Lec11 Convergence of interval's.

we need to have a better and more robust definition of fourier Transform that will allow us to work with the signal's that society need's to function, sint, cost, 1, etc.

- The F.T we have defined will not do the trick.

- we need a more robust definition of F.T to deal with common signal's. one's for which the classical definition is not adequate (Not work). And the issue is exactly the convergence of integral, or if not for the convergence of integral for the function applying Fourier inversion.

<u>two issues:</u>

1. Convergence of integral defining the F.T
2. we want to Fourier inversion to apply.

two way's of dealing with these Problems:-

1. Special techniques (got a Particular Problem atic integral, let's work with that integral), (having trouble with Convergence of a Particular signals having trouble applying Fourier inversion in a Particular case, Okay let's with that, try some trick, some method that applies, not generally, But to that one Problem that we have difficulties with and then we can advance)

2. Second approach, ie to rework the foundation's, to give really a different definition of the F.T that applies more robustly, more equally somehow to all signal's that come up all at once.

(New definition)

Some issues:

The Problem is evident in the very 1st example. $f(t) = \Pi(t)$

$$\mathcal{F}\Pi(s) = \text{sinc}(s) = \frac{\sin \pi s}{\pi s}$$

The Problem is fourier inversion, $\mathcal{F}^{-1}\text{sinc}(t)$, we did this by duality.

$$\mathcal{F}^{-1}(\text{sinc}(s)) = \Pi(t)$$

we used duality To find this

* The Problem is writing down the integral

$$\mathcal{F}^{-1}\text{sinc}(t) = \int_{-\infty}^{+\infty} e^{+2\pi i s t} \frac{\sin \pi s}{\pi s}\, ds$$

it turn's out to be $= \begin{cases} 1 & |s| < 1/2 \\ 0 & |s| \geq \frac{1}{2} \end{cases}$

which is Not correct

* There is a special problem at end point's, $s = \pm 1/2$, The point is that it can be dealt, But its a little bit disconcerting that there simplest examples in the entire subject already poses this problem. The most basic function already require us to do special argument's, just so the Fourier inversion work's.

2nd example of trouble :

There are very simple function's where we can't even started.

$$f(t) = 1$$

No way to make sense of

$$\mathcal{F}1 = \int_{-\infty}^{+\infty} e^{-2\pi i s t} \cdot 1 \cdot dt$$ (it just won't work)

* Likewise for $f(t) = \sin 2\pi t$, or $g(t) = \cos 2\pi t$ we can't make sence of

$$\int_{-\infty}^{+\infty} e^{-2\pi i s t} \sin 2\pi t \, dt, \quad \int_{-\infty}^{+\infty} e^{-2\pi i s t} \cos 2\pi t \, dt$$

These Just won't work, and these are the signal's the society need's.

* The First step in understanding this, is to somehow Back away from trouble and concentrate on what work's well

(Back away from trouble's, what is Best situation)

What is Best situation? in this case what that mean's is, we want to identify class of signal's, which everything we want to be true

* Let's call these class of signal's S

S stand's for Schwarz, the person who isolated these particular class of function's as the best one's upon which to build a theory.

We want 2 thing's:

① if $f(t) \in S$, then $\mathcal{F}f(s) \in S$.

if $f(t)$ is in class S, then it F.T $\mathcal{F}f$ is defined and $\mathcal{F}f$ is in the class S.

* It already tell's us $\Pi(t)$ should not be in the same class, Sin. & cos should not belong to that class. 1 should not be in that class.

(2) Fourier inversion work's.

$$\mathcal{F}^{-1}\mathcal{F} f = f, \quad \mathcal{F}\mathcal{F}^{-1} f = f$$

if we can find a class of function's S that satisfy these two properties, then that's the best class of signal's we hope for, and that's the best classical theory.

There is one other property that comes up, that it always comes up in the discussion, it not a requirement of the theory but it comes up as sort of part of the theory, and it actually extremly useful for the development's.

Further property: Parsevall Identity

for F.T (we have Parsevall identity for Fourier series called Rayleigh's identity)

$$\underbrace{\int_{-\infty}^{+\infty} |\mathcal{F} f(s)|^2 ds}_{\text{energy in freq. Domain}} = \underbrace{\int_{-\infty}^{+\infty} |f(t)|^2 dt}_{\text{energy in time domain}}$$

How to define S?

<u>Solved by Lauram Schwartz</u> :-

S - stand's for Schwartz, not for Signal's, S is Best class of function's for Fourier Analysis, is so called rapidly decreasing function's.

* it's defined by two properties.

① any function f in the class $f \in S$ is infinetly differentiable, it has arbitarly high smoothess.

$f(x)$ is infinitely differentiable (1st property)

(2) for any $m, n \in \mathbb{N}$ (integer's)
$m, n \geq 0$,

$$|x|^n \left| \frac{d^m}{dx^m} f(x) \right| \longrightarrow 0$$

as $x \longrightarrow \pm \infty$

what this say's in word's, any derivative of f tend's to 0, faster than any Power of x, independently.

Ex:

$$|t^{100}| \left| \frac{d^3 f(t)}{dt^3} \right| \longrightarrow 0$$

$$|t^{5\text{mill}}| \left| \frac{d^{50} f(t)}{dt^{50}} \right| \longrightarrow 0$$

in word's, any derivative of $f(t)$ tend's to zero faster than any Power of t independently.

Example of δ function's:

① Gaussian

$$f(x) = e^{-\pi x^2}$$

Another important class of function's, turn out are the smooth function's which vanish identically, as we get close to infinity. Not just tend to 0, they are actually 0 beyond certain interval.

C_c function's, there are infinetly differentiable function's which are identically 0, outside some finite interval.

C = function's of compact support

interval when function is NON-zero.

Connection here comes through derivative theorem.

$$\mathcal{F}(f^{(m)})(s) = (2\pi i s)^m \mathcal{F}f(s)$$

It turns differentiation into multiplications

<u>derivation of Parseval's Identity:</u> for functions in $\mathcal{S}$

$$\int_{-\infty}^{+\infty} |\mathcal{F}f(s)|^2 ds = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

Proof: $f, g \in \mathcal{S}$

$$\int_{-\infty}^{+\infty} \mathcal{F}f(s)\, \overline{\mathcal{F}g(s)}\, ds = \int_{-\infty}^{+\infty} f(t)\, \overline{g(t)}\, dt$$

$$g(t) = \int_{-\infty}^{+\infty} e^{2\pi i s t} \mathcal{F}g(s)\, ds$$

$$\overline{g(t)} = \int_{-\infty}^{+\infty} e^{-2\pi i s t} \, \overline{\mathcal{F} g(f)} \, df$$