FOURIER SERIES

Lec 01

- we will start with Fourier series and use them As a transition to Fourier Transform.

- Fourier series is identified with mathematical analysis of Periodic Phenomenon.

- Fourier Transform can be viewed as a Limiting case of fourier series. It is the study of mathematical analysis of non-periodic Phenomenon.

- Fourier Transfor as a Limiting case of Fourier series is concencerned with analysis of non-Perodic Phenomenon. It does'nt say everything but it does say something.

Analysis	Sythesis
- taking a signal or Function and breaking it into its constituent Parts.	- reassembling the signal or function from its constituent Parts.

* Both Analysis and synthesis are accomplished by Linear operation's. This is one of the reason that the subject so Powerful, Because there is such a Body of knowledge, and advanced understanding of Linear operator's. (Linearity)

Linear operation's = Integral's & Series.

=) Both of them are linear operation's

=) The integral of sum is sum of integral's

Simillary with sum's.

we often hear Fourier Analysis is Part Linear system's.

<u>Periodic Phenomenum & Fourier series :-</u>

There is some Pattern that repeats.

and it repeats regularly.

Mathematics and Engineering of a regullarly repeating Pattern's.

Periodic Phenomenum often either Periodicity in time (Ex, Pendulum) or Periodicity in space

* Fourier Analysis is often associated with symmetry.

* Periodic in time, we use "frequency", as the number of cycles in a second.

Reciprocal Relationship:

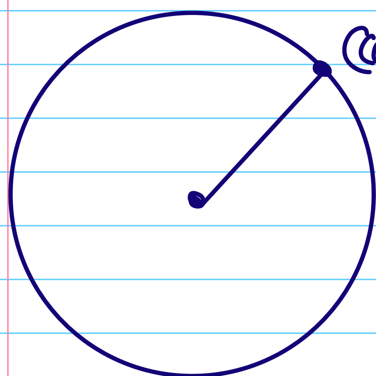
there is Reciprocal relation with frequency and wavelength $\lambda \nu = V$

* Math comes in because there are simple mathematical function's that are Periodic (repeat) can be used to model the Phenomenon. [sin, cosine]

cost, and sint are Periodic of Period 2π

$$\Rightarrow \quad \sin(t+2\pi) = \sin t$$
$$\cos(t+2\pi) = \cos t$$

$\cos t$ and $\sin t$ are Periodic in space



$$(\cos t, \sin t) = [\cos(2\pi+t), \sin(2\pi+t)]$$

$$\rightarrow \quad \cos(t+2\pi n) = \cos t$$
$$\sin(t+2\pi n) = \sin t \qquad n = 0, \pm 1, \pm 2, \ldots$$

These simple function sin, cos can be used to model the most complex behavior.

That is the fundamental discovery of Fourier Series. BASIS OF Fourier Analysis

Lecture 02 : How can we use such simple function's , sin(t) and cos(t) to model such complex periodic phenomenum

we are essentially identifying the subject of fourier series with the Mathematical study of Periodicity.

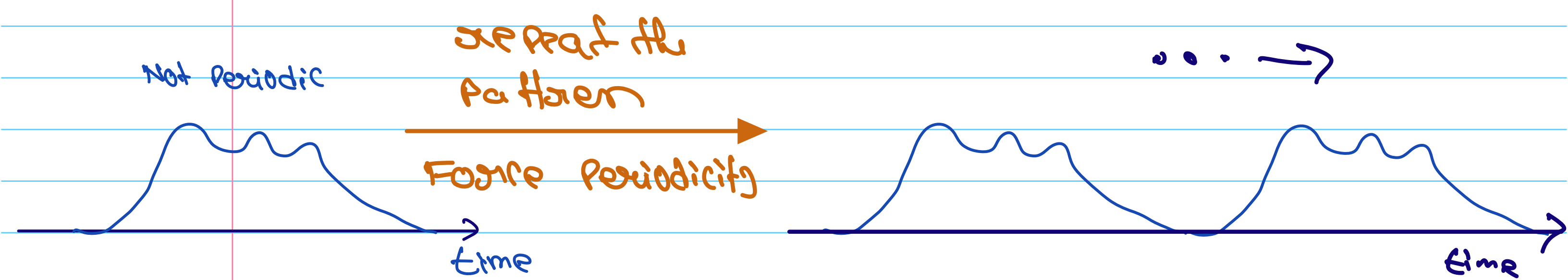
How can we use such simple function's sint , cost to model complex physical phenomenon?

How general?

* Not all phenomenon are periodic. even for periodic phenomenon in time, real phenomenon die out eventually.
* whereas as mathematical function's sin, cosine go on forever. so how can they really be useful to model something that dies out?

we can still apply ideas of periodicity even if only approximation or with extra assumptions

Suppose the signal look's like



For the discussion FIX the Period $(T=1)$

$\Rightarrow$ We use Period 1, so we will consider function's satisfying

$$f(t+1) = f(t) \quad \forall t$$

Model signal's are:

$$\left.\begin{matrix} \sin 2\pi t \\ \cos 2\pi t \end{matrix}\right\} \text{Period } 1$$

* if we "know" the periodic function on an interval of length 1 or any

interval of length , we know it every where. (Because the pattern repeats)

* we can modify and combine $\sin 2\pi t$, $\cos(2\pi t)$ to model general periodic signal's of period 1.

Big Idea:

ONE PERIOD, Many Frequencies.

⇓

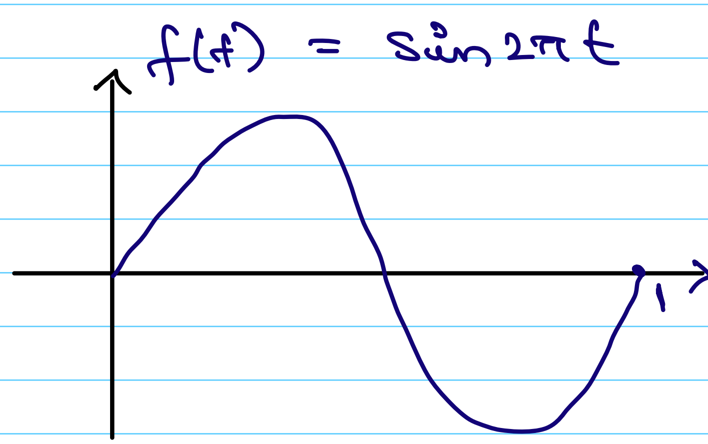
It mean's

Ex:

① $f(t) = \sin 2\pi t$

Period = 1 sec, 2, 3, 4, . . .

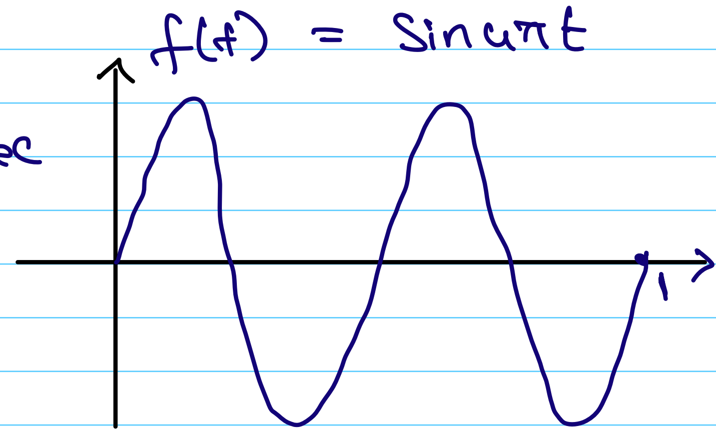
freq = 1



② $f(t) = \sin 4\pi t$

Period $= \frac{1}{2}$ sec, 1 sec, $\frac{3}{2}$ sec

freq $= 2$



we can view one cycly as a signla with Period $= \frac{1}{2}$

and can view two cycly signal with Period $= 2$

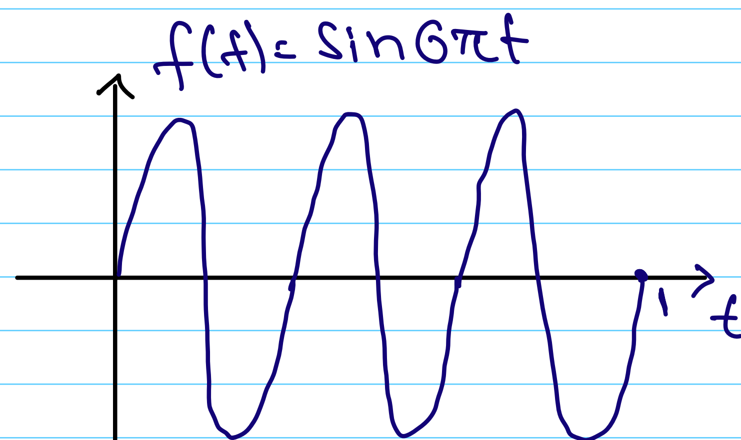
(its true that it repeats on an interval of $1/2$,

But signal also has a longer Period 1,

③ $f(t) = \sin 6\pi t$

Period $= \frac{1}{3}, \frac{2}{3}, 1, \ldots$

freq $= 3$,



But it also has Period $= 1$

ONE Period, Many frequencies

$\sin 2\pi t \Rightarrow T = 1, 2, 3, \ldots$ $\quad f = 1$

$\sin 4\pi t \Rightarrow T = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ $\quad f = 2$

$\sin 6\pi t \Rightarrow T = \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2$ $\quad f = 3$

$\sin 8\pi t \Rightarrow T = \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, \frac{5}{4} \ldots$ $\quad f = 4$

for All these signals period $T = 1$ is in Common.

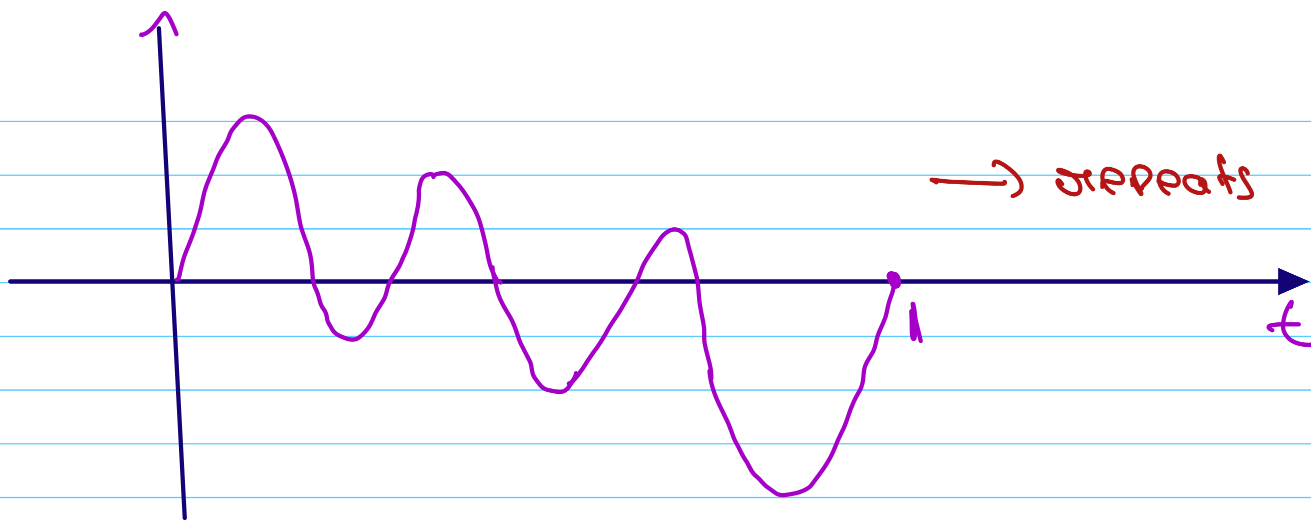
* Even though each signal has its own fundamental period $(1, \frac{1}{2}, \frac{1}{3}, \ldots)$ all of them repeat perfectly within $T = 1$

What about the combination?

* if we combine them together, Ex:

$\sin 2\pi t + \sin 4\pi t + \sin 6\pi t$

$f(t) = \sin 2\pi t + \sin 4\pi t + \sin 6\pi t$



The Period of the sum is still $T = 1$

* because, although the term's of higher freq are repeating more rapidly, the sum cannot go back to where it started until the slowest one get caught up.

(ONE Period & Many frequencies)

There are 3 frequencies in the sum (1, 2, 3) But added together, there is one one Period.

* For Complicated Periodic Phenomenon its really better more revealling to talk in term's of frequencies that might

go into it rather than Periods.

⇒ we are fixing the Period, (length = 1)

But it might have complicated Phenomenon

⇒ That complicated Phenomenon as it turn's out is going to be build up out of sin & cosines of varying frequencies.

Complicated signal with T=1

⇒ frequencies = 1, 2, 3, 4, . . . ∞

⇒ we can modify Amplitues & Phases of each one of those term's.

* To model complicated signal of Period 1 we can modify, Amplitude, freq, Phasses of $2\pi f$ and add up

$$\sum_{K=1}^{N} A_K \sin(2\pi K t + \varphi_K)$$

general kind of sum we can form out of Just sines.

Again, many frequencies are periods, the lowest (longest) period in the sum is when $k=1$ (period $=1$). The higher term's are called Harmonic's.
The higher frequencies have shorter period, but the sum have period $=1$ because the whole pattern can not repeat until the longest period repeat's (completed).

Different way's of writing the sum.

$$\sum_{k=1}^{N} A_k \sin(2\pi k t + \phi_k)$$

$$\Rightarrow \sum_{k=1}^{N} A_k \left(\sin 2\pi k t \cdot \cos\phi_k + \cos 2\pi k t \cdot \sin\phi_k \right)$$

$$\Rightarrow \sum_{k=1}^{N} (A_k \cos\phi_k) \sin 2\pi k t + (A_k \sin\phi_k) \cos 2\pi k t$$

$$\Rightarrow \sum_{k=1}^{N} a_k \cos 2\pi kt + b_k \sin 2\pi kt$$

we can also allow constant term to shift the signal

$$\frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos 2\pi kt + b_k \sin 2\pi kt$$

$\Uparrow$

This is very common way to writing the form of the sum. But by far the most convienient way algebraically and really many way's conceptually is to use complex exponential's to write the sum, Not the real cosines, sines, By far.

By far the Better Representation to represent sin, cosine is via complex exponential's.

$$\boxed{e^{2\pi i k t} = \cos 2\pi kt + i \sin 2\pi kt}$$

$i = \sqrt{-1}$

$$\cos 2\pi kt = \frac{e^{2\pi i kt} + e^{-2\pi i kt}}{2}$$

$$\sin 2\pi kt = \frac{e^{2\pi i kt} - e^{-2\pi i kt}}{2i}$$

we can convert a trignometric sum before to the form

$$\sum_{k=-N}^{N} c_k e^{2\pi i kt} \quad \text{where } c_k = \text{complex.}$$

$$\frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos 2\pi kt + b_k \sin 2\pi kt = \underbrace{\sum_{k=-N}^{N} c_k e^{2\pi i kt}}_{\text{total sum is real.}}$$

<u>Proof</u>

① $k=0 \Rightarrow \frac{a_0}{2} = c_0 e^{2\pi i 0 t} = c_0$

$$\Rightarrow \boxed{c_0 = \frac{a_0}{2}}$$

(2) for arbitrary k (equate $\sin 2\pi kf$, $\cos 2\pi kf$)

$$\Rightarrow \quad a_k \cos 2\pi kt = C_{-k} \cos 2\pi kf + C_k \cos 2\pi kf$$

$$\Rightarrow \quad a_k = C_{-k} + C_k \qquad —(1)$$

$||^{ly}$

$$b_k \sin 2\pi kf = C_{-k} \cdot i \cdot -\sin 2\pi kf$$
$$+ C_k \cdot i \cdot \sin 2\pi kf$$

$$\Rightarrow \quad b_k = i(C_k - C_{-k}) \qquad —(2)$$

from (1) & (2)

$$(1) + -i \cdot (2) \Rightarrow a_k - ib_k = \frac{C_k}{2}$$

$$\Rightarrow \quad C_k = \frac{a_k - ib_k}{2}$$

$$(1) + i(2) \Rightarrow \quad C_{-k} = \frac{a_k + ib_k}{2}$$

$$\Rightarrow \quad \boxed{\begin{aligned} C_k &= \frac{a_k - ib_k}{2} \\ C_{-k} &= \frac{a_k + ib_k}{2} \end{aligned}}$$

$a_k \cos 2\pi k t + b_k \sin 2\pi k t$

$$= \left(\frac{a_k - i b_k}{2}\right) e^{2\pi i k t} + \left(\frac{a_k + i b_k}{2}\right) e^{-2\pi i k t}$$

They satisfy symmetric property.

$$\boxed{C_{-k} = \overline{C_k}}$$

because of symmetric property the total sum is real.

* This is very important identity that's satisfied by the co-efficients for a real signal.

we can now ask fundamental question?

$f(t)$ is periodic signal with period 1

Can we write $f(t)$ as

$$f(t) = \sum_{k=-N}^{N} C_k e^{2\pi i k t} \quad ??$$

Can we express in term's of sine's & cosines?
Can we express in term's of fundamental Building Blocks?

* Linearity Play's a role here, we are considering linear combination's of Basic Building Blocks. $\left(e^{2\pi i k t}\right)$

Secret of universe:

Can I do something like this?
if $f(t)$ is period signal with $T=1$
can we represent

$$f(t) = \sum_{k=-n}^{n} C_k e^{2\pi i k t}$$

Mathematical approach:

Can I do something like this?

Often the 1st step is to suppose that

you can do, see what the consequences are. Later on, say alright, we go backwards.

Suppose we can write

$$f(t) = \sum_{k=-n}^{n} c_k e^{2\pi i k t}$$

what has to happen?

What are the mystery coefficients c_k in term's of $f(t)$?

isolate the unknown, Isolate c_m

$$f(t) = \dots + \dots + c_m e^{2\pi i m t} + \dots$$

$$\Rightarrow c_m e^{2\pi i m t} = f(t) - \sum_{\substack{k=-n \\ k \neq m}}^{n} c_k e^{2\pi i k t}$$

$$\Rightarrow c_m = f(t) e^{-2\pi i m t} - \sum_{\substack{k=-n \\ k \neq m}}^{n} c_k e^{2\pi i (k-m) t}$$

$\Rightarrow$ Integrate both sides 0 to 1

$$C_m \cdot 1 = \int_0^1 f(t) \cdot e^{-2\pi i m t} - \sum_{\substack{k=-n \\ k \neq m}}^{n} \int_0^1 e^{-2\pi i (k-m) t}\, dt$$

$$\int_0^1 e^{2\pi i (k-m) t}\, dt = \left. \frac{e^{2\pi i (k-m) t}}{2\pi i (k-m)} \right]_0^1$$

$$= \frac{1}{2\pi i (k-m)} e^{2\pi i (k-m)}$$

$$= \frac{1}{2\pi i (k-m)} \left(\cos 2\pi (k-m) + i \sin 2\pi (k-m) \right)$$

$$= \frac{1}{2\pi i (k-m)} (0 + i0) = 0$$

$$\Rightarrow \int_0^1 e^{2\pi i (k-m) t}\, dt = \begin{cases} 1 & k = m \\ 0 & k \neq m \end{cases}$$

$$\Rightarrow \quad C_k = \int_0^1 f(t)\, e^{-2\pi i k t}\, dt$$

if $f(t)$ is real signal

then $\quad C_{-k} = \overline{C_k}$

$$\Rightarrow \quad C_{-k} = \int_0^1 \overline{f(t)}\, \overline{e^{-2\pi i k t}}\, dt$$

$$\Rightarrow \quad C_{-k} = \int_0^1 f(t) \cdot e^{2\pi i k t}\, dt$$

Lecture 3: Analysis of Periodic Phenomenon and How it is represented

Last time if $f(t)$ is periodic signal with period 1.

then $f(t) = \sum_{k=-n}^{n} C_k e^{2\pi i k t}$

We have not shown that every periodic function can be expressed this way.

Suppose we can write

$$f(t) = \sum_{k=-n}^{n} C_k e^{2\pi i k t}$$

$$C_k = \int_0^1 f(t)\, e^{-2\pi i k t}$$

if $f(t)$ is real then $C_{-k} = \overline{C_k}$

Define the co-efficients as

$$\hat{f}(k) = \int_0^1 e^{-2\pi i k t} f(t)\, dt$$

kth fourier co-efficient of f(t)

Step-1 :- Suppose if we can write function f(t) in this form, then the co-efficients have to given by

$$\hat{f}(k) = \int_0^1 e^{-2\pi i k t} f(t)\, dt$$

$\Uparrow$

Suppose we can write

$$f(t) = \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi i k t}$$

Step-2 turn that around and ask the following, when is that possible?

Given periodic signal f(t) period 1

Define $\hat{f}(k) = \int_0^1 e^{-2\pi i k t} f(t)\, dt$

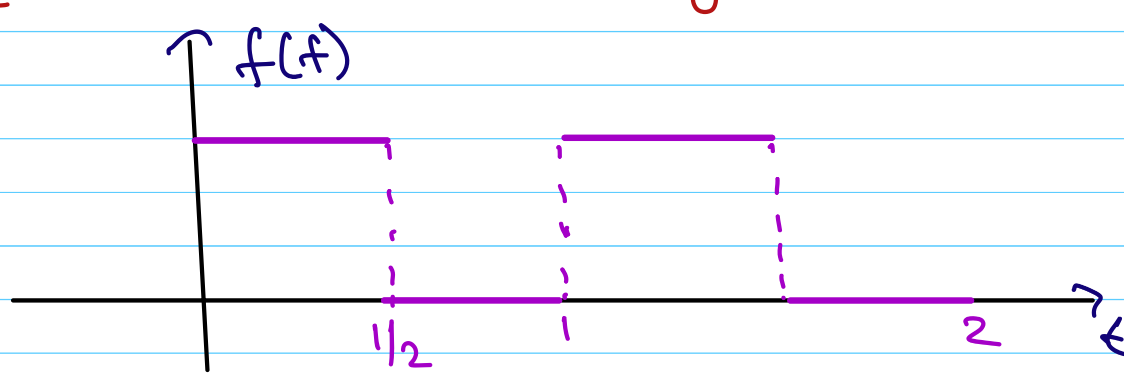
The Question is Can we write

$$f(t) = \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi i k t} \quad \text{for some } n.$$

How general we expect this to be if it works?

All this is fine, but does it really work? That is given a Periodic function can we expect to write it as a sum of exponentials in the way we have described? Let's look at an example.

Example: Square wave of Period 1.



$$f(t) = \begin{cases} +1 & 0 \le t < 1/2 \\ 0 & \frac{1}{2} \le t < 1 \end{cases}$$

Swith (on for 1/2 sec, off for 1/2 sec)

we can compute $\hat{f}(k)$ easily.

$$\hat{f}(k) = \int_0^1 e^{-2\pi i k t} f(t)\, dt = \int_0^{1/2} e^{-2\pi i k t}\, dt$$

$$= \left. \frac{e^{-2\pi i k t}}{-2\pi i k} \right]_0^{1/2}$$

$$\hat{f}(k) = \frac{1 - e^{-\pi i k}}{2\pi i k}$$

But can we write

$$f(t) = \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi i k t} \quad \text{for some } n$$

NO

NO, At least Not for finite sum

Reason:

⇒ Sin & cosine's are continuous

⇒ Sum of finite number of continuous function's is continuou's. It cannot possible represent discontinuous phenomenon.

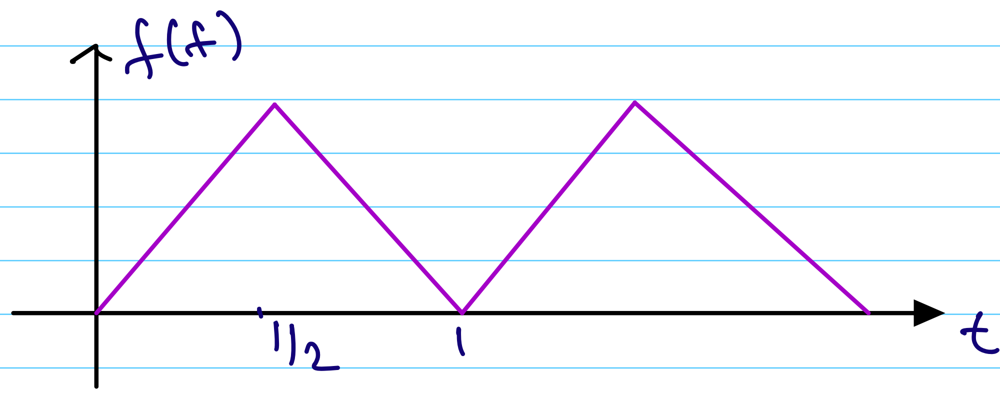
⇒ we cannot represent Discontinuous Phenomenon by a continuous Phenomenon

Theorem BACK in Calculus :-

Sum of 2 continuous function = Continuous.

what if $f(t)$ is continuous function, Can we represent with finite sin, cosine?

Example:



we can easily compute $\hat{f}(k)$, Fourier co-efficient's, But can we represent

$$f(t) = \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi i k t} \quad ?$$

NO, again, Not for finite sum

Reason:

$\Rightarrow$ if two function's are differentiable, then the sum of two function's is differentiable

$\Rightarrow$ $f(t)$ here is not differentiable

$\Rightarrow$ $\sin 2\pi k t$, $\cos 2\pi k t$ is differentiable.

* may be 1st, 2nd derivatives are fine, what if there is discontinuity in 3rd derivative and so on, so on.

* No matter how smooth is a corner, if there is some discontinuty in some high derivative, we are skewed.

* Any discontinuty in any derivative precludes writing

$$f(t) = \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi i k t}$$

So this great idea, we might quit as well, because it does not very general at all.

* if we cannot represent as finite sum then we have to turn to infinite sum's.

* $$f(t) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{+2\pi i k t}$$

MAXIMA :-

It takes high frequencies to make sharp corner or any corner's for that matter.

Any kind of discontinuity in high derivative that mean's that we are gonna trouble representing that phenomenon as finite sum, we are going to have to take n larger and larger to represent it to more accuratly and its takes higher freq to make that bend.

* To represent the general periodic signal's we have to consider infinite sum's.

$$f(t) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi i k t}$$

infinite sums.

How to deal with convergence of infinite sum's? (Later in the course)

Need conspiracy of cancellation's to make such a series convergence.

Jump discontinuity

if t_0 is at the Jump discontinuity then it converges to mid value.

$$f(t_0) = \frac{1}{2}\left(f(t_0^+) + f(t_0^-)\right)$$

* Fourier (scientist) really set his sight very high. that any Periodic Phenomenon could be represented by a fourier series (infinite sum)

General case:

we need, we learned after decades, centuries of bitter experience, we need a different notion of convergence

$\Rightarrow$ Learned not to ask for convergence of $\sum_{k=-\infty}^{\infty} c_k e^{2\pi i k t}$, at Particular Point, rather we need to ask convergence in the mean.

Convergence in mean

(Convergence in energy)

* Suppose $f(t)$ is Periodic with Period 1,

* Suppose we have another property, the integral is square integrable.

$$\int_0^1 |f(t)|^2 dt < \infty$$

(in some areas the integral of the square is identified with power)

energy of the signal

$$= \int_0^1 |f(t)|^2 dt$$

if energy $< \infty \Rightarrow$ finite energy

(Hypothesis of finite energy)

Lecture 4 :

Fourier Series finish:

want to make sense of infinite sum's of complex exponential's.

$$\sum_{k=-\infty}^{\infty} C_k e^{2\pi i k t}$$

* The Problem is general, How do you make sense of such an infinite sum's

* The tricky thing about it is that, if we think in term's of sine's and cosines , these function's are ossillating.

=) Sin's and cosines are ossilating b/w +ve and -ve , so for this thing to converge there got to be some sort of conspiracy of cancellation's that's making it work.

$\Rightarrow$ Ofcourse the size of the co-efficients play a role, But its more than that.

if $f(t)$ is Periodic with $T=1$

we want to write with some Confidence that it's equal to its fourier series.

$$f(t) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{2\pi i k t}$$

Any lack of smoothness forces an infinite sum.

where $$\hat{f}(k) = \int_0^1 e^{-2\pi i k t} f(t)\, dt$$

if $f(t)$ is continuous, smooth, then we get satisfactory convergence results.

Important condition: is Integrability.

Say $f(t)$ is square integrable

$f \in L^2([0,1])$ 2 - stands for square

L - Lebesgue

if $\int_0^1 |f(t)|^2 \, dt < \infty$ (finite energy condition)

where $|f(t)|$ = absolute value squared

$\Rightarrow |f(t)| = f(t)$ if $f(t)$ real function

* if $f(t)$ is periodic, square integrable, then we form fourier coefficients

$$\hat{f}(k) = \int_0^1 e^{-2\pi i k t} f(t) \, dt$$

we have

$$\int_0^1 \left| f(t) - \sum_{k=-n}^{n} \hat{f}(k) e^{2\pi i k t} \right|^2 dt \xrightarrow[n \to \infty]{} 0$$

$$\|f\| = \left(\int_0^1 |f(t)|^2 \, dt \right)^{1/2}$$

(Length of a function)

Properties of Norm:

① Positive definitness

$$\|f\| = 0$$

$$\Rightarrow \left(\int_0^1 |f(t)|^2 \, dt \right)^{1/2} = 0$$

$$\Rightarrow \quad f(t) = 0 \quad \forall t \in (0,1)$$

② Homogeneity

$$\|\alpha f\| = \|\alpha\| \, \|f\|$$

③ triangle inequality

$$\|f+g\| \leq \|f\| + \|g\|$$

$$\Rightarrow \left(\int_0^1 |f(t)+g(t)|^2 dt \right)^{1/2}$$

$$\leq \left(\int_0^1 |f(t)|^2 dt \right)^{1/2} + \left(\int_0^1 |g(t)|^2 dt \right)^{1/2}$$

Distance b/w function's

$$\|f-g\| = \left(\int_0^1 |f(t)-g(t)|^2 dt \right)^{1/2}$$

$\Rightarrow$ if $\|f-g\|=0 \Rightarrow f-g=0$

$\Rightarrow f=g$

Now get this:

The length of a vector is the square root of the sum of the squares of its components.

$\Rightarrow$ The norm defined by integral is the continuous analog of that, and so is the definition of distance.

$$\Rightarrow \int_0^1 |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$$

(will derive later in course)

* Fundamental Property of complex exponential's we used orginaly when we solved fourier co-efficients

* Remember in solving the fourier co-efficient we used

$$\int_0^1 e^{2\pi i n t} \cdot e^{-2\pi i m t} dt = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

* This simple calculus fact emerges, turn's out to be corner stone for understanding these spaces of square integral functions

* introducing geomentry into those space.

⟹ This simple fact is the cornerstone, for introducing "geometry" into $L^2([0,1])$

⟹ The thing that makes geometry geometry is the notion of Perpendicularity.

⟹ Allow's one to define orthogonality via inner product or dot product.

if f and g are square integrable on $(0,1)$, then their inner product (generalization of dot product for vectors)

$$\langle f, g \rangle = \int_0^1 f(t)\,\overline{g(t)}\,dt$$

Continuous, infinite dimentional generalization of the dot product of two vector's.

in Eucliedian vector spac $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$\langle \vec{a}, \vec{b} \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$

Similarly

$$\langle f, g \rangle = \int_0^1 f(t)\, \overline{g(t)}\, dt$$

multi at each time t b/w 0, 1

and then integrate (similar to sum in [a, b])

$\Rightarrow$ f, g are orthogonal if

$$\langle f, g \rangle = 0$$

$\Rightarrow$ Norm of a function (length) in terms of inner product

$$\|f\| = \sqrt{\langle f, f \rangle}$$

$$= \left(\int_0^1 f(t)\, \overline{f(t)}\, dt \right)^{1/2}$$

$$= \left(\int_0^1 |f(t)|^2\, dt \right)^{1/2}$$

The Pythagorem theorm is f is orthogonal to g if and only if

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2$$

$$\Rightarrow \int_0^1 (f+g)\overline{(f+g)}\,dt$$

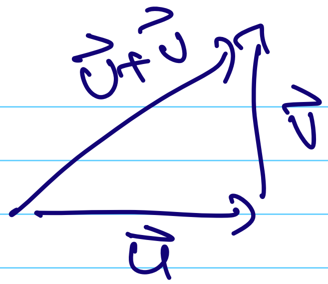
$$= \int_0^1 |f|^2 dt + \int_0^1 |g|^2 dt$$

$$\Rightarrow \cancel{\int_0^1 |f|^2 dt} + \cancel{\int_0^1 |g|^2 dt} + 2\int_0^1 f(t)\overline{g(t)}\,dt$$

$$= \cancel{\int_0^1 |f|^2 dt} + \cancel{\int_0^1 |g|^2 dt}$$

$$\Rightarrow \int_0^1 f(t)\overline{g(t)}\,dt = 0$$

$$\Rightarrow \langle f, g \rangle = 0$$



$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

* The trick is extending that vector's to function's, and reasoning by analogy, The analogy is very strong, in many our geometric intution for what happen's for vector's (we can draw) carrie over atleast algebraically to this more general situation.

* These complex exponential's are exactly orthogonal function's of length 1.

$$\langle e^{2\pi i n t}, e^{2\pi i m t} \rangle = \int_0^1 e^{2\pi i n t} \cdot \overline{e^{2\pi i m t}}\, dt$$

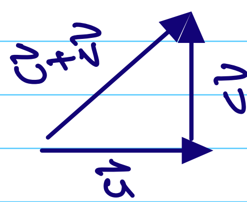
$$= \int_0^1 e^{2\pi i n t} \cdot e^{-2\pi i m t}\, dt = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

$\Rightarrow \quad \langle e^{2\pi i n t}, e^{2\pi i m t} \rangle = 0 \quad$ if $m \neq n$

(orthogonal)

$\Rightarrow \quad \langle e^{2\pi i n t}, e^{2\pi i n t} \rangle = 1 \quad (\text{norm} = 1)$

We can visualize what it mean's for vector's to be Perpendicular, we can draw this picture



But we cannot visualize what it mean's for complex exponential's to be orthogonal.

* The Fourier co-efficients are Projection's of the function on to these complex exponential's. Use inner product to define and comput Projection's

if $\|\vec{u}\| = 1 \quad \|\vec{v}\| = 1$ the $\langle \vec{u}, \vec{v} \rangle$ is Projection v with u

Fourier co-efficient is exactly a projection.

$$\langle f(t), e^{2\pi i n t} \rangle = \int_0^1 f(t) \cdot \overline{e^{2\pi i n t}}\, dt$$

$$= \int_0^1 f(t) \cdot e^{-2\pi i n t}\, dt$$

$$= \hat{f}(n)$$ n^{th} Fourier co-efficient

$\Rightarrow$ The n^{th} Fourier co-efficient is the exact projection of function onto the n^{th} complex exponential.

To write

$$f(t) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\, e^{2\pi i k t}$$ is to write

$$f(t) = \sum_{k=-\infty}^{\infty} \langle f(t), e^{2\pi i k t} \rangle\, e^{2\pi i k t}$$

To write this is to say, the complex exponential's

$$\left\{ e^{2\pi i k t} \right\}_{-\infty < k < \infty}$$ form

orthonormal BASIS for these square integrable periodic function's.

* wavelets $\rightarrow$ different kind of BASIS

Rayleigh's Identity:

$$\int_0^1 |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$$

Proof:

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n t}$$

$$= \sum_{n=-\infty}^{\infty} \langle f(t), e^{2\pi i n t} \rangle e^{2\pi i n t}$$

Then

$$\int_0^1 |f(t)|^2 dt = \|f\|^2 = \langle f, f \rangle$$

$$= \left\langle \sum_{n=-\infty}^{\infty} \langle f(t), e^{2\pi i n t} \rangle e^{2\pi i n t}, \sum_{m=-\infty}^{\infty} \langle f(t), e^{2\pi i m t} \rangle e^{2\pi i m t} \right\rangle$$

$$= \sum_{n,m} \langle f(t), e^{2\pi i n t} \rangle \overline{\langle f(t), e^{2\pi i m t} \rangle} \, e^{2\pi i n t} \cdot \overline{e^{2\pi i m t}}$$

$$= \sum_{n,m} \langle f(t), e^{2\pi i n t} \rangle \overline{\langle f(t), e^{2\pi i m t} \rangle} \delta_{nm}$$

$$= \sum_{n=-\infty}^{\infty} \langle f(t), e^{2\pi i n t} \rangle \overline{\langle f(t), e^{2\pi i n t} \rangle}$$

$$= \sum_{n=-\infty}^{\infty} \left| \langle f, e^{2\pi i n x} \rangle \right|^2$$

$$= \sum_{n=-\infty}^{\infty} \left| \hat{f}(n) \right|^2$$