Lec 25 :-

Connection b/w LTI & Fourier Transform.

* General structure of linear system.

$\Rightarrow$ L is a Linear System

$\Rightarrow$ then we introduced impulse response

$$h(x,y) = L\,\delta(x-y)$$

$\Rightarrow$ The output of the system can be given as integrating impulse response against the input.

$$w(x) = Lv(x)$$

$$= \int_{-\infty}^{+\infty} h(x,y)\, v(y)\, dy$$

we say that L is time-invariant or shift invariant if

$$W(x) = L\, V(x)$$

then $W(x-y) = L\, V(x-y)$

The Linear system is Time-invariant

$\Longleftrightarrow$ The system is given by Convolution.

$\Rightarrow$ Then the Impulse response for LTI system's $h(x) = L\, \delta(x)$

By Time-invariance $h(x-y) = L\, \delta(x-y)$

* system is given by

$$W(x) = \int_{-\infty}^{+\infty} h(x-y)\, V(y)\, dy$$

$$= h * V\ (x)$$

⟹ A satisfactory state of Affaire's as far as the structure of Linear system's goes.

(1) Any Linear System is given by integration against Impulse response

(2) if the system is time invariant ⟺ The integration reduces to convolution.

(3) Same Consideration's hold for discrete System's. Any discrete system remember's multiplication by matrix

if $W = LV$ given by multiplication by matrix $W = AV$

L is LTI $\Longleftrightarrow$ $W = h * V$

* if we write a system as a matrix multiplicati $W = AV$ then A is a special form of time-invariance System's.

Ex:

$$h = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

if $w = Av = h * v$, what is A?

How do we find matrix A? we have to find the image of Basis vector's.

$\Rightarrow$ The coloum's of A are image of Basis vector's

$A\delta_0$, $\delta_0 = (1, 0, 0, 0)$

$A\delta_1$, $\delta_1 = (0, 1, 0, 0)$

$A\delta_2$, $\delta_2 = (0, 0, 1, 0)$

$A\delta_3$, $\delta_3 = (0, 0, 0, 1)$

How do we compute all these things?

we do so by convolution.

* By definion the system is given to as convolution to a vector h

$$A_1 = A\,\delta_0 = h * \delta_0$$

$$A_1[m] = \sum_{n=0}^{3} h[m-n] \cdot \delta_0[n]$$

$$\Rightarrow A_1[0] = \sum_{n=0}^{3} h[0-n]\,\delta_0[n]$$

$$= h[0] = 1$$

Simillar $A_1[1] = h[1] = 2$ $A_1[2] = 3$
$A_1[3] = 4$

Simillary

$$A\,\delta_1 = h * \delta_1$$

$$= (h * \delta_1)[m]$$

$$= h[m-1] = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

(circulant matrix)

Therefore $A = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}$

The system is given by convolution, But it also given by matrix multiplicat

Bring in the F.T

* we have convolution only for LTI system

$$W = h * v$$

$$\mathcal{F} W = \mathcal{F}\, h * v$$

$$\Rightarrow \quad W(s) = H(s)\, V(s)$$

$H(s)$ = Transfer function

$h(t)$ = impulse response

$H(s)$ = Transfer function
System function.

* There is a beautiful structure involved in Linear system's. that the most Basic example of Linear System is the relation with direct Proportionality.

* for LTI system's in freq domain it's exactly described by multiplication
It's exactly described by direct Proportionality

* In freq domain, the system is Given by direct Proportion

$$W(s) = H(s)\,V(s)$$

* Time domain & freq domain are equivilant. we can Pass Back & Forth b/w them

* They are different Picture's of same thing. different view's of same Phenomenon. we can use one to study other's.

* The importance of Drinsins in the F.T to LTI is the fact that complex exponential's are eigen function's for LTI System's.

(Last Great fact on LTI system's is Complex exponential's are eigen-function's)

$$w = Lv = h * v$$

$$W(s) = H(s)V(s)$$

what happen's if we input Complex exponential?

$\Rightarrow \quad v(x) = e^{2\pi i \nu x}$

What is $Lv(x)$? Sometime people call this as freq response, because we are inputting a pure freq, pure harmonic.

$$v(x) = e^{2\pi i \nu x}$$

$$V(s) = \mathcal{F}v(s) = \int_{-\infty}^{+\infty} e^{2\pi i \nu x} e^{-2\pi i s x} dx$$

$$= \int_{-\infty}^{\infty} e^{2\pi i (\nu - s) x} dx$$

$$= \begin{cases} 1 & \text{if } s = \nu \\ 0 & \text{if } s \neq \nu \end{cases}$$

$$V(s) = \delta(s - \nu)$$

$$\Rightarrow \quad W(s) = H(s)\, V(s)$$

$$W(s) = H(s)\, \delta(s-\nu)$$

$$\Rightarrow \quad W(s) = H(\nu)\, \delta(s-\nu)$$

$$\Big\downarrow \mathcal{F}^{-1}$$

$$w(t) = H(\nu)\, e^{2\pi i \nu x}$$

$$\Rightarrow \quad e^{2\pi i \nu x} \longrightarrow \boxed{H(s)} \longrightarrow H(\nu)\, e^{2\pi i \nu x}$$

$\Rightarrow$ $e^{2\pi i \nu x}$ is an eigen function with eigen value $H(\nu)$

$\Rightarrow$ Any LTI System's have complex exponential's as BASIS of eigen function's. (Not sin, cos)

* for any LTI system's, complex exponential's act as BASIS of eigen function's $\Rightarrow$ allow's us to diagonalize The operator's

let $v(x) = \cos(2\pi \nu t)$

$$= \frac{1}{2} e^{2\pi i \nu t} + \frac{1}{2} e^{-2\pi i \nu t}$$

$$Lv(t) = \frac{1}{2} H(\nu) e^{2\pi i \nu t} + \frac{1}{2} H(-\nu) e^{-2\pi i \nu t}$$

if $v(t)$ is real then

$$= \frac{1}{2} H(\nu) e^{2\pi i \nu t} + \frac{1}{2} \overline{H(\nu)} e^{-2\pi i \nu t}$$

$$= \frac{1}{2} H(\nu) e^{2\pi i \nu t} + \frac{1}{2} \overline{H(\nu) e^{2\pi i \nu t}}$$

$$= \operatorname{Re}\{ H(\nu) e^{2\pi i \nu t} \}$$

$$= |H(\nu)| \cos(2\pi\nu t + \phi_H(\nu))$$

$$\Rightarrow \quad \cos(2\pi\nu t) \longrightarrow \boxed{H(s)} \longrightarrow \underbrace{|H(\nu)| \cos(2\pi\nu t + \phi_H(\nu))}_{\text{there is a phase shift}}$$

This show's cos , sin are not eigen function's of LTI System's.

$\Rightarrow$ we are (sort of) back to where we started in the course - with complex exponential's as a Basis for decomposing a signal, and now for decomposing a Operator.