Lec03: Analysis of Periodic Phenomenon and How it is represented

if we have a signal with Time period 1, then we can represent it with sin, cos of Period's $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

$\Rightarrow$ freq $= 1, 2, 3, 4, \ldots$

$f(t) \quad T = 1$ sec

$$\Rightarrow \quad T = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

$$f = 1, 2, 3, 4, \ldots$$

$f(t) \quad T = 3$ sec

$$T = 3, \frac{3}{2}, \frac{3}{3}, \frac{3}{4}, \frac{3}{5}, \ldots$$

$$f = \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, \ldots$$

Last time we took 1st step in analysing general periodic phenomenon via, Linear combination of simple building blocks.

(given)

* $f(t)$ is periodic signal, period 1

* Suppose we can write $f(t)$ as a sum of complex exponential's

$$f(t) = \sum_{k=-n}^{n} C_k e^{2\pi i k t}$$

$$C_k = \overline{C_{-k}}$$

Then co-efficients given by

$$C_k = \int_0^1 e^{-2\pi i k t} f(t)\, dt$$

dependend on $\int_0^1 e^{2\pi i n t} \cdot e^{-2\pi i m t} \, dt = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$

Step 1 Suppose if we can write function $f(t)$ in this form, then the co-efficient's have to given by

$$\hat{f}(k) = \int_0^1 e^{-2\pi i k t} f(t) \, dt$$

$$\Uparrow$$

Suppose if we can write

$$f(t) = \sum_{k=-N}^{N} c_k e^{2\pi i k t}$$

Step-2 turn that around and ask the following when is that possible? Given periodic signal $f(t)$ periodic 1, Define

$$\hat{f}(k) = \int_0^1 e^{-2\pi i k t} f(t) \, dt$$

k^{th} Fourier co-efficient

The question is can we write

$$f(t) = \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi i k t} \quad \text{for some } n.$$

How general we expect this to be if it works?

General period function can be decomposed in this way into very simple term's. Analyzing a very complex system of periodic input's, periodic outputs might be possible to do by analyzing what the system does to the relatively simple input's, output's given by complex exponential's.

* if so we can analyze complex system's by simple building blocks.

* This method is only going to be really helpful if its fairly general.

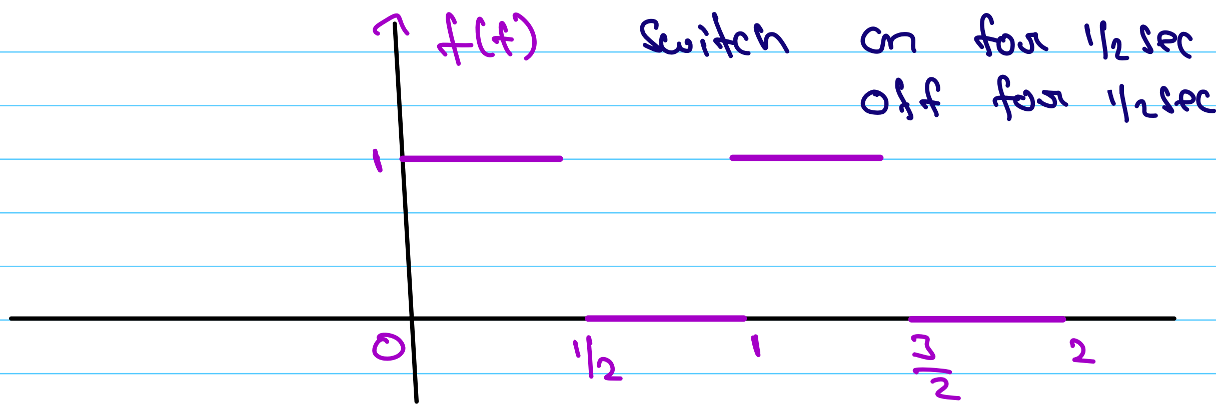
So, How general is this?

How general can we expect this to be?

High Stakes Question.

Look at some examples:

Ex:



$$f(t) = \begin{cases} 1 & 0 \le t < 1/2 \\ 0 & \frac{1}{2} \le t < 1 \end{cases}$$

Switch function (on for 1/2 sec, off for 1/2 sec)

$$\hat{f}(k) = \int_0^1 e^{-2\pi i k t} f(t)\, dt$$

$$\Rightarrow \hat{f}(k) = \int_0^{1/2} e^{-2\pi i k t}\, dt$$

$$= \frac{e^{-2\pi i k t}}{-2\pi i k}\Bigg]_0^{1/2}$$

$$\hat{f}(k) = \frac{1 - e^{-\pi i k}}{2\pi i k}$$

But can we write

$$f(t) = \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi i k t}\, dt$$

for some n ?

NO

No, atleast Not for finite sum.

Reason:

① Sin's and cosines are continuous

② Sum of finite number of continuous function's is Continuous. It cannot possible represent discontinuous function.

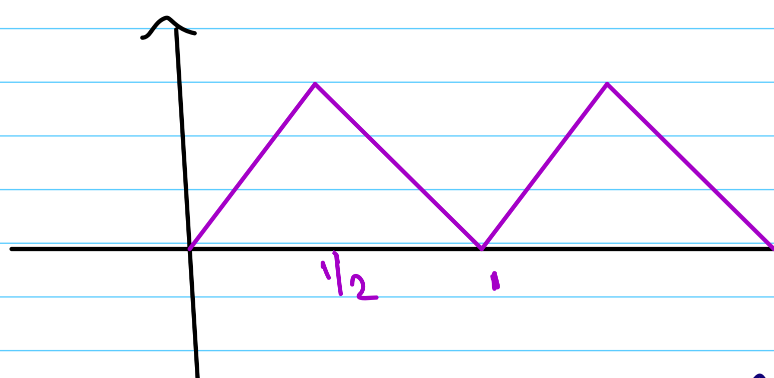
③ We cannot represent Discontinuous phenomenon by continuous Phenomenon.

Theorum, Basic in calculus:-

Sum of 2 continuous function = continuous.

What if f(t) is continuous function, Can we represent with finite sine, cos, function's?

Ex:



We can easily compute $\hat{f}(k)$, Fourier co-efficient's

$$\hat{f}(k) = \int_0^1 e^{-2\pi i k t} f(t)\, dt$$

But can we represent

$$f(t) = \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi i k t}$$

No, again, Not for finite sum.

Reason:

=> if two function's are differentiable, then the sum of two function's is differentiable

=> f(t) here is not differentiable.

* May be 1st, 2nd derivatives are fine what if there is discontinuity in 3rd derivative and so on, so on.

* No matter how smooth is a corner if there is some discontinuity in some high derivative, we are skrewed.

Any discontinuity in any derivative precludes writing

$$f(t) = \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi i k t}$$

So, this great idea, we might quit as well, because it does not very general at all.

if we cannot represent as finite sum's, then we have to turn ito infinite sum's.

$$f(t) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{\cdots}$$

or at least larger and larger finite sum's, sum's of more and more term's.

* It takes high frequencies to make sharp corner's or any corner's for that matter.

Maxim :- it takes high frequencies to make sharp corner's.

we have consider infinite sum's

To represent General Periodic signal's we have to consider infinite sum's

$$\sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi i k t}$$

* Any non smooth Phenomenon signal will generate infinitely many fourier Co-efficient's.

* The only way we could possible have finite fourier series is if the function is start out with infinitely smooth.

+ we have to deal with issues of convergence.

* Need a conspiracy of cancelation to make such series to converge. The stake's are high, and issues are real.

Summary of the main result's,

- Convergence when the function is continuous or smooth

- Convergence when there is jump discontinuty.

- Convergence issues in general.

  - needs fundamental change in perspective

(1) Continuous case:

$$\sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi i k t}$$

converges for each t to $f(t)$

* So if the function $f(t)$ is continuous then we know that series gonna converge to $f(t)$.

$\Rightarrow$ Smooth case: $f(t)$ is differentiable

which mean's the $f(t)$ is continuous and we know that series converge's

$$\sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi i k t}$$

But there is actually more to it.

we actually get uniform convergence

, $\Rightarrow$ for different values of t, we can control, the rate at which series converges.

② Jump discontinuity:

if t_0 is the point at Jump discontinuity

Then $\sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi i k t}$ converges at t_0

to the average value, middle of the Jump i.e $\frac{1}{2}\left[f(t_0^+) + f(t_0^-)\right]$

③ General case:

need a different kind of convergence

* Learned not to ask for convergence of

$\sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi i k t}$ at particular point's

at values of t

rather ask for convergence in the average (convergence in mean.)
(convergence in energy)

⇒ suppose $f(t)$ is periodic with 1

⇒ suppose we have another property, the integral is square integrable

$$\int_0^1 |f(t)|^2 \, dt < \infty$$

(in some areas the integral of the square is identified with power)

energy of the signal

$$= \int_0^1 |f(t)|^2 \, dt$$

if energy $< \infty$ ⇒ finite energy

(hypothesis of finite energy)

if we have finite energy then we can form

$$\hat{f}(k) = \int_0^1 e^{-2\pi i k t} f(t)\, dt$$

we will get convergence in average sense.

Then $$\int_0^1 \left| \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi i k t} - f(t) \right|^2 dt \xrightarrow{n \to \infty} 0$$

convergence in the mean.