Lec 12: Generalized function's

The Best class of function's for Fourier Transform is rapidly decreasing function's, and they are characterized by two properties

① They are infinitely differentiable

② and any derivative decay's faster than any power of x, independefy

$$|x^n| \left| \frac{d^m f(x)}{dx^m} \right| \longrightarrow 0$$

n, m are independently.

Why the Best for F.T?

① if $\varphi(t) \in S \implies \mathcal{F}\varphi(s) \in S$

② Fourier inversion work's

$\varphi(t) \in S \Rightarrow \mathcal{F}^{-1}\mathcal{F}\varphi = \varphi = \mathcal{F}\mathcal{F}^{-1}\varphi$

But if S is so good, ...

$$\Pi \notin S$$

* Π does not in S, because its not continuous.

  $\Pi(t)$ is not even continuous.

* $\Lambda(t)$ is not in S, its continuous, But not differentiable.

* $\sin t$, $\cos t$, I, and many are not in S.

So how do we get to defining F.T for these function's?

What is gained by defining S class?

How can we be sured we haven't lost anything, and have gained even greater generality.

To Answer this, we have to Pick up another line of development.

$\Rightarrow$ That is the idea of generalized function's also refffered to (known as) "distribution"

This use of the word "distribution" has nothing to do with the "Probability distribution".

typified by Dirac Delta function ($\delta(t)$) also called Heavyside Delta function.

Generalized function's (Distribution's)

Def: A generalized function, also known as distribution, is a mathematical object that extend's the concept of a function. It allow's operation's (Like differention or convolution) to be defined even when the classical definition fail's.

Many important "function's" in engineering and Physics - like the Dirac delta function are not well defined in the classical sense

For Ex:

* $$\delta(t) = \begin{cases} \infty & \text{at } t = 0 \\ 0 & \text{at } t \neq 0 \end{cases}$$

* $$\int_{-\infty}^{+\infty} \delta(t)\, dt = 1$$

This is not a function, But we can make sense of it under an integral.

Core Idea:

A generalized function T is defined by how it acts on test function $\phi(t)$

$$T(\phi) = \int_{-\infty}^{+\infty} T(t) \cdot \phi(t)\, dt$$

This is called the action of the distribution on the test function.

A generalized function is not a function in the usual sense, but a rule that tell's you how to act on other function, especially under an integral.

It allow's us to describe idealized signal's like spikes, jump's, or impulses that can't be written as ordinary function's.

Delta function one of the function we learned to use operationally.

Dirac delta $\delta(x)$

ⓐ $$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

ⓑ $$\int_{-\infty}^{+\infty} \delta(x)\, dx = 1$$

ⓒ $$\int_{-\infty}^{+\infty} \delta(x)\, \varphi(x) = \varphi(0)$$

Quite remarkable, every one of these statements is a complete Bullshit, it doesn't make sense, But there is something there, People use it.

So when we say Operationally, we don't need to rigorously define $\delta(t)$, as a real function (because its not - its a distribution)

* instead, we work with it via the rules above.

* operationally, mean's using Dirac delta according to its behavior (shifting, integration etc..) without worrying about its technical definition as a distribution.

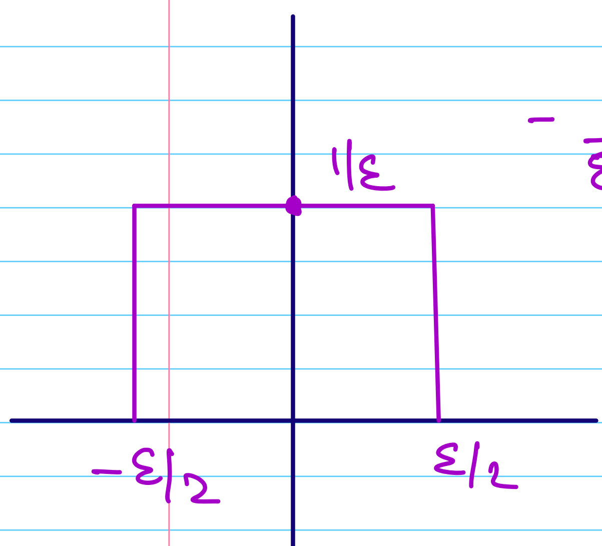
δ is supposed to represent a function which is concentrated at a Point. There are various way of approaching this, But it was always via Limiting Process.

Limiting Process mean's, what we do is consider typically a family's of legitimate function's that are getting narrow &

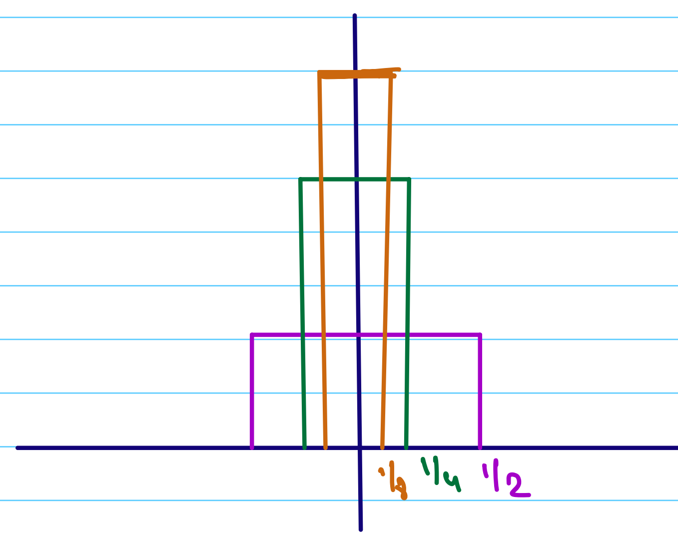
narrow and still satisfy these BASIC properties.

Ex: One parameter rectangular family of functions

$$\frac{1}{\varepsilon}\Pi_{\varepsilon}(x) = \frac{1}{\varepsilon}\Pi\left(\frac{x}{\varepsilon}\right) = \begin{cases} \frac{1}{\varepsilon} & |x| < \frac{\varepsilon}{2} \\ 0 & |x| \geq \frac{\varepsilon}{2} \end{cases}$$



$- \frac{1}{\varepsilon}\Pi_{\varepsilon}(x)$



δ is getting smaller and smaller and it becomes a concentrated function

$$\int_{-\infty}^{+\infty} \frac{1}{\varepsilon}\Pi_{\varepsilon}(x)\,dx = \frac{1}{\varepsilon}\int_{-\varepsilon/2}^{\varepsilon/2} 1\,dx = 1$$

$$* \quad \int_{-\infty}^{\infty} \frac{1}{\varepsilon} \Pi_{\varepsilon}(x)\, \varphi(x)\, dx$$

Assume $\varphi(x)$ is a smooth function,

and expand $\varphi(x)$ using taylor series.

$$\varphi(x) = \varphi(0) + \varphi'(0)\, x + \varphi''(0) \frac{x^2}{2} + \text{H.O.T}$$

assuming $\varphi(x)$ is smooth around '$x=0$'

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{1}{\varepsilon} \Pi_{\varepsilon}(x)\, \varphi(x)\, dx = \int_{-\infty}^{+\infty} \frac{1}{\varepsilon} \Pi_{\varepsilon}(x) \left[\varphi(0) + \varphi'(0)\, x + \varphi''(0) \frac{x^2}{2} + \cdots \right] dx$$

$$= \frac{1}{\varepsilon} \int_{-\infty}^{+\infty} \left(\varphi(0) + \varphi'(0)\, x + \varphi''(0) \frac{x^2}{2} + \cdots \right) dx$$

$$= \frac{1}{\varepsilon}\left[\int_{-\varepsilon}^{\varepsilon}\varphi(0)\,dx + \int_{-\varepsilon}^{+\varepsilon}\varphi'(0)\,x\,dx + \int_{-\varepsilon}^{+\varepsilon}\varphi''(0)\frac{x^2}{2}\,dx + \cdots\right]$$

$$= \frac{1}{\varepsilon}\left[\varphi(0)\cdot\varepsilon + \varphi'(0)\cdot 0 + \varphi''(0)\frac{\varepsilon^3}{6} + \cdots\right]$$

$$= \varphi(0) + O(\varepsilon) \quad \text{(higher order terms)}$$

as $\varepsilon \to 0 \implies O(\varepsilon) = 0$

$$\implies \lim_{\varepsilon \to 0}\int_{-\infty}^{+\infty}\frac{1}{\varepsilon}\Pi_\varepsilon(x)\,\varphi(x)\,dx = \varphi(0)$$

Concentration via a Limiting Process.

$\lim_{\varepsilon \to 0}\frac{1}{\varepsilon}\Pi_\varepsilon(x) \implies$ To consider this Limit makes no sense.

But to consider operationally

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} \frac{1}{\varepsilon} \Pi_\varepsilon(x)\, \varphi(x)\, dx$$ does makes sense.

* By experience, the way's that delta appered in application's werent so much this way $\left(\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Pi_\varepsilon(x) \right.$, naked$\left.\right)$

* The limit of th seq^n, really appeared operationally, when it was Paired with a legitimate function. The idea was by limiting Process we were concentrating thing's we are pulling out the value at origin. That's really operationally How it appear, and that was a extreamly important thing to realize.

The above statement's (rule's of $\delta(t)$ function) somehow individually doesn't make sense, but in practice the way it was used, we replace $\delta(x)$ in

$$\int_{-\infty}^{+\infty} \delta(x)\, \varphi(x)\, dx$$

with seq^n of function's which are concentrating, and we consider them as paired with a $\varphi(x)$ via integration

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} \frac{1}{\varepsilon} \Pi_{\varepsilon}(x)\, \varphi(x)\, dx$$

This was a Big Conceptual step, it follow's the mathematical modus operandi, of turning the solution to a problem, into a definition.

so, instead of concentrating on somehow the limiting behaviour and so on, and so on, the idea is to concentrate on the operational effect or outcome.

Fundamental Change of Point of view:-

To capture this idea, and to include more, we need a fundamental change of view.

It would become operation or outcome rather than process.

The focus is on outcome rather than process

what we mean by this is in the case of δ, The outcome was it concentrated in pairing, approximated seq^n of functions,

gave us the value of the function $\varphi(0)$,
The process was taking the limit.

we want to concentrate on the outcome actually getting $\varphi(0)$, rather than the process. Here is how we set that up.

Definition of Generalized functions
(Distributions)

① Start with class of "test functions".

when F.T comes back into picture, this is going to be the class of "rapidly decreasing function". But for other problems we might consider different class of functions. But generally speaking, these are the best functions for the properties we are worried about.

$\varphi(x)$, Best function's for the area given for F.T These are S class

(2) Associated with test function's, is a class called Generalized distribution function's. A distribution T is a linear operator operating on test function that produces a Number

distribution T is a linear functional on Test function

$\Rightarrow$ for a test function $\varphi(x)$, $T(\varphi)$ is a number. T is linear

$$T(\alpha\,\varphi_1(x) + \beta\,\varphi_2(x))$$

$$= \alpha\,T(\varphi_1) + \beta\,T(\varphi_2)$$

Ex: $$\int_{-\infty}^{+\infty} \delta(x)\left[a f(x) + b g(x)\right] dx$$

$$= a \int_{-\infty}^{+\infty} \delta(x) f(x)\, dx + b \int_{-\infty}^{+\infty} \delta(x) g(x)\, dx$$

$$= a f(0) + b g(0)$$

$$= a T(f) + T(g)$$

③ Continuity Property:-

These linear operator's are continuous

if $\varphi_n(x) \xrightarrow[n\to\infty]{} \varphi(x)$

then $\Rightarrow T(\varphi_n(x)) \xrightarrow[n\to\infty]{} T(\varphi)$

we often say that the distribution is paired with Test function

often write (Notation for pairing)

$$\langle T, \varphi \rangle \quad , T(\varphi)$$

* This is not an inner product, it supposed to indicate that T is somehow operating on φ to produce a number and the operation is linear.

recover δ in the content of this definition.

what is δ doing operationally, Delta operationally is what is the outcome of applying Delta, it is to pull out value of the function at 0.

Operationally, the effect of delta is to pull out the value at the origin (evaluate the function at the origin)

Turn this into definition.

define $\delta(x)$ by δ paired with φ

$$\langle \delta, \varphi \rangle = \varphi(0) \quad \text{(Definition)}$$

$\varphi(x)$: Class of Test Function's

what δ does to these Test Function

$$\langle \delta, \varphi \rangle = \varphi(0)$$

<u>Linear:</u>

* $\langle \delta, \varphi_1 + \varphi_2 \rangle = \langle \delta, \varphi_1 \rangle + \langle \delta, \varphi_2 \rangle$
  $= \varphi_1(0) + \varphi_2(0)$

* $\langle \delta, \alpha\varphi_1 + \beta\varphi_2 \rangle = \alpha \langle \delta, \varphi_1 \rangle + \beta \langle \delta, \varphi_2 \rangle$

* if $\varphi_n(x) \xrightarrow[n\to\infty]{} \varphi(x)$

then $\Rightarrow$ $\langle \delta, \varphi_n \rangle \xrightarrow[n\to\infty]{} \langle \delta, \varphi \rangle$

$\Rightarrow$ $\varphi_n(0) \xrightarrow[n\to\infty]{} \varphi(0)$

seq^n of numbers $\varphi_n(0)$ converging to $\varphi(0)$

<u>Another Example:</u>

Shifted δ function.

$$\int_{-\infty}^{+\infty} \delta(x-y) f(y)\, dy = f(x)$$

we want to define a shifted delta function operationally. we want to define in a way, we give you $\varphi(x)$, and we need to get what the value of Test function $\varphi(x)$ at number $a \Rightarrow \varphi(a)$

__Define__: we are defining $\delta_a(x)$ as distribution operating on test function $\varphi(x)$

$$\langle \delta_a, \varphi \rangle = \varphi(a)$$

$$\Rightarrow \int_{-\infty}^{+\infty} \delta(x-a)\,\varphi(x)\,dx = \delta(a)$$

we have defined δ, But have we lost anything?

we gained δ, But have we lost anything?

$\Rightarrow$ what does this mean is, ultimately The Test function $\varphi(x)$ are very restrictive, (rapidly decreasing functions) They don't include, constant function, $\Lambda(t)$, $\Pi(t)$, so on,...

How are those functions going to get BACK into the scene.

How are Ex: $\Pi(t)$, $\Lambda(t)$, sin, cos, 1

are going to come BACK in?

+ Can we consider those in the context of generalized function's? and more to the point, when we get to it, can we consider those in the context of taking F.T.

F.T $\mathcal{S}$ gonna make sense,

$\mathcal{F}\Lambda(t)$, $\mathcal{F}\Pi(t)$, $\mathcal{F}\sin$, $\mathcal{F}\cos$,

$\mathcal{F}1$ gonna makes sense, How can we do that in this contest.

How generalized function's include in a natural way the sort of ordinary function's.

How to consider ordinary functions in this context.

Ex: 1 as a distribution.

$$\langle 1, \varphi \rangle = 1(\varphi) = \int_{-\infty}^{+\infty} 1\, \varphi(x)\, dx$$

Likewise for ex $\Pi(f)$.

$$\langle \Pi, \varphi \rangle = \int_{-\infty}^{+\infty} \Pi(x)\, \varphi(x)\, dx$$

Ex: $\sin 2\pi x$ as distribution

$$\langle \sin 2\pi x, \varphi(x) \rangle = \int_{-\infty}^{+\infty} \sin 2\pi x\, \varphi(x)\, dx$$

if $f(x)$ is any function (many of them) we can consider $f(x)$ as a generalized function by defining

$$\langle f, \varphi \rangle = \int_{-\infty}^{+\infty} f(x)\, \varphi(x)\, dx$$