7.3 Method's for Evaluating Estimator's

The general topic of evaluating statistical Procedures is Part of the Branch of statistics known as decision theory,

7.3.1 MEAN Squared error

Definition 7.3.1 :

The mean squared error (MSE) of an estimator W of a Parameter θ is the function of θ defined by $\mathbb{E}_\theta\left[(W-\theta)^2\right]$

$$\mathbb{E}_\theta\left[(W-\theta)^2\right] = \operatorname{Var}_\theta W + \left(E_\theta[W]-\theta\right)^2$$

$$= \operatorname{Var}_\theta W + \left(\operatorname{Bias}_\theta W\right)^2$$

Definition 7.3.2 :-

The Bias of a point estimator W of a parameter θ is the difference b/w the Expected value of W and θ; that is,

$\text{Bias}_\theta W = E_\theta W - \theta$. An Estimator whose Bias is identically (in θ) equal to 0 is called unbaised and satisfies $E_\theta[W] = \theta$ for all θ.

$$MSE = \text{Var} + (\text{Bias})^2$$

measuring the variability of the Estimator (Precision)

measuring its BIAS (accuracy)

$\Rightarrow$ An Estimator that has good MSE Properties has small combined variance and Bias.

$\Rightarrow$ To find an Estimator with good MSE

Properties, we need to find Estimator's that control both variance and Bias.

Example 7.3.3 (Normal MSE):-

$X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$

$(\bar{X}, S^2)$ Both are unbiased Estimator's

$E[\bar{X}] = \mu, \quad E[S^2] = \sigma^2$

$$\mathbb{E}[(\bar{X}-\mu)^2] = var(\bar{X}) = \frac{\sigma^2}{n}$$

$$\mathbb{E}\{(S^2-\sigma^2)^2\} = var(S^2) = \frac{2\sigma^4}{n-1}$$

Ex 7.3.4 :-

$x_1, x_2, \ldots, x_n \sim N(\mu, \sigma^2)$

$$\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 = \frac{n-1}{n}S^2$$

$$\mathbb{E}[\hat{\sigma}^2] = \mathbb{E}\left[\frac{n-1}{n} S^2\right] = \frac{n-1}{n}\sigma^2$$

$\Rightarrow$ $\hat{\sigma}^2_{MLE}$ is Biased Estimator.

$$\mathrm{var}(\hat{\sigma}^2) = \mathrm{var}\left(\frac{n-1}{n} S^2\right)$$

$$= \left(\frac{n-1}{n}\right)^2 \mathrm{var}(S^2)$$

$$= \left(\frac{n-1}{n}\right)^2 \cdot \frac{2\sigma^4}{n-1}$$

$$= \frac{2(n-1)\sigma^4}{n^2}$$

MSE

$$\mathbb{E}\left[(\hat{\sigma}^2 - \sigma^2)^2\right] = \mathrm{var} + \mathrm{Bias}^2$$

$$= \frac{2(n-1)\sigma^4}{n^2} + \left(\frac{n-1}{n}\sigma^2 - \sigma^2\right)^2$$

$$= \left(\frac{2n-1}{n^2}\right)\sigma^4. < \mathbb{E}\left((S^2 - \sigma^2)^2\right)$$