This document contain's solution's to Exercises 7.1 to 7.15 from chapter 6, of Statistical inference by Casella and Berger, focusing on METHOD of MOMENTS & MLE Estimator's

Some problem's are still in Progress, and the document will be updated with their solution's soon.

7.1 One observation is taken on a discrete random variable X with pmf $f(x|\theta)$, where $\theta \in \{1,2,3\}$. Find the MLE of θ

x	$f(x\|1)$	$f(x\|2)$	$f(x\|3)$
0	1/3	1/4	0
1	1/3	1/4	0
2	0	1/4	1/4
3	1/6	1/4	1/2
4	1/6	0	1/4

Sol^n

$\Theta = \{1,2,3\}$

if $x_1 = 0$ then $\hat{\theta} = 1$

$x_1 = 1$ then $\hat{\theta} = 1$

$x_1 = 2$ then $\hat{\theta} = 2$ or 3

$x_1 = 3$ then $\hat{\theta} = 3$

$x_1 = 4$ then $\hat{\theta} = 3$

7.2 Let $X_1, X_2, \ldots X_n$ be a random sample from a gamma (α, β) population.

(a) find the MLE of β, assuming α is known

$$f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\beta}}$$

$\alpha > 0$

$\beta > 0$

$0 < x < \infty$

$$L(\beta|x,\alpha) = f_n(X|\alpha,\beta) = \left(\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\right)^n \left(\prod x_i\right)^{\alpha-1} e^{-\frac{1}{\beta}\sum x_i}$$

$$\ell(\beta|x,\alpha) = -n\log\Gamma(\alpha) - n\alpha\log\beta + (\alpha-1)\sum\log x_i - \frac{1}{\beta}\sum x_i = 0$$

$$\hat{\beta} = \underset{\beta}{\operatorname{argmax}}\ \ell(\beta|x,\alpha)$$

$$= \underset{\beta}{\operatorname{argmax}}\ -n\log\Gamma(\alpha) - n\alpha\log\beta + (\alpha-1)\sum\log x_i - \frac{1}{\beta}\sum x_i$$

$$\frac{\partial l(\beta|x,\alpha)}{\partial \beta} = 0 - \frac{n\alpha}{\beta} + \frac{\sum x_i}{\beta^2} = 0$$

$$\Rightarrow \quad \frac{n\alpha}{\beta} = \frac{\sum x_i}{\beta^2}$$

$$\Rightarrow \quad \boxed{\hat{\beta} = \frac{\sum x_i}{n\alpha}}$$

need to check whether it is maxima or minima

$$\frac{\partial^2 l(\beta|x,\alpha)}{\partial \beta} = +\frac{n\alpha}{\hat{\beta}^2} - 2\frac{\sum x_i}{\hat{\beta}^3}$$

$$= \frac{n\alpha}{\left(\frac{\sum x_i}{n\alpha}\right)^2} - 2\frac{\sum x_i}{\left(\frac{\sum x_i}{n\alpha}\right)^3}$$

$$= \frac{(n\alpha)^3}{(\sum x_i)^2} - 2\frac{(n\alpha)^3}{(\sum x_i)^2}$$

$$= -\frac{(n\alpha)^3}{(\sum x_i)^2} < 0$$

Hence $\hat{\beta}_{MLE} = \dfrac{\sum x_i}{n\alpha}$

7.2 (b) if α and

will come back again

(7.3) Given a random sample $X_1, X_2, .. X_n$ from a population with pdf $f(x|\theta)$, show that maximizing the likelihood function $L(\theta|x)$ as a function of θ is equivalent to maximizing $\log L(\theta|x)$

Sol^n

monotone increasing function $g(x)$ preserve's the location of the maximum of a function $f(x)$ because it does not change the ordering of values.

That is if $x_1 > x_2$, then $g(x_1) > g(x_2)$ ensuring that the point maximizing $f(x)$

also maximizes $g(f(x))$

here log is a monotone increasing function

$\Rightarrow$ if $L(\theta_1|x) > L(\theta_2|x)$

then $\log L(\theta_1|x) > \log L(\theta_2|x)$

$$\Rightarrow \arg\max_{\theta} L(\theta|x) = \arg\max_{\theta} \log L(\theta|x)$$

76 Let $X_1, X_2, \ldots, X_n$ be a random sample from the pdf

$$f(x|\theta) = \theta x^{-2} \qquad 0 < \theta \le x < \infty$$

(a) What is a sufficient statistic for θ?

Sol^n

$$f(x|\theta) = \begin{cases} \theta \cdot x^{-2} & \text{if } 0 < \theta \le x < \infty \\ 0 & \text{o.w} \end{cases}$$

$$f_n(x|\theta) = \begin{cases} \dfrac{\theta^n}{\prod_{i=1}^{n} x_i^2} & \text{if } 0 < \theta \le x_i < \infty \ \forall i \\ 0 & \text{o.w} \end{cases}$$

$$\Rightarrow \quad f_n(x|\theta) = \frac{\theta^n}{\prod_{i=1}^{n} x_i^2} \, I_{(\theta,\infty)}(\min x_i)$$

$$= \underbrace{\frac{1}{\prod_{i=1}^{n} x_i^2}}_{h(x)} \; \underbrace{\theta^n I_{(\theta,\infty)}(\min x_i)}_{g(T(x)\,|\,\theta)}$$

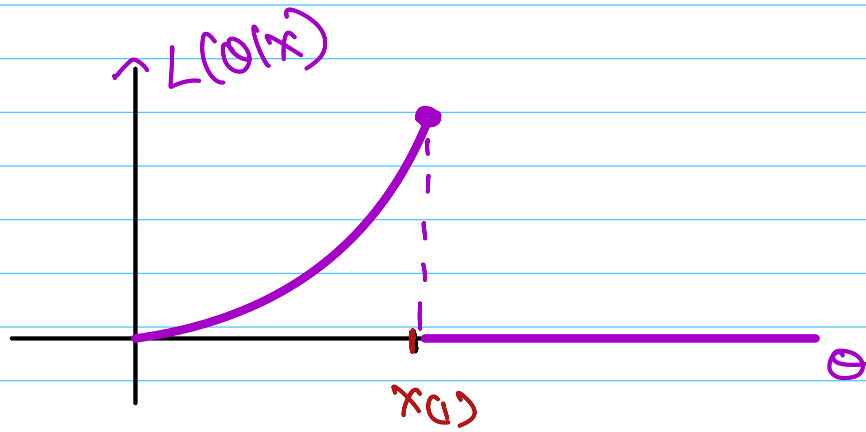
Hence $T(x) = \min x_i = x_{(1)}$

(b) find the MLE of θ.

$$L(\theta|x) = f_n(x|\theta) = \frac{\theta^n}{\prod_{i=1}^{n} x_i^2} \, I_{(\theta,\infty)}(\min_i x_i)$$

$$= 0 \qquad \text{o.w}$$

$$L(\theta|x) = \begin{cases} \dfrac{\theta^n}{\prod_{i=1}^{n} x_i^2} & \text{if } \theta \leq x_{(1)} \\ 0 & \text{if } \theta > x_{(1)} \end{cases}$$



$$\hat{\theta}_{MLE} = x_{(1)}$$

(c) Find the Method of Moment's estimator of θ.

$$\mathbb{E}[X] = \int_{\theta}^{\infty} x \cdot \frac{\theta}{x^2} \, dx = \int_{\theta}^{\infty} \frac{\theta}{x} \, dx$$

$$= \theta \ln|x| \Big]_{\theta}^{\infty}$$

$$\mathbb{E}[X] = \infty \qquad = \infty$$

Method of Moment's Estimator does not exist's.

7.7 let $x_1, x_2, \dots x_n$ be iid with one of two Pdf's. if $\theta = 0$, then

$$f(x|\theta) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{o.w} \end{cases}$$

While if $\theta = 1$, then

$$f(x|\theta) = \begin{cases} \frac{1}{2\sqrt{x}} & \text{if } 0 < x < 1 \\ 0 & \text{o.w} \end{cases}$$

Find the MLE of θ

sol'n

$$f(x|\theta) = \left(\frac{1}{2\sqrt{x}}\right)^{\theta} \quad \text{if} \quad 0 < x < 1$$

$$L(\theta|x) = f_n(x|\theta)$$

$$= \prod_{i=1}^{n} \left(\frac{1}{4x_i}\right)^{\frac{\theta}{2}} = \left(\frac{1}{4^n \prod x_i}\right)^{\frac{\theta}{2}}$$

$$\Rightarrow \quad L(\theta=0|x) = 1$$

$$L(\theta=1|x) = \left(\frac{1}{4^n \prod x_i}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2^n} \cdot \frac{1}{(\prod x_i)^{1/2}}$$

if $\frac{1}{2^n} \cdot \frac{1}{(\prod x_i)^{1/2}} < 1$

then $\hat{\theta}_{MLE} = 0$

if $\frac{1}{2^n} \cdot \frac{1}{(\prod x_i)^{1/2}} > 1$

then $\hat{\theta}_{MLE} = 1$

7.8 One observation, X, is taken from a $n(0,\sigma^2)$ population.

(a) Find an unbaised estimator of σ^2

$$L(\sigma^2|x) = f(x|\sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{-\frac{1}{2\sigma^2} x^2\right\}$$

$$\Rightarrow \log L(\sigma^2|x) = -\frac{1}{2}\log 2\pi - \frac{1}{2}\log \sigma^2 - \frac{1}{2\sigma^2} x^2$$

$$\Rightarrow \frac{\partial l(\sigma^2|x)}{\partial \sigma^2} = 0 - \frac{1}{2\sigma^2} - \frac{x^2}{2} \cdot \frac{-1}{\sigma^4} = 0$$

$$\Rightarrow \frac{x^2}{2\sigma^4} = \frac{1}{2\sigma^2}$$

$$\Rightarrow \tilde{\sigma}^2 = x^2$$

$$\mathbb{E}[\tilde{\sigma}^2] = \mathbb{E}[x^2]$$

$$= \text{var}(x) + E(x)^2$$

$$= \sigma^2$$

Hence $\boxed{\tilde{\sigma}^2_{MLE} = x^2}$ unbaised estimator.

(b) find the MLE of σ

$$\log L(\sigma|x) = -\frac{1}{2}\log 2\pi - \frac{1}{2}\log\sigma^2 - \frac{1}{2\sigma^2}x^2$$

σ^{-2}

$$\frac{\partial}{\partial\sigma}\log L(\sigma|x)$$

$$\Rightarrow -\frac{1}{2\sigma^2}\cdot 2\sigma - \frac{x^2}{2}\cdot\frac{-2\cdot}{\sigma^3} = 0$$

$$\Rightarrow -\frac{1}{\sigma} + \frac{x^2}{\sigma^2} = 0$$

$$\Rightarrow \sigma^2 = x^2 \Rightarrow \sigma = |x|$$

$$\boxed{\hat{\sigma}_{MLE} = |x|}$$

(c) Discuss how the method of moment's of σ might be found.

Sol^n

$$\mathbb{E}[x] = \bar{x} = x = 0$$

$$\mathbb{E}[x^2] = \frac{1}{n}\sum x^2 = x^2$$

$$\sigma^2 + 0^2 = x^2$$

$$\Rightarrow \sigma^2 = x^2 \Rightarrow \boxed{\hat{\sigma} = |x|}$$

(7.9) Let $X_1, X_2, \ldots, X_n$ be iid with pdf

$$f(x|\theta) = \frac{1}{\theta}, \quad 0 \le x \le \theta, \quad \theta > 0$$

Estimate θ using both the method of moments and MLE. Calculate the mean's and variance of the two estimator's.
Which one should be preferred and why?

Sol^n

$$f(x|\theta) = \begin{cases} \frac{1}{\theta} & 0 \le x \le \theta \quad \theta > 0 \\ 0 & \text{o.w} \end{cases}$$

$$f_n(x|\theta) = \begin{cases} \frac{1}{\theta^n} & 0 \le x_i \le \theta \quad \forall i \\ 0 & \text{o.w} \end{cases}$$

$$\Rightarrow \quad f_n(X|\theta) = \frac{1}{\theta^n} \cdot I_{[0,\theta]}(\max x_i)$$

$$L(\theta|X) = 0 \quad \text{if} \quad \theta > X_{(n)}$$

$$= \frac{1}{\theta^n} \quad \text{if} \quad \theta \le X_{(n)}$$

$$\Rightarrow \boxed{\hat{\theta}_{MLE} = X_{(n)}}$$

$$\mathbb{E}[X] = \frac{\theta}{2} = \bar{X}$$

$$\boxed{\Rightarrow \hat{\theta}_{MoM} = 2\bar{X}}$$

<u>for MLE Estimator:</u>

$$\hat{\theta}_{MLE} = X_{(n)}$$

$$f_{X_{(n)}}(x) = \frac{n!}{(n-1)!} f_X(x) \, F_X(x)^{n-1}$$

$$= n \, f_X(x) \, F_X(x)^{n-1}$$

$$0 < x_n < \theta$$

$$f_X(x) = \frac{1}{\theta} \quad 0 < x < \theta$$

$$F_X(x) = \frac{x}{\theta} \quad 0 < x < \theta$$

$$f_{X_{(n)}}(x_{(n)}) = n \cdot \frac{1}{\theta} \cdot \left(\frac{x_{(n)}}{\theta}\right)^{n-1} \quad 0 < x_{(n)} < \theta$$

$$f_{X_{(n)}}(x_{(n)}) = \frac{n}{\theta^n}\left(x_{(n)}\right)^{n-1} \qquad 0 < x_{(n)} < \theta$$

$$\mathbb{E}[X_{(n)}] = \int_0^\theta \frac{n}{\theta^n}\left(x_{(n)}\right)^{n} dx$$

$$= \frac{n}{\theta^n} \cdot \frac{\left(x_{(n)}\right)^{n+1}}{n+1}$$

$$\mathbb{E}[X_{(n)}] = \frac{n}{n+1}\theta$$

$$\text{Var}(X_{(n)}] = \frac{n}{n+2}\cdot\theta^2 - \frac{n^2\theta^2}{(n+1)^2}$$

$$\boxed{\text{Var}[X_{(n)}] = \frac{n\theta^2}{(n+2)(n+1)^2}}$$

<u>Method of Moment's</u>

$$\hat{\theta}_{MOM} = 2\bar{X}$$

$$\mathbb{E}[2\bar{X}] = 2\mathbb{E}[\bar{X}] = 2\cdot\frac{\theta}{2} = \theta$$

$$\text{Var}(2\bar{X}) = 4\,\text{var}(\bar{X}) = 4\cdot\frac{\theta^2}{12n} = \frac{\theta^2}{3n}$$

$\hat{\theta}_{MLE}$ is Biased Estimator of θ

$\hat{\theta}_{MoM}$ is unbiased Estimator of θ

$Var(\hat{\theta}_{MLE}) < Var(\hat{\theta}_{MoM})$ for large n

$\Rightarrow$ $\hat{\theta}_{MLE}$ is Preferable.

7.10 The independent r.v's $X_1, X_2, \ldots, X_n$ have the common distribution.

$$\mathbb{P}(X_i \le x \mid \alpha, \beta) = \begin{cases} 0 & \text{if } x < 0 \\ \left(\frac{x}{\beta}\right)^{\alpha} & \text{if } 0 \le x < \beta \\ 1 & \text{if } x > \beta \end{cases}$$

$\alpha > 0, \ \beta > 0$

(a) Find a two-Dimensional sufficient statistic for (α, β)

Sol^n

$$F_X(x) = \mathbb{P}(X_i \le x \mid \alpha, \beta) = \begin{cases} 0 & \text{if } x < 0 \\ \left(\frac{x}{\beta}\right)^{\alpha} & 0 \le x \le \beta \\ 1 & x > \beta \end{cases}$$

$$f_X(x) = \frac{\partial}{\partial x} F_X(x) = \frac{\alpha \cdot x^{\alpha-1}}{\beta^\alpha} \qquad 0 \le x \le \beta$$

$$\Rightarrow \quad f_X(x) = \frac{\alpha}{\beta^\alpha} \cdot x^{\alpha-1} \qquad 0 \le x \le \beta$$

$$f_n(x|\alpha,\beta) = \frac{\alpha^n}{\beta^{\alpha n}} \cdot \left[\prod x\right]^{\alpha-1} I_{[0,\beta)}(x_{(n)})$$

therefore $\left(\prod_{i=1}^{n} x, \; x_{(n)}\right)$

are sufficient statistic for (α, β)

(b) Find the MLE of α, β

$$L(\alpha,\beta|x) = \frac{\alpha^n}{\beta^{\alpha n}} \left(\prod x_i\right)^{\alpha-1} \qquad 0 \le x_{(n)} \le \beta$$

$$L(\alpha, \beta | x) = 0 \quad \text{if } \beta < x_{(n)}$$

& decreasing function as β

$$\Rightarrow \quad \hat{\beta}_{MLE} = x_{(n)}$$

$$l(\alpha, \beta | x) = n \log \alpha - \alpha n \log \beta + (\alpha - 1) \sum \log x_i$$

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - n \log \beta + \sum \log x_i = 0$$

$$\frac{n}{\alpha} = n \log x_{(n)} - \sum \log x_i$$

$$\Rightarrow \hat{\alpha}_{MLE} = \frac{n}{n \log x_{(n)} - \sum \log x_i}$$

$$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{n}{\alpha^2} < 0 \Rightarrow \hat{\alpha} \text{ is maximum point}$$

$$\Rightarrow \boxed{\begin{aligned} \hat{\alpha}_{MLE} &= \frac{n}{n \log x_{(n)} - \sum \log x_i} \\ \hat{\beta}_{MLE} &= x_{(n)} \end{aligned}}$$

(c)

Data:- 22, 23.9, 23.8, 25, 24, 21.7, 23.8

22.8, 23.1, 23.1, 23.5, 23, 23

find the MLE's of α and β

$n = 14$

Sol^n

$$\hat{\beta}_{MLE} = X_{(n)} = 25$$

$$\hat{\alpha}_{MLE} = \frac{14}{14 \cdot \log 25 - \log \prod x_i} = 12.594$$

$$\Rightarrow \boxed{\begin{array}{l} \hat{\beta} = 25 \\ \hat{\alpha} = 12.594 \end{array}}$$

7.11 Let $X_1, \ldots, X_n$ be iid with pdf

$$f(x|\theta) = \theta \cdot x^{\theta - 1} \qquad 0 \leq x \leq 1$$
$$0 < \theta < \infty$$

(a) find the MLE of θ, and show that its variance $\longrightarrow 0$ as $n \to \infty$

Sol^n

$$f_n(x|\theta) = L(\theta|x) = \theta^n \cdot \left(\prod x_i\right)^{\theta - 1}$$

$$l(\theta|x) = n \log \theta + (\theta - 1) \sum \log x_i$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} + \sum \log x_i = 0$$

$$\Rightarrow \quad \hat{\theta}_{MLE} = \frac{-n}{\sum \log x_i} = \frac{-n}{\log(\prod x_i)}$$

$$\frac{\partial^2 l}{\partial \theta^2} = \frac{-n}{\theta^2} < 0 \Rightarrow \text{maximum}$$

$$\boxed{\hat{\theta}_{MLE} = \frac{-n}{\log(\prod x_i)} = \frac{-n}{\sum \ln x_i}}$$

Finding variance of $\hat{\theta}_{MLE}$

$$f_X(x|\theta) = \theta \cdot x^{\theta-1}$$

$$= \frac{\Gamma(\theta+1)}{\Gamma(\theta)\cdot\Gamma(1)} x^{\theta-1}(1-x)^{1-1}$$

$$X \sim \text{Beta}(\theta, 1)$$

$Y_i = -\ln X_i$ $\quad$ $-\ln x$ is monotone decreasing function.

$0 \leq y_i \leq \infty$

$x = e^{-y}$ $\quad \frac{dx}{dy} = e^{-y}$

$$f_Y(y) = \theta \cdot e^{-y(\theta-1)} \cdot e^{-y}$$

$$= \theta \cdot e^{-y\theta + y - y}$$

$$= \theta \cdot e^{-y\theta} \qquad 0 \leq y \leq \infty$$

$Y = -\log x \sim \text{exponentia}(1/\theta)$

$\Rightarrow$ $Y \sim \text{Beta} \sim \text{gamma}(1, 1/\theta)$

the $Y_1 + Y_2 + \cdots + Y_n \sim \text{gamma}(n, 1/\theta)$

we can use MGF to

prove this.

<u>Proof</u>

if $Y \sim \text{gamma}(\alpha, \beta)$

$$M_Y(t) = \left(\frac{1}{1-\beta t}\right)^{\alpha}$$

$$M_{Y_1+Y_2+\dots Y_n}(t) = \mathbb{E}\left[e^{t(Y_1+Y_2+\dots+Y_n)}\right]$$

$$= \mathbb{E}\left[e^{tY_1}\right]\mathbb{E}\left[e^{tY_2}\right]\dots\mathbb{E}\left[e^{tY_n}\right]$$

$$= \mathbb{E}\left[e^{tY}\right]^n$$

$$= \left(\frac{1}{1-\beta t}\right)^{\alpha n}$$

$$\Rightarrow \quad Y_1+Y_2+\dots+Y_n \sim \text{gamma}(\alpha n, \beta)$$

$$\mathbb{E}\left[\hat{\theta}_{MLE}\right] = \mathbb{E}\left[\frac{n}{-\sum \log x_i}\right]$$

$$= \mathbb{E}\left[\frac{n}{Y_1+Y_2+\dots Y_n}\right]$$

$$= n\,\mathbb{E}\left[\frac{1}{Y_1+Y_2+\dots Y_n}\right]$$

$$= \frac{1}{\Gamma(n)\left(\frac{1}{\theta}\right)^n}\int_0^\infty \frac{1}{y}\, y^{n-1} e^{-\theta y}\, dt$$

$$= \frac{\theta^n}{\Gamma(n)}\int_0^\infty y^{n-1-1}\cdot e^{-\theta y}\, dt$$

$$= \frac{\theta^n}{\Gamma(n)} \Gamma(n-1) \cdot \left(\frac{1}{\theta}\right)^{n-1}$$

$$\mathbb{E}\left[\hat{\theta}_{MLE}\right] = \frac{\theta}{n-1}$$

$$\mathbb{E}\left[\hat{\theta}^2_{MLE}\right] = \frac{\theta^n}{\Gamma(n)} \Gamma(n-2) \left(\frac{1}{\theta}\right)^{n-2}$$

$$= \frac{\theta^2}{(n-1)(n-2)}$$

$$\mathrm{var}\left(\hat{\theta}_{MLE}\right) = \frac{\theta^2}{(n-1)(n-2)} - \frac{\theta^2}{(n-1)^2}$$

$$= \frac{\theta^2}{(n-1)^2(n-2)} \left(n-1 - n+2\right)$$

$$\mathrm{var}\left(\hat{\theta}_{MLE}\right) = \frac{\theta^2}{(n-1)^2(n-2)}$$

$$\boxed{\lim_{n\to\infty} \mathrm{var}\left(\hat{\theta}_{MLE}\right) = \lim_{n\to\infty} \frac{\theta^2}{(n-1)^2(n-2)} = 0}$$

(b) Find the method of moments estimator of θ

$X \sim Beta(\theta, 1)$

$$\mathbb{E}[X] = \frac{\alpha}{\alpha+\beta} = \frac{\theta}{\theta+1} = \bar{X}$$

$$\Rightarrow \quad \theta = \bar{X}\theta + \bar{X}$$

$$\Rightarrow \quad \boxed{\theta = \frac{\bar{X}}{1-\bar{X}}}$$

7.12 Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with pmf

$$P_\theta(X=x) = \theta^x (1-\theta)^{1-x}, \quad x = 0 \text{ or } 1$$
$$0 \le \theta \le 1/2$$

(a) Find the method of moment's estimator and MLE of θ

$X \sim \text{Bernoilli}(\theta)$

$\underline{\underline{Sol^n}}$ $\quad \mathbb{E}[X] = \theta = \bar{X} \quad \text{if} \quad \bar{X} \leq 1/2$

$$f_n(x|\theta) = \theta^{\Sigma x_i} (1-\theta)^{n - \Sigma x_i}$$

$$l(\theta|x) = \Sigma x_i \log\theta + (n - \Sigma x_i) \log(1-\theta)$$

$$\frac{\partial l}{\partial \theta} = \frac{\Sigma x_i}{\theta} - \frac{n - \Sigma x_i}{1-\theta} = 0$$

$$\Rightarrow \quad \frac{1-\theta}{\theta} = + \frac{(n - \Sigma x_i)}{\Sigma x_i}$$

$$\hat{\theta}_{MLE} = \bar{X} \quad \text{if} \quad \bar{X} \leq 1/2$$

7.13 Let $X_1, X_2, \ldots, X_n$ be a sample from a population with double exponential pdf

$$f(x|\theta) = \frac{1}{2} e^{-|x-\theta|}$$

$$-\infty < x < \infty$$
$$-\infty < \theta < \infty$$

Sol^n

$$L(\theta|x) = \frac{1}{2^n} \prod e^{-|x_i-\theta|}$$

$$= \frac{1}{2^n} e^{-\sum_{i=1}^{n} |x_i-\theta|}$$

(7.14) Let X and Y be independent exponential r.v with

$$f(x|\lambda) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \quad , x > 0$$

$$f(x|\mu) = \frac{1}{\mu} e^{-\frac{y}{\mu}} \quad , y > 0$$

we observe Z and W with

$$Z = \min(X, Y) \quad \text{and} \quad W = \begin{cases} 1 & \text{if } Z = X \\ 0 & \text{if } Z = Y \end{cases}$$

Sol^n

$X \sim \exp(\lambda) \qquad Y \sim \exp(\mu)$

$$Z = \min(X, Y) \quad \text{and} \quad W = \begin{cases} 1 & \text{if } Z = X \\ 0 & \text{if } Z = Y \end{cases}$$

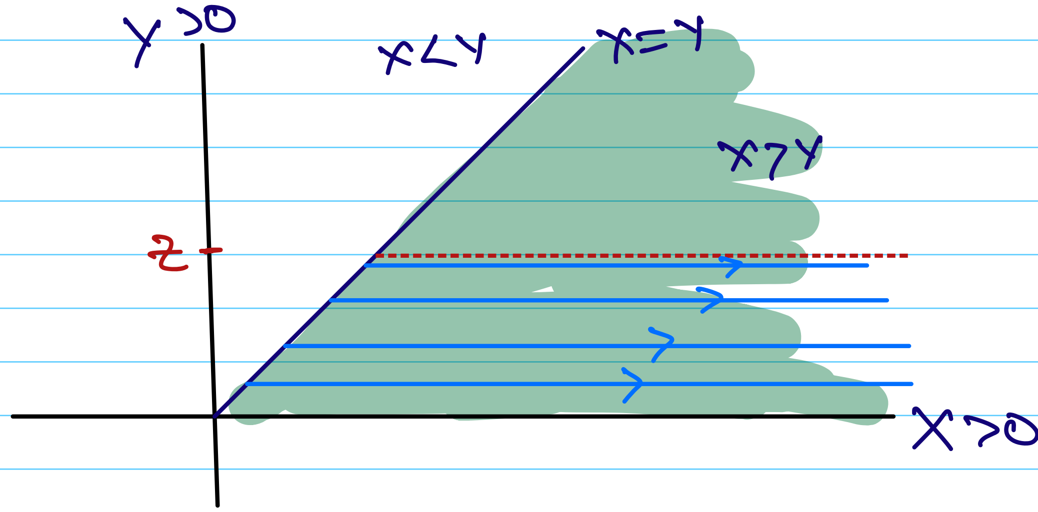
if $X < Y$ then $Z = X$
$W = 1$

if $X > Y$ then $Z = Y$
$W = 0$

Joint CDF:

$$\mathbb{P}(Z < z,\ W = 0) = \mathbb{P}(\min(X, Y) < z\ ,\ Z = Y)$$

$$= \mathbb{P}(\ Y < z,\ Y < X)$$



$$X : Y \longrightarrow \infty$$

$$Y : 0 \longrightarrow z$$

$$\Rightarrow \mathbb{P}(\ Y < z,\ Y < X) = \int_{y=0}^{z} \int_{x=y}^{\infty} \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \cdot \frac{1}{\mu} e^{-\frac{y}{\mu}}\, dx\, dy$$

$$= \frac{1}{\lambda\mu} \int_{0}^{z} e^{-\frac{y}{\mu}} \int_{y}^{\infty} e^{-\frac{x}{\lambda}}\, dx\ \ dy$$

$$= \frac{1}{\lambda\mu}\int_0^z e^{-\frac{y}{\mu}} \cdot -\lambda e^{-\frac{x}{\lambda}}\Big]_y^{\infty} dy$$

$$= \frac{1}{\lambda\mu}\int_0^z e^{-\frac{y}{\mu}} \left[\lambda e^{-\frac{y}{\lambda}} - 0\right] dy$$

$$= \frac{1}{\mu}\int_0^z e^{-y\left(\frac{1}{\mu}+\frac{1}{\lambda}\right)} dy$$

$$= \frac{1}{\mu} \cdot \frac{e^{-y\left(\frac{1}{\mu}+\frac{1}{\lambda}\right)}}{-\left(\frac{1}{\mu}+\frac{1}{\lambda}\right)}\Bigg]_0^z$$

$$P(Z \le z, W=0) = \frac{\lambda}{\mu+\lambda}\left[1 - e^{-z\left(\frac{1}{\mu}+\frac{1}{\lambda}\right)}\right]$$

$$\frac{\partial}{\partial z} P(Z \le z, W=0) = P(Z=z, W=0)$$

$$= \frac{-\lambda}{\mu+\lambda} \cdot e^{-z\left(\frac{1}{\mu}+\frac{1}{\lambda}\right)} \cdot -\left(\frac{1}{\mu}+\frac{1}{\lambda}\right)$$

$$\boxed{P(Z=z, W=0) = \frac{1}{\mu} e^{-z\left(\frac{1}{\mu}+\frac{1}{\lambda}\right)}}$$

Simillarly

$$\mathbb{P}(Z \le z,\ W=1) = \frac{\mu}{\mu+\lambda}\left(1-e^{-z\left(\frac{1}{\mu}+\frac{1}{\lambda}\right)}\right)$$

$$\boxed{\mathbb{P}(Z=z,\ W=1) = \frac{1}{\lambda}e^{-z\left(\frac{1}{\mu}+\frac{1}{\lambda}\right)}}$$

$$\Rightarrow \quad \mathbb{P}(Z_i=z,\ W_i=w) = \left(\frac{1}{\lambda}\right)^{w_i}\left(\frac{1}{\mu}\right)^{1-w_i} e^{-z\left(\frac{1}{\mu}+\frac{1}{\lambda}\right)}$$

if $(Z_1,W_1),(Z_2,W_2)\ \ldots\ (Z_n,W_n) \overset{iid}{\sim} (Z,W)$

then $f_n(Z=z,\ W=w)$

$$L(\lambda,\mu \mid z,w) = \left(\frac{1}{\lambda}\right)^{\sum w_i}\left(\frac{1}{\mu}\right)^{n-\sum w_i} e^{-\left(\frac{1}{\mu}+\frac{1}{\lambda}\right)\cdot\sum z_i}$$

$$\ell(\lambda,\mu \mid z,w) = -\sum w_i \log\lambda - \left(n-\sum w_i\right)\log\mu - \left(\frac{1}{\mu}+\frac{1}{\lambda}\right)\sum z_i$$

$$\frac{\partial l}{\partial \lambda} = -\frac{\sum w_i}{\lambda} + \frac{\sum z_i}{\lambda^2} = 0$$

$$\Rightarrow \quad \frac{\sum z_i}{\lambda} = \frac{\sum w_i}{1}$$

$$\Rightarrow \quad \hat{\lambda} = \frac{\sum z_i}{\sum w_i}$$

$$\frac{\partial l}{\partial \mu} = -\frac{(n - \sum w_i)}{\mu} + \frac{\sum z_i}{\mu^2} = 0$$

$$\hat{\mu} = \frac{\sum z_i}{n - \sum w_i}$$

therefore

$$\boxed{\hat{\lambda} = \frac{\sum z_i}{\sum w_i} \qquad \hat{\mu} = \frac{\sum z_i}{n - \sum w_i}}$$

7.15 Let $x_1, x_2, \ldots x_n \overset{iid}{\sim}$ inverse Gaussian pdf

$$f(x|\mu,\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left\{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right\} \quad x>0$$

(a) Show that the MLE's of μ & λ are

$$\hat{\mu}_n = \bar{x} \quad \text{and} \quad \hat{\lambda}_n = \frac{n}{\sum \frac{1}{x_i} - \frac{1}{\bar{x}}}$$

Sol^n

$$f(x|\mu,\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{\frac{1}{2}} \exp\left\{-\lambda \frac{(x-\mu)^2}{2\mu^2 x}\right\}$$

$$f_n(x|\mu,\lambda) = \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \cdot \left(\frac{1}{\prod x_i}\right)^{\frac{3}{2}} \cdot$$

$$L(\mu,\lambda|x) = \exp\left\{-\frac{\lambda}{2\mu^2} \sum \frac{(x_i-\mu)^2}{x_i}\right\}$$

$$l(\mu,\lambda|x) = \frac{n}{2}\log\frac{\lambda}{2\pi} - \frac{3}{2}\log\prod x_i - \frac{\lambda}{2\mu^2}\sum\frac{(x_i-\mu)^2}{x_i}$$

$$= \frac{n}{2}\log\frac{\lambda}{2\pi} - \frac{3}{2}\log\prod x_i - \frac{\lambda}{2\mu^2}\sum x_i - 2\mu + \frac{\mu^2}{x_i}$$

$$\frac{\partial l}{\partial \mu} = 0$$

$$\Rightarrow \sum\frac{(x_i-\mu)^2}{x_i}\cdot\frac{\lambda}{\mu^{\cancel{3}}} - \frac{\lambda}{\cancel{2}\cancel{\mu}\mu^2}\cdot\cancel{2\mu}\sum\frac{1}{x_i} = 0$$

$$\Rightarrow \frac{\lambda}{\mu^2}\sum\frac{(x_i-\mu)^2}{x_i} - \lambda\cdot\sum\frac{1}{x_i} = 0$$

$$\Rightarrow \frac{1}{\mu^2}\sum\frac{(x_i-\mu)^2}{x_i} - \sum\frac{1}{x_i} = 0$$

$$\Rightarrow \frac{1}{\mu^2}\sum x_i - 2\mu + \frac{\mu^2}{x_i} - \sum\frac{1}{x_i} = 0$$

$$\Rightarrow \frac{\sum x_i}{\mu^2} - \frac{2n}{\mu} + \cancel{\sum\frac{1}{x_i}} - \cancel{\sum\frac{1}{x_i}} = 0$$

$$\boxed{\hat{\mu} = \frac{\overline{X}}{2}}$$

some calculation mistake

$$l(\mu, \lambda | x) = \frac{n}{2} \log \frac{\lambda}{2\pi} - \frac{3}{2} \log \prod x_i - \frac{\lambda}{2\mu^2} \sum \frac{(x_i - \mu)^2}{x_i}$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{2\lambda} - \frac{1}{2\mu^2} \sum \frac{(x_i - \mu)^2}{x_i} = 0$$

$$\Rightarrow \quad \frac{1}{\mu^2} \sum \frac{(x_i - \mu)^2}{x_i} = \frac{n}{\lambda}$$

$$\Rightarrow \quad \lambda = \frac{n\mu^2}{\sum \frac{(x_i - \mu)^2}{x_i}}$$

$$\Rightarrow \quad \lambda = \frac{n \overline{X}^2}{\sum \left(x_i - 2\overline{X} + \frac{\overline{X}^2}{x_i}\right)}$$

$$\Rightarrow \quad \lambda = \frac{n\bar{x}^2}{n\bar{x} - 2n\bar{x} + \bar{x}^2 \cdot \sum \frac{1}{x_i}}$$

$$\Rightarrow \quad \lambda = \frac{n}{\sum \frac{1}{x_i} - \frac{n}{\bar{x}}}$$

$$\Rightarrow \quad \boxed{\hat{\lambda} = \frac{n}{\sum \frac{1}{x_i} - \frac{1}{\bar{x}}}}$$

7.15 b

(i) Show that $\hat{\mu}_2$ has a inverse Gaussian with $\mu, 2\lambda$, $\frac{2\lambda}{\hat{\lambda}_2}$ has a χ_1^2 distribution, and they are independent.

Soln

$$\hat{\mu}_2 = \bar{x} = \frac{x_1 + x_2}{2}$$