Lec 04: Fourier series finis

we want to make sense of infinite sum's $\sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi i k t}$

* The fourier coefficient's $\hat{f}(k)$. The Problem's in general, how do we make sense of such infinite sum? The tricky thing about it is that, if we think in term's of sin's & cosines these term's, function's are oscillating b/w +ve and -ve, so for this thing to converge there is got to be some conspiracy cancellation's that's making it work, ofcourse the size of the co-efficient's is gonna play a role, But its more than that, oscillating that make's it trickier to study.

if $f(t)$ is periodic with period 1

we want to write with some confidence that

$$f(t) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi i k t}$$

* Any small lack of smoothness in the function or in any of its derivatives, gonna force to infinite term's.

* A finite number of term's will be infinitely smooth, the function and all its derivatives is gonna be continuous.

$\Rightarrow$ Any Lack of smoothness forces infinite term's.

where $\hat{f}(k) = \int_0^1 e^{-2\pi i k t} f(t)\, dt$

if $f(t)$ is continuous, smooth, then we get satisfactory convergence results.

Important condition: is integrability,

instead of smoothness & differentiability

we care about integrability of $f(t)$

Say $f(t)$ is square integrable

$$f \in L^2([0,1])$$ Lebesgue square

if $$\int_0^1 |f(t)|^2 \, dt < \infty$$ (finite energy condition)

where $|f(t)|^2 =$ absolute value squared

$\Rightarrow$ $|f(t)| = f(t)$ for real function.

if $f(t)$ is periodic, square integrable, then

we have from fourier co-efficients

$$\hat{f}(k) = \int_0^1 e^{-2\pi i k t} f(t) \, dt$$

we have

$$\int_0^1 \left| f(t) - \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi i k t} \right|^2 dt \xrightarrow[n\to\infty]{} 0$$

Remember in solving fourier co-efficient's we used

$$\int_0^1 e^{2\pi i n t} \cdot e^{-2\pi i m t}\, dt = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

Cornerstone for understanding these spaces, and to introducing geometry into those space.

This simple fact is the cornerstone for introducing "geometry" into space of square integrable function $L^2([0,1])$

Geometry: Allow's one to define orthogonality via inner product or Dot Product.

f, g square - integrable on $[0,1]$

inner Product: generalization of the dot product for vector's.

$$\langle f, g \rangle = \int_0^1 f(t)\,\overline{g(t)}\,dt$$

$\Rightarrow$ f, g are orthogonal if

$$\langle f, g \rangle = \int_0^1 f(t)\,\overline{g(t)}\,dt = 0$$

Norm of f is

$$\|f\|_2 = \langle f, f \rangle^{1/2}$$

$$= \left[\int_0^1 |f(t)|^2\,dt \right]^{1/2}$$

Complex exponential's are orthonormal function's of length 1

$$\langle e^{2\pi i n t}, e^{2\pi i m t} \rangle = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

Use inner product to define and compute projection.

Fourier coefficient is exactly projection onto complex exponential

$$\hat{f}(k) = \langle f(t), e^{2\pi i k t} \rangle = \int_0^1 f(t)\, e^{2\pi i k t}\, dt$$

$$\Rightarrow \quad \hat{f}(k) = \int_0^1 f(t) e^{-2\pi i k t}\, dt$$

$$\Rightarrow \quad f(t) = \sum_{k=-\infty}^{\infty} \langle f, e^{2\pi i k t} \rangle e^{2\pi i k t}$$

$\Rightarrow$ The complex exponential functions form a orthonormal BASIS for the square integrable functions. $L^2([0,1])$

$\{ e^{2\pi i k t} \}$ form orthonormal BASIS.

Rayleigh's Identity :-

$$\int_0^1 |f(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$$

Proof:

$$\int_0^1 |f(t)|^2 \, dt = \|f\|^2$$

$$= \langle f, f \rangle$$

$$= \left\langle \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n , \sum_{m=-\infty}^{\infty} \langle f, e_m \rangle e_m \right\rangle$$

$$= \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$$