Lec 06:- System's represented by Differential eqn

A Particularly important set of system's which are LTI are those that are represented by linear constant co-eff differential & difference eqn.

* Solution's for linear constant co-eff differential eqn are Particular soln, homogenous soln with initial condition's.

When they do and don't correspond to LTI system's?

nth order

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

nth order difference eqⁿ

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

This is Linear because its Linear combination of derivatives, not because it correspond's to Linear system.

=) in fact we will see this eqn may or may not infact correspond to Linear system.

* in discrete time case

nth order

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Linear combination of delayed versions of the output = Linear combination of delayed version's of the input.

n^{th} order difference eqn.

Given $x(t)$, if $x_p(t)$ satisfies the eqn then we can add any other soln which satisfies homogenous's eqn $\left(\sum_{k=0}^{N} a_k \frac{d^k y_n(t)}{dt^k} = 0 \right)$

in fact the differential eqn

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

is not a unique specification of the system. because if we have any solution $y_p(t)$ then we

Can add to that sol^n any other sol^n which satisfies the homogeneous eq^n and sum of those two will likewise be a sol^n

The homogeneous solution to

$$\sum_{k=0}^{N} a_k \frac{d^k y_n(t)}{dt^k} = 0$$ ie of the

form $y_n(t) = Ae^{st}$

$$\sum_{k=0}^{K} a_k A s^k e^{st} = 0$$

$$\sum_{k=0}^{N} a_k s^k = 0$$ N roots s_i $i=1,2,..N$

$$y_n(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + .. + A_N e^{s_N t}$$

Need n initial condition

$$y(t_0),\ y'(t_0),\ \ldots\ \frac{d y^{n-1}}{dt^{n-1}}(t_0)$$

* Linear system $\Longleftrightarrow$ auxillary condition $=0$

* Causal, LTI $\Longleftrightarrow$ initial rest,

if $x(t)=0 \quad t<t_0$

then $y(t)=0 \quad t<t_0$

Example:

$$\frac{dy(t)}{dt} + a y(t) = x(t)$$

homogenous sol^n:

$$\frac{dy_h(t)}{dt} + a\, y_h(t) = 0$$

"guess" $\quad y_h(t) = A e^{st}$

$$Ase^{st} + aAe^{st} = 0$$

$$\Rightarrow \quad s = -a$$

$$y_n(t) = Ae^{-at}$$

let's take $x(t) = K u(t)$

Particular solution will be of form

$$y_p(t) = K' u(t)$$

$$\Rightarrow \quad y_p(t) = \frac{K}{a} u(t)$$

$$\Rightarrow \quad y(t) = y_n(t) + y_p(t)$$

$$y(t) = \left(Ae^{-at} + \frac{K}{a}\right) u(t)$$

Causal LTI $\Longleftrightarrow$ initial rest

$y(0) = 0$

$$\Rightarrow \quad y(0) = \left(A + \frac{K}{a}\right) = 0$$

$$A = -\frac{k}{a}$$

$$y(t) = \left(-\frac{k}{a}e^{-at} + \frac{k}{a}\right)u(t)$$

$$\Rightarrow \boxed{y(t) = \frac{k}{a}\left(1 - e^{-at}\right)u(t)}$$

* for LTI system's impulse response is derivative of step response

$$u(t) \longrightarrow \boxed{\frac{d}{dt}} \xrightarrow{\delta(t)} \boxed{h(t)} \xrightarrow{h(t)}$$

Both the system's are same

$$u(t) \longrightarrow \boxed{h(t)} \xrightarrow{s(t)} \boxed{\frac{d}{dt}} \xrightarrow{h(t)}$$

Cascade system's are same for LTI system's

for the above example $s(t) = \frac{1}{a}\left(1 - e^{-at}\right)u(t)$

$$\frac{d\,s(t)}{dt} = \frac{d}{dt}\left(\frac{1}{a}\left(1-e^{-at}\right)u(t)\right)$$

$$= \frac{1}{a}\left(a \cdot e^{-at}\right)u(t) + \frac{1}{a}\left(1-e^{-at}\right)\delta(t)$$

$$= e^{-at}u(t) + \underbrace{\frac{1}{a}\left(1-e^{-at}\right)\delta(t)}_{t=0}$$

$$= e^{-at}u(t) + \frac{1}{a}\left(1-\cancelto{0}{e^{-a\cdot 0}}\right)$$

$$h(t) = \frac{d}{dt}s(t) = e^{-at}u(t)$$

$\Rightarrow$ stable $\Leftrightarrow a > 0$

<u>difference eqn:</u>

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

$$\sum_{k=0}^{N} a_k\, y_h[n-k] = 0 \quad \text{(Homogenous soln)}$$

⇒ Linear combination of delayed version's of the output = Linear combination of delayed version's of the input.

$$y[n] = y_p[n] + y_h[n]$$

Homogenous sol^n $\quad y_h[n] = Az^n$

N auxillary root's are

$$y[n_0],\ y[n_0-1],\ y[n_0-2], \dots\ y[n_0-N+1]$$

Linear system $\Longleftrightarrow$ auxillary condition $= 0$

causal, LTI $\Longleftrightarrow$ initial rest

if $x[n] = 0 \quad n < n_0$

$y[n] = 0 \quad n < n_0$

Example:

$$y[n] - a\,y[n-1] = x[n]$$

causal, LTI $\Longleftrightarrow$ initial rest

$$y[n] = x[n] + a\,y[n-1]$$

$$x[n] = \delta[n] \qquad y[n] = 0 \quad n < 0$$

$$h[n] = \delta[n] + a\,h[n-1]$$

$h[n] = 0 \quad n < 0$ (initial rest)

$$h[0] = \delta[0] + a\,h[-1]$$

$$h[0] = 1$$

$$h[1] = \delta[1] + a\,h[0] = a$$

$$h[2] = \delta[2] + a\,h[1] = a^2$$

$$h[n] = a^n\,u[n]$$

stable if $a < 1$

Causal LTI

$$\delta[n] \longrightarrow a^n u[n]$$

family of solution's

$$\delta[n] \longrightarrow a^n u[n] + y_h[n]$$

$$y_h[n] = Az^n$$

$$\Rightarrow \quad y_h[n] - a\,y_h[n-1] = 0$$

$$\Rightarrow \quad Az^n - aAz^{n-1} = 0$$

$$\Rightarrow \quad z = a$$

$$\Rightarrow \quad y_h[n] = Aa^n$$

$$\delta[n] \longrightarrow a^n u[n] + Aa^n$$

Block diagram:

$$y[n] = x[n] + a\,y[n-1]$$