lec14 :- 11:32 AM

* It's possible define the derivative of distribution.

* distribution's turn's out infinitely differentiable.

Derivate of a distribution:-

T is a distribution. How to define T' ?

Have to define what the pairing is $\langle T', \varphi \rangle$, φ is a test function.

If T' is given by a function & pairing is integration

define: T' : $\langle T', \varphi \rangle = \int_{-\infty}^{+\infty} T'(x)\, \varphi(x)\, dx$

$$\Rightarrow \int_{-\infty}^{+\infty} T'(x)\,\varphi(x)\,dx$$ use integration by part's.

$$\int u\,dv = uv - \int v\,du$$

$$\Rightarrow \int_{-\infty}^{+\infty} T'(x)\,\varphi(x)\,dx = T(x)\,\varphi(x)\Big]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} T(x)\,\varphi'(x)\,dx$$

use Test function $\varphi(x)$, such that

$\varphi(x)$ at $\pm\infty = 0$

$$\Rightarrow \int_{-\infty}^{+\infty} T'(x)\,\varphi(x)\,dx = -\int_{-\infty}^{+\infty} T(x)\,\varphi'(x)\,dx$$

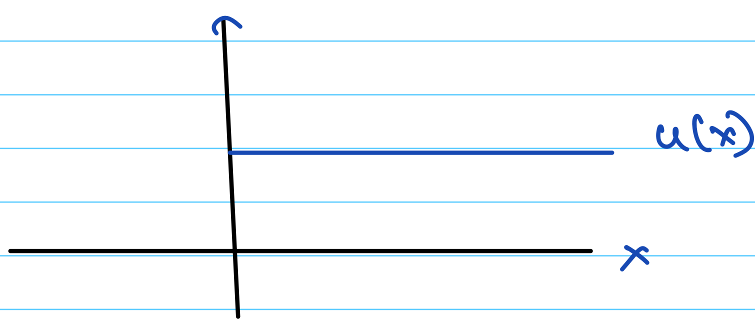
$$= -\langle T, \varphi' \rangle$$

$$\Rightarrow \quad \boxed{\langle T', \varphi \rangle = -\langle T, \varphi' \rangle}$$ if $\varphi(x)$ at $\pm\infty = 0$

Turn this into a definition.

Define T' by $\langle T', \varphi \rangle = -\langle T, \varphi' \rangle$

__Example:__ $T = u(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$



u(x) defines a distribution because

$$\langle u, \varphi \rangle = \int_{-\infty}^{+\infty} \varphi(x)\,dx$$ (integration exists)

* so u' exists as a distribution.

we know $u' = \delta$

$$\Rightarrow \quad \langle u', \varphi \rangle = -\langle u, \varphi' \rangle$$

$$= -\int_{-\infty}^{+\infty} u(x)\, \varphi'(x)\, dx$$

$$= -\int_{0}^{\infty} \varphi'(x)\, dx$$

$$= -\left\{ \Big[\varphi(x) \Big]_0^\infty \right\}$$

$$= -\left\{ \varphi(\infty)^{\nearrow 0} - \varphi(0) \right\}$$

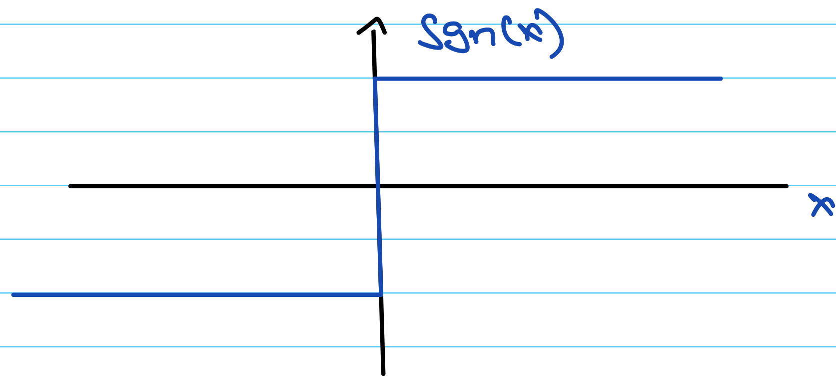
$$= \varphi(0)$$

$$= \langle \delta, \varphi \rangle$$

$$\Rightarrow \quad \langle u', \varphi \rangle = \langle \delta, \varphi \rangle$$

$$\Rightarrow \quad \boxed{u'(x) = \delta(x)}$$

Example: $\operatorname{Sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$



we know $\frac{d}{dx} \operatorname{Sgn}(x) = 2\,\delta(x)$

Proof:

Define $\operatorname{Sgn}(x)$ as a distribution.

$$\langle \operatorname{Sgn}', \varphi(x) \rangle = -\langle \operatorname{Sgn}, \varphi' \rangle$$

$$= - \int_{-\infty}^{+\infty} \operatorname{sgn}(x)\, \varphi'(x)\, dx$$

$$= \int_{-\infty}^{0} \varphi'(x)\, dx - \int_{0}^{\infty} \varphi'(x)\, dx$$

$$= \varphi(x)\Big]_{-\infty}^{0} - \varphi(x)\Big]_{0}^{\infty}$$

$$= \varphi(0) + \varphi(0) = 2\varphi(0)$$

$$= \langle 2\delta, \varphi \rangle$$

Therefore

$$\langle \operatorname{sgn}', \varphi \rangle = \langle 2\delta, \varphi \rangle$$

$$\Rightarrow \boxed{\frac{d}{dx} \operatorname{sgn}(x) = 2\delta(x)}$$

Application to F.T :-

find F.T of $Sgn(x)$, $U(x)$.

* we need derivative theorem.

$$\mathcal{F}(T') = 2\pi i s \, \mathcal{F}T$$

Turn's differentiation into multiplication.

$$(\mathcal{F}T)' = \mathcal{F}(-2\pi i t T)$$

Use this To find F.T of $Sgn(x)$.

we derived $Sgn'(x) = 2\,\delta(x)$

$$\Rightarrow \quad \mathcal{F}(Sgn'(x)) = \mathcal{F}(2\,\delta(x))$$

$$\Rightarrow \mathcal{F}(\operatorname{sgn}'(x)) = 2\pi i s\, \mathcal{F}(\operatorname{sgn}(x))$$

$$\Rightarrow 2\pi i s\, \mathcal{F}(\operatorname{sgn}(x)) = \mathcal{F}(2\delta(x))$$

$$\Rightarrow 2\pi i s\, \mathcal{F}(\operatorname{sgn}(x)) = 2$$

$$\Rightarrow \boxed{\mathcal{F}(\operatorname{sgn}(x)) = \frac{1}{\pi i s}}$$

$$\mathcal{F}u(x) = ?$$

Sol^n

$$u(x) = \frac{1}{2}(1 + \operatorname{sgn}(x))$$

$$\mathcal{F}u(s) = \mathcal{F}\frac{1}{2} + \mathcal{F}\frac{\operatorname{sgn}}{2}$$

$$= \frac{\delta(s)}{2} + \frac{1}{2}\left(\frac{1}{\pi i s}\right)$$

$$\Rightarrow \mathcal{F} u(s) = \frac{1}{2}\left(\delta(s) + \frac{1}{\pi i s}\right)$$

Multiplication and Convolution:

multiplication of function's does not carry over to multiplication of distribution's.

* if S, T are given distribution, in general $S.T$ is not defined.

* Convolution is also little complicated for distribution's.

What is defined in most cases is

$f.T$ where f is a function.

we have to define $\langle f.T, \varphi \rangle$

$$\langle f\cdot T, \varphi \rangle = \int_{-\infty}^{+\infty} f(x)\, T(x)\, \varphi(x)\, dx$$

$$= \int_{-\infty}^{+\infty} T(x)\, \varphi(x) f(x)\, dx$$

$$= \langle T, \varphi \cdot f \rangle$$

$$\Rightarrow \boxed{\langle f\cdot T, \varphi \rangle = \langle T, \varphi \cdot f \rangle}$$

so in general we define $f\cdot T$ by

$$\langle f\cdot T, \varphi \rangle = \langle T, f\cdot \varphi \rangle$$

This again work's only when $f\cdot\varphi$ is a Test function.

* we use this when we wrote $\mathcal{F}(T') = \underbrace{2\pi i s}_{\text{function}}\, \mathcal{F}T$ $\Downarrow$ distribution.

what happen's in the case of δ function.

Special case:

define $f\cdot\delta$: $\langle f\cdot\delta, \varphi\rangle = \langle \delta, f\varphi\rangle$

$$= f\varphi(0)$$

$\Rightarrow$ $\langle f\cdot\delta, \varphi\rangle = f\cdot\varphi\ (0)$

$$= f(0)\cdot\varphi(0)$$

$$= \langle f(0)\,\delta\ , \varphi\rangle$$

$\Rightarrow$ $$\boxed{f(x)\cdot\delta(x) = f(0)\cdot\delta(x)}$$

litel Bit more generally

$$\langle f\cdot\delta_a, \varphi\rangle = \langle \delta_a, f\cdot\varphi\rangle$$

$$= f(a)\,\varphi(a) = \langle f(a)\,\delta_a, \varphi\rangle$$

$$\Rightarrow \quad \boxed{f(x)\,\delta_a(x) = f(a)\,\delta_a(x)}$$

This is called the sampling property of delta function.

* for us to sample mean's multiply by delta function.

Convolution:-

S, T are distribution, how to define their convolution. It's not always's defined, there are restriction's.

* Need extra restriction's of S, T

many cases when all is well,

$f * T$ often makes sense

when f is a function. and Convolution theorem Hold's

$$\mathcal{F}(f * T) = (\mathcal{F}f)(\mathcal{F}T)$$

Special case: $T = \delta$

$$\boxed{f * \delta = f}$$

Shifting property.

more generally

$$(f * \delta_a)(x) = f(x-a)$$

Ex: Can convolve δ with itself.

$$\delta_a * \delta_b = \delta_{a+b}$$

Note:
$$(f * \delta_a) * \delta_b$$
$$= f(x-a) * \delta_b$$
$$= f(x-a-b)$$
$$= f * \delta_{a+b}$$

Scaling property of delta function:-

what is $\delta(ax)$?

Defined by defining the scaling operation on distribution's.

$$\langle \delta(ax), \varphi(x) \rangle$$

$$= \int_{-\infty}^{+\infty} \delta(ax)\varphi(x)\, dx$$

$u = ax \Rightarrow du = a\,dx$
$a > 0$

$$= \int_{-\infty}^{+\infty} \frac{1}{a} \delta(u)\, \varphi\left(\frac{u}{a}\right) du$$

$$= \frac{1}{a} \int_{-\infty}^{+\infty} \delta(u)\, \varphi\left(\frac{u}{a}\right) du$$

$$= \frac{1}{a} \left\langle \delta, \varphi\left(\frac{u}{a}\right) \right\rangle = \frac{1}{a} \varphi(0)$$

$$= \left\langle \frac{1}{a}\delta , \psi \right\rangle$$

$$\Rightarrow \quad \langle \delta(ax), \varphi(x) \rangle$$

$$= \left\langle \frac{1}{a}\delta(x), \varphi(x) \right\rangle$$

$$\Rightarrow \quad \boxed{\delta(ax) = \frac{1}{a}\delta(x) \quad , a>0}$$

$$\boxed{\delta(ax) = -\frac{1}{a}\delta(x) \qquad a<0}$$

$$\Rightarrow \quad \boxed{\delta(ax) = \frac{1}{|a|}\delta(x)}$$