Lec 07

Fourier transform and its inverse

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\, dt$$

$$\mathcal{F}^{-1}f(t) = \int_{-\infty}^{\infty} e^{2\pi i s t} f(s)\, ds$$

Question is existence of integral (convergence of integral) (more on this later)

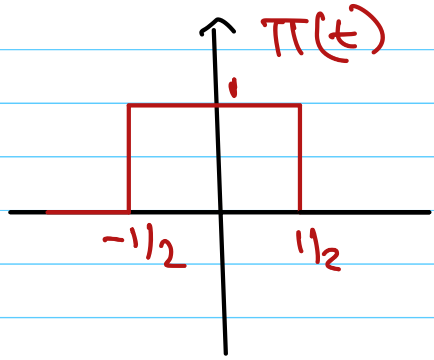
* The set $S \in \mathbb{R}$ for which fourier transform is defined (exists) is called spectrum.

Every signal has a spectrum and is determined by its spectrum. we can analyze the signal either in the time (or spatial) domain or in the frequency domain.

Examples:

① $f(t) = \Pi(t)$ $\qquad\Longleftrightarrow\qquad$ $F(s) = \text{Sinc}\, S$

$$= \frac{\sin \pi s}{\pi s}$$



② $f(t) = \Lambda(t)$ $\qquad$ $\mathcal{F} f(s) = \text{sinc}^2 s$

$$= \begin{cases} 1-|t| & t \le 1 \\ 0 & \text{o.w} \end{cases} \quad\Longleftrightarrow\quad = \left(\frac{\sin \pi s}{\pi s}\right)^2$$

③

Gaussian curve

$f(x) = e^{-x^2}$ $\qquad$ $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} = I$ (assume)

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy$$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

$x = r\cos\theta$

$y = r\sin\theta$

$$\Rightarrow I^2 = \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} e^{-r^2}\, r\,dr\,d\theta \qquad dx\,dy = r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} -\frac{e^{-r^2}}{2}\Big]_0^{\infty} = \frac{2\pi}{2} = \pi$$

$$\Rightarrow I^2 = \pi \Rightarrow I = \sqrt{\pi}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Fourier transform of Gaussian

Gaussian is an important function in many applications and it has remarkable property, with regard to F.T

$$f(x) = e^{-\pi x^2} \qquad \int_{-\infty}^{\infty} f(x) = 1$$

Somehow the Gaussian is equally spreadout in time domain & freq domain.

The Fourier Transform of Gaussian is itself.

$$f(t) = e^{-\pi t^2}$$

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\, dt$$

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-2\pi i s t}\, dt$$

$$\frac{d}{ds}\mathcal{F}f(s) = \frac{d}{ds}\int_{-\infty}^{\infty} e^{-\pi t^2} e^{-2\pi i s t}\,dt$$

$$= \int_{-\infty}^{\infty} e^{-\pi t^2}(-2\pi i t)\,e^{-2\pi i s t}\,dt$$

$$= i \cdot \int_{-\infty}^{\infty} e^{-2\pi i s t}\cdot\left(-2\pi t e^{-\pi t^2}\right)dt$$

$$= i \int_{-\infty}^{\infty} e^{-2\pi i s t}\,d\left(e^{-\pi t^2}\right)$$

$$\int_a^b u\,dv = uv\Big]_a^b - \int_a^b v\,du$$

$$v = e^{-\pi t^2} \qquad u = e^{-2\pi i s t}$$

$$du = -2\pi i s\, e^{-2\pi i s t}\,dt$$

$$= i\left[\underbrace{\left[e^{-\pi t^2} e^{-2\pi i s t}\right]_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} e^{-\pi t^2}(-2\pi i s)\,e^{-2\pi i s t}\,dt\right.$$

$$\Rightarrow \frac{d}{ds} F(s) = -2\pi s \int_{-\infty}^{\infty} e^{-\pi t^2} \cdot e^{-2\pi i s t} \, dt$$

$$\Rightarrow \frac{d}{ds} F(s) = -2\pi s F(s)$$

(we know $\frac{dx}{dt} = ax \Rightarrow x(t) = x(0) \cdot e^{at}$)

$$F(s) = F(0) \, e^{-\pi s^2}$$

$$\Downarrow$$

$$1$$

$$F(s) = e^{-\pi s^2}$$

$$f(t) = e^{-\pi t^2} \Longleftrightarrow F(s) = e^{-\pi s^2}$$

General properties of F.T

(1) Fourier transform duality

Exploit the simillarities in the formula's for the Fourier transform and its inverse.

* The simillarities b/w $\mathcal{F}$ & $\mathcal{F}^{-1}$

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\,dt$$

* The F.T is an operation that turn's one function into another function. To evaluate the Transform, we have to evaluta at a variable (s)

$$\mathcal{F}f(-s) = \int_{-\infty}^{\infty} e^{2\pi i s t} f(t)\,dt$$

$\Rightarrow \quad \mathcal{F} f(-s) = \mathcal{F}^{-1} f(s)$

* F.T is an Formula, an operation that turn's one function into another function.
* To write down the formula, we have to evalual the operation at a variable.

write more neatly. introduce reverse signal

if $f(t)$ is a signal then define

$$f^{-}(t) = f(-t) \quad \left(\text{flipping along origin}\right)$$

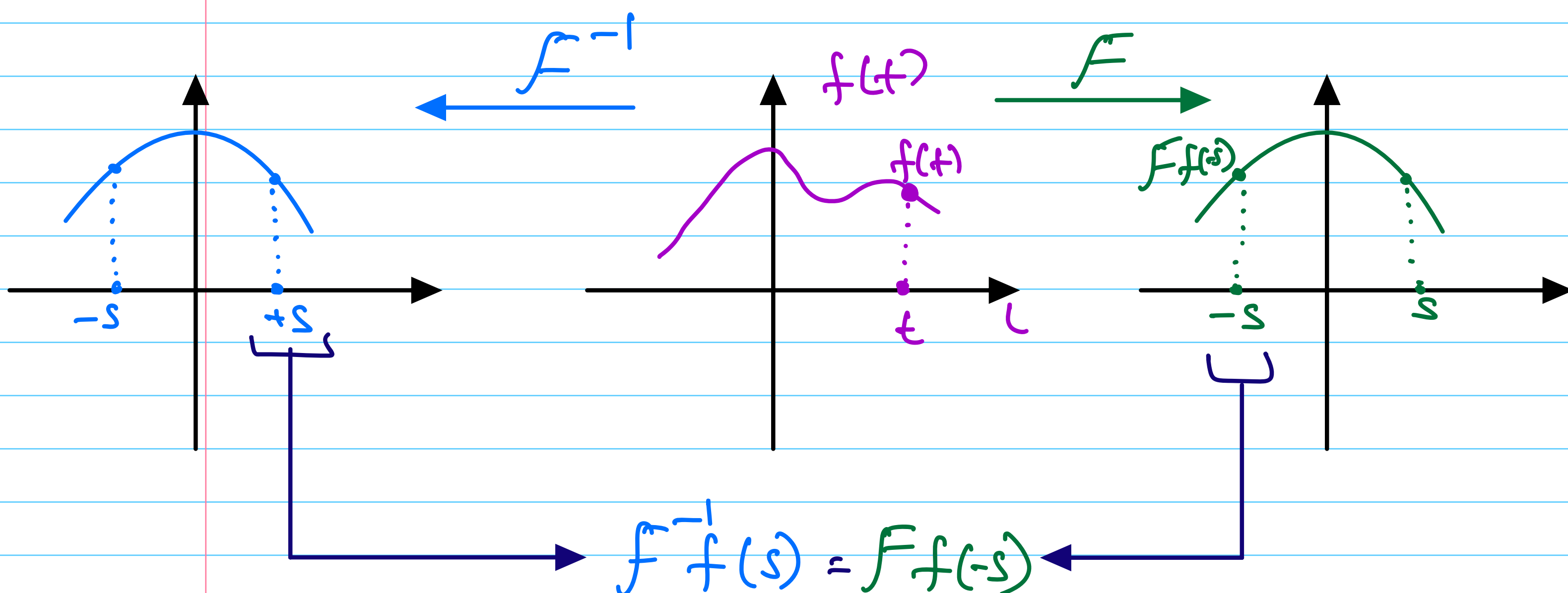
f is even $\Longrightarrow f^{-}(t) = f(t)$

f is odd $\Rightarrow f^{-}(t) = -f(t)$

① $\mathcal{F}f(-s) = (\mathcal{F}f)(-s) = \int_{-\infty}^{\infty} e^{-2\pi i(-s)t} f(t)\, dt$

$$= \int_{-\infty}^{\infty} e^{2\pi i s t} f(t)\, dt$$

$$= \mathcal{F}^{-1}f(s) = (\mathcal{F}^{-1}f)(s)$$



$$(\mathcal{F}f)(-s) = (\mathcal{F}^{-1}f)(s)$$

$$\Rightarrow \quad (\mathcal{F}f)^{-}(s) = (\mathcal{F}^{-1}f)(s)$$

$$\Rightarrow \quad (\mathcal{F}f)^{-} = \mathcal{F}^{-1}f \quad \text{—} \quad (1)$$

$$\Downarrow$$

Take the Fourier transform of function f and flip it around the origin $\iff$ take inverse fourier Transform of function f

$\Rightarrow$ The reverse of the F.T of f is the inverse fourier transform

F.T of f^{-} (F.T of reverse signal)

$$\mathcal{F}f^{-}(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f^{-}(t)\, dt$$

$$= \int_{-\infty}^{\infty} e^{-2\pi i s t} f(-t)\, dt$$

$u = -t \Rightarrow du = -dt$

$$= -\int_{\infty}^{-\infty} e^{2\pi i s u} f(u)\, du$$

$$= \int_{-\infty}^{+\infty} e^{2\pi i s u} f(u)\, du$$

$$= \mathcal{F}^{-1} f(s)$$

$$(\mathcal{F} f^{-})(s) = (\mathcal{F}^{-1} f)(s)$$

$$\Rightarrow \quad \mathcal{F} f^{-} = \mathcal{F}^{-1} f \quad -\ (2)$$

Fourier transform of reverse signal $f^{-}(t) \Longleftrightarrow$ inverse fourier transform of signal f

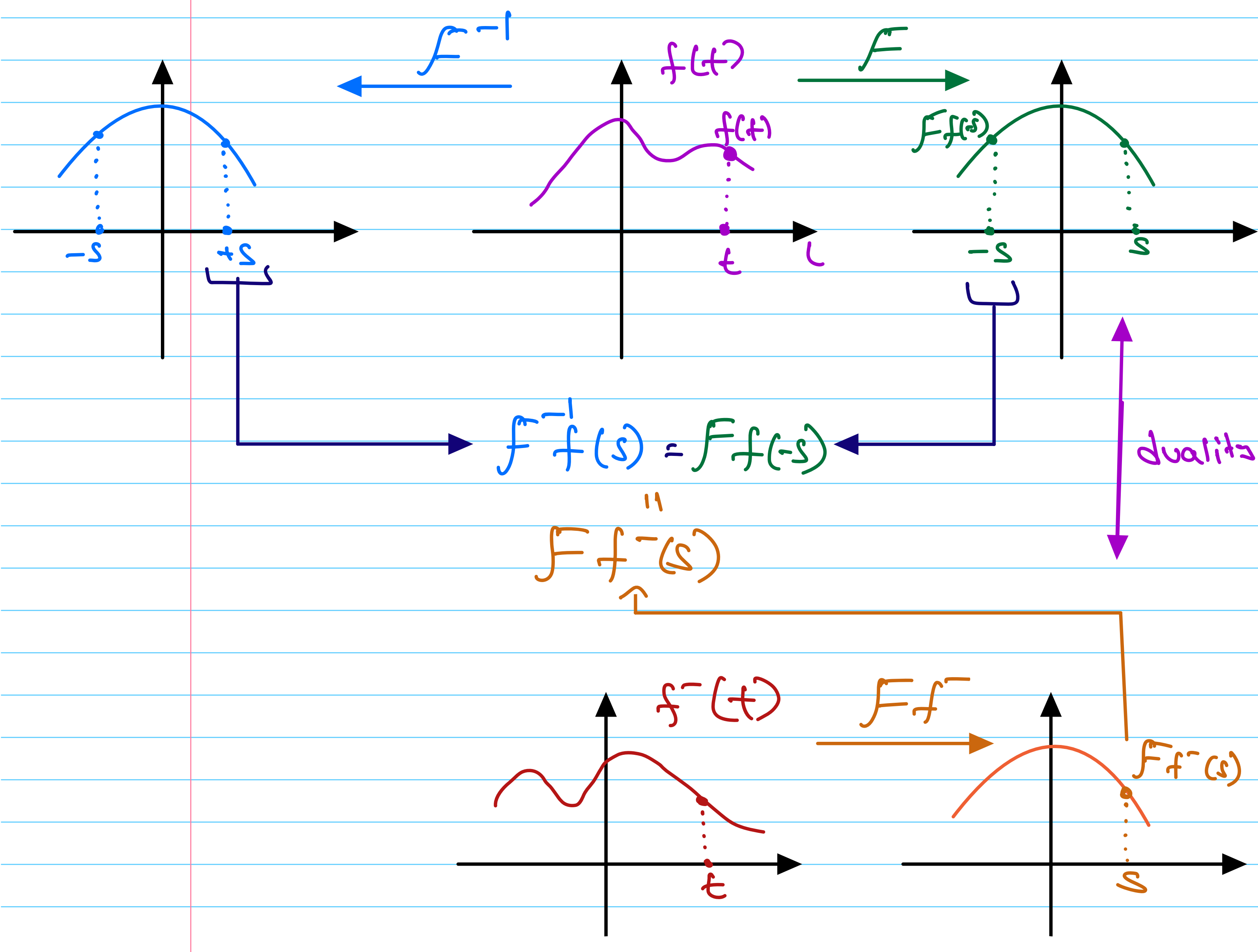
from (1) & (2) we can write

$$\mathcal{F} f^{-} = (\mathcal{F} f)^{-}$$

⟹ The F.T of reverse signal $f(t)$

⟺ reverse of Fourier transform of signal.

$$(\mathcal{F}f)^{-} = \mathcal{F}^{-1}f = \mathcal{F}f^{-}$$



duality:

The Fourier Transform reversed signal = inverse Fourier transform.

+ The Fourier Transform of reverse signal is the reverse of Fourier Transform.

one more duality

$$\mathcal{F}\mathcal{F}f = f^{-}$$

derive this.

$$\mathcal{F}f(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\, dt$$

$$\mathcal{F}\mathcal{F}f(r) = \int_{-\infty}^{\infty} e^{-2\pi i s r} \int_{-\infty}^{\infty} e^{-2\pi i s t} f(t)\, dt\, ds$$

Example:

① $\mathcal{F}\,\mathrm{sinc} = ?$

Sol^n we know $\mathcal{F}\,\Pi = \mathrm{sinc}$

$\Rightarrow \mathcal{F}\mathcal{F}\,\Pi = \mathcal{F}\,\mathrm{sinc}$

$\Rightarrow \Pi^{-}(\cdot) = (\mathcal{F}\,\mathrm{sinc})(\cdot)$

But rectangular function is even

$\Rightarrow \mathcal{F}\,\mathrm{sinc} = \Pi$

* Same duality argument gives

$$\mathcal{F}(\mathrm{sinc})^2 = \Lambda(\cdot)$$