The Likelihood Principle :-

Def 6.3.1 :- Let $f(x|\theta)$ denote the joint pdf or pmf of the sample $X = (X_1, X_2, .. X_n)$. Then, given that $X = x$ is observed, the function of θ defined by

$$L(\theta|x) = f(x|\theta)$$

is called a Likelihood function.

let X is a random vector

then $L(\theta|x) = \mathbb{P}_\theta(X = x)$

if $\mathbb{P}_{\theta_1}(X = x) = L(\theta_1|x) > L(\theta_2|x) = P_{\theta_2}(\theta_2|x)$

$\Rightarrow$ The sample we observed $X = x$ is more likely to have occured if $\theta = \theta_1$ than if $\theta = \theta_2$, which can be interpreted as saying that

θ_1 is more plausible value for the true value of θ than is θ_2.

* The Likelihood function measures how well different values of θ explain the observed data.

* It compares different parameter value's based on how well they explain the data, but does not provide the probability of θ itself.

Example 6.3.2 (Negative Binomial Likelihood)

$X \sim$ negative Binomial $(r=3, p)$

x = number of failure before $r=3$ number of success.

let $x=2$

$$\Rightarrow \quad L(P|x=2) = \mathbb{P}(X=2|P)$$

$$= \binom{r+x-1}{x} p^r (1-p)^x$$

$$= \binom{3+2-1}{2} p^3 (1-p)^2$$

$$L(P|2) = \binom{4}{2} p^3 (1-p)^2$$

* The Likelihood Principle specifies how the likelihood function should be used as a data reduction device.

LIKELIHOOD PRINCIPLE :

if x and y are two sample points such that $L(\theta|x)$ is proportional to $L(\theta|y)$, that is, there exists a constant $C(x,y)$ such that

$$L(\theta|x) = C(x,y)\, L(\theta|y) \quad \forall \theta$$

then the conclusion's drawn from x and y should be identical.

fiducial inference:

Sometimes interprets likelihoods as probabilities for θ.

$$L(\theta|x) \cdot M(x)$$

where $M(x) = \left(\int_{-\infty}^{\infty} L(\theta|x)\, d\theta \right)^{-1}$

$$\Rightarrow \left. \frac{L(\theta|x)}{\int_{-\infty}^{\infty} L(\theta|x)\, d\theta} \right\} \text{interpreted as pdf for } \theta.$$

$\Rightarrow$ if $L(\theta|x) = C(x,y)\, L(\theta|y) \quad \forall \theta$

(Likelihood Principle)

then both $\dfrac{L(\theta|x)}{\int_{-\infty}^{\infty} L(\theta|x)\, d\theta} = \dfrac{L(\theta|y)}{\int_{-\infty}^{\infty} L(\theta|y)\, d\theta}$

Both yield same pdf.

Example 6.3.3 (Normal fiducial distribution)

Let $x_1, x_2, \ldots x_n \sim N(\mu, \sigma^2)$ σ^2 known.

$$f(x|\mu) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(\frac{-1}{2\sigma^2}\left((n-1)s^2 + n(\bar{x}-\mu)^2\right)\right)$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[\frac{-1}{2\sigma^2}\left(\sum x_i^2 - \frac{1}{n}\left(\sum x_i\right)^2 + \frac{1}{n}\left(\sum x_i - n\mu\right)^2\right)\right]$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left[\frac{-1}{2\sigma^2}\left(\sum x_i^2 - \frac{1}{n}\cancel{\left(\sum x_i\right)^2} + \frac{\cancel{\left(\sum x_i\right)^2}}{n} + n\mu^2 - 2\mu \sum x_i\right)\right]$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(\frac{-1}{2\sigma^2}\left(\sum x_i^2 + n\mu^2 - 2\mu \sum x_i\right)\right)$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(\frac{-n\mu^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{\sum x_i^2}{2\sigma^2} + \frac{\mu}{\sigma^2} \sum x_i\right)$$

$$\frac{L(\theta|x)}{L(\theta|y)} = \frac{f(x|\theta)}{f(y|\theta)} = \exp\left[\frac{-1}{2\sigma^2}\left(\sum x_i^2 - \sum y_i^2\right) + \frac{\mu}{\sigma^2}\left(\sum x_i - \sum y_i\right)\right]$$

if $\bar{x} = \bar{y}$ (Likelihood Principle)

$$\frac{L(\mu|x)}{L(\mu|y)} = \exp\left(\frac{-1}{2\sigma^2}\left(\sum x_i^2 - \sum y_i^2\right)\right)$$

independent of μ

$$C(x,y) = \exp\left(\frac{-1}{2\sigma^2}\left(\sum x_i^2 - \sum y_i^2\right)\right)$$

$$= \exp\left(\frac{-1}{2\sigma^2}\left(\sum x_i^2 - n\bar{x}^2 - \sum x_i^2 + n\bar{y}^2\right)\right)$$

$$= \exp\left(\frac{-1}{2\sigma^2}\left(\sum (x_i-\bar{x})^2 - \sum (y_i-\bar{y})^2\right)\right)$$

$$= \exp\left(-\frac{\sum (x_i-\bar{x})^2}{2\sigma^2} + \frac{\sum (y_i-\bar{y})^2}{2\sigma^2}\right)$$