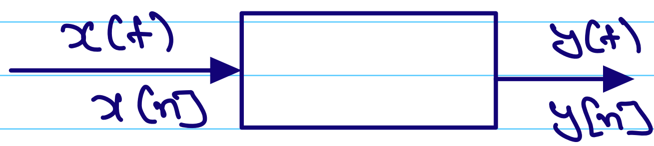
Lec 04: Convolution

System Properties



- memory
- Invertibility
- Causality
- Stability
- Time-invariance
- Linearity

we will be discussing Linear Time-invariant system's.

Time-invariance:

C-T $\quad x(t) \longrightarrow y(t)$

then $x(t-t_0) \longrightarrow y(t-t_0)$ for any t_0

D-T $\quad x[n] \longrightarrow y[n]$

then $\quad x[n-n_0] \longrightarrow y[n-n_0]$ for any n_0

Time-invariance is a property that applied to Both C.T, D-T, for any Given input-Output relationship, if we simple shift the input then the output shift by same amount.

* Time-invariance is a property that said that the system did'nt care about what the time origin of the signal is

Linearity:

$$\phi_K \longrightarrow \varphi_K$$

Then

$$a_1\phi_1 + a_2\phi_2 + \cdots \longrightarrow a_1\varphi_1 + a_2\varphi_2 + \cdots$$

* if we have a set of output's associated with given set of input's, $(\phi_K \longrightarrow \varphi_K)$ then the property of Linearity states that

if we have an put which is a linear combination of input's, then the output is linear combination of associated outputs.

The question is how can we exploit the properties of linearity & Time invariance

Stategy:-

* decompose a signal (C-T, D T) into a set of Basic signal's.

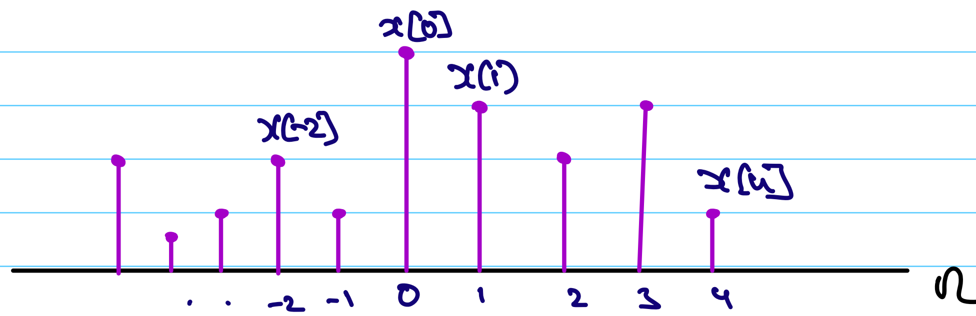
* Choose basic signal's so that response easy to compute

2 classes of signal's that are particularly suited for that stategy.

① delayed impulse :- decomposing a signal into linear combination of delayed

impulses $\Longleftrightarrow$ that lead's to repre-
sentation for L.T.I Convolution

(2) decompose the input into linear combination of complex exponential's $\Longleftrightarrow$ that lead's to representation of signal's & system's what we will refer to as Fourier analysis.



* we talked about representing unit step in term's of impulses, we can think of general seq of a seq^n of impulses, delayed (namely accuring at appropriate time instant) and with appropriate amplitude

* we can think of this general seq^n

an impulse occurring at $n=0$, with a height of $x[0]$ + impulse of height of $x[1]$ occurring at $n=1$ + . . .

$$x[n] = x[0]\,\delta[n] + x[1]\,\delta[n-1] + x[-1]\,\delta[n+1] + \cdots$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

input = (Linear combination of delayed Impulses)

Linear System:

The response to the linear combination = a linear combination of the responses.

$$\delta[n-k] \longrightarrow \boxed{\text{Linear}} \longrightarrow h_k[n]$$

then response to $x[n] = \displaystyle\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h_k[n]$$

+ Time- invariant

$$\delta[n-k] \longrightarrow \boxed{LTI} \longrightarrow h_k[n] = h_0[n-k]$$

Respose to an delayed impulse at time k $(\delta[n-k])$ is exactly the same as response to an impulse at t=0 shifted to time k

$$h_k[n] = h_0[n-k]$$

When $\delta[n] \longrightarrow h_0[n]$

$$\Rightarrow \quad h[n] = h_0[n]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Convolution sum

in Continuous time

decomposing continuous time into succession of arbitrarily narrow rectangle's, as $\Delta \to 0$ the approximation gets better.

$$x(t) \cong x(0)\,\delta_\Delta(t)\,\Delta + x(\Delta)\,\delta_\Delta(t-\Delta)\,\Delta$$

$$+ x(-\Delta)\,\delta_\Delta(t+\Delta)\,\Delta + \cdots$$

$$x(0) = \underbrace{x(0)}_{\substack{\text{height} \\ \text{of } x(t) \text{ at} \\ t=0}}\ \underbrace{\delta_\Delta(t)}_{\text{has height } \frac{1}{\Delta}}\ \Delta \rightarrow \Delta$$

*

$$x(t) \cong \sum_{k=-\infty}^{\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\,\Delta$$

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\,\Delta$$

$$= \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

Shifting integral.

an integral that tells us how $x(t)$ can be described as a sum of linear combination involving impulses.

Linear system:

$$\delta_\Delta(t-k\Delta) \longrightarrow h_{k\Delta}(t)$$

describe a time function as a linear combination of weighted delayed impulses.

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\, \delta_\Delta(t-k\Delta)\, \Delta$$

response to the linear syst

Linear system:

$$y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\, h_{k\Delta}(t)\, \Delta$$

sum of responses to the delayed impulses.

$\Rightarrow$ weighted sum of response's to the delayed impulses.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h_\tau(t)\, d\tau$$

Time invariant

$$\delta_0(t) \longrightarrow h_0(t)$$

$$\delta_{k\Delta}(t) \longrightarrow h_{k\Delta}(t) = h_0(t - k\Delta)$$

if the system is time-invariant, then the response to each of these delayed impulse is simple the delayed version of the Impulse response.

LTI :

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Convolution integral.

What we have managed to accomplish is to exploit the properties of Linearity and time-invariance, so that the system could be represented in term's only its response to Impulse at time 0.

$\Rightarrow$ for LTI system's, if we know its response to an impulse at $t=0$ or $n=0$, then infact through the convolution-sum (in DT), Convolution-integral (in CT) we can generate response to arbitrary inputs.

Convolution sum:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\, \delta[n-k]$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k] = x[n] * h[n]$$

Convolution - integral

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau = x(t) * h(t)$$