Lec 13 :- Fourier Transform of Generalized function's.

Today we will be finding F.T of some well known function's.

* Fourier Transform of a distribution.

$\Rightarrow$ To define a distribution we need a class of test function's

<u>Setup</u>:-

1. define a class of test function's, usually which has particularly nice properties for a given problem at hand, and vary problem to problem,

$\Rightarrow$ for F.T "rapidly decreasing function's"

(2) A distribution (generalized function)

a continuous linear functional on test functions

T is a way of measuring device.

φ is the thing we are measuring.

$\Rightarrow$ φ is a test function

T is a generalized function (distribution)

T operating on φ

$$T(\varphi) = \langle T, \varphi \rangle$$

(1) T is linear $\Rightarrow \langle T, \varphi_1 + \varphi_2 \rangle = \langle T, \varphi_1 \rangle + \langle T, \varphi_2 \rangle$

Scaling:

$$\langle T, \alpha \varphi_1 + \beta \varphi_2 \rangle = \alpha \langle T, \varphi_1 \rangle + \beta \langle T, \varphi_2 \rangle$$

(2) Continuity

$$\varphi_n(x) \xrightarrow[n\to\infty]{} \varphi(x)$$

Convergence of functions

$$\langle T, \varphi_n \rangle \xrightarrow[n\to\infty]{} \langle T, \varphi \rangle$$

Converges of numbers.

" Distribution's in dual space of Test function's"

Example: δ defined as distribution's

and its the simplest distribution. This misterious δ function, which captures this property of concentration at a point emerges as simple evaluation at origin

definition: δ : $\langle \delta, \varphi \rangle = \varphi(0)$

(Pairing)

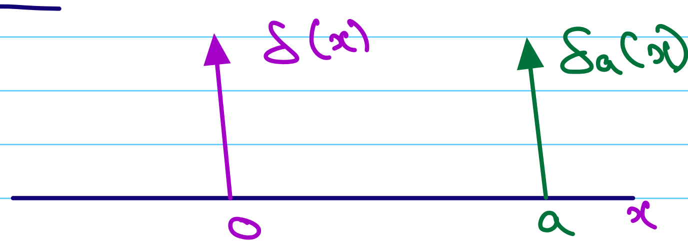
we are gonna define a distribution, $\Rightarrow$ what does that mean?

That mean's you give me a Test function, I have to tell you how to operate on it. How δ operates on $\varphi(x) \Rightarrow$ Just $\varphi(0)$ (gives $\varphi(0)$)

Example: Shifted $\delta(t)$

definition $\delta_a(x)$: $\langle \delta_a, \varphi \rangle = \varphi(a)$

Graphical:



$$\langle \delta, \varphi \rangle = \int_{-\infty}^{+\infty} \delta(x)\, \varphi(x)\, dx = \varphi(0)$$

$$\langle \delta_a, \varphi \rangle = \int_{-\infty}^{+\infty} \delta(x-a)\, \varphi(x)\, dx = \varphi(a)$$

2nd example:

distribution induced by functions.

$\Rightarrow$ if $f(x)$ a function, s.t

$$\underbrace{\int_{-\infty}^{+\infty} f(x)\,\varphi(x)\,dx}_{}$$

if this integral converges.

$\Rightarrow$ $\varphi(x)$ is such a nice function, that even if $f(x)$ is bad function, The Product is such that the integral converges.

* if $\int_{-\infty}^{+\infty} f(x)\,\varphi(x)\,dx$ makes sense

then that defines Pairing f & φ

define it : $\langle f, \varphi \rangle = \int_{-\infty}^{+\infty} f(x)\varphi(x)\,dx$

Ex: $1: \int_{-\infty}^{+\infty} 1 \cdot \varphi(x)\, dx = \int_{-\infty}^{+\infty} \varphi(x)\, dx$

Ex: $e^{2\pi i a x}$ might be ok if

$$\int_{-\infty}^{+\infty} e^{2\pi i a x} \varphi(x)\, dx = \langle e^{2\pi i a x}, \varphi \rangle$$

Fourier Transform of Distribution's:-

→ The test function's are gonna be Schwartz function's : S (rapidly decreasing functions)

* Take test function's to be S

why S are good for F.T?

① if $\varphi(x) \in S \Rightarrow \mathcal{F}\varphi(s) \in S$

$$\mathcal{F}^{-1}\varphi \in S$$

(2) Fourier inversion work's

$$\mathcal{F}^{-1}\mathcal{F}\varphi = \varphi \,, \quad \mathcal{F}\mathcal{F}^{-1}\varphi = \varphi$$

* The corresponding class of distribution, is called Tempered distribution's.

we will show how to define F.T of Tempered Distribution to be another Tempered Distribution.

* if T is a tempered distribution, we want to define its F.T $\Rightarrow$ its another Tempered distribution.

<u>Define</u> $\mathcal{F}T$: $\langle \mathcal{F}T, \varphi \rangle$

what if the pairing is by integration

let's say

$$\langle \mathcal{F}T, \varphi \rangle = \int_{-\infty}^{+\infty} \mathcal{F}T(x)\, \varphi(x)\, dx$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-2\pi i x y}\, T(y)\, dy \cdot \varphi(x)\, dx$$

swap the order of integration

$$= \int_{-\infty}^{+\infty} T(y) \left[\int_{-\infty}^{+\infty} e^{-2\pi i x y}\, \varphi(x)\, dx \right] dy$$

$$= \int_{-\infty}^{+\infty} T(y)\, \mathcal{F}\varphi(y)\, dy$$

$$= \langle T, \mathcal{F}\varphi \rangle$$

$$\boxed{\langle \mathcal{F} T, \varphi \rangle = \langle T, \mathcal{F}\varphi \rangle}$$

Turn this into definition:

+ T is a tempered distribution.

Define $\mathcal{F}T$ by $\langle \mathcal{F} T, \varphi \rangle = \langle T, \mathcal{F}\varphi \rangle$

How should we define inverse F.T of distribution, $\mathcal{F}^{-1}T$?

+ $\langle \mathcal{F}^{-1} T, \varphi \rangle = \langle T, \mathcal{F}^{-1}\varphi \rangle$

Now Prove Fourier inversion:

1. $\mathcal{F}^{-1}\mathcal{F} T = T$
2. $\mathcal{F}\mathcal{F}^{-1} T = T$

$$\langle \mathcal{F}^{-1}\mathcal{F}T, \varphi \rangle = \langle \mathcal{F}T, \mathcal{F}^{-1}\varphi \rangle$$

$$= \langle T, \mathcal{F}\mathcal{F}^{-1}\varphi \rangle$$

$$= \langle T, \varphi \rangle$$

F.T of δ:

$$\langle \mathcal{F}\delta, \varphi \rangle = \langle \delta, \mathcal{F}\varphi \rangle$$

$$= \mathcal{F}\varphi(0)$$

$$= \int_{-\infty}^{+\infty} e^{-2\pi i\, 0.x}\, \varphi(x)\, dx$$

$$= \int_{-\infty}^{+\infty} 1 . \varphi(x)\, dx$$

$$= \langle 1, \varphi \rangle$$

$$\Rightarrow \quad \langle \mathcal{F}\delta, \varphi \rangle = \langle 1, \varphi \rangle$$

$$\Rightarrow \quad \boxed{\mathcal{F}\delta = 1}$$

The simplest of all distribution (δ) have the simplest of all FT (1)

$$\boxed{\mathcal{F}\delta = 1}$$

* Extreme example of sort of dual relationship b/w concentrated in one domain is spread out in other domain.

* δ is infinitely concentrated

* $\mathcal{F}\delta = 1$ is uniformly spreadout.

Example: $\mathcal{F}\delta_a$?

$$\langle \mathcal{F}\delta_a, \varphi \rangle = \langle \delta_a, \mathcal{F}\varphi \rangle$$

$$= \mathcal{F}\varphi(a)$$

$$= \int_{-\infty}^{+\infty} e^{-2\pi i a x} \varphi(x)\, dx$$

$$\langle \mathcal{F}\delta_a, \varphi \rangle = \langle e^{-2\pi i a x}, \varphi \rangle$$

$$\Rightarrow \quad \boxed{\mathcal{F}\delta_a = e^{-2\pi i a x}}$$

Ex: $\mathcal{F} e^{2\pi i a x}$?

$$\langle \mathcal{F} e^{2\pi i a x}, \varphi \rangle = \langle e^{2\pi i a x}, \mathcal{F}\varphi \rangle$$

$$= \int_{-\infty}^{+\infty} e^{2\pi i a x} \mathcal{F}\varphi(x)\, dx$$

$$= (\mathcal{F}^{-1}\mathcal{F}\varphi)(a)$$

$$= \varphi(a)$$

$$= \langle \delta_a, \varphi \rangle$$

$$\Rightarrow \langle \mathcal{F} e^{2\pi i a x}, \varphi \rangle = \langle \delta_a, \varphi \rangle$$

$$\Rightarrow \boxed{\mathcal{F} e^{2\pi i a x} = \delta_a}$$

when $a = 0$

$$\mathcal{F} e^{2\pi i 0 \cdot x} = \delta(x - 0)$$

$$\Rightarrow \mathcal{F} 1 = \delta(x)$$

$$\Rightarrow \boxed{\begin{aligned} \mathcal{F} \delta &= 1 \\ \mathcal{F} 1 &= \delta \end{aligned}}$$

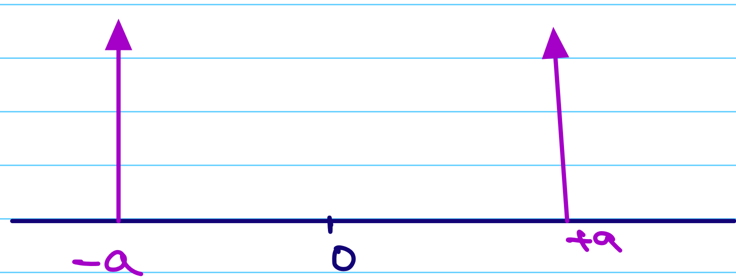
How about sin & cos?

$\mathcal{F} \cos 2\pi a x$?

$\Rightarrow \quad \cos 2\pi a x = \frac{1}{2}\left(e^{2\pi i a x} + e^{-2\pi i a x}\right)$

$\Rightarrow \quad \mathcal{F} \cos 2\pi a x = \frac{1}{2} \mathcal{F} e^{2\pi i a x} + \frac{1}{2} \mathcal{F} e^{-2\pi i a x}$

$= \frac{1}{2} \delta(x-a) + \frac{1}{2} \delta(x+a)$



$\Rightarrow \quad \mathcal{F} \sin 2\pi a x = \frac{1}{2i}\left(\mathcal{F} e^{2\pi i a x} - \mathcal{F} e^{-2\pi i a x}\right)$

$= \frac{1}{2i}\left(\delta(x-a) - \delta(x+a)\right)$

The Problem with classical F.T is that it didn't make sense on Function's that really wanted to make sense on, Ex:- 1, Sin, cos, sinc . .

* Now all those thing's work.