Field's

A field consists of a set F of scalar's and two operator's : addition "+" and multiplication "$\cdot$". Such that

① $\mathcal{F}$ is closed under addition and multiplication.

$\forall \alpha, \beta \in \mathcal{F}$, then $\alpha + \beta \in \mathcal{F}$, $\alpha \cdot \beta \in \mathcal{F}$

② Addition and multiplication are commutative

$\forall \alpha, \beta \in \mathcal{F}$, $\alpha + \beta = \beta + \alpha$, $\alpha \cdot \beta = \beta \cdot \alpha$

③ Addition and multiplication are assosiative

$\alpha, \beta, \gamma \in \mathcal{F}$ $\quad \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

$$\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$$

④ Multiplication is distributive over addition

$\forall \alpha, \beta, \gamma \in \mathcal{F}$, $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$

⑤ $\mathcal{F}$ contains additive (0) and multiplicative (1) identity elements such that

$\forall \alpha \in \mathcal{F}$, $\alpha + 0 = \alpha$, $1 \cdot \alpha = \alpha$

⑥ Each element has an additive inverse

$\forall \alpha \in \mathcal{F}$, $\exists \beta \in \mathcal{F}$ (called additive inverse of α) s.t $\alpha + \beta = 0$

⑦ Each element (except for 0) has a multiplicative inverse

$\forall \alpha \in \mathcal{F} \setminus \{0\}$, $\exists \gamma \in \mathcal{F}$ (called the multiplicative inverse of α) s.t $\alpha \cdot \gamma = 1$

Ex: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$

Vector spaces

A linear (vector) space over a field $\mathcal{F}$, denoted by $(\mathcal{X}, \mathcal{F})$ consists of a set $\mathcal{X}$ of vector's, a field $\mathcal{F}$, and two operation's vector addition and scalar multiplication. Such that.

$\mathcal{X}$ a set, $\mathcal{F}$ a field $(\mathcal{X}, \mathcal{F})$

(1) $\mathcal{X}$ is closed under vector addition

$$\forall x_1, x_2 \in \mathcal{X}, \quad x_1 + x_2 \in \mathcal{X}$$

(2) vector addition is commutative

$$\forall x_1, x_2 \in \mathcal{X}, \quad x_1 + x_2 = x_2 + x_1$$

(3) $\mathcal{X}$ consists a zero vector 0 (origin of vector space)

$$\forall x \in \mathcal{X}, \quad x + 0 = x$$

(5) Each element of $\mathcal{X}$ has an additive inverse

$$\forall x \in \mathcal{X}, \quad \exists \bar{x} \in \mathcal{X} \text{ s.t } x + \bar{x} = 0$$

(6) $\mathcal{X}$ closed under scalar multiplication for

any $\alpha \in \mathcal{F}$ $\quad \forall x \in \mathcal{X}, \forall \alpha \in \mathcal{F}, \quad \alpha \cdot x \in \mathcal{X}$

(7) Scalar multiplication is associative

$\forall \alpha, \beta \in \mathcal{F}, \quad \forall x \in \mathcal{X} \quad (\alpha \cdot \beta) x = \alpha \cdot (\beta \cdot x)$

(8) scalar multiplication is distributive over vector addition

$\forall \alpha \in \mathcal{F}, \quad \forall x_1, x_2 \in \mathcal{X}, \quad \alpha(x_1 + x_2) = \alpha \cdot x_1 + \alpha \cdot x_2$

(9) Scalar multiplication is distributive over scalar addition.

$\forall \alpha, \beta \in \mathcal{F}, \forall x \in \mathcal{X} \quad (\alpha + \beta) x = \alpha \cdot x + \beta \cdot x$

(10) For any $x \in \mathcal{X}$. $1 \cdot x = x$ where 1 is the multiplicative identity in $\mathcal{F}$

Ex: $(\mathbb{R}^n, \mathbb{R})$ vector space

$(\mathbb{R}, \mathbb{R})$ vector space

$(\mathcal{F}, \mathcal{F})$ vector space

$(\mathbb{R}^{2 \times 3}, \mathbb{R})$ vector space

Ex: $\mathbb{F} = \mathbb{R}$ take $D \subset \mathbb{R}$

(ex. $D = [a,b]$, $D = [0,\infty)$ or $D = \mathbb{R}$)

Define $X = \{ f : D \longrightarrow \mathbb{R} \}$

$= \{$ set of function's from D to $\mathbb{R} \}$

$\Rightarrow \forall f, g \in X, x \in D, \quad (f+g)(x) = f(x) + g(x)$

$\Rightarrow \forall \alpha \in \mathbb{R}, \forall f \in X \quad \alpha \cdot f : \forall x \in D$

$(\alpha \cdot f)(x) = \alpha \cdot f(x)$

* Ex: Lebesgue space $L^2([0,1])$

all the function's $f : [0,1] \longrightarrow \mathbb{R}$

with finite energy $\int_0^1 |f(x)|^2 dx < \infty$

Linear Operator:

Let $(\mathcal{X}, \mathcal{F})$ and $(\mathcal{Y}, \mathcal{F})$ be vector spaces

$L : \mathcal{X} \longrightarrow \mathcal{Y}$ is a linear operator

(mapping, transformation) if $\forall x, \bar{x} \in \mathcal{X}$
$\forall \alpha, \bar{\alpha} \in \mathcal{F}$

① $L(x + \bar{x}) = L(x) + L(\bar{x})$

② $L(\alpha x) = \alpha L(x)$

$\Rightarrow L(\alpha x + \bar{\alpha} \bar{x}) = \alpha L(x) + \bar{\alpha} L(\bar{x})$

Norm's:

$\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$ $(\mathcal{X}, \mathcal{F})$ is a vector space

Def: $\| \cdot \| : \mathcal{X} \longrightarrow \mathbb{R}$ is a norm

if

(a) $\forall x \in \mathcal{X}$, $\|x\| \geq 0$ and

$\|x\| = 0 \iff x = 0$

(positive definiteness)

(b) [Triangular inequality] $\forall x, y \in \mathcal{X}$

$$\|x + y\| \leq \|x\| + \|y\|$$

(c) [Positive homogenity] $\forall \alpha \in \mathcal{F}$, $\forall x \in \mathcal{X}$

$$\|\alpha \cdot x\| = |\alpha| \cdot \|x\|.$$

Ex: $\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$, $\mathcal{X} = \mathcal{F}^n$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

(i) $$\|x\|_2 := \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

for $\mathcal{F} = \mathbb{R}$ this is called Euclidean norm

(ii) $$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$ $\quad 1 \leq p \leq \infty$

p-norm

(iii) $$\|x\|_\infty := \max_{0 \leq i \leq n} |x_i|$$ (known as max-norm, sup-norm, or infinite norm)

(iv) $\left(L^2([0,1]), \|\cdot\|_2 \right)$

(i) $L^2([0,1])$ is a set of square-integrable functions on the interval $[0,1]$, meaning all functions f such that

$$\int_0^1 |f(x)|^2 \, dx < \infty$$

(ii) $\|\cdot\|_2$ is the L^2-norm (or Euclidean norm in function space), defined as

$$\|f\|_2 = \left(\int_0^1 |f(x)|^2 dx \right)^{1/2}$$

Def:- $(X, \mathcal{F}, \|\cdot\|)$ called normed space

if $(X, \mathcal{F})$ is a vector space and $\|\cdot\|$ is a norm

Ex: $\left(L^2([0,1]), \|\cdot\|_2 \right)$

Definition: for $x, y \in X$, the distance from x and y is $d(x,y) := \|x-y\|$
$$= \|y-x\|$$

$$d : X \times X \longrightarrow \mathbb{R}$$

Inner Product:

Let $(X, \mathbb{F})$ be vector space.

A function $\langle \cdot, \cdot \rangle : X \times X \longrightarrow \mathbb{R}$ is an inner product if

ⓐ $\forall x, y \in X$, $\langle x, y \rangle = \langle y, x \rangle$
(symmetry)

ⓑ $\forall \alpha_1, \alpha_2 \in \mathbb{F}$, $\forall x_1, x_2 \in X$, $\forall y \in X$

$\langle \alpha_1 x_1 + \alpha_2 x_2, y \rangle = \alpha_1 \langle x_1, y \rangle + \alpha_2 \langle x_2, y \rangle$
(Linearity)

ⓒ $\forall x \in X$, $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \iff x = 0$

Ex:

ⓐ $(\mathbb{R}^n, \mathbb{R})$, $\langle x, y \rangle = x^T y = \sum_{i=1}^{n} x_i y_i$

(b) $(\mathbb{R}^{n\times m}, \mathbb{R})$, $\langle A, B\rangle = tr(A^T B)$

(c) $(X, \mathbb{R})$ $X = \{ f_b : [a,b] \to \mathbb{R} \mid f \text{ continuous} \}$

$$\langle f, g\rangle = \int_a^b f(y)\, g(y)\, dy$$

(d) $(L^2([0,1]), \mathbb{C})$

$$\langle f, g\rangle = \int_0^1 f(x)\, \overline{g(x)}\, dx$$

Def: Given $(X, \mathbb{F})$ a vector space and $\langle \cdot, \cdot \rangle$ an inner product on it, $(X, \mathbb{F}, \langle \cdot, \cdot \rangle)$ is an inner product space.

induced norm: all inner product induce a norm (but converse is not true)

$$\|x\| := \langle x, x \rangle^{1/2}$$

induced distance:

$$d(f,g) = \|f-g\|$$

$$= \left[\int_0^1 |f(x) - g(x)|^2 \, dx \right]^{1/2}$$

$$d(x,y) := \|x-y\| := \langle x-y, x-y \rangle^{1/2}$$

inner Product $\underset{\not\longleftarrow}{\longrightarrow}$ norms $\underset{\not\longleftarrow}{\longrightarrow}$ distance