This document contain's solution's to Exercises 7.16 to 7.22 from chapter 7, of statistical inference by Casella and Berger, focusing on MLE Estimator's and Bayesian Estimator's ($\hat{\theta}_{MAP}$)

Some problem's are still in progress, and the document will be updated with their solution's soon.

7.17 Suppose $X_1, X_2 \overset{iid}{\sim}$ exponential(θ)

(a) if we observe only X_2, show that the MLE of θ is $\hat{\theta} = X_2$

Sol^n

$$f(x_2|\theta) = \frac{1}{\theta} e^{-\frac{x_2}{\theta}} \quad 0<x<\infty$$

$$L(\theta|x_2) = \frac{1}{\theta} e^{-\frac{x_2}{\theta}}$$

$$\log L(\theta|x_2) = -\log\theta - \frac{x_2}{\theta}$$

$$\frac{\partial}{\partial\theta} \log L(\theta|x_2) = -\frac{1}{\theta} + \frac{x_2}{\theta^2} = 0$$

$$\Rightarrow \quad \frac{1}{\theta} = \frac{x_2}{\theta^2}$$

$$\Rightarrow \quad \boxed{\hat{\theta}_{MLE} = X_2}$$

7.17 b Suppose that we instead observe only $Z = \frac{(X_2 - 1)}{X_1}$. find the joint distribution of (X_1, Z) and integrate out X_1 to get the Likelihood function.

<u>Sol^n</u>

$$\left.\begin{aligned} U &= X_1 \\ V &= \frac{X_2 - 1}{X_1} \end{aligned}\right\} \Rightarrow \begin{aligned} X_1 &= U \\ X_2 &= UV + 1 \end{aligned}$$

$$|J| = \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix} = u$$

$$f(x_1, x_2) = \frac{1}{\theta} e^{-\frac{x_1}{\theta}} \cdot \frac{1}{\theta} e^{-\frac{x_2}{\theta}}$$

$$= \frac{1}{\theta^2} e^{-\frac{x_1 + x_2}{\theta}}$$

$$f_{U,V}(u,v) = \frac{1}{\theta^2} e^{-\frac{u + uv + 1}{\theta}} u$$

$$= \frac{1}{\theta^2} \cdot u \cdot e^{-\frac{u(v+1)}{\theta}} \cdot e^{-\frac{1}{\theta}}$$

$$f_V(v) = \int_{u=0}^{\infty} \frac{1}{\theta^2} \cdot e^{-\frac{1}{\theta}} \cdot u e^{-\frac{u(v+1)}{\theta}} du$$

$$= \frac{1}{\theta^2} \cdot e^{-\frac{1}{\theta}} \int_{u=0}^{\infty} u \cdot e^{-\frac{u(v+1)}{\theta}} du$$

$$= \frac{1}{\theta^2} \cdot e^{-\frac{1}{\theta}} \underbrace{\int_{u=0}^{\infty} u^{2-1} \cdot e^{-\frac{u}{\theta/v+1}} du}_{\text{gamma}\left(2, \frac{\theta}{v+1}\right)}$$

$$f_V(v) = \frac{1}{\theta^2} \cdot e^{-\frac{1}{\theta}} \cdot \underset{\overset{\shortparallel}{1}}{\Gamma(2)} \left(\frac{\theta}{v+1}\right)^2$$

$$f_V(v) = \frac{e^{-\frac{1}{\theta}}}{(v+1)^2}$$ Here $Z = V$

$$\Rightarrow \quad f_Z(z) = \frac{1}{(v+1)^2} e^{-\frac{1}{\theta}} \qquad -\infty < z < \infty$$

7.17 c

Suppose that $x_2 = 1$. Compare the MLE's of θ from Part (a) and (b)

Sol^n

$$\boxed{\hat{\theta}_{MLE}(x_2) = x_2 = 1}$$

$$f_z(z) = \frac{1}{(v+1)^2} e^{-\frac{1}{\theta}}$$

$$\log f_z(z) = -\frac{1}{\theta} - 2\log(v+1)$$

$$\frac{\partial}{\partial z} \log f_z(z) = +\frac{1}{\theta^2} = 0$$

$$\boxed{\hat{\theta}_{MLE}(z) = \infty}$$

7.17 d

Bayesian analysis is not Immune to the Borel Paradox. if $\pi(\theta)$ is a Prior density for θ, Show that the Posterior distribution's at $x_2 = 1$, are different in Part's (a) & (b).

(a) $$\pi(\theta|x) = \frac{f(x|\theta)\,\pi(\theta)}{\int f(x|\theta)\,\pi(\theta)\,d\theta}$$

$$\pi(\theta|x_2) = \frac{\frac{1}{\theta}e^{-\frac{x_2}{\theta}} \cdot \pi(\theta)}{\int \frac{1}{\theta}e^{-\frac{x_2}{\theta}} \cdot \pi(\theta)\,d\theta}$$

(b) $$\pi(\theta|z) = \frac{\frac{1}{(\nu+1)^2}e^{-\frac{1}{\theta}}\ \pi(\theta)}{\int \frac{1}{(\nu+1)^2}e^{-\frac{1}{\theta}}\,\pi(\theta)\,d\theta}$$

Both are different.

(7.18) Let (X_1, Y_1), $(X_2, Y_2) \ldots (X_n, Y_n)$

be iid bivariate r.v (Pairs) where all five Parameter's are unknown.

(a) Show that the M.O.M for u_x, u_y, σ_x^2, σ_y^2, ρ are $\bar{U}_x = \bar{X}$

$\bar{U}_y = \bar{Y}$, $\bar{\sigma}_x^2 = \frac{1}{n} \sum (x_i - \bar{x})^2$

$\sigma_y^2 = \frac{1}{n} \sum (y_i - \bar{y})^2$, $\bar{\rho} = \frac{1}{n} \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\hat{\sigma}_x \hat{\sigma}_y}$.

Sol^n $f(x, y) =$

$$\frac{1}{\left(2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}\right)} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-u_x}{\sigma_x}\right)\left(\frac{y-u_y}{\sigma_y}\right) + \left(\frac{y-u_y}{\sigma_y}\right)^2\right]\right)$$

we know

$$\mathbb{E}[X] = \mu_x = \bar{X}$$

$$\mathbb{E}[Y] = u_y = \bar{Y}$$

$$\mathbb{E}[X^2] = \operatorname{var}(X) + E[X]^2$$

$$\Rightarrow \mathbb{E}[X^2] = \sigma_x^2 + u_x^2$$

$$\Rightarrow \sigma_x^2 = \frac{1}{n} \sum x_i^2 - \bar{X}^2$$

$$\sigma_y^2 = \frac{1}{n} \sum y_i^2 - \bar{Y}^2$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}[(X-\mu_X)(Y-\mu_Y)]$$

$$= \mathbb{E}[XY] - \mathbb{E}[X]\,E[Y]$$

$$\Rightarrow \quad \rho\,\sigma_x\sigma_y = \frac{1}{n}\sum_{i=1}^{n} x_i y_i - \mu_x \mu_y$$

$$\Rightarrow \quad \rho = \frac{\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{x}\,\bar{y}}{\hat{\sigma}_x\,\hat{\sigma}_y}$$

7.19 Suppose that the random variable $Y_1, Y_2, \dots, Y_n$ satisfy

$$Y_i = \beta x_i + \varepsilon_i \qquad i = 1, 2, \dots n$$

where $x_1, x_2, \dots x_n$ are fixed constan's and $\varepsilon_1, \varepsilon_2, \dots \varepsilon_n$ are iid $n(0, \sigma^2)$, σ^2 unknown.

a) find a two-dimensional sufficient statistic for (β, σ^2)

Soln

$$\varepsilon_i \sim N(0, \sigma^2)$$

$$y_i \sim N(\beta x_i, \sigma^2)$$

$$f_Y(y) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{-\frac{1}{2\sigma^2}(y_i - \beta x_i)^2\right\}$$

$$f_n(y) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta x_i)^2\right\}$$

$$= \frac{1}{(2\pi)^{n/2}} \cdot \frac{1}{\sigma^n} \cdot \exp\left\{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n} y_i^2 + \beta^2 \sum_{i=1}^{n} x_i - 2\beta \sum_{i=1}^{n} x_i y_i\right)\right\}$$

$$= \frac{1}{(2\pi)^{n/2}} \cdot \frac{1}{\sigma^n} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2 - \frac{\beta^2}{2\sigma^2}\sum_{i=1}^{n} x_i + \frac{\beta}{\sigma^2}\sum_{i=1}^{n} x_i y_i\right\}$$

$$= \frac{1}{(2\pi)^{n/2}} \frac{1}{\sigma^n} \exp\left\{-\frac{\beta^2}{2\sigma^2}\sum_{i=1}^{n} x_i\right\}$$

$$\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} y_i^2 + \frac{\beta}{\sigma^2}\sum_{i=1}^{n} x_i y_i\right\}$$

$\Rightarrow$ By Factorization theorem

$$T(X) = \left(\sum_{i=1}^{n} y_i^2, \ \sum_{i=1}^{n} x_i y_i \right)$$

(b) Find the MLE of β and show that it is an unbaised estimator of β

Soln $f_n(Y \mid X, \beta, \sigma^2)$

$$= \frac{1}{(2\pi)^{n/2}} \frac{1}{\sigma^n} \exp\left\{ -\frac{\beta^2}{2\sigma^2} \sum_{i=1}^{n} x_i \right\}$$

$$\exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} y_i^2 + \frac{\beta}{\sigma^2} \sum_{i=1}^{n} x_i y_i \right\}$$

$\log L(\beta, \sigma^2 \mid X, Y)$

$$= -\frac{n}{2} \log 2\pi - \frac{n}{2} \log \sigma^2 - \frac{\beta^2}{2\sigma^2} \sum_{i=1}^{n} x_i^2$$

$$- \frac{1}{2\sigma^2} \sum_{i=1}^{n} y_i^2 + \frac{\beta}{\sigma^2} \sum_{i=1}^{n} x_i y_i$$

$$\frac{\partial L}{\partial \beta} = +0+0 - \frac{\cancel{2}\beta}{\cancel{2}\sigma^2}\sum_{i=1}^{n} x_i^2 + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i y_i = 0$$

$$\Rightarrow \quad \beta \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i$$

$$\Rightarrow \quad \boxed{\hat{\beta}_{MLE} = \frac{\sum y_i x_i}{\sum x_i^2}}$$

$$\mathbb{E}\left[\hat{\beta}_{MLE}\right] = \mathbb{E}\left[\frac{\sum x_i y_i}{\sum x_i^2}\right]$$

$$= \frac{1}{\sum x_i^2} \sum_{i=1}^{n} \mathbb{E}[x_i y_i]$$

$$= \frac{1}{\sum x_i^2} \sum_{i=1}^{n} x_i \, \mathbb{E}[y_i]$$

$$= \frac{1}{\sum x_i^2} \sum_{i=1}^{n} x_i \cdot \beta x_i$$

$$= \beta \cdot \frac{\sum x_i^2}{\sum x_i^2} = \beta$$

Hence $\hat{\beta}_{MLE} = \dfrac{\sum x_i y_i}{\sum x_i^2}$ is unbiased Estimator

(c) Find the distribution of the MLE of β

Sol^n $\hat{\beta}_{MLE} = \dfrac{\sum x_i y_i}{\sum x_i^2}$

$$y_i \sim N(\beta x_i, \sigma^2)$$

$$x_i y_i \sim N(\beta x_i^2, x_i^2 \sigma^2)$$

$$\sum x_i y_i \sim N(\beta \sum x_i^2, \sigma^2 \sum x_i^2)$$

$$\frac{\sum x_i y_i}{\sum x_i^2} \sim N\left(\beta, \frac{\sigma^2 \sum x_i^2}{(\sum x_i^2)^2}\right)$$

$$\frac{\sum x_i y_i}{\sum x_i^2} \sim N\left(\beta, \frac{\sigma^2}{\sum x_i^2}\right)$$

$$\boxed{\hat{\beta}_{MLE} \sim N\left(\beta, \frac{\sigma^2}{\sum x_i^2}\right)}$$

7.20 Consider $Y_1, Y_2, \ldots Y_n$ as defined in Exercise 7.19

(a) Show that $\frac{\sum Y_i}{\sum x_i}$ is an unbiased estimator of β

Sol^n

$$\mathbb{E}\left[\frac{\sum Y_i}{\sum x_i}\right] = \frac{1}{\sum x_i}\mathbb{E}\left[\sum Y_i\right]$$

$$= \frac{1}{\sum x_i}\sum E[Y_i]$$

$$= \frac{1}{\sum x_i}\sum \beta x_i = \beta$$

Hence unbiased.

(b) Calculate the exact variance of $\frac{\sum Y_i}{\sum x_i}$ and compare it to variance of the MLE

Sol^n

$$Y_i \sim N(\beta x_i, \sigma^2)$$

$$\sum Y_i \sim N(\beta \sum x_i, n\sigma^2)$$

$$\hat{\beta} = \frac{\sum Y_i}{\sum x_i} \sim N\left(\beta, \frac{n\sigma^2}{(\sum x_i)^2}\right)$$

$$\Rightarrow \operatorname{var}(\hat{\beta}) = \frac{n\sigma^2}{(\sum x_i)^2} > \operatorname{var}(\hat{\beta}_{MLE}) = \frac{\sigma^2}{(\sum x_i)^2}$$

7.21 Again $Y_1, Y_2, \ldots Y_n$ defined in Exercise 7.19

(a) Show that $\frac{\sum \frac{Y_i}{x_i}}{n}$ is also an unbiased estimator of β

Sol^n

$$\mathbb{E}\left[\frac{\sum \frac{Y_i}{x_i}}{n}\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\frac{Y_i}{x_i}\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\mathbb{E}[Y_i]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\beta x_i = \beta\frac{n}{n}$$

$$= \beta$$

Hence unbiased estimator.

(b) Calculate exact variance of $\frac{\sum \frac{Y_i}{x_i}}{n}$ and compare it to the variance of the estimator's in the previous two exercises

Sol^n

$$\frac{Y_i}{x_i} \sim N\left(\beta, \frac{\sigma^2}{x_i^2}\right)$$

$$\Rightarrow \sum \frac{Y_i}{x_i} \sim N\left(n\beta, \sigma^2 \sum \frac{1}{x_i^2}\right)$$

$$\Rightarrow \frac{1}{n} \sum \frac{Y_i}{x_i} \sim N\left(\beta, \frac{\sigma^2}{n^2} \sum \frac{1}{x_i^2}\right)$$

we know $\frac{1}{n \bar{x}^2} \leq \frac{1}{n^2} \sum \frac{1}{x_i^2}$

(convex function)

$$\Rightarrow \operatorname{var}(\hat{\beta}) < \operatorname{var}\left(\frac{\sum \frac{Y_i}{x_i}}{n}\right)$$

7.22 $X_1, X_2, \ldots, X_n \overset{iid}{\sim} n(\theta, \sigma^2)$

$\pi(\theta) \sim N(\mu, \tau^2)$

σ^2, μ, τ^2 are all known

(a) Find the Joint Pdf of $\bar{X}$ and θ

Sol^n

$$f(\bar{x}, \theta) = f_n(\bar{x}|\theta)\, \pi(\theta)$$

$$\bar{X}|\theta \sim N\left(\theta, \frac{\sigma^2}{n}\right)$$

$$f(\bar{x}|\theta) = \frac{1}{\left(2\pi \frac{\sigma^2}{n}\right)^{1/2}} \exp\left\{ -\frac{1}{2\frac{\sigma^2}{n}} (\bar{x}-\theta)^2 \right\}$$

$$\pi(\theta) = \frac{1}{(2\pi\tau^2)^{1/2}} \exp\left\{ -\frac{1}{2\tau^2} (\theta-\mu)^2 \right\}$$

$f(\bar{x}, \theta) =$

$$\frac{1}{\left(2\pi\frac{\sigma^2}{n}\right)^{1/2}} \cdot \frac{1}{(2\pi\tau^2)^{1/2}} \exp\left\{ -\frac{1}{2\frac{\sigma^2}{n}} (\bar{x}-\theta)^2 - \frac{1}{2\tau^2}(\theta-\mu)^2 \right\}$$

(b) Show that $m(\bar{x} \mid \sigma^2, \mu, \tau^2)$ the marginal distribution of $\bar{x}$ is $N(\mu, \frac{\sigma^2}{n} + \tau^2)$

$$m(\bar{x}) = \int_{\Theta} f(\bar{x}, \theta)\, d\theta$$

$$= \int_{\Theta} f(\bar{x} \mid \theta)\, \pi(\theta)\, d\theta$$

$$= \frac{1}{\left(2\pi \frac{\sigma^2}{n}\right)^{1/2}} \frac{1}{(2\pi\tau^2)^{1/2}} \int_{-\infty}^{\infty} \exp\left\{ \frac{-1}{2\frac{\sigma^2}{n}} (\bar{x}-\theta)^2 - \frac{1}{2\tau^2}(\theta-\mu)^2 \right\} d\theta$$

$$\frac{-1}{2\frac{\sigma^2}{n}} (\bar{x}-\theta)^2 - \frac{1}{2\tau^2}(\theta-\mu)^2 = \frac{-1}{2\frac{\sigma^2}{n}} (\bar{x}^2 + \theta^2 - 2\theta\bar{x})$$

$$- \frac{1}{2\tau^2}(\theta^2 + \mu^2 - 2\theta\mu)$$

$$= -\theta^2 \left[\frac{1}{2\frac{\sigma^2}{n}} + \frac{1}{2\tau^2} \right] + 2\theta \left[\frac{\bar{x}}{2\frac{\sigma^2}{n}} + \frac{\mu}{2\tau^2} \right]$$

$$- \frac{\bar{x}^2}{2\frac{\sigma^2}{n}} - \frac{\mu^2}{2\tau^2}$$

$$= -\left[\frac{1}{2\frac{\sigma^2}{n}} + \frac{1}{2\tau^2}\right]\left[\theta^2 - 2\theta\,\frac{\left[\frac{\bar{X}}{2\frac{\sigma^2}{n}} + \frac{\mu}{2\tau^2}\right]}{\frac{1}{2\frac{\sigma^2}{n}} + \frac{1}{2\tau^2}}\right]$$

$$- \left[\frac{\bar{X}^2}{2\frac{\sigma^2}{n}} + \frac{\mu^2}{2\tau^2}\right]$$

$$= -\frac{1}{2}\left(\frac{1}{\frac{\sigma^2}{n}} + \frac{1}{\tau^2}\right)\left[\theta^2 - \frac{2\theta\left[\frac{\bar{X}}{2\frac{\sigma^2}{n}} + \frac{\mu}{2\tau^2}\right]}{\frac{1}{2\frac{\sigma^2}{n}} + \frac{1}{2\tau^2}} + \left(\frac{\cancel{2}}{\cancel{2}}\,\frac{\left[\frac{\bar{X}}{\frac{\sigma^2}{n}} + \frac{\mu}{\tau^2}\right]}{\frac{1}{\frac{\sigma^2}{n}} + \frac{1}{\tau^2}}\right)^2\right]$$

$$+ \frac{1}{2}\,\frac{\left(\frac{\bar{X}}{\frac{\sigma^2}{n}} + \frac{\mu}{\tau^2}\right)^2}{\frac{1}{\frac{\sigma^2}{n}} + \frac{1}{\tau^2}} - \frac{1}{2}\left[\frac{\bar{X}^2}{\frac{\sigma^2}{n}} + \frac{\mu^2}{\tau^2}\right]$$

$$= -\frac{1}{2}\,\frac{1}{\left(\frac{1}{\frac{\sigma^2}{n}} + \frac{1}{\tau^2}\right)^{-1}}\left(\theta - \frac{\frac{\bar{X}}{\frac{\sigma^2}{n}} + \frac{\mu}{\tau^2}}{\frac{1}{\frac{\sigma^2}{n}} + \frac{\mu}{\tau^2}}\right)^2$$

$$+ \frac{1}{2}\,\frac{\left(\frac{\bar{X}}{\frac{\sigma^2}{n}} + \frac{\mu}{\tau^2}\right)^2}{\frac{1}{\frac{\sigma^2}{n}} + \frac{1}{\tau^2}} - \frac{1}{2}\left[\frac{\bar{X}^2}{\frac{\sigma^2}{n}} + \frac{\mu^2}{\tau^2}\right]$$

$m(\bar{x}) =$

$$= \frac{1}{\left(2\pi \frac{\sigma^2}{n}\right)^{1/2}} \frac{1}{\left(2\pi \tau^2\right)^{1/2}} \left(\frac{2\pi}{\frac{1}{\frac{\sigma^2}{n}} + \frac{1}{\tau^2}} \right)^{1/2} \cdot \exp\left(-\frac{1}{2} \left[\frac{\bar{x}^2}{\frac{\sigma^2}{n}} + \frac{\mu^2}{\tau^2} \right] + \frac{1}{2} \frac{\left(\frac{\bar{x}}{\frac{\sigma^2}{n}} + \frac{\mu}{\tau^2} \right)^2}{\frac{1}{\frac{\sigma^2}{n}} + \frac{1}{\tau^2}} \right)$$

$$= \frac{1}{(2\pi)^{1/2}} \frac{1}{\left(\frac{\sigma^2}{n} \cdot \tau^2\right)^{1/2}} \left(\frac{\frac{\sigma^2}{n} \tau^2}{\frac{\sigma^2}{n} + \tau^2} \right)^{1/2} \exp\left(-\frac{1}{2} \left[\frac{\bar{x}^2}{\frac{\sigma^2}{n}} + \frac{\mu^2}{\tau^2} \right] + \frac{1}{2} \frac{\left(\frac{\bar{x}}{\frac{\sigma^2}{n}} + \frac{\mu}{\tau^2} \right)^2}{\frac{1}{\frac{\sigma^2}{n}} + \frac{1}{\tau^2}} \right)$$

$$= \frac{1}{(2\pi)^{1/2}} \frac{1}{\left(\frac{\sigma^2}{n} + \tau^2\right)^{1/2}} \cdot$$

$$\exp\left\{ -\frac{1}{2} \left[\frac{\bar{x}^2}{\frac{\sigma^2}{n}} + \frac{\mu^2}{\tau^2} - \frac{\left(\frac{\sigma^2}{n} \tau^2\right)}{\frac{\sigma^2}{n} + \tau^2} \left(\frac{\bar{x}^2}{\left(\frac{\sigma^2}{n}\right)^2} + \frac{\mu^2}{(\tau^2)^2} + \frac{2 \bar{x} \mu}{\frac{\sigma^2}{n} \tau^2} \right) \right] \right\}$$

$$\exp\left\{ \frac{-1}{2\left(\frac{\sigma^2}{n} + \tau^2\right)} \left[\frac{\left(\frac{\sigma^2}{n} + \tau^2\right)}{\frac{\sigma^2}{n}} \bar{x}^2 + \frac{\left(\frac{\sigma^2}{n} + \tau^2\right)}{\tau^2} \mu^2 \right. \right.$$

$$-\frac{\sigma^2}{n}\tau^2 \frac{\bar{x}^2}{\left(\frac{\sigma^2}{n}\right)^2} - \left(\frac{\sigma^2}{n}\tau^2\right)\frac{\mu^2}{(\tau^2)^2}$$

$$\left.\left. - 2\cancel{\left(\frac{\sigma^2}{n}\tau^2\right)} \frac{\bar{x}\mu}{\cancel{\left(\frac{\sigma^2}{n}\tau^2\right)}} \right\} \right\}$$

$$\exp\left\{-\frac{1}{2\left(\frac{\sigma^2}{n}+\tau^2\right)}\left[\bar{x}^2 + \mu^2 - 2\bar{x}\mu\right]\right\}$$

$$\Rightarrow \quad m(\bar{x}) = \frac{1}{\left(2\pi\left(\frac{\sigma^2}{n}+\tau^2\right)\right)^{1/2}} \exp\left\{-\frac{1}{2\left(\frac{\sigma^2}{n}+\tau^2\right)}(\bar{x}-\mu)^2\right\}$$

$$\Rightarrow \quad \bar{X}^2 \sim N\left(\mu, \frac{\sigma^2}{n}+\tau^2\right)$$

7.22

(c) Show that $\pi(\theta|\bar{x}, \sigma^2, \mu, \tau^2)$, the posterior distribution of θ, is normal with mean and variance given by 7.2.10

Sol^n $$\pi(\theta|\bar{x}, \sigma^2, \mu, \tau^2) = \frac{f(\bar{x}|\theta, \sigma^2)\,\pi(\theta|\mu, \tau^2)}{f(\bar{x}|\sigma^2, \mu, \tau^2)}$$

$$= \frac{f(\bar{x} \mid \theta, \sigma^2)\, \pi(\theta \mid \mu, \gamma^2)}{m(\bar{x} \mid \sigma^2, \mu, \gamma^2)}$$

$$= \frac{N\left(\theta, \frac{\sigma^2}{n}\right) N(\mu, \gamma^2)}{N\left(\mu, \frac{\sigma^2}{n} + \gamma^2\right)}$$

$$= \frac{\frac{1}{\left(2\pi \frac{\sigma^2}{n}\right)^{1/2}} \cdot \frac{1}{(2\pi \gamma^2)^{1/2}}}{\left(\frac{1}{2\pi\left(\frac{\sigma^2}{n} + \gamma^2\right)}\right)^{1/2}} \quad \frac{\exp\{\ \}\ \exp\{\ \}}{\exp\{\ \}}$$

$$= \frac{1}{(2\pi)^{1/2}} \left(\frac{\frac{\sigma^2}{n} + \gamma^2}{\frac{\sigma^2}{n} \gamma^2} \right)^{1/2}$$

very long

7.23 we know $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

The conjugate prior for σ^2 is the inverted gamma pdf $IG(\alpha,\beta)$, given by

$$\pi(\sigma^2) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \frac{1}{(\sigma^2)^{\alpha+1}} e^{-\frac{1}{\beta\sigma^2}}$$

$0 < \sigma^2 < \infty$

Sol^n

$$f(y|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \frac{1}{y^{\alpha+1}} e^{-\frac{1}{\beta y}}$$

$\alpha>0, \beta>0$

$$\mathbb{E}[Y] = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} \frac{y}{y^{\alpha+1}} e^{-\frac{1}{\beta y}} dy$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty \frac{1}{y^{(\alpha-1)+1}} e^{-\frac{1}{\beta y}} dy$$

$$= \frac{\Gamma(\alpha-1)\beta^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} = \frac{\beta}{\alpha-1}$$

$$\mathbb{E}[y^2] = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty \frac{1}{y^{(\alpha-2)+1}} e^{-\frac{1}{\beta y}} dy$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \Gamma(\alpha-2)\beta^{\alpha-2} = \frac{\beta^2}{(\alpha-1)(\alpha-2)}$$

$$var(x) = \mathbb{E}[y^2] - E[y]^2$$

$$= \frac{\beta^2}{(\alpha-1)(\alpha-2)} - \frac{\beta^2}{(\alpha-1)^2}$$

$$= \frac{\beta^2}{(\alpha-1)}\left[\frac{1}{\alpha-2} - \frac{1}{\alpha-1}\right]$$

$$var(x) = \frac{\beta^2}{(\alpha-1)}\left[\frac{1}{(\alpha-1)(\alpha-2)}\right] = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)}$$

$\alpha > 2$

sol^n

$$\chi^2_{n-1} \sim gamma\left(\frac{n-1}{2}, 2\right)$$

$$f(\chi^2_{n-1}) = \frac{1}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-1}{2}}} x^{\frac{n-1}{2}-1} e^{-\frac{x}{2}}$$

$0 < x < \infty$

let $X \sim \chi^2_{n-1}$ $\qquad Y = S^2$

$$Y = \frac{\sigma^2 X}{(n-1)}$$

$$X = \frac{(n-1)Y}{\sigma^2}$$

$$\frac{dx}{dy} = \frac{n-1}{\sigma^2}$$

$$f_Y(y|\sigma^2) = \frac{1}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-1}{2}}} \left(\frac{(n-1)}{\sigma^2} y\right)^{\frac{n-1}{2}-1} e^{-\frac{n-1}{2\sigma^2} y} \cdot \frac{n-1}{\sigma^2}$$

$0 < y < \infty$

$$\pi(\sigma^2 | \text{data}) \propto L(\sigma^2 | \text{data})\, \pi(\sigma^2)$$

$$\propto \frac{y^{\frac{n-1}{2}-1}}{(\sigma^2)^{\frac{n-1}{2}}} e^{-\frac{(n-1)y}{2\sigma^2}} \cdot \frac{1}{(\sigma^2)^{\alpha+1}} e^{-\frac{1}{\beta\sigma^2}}$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{n-1}{2}+\alpha+1} e^{-\frac{1}{\sigma^2}\left(\frac{1}{\beta}+\frac{(n-1)s}{2}\right)}$$

$$\pi(\sigma^2 \mid \text{Data}) \sim IG\left[\alpha+\frac{n-1}{2}, \left(\frac{(n-1)s^2}{2}+\frac{1}{\beta}\right)^{-1}\right]$$

$$\mathbb{E}\left[\pi(\sigma^2 \mid \text{Data})\right]$$

$$= \frac{1}{\alpha+\frac{n-1}{2}-1}\left(\frac{(n-1)s^2}{2}+\frac{1}{\beta}\right)$$

$$E\left[\pi(\theta \mid \text{Data})\right] = \frac{\frac{(n-1)s^2}{2}+\frac{1}{\beta}}{\alpha+\frac{n-1}{2}+1}$$

7.24 Let $x_1, x_2, \ldots x_n \sim \text{Poisson}(\lambda)$, and let λ have a gamma(α, β) distribution, the Conjugate family for the Poission.

(a) Find the posterior distribution of λ

Sol'n

$$L(\lambda|x) = \prod_{i=1}^{n} e^{-\lambda} \frac{\lambda^{x_i}}{x_i!}$$

$$= \frac{e^{-n\lambda} \lambda^{\sum x_i}}{\prod x_i!}$$

$$\pi(\lambda) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \lambda^{\alpha-1} e^{-\frac{\lambda}{\beta}}$$

$$\pi(\lambda|x) \propto L(\lambda|x)\pi(\lambda)$$

$$\propto \lambda^{\sum x_i + \alpha - 1} e^{-\lambda\left(n + \frac{1}{\beta}\right)}$$

$$\pi(\lambda|x) \sim \text{gamma}\left(\alpha + \sum x_i, \frac{\beta}{n\beta+1}\right)$$

(b) Calculate the posterior mean & var

$$\mathbb{E}[\pi(\lambda|x)] = (\alpha + \sum x_i)\left(\frac{\beta}{n\beta+1}\right)$$

$$\text{var}(\pi(\lambda|x)] = (\alpha + \sum x_i)\left(\frac{\beta^2}{(n\beta+1)^2}\right)$$

7.25

Suppose we observe $x_1, x_2, \ldots x_n \sim N(\theta_i, \sigma^2)$
$i = 1, 2, \ldots, n$ independent.
$\theta_i \sim n(\mu, \tau^2)$ $i = 1, 2, \ldots, n$ independ

(a) Show that the marginal distribution of $x_i \sim N(\mu, \sigma^2 + \tau^2)$

Sol^n we proved in 7.22 b

7.22 b if $\bar{x} | \theta \sim N(\theta, \frac{\sigma^2}{n})$
& $\theta \sim N(\mu, \tau^2)$

then $\bar{x} \sim N(\mu, \frac{\sigma^2}{n} + \tau^2)$

Simillarl if $x_i | \theta_i \sim N(\theta_i, \sigma^2)$
$\theta_i \sim N(\mu, \tau^2)$

then $x_i \sim N(\mu, \sigma^2 + \tau^2)$

7.26 let $X_1, X_2, \ldots X_n \sim N(\theta, \sigma^2)$ and θ have a double exponential distribution $\pi(\theta) = \frac{e^{-\frac{|\theta|}{a}}}{2a}$ a known.

Find the mean of the posterior distribution of θ.

Sol^n

$$L(\theta, \sigma^2 | X) = f(x_1 x_2 x_3 \ldots x_n | \theta, \sigma^2)$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left\{-\frac{1}{2\sigma^2} \sum (x_i - \mu)^2\right\}$$

$$\pi(\theta) = \frac{1}{2a} e^{-\frac{|\theta|}{a}}$$

$$f(x_1, x_2, \ldots x_n, \theta | \sigma^2)$$

$$= \frac{1}{(2\pi\sigma^2)^{n/2}} \frac{1}{2a} \exp\left\{-\frac{1}{2\sigma^2} \sum (x_i - \theta)^2\right\} \exp\left\{-\frac{|\theta|}{a}\right\}$$

this is gonna be long

⋮

Skipping EM algorithm problem's

7.33 in example 7.2.5 the MSE of the Bayes Estimator, $\hat{p}_B$, of a success Probability was calculated. Show that the choice $\alpha = \beta = \sqrt{\frac{n}{4}}$ yield a constant MSE of $\hat{p}_B$

Sol^n

$$MSE(\hat{p}_B) = \mathbb{E}[(\hat{p}_B - p)^2]$$

$$= var(\hat{p}_B) + Bias(\hat{p}_B)^2$$

$$= var_p\left(\frac{Y+\alpha}{\alpha+\beta+n}\right) + \left(E_p\left[\frac{Y+\alpha}{\alpha+\beta+n}\right] - p\right)^2$$

$$MSE(\hat{p}_B) = \frac{np(1-p)}{(\alpha+\beta+n)^2} + \left(\frac{np+\alpha}{\alpha+\beta+n} - p\right)^2$$

for $\alpha = \beta = \sqrt{\frac{n}{4}}$

$$MSE(\hat{p}_B) = \frac{np(1-p)}{(\sqrt{n}+n)^2} + \left(\frac{np+\frac{\sqrt{n}}{2}}{\sqrt{n}+n} - p\right)^2$$

$$= \frac{np(1-p)}{n+n^2+2n\sqrt{n}} + \frac{\left(\frac{\sqrt{n}}{2} - p\sqrt{n}\right)^2}{n+n^2+2n\sqrt{n}}$$

$$= \frac{np(1-p) + \frac{n}{4} + p^2 n - pn}{(n+\sqrt{n})^2}$$

$$= \frac{\frac{n}{4}}{(n+\sqrt{n})^2} = \frac{n}{4(n+\sqrt{n})^2}$$

7.34 Let $X_1, X_2, \ldots X_n$ be a random sample from a binomial (n, p). We want to find equivariant point estimator's of p using the group described in Example 6.4.1

(a) Find the class of Estimator's that are equivariant with respect to this Group

(I haven't read Equivariant)