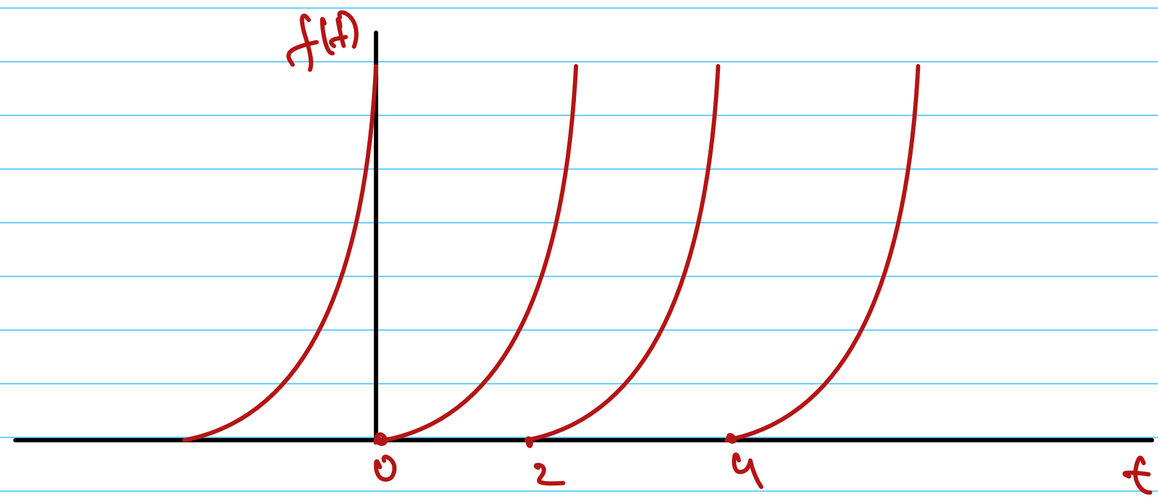
Problem set 02

① Let $f(t)$ be a function of Period $T=2$ with

$$f(t) = t^2 \quad \text{if} \quad 0 \le t < 2$$



ⓐ find the fourier series co-efficient's c_n of $f(t)$

Sol^n

$$f(t) = \sum_{k=-\infty}^{+\infty} c_k e^{2\pi i \frac{k}{T} t}$$

$$c_k = \frac{1}{T} \int_T f(t) e^{-2\pi i \frac{k}{T} t} \, dt$$

$$C_k = \frac{1}{2}\int_0^2 t^2 e^{-2\pi i \frac{k}{2} t}\, dt = \frac{1}{2}\int_0^2 t^2 e^{-\pi i k t}\, dt$$

$$= \frac{1}{2}\left[\left[t^2 \frac{e^{-\pi i k t}}{-\pi i k}\right]_0^2 - \int_0^2 2t\, \frac{e^{-\pi i k t}}{-\pi i k}\, dt\right]$$

$$= \frac{1}{2}\left[\frac{-4e^{-2\pi i k}}{\pi i k} + \frac{2}{\pi i k}\int_0^2 t e^{-\pi i k t}\, dt\right]$$

$$e^{-2\pi i k} = \cos(2\pi k) - i \sin(2\pi k)$$
$$= 1$$

$$= \frac{1}{2}\left[\frac{-4}{\pi i k} + \frac{2}{\pi i k}\int_0^2 t e^{-\pi i k t}\, dt\right]$$

$$= \frac{1}{2}\left\{\frac{-4}{\pi i k} + \frac{2}{\pi i k}\left[\left[t \frac{e^{-\pi i k t}}{-\pi i k}\right]_0^2 - \int_0^2 \frac{e^{-\pi i k t}}{-\pi i k}\, dt\right]\right]$$

$$= \frac{1}{2}\left[\frac{-4}{\pi i k} + \frac{2}{\pi i k}\left[-\frac{2}{\pi i k} e^{-2\pi i k} - \left[\frac{e^{-\pi i k t}}{(\pi i k)^2}\right]_0^2\right]\right]$$

$$= \frac{1}{2}\left[\frac{-4}{\pi i k} - \frac{4}{(\pi i k)^2} - \frac{2}{\pi i k}\left[\cancel{\frac{1}{(\pi i k)^2}} - \cancel{\frac{e^{-2\pi i k}}{(\pi i k)^2}}\right]\right]$$

$$e^{-2\pi i k} = 1$$

$$= -2\left[\frac{1}{\pi i k} + \frac{1}{(\pi i k)^2}\right]$$

$$= \frac{2i}{\pi k} + \frac{2}{(\pi k)^2} = 2\left(\frac{1 + i\pi k}{(\pi k)^2}\right)$$

$$\boxed{C_k = \frac{2(1 + i\pi k)}{\pi^2 k^2} \quad k \neq 0}$$

$$C_0 = \frac{1}{2}\int_0^2 t^2\,dt = \frac{1}{2}\left[\frac{t^3}{3}\right]_0^2$$

$$= \frac{1}{2}\,\frac{8}{3} = \frac{4}{3}$$

$$\boxed{C_0 = \frac{4}{3}}$$

(b) Prove $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

we know $t^2 = \sum_{n=-\infty}^{+\infty} C_n e^{2\pi i \frac{n}{2} t}$

$$\Rightarrow \quad t^2 = \sum_{n=-\infty}^{+\infty} \frac{2(1+i\pi n)}{\pi^2 n^2} e^{2\pi i \frac{n}{2} t}$$

at $t=0$ the function converges to average of jump if at $t=0$, the f.S converges to $\frac{0+2^2}{2} = 2$

$$2 = \sum_{n=-\infty}^{-1} \frac{2(1+i\pi n)}{\pi^2 n^2} + C_0 + \sum_{n=1}^{\infty} \frac{2(1+i\pi n)}{\pi^2 n^2}$$

$$\Rightarrow \quad 2 = \sum_{n=-\infty}^{-1} \frac{2(1+i\pi n)}{\pi^2 n^2} + \frac{4}{3} + \sum_{n=1}^{\infty} \frac{2(1+i\pi n)}{\pi^2 n^2}$$

$$\Rightarrow \quad 1 = \frac{2}{3} + \sum_{n=-\infty}^{-1} \frac{(1+i\pi n)}{\pi^2 n^2} + \sum_{n=1}^{\infty} \frac{(1+i\pi n)}{\pi^2 n^2}$$

$$\Rightarrow \quad \frac{1}{3} = \sum_{n=1}^{\infty} \frac{1+i\pi n + (1-i\pi n)}{\pi^2 n^2}$$

$$\Rightarrow \quad \frac{1}{3} = \sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2}$$

$$\boxed{\Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

$$C_0 + \sum_{n=-\infty}^{-1} C_k e^{\frac{2\pi i n}{2} t} + \sum_{n=1}^{\infty} C_k e^{\frac{2\pi i n}{2} t} = t^2$$

$$\Rightarrow \quad \frac{4}{3} + \sum_{n=-\infty}^{-1} \frac{2(1+i\pi n)}{\pi^2 n^2} e^{\frac{2\pi i n}{2} t} + \sum_{n=1}^{\infty} \frac{2(1+i\pi n)}{\pi^2 n^2} e^{\frac{2\pi i n t}{2}} = t^2$$

$$\Rightarrow \frac{4}{3} + \sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2} \left[e^{\frac{2\pi i n t}{2}} (1 + i\pi n) + e^{-\frac{2\pi i n t}{2}} (1 - i\pi n) \right] = t^2$$

$$\Rightarrow \frac{4}{3} + \sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2} \left[2\cos \pi n t + i\pi n \left[e^{\pi i n t} - e^{-\pi i n t} \right] \right]$$

$$\Rightarrow \frac{4}{3} + \sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2} \left[2\cos \pi n t - 2\pi n \sin \pi n t \right] = t^2$$

at $t = 1$

$$\Rightarrow \frac{4}{3} + \sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2} \left[2\cos \pi n - 2\pi n \cancelto{0}{\sin \pi n} \right] = 1$$

$$\Rightarrow \frac{4}{3} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} (-1)^n = 1$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} (-1)^n = -\frac{1}{3}$$

i)
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

ii)
$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}}$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

(2) Whither Rayleigh? What happen's to Rayleigh's identity if $f(t)$ is periodic of periodic $T \neq 1$?

Sol^n

$$f(t) = \sum_{k=-\infty}^{+\infty} \hat{f}(k) e^{2\pi i \frac{k}{T} t}$$

$$\hat{f}(k) = \frac{1}{T} \int_T f(t) e^{-2\pi i \frac{k}{T} t} dt$$

$$\Rightarrow \quad \hat{f}(k) = \frac{1}{T} \int_T f(t) \, \overline{e^{2\pi i \frac{k}{T} t}} \, dt$$

$$\Rightarrow \quad \hat{f}(k) = \frac{1}{T} \langle f, e_k \rangle$$

$$\int_T |f(t)|^2 = \|f\|^2$$

$$= \langle f, f \rangle$$

$$= \left\langle \sum_{n=-\infty}^{+\infty} \frac{1}{T} \langle f, e_n \rangle e_n \, , \sum_{m=-\infty}^{+\infty} \frac{1}{T} \langle f, e_m \rangle e_m \right\rangle$$

$$= \frac{1}{T^2} \left\langle \sum_{n=-\infty}^{+\infty} \langle f, e_n \rangle e_n , \sum_{m=-\infty}^{+\infty} \langle f, e_m \rangle e_m \right\rangle$$

$$= \frac{1}{T^2} \left(\sum_{n,m} \langle f, e_n \rangle \overline{\langle f, e_m \rangle} \langle e_n, e_m \rangle \right]$$

$$= \frac{1}{T^2} \left(\sum_{n=-\infty}^{+\infty} \langle f, e_n \rangle \overline{\langle f, e_n \rangle} \langle e_n, e_n \rangle \right)$$

$$= \frac{1}{T^2} \left(\sum_{n=-\infty}^{+\infty} \langle f, e_n \rangle \overline{\langle f, e_n \rangle} \, T \right)$$

$$= \frac{1}{T} \sum_{n=-\infty}^{+\infty} \left| \langle f, e_n \rangle \right|^2$$

$$= \frac{1}{T} \sum_{n=-\infty}^{+\infty} T^2 \left| \hat{f}(n) \right|^2$$

$$= T \sum_{n=-\infty}^{+\infty} \left| \hat{f}(n) \right|^2$$

$$\Rightarrow \boxed{\frac{1}{T} \int_0^T |f(t)|^2 \, dt = \sum_{n=-\infty}^{+\infty} |c_n|^2}$$

4

(a) find the Fourier Transform of the following signal

$$\Lambda(t) = \begin{cases} 1 - |t| & |t| \leq 1 \\ 0 & \text{o.w} \end{cases}$$

$$\Lambda_a(t) = \begin{cases} 1 - \frac{|t|}{a} & |t| \leq a \\ 0 & \text{o.w} \end{cases}$$

$$\Lambda_a(t - t_0) = \begin{cases} 1 - \frac{|t-t_0|}{a} & |t-t_0| \leq a \\ 0 & \text{o.w} \end{cases}$$

$$b\Lambda_a(t - t_0) = \begin{cases} b\left(1 - \frac{|t-t_0|}{a}\right) & |t-t_0| \leq a \\ 0 & \text{o.w} \end{cases}$$

Here $f(t) = 2\Lambda_2(t-2)$

$+ \; 2.5\,\Lambda_2(t-4)$

we can use stretch & shifting theorem

$$\Lambda(t) = \begin{cases} 1 - |t| & |t| \le 1 \\ 0 & \text{o.w} \end{cases}$$

$$\mathcal{F}\Lambda(s) = \int_{-\infty}^{+\infty} \Lambda(t) e^{-2\pi i s t}\, dt$$

$$= \int_{-1}^{0} (1+t) e^{-2\pi i s t}\, dt + \int_{0}^{1} (1-t) e^{-2\pi i s t}\, dt$$

$$= \frac{(1+t)\, e^{-2\pi i s t}}{-2\pi i s}\Bigg]_{-1}^{0} - \int_{-1}^{0} \frac{e^{-2\pi i s t}}{-2\pi i s}\, dt$$

$$+ \frac{(1-t)\, e^{-2\pi i s t}}{-2\pi i s}\Bigg]_{0}^{1} + \int_{0}^{1} \frac{e^{-2\pi i s t}}{-2\pi i s}\, dt$$

$$= \cancel{\frac{1}{-2\pi i s}} - \frac{e^{-2\pi i s t}}{(2\pi i s)^2}\Bigg]_{-1}^{0} + \cancel{\frac{1}{2\pi i s}} + \frac{e^{-2\pi i s t}}{(2\pi i s)^2}\Bigg]_{0}^{1}$$

$$= \frac{e^{2\pi i s}}{(2\pi i s)^2} - \frac{1}{(2\pi i s)^2} + \frac{e^{-2\pi i s}}{(2\pi i s)^2} - \frac{1}{(2\pi i s)^2}$$

$$= \left(\frac{\sin \pi s}{\pi s}\right)^2 = \operatorname{sinc}^2 s$$

$$\Lambda(t) \rightleftharpoons \operatorname{sinc}^2 s$$

$$\Lambda\left(\frac{t-2}{2}\right) \rightleftharpoons 2e^{-2\pi i s \cdot 2} F(2s)$$

$$\Rightarrow \Lambda\left(\frac{t-2}{2}\right) \rightleftharpoons 2e^{-4\pi i s}(\operatorname{sinc}(2s))^2$$

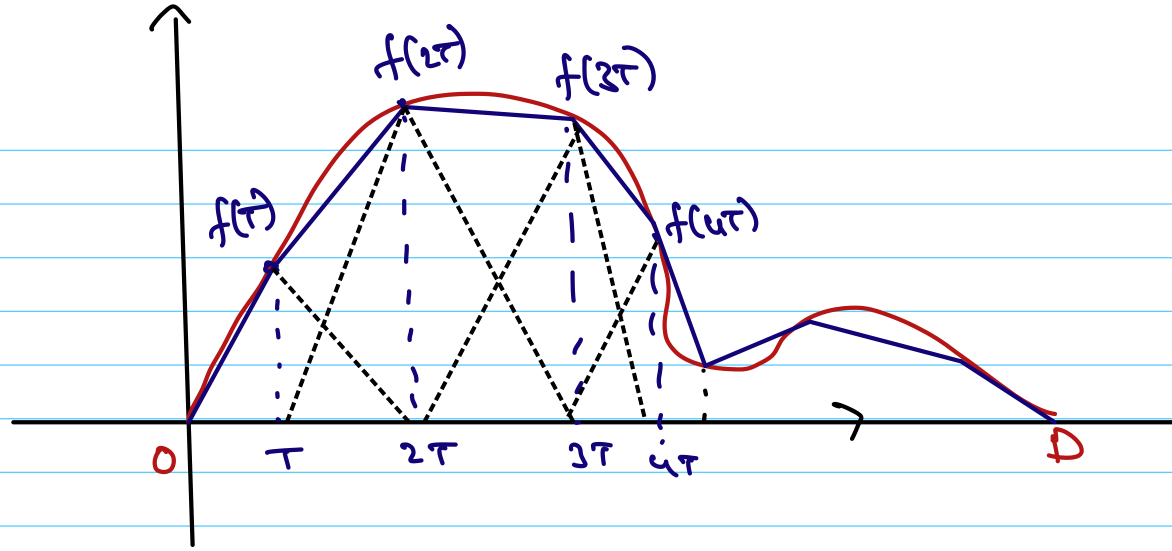
$$\Lambda\left(\frac{t-4}{2}\right) \rightleftharpoons 2e^{-8\pi i s}(\operatorname{sinc}(2s))^2$$

$$2\Lambda\left(\frac{t-2}{2}\right) + 2.5\Lambda\left(\frac{t-4}{2}\right) \rightleftharpoons 4e^{-4\pi i s}(\operatorname{sinc} 2s)^2$$

$$5e^{-8\pi i s}(\operatorname{sinc} 2s)^2$$

$$\mathcal{F} f(t) = \left(4e^{-4\pi i s} + 5e^{-8\pi i s}\right) \operatorname{sinc}^2 2s$$

4 (b)



g(t) can be approximated by (n-1) shifted, stretched $\Lambda(t)$ function's ..

$$\Rightarrow \quad g(t) = f(T)\,\Lambda_T(t-T) + f(2T)\,\Lambda_T(t-2T) + \dots f((n-1)T)\,\Lambda_T(t-(n-1)T)$$

$$f(t) = \sum_{k=1}^{n-1} f(kT)\,\Lambda_T(t-kT)$$

$$\mathcal{F} f(s) = \mathcal{F}\left(\sum_{k=1}^{n-1} f(kT)\,\Lambda_T(t-kT) \right)$$

$$= \sum_{k=1}^{n-1} f(kT)\, \mathcal{F}\left(\Lambda_T (t-kT)\right)$$

$$= \sum_{k=1}^{n-1} f(kT)\, T e^{-2\pi i s kT} \operatorname{sinc}^2 Ts$$

$$\boxed{\mathcal{F}f(s) = T \operatorname{sinc}^2 Ts \sum_{k=1}^{n-1} f(kT) e^{-2\pi i s kT}.}$$

(5) The modulation property of the Fourier Transform.

(a) let $f(t)$ be a signal, s_0 a number and define

$$g(t) = f(t)\cos(2\pi s_0 t)$$

Show that

$$\mathcal{F}g(s) = \frac{1}{2}\mathcal{F}f(s-s_0) + \frac{1}{2}\mathcal{F}f(s+s_0)$$

Sol^n

$$g(t) = f(t)\cos(2\pi s_0 t)$$

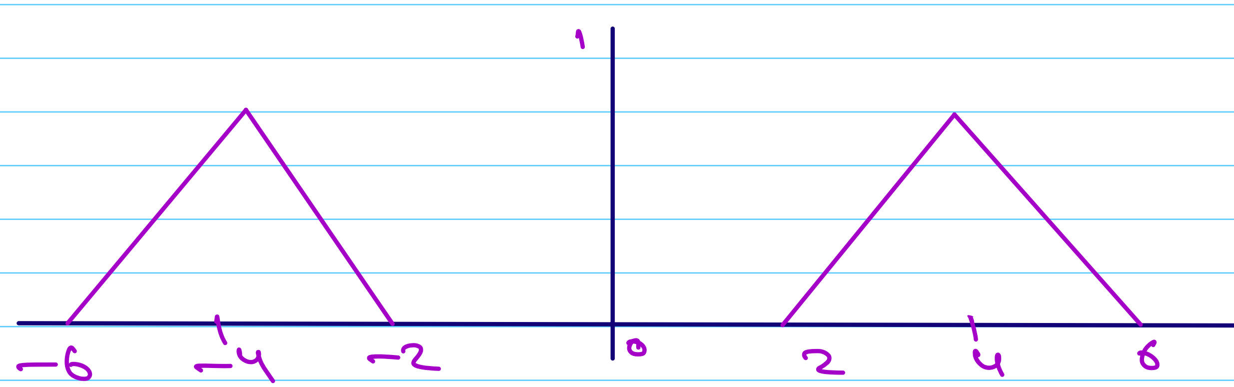
$$= f(t) \cdot \frac{e^{2\pi i s_0 t} + e^{-2\pi i s_0 t}}{2}$$

$$g(t) = \frac{1}{2} f(t) e^{2\pi i s_0 t} + \frac{1}{2} f(t) e^{-2\pi i s_0 t}$$

$$\mathcal{F}g(s) = \frac{1}{2}\mathcal{F}\left(f(t)e^{2\pi i s_0 t}\right) + \frac{1}{2}\mathcal{F}\left(f(t)e^{-2\pi i s_0 t}\right)$$

$$\mathcal{F}g(s) = \frac{1}{2}\mathcal{F}f(s_0 - s_0) + \frac{1}{2}\mathcal{F}f(s_0 + s_0)$$

(b)



$$\mathcal{F}g(s) = \frac{1}{2}\left(2\Lambda_2(s+4) + 2\Lambda_2(s-4)\right)$$

form the above Problem 5a

we can write

$$g(t) = f(t)\cos(2\pi s_0 t)$$

$s_0 = 4$

$$= f(t)\cos(8\pi t)$$

$$\mathcal{F}f(s) = 2\Lambda_2(s)$$

we know that $\mathcal{F}\mathcal{F}f = f^-$

$$\mathcal{F}\mathcal{F}f(s) = \mathcal{F}(2\Lambda_2(s))$$

$$= 2\,\mathcal{F}(\Lambda_2(s))$$

$$= 2 \cdot 2\,\mathrm{sinc}^2 2t$$

$$f^- = 4\,\mathrm{sinc}^2 2t$$

$$f = 4\,\text{sinc}^2 2t$$

$$\Rightarrow \quad \boxed{g(t) = 4\,\text{sinc}^2\, 2t \cdot \cos(8\pi t)}$$

6 Suppose the function $f(t)$ is zero outside the interval $-1/2 \le t \le 1/2$. We form a function $g(t)$ which is a periodic version of $f(t)$ with period 1 by the formula.

$$g(t) = \sum_{k=-\infty}^{+\infty} f(t-k)$$

The Fourier Series representation of $g(t)$ is given by

$$g(t) = \sum_{k=-\infty}^{+\infty} \hat{g}(n)\, e^{2\pi i n t}$$

Find the relation b/w the F.T of $f(t)$ and the Fourier series coefficients $\hat{g}(n)$

Sol^n

$$g(t) = \sum_{k=-\infty}^{+\infty} f(t-k)$$

$f(t)$ is a function b/w $-1/2$ to $1/2$

$$\Rightarrow \quad \mathcal{F}f(s) = \int_{-1/2}^{1/2} f(t) e^{-2\pi i s t} dt$$

$$g(t) = \sum_{n=-\infty}^{+\infty} \hat{g}(n) e^{2\pi i n t}$$

$$\hat{g}(n) = \int_{-1/2}^{1/2} g(t) e^{-2\pi i n t} dt$$

$$\hat{g}(n) = \int_{-1/2}^{1/2} f(t) e^{-2\pi i n t} dt$$

$$\hat{g}(n) = \int_{-\infty}^{+\infty} f(t) e^{-2\pi i n t} dt = \mathcal{F}f(n)$$

$$\Rightarrow \quad \hat{g}(n) = \mathcal{F}f(s)\Big|_{s=n}$$

7 Consider the function's $g(x)$ & $h(x)$, shown that