Problem set 1

(1) some Practice with geometric series and complex exponentials

(a) if ω is a complex number, $\omega \neq 1$ and p and q are any integer's, show that

$$\sum_{n=p}^{q} \omega^n = \frac{\omega^p - \omega^{q+1}}{1-\omega}$$

Discuss the cases when $p=-\infty$ or $q=\infty$. what about $p=-\infty$ and $q=+\infty$?

Sol^n

$$\sum_{n=p}^{q} \omega^n = f$$

$$f - \omega f = \sum_{n=p}^{q} \omega^n - \sum_{n=p}^{q} \omega^{n+1}$$

$$= \omega^p - \omega^{q+1}$$

$$\Rightarrow \quad f(1-\omega) = \omega^p - \omega^{q+1}$$

$$\Rightarrow \quad f = \frac{\omega^p - \omega^{q+1}}{1-\omega}$$

$$\Rightarrow \quad \sum_{n=p}^{q} \omega^n = \frac{\omega^p - \omega^{q+1}}{1-\omega}$$

<u>Case 1:</u> $p = -\infty$

if $\left|\frac{1}{\omega}\right| < 1$ i.e $|\omega| > 1$

then $\omega^p = \omega^{-\infty} = \left(\frac{1}{\omega}\right)^{\infty} = 0$

i.e $\displaystyle\sum_{n=-\infty}^{q} \omega^n = \frac{\omega^{-\infty} - \omega^{q+1}}{1-\omega} = -\frac{\omega^{q+1}}{1-\omega}$

$|\omega| > 1$

$$= \frac{\omega^{q+1}}{\omega - 1}$$

<u>Case 2:</u> $q = \infty$

then for the series to converge $|\omega| < 1$

$$\omega^q \longrightarrow 0 \quad \Rightarrow \quad \sum_{n=p}^{+\infty} \omega^n = \frac{\omega^p}{1-p} \quad |\omega| < 1$$

CASE 3 $\quad p = -\infty$ and $q = +\infty$

$$\sum_{n=-\infty}^{+\infty} \omega^n = \frac{\omega^p - \omega^{q+1}}{1-\omega} = \frac{\omega^{-\infty} - \omega^{+\infty}}{1-\omega}$$

The series does not converge

(b) Find the sum

$$\sum_{n=0}^{N-1} e^{2\pi i n/N}$$

and explain your answer geometrically.

Sol^n

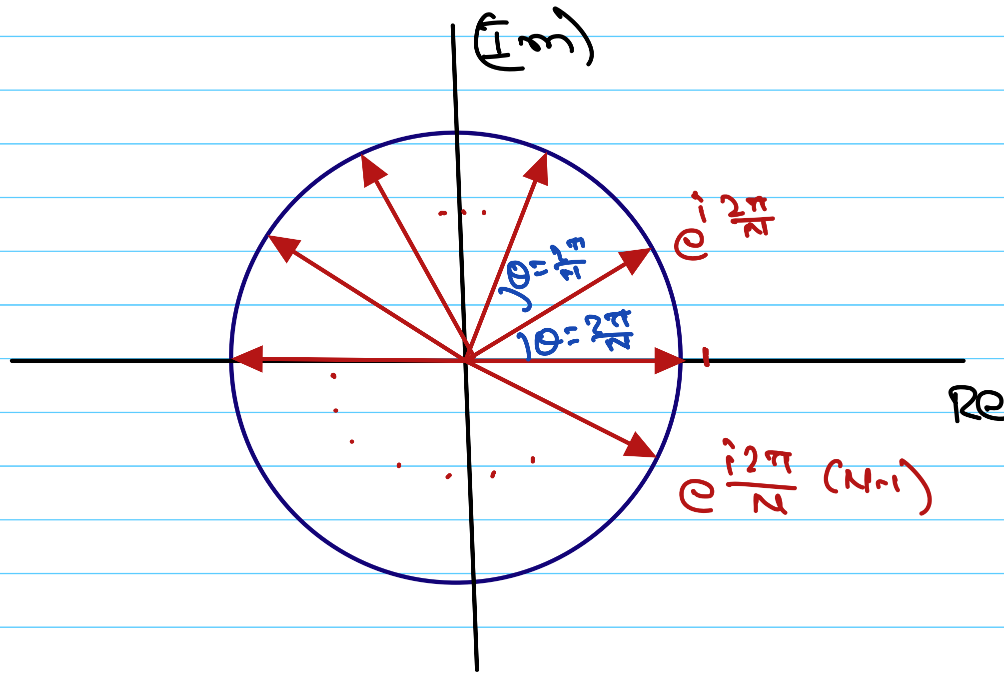
$$\sum_{n=0}^{N-1} e^{2\pi i n/N} = \frac{1 - e^{2\pi i \frac{N}{N}}}{1-e^{\frac{2\pi i}{N}}}$$

$$= \frac{1-e^{2\pi i}}{1-e^{\frac{2\pi i}{N}}}$$

$$= \frac{1-1}{1-e^{\frac{2\pi i}{N}}} = 0$$

$$\sum_{n=0}^{N-1} e^{\frac{2\pi i n}{N}} = 1 + e^{\frac{2\pi i}{N}} + e^{\frac{2\pi i}{N}\cdot 2} + \dots e^{\frac{2\pi i}{N}\cdot N-1}$$

$$= 1 e^{i\cdot 0} + |1| e^{i\frac{2\pi}{N}} + |1| e^{i\frac{2\pi}{N}\cdot 2} + \dots + |1| e^{i\frac{2\pi}{N}N-1}$$



sum of all these vector's = 0

(c) Derive the formula

$$\sum_{k=-N}^{N} e^{2\pi i N t} = \frac{\sin(2\pi t(N+1/2))}{\sin(\pi t)}$$

$\underline{\underline{\text{Sol}^n}}$

$$\sum_{k=-N}^{N} \left(e^{2\pi i t}\right)^k = \frac{\left(e^{2\pi i t}\right)^{-N} - \left(e^{2\pi i t}\right)^{N+1}}{1 - e^{2\pi i t}}$$

$$= \frac{e^{-2\pi i t N} - e^{2\pi i t (N+1)}}{1 - e^{2\pi i t}}$$

$$= \frac{e^{-2\pi i t N} - e^{2\pi i t N + 2\pi i t}}{1 - e^{2\pi i t}}$$

$$= \frac{e^{-2\pi i t (N+1/2) + 2\pi i \frac{t}{2}} - e^{2\pi i t (N+1/2) + 2\pi i \frac{t}{2}}}{1 - e^{2\pi i t}}$$

$$= \frac{2i \cdot e^{\frac{2\pi i t}{2}}}{1 - e^{2\pi i t}} \left(\frac{e^{-2\pi i t (N+1/2)} - e^{2\pi i t (N+1/2)}}{2i} \right)$$

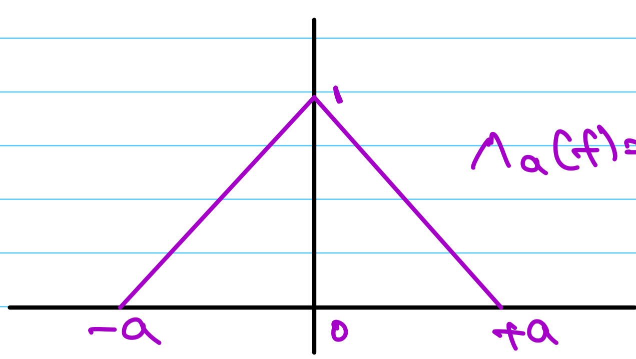
$$= \frac{2i e^{\pi i t}}{1-e^{2\pi i t}} \cdot -\sin\left(2\pi t\left(N+1/2\right)\right)$$

$$= \frac{2i}{e^{-\pi i t} - e^{\pi i t}} \cdot -\sin\left(2\pi t\left(N+1/2\right)\right)$$

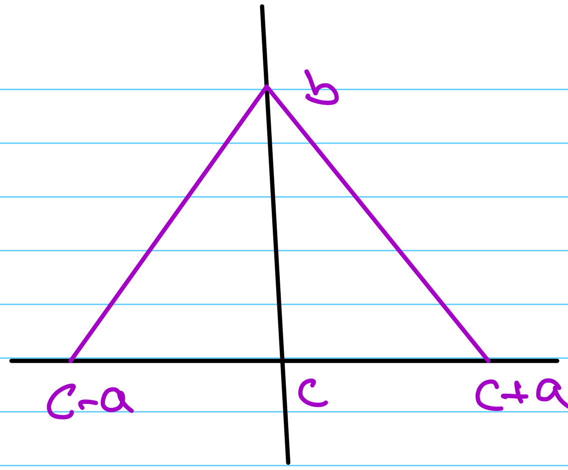
$$= \frac{-\sin\left(2\pi t\left(N+1/2\right)\right)}{-\sin(\pi t)}$$

$$\Rightarrow \sum_{k=-N}^{N} e^{2\pi i k t} = \frac{\sin\left(2\pi t\left(N+1/2\right)\right)}{\sin(\pi t)}$$

(2) Some Practicing Combining Simple signals



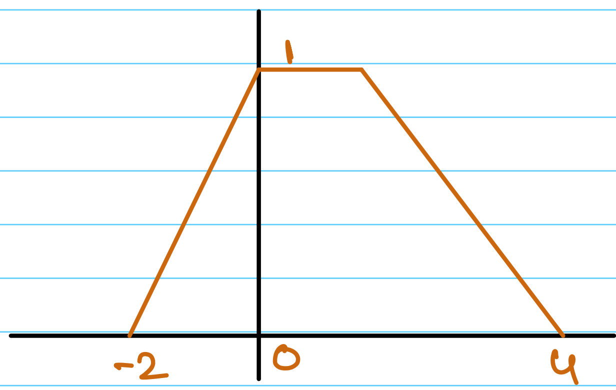
$$\Lambda_a(t) = \Lambda(t/a) = \begin{cases} 1 - \frac{1}{a}|t| & |t| \leq a \\ 0 & |t| > a \end{cases}$$



$b\Lambda_a(t-c)$

shifting + scaling

(a)

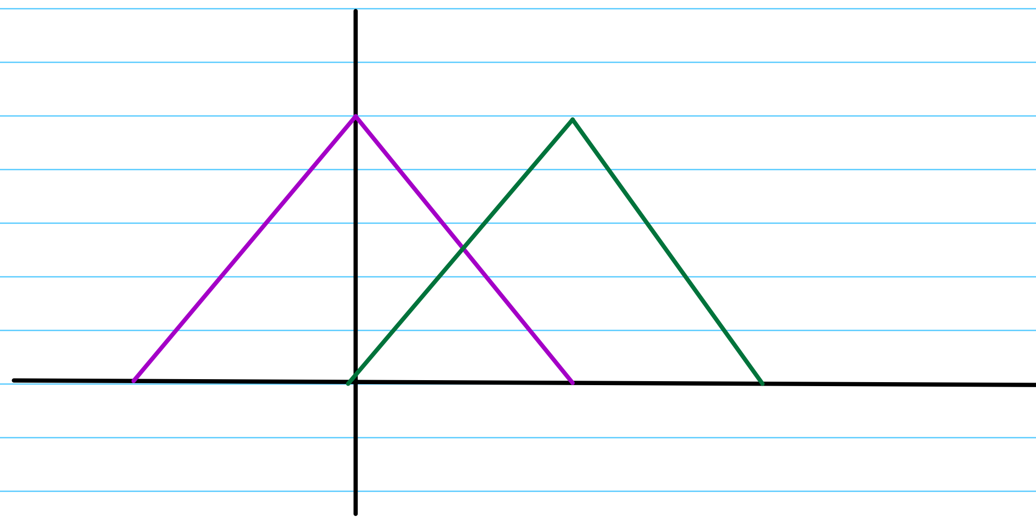


this can be expressed as

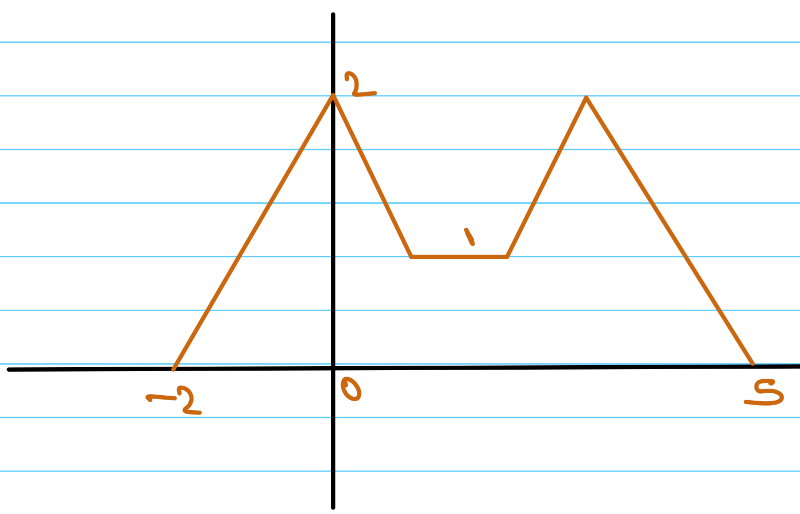
$$\Lambda_2(t) + \Lambda_2(t-2)$$

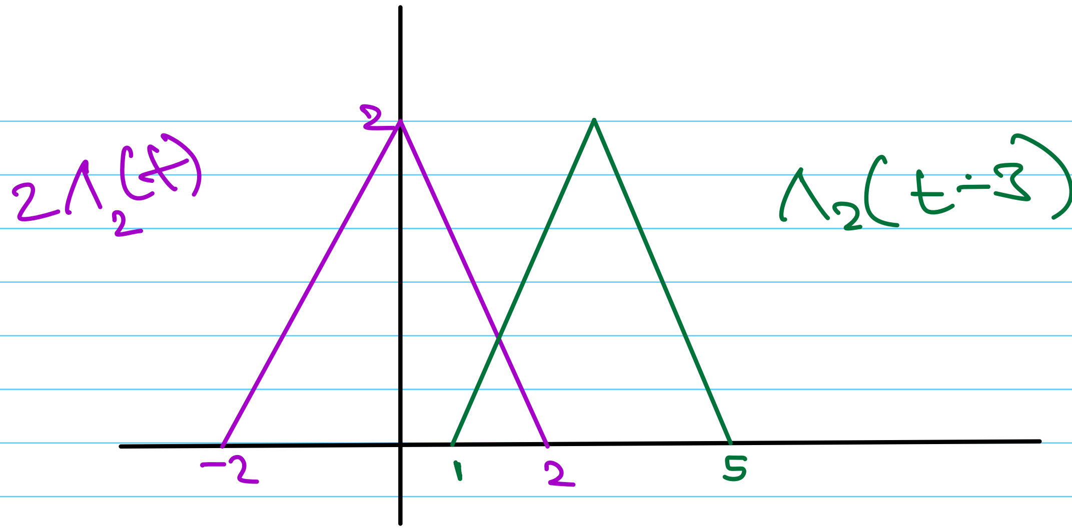
$$= \begin{cases} 1-\frac{1}{2}|t| & |t| \le 2 \\ 0 & |t| > 2 \end{cases} + \begin{cases} 1-\frac{1}{2}|t-2| & |t-2| \le 2 \\ 0 & |t-2| > 2 \end{cases}$$

$$= \begin{cases} 1 + \frac{1}{2} t & -2 < t < 0 \\ 1 & 0 \leq t < 2 \\ 1 - \frac{1}{2}(t-2) & 2 \leq t < 4 \end{cases}$$

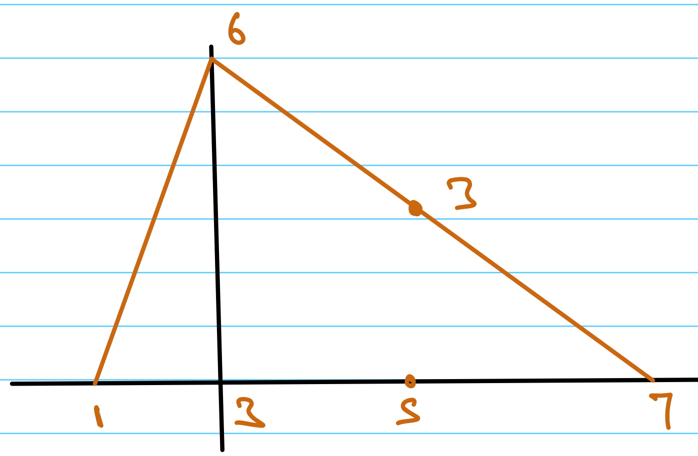


b





(c)



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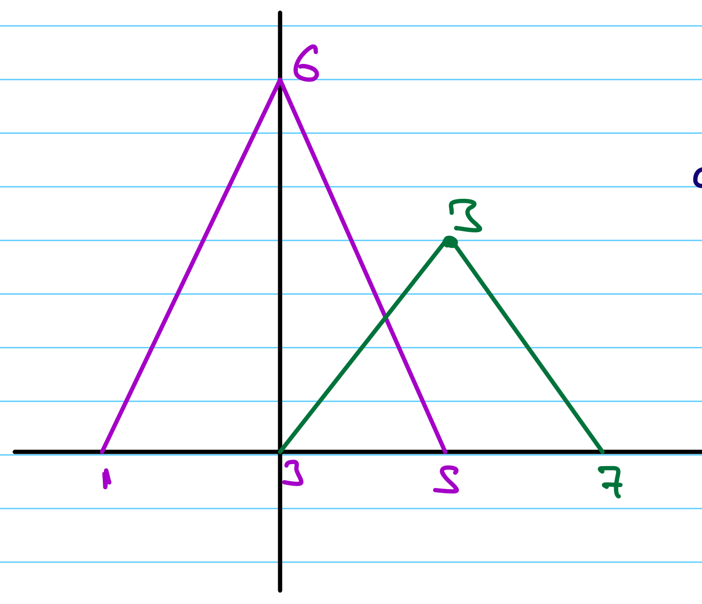


at $t=5$
height $=3$

$$= 6\Lambda_2(t-3) + 3\Lambda_2(t-5)$$

③ Creating Periodic functions

let $f(t)$ be a function, defined for all t, and let $T > 0$. define

$$g(t) = \sum_{n=-\infty}^{+\infty} f(t-nT)$$

ⓐ Provided the sum converges, show that $g(t)$ is periodic with period T. One sometimes says that $g(t)$ is the periodization of $f(t)$

sol^n

$$g(t) = \sum_{n=-\infty}^{+\infty} f(t-nT)$$

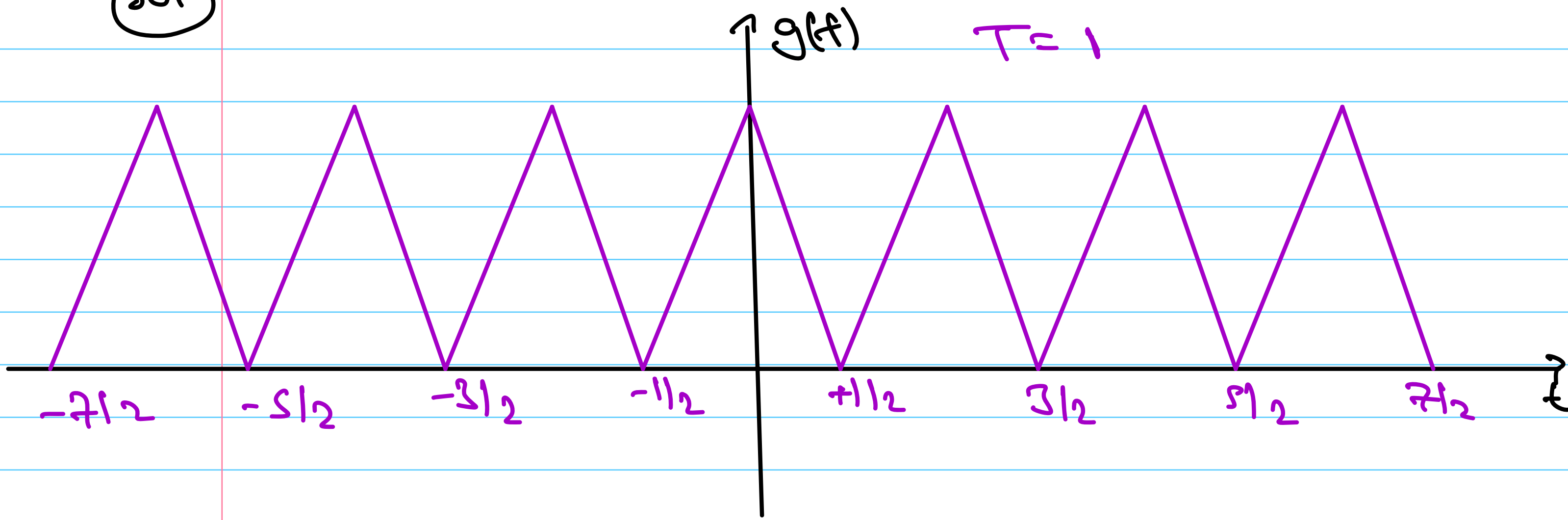
$$g(t+T) = \sum_{n=-\infty}^{+\infty} f(t+T-nT)$$

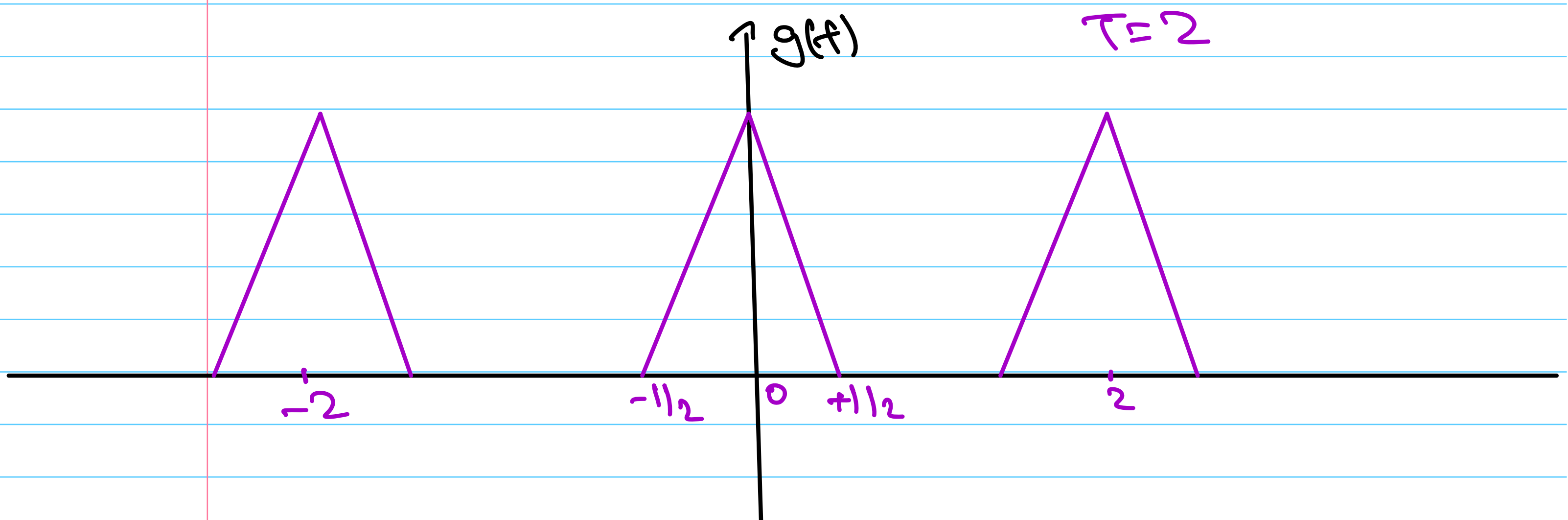
$$= \sum_{n=-\infty}^{+\infty} f(t-(n-1)T) \quad \boxed{n-1=n}$$

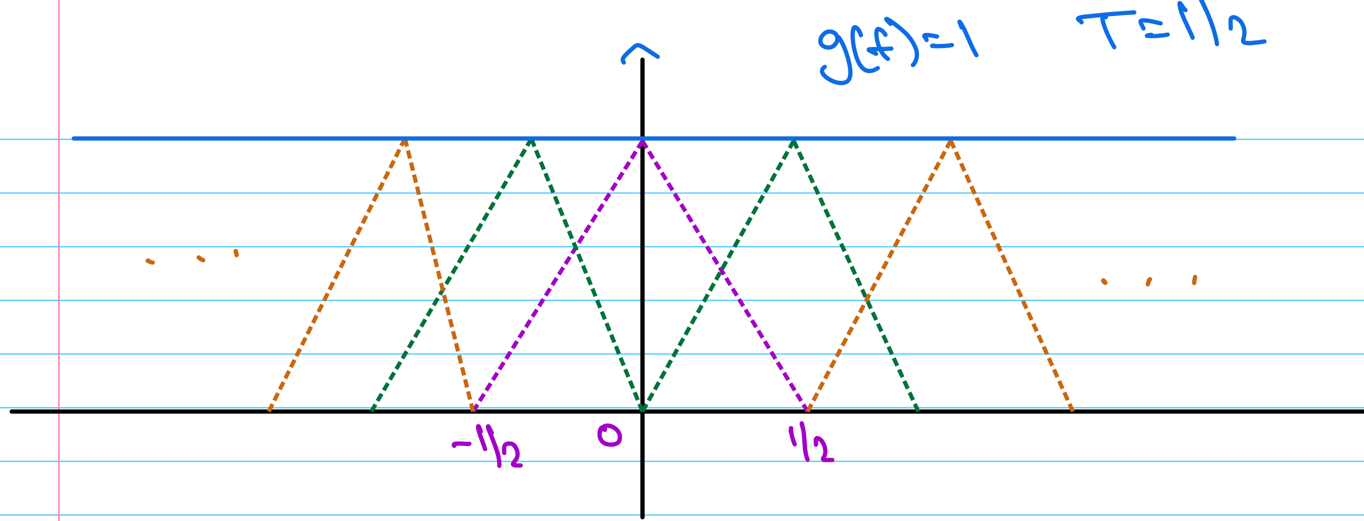
$$= \sum_{n=-\infty}^{+\infty} f(t-nT)$$

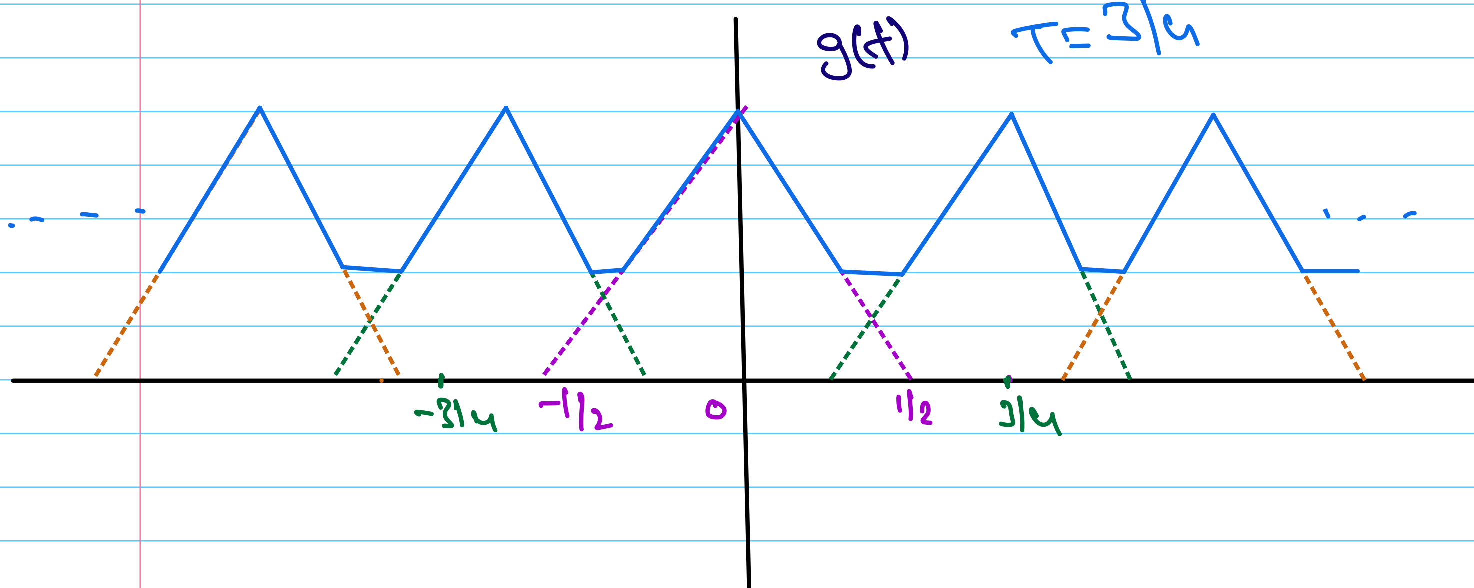
(b) let $f(t) = \Lambda_{1/2}(t)$. Sketch the periodization's $g(t)$ of $f(t)$ for $T = 1/2$, $T = 3/4$, $T=1$, $T=2$

Sol









(c) if the function is already Periodic, is it equal to its own Periodization? Explain?

④ ⓐ $f(x) = \sin(2\pi m x) + \sin(2\pi n x)$

n, m are +ve integer's.

is $f(x)$ Periodic? if so, what is the period?

Sol^n

$$\sin(2\pi m x) \Rightarrow T_1 = \frac{2\pi}{2\pi m}$$

$$\Rightarrow T_1 = \frac{1}{m}$$

$$\sin(2\pi n x) \Rightarrow T_2 = \frac{2\pi}{2\pi n}$$

$$T_2 = \frac{1}{n}$$

Time Period of $f(x) = \text{lcm}\left(\frac{1}{m}, \frac{1}{n}\right)$

$$= \frac{1}{\gcd(m,n)}$$

(b) $g(x) = \sin(2\pi p x) + \sin(2\pi q x)$

$p = m/r \qquad q = n/s$

is $g(x)$ periodic? If so, what is its period?

Sol^n

$g(x)$ is periodic with period

$$\mathrm{LCM}\left(\frac{1}{p}, \frac{1}{q}\right)$$

$$\Rightarrow \mathrm{LCM}\left(\frac{r}{m}, \frac{s}{n}\right)$$

$$\Rightarrow \frac{\mathrm{LCM}(r, s)}{\gcd(m, n)}$$

(c) $f(t) = \cos t + \cos\sqrt{2}t$ (Not Periodic)

$\sqrt{2}$ is irrational

5 $f(t)$, $g(t)$ be two signal's

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} f(t)\,\overline{g(t)}\,dt$$

Reversed signal $\quad f^{-}(t) = f(-t)$

Delay operator $\quad T_a f(t) = f(t-a)$

a if both $f(t)$ & $g(t)$ are reversed, what happend's to their inner product?

Sol^n

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} f(t)\,\overline{g(t)}\,dt$$

$$\langle f^{-}, g^{-} \rangle = \int_{-\infty}^{+\infty} f^{-}(t)\,\overline{g^{-}(t)}\,dt$$

$$= \int_{-\infty}^{+\infty} f(-t)\,\overline{g(-t)}\,dt$$

$y = -t \Rightarrow dy = -dt$

$$= \int_{y=+\infty}^{-\infty} - f(-y)\, \overline{g(-y)}\, dy$$

$$= \int_{y=-\infty}^{+\infty} f(-y)\, \overline{g(-y)}\, dy$$

$\langle \bar{f}, \bar{g} \rangle = \langle f, g \rangle$

Same inner Product

(b) if one of $f(t)$ or $g(t)$ reversed, what happen's to their inner Product.

Sol^n^ $\langle \bar{f}, g \rangle = \int_{-\infty}^{+\infty} f(-t)\, \overline{g(t)}\, dt$

$y = -t$

$$= \int_{-\infty}^{+\infty} f(y)\, \overline{g(-y)}\, dy$$

$$= \langle f, \bar{g} \rangle$$

(c) If Both $f(t)$ & $g(t)$ are

Shifted By same amount,

what happen's to their inner product?

Sol

$$\langle f(t-a), g(t-a) \rangle$$

$$= \int_{-\infty}^{+\infty} f(t-a)\, \overline{g(t-a)}\, dt$$

$$t - a = y$$

$$dt = dy$$

$$= \int_{-\infty}^{+\infty} f(y)\, \overline{g(y)}\, dy$$

$$= \langle f, g \rangle$$

Same inner Product.

(d) One of the $f(t)$ and $g(t)$ is shifted.

Sol $\langle f(t-a), g(t) \rangle = \int_{-\infty}^{+\infty} f(t-a)\,\overline{g(t)}\,dt$

$t-a = y$

$dt = dy$

$$= \int_{-\infty}^{+\infty} f(y)\,\overline{g(y+a)}\,dy$$

$$= \langle f(t), g(t+a) \rangle$$

$$\Rightarrow \quad \langle T_a f, g \rangle = \langle f, T_{-a} g \rangle$$

(e) Both $f(t)$ & $g(t)$ shifted by different amount's,

Sol: $\langle T_a f, T_b g \rangle = \int_{-\infty}^{+\infty} f(t-a)\,\overline{g(t-b)}\,dt$

$\Rightarrow \quad t-a = y$

$dt = dy$

$$= \int_{-\infty}^{\infty} f(y)\, \overline{g(y+a-b)}\, dy$$

$$= \langle f, T_{(b-a)} g \rangle$$

$$= \langle T_{(a-b)} f, g \rangle$$

$$\Rightarrow \langle T_a f, T_b f \rangle = \langle f, T_{(b-a)} g \rangle = \langle T_{(a-b)} f, g \rangle$$

(f) f, g are periodic with period 1

$$\Rightarrow \langle f, g \rangle = \int_0^1 f(t)\, \overline{g(t)}\, dt$$