Lec 24:-

* The state of affair's we will see today when we wrap up the discussion is really quite satisfactory in understanding Linear system's & its structure.

Linear system's II

1. in discrete case any Linear system has to be given by multiplication by a matrix. (finite dimentional case)

2. An analogous hold's in Continuou's case. any infinite dimentional continuous case (input's & output's are function's) any Linear system has to be given by integration against kernal.

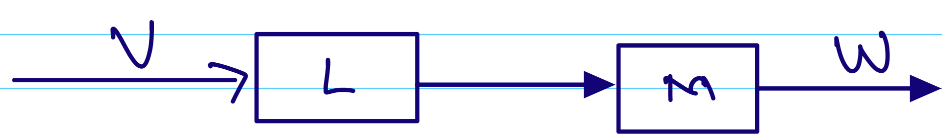
* A linear system is given by integration against a kernal.

$$Lv(x) = \int_{-\infty}^{\infty} \underbrace{K(x,y)}_{\text{Play's the role of matrix}} \overbrace{v(y)}^{\text{Input}} \, dy$$

This is not just a good example of linear system, This is the only example of linear system. Any linear system essentially look's like this.

* we have to produce a $K(x,y)$ for general linear system.

* To do this we need a breif digression into the idea of cascading linear system's.



if L and M are linear system's , L

followed by M

$W = MLV$ is Linear.

<u>Special case:</u>

$$Lv = \int_{-\infty}^{\infty} k(x,y)\, v(y)\, dy$$

what is M applied to L at v?

$MLv(x)$?

* M applied to L at v(x) is given by

$$MLv(x) = \int_{-\infty}^{\infty} M_x K(x,y)\, v(y)\, dy$$

* But why does this work? The idea is we approximate the integral by sum and then apply ordinary

Linearity.

$$\int_{-\infty}^{\infty} K(x,y)\, V(y)\, dy \approx \sum_i K(x,y_i)\, V(y_i)\, \Delta y_i$$

x = continuous

y = discrete set of measurements

Operate with M on sum

$$M\left(\sum_i K(x,y_i)\, V(y_i)\, \Delta y_i\right)$$

$$= \sum_i M\left(K(x,y_i)\, V(y_i)\, \Delta y_i\right)$$

M operates on $x \Rightarrow V(y_i)\, \Delta y_i$ are constants

$$= \sum_i M_x\left(K(x,y_i)\right)\, V(y_i)\, \Delta y_i$$

$$\approx \int_{-\infty}^{\infty} M_x\left(K(x,y)\right)\, V(y)\, dy$$

$$Lv(x) = \int_{-\infty}^{\infty} K(x,y)\, v(y)\, dy$$

$$MLv(x) = \int_{-\infty}^{\infty} M_x\big(K(x,y)\big)\, v(y)\, dy$$

* Any linear system is some integration against some kernal.

* The main plot is for any linear system is given by integration against kernal.

Discretize both x & y axis

$$Lv(x) = \int_{-\infty}^{\infty} K(x,y)\, v(y)\, dy$$

$$\begin{bmatrix} K(x_1,y_1) & K(x_1,y_2) & K(x_1,y_3) & \cdots & K(x_1,y_\infty) \\ K(x_2,y_1) & K(x_2,y_2) & K(x_2,y_3) & \cdots & K(x_2,y_\infty) \\ K(x_3,y_1) & \vdots & \vdots & & \\ \vdots & \vdots & \vdots & & \\ K(x_\infty,y_1) & K(x_\infty,y_2) & K(x_\infty,y_3) & \cdots & K(x_\infty,y_\infty) \end{bmatrix} \begin{bmatrix} v(y_1) \\ v(y_2) \\ v(y_3) \\ \vdots \\ v(y_\infty) \end{bmatrix}$$

$$= \begin{bmatrix} \int_{-\infty}^{\infty} K(x_1, y)\, V(y)\, dy \\ \int_{-\infty}^{\infty} K(x_2, y)\, V(y)\, dy \\ \vdots \\ \int_{-\infty}^{\infty} K(x_\infty, y)\, V(y)\, dy \end{bmatrix}$$

$$= \int_{-\infty}^{\infty} K(x, y)\, V(y)\, dy$$

Delta Function: $\delta(t)$

any input function can be decomposed into

$$\underbrace{V(x) = \int_{-\infty}^{+\infty} \delta(x-y)\, V(y)\, dy}$$

* Continuous analog of expressing a vector in

terms of sum of its components.

* if we think Delta function as one in one slot & 0 in all other slots, then writing a vector as $ve_1 + ve_2 + \dots + ve_n$ is exactly continuous analog of writing $v(x)$

$$v(x) = \int_{-\infty}^{\infty} \delta(x-y)\, v(y)\, dy \quad \text{(shifting property)}$$

Ex:

$$\begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

* Now if L is any linear system, then apply L to $v(x)$

$$\xrightarrow{v(x)} \boxed{L} \longrightarrow ?$$

$$Lv(x) = L \int_{-\infty}^{\infty} \delta(x-y)\, v(y)\, dy$$

$$= \int_{-\infty}^{\infty} L_x \delta(x-y)\, V(y)\, dy$$

Now set $\quad K(x,y) = L_x \delta(x-y)$

$$\Rightarrow \quad LV(x) = \int_{-\infty}^{\infty} K(x,y)\, V(y)\, dy$$

* any Linear system is given by integration against kernal. The kernal is what the system does when we feed it a delta function.

$\Rightarrow$ $\delta(x)$ is called impulse

$\Rightarrow$ when we feed a impulse as input the system respond's and we call it as $K(x,y)$ impulse response

$K(x,y)$: Impulse response :- how a system respond's to feeding it with an Impulse analog:

① $V = \begin{bmatrix}1\\5\\3\end{bmatrix} = 1\begin{bmatrix}1\\0\\0\end{bmatrix} + 5\begin{bmatrix}0\\1\\0\end{bmatrix} + 3\begin{bmatrix}0\\0\\1\end{bmatrix}$

$$\Longleftrightarrow \quad v(x) = \int_{-\infty}^{\infty} \delta(x-y)\, v(y)\, dy$$

②

$$Av = A\begin{bmatrix}1\\5\\3\end{bmatrix}$$

$$= 1 \cdot A\begin{bmatrix}1\\0\\0\end{bmatrix} + 5 \cdot A\begin{bmatrix}0\\1\\0\end{bmatrix} + 3 \cdot A\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\Longleftrightarrow \quad Lv(x) = L\int_{-\infty}^{\infty} \delta(x-y)\, v(y)\, dy$$

$$= \int_{-\infty}^{\infty} L_x\, \delta(x-y)\, v(y)\, dy$$

$$= \int_{-\infty}^{\infty} K(x,y)\, v(y)\, dy$$

Finite dimentional discrete case

$LV = AV$

what is the Impulse response

$$V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Downarrow$$

$$v(x) = \int_{-\infty}^{+\infty} \delta(x-y)\, v(y)\, dy$$

so, the delta function are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

apply linear operator L to these delta function's Le_1, Le_2, Le_3

$$L[e_1\ e_2\ e_3] = A[e_1\ e_2\ e_3] = \underset{\Uparrow}{A}$$

$\Rightarrow$ A is Impulse response

$$LV = L(v_1 e_1 + v_2 e_2 + v_3 e_3) = v_1 L e_1 + v_2 L e_2 + v_3 L e_3$$

if is the system L respond's to the input impulse shifted to y. $\delta(x-y)$

$$L_x \, \delta(x-y)$$

Schwatz kernal theorem:- deepest fact in the entire theory of distribution's

* if L is a Linear operator on distribution then there is a unique kernal K which is another distribution so that

$$LV = \langle K, V \rangle$$

where K is L applied to Delta function.

Example's :-

what is the impulse response for F.T? viewed as Linear system.

* By definicn Impulse response is system applied to shifted Delta function's.

$$\mathcal{F}(\delta(x-y)) = e^{-2\pi i x y}$$

$$\mathcal{F} f(x) = \int_{-\infty}^{\infty} e^{-2\pi i x y} f(y)\,dy$$

=) This exibit's F.T as integration against kernel. The F.T is a Linear system with impulse response (kernel) as $e^{-2\pi i x y}$

$$h(x,y) = e^{-2\pi i x y}$$

* we know that any Linear operator has to be given by integration against the impulse response, Furthermore impulse response is unique. if we can express our Linear operator as integration, then we have found

impulse response

$\Downarrow$

$$\mathcal{F} f(s) = \int_{-\infty}^{+\infty} \underbrace{e^{-2\pi i s t}}_{\Downarrow} f(t)\, dt$$

This has to be impulse response

i.e $\mathcal{F} \delta(s-t) = e^{-2\pi i s t}$

Example:

Finite dimential (discrete case)

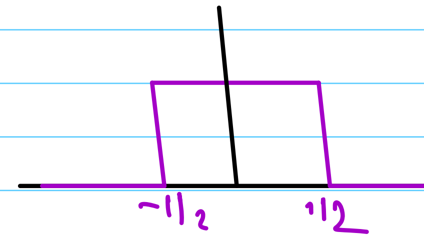
$$LV(t) = A(t)\, V(t)$$

* Any Linear system (finite dimensional linear system) Given by $A(t)\, v(t)$

What is impulse response? $A(t)$

Example: Switch

$Lv = \Pi v$



what is the impulse response?

$$L(\delta(x-y)) = \Pi(x)\,\delta(x-y)$$

$$= \Pi(y)\,\delta(x-y)$$

$$\Rightarrow \quad h(x,y) = \Pi(y)\,\delta(x-y)$$

(Impulse response of switch)

$$Lv(x) = \int_{-\infty}^{\infty} h(x,y)\,v(y)\,dy$$

$$= \int_{-\infty}^{\infty} \Pi(y)\,\delta(x-y)\,v(y)\,dy$$

$$= \int_{-\infty}^{\infty} \delta(x-y)\,\Pi(y)\,v(y)\,dy$$

$$Lv(x) = \Pi(x)\,v(x)$$

Spectial Case : Convolution :

everything we said before applies in great generality. Now we want to look at a special case, where system is given by convolution.

$$Lv(x) = (h * v)(x)$$

$$\Rightarrow \quad Lv = h * v$$

$$\Rightarrow \quad Lv(x) = \int_{-\infty}^{+\infty} h(x-y)\, v(y)\, dy$$

Now we can appeal Schwartz theorem that a linear system given as a integration agaist a kernal, that kernal must be impulse response

$$\Rightarrow \quad h(x-y) = \text{impulse response}$$

from that theorem we concelde

$$L(\delta(x-y)) = h(x-y)$$

Proof:

$$L(\delta(x)) = \int_{-\infty}^{\infty} h(x-y)\, \delta(y)\, dy$$

$$= h(x-y)$$

There is a special property of convolution, that makes it particularly important for Linear system's, is the relationship b/w convolution & delay that turn's out to be worth singling out.

Delay operator:

$$T_a\, v(x) = v(x-a)$$

$T_a\, v(x) = v(x-a)$ delay operator is
an linear operator

What its impulse response ?

$T_a\, \delta(x) = \delta(x-a)$

Convolution of a delayed signal is the delay of the convolution of the signal.

$$h * (T_a v) = T_a (h * v)$$

we interpret in form's of linear system

LTI System's

The time-invariant property is that a shift in time of the input's should result in an identical shif in time of the output's.

$$\text{if} \quad w(t) = Lv(t)$$

if L is a LTI system

$$Lv(t) = w(t)$$

then $$Lv(t-\tau) = w(t-\tau)$$

What about the Impulse response of LTI system?

$$h(t,\tau) = L\,\delta(t-\tau)$$

$\Rightarrow$ the response can have different forms for impulses at different times.

But this isn't the case for an LTI system. Let

$$L\,\delta(t) = h(t) \quad \text{(impulse response at } \tau = 0\text{)}$$

Then by Time-invariance

$$L\,\delta(t-\tau) = h(t-\tau)$$

That is impulse response does not depend independently on t and τ, but rather only on their difference, $t-\tau$.

$$w(t) = Lv(t) = \int_{-\infty}^{\infty} h(t-\tau)\, v(\tau)\, d\tau$$

$$= (h * v)(t)$$

Conversely, let's show that a linear system given by a convolution integral is time invariant. Suppose

$$w(t) = Lv(t) = (g * v)(t) = \int_{-\infty}^{\infty} g(t-y)v(y)\,dy$$

then $$L(v(t-t_0)) = \int_{-\infty}^{\infty} g(t-y\quad)\, v(y-t_0)\,dy$$

$y - t_0 = s$

$$= \int_{-\infty}^{\infty} g(t-t_0-s)\,v(s)\,ds$$

$$= (g * v)(t-t_0) = w(t-t_0$$

$$L\delta(t) = (g * \delta)(t) = g(t)$$

$$L\delta(t-y) = g(t-y)$$

$\Rightarrow$ $g(t-y)$ is Impulse response

A system is Time-invariant $\Longleftrightarrow$ it is a convolution.

$$* \quad h * (T_a v) = T_a(h * v)$$

convolution with a delayed input = delay of the convolution of the signal.

"Re interpret this in term's of Linear system's"

$\Longrightarrow$ if the system is given by convolution

$$w = Lv = h * v$$

Delay of the input $\quad v \longrightarrow v(x-a)$

causes an identical delay of output

$$\begin{tikzcd}
v \arrow[r, "T_a v"] \arrow[d, "L"', "h * v"] & v(t-a) \arrow[d, "\boxed{h * v(t-a) = T_a(h * v)}"] \\
w \arrow[r, "T_a w"'] & w(t-a)
\end{tikzcd}$$

$$h * (T_a v) = T_a (h * v)$$

fundamental relationship b/w delay and convolution. The convolution of delay is the same as delay of the convolution.

* delayed input's goes to delayed output's

* we say that L is a Time-invariant.

$\Rightarrow$ if $\quad w(x) = Lv(x)$

then $\quad w(x-a) = Lv(x-a)$

* if the system is given by convolution then the the system is Time-invariant.

* The converse is also true.

<u>Converse:</u> if the system is Time-invariant $\Longrightarrow$ The system should be given by convolution.

we know that

$$LV(x) = \int_{-\infty}^{+\infty} K(x,y)\, V(y)\, dy$$

where $K(x,y)$ is impulse response of delayed impulse

$$L_x(\delta(x-y)) = K(x,y)$$

$$\Rightarrow LV(x) = \int_{-\infty}^{+\infty} L_x(\delta(x-y))\, V(y)\, dy$$

let $L\delta(x) = h(x)$

Time-invariant system. then $L\delta(x-y) = h(x-y)$

$$\Rightarrow LV(x) = \int_{-\infty}^{\infty} h(x-y)\, V(y)\, dy$$

The system is Time invariant $\Longleftrightarrow$ The system is given by convolution.

2 thing's we discussed today.

① any linear system is integration against the kernal. and the kernel is impulse response

② The special case of the Time-invariant system is also against kernel, but its a special integration, its convolution.

The system is time invariant $\Longleftrightarrow$ The system given by convolution.

it's yet another indication of How fundamental and operation convolution is.