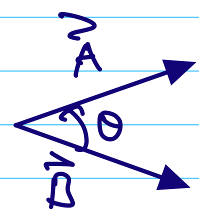
Inner Product or Dot Product or Scalar Product:

way of multiplying two vector's to get a scalar.

$$\langle \vec{A}, \vec{B} \rangle = \vec{A} \cdot \vec{B} = \sum a_i b_i$$

$$= \|\vec{A}\|_2 \, \|\vec{B}\|_2 \cos\theta$$

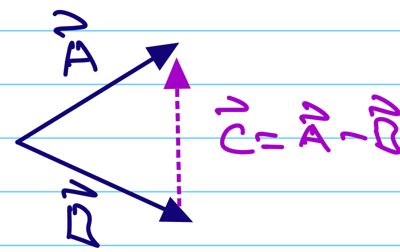


It tell's us at the same time about length's & angle's.

① $\langle \vec{A}, \vec{A} \rangle = \|\vec{A}\|_2 \, \|A\|_2 \cos 0$

$$= \|A\|_2^2$$

② Law of cosine's



$$\|\vec{C}\|_2^2 = \|\vec{A}\|_2^2 + \|\vec{B}\|_2^2 - 2\|\vec{A}\|_2 \|\vec{B}\|_2 \cos\theta \quad \text{— (1)}$$

Dot

$$\|\vec{C}\|_2^2 = \langle \vec{C}, \vec{C} \rangle$$
$$= \langle \vec{A}-\vec{B}, \vec{A}-\vec{B} \rangle$$
$$= \vec{A}\cdot\vec{A} - 2\vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{B}$$
$$= \|\vec{A}\|_2^2 + \|\vec{B}\|_2^2 - 2\vec{A}\cdot\vec{B}$$

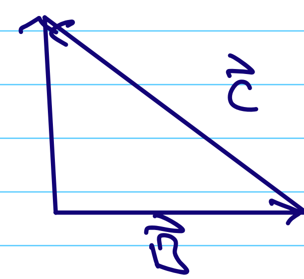
$$\vec{A}\cdot\vec{B} = \|\vec{A}\|_2 \, \|\vec{B}\|_2 \cos\theta \quad (2)$$

Application's

(1) Compute length's and angle's

(2) Defect orthogonality

Pythogerous theorem

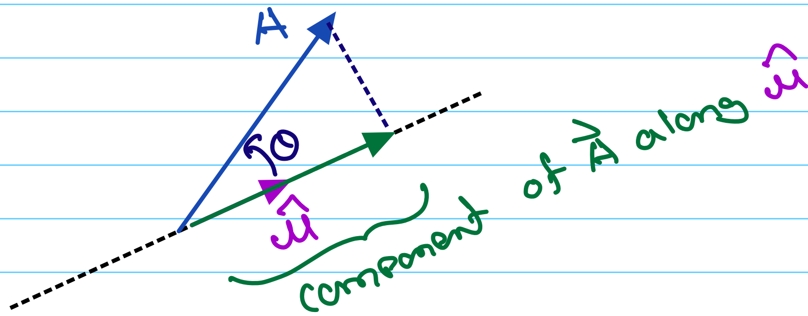


$$\|\vec{C}\|_2^2 = \|\vec{A}\|_2^2 + \|\vec{B}\|_2^2$$

$$\Rightarrow \cancel{\|\vec{A}\|_2^2} + \cancel{\|\vec{B}\|_2^2} - 2\vec{A}\cdot\vec{B}$$
$$= \cancel{\|\vec{A}\|_2^2} + \cancel{\|\vec{B}\|_2^2}$$

$$\vec{A}\cdot\vec{B} = 0 \Rightarrow \langle \vec{A}, \vec{B} \rangle = 0$$

③ Component's of $\vec{A}$ along direction $\hat{u}$ (unit vector)



$$= \|\vec{A}\|_2^2 \cos\theta$$

$$= \|\vec{A}\|_2 \cdot 1 \cdot \cos\theta$$

$$= \|\vec{A}\|_2 \cdot \|\hat{u}\|_2 \cos\theta$$

$$= \langle \vec{A}, \hat{u} \rangle$$

Projection of vector $\vec{v}$ onto $\vec{w}$

$$\frac{\langle \vec{v}, \vec{w} \rangle}{\langle \vec{w}, \vec{w} \rangle} w = \frac{\|v\|_2 \|w\|_2 \cos\theta}{\|w\|_2^2} \|w\|_2 \hat{w}$$

$$= \|v\|_2 \cos\theta \; \hat{w}$$

Function's

$L^2([0,1])$ – lebesgue space of square-integrable function's defined on the interval $[0,1]$

(i) $L^2([0,1])$ consists of all measurable function's $f: [0,1] \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) such that the integral of their square is finite

$$\int_0^1 |f(x)|^2 dx < \infty$$

Orthonormal function's

$$\| f + g \|_2^2 = \| f \|_2^2 + \| g \|_2^2$$

$$\int_0^1 \left[f(t) + g(t)\right]^2 dt = \int_0^1 f(t)^2 dt + \int_0^1 g(t)^2 dt$$

$$\Rightarrow \quad 2\int_0^1 f(t)\, g(t) = 0$$

$$\Rightarrow \quad \langle f, g \rangle = 0$$

for complex valued function's

$$\langle f, g \rangle = \int_0^1 f(t)\, \bar{g}(t)\, dt$$

The complex exponential's are Orthonormal Basis

$$e_n(t) = e^{2\pi i n t}$$

$$\langle e_n(t), e_m(t) \rangle = \int_0^1 e_n(t)\, \overline{e_m(t)}\, dt$$

$$= \int_0^1 e^{2\pi i n t}\, e^{-2\pi m t}\, dt$$

$$= \int_0^1 e^{2\pi i(n-m)t}\, dt$$

$$\langle e_n, e_m \rangle = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

Now we are drawing a analogy b/w Linear algebra

if $Q = \begin{bmatrix} q_1 & q_2 \cdots & q_n \end{bmatrix}$ orthonormal matrix

$$\langle q_n, q_m \rangle = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

then for $Qx = b$

$$\Rightarrow \quad x = Q^{-1} b = Q^T b$$

$$\Rightarrow \quad x = \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix} \begin{bmatrix} b \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} q_1^T b \\ q_2^T b \\ \vdots \\ q_n^T b \end{bmatrix} = \begin{bmatrix} \langle b, q_1 \rangle \\ \langle b, q_2 \rangle \\ \vdots \\ \langle b, q_n \rangle \end{bmatrix}$$

$$\Rightarrow \quad b = Qx$$

$$\Rightarrow \quad b = x_1 \vec{q}_1 + x_2 \vec{q}_2 + \cdots + x_n \vec{q}_n$$

$$\Rightarrow \quad b = \langle b, q_1 \rangle \vec{q}_1 + \langle b, q_2 \rangle \vec{q}_2 + \cdots + \langle b, \vec{q}_n \rangle \vec{q}_n$$

$$b = \sum_{i=1}^{n} \langle b, \vec{q}_i \rangle \vec{q}_i$$

decomposition of b into orthonormal vector's $q_1, q_2, \ldots q_n$ using projection's

Simillary in $L^2([0,1])$

what is the component of a function $f(t)$ "in the direction" $e_n(t)$?

$$\langle f, e_n \rangle = \int_0^1 f(t)\,\overline{e_n(t)}\,dt$$

$$\hat{f}(n) = \int_0^1 f(t)\, e^{-2\pi i n t}\,dt$$

$$\Rightarrow \quad f(t) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\, e^{2\pi i k t}$$

$$= \sum_{k=-\infty}^{\infty} \langle f, e_k \rangle\, e^{2\pi i k t}$$

infinite BASIS

$$\begin{bmatrix} \vdots \\ \hat{f}(-1) \\ \hat{f}(0) \\ \hat{f}(1) \\ \vdots \\ \hat{f}(n) \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \langle f, e_{-1} \rangle \\ \langle f, e_0 \rangle \\ \langle f, e_1 \rangle \\ \vdots \\ \langle f, e_n \rangle \\ \vdots \end{bmatrix}$$