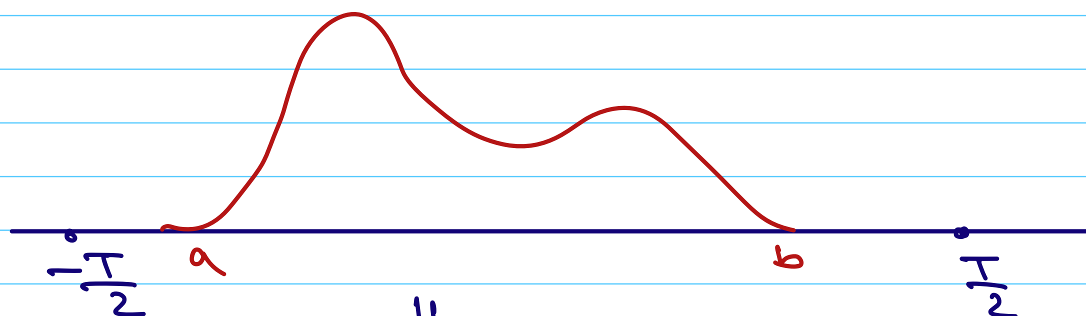
Lec06:

About to get the Fourier Transform as a limiting case of Fourier Series.

Take case



$\Downarrow$

Periodize (pattern repeats

$$C_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-2\pi i \frac{k}{T} t} f(t)\, dt$$

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{2\pi i \frac{k}{T} t}$$

We would like to just let $T \to \infty$

(But this does not quite work)

because $C_k \longrightarrow 0$ as $T \longrightarrow \infty$

* Scale up the co-efficient's by T

$$C_k \cdot T = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-2\pi i \frac{k}{T} t} f(t)\, dt$$

$$\Rightarrow \mathcal{F} f\left(\frac{k}{T}\right) = \int_{\frac{T}{2}}^{\frac{T}{2}} e^{-2\pi i \frac{k}{T} t} f(t)\, dt$$

$$f(t) = \sum_{k=-\infty}^{\infty} \mathcal{F} f\left(\frac{k}{T}\right) e^{2\pi i \frac{k}{T} t} \frac{1}{T}$$

Now let $T \to \infty$, for fixed k $\frac{k}{T} \to 0$

But the idea is k is also going from $-\infty$ to ∞.

the discrete variable $\frac{K}{T}$ is approach a continuous variable s

The Discrete variable $\frac{k}{T}$ $\longrightarrow$ continuous variable s, s ranges from $-\infty$ to ∞.

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\, dt$$

Fourier series

$$f(t) = \sum_{k=-\infty}^{\infty} \mathcal{F}f(s) e^{+2\pi i s t} \frac{1}{T}$$

$$\downarrow \qquad \qquad \downarrow$$

$$\int_{s=-\infty}^{\infty} \qquad \qquad ds$$

$$\Rightarrow \quad f(t) = \int_{-\infty}^{\infty} \mathcal{F}f(s)\, e^{2\pi i s t}\, ds$$

discrete

$$\frac{k}{T} = \cdots -\frac{5}{T}, -\frac{4}{T}, -\frac{3}{T}, -\frac{2}{T}, -\frac{1}{T}, 0, \frac{1}{T}, \frac{2}{T}, \frac{3}{T}, \cdots$$

$$\downarrow \qquad \qquad \downarrow$$

continuous

$$s = \longleftrightarrow s$$

$$f(t) = \sum_{k=-\infty}^{\infty} e^{2\pi i \left(\frac{k}{T}\right) t} \, \mathcal{F}\hat{f}\left(\frac{k}{T}\right) \frac{1}{T}$$

$$f(t) = \int_{s=-\infty}^{\infty} e^{2\pi i s t} \, \mathcal{F}\hat{f}(s) \, ds$$

* victory:

if $f(t)$ is a function define in whole interval $-\infty < t < \infty$, we define its fourier transform by

$$F(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t) \, dt$$

$-\infty < s < \infty$ (real variable)

But The Fourier transform $F(s)$ is complex value because we are integr-ating real function $f(t)$ with

Complex function.

* Fourier Transform analzes $f(t)$ into its constituent part's.

* Fourier inverse say's that we can sythesize $f(t)$ from its constituent parts

$$f(t) = \int_{-\infty}^{\infty} e^{2\pi i s t} \mathcal{F}f(s)\, ds$$

* The Fourier Transform analyzes the signal, the non-periodic signal into its component parts

$\downarrow$

a continuous family of exponential's $e^{2\pi i s t}$ (Not discrete)

$f(t) \longrightarrow$ time domain

$\mathcal{F}f(s) \longrightarrow$ freq domain

Major secret of universe :-

Every signal has a spectrum and the spectrum determines the signal.

* The analysis and synthesis of a function are two way's of seeing the same thing. we can look at the function in the freq domain F.T and we can look in time domain Inverse F.T.

* if we have two representation's of same thing then we have tremendoes power over it. They are equivalent. Knowledge of one is equivalent to knowledge of the other.

$$\mathcal{F} f(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\, dt$$

$$\mathcal{F}^{-1} g(t) = \int_{-\infty}^{\infty} e^{2\pi i s t} g(s)\, ds$$

Fourier inversion say's

$$\mathcal{F}^{-1} \mathcal{F} f(s) = f(s)$$

$$\mathcal{F} \mathcal{F}^{-1} g(t) = g(t)$$

$$\mathcal{F} f(0) = \int_{-\infty}^{\infty} e^{-2\pi i 0 t} f(t)\, dt$$

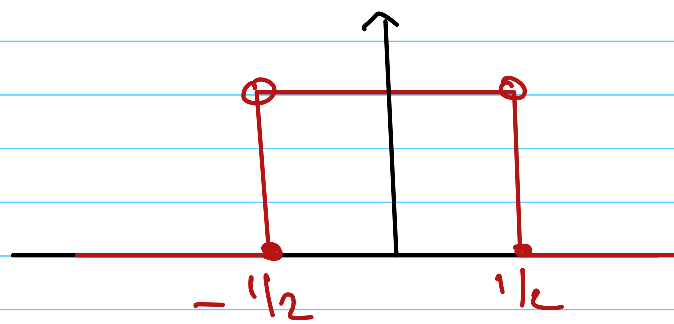
$$= \int_{-\infty}^{\infty} f(t)\, dt \Rightarrow \text{average value}$$

(area under the curve)

$$\mathcal{F}^{-1} g(0) = \int_{-\infty}^{\infty} \mathcal{F} g(s)\, ds$$

Example: rectangle function:

$$\Pi(t) = \begin{cases} 1 & |t| < 1/2 \\ 0 & |t| > 1/2 \end{cases}$$



$$\mathcal{F}\Pi(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t}\, \pi(t)\, dt$$

$$= \int_{-1/2}^{1/2} e^{-2\pi i s t}\, dt$$

$$= \left. \frac{e^{-2\pi i s t}}{-2\pi i s} \right]_{-1/2}^{1/2}$$

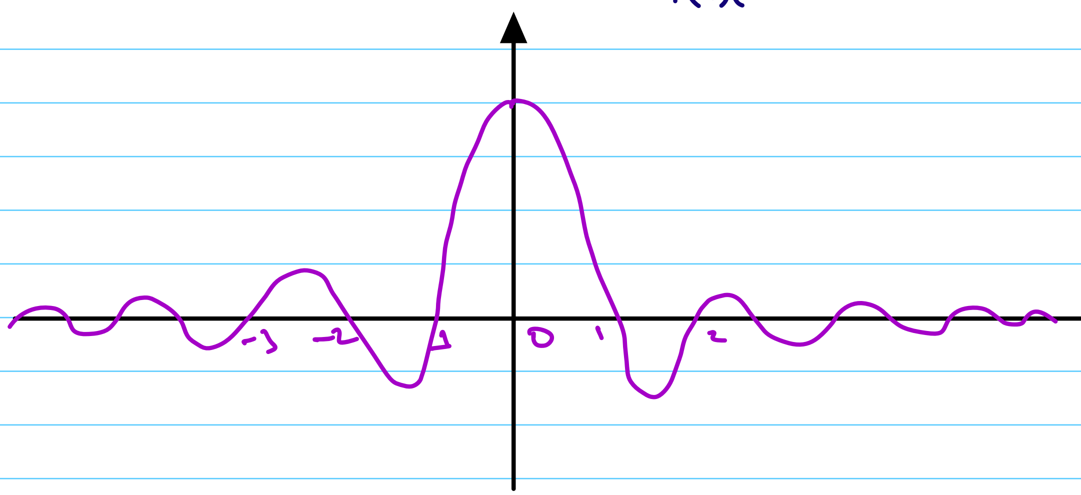
$$= \frac{-1}{2\pi i s}\left[e^{-\pi i s} - e^{\pi i s} \right]$$

$$= \frac{1}{2\pi i s}\left[e^{\pi i s} - e^{-\pi i s} \right]$$

$$= \frac{1}{\pi s}\left[\frac{e^{\pi i s} - e^{-\pi i s}}{2i}\right]$$

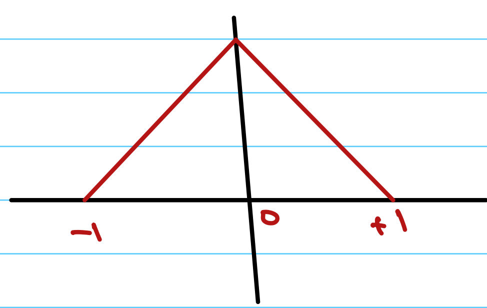
$$\mathcal{F}\Pi(s) = \frac{\sin \pi s}{\pi s} \quad (\text{sinc function})$$

$$\text{sinc}\, x = \frac{\sin \pi x}{\pi x} \quad (\text{sinc function})$$



E2:

$$\Lambda(t) = \begin{cases} 1 - |t| & |t| \le 1 \\ 0 & \text{o.w} \end{cases}$$



$$\mathcal{F}\Lambda(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t}\, \Lambda(t)\, dt$$

$$\hat{F}(s) = \int_{-1}^{0} (1+t)\, e^{-2\pi i s t}\, dt + \int_{0}^{1} (1-t)\, e^{-2\pi i s t}\, dt$$

$$= (1+t) \int e^{-2\pi i s t}\, dt \Big]_{-1}^{0} - \int_{-1}^{0} \int e^{-2\pi i s t}\, dt$$

$$+ (1-t) \int e^{-2\pi i s t}\, dt \Big]_{0}^{1} + \int_{0}^{1} \int e^{-2\pi i s t}\, dt$$

$$= \frac{(1+t)\, e^{-2\pi i s t}}{-2\pi i s} \Big]_{-1}^{0} - \frac{1}{(2\pi i s)^2} e^{-2\pi i s t} \Big]_{-1}^{0}$$

$$+ \frac{(1-t)\, e^{-2\pi i s t}}{-2\pi i s} \Big]_{0}^{1} + \frac{1}{(-2\pi i s)^2} e^{-2\pi i s t} \Big]_{0}^{1}$$

$$= \cancel{\frac{-1}{2\pi i s}} + \frac{1}{4\pi^2 s^2} \left(1 - e^{2\pi i s}\right)$$

$$+ \cancel{\frac{1}{2\pi i s}} + \frac{1}{4\pi^2 s^2} \left(1 - e^{-2\pi i s}\right)$$

$$= \frac{1}{4\pi^2 s^2}\left[2 - \left(e^{2\pi i s} + e^{-2\pi i s}\right)\right]$$

$$= \frac{1}{4\pi^2 s^2}\left[2 - 2\cos 2\pi s\right]$$

$$= \frac{1}{2\pi^2 s^2}\left(1 - \cos 2\pi s\right) = \frac{1}{2\pi^2 s^2}\left(1 - (1 - 2\sin^2 \pi s)\right)$$

$$= \left(\frac{\sin \pi s}{\pi s}\right)^2$$