Lec 02: How can we use such simple function's $\sin(t)$ and $\cos(t)$ to model such complex periodic phenomenum.

we are essentially identifying the subject of Fourier Series with the study of mathematical periodicity.

* we made a distinction, between periodicity in time and periodicity in space.

* mathematical approach to periodicity is possible because there are very simple mathematical function's that exhibit periodic behaviour, namely $\sin t$, $\cos t$. But that's also the problem because, periodic

Phenomenum can be very general, very complicated, sin, cos are so simple.

So How can you really expect to use sin & cosine to model very general periodic phenomenum.

How can we use such simple function's $\sin t$, $\cos t$ to model complex periodic phenomenum?

* 1st General mark: How high should we aim here, How general can we expect this to be? That's really fundamental question here.

(we are really aiming quite high here)

we are really hoping to apply these ideas in quite general circumstances.

* Not all phenomenon are periodic.
* And even in the case of periodic phenomenun it may not be realistic assumption, I think its important to realize that what the limits may be, or how far the limits can be pushed.

Not all phenomenon are naturally periodic, although many interesting ones are periodic.

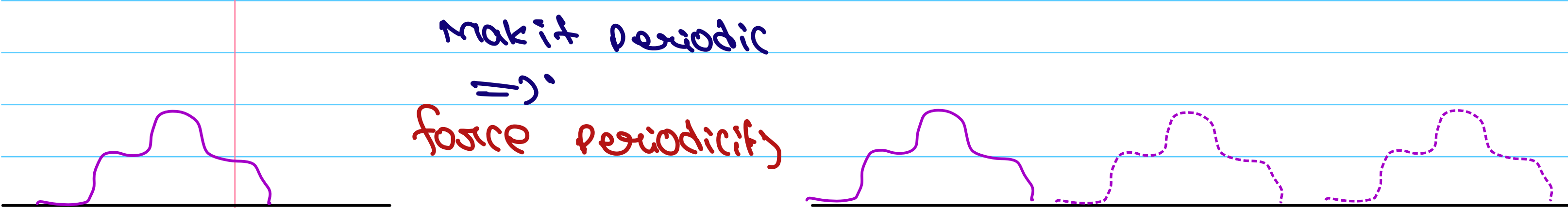
Even for periodic phenomena in time, the real phenomenum die out eventually, we only observe something will refine refine period of time, where as sin, cos go on forever.

So How can they be really usefol to

model something that dies out?

we can still apply the ideas of periodicity with some extra assumptions.

suppose the function is




Force periodicity by repeating the pattern. (periodization of the signal)

Used to study signal's which are non-periodic too in term's of method's of Fourier series.

so, the study can be pretty general.

For the discussion "Fix the Period"

Natural choice would be $2\pi$, But its convenient to fix the Period to "1" (we use functions with Period 1)

$\Rightarrow$ $f(t) = f(t+1) \quad \forall t$

$f(t) = f(t+n) \quad \forall n = \pm 1, \pm 2, \ldots$

Model signal's

$$\left.\begin{array}{l}\sin 2\pi t \\ \cos 2\pi t\end{array}\right\} \text{Period } 1$$

Simple enough.

* if we know what the function is on a interval of length 1, or any interval of length 1, we know it

everywhere, Because the Pattern repeat's.

we can modify & combine, $\sin 2\pi t$, $\cos 2\pi t$, to model very general Periodic signal's of Period 1.

## Big Idea:

"one Period , many frequencies"

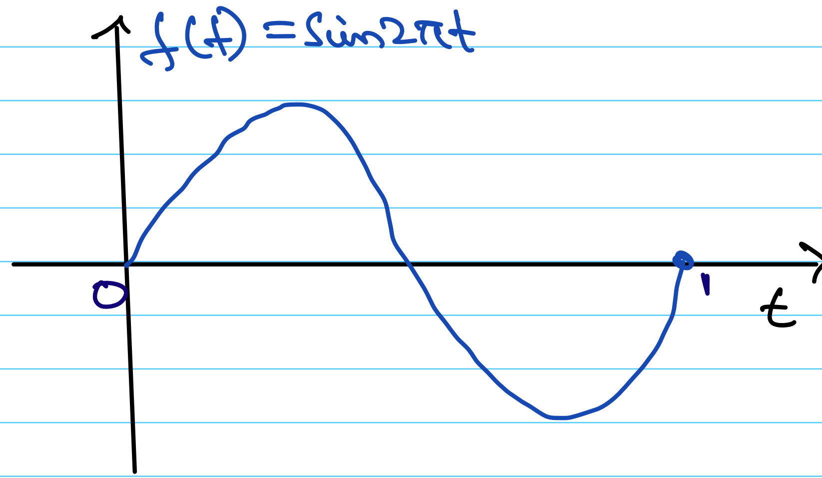
↳ in mean's

**Ex:**

① $f(t) = \sin 2\pi t$

Period := 1 , 2, 3, . . .

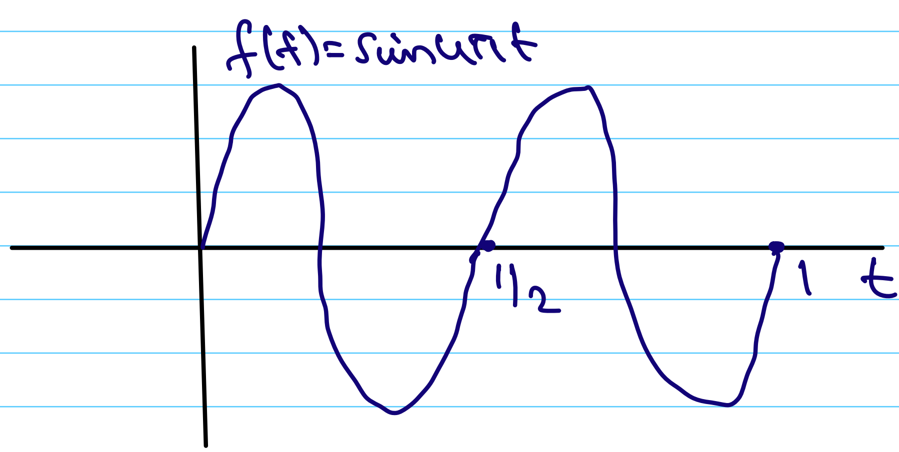
freq := 1




② $f(t) = \sin 4\pi t$

Period := 1/2 , 1 , 1 1/2 , 2 . .

freq : 2



Period 1/2 also mean's Period 1

$\Rightarrow$ Because if we consider entire signal Until time 1 as Base signal, then that pattern repeats for every sec.

Shoter Period = 1/2

longer Period = 1

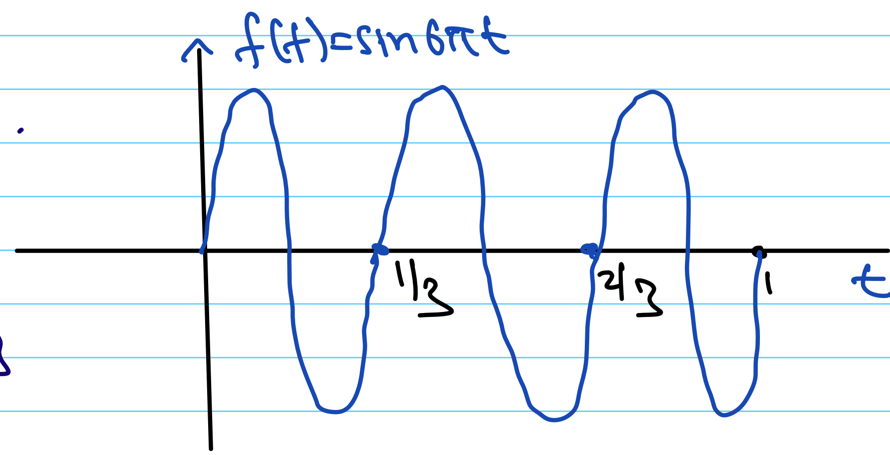
(3) $f(t) = \sin 6\pi t$

Period: 1/3, 2/3, 1 ..

freq: 3,

Short Period = 1/3

long Period = 1




One Period, Many frequencies.

$\sin 2\pi t \Rightarrow$ T = 1, 2, 3, .. f = 1

$\sin 4\pi t \Rightarrow$ T = 1/2, 1, 1 1/2, 2, 2 1/2, .. f = 2

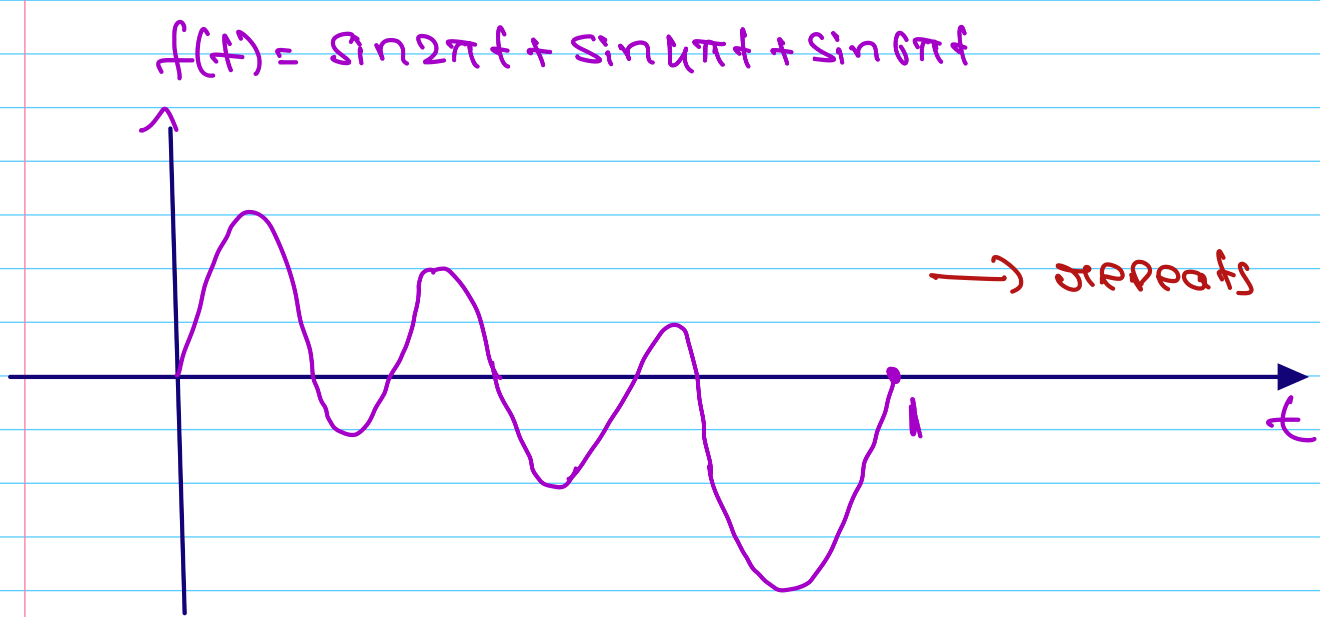
$\sin 6\pi t \Rightarrow$ T = $\frac{1}{3}, \frac{2}{3}, 1, 1\frac{1}{3}, 1\frac{2}{3}, 2, ..$ f = 3

$\sin 8\pi t \Rightarrow$ T = 1/4, 2/4, 3/4, 1, $1\frac{1}{4}$, .. f = 4 for all these signal's $T=1$ is common.

Even though each signal has its own fundamental period $(1, 1/2, \frac{1}{3}, ..)$ all of them repeat perfectly within $T=1$

* if we combine them together, what about the combination

Ex: $\sin 2\pi t + \sin 4\pi t + \sin 6\pi t$




The period is still $T=1$

* Because although the term's of higher freq are repeating more rapidly, the sum cannot go back to where it started until the slowest one get's caught up.

(one period, many frequencies)

There are 3 frequencies in the sum, (1, 2, 3) but added together there is only one period.

* So for complicated signal's, it's really better, more revealing to talk in term's of frequencie that might go into it rather than the period alone. we are fixing the period of length 1. But it might have

complicated phenomenum, is gonna be
Built up out of sin, cos of varying
freq.

Complicated signal with $T=1$

$\Rightarrow$ freq $= 1, 2, 3, \dots \infty$

$\Rightarrow$ we can modify amplitudes, phase
of each one of those term's.

$$\sum_{k=1}^{N} A_k \sin(2\pi k t + \varphi_k)$$

general kind of sum we
can form out of just sines,

Again, many frequencies, one period
The lowest (longest) period in the
sum is when $k=1$, the higher
term's called harmonics, have shorter
Period, But the sum have period 1

Because the whole pattern cannot repeat until the longest period repeat's, until the longest pattern completed.

## Different way's of writing this sum

$$\sum_{k=1}^{N} A_k \sin(2\pi kt + \varphi_k)$$

$$= \sum_{k=1}^{N} A_k \left( \sin 2\pi kt \cos\varphi_k + \cos 2\pi kt \sin\varphi_k \right)$$

$$= \sum_{k=1}^{N} (A_k \cos\varphi_k) \sin 2\pi kt + (A_k \sin\varphi_k) \cos 2\pi kt$$

$$\Rightarrow \sum_{k=1}^{N} a_k \cos 2\pi kt + b_k \sin 2\pi kt$$

we can also allow constant term to shift the signal

$$\frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos 2\pi k t + b_k \sin 2\pi k t$$

$\Uparrow$

This is very common way to writing the form of the sum. But by far the most convenient way algebraically and really many way's conceptually is to use complex exponential's to write the sum, Not the real cosine, sines, by far.

By far the Better Representation to represent sin, cos is via complex exponential's.

$$\boxed{e^{2\pi i k t} = \cos 2\pi k t + i \sin 2\pi k t}$$

$i = \sqrt{-1}$

$$\cos 2\pi kt = \frac{e^{2\pi ikt} + e^{-2\pi ikt}}{2}$$

$$\sin 2\pi kt = \frac{e^{2\pi ikt} - e^{-2\pi ikt}}{2i}$$

we can convert a trignometric sum before to the form

$$\sum_{k=-N}^{N} C_k e^{2\pi ikt}$$ where $C_k$ = complex.

$$\frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos 2\pi kt + b_k \sin 2\pi kt$$

$$= \underbrace{\sum_{k=-N}^{N} C_k e^{2\pi ikt}}_{\text{total sum is real}}$$

But

$C_k$ is complex

$e^{2\pi ikt}$ is complex.

Proof:

$$c_0 = \frac{a_0}{2} \quad \text{— } 1$$

$$\frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos 2\pi k t + b_k \sin 2\pi k t$$

$$= \sum_{k=-N}^{N} c_k e^{2\pi i k t}$$

$$\frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos 2\pi k t + b_k \sin 2\pi k t$$

$$= \sum_{k=-N}^{N} c_k \left(\cos 2\pi k t + i \sin 2\pi k t\right)$$

Using cosine co-efficient

$$a_k \cos 2\pi k t = c_k \cos 2\pi k t$$
$$+ c_{-k} \cos 2\pi - k t$$

$$\Rightarrow a_k \cos 2\pi k t = \left(c_k + c_{-k}\right) \cos 2\pi k t$$

$$\Rightarrow a_k = c_k + c_{-k} \quad \text{— } 2$$

Simillarly

$$b_k \sin 2\pi k t = i\left(c_k \sin 2\pi k t\right.$$
$$\left. - c_{-k} \sin 2\pi k t\right)$$

$$\Rightarrow \quad b_k = i(c_k - c_{-k}) \quad - (3)$$

$$\Rightarrow \quad c_k + c_{-k} = a_k$$

$$c_k - c_{-k} = -i b_k$$

$$\Rightarrow$$

$$c_0 = \frac{a_0}{2}$$

$$c_k = \frac{a_k - i b_k}{2}$$

$$c_{-k} = \frac{a_k + i b_k}{2}$$

Symmetric property

$$c_k = \overline{c_{-k}} \quad \text{(for real signal)}$$

$a_k \cos 2\pi k t + b_k \sin 2\pi k t$

$$= \left(\frac{a_k + i b_k}{2}\right) e^{-2\pi i k t} + \left(\frac{a_k - i b_k}{2}\right) e^{2\pi i k t}$$

Because of symmetric property the total sum is real.

* This is very important identity that's satisfies by the co-efficient for a real signal.

we are ready at last at least ask the question, that's really heart of all of this, How general can this be?

Now we are in form now, algebraically writing sum's of $\sum_{k=-N}^{N} c_k e^{2\pi i k t}$ is for the easist way to approach.

Can now ask fundamental question?

$f(t)$ is periodic function of period 1, can we write it as a $f(t) = \sum_{k=-N}^{N} c_k e^{2\pi i k t}$ trigonometric

sum? (Next lecture)

Linearity play's a role here, we are considering linear combination's of the basic building blocks.

Secret of the universe:

When we try to apply math to various problem's, we often hae a question like this, is it possible to write something like this?

Often the 1st step is suppose we can? and see what the consequence are, and later on we go backward's.

1st step:- suppose the problem is solved, what has to happen?

Suppose we can write

$$f(t) = \sum_{k=-N}^{N} C_k e^{2\pi i k t}$$

what has to happen?

then what are the mystery co-eff $C_k$ in term's of $f(t)$.

Unknown's here is co-efficients $C_k$

suppose if we can write it like this, can we solve for the co-efficient's?

$$f(t) = \ldots + C_m e^{2\pi i m t} + \ldots$$

$$C_m e^{2\pi i m t} = f(t) - \sum_{k \neq m} C_k e^{2\pi i k t}$$

$$\Rightarrow \quad c_m = f(t)\, e^{-2\pi i m t} - \sum_{k \neq m} c_k e^{2\pi i k t} e^{-2\pi i m t}$$

$$\Rightarrow \quad c_m = f(t)\, e^{-2\pi i m t} - \sum_{k \neq m} c_k e^{2\pi i (k-m) t}$$

$$\int_0^1 c_m = \int_0^1 e^{-2\pi i m t} f(t) - \sum_{k \neq m} c_k \int_0^1 e^{2\pi i (k-m) t}\, dt$$

$$\int_0^1 e^{2\pi i (k-m) t}\, dt \qquad \text{when } k \neq m$$

$$= \left. \frac{e^{2\pi i (k-m) t}}{2\pi i (k-m)} \right]_0^1$$

$$= \frac{1}{2\pi i (k-m)} \left( e^{2\pi i (k-m) t} - e^0 \right)$$

$$= \frac{1}{2\pi i(k-m)} (1-1) = 0$$

$$\Rightarrow \quad C_m = \int_0^1 e^{-2\pi i m t} f(t)\,dt + 0$$

$$\Rightarrow \quad \boxed{C_m = \int_0^1 e^{-2\pi i m t} f(t)\,dt}$$

Given $f(t)$ Period 1, suppose if we can write

$$f(t) = \sum_{k=-N}^{N} C_k e^{2\pi i k t} \quad \text{then}$$

$$C_k = \int_0^1 e^{-2\pi i k t} f(t)\,dt$$

Next time, we will ask, suppose we are given these co-efficient's by the formula do we have something like

$$f(t) = \sum_{k=-N}^{N} c_k e^{2\pi i k t}$$ , and

in what sense.