Lec08 :

* General Properties of F.T & other Properties.

Three Big item's today

1. Time delay signal (Shifted)
2. Signal Stretch
3. Convolution

1. The Shift theorem

if the signal is delayed (shifted) by amount b what happen't's to Fourier transform?

$$f(t) \longleftrightarrow F(s)$$

$$f(t-b) \longleftrightarrow ?$$ (what happen's)

$$F(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\, dt$$

F.T of $f(t-b)$ is

$$= \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t-b)\, dt$$

$t-b = u$

$dt = du$

$$= \int_{-\infty}^{\infty} e^{-2\pi i s (u+b)} f(u)\, du$$

$$= e^{-2\pi i s b} \int_{-\infty}^{\infty} e^{-2\pi i s u} f(u)\, du$$

F.T of $f(t-b) = e^{-2\pi i s b} F(s)$

$\Rightarrow$

$$f(t) \Longleftrightarrow F(s)$$

$$f(t-b) \Longleftrightarrow e^{-2\pi i s b} F(s)$$

(Shift theorem, Delay theorem)

in general

$$f(t \pm b) \Longleftrightarrow e^{\pm 2\pi i s b} F(s)$$

Interpretation:

$\Rightarrow$ Shift in time correspond's to a phase shift in frequency.

Notice that, as promised, the magnitude of the F.T has not changed under a time shift because the factor out front has magnitude 1

$$\left| e^{\pm 2\pi i s b} F(s) \right| = \left| e^{\pm 2\pi i s b} \right| |F(s)|$$

$$= |F(s)|$$

Fourier transform is a complex number. it has magnitude + phase

$$\Rightarrow \quad F(s) = |F(s)| e^{2\pi i \theta(s)}$$

$$\Rightarrow \quad F(s) e^{-2\pi i b s} = |F(s)| e^{2\pi i \theta(s)} e^{-2\pi i b s}$$

$$= |F(s)| e^{2\pi i (\theta(s) - bs)}$$

The magnitude stay's same, But the Phase change. there is a Phase shift.

②

The stretch (similarity) theorem

$$f(t) \rightleftharpoons F(s)$$

$$f(at) \rightleftharpoons ?$$

How does the Fourier Transform change if we stretch or shrink the variable in the time domain?

$$\int_{-\infty}^{\infty} e^{-2\pi i s t} f(at)\, dt$$

$$at = u$$

$$a\,dt = du$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-2\pi i s \frac{u}{a}} f(u) \frac{du}{a}$$

$$\Rightarrow \frac{1}{|a|} \int_{-\infty}^{\infty} e^{-2\pi i \left(\frac{s}{a}\right) u} f(u)\, du$$

$$\Rightarrow \frac{1}{|a|} F\left(\frac{s}{a}\right)$$

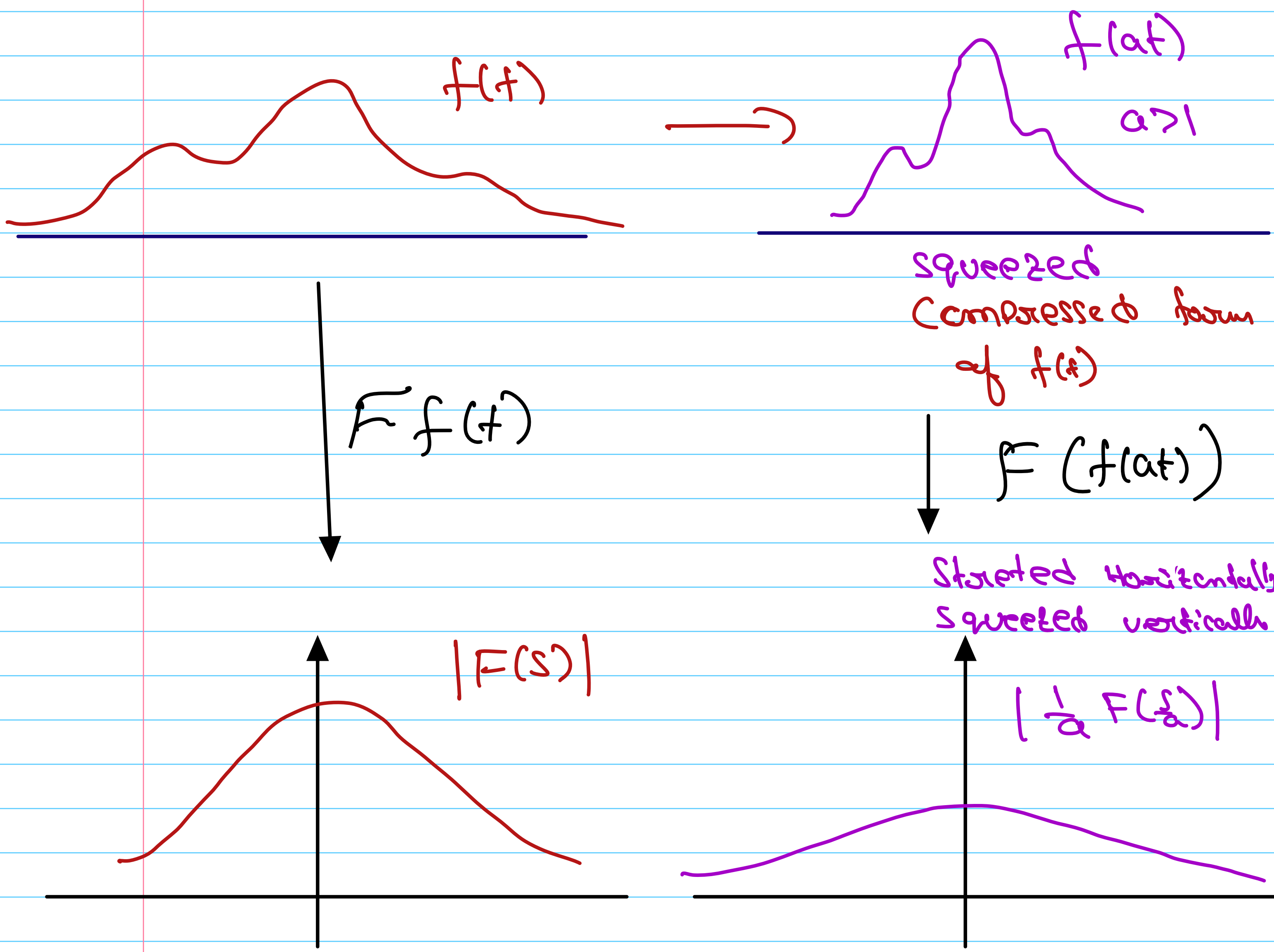
$$\Rightarrow \quad f(t) \rightleftharpoons F(s)$$

$$f(at) \rightleftharpoons \frac{1}{|a|} F\left(\frac{s}{a}\right)$$

<u>Interpretation:</u>

<u>Case 1:</u> a is $+ve$, $a > 1$

then the graph of $f(at)$ is squeezed horizontally compared to $f(t)$

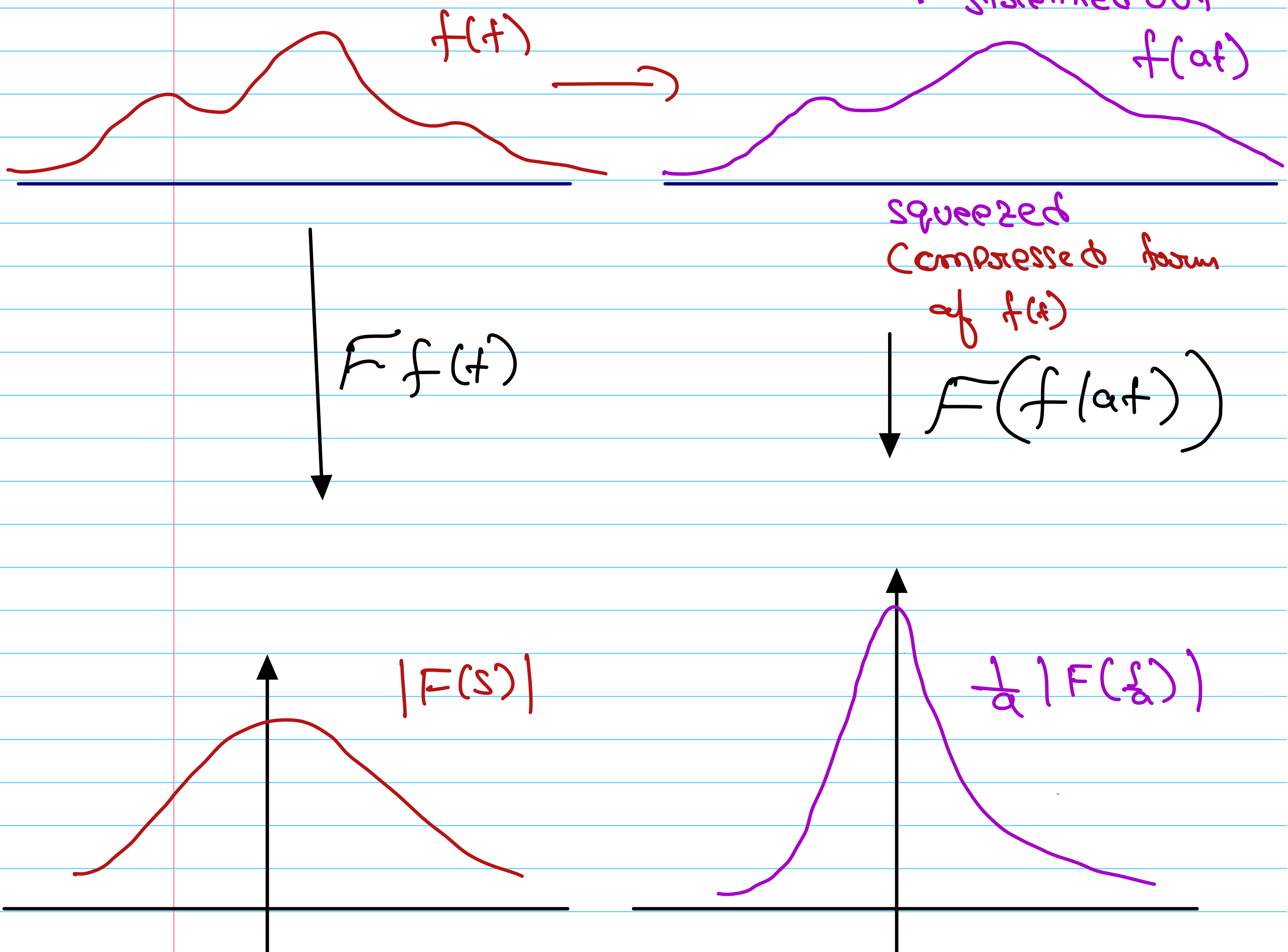


* we can't draw the graph of $F(s)$, Fourier transform, because F.T is complex. all we can draw is magnitude

if $a>1$, $\frac{1}{a}F\left(\frac{s}{a}\right)$ is stretched horizontally, squashed vertically.

CASE 2

$$0<a<1$$



* compressed horizontally
* stretched vertically

* if $a >> 1$ Getting larger & larger than 1. then the signal is getting more, and more compressed, it is getting more localized in time

* The Fourier Transform for $(a >> 1)$ is getting more and more stretched out horizontally, squeezed down in vertical direction.

Reciprocal relationship.

=) we can't have a signal's which is both localized in time and localized in freq.

=) To localize the signal in time, is to squeeze it down in time, but that has a consequence in freq of stretching things out

* if we want to concentrate signal in time, the Fourier transform is going to stretch out & reverse situation also hold's.

* Signal cannot be localized in Both time & frequency domain

F.T of Gaussian $e^{-\pi t^2}$ is itself.

* $f(t) = e^{-\pi t^2} \overset{\mathcal{F}}{\rightleftharpoons} F(s) = e^{-\pi s^2}$

$\Rightarrow$ Gaussian is Perfectly Balanced Somehow in Both time and freq.

* It is Perfectly Balanced
* It's Just concentrated enough in time and Just concentrated enough in freq domain.

What happen's when we combein shift's & stretches ?

Intro to convolution:

* Convolution, is probably most important operation in signal processing.
* most frequently used, in modifying the spectrum of signal.

<u>Signal Processing:</u> How can we use one function to modify another. How can we use one signal to modify another. (BASIC Question of all signal processing)

* Its probably more common to address this question in the freq domain.
* when we talk about modifying a signal → most often we are taking about modifying its spectrum of signal

Ex: for example, Linearity gives you in some sense the simplest operation of signal processing.

Ex: Linearity

$$\mathcal{F}(f+g) = \mathcal{F}f + \mathcal{F}g$$

modify spectrum of $\mathcal{F}f$, by adding spectrum of $\mathcal{F}g$.

* The very natural question is what if we multiply

$$(\mathcal{F}f) \cdot (\mathcal{F}g) \ ?$$

So, we are scaling each individual freq, by function $\mathcal{F}g$.

Question is how does that come out in time domain? so that in freq domain we are scaling spectrum.

$$\mathcal{F} g(s) \cdot \mathcal{F} f(s)$$

$$= \left(\int_{-\infty}^{\infty} e^{-2\pi i s t} g(t)\, dt \right) \left(\int_{-\infty}^{\infty} e^{-2\pi i s x} f(x)\, dx \right)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi i s (t+x)} g(t) f(x)\, dt\, dx$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-2\pi i s (t+x)} g(t)\, dt \right) f(x)\, dx$$

$$u = t + x \Rightarrow t = u - x$$

$$du = dt$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-2\pi i s u} g(u-x)\, du \right) f(x)\, dx$$

swap order of integration

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(u-x) f(x)\, dx \right) e^{-2\pi i s u}\, du$$

Define

$$h(u) = \int_{-\infty}^{\infty} g(u-x) f(x)\, dx$$

$$\Rightarrow \int_{-\infty}^{\infty} h(u)\, e^{-2\pi i s u}\, du = \mathcal{F}u(s)$$

we discovered The F.T of $\mathcal{F}g(s) \cdot \mathcal{F}f(s)$

$$= \mathcal{F}h(s)$$

$$(g * f)(t) = \int_{-\infty}^{\infty} g(t-x) f(x)\, dx$$

$$\mathcal{F}(g * f)(s) = \mathcal{F}g(s) \cdot \mathcal{F}f(s)$$

Convolution in the time domain Corres-pond's to multiplication in the freq domain.