7.2.3 Baye's Estimator

Bayesian approach in statistics

⇒ in the classical approach the parameter θ, is thought to be an unknown, But fixed, quantity.

⇒ A random sample $x_1, x_2, \ldots, x_n$ is drawn from a population indexed by θ and, Based on the observed values in the sample, knowledge about the value of θ is obtained.

⇒ in the Bayesian approach θ is considered to be a quantity whose variation can be described by a probability distribution (called the Prior distribution)

$$\pi(\theta|x) = \frac{f(x|\theta)\,\pi(\theta)}{\int f(x|\theta)\,\pi(\theta)\,d\theta}$$

Example 7.2.14 (Binomial Bayes estimation):-

$X_1, X_2, \ldots, X_n \overset{iid}{\sim} Ber(p)$

$Y = \sum X_i \sim Binomial(n, p)$

$p \sim Beta(\alpha, \beta)$ Prior distribution.

$$f(y,p) = \binom{n}{y} p^y (1-p)^{n-y} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1} (1-p)^{\beta-1}$$

$$= \binom{n}{y} p^{\alpha+y-1} (1-p)^{n+\beta-y-1} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$$

The marginal pdf of Y is

$$f(y) = \int_0^1 f(y,p)\,dp = \int_0^1 \binom{n}{y} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha+y-1} \cdot (1-p)^{n+\beta-y-1}\,dp$$

$$= \binom{n}{y} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+y)\,\Gamma(n+\beta-y)}{\Gamma(\alpha+\beta+n)} \int_0^1 Beta(\alpha+y, n+\beta-y)\,dp$$

$$f(y) = \binom{n}{y} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+y)\,\Gamma(n+\beta-y)}{\Gamma(\alpha+\beta+n)}$$

$$\pi(p|y) = \frac{f(y,p)}{f(y)} = \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+y)\,\Gamma(n+\beta-y)} \, p^{y+\alpha-1} \cdot (1-p)^{n-y+\beta-1}$$

$$p|y \sim \text{Beta}(\alpha+y,\ n+\beta-y)$$

$$\hat{p} = \mathbb{E}[p|y] = \frac{\alpha+y}{\alpha+\beta+n}$$

in general, for any sampling distribution, there is a natural family of prior distribution, called the conjugate family.

Definition 7.2.15 :- Let $\mathcal{F}$ denote the class of pdf's and pmf's $f(x|\theta)$ (indexed by θ). A class Π of prior distribution is a conjugate family for $\mathcal{F}$ if the posterior distribution is in the class Π for all $f \in \mathcal{F}$, all Prior's in Π, and all $x \in \mathcal{X}$.

Example 7.2.18 (Normal Bayes Estimator's)

let $X \sim n(\theta, \sigma^2)$

$\pi(\theta) \sim n(\mu, \tau^2)$

σ^2, μ, τ^2 are known

θ is unknown

$$\theta | x \sim n\left(\frac{\tau^2}{\tau^2+\sigma^2} x + \frac{\sigma^2}{\sigma^2+\tau^2} \mu , \frac{\sigma^2 \tau^2}{\sigma^2+\tau^2} \right)$$