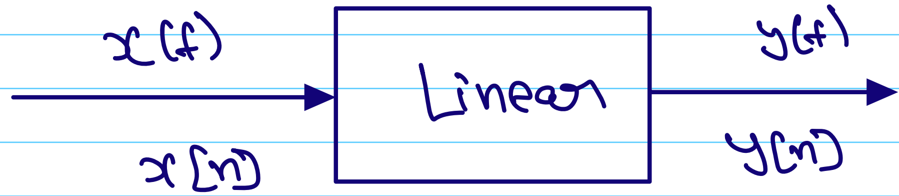
Lec07 :- Continuous Time Fourier series

* Last few lectures, we have dealt with the representation of LTI through convolution

BASIC Stategy :-

* essentially the idea was to exploit the notion of linearity by decomposing the input into some of BASIC inputs & using linearity to tell us that the output can be represented as the corresponding linear combination of associated outputs.



if $x(t) = a_1 \phi_1(t) + a_2 \phi_2(t) + \cdots$

$$\phi_k(t) \longrightarrow \psi_k(t)$$

and system is linear.

Then $y(t) = a_1 \psi_1(t) + a_2 \psi_2(t) + \cdots$

How to choose BASIC signal's $\phi_k(t)$?

1. a broad class of signal's can be constructed as a linear combination of ϕ_k's

2. response to ϕ_k's easy to compute

LTI System's

$$\phi_k[n] = \delta[n-k]$$

$$\psi_k[n] = h[n-k]$$

$\Rightarrow$ convolution sum

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\,\delta[n-k]$$

Convolution sum

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\,h[n-k]$$

C-T:

$$\phi_k(t) = \delta(t-\tau)$$

$$\psi_k(t) = h(t-\tau)$$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

Convolution integral

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\,h(t-\tau)\,d\tau$$

in general sense we will be decomposing our signal's into linear combination of complex exponential's

$$\phi_k(t) = e^{s_k t} \qquad s_k \text{ complex}$$

$$\phi_k[n] = z_k^{\,n} \qquad z_k \text{ complex}$$

Fourier Analysis

C-T : $S_k = j\omega_k$ $\qquad \phi_k(t) = e^{j\omega_k t}$

D-T : $|z_k| = 1$ $\qquad \phi_k[n] = e^{j\Omega_k n}$

S_k complex $\Rightarrow$ Laplace transform

z_k complex $\Rightarrow$ z-transform.

eigen function property of complex exponent:

$$\phi_k(t) = e^{j\omega_k t}$$

* for LTI the response to complex exponential's is of exactly same form just simple multiplied by complex factor. That complex factor depend on freq ω_k

$$e^{j\omega_k t} \longrightarrow H(j\omega)\, e^{j\omega_k t}$$

<u>Proof:-</u>

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

if $\quad x(t) = e^{j\omega_k t}$

then $\quad y(t) = \int_{-\infty}^{+\infty} e^{j\omega_k \tau}\, h(t-\tau)\, d\tau$

$$= \int_{-\infty}^{+\infty} e^{j\omega_k (t-\tau)}\, h(\tau)\, d\tau$$

$$= e^{j\omega_k t} \int_{-\infty}^{+\infty} e^{-j\omega_k \tau}\, h(\tau)\, d\tau$$

$$= \underbrace{e^{j\omega_k t}}_{\text{eigen function}} \quad \underbrace{H(j\omega_k)}_{\text{eigen value.}}$$

Because of the eigenfunction property that, complex exponential's particulary convinents as a building block's (basic signal's)

Periodic signal's

— Fourier series

Aperiodic Signal's

— Fourier transform.

C–T Fourier series

$$x(t) = x(t+T_0)$$

$$\omega_0 = \frac{2\pi}{T_0} = 2\pi f_0$$

$$e^{j\omega_0 t} \qquad T_0 = \frac{2\pi}{\omega_0}$$

$$e^{jk\omega_0 t} \qquad \frac{T_0}{k} = \frac{2\pi}{k\omega_0}$$

* $e^{j\omega_0 t}$ is the complex exponential which has t_0 as fundamental frequency.

* But there are Harmonically related complex exponential's that also have T_0 as Period, although infact their fundamental Period is shorter.

$$e^{jk\omega_0 t} \Rightarrow \underbrace{\frac{T_0}{\omega} = \frac{2\pi}{k\omega_0}}_{\text{fundamental Period}}$$

It has a T_0 Period as harmonic.

$$\left.\begin{array}{l} e^{j\omega_0 t} \\ e^{jk\omega_0 t} \\ k = \text{integer} \end{array}\right\}$$ Harmonically related complex exponential's.

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \quad \Big) \text{ fourier series.}$$

if we have very general periodic signal, we can represent it as a linear combination of these harmonically-related complex exponential's.

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Fourier series.

Period $T_0 = \frac{2\pi}{\omega_0}$

fundamental freq $= \omega_0 = \frac{2\pi}{T_0}$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$\int_{T_0} e^{jm\omega_0 t} \, dt = \begin{cases} T_0 & m=0 \\ 0 & m \neq 0 \end{cases}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$x(t)\, e^{-jn\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}$$

$$\int_{T_0} x(t) e^{-jn\omega_0 t}\, dt = \sum_{k=-\infty}^{+\infty} a_k \int_{T_0} e^{j(k-n)\omega_0 t}\, dt$$

$$\int_{T_0} e^{j(k-n)\omega_0 t}\, dt = \begin{cases} 0 & n \neq k \\ \frac{1}{T} & n = k \end{cases}$$

$$\Rightarrow \quad a_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t}\, dt$$

Synthesis

① $x(t) = \sum_{-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ (How to Build x(t) out of these complex exponential's)

② analysis

$$a_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt$$

* with low-freq what we are tending to build was the general behavior, & as higher freq came in that tending to contribute discontinuity.

let $f(t)$ be a C-T Periodic signal real signal

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(2\pi \frac{k}{T_0} t\right) + b_k \sin\left(2\pi \frac{k}{T_0} t\right)$$

$$= \sum_{k=-\infty}^{+\infty} c_k e^{2\pi i \frac{k}{T_0} t}$$

<u>Properties:</u>

$f(t)$ is real.

$$c_0 = \frac{a_0}{2} \quad -(1)$$

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos 2\pi \frac{k}{T_0} t + b_k \sin 2\pi \frac{k}{T_0} t$$

$$= \sum_{k=-\infty}^{+\infty} c_k e^{2\pi i \frac{k}{T_0} t}$$

equate cosines :

$$a_k \cos 2\pi \frac{k}{T_0} t = C_{-k} \cos 2\pi \frac{k}{T_0} t + C_k \cos 2\pi \frac{k}{T_0} t$$

$$\Rightarrow \quad a_k = C_k + C_{-k}$$

Similary equate sine term's

$$b_k \sin 2\pi \frac{k}{T_0} t = i C_k \sin 2\pi \frac{k}{T_0} t - i C_{-k} \sin 2\pi \frac{k}{T_0} t$$

$$\Rightarrow \quad b_k = i\left(C_k - C_{-k}\right)$$

$$\boxed{\begin{aligned} C_0 &= \frac{a_0}{2} \\ C_k &= \frac{a_0 - i b_k}{2} \\ C_{-k} &= \frac{a_k + i b_k}{2} \end{aligned}}$$

real function $f(t)$

$$\Rightarrow \quad \boxed{C_k = \overline{C_{-k}}} \quad \text{conjugate term's}$$

⇒ if $f(t)$ is even function

$$\Rightarrow \quad f(t) = f(-t)$$

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos 2\pi \frac{k}{T_0} t + b_k \sin 2\pi \frac{k}{T_0} t$$

$$= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos 2\pi \frac{k}{T_0} t - b_k \sin 2\pi \frac{k}{T_0} t$$

$$\Rightarrow \quad b_k = 0$$

only cosine term's

⇒ if $f(t)$ is odd function's

$$f(t) = -f(-t) \Rightarrow a_k = 0 \quad \forall k$$

⇒ only sine term's.