

Lec 25:

Connection b/w LTI & Fourier Transform.

* General structure of linear system.

\Rightarrow L is a Linear System

\Rightarrow then we introduced impulse response

$$h(x, y) = L \delta(x-y)$$

\Rightarrow The output of the system can be given as integrating impulse response against the input.

$$\begin{aligned} W(x) &= L V(x) \\ &= \int_{-\infty}^{+\infty} h(x, y) v(y) dy \end{aligned}$$

we say that L is time-invariant or shift invariant if

$$W(x) = L V(x)$$

$$\text{then } W(x-y) = L V(x-y)$$

The Linear system is Time-invariant

\iff The system is given by Convolution.

\Rightarrow Then the Impulse response for LTI system's $h(x) = L \delta(x)$

$$\text{By Time-invariance } h(x-y) = L \delta(x-y)$$

* system is given by

$$\begin{aligned} W(x) &= \int_{-\infty}^{+\infty} h(x-y) V(y) dy \\ &= h * V(x) \end{aligned}$$

\Rightarrow A satisfactory state of Affairs as far as the structure of Linear System's goes.

① Any Linear System is given by integration against Impulse response

② if the system is time invariant \Leftrightarrow
The integration reduces to convolution.

③ Same Consideration's hold for discrete System's. Any discrete system remembers multiplication by matrix

if $W = LV$ given by multiplication
by matrix $W = AV$

$$L \text{ is LTI} \Leftrightarrow W = h * V$$

* if we write a system as a matrix multiplication $W = AV$ then A is a special form of time-invariant System's.

Ex: $h = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

if $w = Av = h * v$, what is A ?

How do we find matrix A ? we have to find the image of basis vectors.

\Rightarrow The columns of A are image of basis vectors

$$A\delta_0, \quad \delta_0 = (1, 0, 0, 0)$$

$$A\delta_1, \quad \delta_1 = (0, 1, 0, 0)$$

$$A\delta_2, \quad \delta_2 = (0, 0, 1, 0)$$

$$A\delta_3, \quad \delta_3 = (0, 0, 0, 1)$$

How do we compute all these things?
we do so by convolution.

* By definition the system is given to as convolution to a vector h

$$A_1 = A \delta_0 = h * \delta_0$$

$$A_1[m] = \sum_{n=0}^3 h[m-n] \delta_0[n]$$

$$\Rightarrow A_1[0] = \sum_{n=0}^3 h[0-n] \delta_0[n]$$

$$= h[0] = 1$$

$$\text{Similarly } A_1[1] = h[1] = 2 \quad A_1[2] = 3 \\ A_1[3] = 4$$

Similarly

$$A \delta_1 = h * \delta_1$$

$$= (h * \delta_1)[m]$$

$$= h[m-1] = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

(Circulant matrix)

Therefore $A = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$

The system is given by convolution,
But it also given by matrix multiplication

Bring in the F.T

* we have convolution only for LTI system

$$W = h * v$$

$$F W = F h * v$$

$$\Rightarrow W(s) = H(s) V(s)$$

$H(s)$ = Transfer function

$h(t)$ = impulse response

$H(s)$ = Transfer function
System function.

* There is a beautiful structure involved in Linear systems. That the most basic example of Linear system is the relation with direct proportionality

* For LTI systems in freq domain it's exactly described by multiplication
It's exactly described by direct proportionality

* In freq domain, the system is given by direct proportion

$$W(s) = H(s) U(s)$$

* Time domain & freq domain are equivalent, we can pass back & forth b/w them

* They are different Pictures of same thing. different views of same phenomenon. we can use one to study other's.

* The importance of being in the F-T to LTI is the fact that complex exponential's are eigen function's for LTI system's.

(Last great fact on LTI system's is complex exponential's are eigen function's)

$$W = Lv = h * v$$

$$W(s) = H(s)V(s)$$

what happen's if we input complex exponential?

$$\Rightarrow V(\omega) = e^{2\pi i \omega x}$$

What is $L V(x)$? Sometimes people call this a freq response, because we are inputting a pure freq, pure harmonic.

$$V(x) = e^{2\pi i \omega x}$$

$$V(s) = F V(x) = \int_{-\infty}^{+\infty} e^{2\pi i \omega x} e^{-2\pi i s x} dx$$

$$= \int_{-\infty}^{\infty} e^{2\pi i (\omega - s) x} dx$$

$$= \begin{cases} 1 & \text{if } s = \omega \\ 0 & \text{if } s \neq \omega \end{cases}$$

$$V(s) = \delta(s - \omega)$$

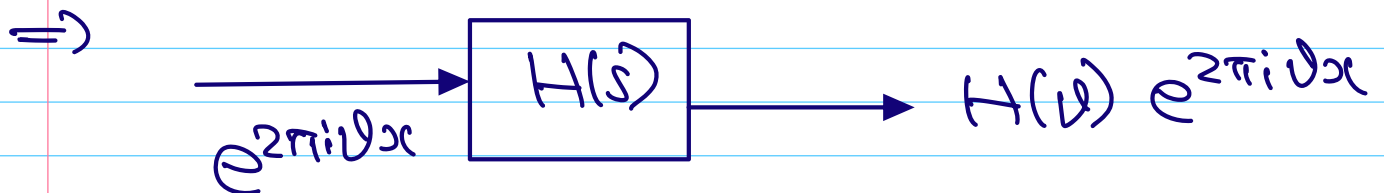
$$\Rightarrow W(s) = H(s) V(s)$$

$$W(s) = H(s) \delta(s - \nu)$$

$$\Rightarrow W(s) = H(\nu) \delta(s - \nu)$$

\downarrow
 \mathcal{F}^{-1}

$$w(t) = H(\nu) e^{2\pi i \nu t}$$



$\Rightarrow e^{2\pi i \nu t}$ is an eigen function with eigen value $H(\nu)$

\Rightarrow Any LTI System's have complex exponential's as basis of eigen function's. (Not sin, cos)

* for any LTI System's, Complex exponential's act as Basis of eigen function's \Rightarrow allow's us to diagonalize The operator's

$$\text{let } v(t) = \cos(2\pi\theta t)$$

$$= \frac{1}{2} e^{2\pi i\theta t} + \frac{1}{2} e^{-2\pi i\theta t}$$

$$Lv(t) = \frac{1}{2} H(\theta) e^{2\pi i\theta t} + \frac{1}{2} H(-\theta) e^{-2\pi i\theta t}$$

if $v(t)$ is real then

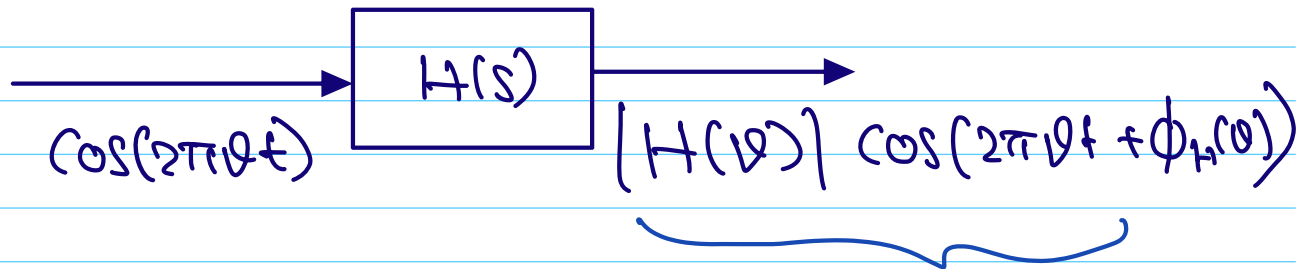
$$= \frac{1}{2} H(\theta) e^{2\pi i\theta t} + \frac{1}{2} \overline{H(\theta)} e^{-2\pi i\theta t}$$

$$= \frac{1}{2} H(\theta) e^{2\pi i\theta t} + \frac{1}{2} \overline{H(\theta) e^{2\pi i\theta t}}$$

$$= \operatorname{Re} \{ H(\theta) e^{2\pi i\theta t} \}$$

$$= |H(\omega)| \cos(2\pi\omega t + \phi_H(\omega))$$

\Rightarrow



there is a phase shift

This shows \cos , \sin are not eigenfunctions of LTI systems.

\Rightarrow we are (sort of) back to where we started in the course - with complex exponential's as a basis for decomposing a signal, and now for decomposing an operator.