Lec09 :- Convolution

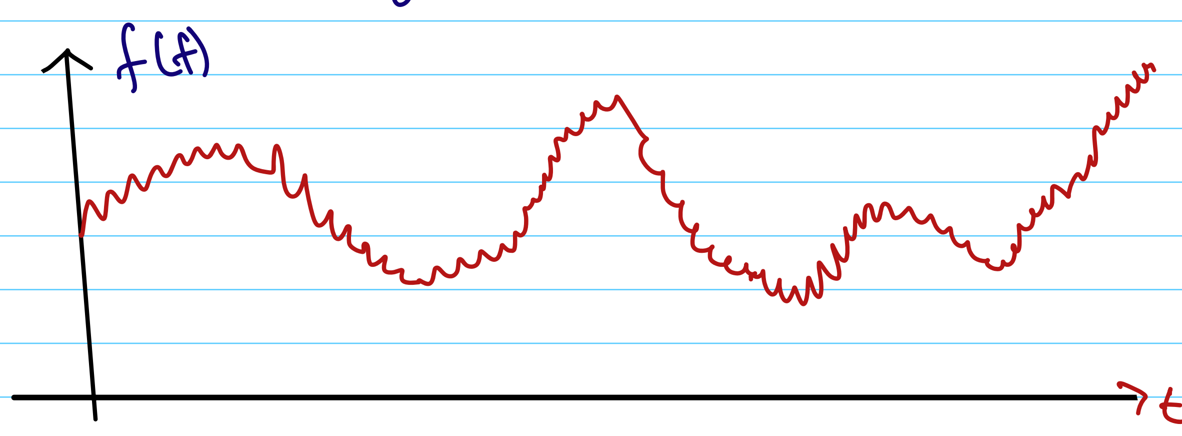
* How do we combine two signal's in such a way that their fourier Transform multiply.
* It led to convolution.

$$\mathcal{F}\, f * g = (\mathcal{F} f)(\mathcal{F} g)$$

$$(g * f)(x) = \int_{-\infty}^{\infty} g(x-y)\, f(y)\, dy$$

$$\mathcal{F}(g * f) = (\mathcal{F} g) \cdot (\mathcal{F} f)$$

Example of covolution in Filtering :-



we see lot of Jaggedness in the signal,

* The input signal f(t) is noisy, Jaggedness, and we want get rid of noise, Jaggedness of signal. The way to remove noise, Jaggedness The way to smoothout the data a little bit is not in the time domain, But to do it in freq domain.

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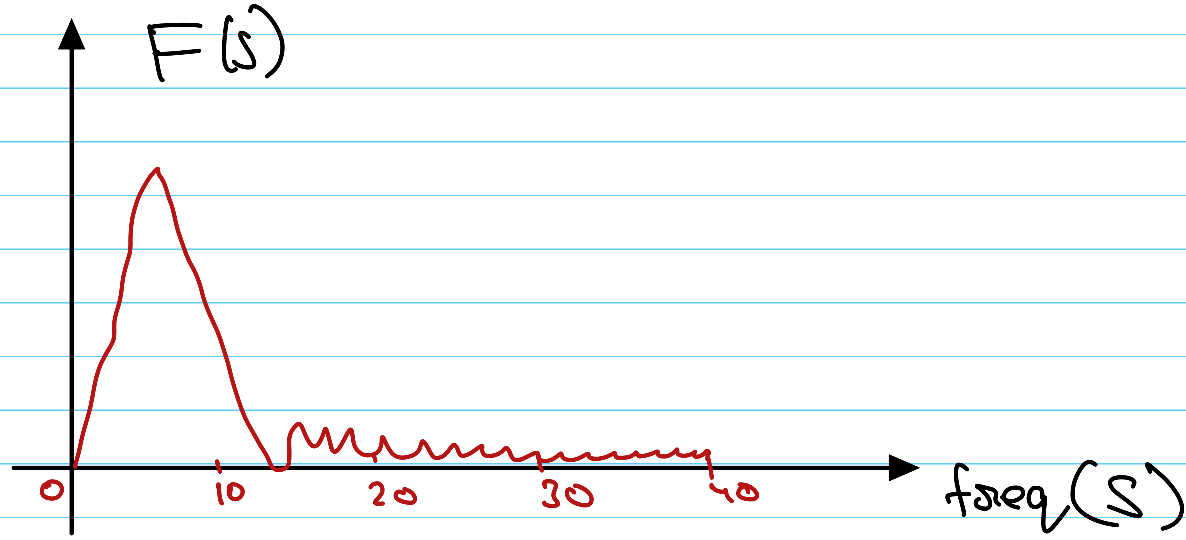
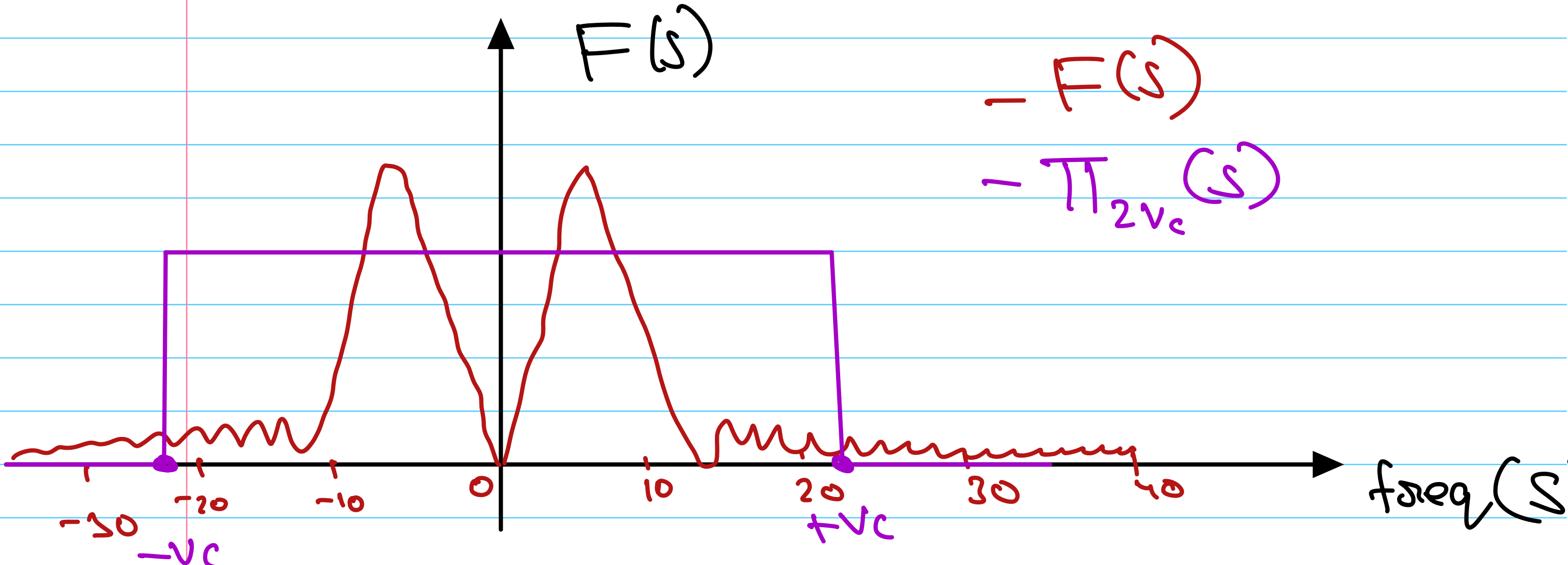


The High freq is what causing the Jaggedness, Just as in the Fourier series, the high Harmonics are causing the signal to oscillate quickly.

what do we want to do to smoot it?

The natural thing to do is to kill off the high freq. How do we kill off High freq? we do it by multiplying by a rectangular Function in freq domain

Kill off High freq by multiplying in freq domain by a scaled - rectangular function.



eliminated all freq below & above $|V_c|$. & keep the freq in b/w.

$\Rightarrow$ Passing the low freq, eliminating the high freq $\Rightarrow$ low pass Filter.

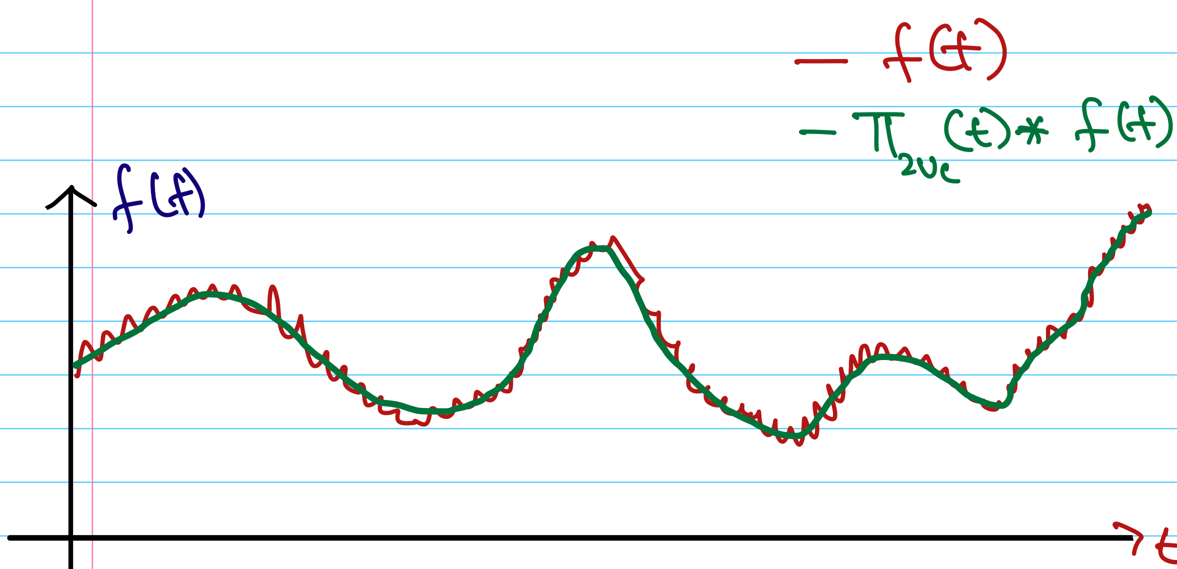
in freq domain

$$\Pi_{2\nu_c}(s) \cdot F(s) \quad \text{freq domain}$$

$$\Big\downarrow \mathcal{F}^{-1}$$

$$\Pi_{2\nu_c}(t) * f(t) = 2\nu_c \operatorname{Sinc}(2\nu_c t) * f(t)$$

in time domain this is convolution.



* This is Probably one of the Filtering

* Filtering is often always synonomous with convolution

* Filter is defined by sort of a fixed function that we are convolving with or in the freq domain a fix function we are multiplying the Fourier Transform's with

* The input's vary but the filter function's stay's the same

* System is that convolve's an input (which can vary) with a fixed function.

* The fixed Signal is called Impulse Response. (we will understand, a little bit more about Delta function's, Linear system, so on)

(in time domain)

$$g = f * h$$

g ↑ output, f ↑ input, h ↑ impulse response (fixed)

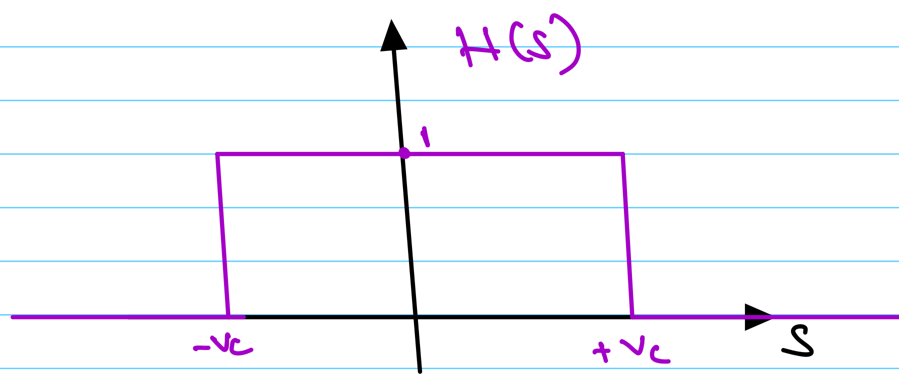
(in freq domain)

$$G(S) = F(S) \cdot H(S)$$

transfer function.

* so to design a filter, then it is often to design the appropriate transfer function $H(s)$

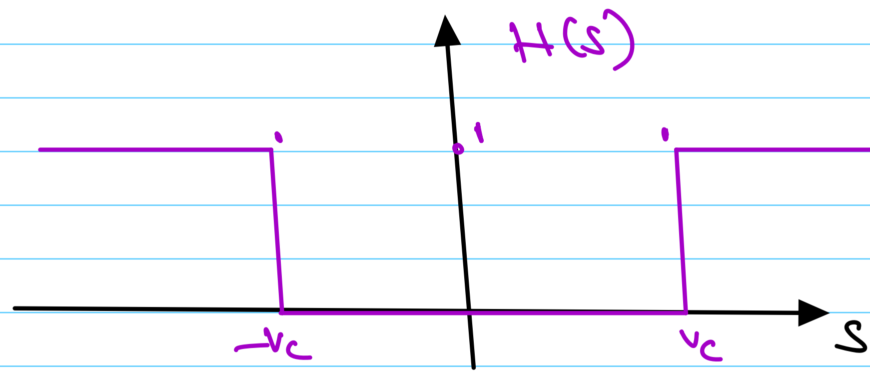
Low pass Filter:



Problem with low pass filter is the sharp cut-off, it cut's off exactly at the freq.

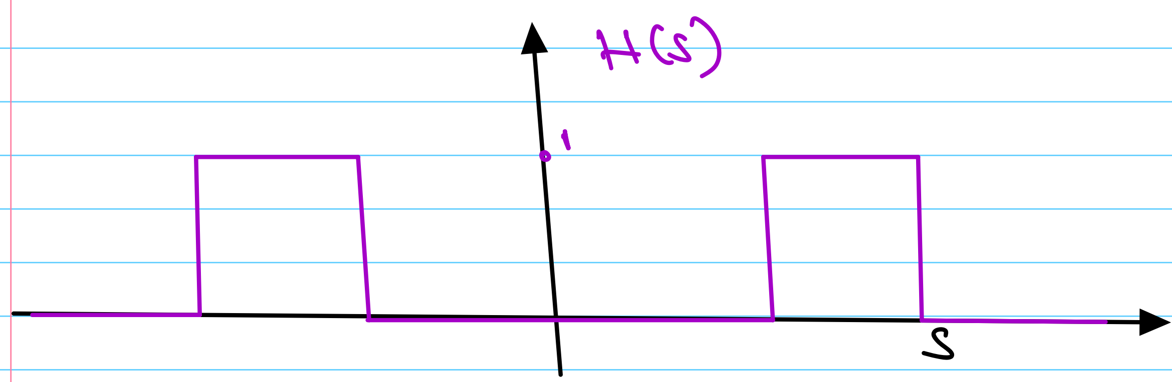
High pass Filter :-

Pass the high freq & cut-off low freq.



Purpose: **Ex:** edge detection

BAND PASS Filtering



The Whole idea of Filtering, the whole idea of computing convolution's in the time domain is to see what happen's to the signal in discre & analog is a big industry.

EASY to understand Filtering (convolution) in freq domain, Not so easy in time.

* Convolution is used in many way's, not subject to single interpretation.
* in many contest, Convolution is interpreted as Smoothing or averaging.

Properties of Convolution in general

in general, $f * g$ has the Best Properties of f and g seperately.

⇒ $f * g$ is usally smoother, than f, and g seperately.

Ex:

$$\underbrace{\Pi(t) * \Pi(t)}_{\text{discontinuous}} = \underbrace{\Lambda(t)}_{\text{continuous}}$$

$$\mathcal{F}(\Pi * \Pi) = \mathcal{F}\Pi \cdot \mathcal{F}\Pi$$

$$= (\mathrm{sinc})^2$$

* f is differentiable, g is not

① f * g is differentiable

② $(f*g)' = f' * g$

Convolution & Diff eq^n

Need the derivative theorem for Fourier transform.

$$(\mathcal{F}f')(s) = 2\pi i s\, \mathcal{F}f(s)$$

$$(\mathcal{F}f^{n})(s) = (2\pi i s)^n\, \mathcal{F}f(s)$$

A F.T turn's differentiation into multiplication. (Fundamental property of F.T)

* Derivation in the case when $f(t) \to 0$ as $t \longrightarrow \pm\infty$

$$(\mathcal{F} f')(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f'(t)\, dt$$

$$= e^{-2\pi i s t} \int f'(t)\, dt \Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -2\pi i s\, e^{-2\pi i s t} \int f'(t)\, dt$$

$$= \underbrace{e^{-2\pi i s t} f(t) \Big]_{-\infty}^{\infty}}_{0} + 2\pi i s \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\, dt$$

$$(\mathcal{F} f')(s) = 2\pi i s\, \mathcal{F} f(s)$$

Fourier transform turn's differentiation into multiplication.

Ex:

$$u'' - u = -f$$

$$\Rightarrow \quad \frac{d^2u(t)}{dt} - u(t) = -f(t)$$

$$\mathcal{F}\left(\frac{d^2u(t)}{dt} - u(t) = -\mathcal{F}(f(t))\right)$$

$$\Rightarrow \quad (2\pi i s)^2 \mathcal{F}u - \mathcal{F}u = -\mathcal{F}f$$

$$\Rightarrow \quad \mathcal{F}u = \frac{-\mathcal{F}f}{(2\pi i s)^2 - 1}$$

$$\Rightarrow \quad \mathcal{F}u = \frac{\mathcal{F}f}{1+4\pi^2 s^2}$$

$$\Rightarrow \quad \mathcal{F}u = \mathcal{F}\left(\frac{1}{2}e^{-|t|}\right)\mathcal{F}f$$

$$u(t) = \frac{1}{2}e^{-|t|} * f(t)$$

$$u(t) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|t-y|} f(y)\, dy$$