input signal



$x(t) =$ impulse response

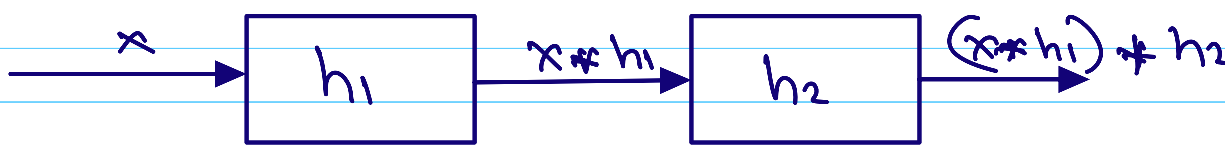
$h(t) =$ input signal

The commutative property tell us for LTI
The system output is independent of which function we call the input, which function we call the Impulse response

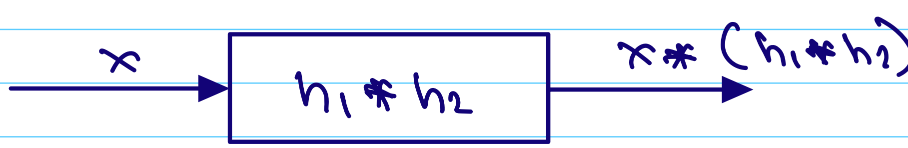
⇒ we can interchange the role of input & impulse response, from output point of view the output of system doesn't care.

Commutative + associative:

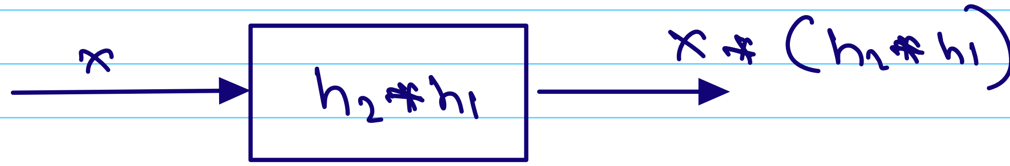
LTI can be cascaded in any order



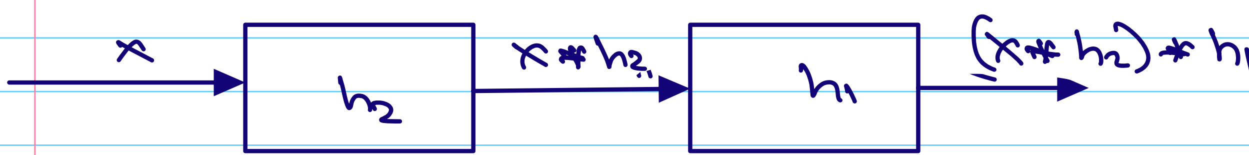
Associative:



Commutative:



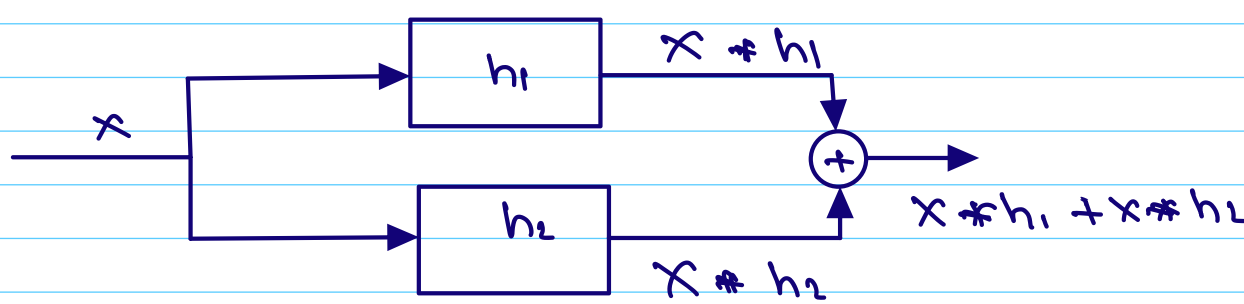
Associative:



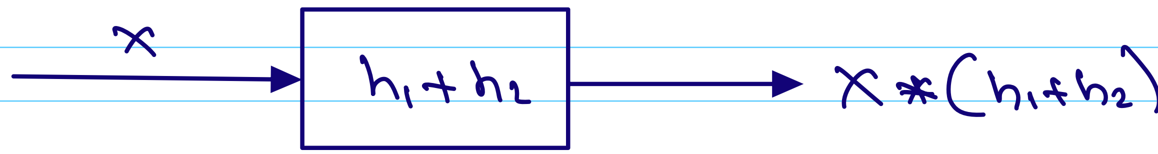
The result say's, if we have 2 LTI system's that are cascade, we can cascade in any order & the result is same.

Distributive:

inter-connection of system's



Convolution distributes over addition



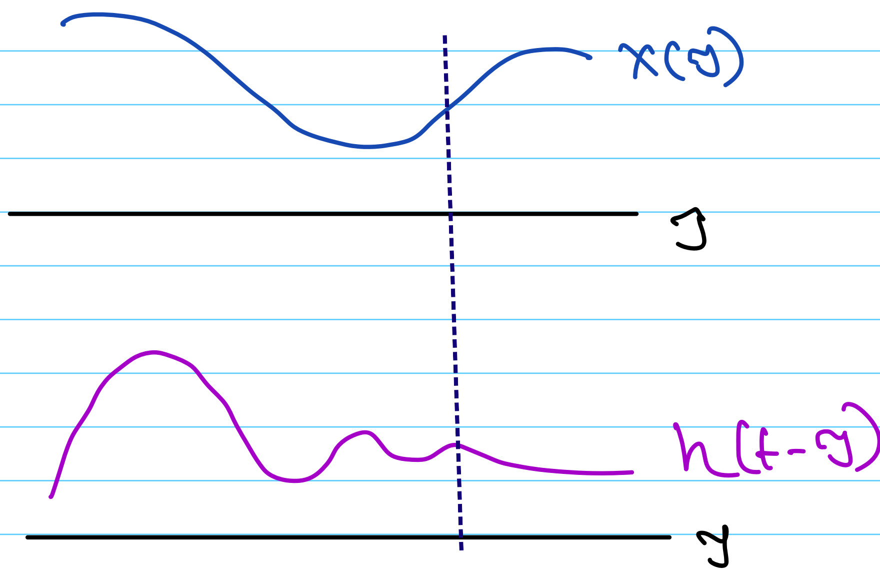
for LTI system's in parallel, we can replace that interconnection by a single system who's impulse response is sum of Impulse responses.

System Properties:

* other system properties and see how for LTI in particular, other system properties can be associated with particular properties are characteristics of system impulse response.

Memory:

what are the implication's for system impulse response for LTI system's if the does or does not have memory.



$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

what can we say about $h(t)$ in order to gaurante that output $y(t)$ depend's only on input at time t

from graph's, if we only want the output to depend on $x(\tau)$ at $\tau = t$, then $h(t-\tau)$ is better non zero at $\tau = t$

$$\Rightarrow \quad y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$\Rightarrow \quad y(t) = \int_{t-\varepsilon}^{t+\varepsilon} x(t)\, h(t-\tau)\, d\tau$$

we want impulse response to be non zero at one point $\Rightarrow h(t) = K\,\delta(t)$

(scaled impulse

$$y(t) = x(t) \cdot K$$

$\Rightarrow$ LTI + memoryless

$$h(t) = K\,\delta(t) \implies y(t) = K\,x(t)$$

$$h[n] = K\,\delta[n] \implies y[n] = K\,x[n]$$

Invertability:

The inverse of a system, is a system which when we cascaded with the one we are enquiring about, the overall cascade is the identity system (output = input)

$$x \longrightarrow \boxed{h} \longrightarrow \boxed{h_i} \longrightarrow y$$

$$y = x * (h * h_i) = x$$

$$h_i = h^{-1}$$

$$x * \underbrace{(h * h_i)}_{\delta} = x$$

$$h_i = h^{-1}$$

Stability:

Every Bounded input $\Longrightarrow$ Bounded output

(BIBO)

Stable: $$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

$$\int_{-\infty}^{+\infty} |h(\tau)|\, d\tau < \infty$$

Zero Input Response:

$$x(t) = 0 \quad \forall t$$
$$\downarrow$$
$$y(t) = 0 \quad \forall t$$

for a linear system - (whether it is Linear Time variant or invariant), this applies

$\Rightarrow$ if you put nothing into it, you get nothing out of it.

$\Rightarrow$ if we have an input $x(t) = 0 \quad \forall t$

and if the output $y(t)$ then

if $x(t) = 0 \quad \forall t \Rightarrow y(t) = 0 \quad \forall t$

$x[n] = 0 \quad \forall n \Rightarrow y[n] = 0 \quad \forall n$

true for all linear systems

$$x(t) \longrightarrow y(t)$$

Then

$$a\,x(t) \longrightarrow a\,y(t)$$

we can simple choose $a = 0$

$$\Rightarrow \quad 0 \Longrightarrow 0$$

This has some important implication in term's of causality.

Causality:

The system can't anticipate the inputs

$y(t_0)$ only depend's on the $x(t)$ for $t < t_0$. It cannot anticipate the future value's of input to produce the output at a given time.

if two input's identical up until some time, then the output's must be identical up until the same time.

$\Rightarrow$ If the system causal, it can't anticipate the future

if $x_1(t) = x_2(t) \quad t < t_0$

Then $y_1(t) = y_2(t) \quad t < t_0$

Same for D.T

for Linear system:

if $x(t) = 0 \quad t < t_0 \implies y(t) = 0 \quad t < t_0$

what that in effect say's that, for the Linear system to be Causal, it must have have the Property called initial rest. meaning It doesn't respond until some input that happen's

* It's initially at rest until the input become's non-Zero

$\Rightarrow$ if we have an input that's 0 for $t < t_0$, and the system can't anticipate whether that input is going to change from 0 or not, then The system must generate an output $y(t) = 0$ for $t < t_0$

Linear system:

if $x(t) = 0 \quad t < t_0 \implies y(t) = 0 \quad t < t_0$

Same for D.T

Consequence of Zero in $\longrightarrow$ Zero out.

this tells us how to interpret causality for Linear system's, now let's Proceed to LTI system's

* in Particular, the necessary and sufficient condition for causality in case of LTI systems, is that the impulse response $h(t) = 0$ for $t<0$

$$\text{Causality} \iff \begin{cases} h(t) = 0 & t<0 \\ h[n] = 0 & n<0 \end{cases}$$

$$\left.\begin{array}{ll} \Rightarrow \delta(t) = 0 & t<0 \\ \Rightarrow \delta[n] = 0 & n<0 \end{array}\right\} \text{output need's to be 0 until } t<0$$

$$\Rightarrow h(t) = 0 \quad \forall\, t<0$$

$$h[n] = 0 \quad \forall\, n<0$$

$$\Longleftarrow$$

$$y[n] = \sum_{k=-\infty}^{\cancel{+\infty}\; n} x[k]\, h[n-k]$$

$$y(t) = \int_{-\infty}^{\cancel{+\infty}\; t} x(\tau)\, h(t-\tau)\, d\tau$$

Example:

Accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

Accumulated value of the input.

* Accumulator system is LTI

$x(t) = \delta(t)$ then $h(t) = \int_{-\infty}^{t} \delta(t) = \begin{cases} 0 & t<0 \\ 1 & t \geq 0 \end{cases}$

$$h(t) = u(t)$$

$\Rightarrow$ $h[n] = u[n]$ impulse response

(i) $h[n] \neq k\,\delta[n]$ $\Rightarrow$ memory

(ii) $h[n] = 0 \quad n<0$ $\Rightarrow$ causal

(iii) $\sum_{n=-\infty}^{+\infty} |h[n]| = +\infty$ $\Rightarrow$ Not stable.

we can re-write the eq^n for accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[n] = \sum_{k=-\infty}^{n-1} x[n] + x[n]$$

$$= y[n-1] + x[n]$$

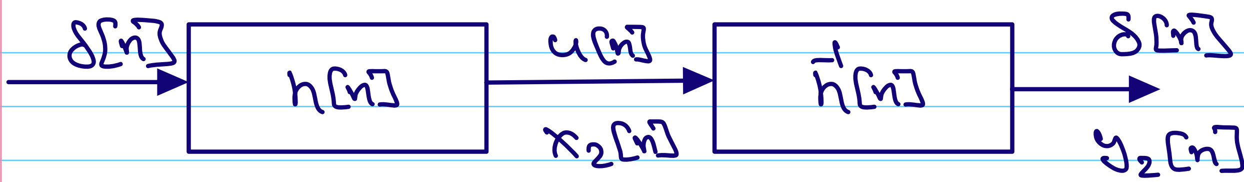
$$\Rightarrow \quad y[n] - y[n-1] = x[n]$$

(recursive difference eqn)

inverse of Accumulator :-

$$h[n] = u[n]$$

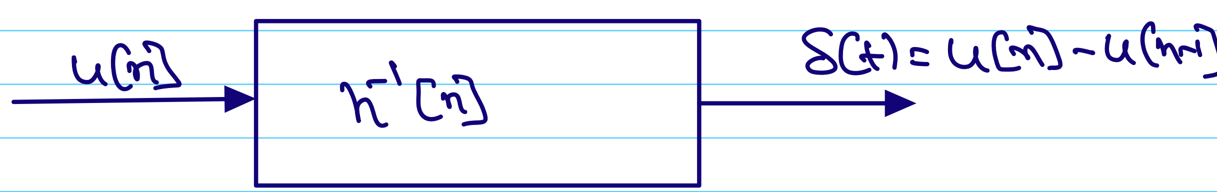
$$h^{-1}[n] = ?$$



we know $u[n] - u[n-1] = \delta[n]$

$\Uparrow$

if the system $h^{-1}[n]$ does this, the output will be $\delta[n]$



then the impulse response of

$h^{-1}[n]$ system is

$$y_2[n] = x_2[n] - x_2[n-1]$$

Impulse response

$$h^{-1}[n] = \delta[n] - \delta[n-1]$$

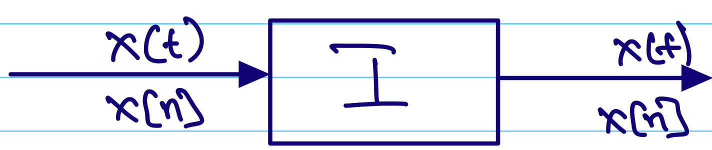
Example:

$$y[n] - a y[n-1] = x[n]$$

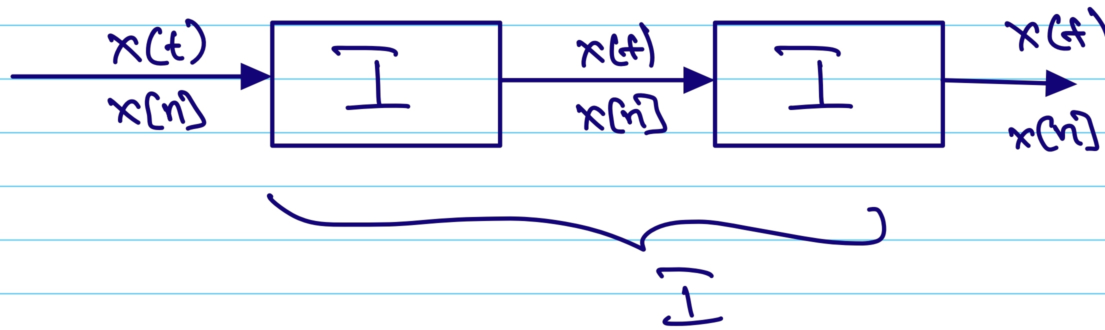
initial rest $\Longrightarrow$ LTI

$$x[n] = \delta[n] \Rightarrow h[n] = a^n u[n]$$

* How to deal with some of the mathematical difficulties associated with Impulses & step's.

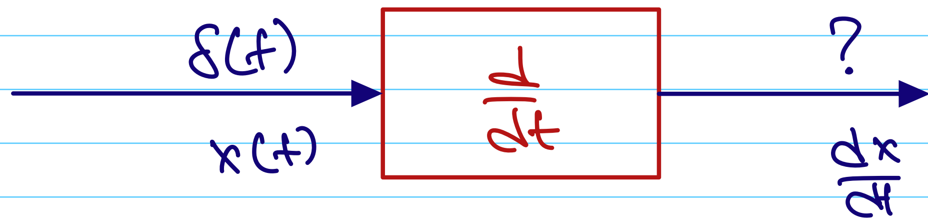


$$h(t) = \delta(t) \quad \& \quad h[n] = \delta[n]$$



$$\delta(t) * \delta(t) = \delta(t)$$

$$\delta[n] * \delta[n] = \delta[n]$$

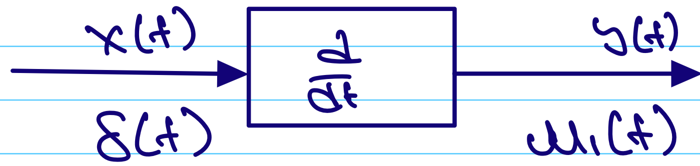


Operational Definition:

$$x(t) * \delta(t) = x(t)$$

The operation definition is related, Not to what the impluse is, But to what Impulse does.

$$x(t) * \delta(t) = x(t) \quad \text{(def of Impulse)}$$



$$x(t) * u_1(t) = \frac{d\,x(t)}{dt}$$