Lec 23

Linear system's

- An Appreciation.

* we want to see how the Fourier Transform applies to linear system's.

* The main idea's are about the impulse response and the Transfer Function,

* complex exponential's appearing as eigen function's of LTI system's.

BASIC Definition's

So a linear system for us is a method of associating an output to an input that satisfies the principal of superposition.

* A Mapping from input's to output's that satisfies the superposition principle.

$$v \longrightarrow \boxed{L} \longrightarrow w$$

$$\left.\begin{array}{ll} ① & L v_1 + L v_2 = L(v_1 + v_2) \\ ② & L \alpha v = \alpha L v \end{array}\right\} \text{Linearity}$$

$$L\left(\sum_{i=1}^{n} \alpha_i v_i\right) = \sum_{i=1}^{n} \alpha_i L v_i$$

we can extend this idea to infinite sum's & integral's but generally that requires assumption's on the operator L

* we assume some kind of continuity.

Ex: Linear system

* it is the relationship with direct proportionality.

$$Lv = \alpha v$$

$\Rightarrow$ $L(v_1+v_2) = \alpha Lv_1 + \alpha Lv_2 = \alpha(Lv_1 + Lv_2)$

* all the linear system's essentially can be understood in term's of direct proportionality.

$\Rightarrow$ all linear system's hearken back somehow to the operation of direct proportionality.

$\Rightarrow$ direct proportionaly also known as multiplication

$\Rightarrow$ any linear system that is given by multiplication (or any system that is given by multiplication) is a linear system

$\Rightarrow$ a Little more generally is a multiplication

$\Rightarrow$ if the input signal is a function of t or x, then we can multiply by another function.

$$Lv(t) = \alpha(t)\, v(t) \quad (\text{Linear})$$

Ex:

Switching on and off for a duration a

$$Lv(t) = \Pi_a(t)\, v(t)$$

* Sampling is a Linear Operation.

$$Lv(t) = v(t)\, \text{III}_p(t)$$

$$= \sum_{k=-\infty}^{\infty} v(kp)\, \delta(t-kp)$$

A slight but important generalization is ① direct proportion + adding

i.e matrix multiplication

$A_{n \times m}$

$\Rightarrow$ $w = Av$ who $v \in \mathbb{R}^m$
$w \in \mathbb{R}^n$

① $A(v_1 + v_2) = Av_1 + Av_2$

② $A \alpha v_1 = \alpha A v_1$

Ex: EE 263 $\dot{x} = Ax$

$x(0) = v$ initial conditio

then $x(t) = e^{At} v = e^{At} x(0)$

* Linear system's with special properties derived from special properties of matrix.

① if A is symmetric $A = A^T$

that's a special type of system,

② A is unitary or orthogonal. (A $n \times n$)

$$AA^* = I \qquad QQ^T = I$$

special linear system

we often look for eigen value & eigen vector's of matrix A.

$\Rightarrow$ V is an eigenvector, $v \neq 0$ if

$$Av = \lambda v.$$

Now it may be if we have the whole family of eigen vector's which span the set of all possible input's (eigenvector's are basis for input function)

* if we have eigen vector's

$v_1, v_2, \ldots \quad v_n$ with corresponding

eigen value's $\lambda_1, \lambda_2, \ldots \quad \lambda_n$, that form a

BASIS for all the input's. Then we can analyze the action of A easily. Because let's v be any input, then, we can write

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n$$

then A operating on v, by linearity

$$Av = \alpha_1 Av_1 + \alpha_2 Av_2 + \dots + \alpha_n Av_2$$

$$= \sum_{i=1}^{n} \alpha_i A v_i$$

$$= \sum_{i=1}^{n} \alpha_i \lambda_i v_i$$

$\Rightarrow$ The action of A on a arbitrary input V is really in direct proportionality + Adding.

Question is when do linear system's have a BASIS of eigen vector's?

When can we do this? that's when these special properties come in. orthonormal, symmetric etc..

* The spectral theorem in finite dimension for matrice say's if A is a Hermissan operator or symmetric operator (in real case) then it has basis of eigen vector's.

* when we talked about Fourier series, complex exponential's forming orthonormal basis for the signal's. and the whole idea of Fourier Transform & diagonolizing the F.T and finding eigen vector's & eigen function in that case is exactly in matrix form

* These are simple idea's with what we started with, The idea of superposition. Sum of the input's goes to the sum of the output's. Scaled input's goes to the scaled output's.

All these thing's have some analog in continuous case, in infinite dimentional continuous case.

One more property in finite dimentional (matrix multiplication)

* its not just direct proportionality as an example of linear system's, direct proportionality is the only example of linear system.

* It's Not Just that matrix

multiplication is a good (Natural) example of finite dimentional Linear system. It's the only example.

* Any Linear Operator on a finite dimentional space can be realized as matrix multiplication.

* Any finite dimentional (finite number of D.O.F, finite number of describing any input, described by finite set of vector's) Linear system can be realized as matrix multiplication.

(one of the fundamental results of linear algebra)

Example: All Polynomial's of degree $\le n$

input : $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

a familiar Linear operator is differentiator

$L = \frac{d}{dx} \Rightarrow$ L can be described

as $(n+1) \times (n+1)$ matrix.

$\Rightarrow$ any Linear operator on a finite dimentional space can be described as matrix multiplication.

* and in fact we are going to see that same statement hold's in the infinite dimensional continuous case.

* we will see very similar (analogous) statement for infinite dimentional continuous case.

* The example that generalize's the matrix multiplication is integration against kernal.

+ Input's is a function $v(x)$
+ Kernal is a function $k(x,y)$

$$w(x) = Lv(x) = \underbrace{\int_{-\infty}^{+\infty} k(x,y)\, v(y)\, dy}_{\text{integration against kernal}}$$

* the kernal $k(x,y)$ defines the operations define's the Linear system., because the integration is Linear.

L is Linear system

* we can think of this, infinite dimentional continuous analog of matrix multiplication

It's like to think v is some how an infinite continuous column vector.

$$\int_{-\infty}^{\infty} K(x,y)\, v(y)\, dy$$

column vector

like a matrix
(a doubly infinite continuous matrix)

x : index of row
y : index of column

integration is like sum.

what else is true?

* in finite dimentional case there are special linear system's, that are characterized by special properties of the matrix

* so too in the infinite dimensional continuous case there are spectial properties of system that are characterized by spectial properties of kernal.

→ Special Linear system's are arised by special extra assumption's on the kernal.

① $K(x,y) = K(y,x)$ (symmetric kernal)

one special property

② $K(x,y) = \overline{K(y,x)}$ (Hermission property)

<u>Example:</u>

① Fourier transform

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\, dt$$

$k(s,t) = e^{-2\pi i s t}$ (kernal)

② Convolution.

fix a function $h(t)$

$Lv(t) = (h * v)(t)$

$$Lv(t) = \int_{-\infty}^{\infty} h(t-y)\, v(y)\, dy$$

this is special one, because $h(t-y)$, kernal doesnot depend on seperatly on t, y, it depend's on its difference.

* for convolution the kernal = $h(t-y)$ depend's on $t-y$, But not on t, y seperately. (Time invariance property)

* if we shift t, y by the same amount

$$t \to t-a$$
$$y \to y-a$$

then $t-y \to t-y$ (unchanged)

so the convolution is unchanged

* It's Not Just this is a good idea, it's not just that this is a good example of Linear system's. It is the only example.