This document contain's solution's to Exercises 7.37 to 7.49 from chapter 7, of statistical inference by Casella and Berger, focusing on Cramer-Rao inequality and sufficiency & unbaisedness

Some problem's are still in progress, and the document will be updated with their solution's soon.

7.37 let $X_1, \ldots X_n$ be a random sample from a population with pdf

$$f(x|\theta) = \frac{1}{2\theta} \qquad -\theta < x < \theta \ , \theta > 0$$

Find, if one exists, a best unbiased estimator of θ

Sol^n

$$f_X(x|\theta) = \frac{1}{2\theta} \ , \ -\theta < x < \theta \ , \theta > 0$$

$$f_n(x|\theta) = \left(\frac{1}{2\theta}\right)^n I_{[0,\theta)}\left(\max_i |x|\right)$$

$\Rightarrow T(x) = \max_i |x|$

sufficient statistic

let $Y = |x|$

$\Rightarrow A_1 = (-\theta, 0) \quad Y = -X \quad x = -y$

$\Rightarrow \left|\frac{dx}{dy}\right| = |-1| = 1$

$A_2 = (0, \theta) \quad Y = X \quad x = y$

$\frac{dx}{dy} = 1$

$$f_Y(y|\theta) = \frac{1}{2\theta} \cdot 1 + \frac{1}{2\theta} \cdot 1 = \frac{1}{\theta}$$

$Y_1, Y_2, \ldots, Y_n \sim Y$

distoribotion of $Y_{(n)}$

$$f_Y(y|\theta) = \frac{1}{\theta} \quad 0 < y < \theta \quad F(y|\theta) = \frac{y}{\theta}$$

$$f_{Y_{(n)}}(y_{(n)}) = \frac{n!}{(n-1)!} \frac{1}{\theta} \left(\frac{y}{\theta}\right)^{n-1}$$

$$f_{Y_{(n)}}(y_{(n)}) = n \frac{y^{n-1}}{\theta^n}$$

The sufficient statistic

$$f_T(t|\theta) = n \frac{t^{n-1}}{\theta^n} \quad 0 < t < \theta$$

Check if the sufficient statistic is complete

$$0 = \mathbb{E}_\theta[g(T)] = \int_0^\theta g(t) \cdot n \frac{t^{n-1}}{\theta^n} dt$$

$$= \frac{n}{\theta^n} \int_0^\theta g(t) t^{n-1} dt$$

$$\frac{d}{d\theta} \mathbb{E}_\theta[g(T)] = 0 \Rightarrow g(\theta) \theta^{n-1} = 0$$

$$\Rightarrow g(\theta) = 0 \quad \forall \theta$$

$\Rightarrow T = \max_i |x_i|$ is sufficient statistic.

$$\mathbb{E}[T(X)] = \int_0^\theta t \cdot n \frac{t^{n-1}}{\theta^n} dt$$

$$= \frac{n}{\theta^n} \left. \frac{t^{n+1}}{n+1} \right]_0^\theta = \frac{n}{\theta^n} \frac{\theta^{n+1}}{n+1}$$

$$= \theta \cdot \frac{n}{n+1}$$

$$\Rightarrow \quad \frac{n+1}{n} T(X) \Rightarrow \frac{n+1}{n} \max_i |X_i|$$

is the Best unbaised estimator of θ.

(7.38) for each of the following distribution's, let $X_1, X_2, \dots, X_n$ be a random sample. is there a function of θ, say $g(\theta)$, for which there exists an unbaised estimator whose variance attain's the cramer-Rao lower bound? if so, find it, if not, show why not.

(a) $f(x|\theta) = \theta x^{\theta-1} \quad 0<x<1, \theta>0$

Sol^n $f_n(x|\theta) = \theta^n \left(\prod x_i\right)^{\theta-1} \quad 0<x_i<1$
$\theta>0$

$\Rightarrow T(x) = \prod x_i$ is sufficient statistic.

$$L(\theta|x) = \theta^n \left(\prod x_i\right)^{\theta-1}$$

$$\log L(\theta|x) = n \log \theta + (\theta-1) \log \prod x_i$$

$$\frac{\partial}{\partial \theta} \log L(\theta|x) = \frac{n}{\theta} + \log \prod x_i$$

$$= \sum \log x_i + \frac{n}{\theta} = -n\left[\frac{-\sum \log x_i}{n} - \frac{1}{\theta}\right]$$

$$\Rightarrow \mathbb{E}[-\log x_i] = \frac{1}{\theta} \qquad \Rightarrow \mathbb{E}\left[-\frac{\sum \log x_i}{n}\right] = \frac{1}{\theta}$$

Therefore $T(x) = -\frac{\sum \log x_i}{n}$ attain's CRLB

(b) $f(x|\theta) = \frac{\log(\theta)}{\theta-1}\, \theta^x \qquad 0 < x < 1$, $\theta > 1$

Sol^n

$$f_n(x|\theta) = \prod_{i=1}^{n} \frac{\log(\theta)}{\theta-1}\, \theta^{x_i}$$

$$= \left(\frac{\log(\theta)}{\theta-1}\right)^n \theta^{\sum x_i}$$

$$\log f_n(x|\theta) = n\log\left(\frac{\log(\theta)}{\theta-1}\right) + \Sigma x_i \log\theta$$

$$= n\log(\log\theta) - n\log(\theta-1) + \Sigma x_i \log\theta$$

$$\frac{\partial}{\partial\theta}\log f_n(x|\theta)$$

$$= \frac{n}{\log\theta}\cdot\frac{1}{\theta} - \frac{n}{\theta-1} + \frac{\Sigma x_i}{\theta}$$

$$= \frac{n}{\theta}\left[\frac{\Sigma x_i}{n} - \frac{\theta}{\theta-1} + \frac{1}{\log\theta}\right]$$

$$= \frac{n}{\theta}\left[\frac{\Sigma x_i}{n} - \left(\frac{\theta}{\theta-1} - \frac{1}{\log\theta}\right)\right]$$

$\frac{\Sigma x_i}{n}$ attain's CRLB for

$$\frac{\theta}{\theta-1} - \frac{1}{\log\theta}$$

7.39 Prove Lemma 7.3.11

if $f(x|\theta)$ satisfies

$$\frac{d}{d\theta} E_\theta\left[\frac{\partial}{\partial\theta}\log f(x|\theta)\right]$$

$$= \int \frac{\partial}{\partial\theta}\left[\left(\frac{\partial}{\partial\theta}\log f(x|\theta)\right) f(x|\theta)\right] dx$$

then $$\mathbb{E}_\theta\left[\left(\frac{\partial}{\partial\theta}\log f(x|\theta)\right)^2\right]$$

$$= - E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log f(x|\theta)\right]$$

Sol^n $\Rightarrow$ $$\mathbb{E}\left[\frac{\partial}{\partial\theta}\log f(x|\theta)\right] = 0$$

$$\frac{\partial^2}{\partial\theta^2}\log f(x|\theta) = \frac{\partial}{\partial\theta}\left[\frac{\frac{\partial}{\partial\theta}f(x|\theta)}{f(x|\theta)}\right]$$

$$= \frac{f(x|\theta)\frac{\partial^2}{\partial\theta^2}f(x|\theta) - \left(\frac{\partial}{\partial\theta}f(x|\theta)\right)^2}{f(x|\theta)^2}$$

$$\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\log f(x|\theta)\right] = \int_x \underbrace{\frac{\partial^2}{\partial\theta^2} f(x|\theta)\,dx}_{\Rightarrow 0} - \int_x \frac{\left(\frac{\partial}{\partial\theta} f(x|\theta)\right)^2 f(x|\theta)\,dx}{f(x|\theta)^2}$$

$$\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\log f(x|\theta)\right] = -\int_x \left(\frac{\partial}{\partial\theta}\log f(x|\theta)\right)^2 f(x|\theta)\,dx$$

$$\Rightarrow \mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\log f(x|\theta)\right] = -\mathbb{E}\left[\left(\frac{\partial}{\partial\theta}\log f(x|\theta)\right)^2\right]$$

$$\Rightarrow \mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\log f(x|\theta)\right] + \mathbb{E}\left[\left(\frac{\partial}{\partial\theta}\log f(x|\theta)\right)^2\right] = 0$$

7.40 $X_1, X_2, \dots X_n \sim \mathrm{Ber}(P)$. show that the variance of $\bar{X}$ attains the cramer-Rao lower Bound, hence $\bar{X}$ is the Best unbiased estimator of p

Sol^n

$X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Ber}(p)$

$$f_n(x|p) = \prod_{i=1}^{n} p^{x_i} (1-p)^{1-x_i}$$

$$f_n(x|p) = p^{\sum x_i} (1-p)^{n-\sum x_i}$$

$$\log f_n(x|p) = \sum x_i \log p + (n-\sum x_i) \log(1-p)$$

$$\frac{\partial}{\partial p} \log f_n(x|p) = \frac{\sum x_i}{p} - \frac{n-\sum x_i}{1-p}$$

$$= \frac{(1-p)\sum x_i - np + p\sum x_i}{p(1-p)}$$

$$= \frac{\sum x_i - np}{p(1-p)}$$

$$= \frac{n}{p(1-p)} \left[\frac{\sum x_i}{n} - p \right]$$

Hence $\frac{\sum x_i}{n}$ attains CRLB

(7.41) let $x_1, x_2, \ldots x_n$ be a random sample from a population with mean μ and variance σ^2.

(a) Show that the Estimator $\sum_{i=1}^{n} a_i x_i$ is an unbiased estimator of μ if $\sum_{i=1}^{n} a_i = 1$

<u>sol^n</u>

$$T(x) = \sum_{i=1}^{n} a_i x_i$$

$$\mathbb{E}\left[\sum_{i=1}^{n} a_i x_i\right] = \sum_{i=1}^{n} a_i \mathbb{E}[x_i]$$

$$= \mu \sum_{i=1}^{n} a_i$$

$$= \mu$$

$\Rightarrow \sum_{i=1}^{n} a_i x_i$ is an unbaised Estimator.

(b) among all unbiased estimator's, of this form (called Linear unbiased Estimator's) find the one with minimum variance, and calculate the variance.

sol^n

$$\text{Var}\left(\sum_{i=1}^{n} a_i x_i\right) \quad \text{where } \sum_{i=1}^{n} a_i$$

$$\Rightarrow \sum_{i=1}^{n} a_i^2 \, \text{Var}(x_i) \Rightarrow \sigma^2 \sum_{i=1}^{n} a_i^2$$

$$\min_{\sum a_i = 1} \sum_{i=1}^{n} a_i^2$$

7.42

I will come back again

7.43

7.44 let $X_1, X_2, \ldots X_n$ iid $N(\theta, 1)$. Show that the best unbiased estimator of θ^2 is $\overline{X}^2 - \frac{1}{n}$. Calculate its variance and show that it is greater than the Cramer-Rao Lower Bound.

Sol^n $X_1, X_2 \ldots X_n \sim N(\theta, 1)$

$\overline{X}$ is the best unbiased estimator of θ

we want best unbiased estimator for θ^2

$\Rightarrow$ find an unbiased estimator for θ^2 and apply Rao-Blackwell theorem.

$$\mathbb{E}[X_i^2] = var(x) + E[x]^2$$

$$= 1 + \theta^2 \Rightarrow E[X_i^2 - 1] = \theta^2$$

$$h(x) = x^2 - 1$$

$$\phi(\bar{X}) = \mathbb{E}[X_1^2 - 1 \mid \bar{X}]$$

$$= \mathbb{E}[X_1^2 \mid \bar{X}] - 1$$

$$(n\bar{x})^2 - 2\,\frac{n(n-1)}{2}(\bar{x})^2 =$$

we know that $\bar{X}$ is complete sufficient statistic

Then any function $\bar{X}$ will give Best unbaised Estimator.

$$\mathrm{Var}(\bar{X}^2 - 1/n) \qquad \bar{X} \sim N(0, \tfrac{1}{n})$$

Stein's lemma:

$$\mathbb{E}[g(x)(x-\theta)] = \sigma^2\, \mathbb{E}[g'(x)]$$

we need $\mathrm{var}(\bar{X}^2 - \frac{1}{2}) = \mathrm{Var}(\bar{X}^2)$

let's take $T = \bar{X}$

$$\mathrm{Var}(T^2) = \mathbb{E}[(T^2 - E[T^2])^2]$$

$$= \mathbb{E}[T^4] - \mathbb{E}[T^2]^2$$

$$E[T^4] = \mathbb{E}[T^3(T-\theta)] + \mathbb{E}[T^3\theta]$$

$$= \sigma^2 3\mathbb{E}[T^2] + \theta\,\mathbb{E}[T^3]$$

$$E[T^3] = \mathbb{E}[T^2(T-\theta)] + \theta\,\mathbb{E}[T^2]$$

$$= \sigma^2 2\,E[T] + \theta\,E[T^2]$$

$$= 2\theta\sigma^2 + \theta(\sigma^2 + \theta^2)$$

$$= 3\theta\sigma^2 + \theta^3$$

$$E[T^4] = 3\sigma^2(\sigma^2 + \theta^2) + \theta(3\theta\sigma^2 + \theta^3)$$

$$= \theta^4 + 3\theta^2\sigma^2 + 3\theta^2\sigma^2 + 3\sigma^4$$

$$= \theta^4 + 6\theta^2\sigma^2 + 3\sigma^4$$

$$E[T^2]^2 = (\sigma^2 + \theta^2)^2 = \sigma^4 + \theta^4 + 2\sigma^2\theta^2$$

$$\text{var}(T^2) = E[T^4] - E[T^2]^2$$

$$= 2\sigma^2 + 4\theta^2\sigma^2$$

$$\text{var}(T^2) = \frac{2}{n} + \frac{4\theta^2}{n}$$

Cramer-Rao Lower Bound

$$\frac{\left(\frac{d}{d\theta}\tau(\theta)\right)^2}{I(\theta)}$$

$$f(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\theta)^2\right\}$$

$$\log f(x) = -\frac{1}{2}\log 2\pi\sigma^2 - \frac{1}{2\sigma^2}(x-\theta)^2$$

$$\frac{\partial}{\partial\theta}\log f(x) = -\frac{1}{2\sigma^2}\cdot 2(x-\theta)\cdot -1 = \frac{x-\theta}{\sigma^2}$$

$$\frac{\partial^2}{\partial\theta}\log f(x) = -\frac{1}{\sigma^2}$$

$$I(\theta) = \frac{n}{\sigma^2} = n$$

$$CRLB = \frac{4\theta^2}{n}$$

$$var\left(\bar{x}^2 - \frac{1}{n}\right) = \frac{2}{n} + \frac{4\theta^2}{n} > \frac{4\theta^2}{n} = CRLB$$

$\bar{x}^2 - \frac{1}{n}$ could not able to achieve CRLB

7.45 $x_1, x_2, \ldots, x_n \sim X$ mean $= \theta$, var $= \sigma^2$

(a) Show that for any estimator of the form aS^2, where a is constant.

$$MSE(aS^2) = \mathbb{E}[aS^2 - \sigma^2]^2$$

$$= a^2 \, var(S^2) + (a-1)^2 \sigma^4$$

Soln

$MSE(aS^2)$

$$= \mathbb{E}\left[(as^2-\sigma^2)^2\right]$$

$$= \mathbb{E}\left[a^2\left(s^2-\frac{\sigma^2}{a}\right)^2\right]$$

$$= a^2\,\mathbb{E}\left[\left(s^2-\frac{\sigma^2}{a}\right)^2\right]$$

$$= a^2\,\mathbb{E}\left[\left[(s^2-\sigma^2)+\left(\sigma^2-\frac{\sigma^2}{a}\right)\right]^2\right]$$

$$= a^2\,\mathbb{E}\left[(s^2-\sigma^2)^2+\left(\sigma^2-\frac{\sigma^2}{a}\right)^2 + 2(s^2-\sigma^2)\left(\sigma^2-\frac{\sigma^2}{a}\right)\right]$$

$$= a^2\,\mathbb{E}\left[(s^2-\sigma^2)^2\right]+a^2\left(\sigma^2-\frac{\sigma^2}{a}\right)^2$$

$$+0$$

$$\Rightarrow \quad a^2\,\mathrm{var}(s^2) + \sigma^4 a^2\left(1-\frac{1}{a}\right)^2$$

$$\mathrm{MSE}[as^2] = a^2\,\mathrm{var}(s^2) + \sigma^4(a-1)^2$$

(b) Show that $Var(S^2) = \frac{1}{n}\left(k - \frac{n-3}{n-1}\sigma^4\right)$

where $k = \mathbb{E}[(X-\mu)^4]$

sol^n

$$Var(S^2) = \mathbb{E}\left[\left(S^2 - E[S^2]\right)^2\right]$$

$$S^2 = \frac{1}{n-1}\sum(x-\bar{x})^2$$

$$\Rightarrow \quad S^2 = \frac{1}{n-1}\left(\sum x_i^2 - \bar{x}^2\right)$$

It's a long proof..

7.46 $X_1, X_2, X_3 \sim \text{unif}(\theta, 2\theta)$ $\theta > 0$

(a) Method of moment's

Sol^n

$$E[X] = \int_{\theta}^{2\theta} \frac{x}{\theta}\, dx = \frac{1}{\theta} \cdot \frac{x^2}{2}\Big]_{\theta}^{2\theta}$$

$$= \frac{1}{\theta}\left(\frac{4\theta^2 - \theta^2}{2}\right)$$

$$E[X] = \frac{3}{2}\theta$$

$$\hat{\theta}_{MOM} \cdot \frac{3}{2} = \frac{1}{n} \sum X_i$$

$$\Rightarrow \hat{\theta}_{MOM} = \frac{2}{3}\bar{X} = \frac{2}{9}(X_1 + X_2 + X_3)$$

(b) find the MLE, $\hat{\theta}$, and find a constant k such that $E[k\hat{\theta}] = \theta$

Sol^n

$$f_X(x|\theta) = \begin{cases} \frac{1}{\theta} & \theta < x < 2\theta \\ 0 & \text{o.w} \end{cases}$$

$$= \begin{cases} \frac{1}{\theta} & \theta < x, \ \frac{x}{2} < \theta \\ 0 & \text{o.w} \end{cases}$$

$$L(\theta|x) = f_n(x|\theta) = \begin{cases} \frac{1}{\theta^n} & \frac{x_{(n)}}{2} < \theta < x_{(1)} \\ 0 & \text{o.w} \end{cases}$$

$$L(\theta|x) = \frac{1}{\theta^n} I_{\left(\frac{x_{(n)}}{2},\, x_{(1)}\right)}(\theta)$$

since $\frac{1}{\theta^n}$ is decreasing function.

The maximum will be at $\frac{x_{(n)}}{2}$

$$\hat{\theta}_{MLE} = \frac{X_{(n)}}{2}$$

$$f_X(x|\theta) = \frac{1}{\theta} \qquad \theta < x < 2\theta$$

$$F_X(x|\theta) = \begin{cases} 0 & x < \theta \\ \frac{x-\theta}{\theta} & \theta \leq x < 2\theta \\ 1 & x \geq 2\theta \end{cases}$$

$$f_{X_{(n)}}(x_{(n)}) = n\left(\frac{1}{\theta}\right)\left(\frac{x_{(n)}-\theta}{\theta}\right)^{n-1}$$

$$\Rightarrow \quad f_{X_{(n)}}(X_{(n)}) = \frac{n}{\theta^n}(X_{(n)}-\theta)^{n-1}$$

$$E[X_{(n)}] = \int_{\theta}^{2\theta} \frac{n}{\theta^n}(x-\theta)^{n-1} x \, dx$$

$y = x-\theta$

$dy = dx$

$$= \int_0^{\theta} \frac{n}{\theta^n} y^{n-1}(y+\theta)\, dy$$

$$= \frac{n}{\theta^n}\int_0^{\theta} y^n dy + \frac{n}{\theta^n}\int_0^{\theta} \theta \cdot y^{n-1} dy$$

$$= \frac{n}{\theta^n} \left.\frac{y^{n+1}}{n+1}\right]_0^{\theta} + \frac{\cancel{n}}{\theta^{n-1}} \left.\frac{y^n}{\cancel{n}}\right]_0^{\theta}$$

$$E[X_{(n)}] = \frac{n\theta}{n+1} + \theta = \frac{2n+1}{n+1}\theta$$

$$\Rightarrow \quad \mathbb{E}\left[\frac{2n+2}{2n+1}\hat{\theta}_{MLE}\right] = \theta$$

$$\Rightarrow \quad k = \frac{2n+2}{2n+1}$$

(c) Which of the two estimators can be improved by using sufficiency? How?

Soln

$$f_n(x|\theta) = \left(\frac{1}{\theta}\right)^n I_{[\theta,2\theta)}(x_{(1)})\, I_{(\theta,2\theta)}(x_{(n)})$$

$T(x) = (x_{(1)}, x_{(n)})$ is minimal sufficient statistic

$$\hat{\theta}_{MLE} \cdot \frac{2n+2}{2n+1} = \frac{x_{(n)}}{2} \frac{2n+2}{(2n+1)}$$

$$\hat{\theta}_{MOM} = \frac{2}{3}\bar{x}$$

Both of them are unbiased estimator

The best unbiased estimator is unique & function of sufficient statistic

$\Rightarrow$ $\hat{\theta}_{MLE}$ is function of sufficient statistic

$\Rightarrow$ $\hat{\theta}_{MLE} \cdot \frac{2n+2}{2n+1}$ is Best unbiased Estimator

$\Rightarrow$ $\hat{\theta}_{MOM}$ can be improved using sufficiency.

(d) find the MOM and the MLE of θ based on the data 1.29, 0.86, 1.33

Soln

$$\hat{\theta}_{MOM} = \frac{2}{3}\bar{X} = \frac{2}{9}(1.29+0.86+1.33)$$

$$\hat{\theta}_{MOM} = 0.733$$

$$\hat{\theta}_{MLE} = \frac{X_{(n)}}{2} = \frac{1.33}{2} = 0.665$$

7.47 $\varepsilon \sim N(0,\sigma^2)$. n independent measurement's are made, find an unbiased estimator of the area of the circle. is it best unbiased? (assuming σ^2 = known)

$\underline{\underline{\text{Sol}^n}}$ $A = \pi R^2$ $R \sim N(r, \sigma^2)$

we want to find an unbaised estimator for πR^2

$\hat{r}_{MLE} = \bar{X}$ unbaised estimator for radius.

$$\bar{X} \sim N\left(r, \frac{\sigma^2}{n}\right)$$

$$\mathbb{E}[\bar{X}^2] = r^2 + \frac{\sigma^2}{n}$$

$$\Rightarrow \mathbb{E}\left[\bar{X}^2 - \frac{\sigma^2}{n}\right] = r^2$$

$$\mathbb{E}\left[\pi\left(\bar{X}^2 - \frac{\sigma^2}{n}\right)\right] = \pi r^2$$

$\pi\left(\bar{X}^2 - \frac{\sigma^2}{n}\right)$ is an unbaised estimator of Area πr^2

$\pi\left(\bar{X}^2 - \frac{\sigma^2}{n}\right)$ is a function of sufficient statistic $\Longrightarrow$ unique & Best unbaised Estimator

7.48 Suppose that $x_i, i=1,2,\dots,n$ are iid Ber (p)

(a) Show that the variance of the MLE of p attains the CRLB.

Sol^n

$$x_1, x_2, \dots x_n \sim \text{Ber}(p)$$

$$f_n(x|p) = p^{\sum x_i}(1-p)^{n-\sum x_i}$$

$$T(x) = \sum x_i \quad \text{sufficient statistic}$$

$$\log f_n(x|p) = \sum x_i \log p + (n-\sum x_i)\log(1-p)$$

$$\frac{\partial l}{\partial p} = \frac{\sum x_i}{p} - \frac{n-\sum x_i}{1-p} = 0$$

$$\Rightarrow \hat{p}_{MLE} = \frac{\sum x_i}{n}$$

$\Rightarrow$ $\hat{p}_{MLE}$ is the Best Unbiased Estimator of p (function of sufficient statistic)

$$\text{Var}(\hat{p}_{MLE}) = \frac{1}{n^2} \cdot n \,\text{Var}(x)$$

$$= \frac{p(1-p)}{n}$$

$$CRLB = \frac{\left(\frac{d}{d\theta}\tau(\theta)\right)^2}{I(\theta)}$$

$$\frac{\partial l}{\partial p} = \frac{\sum x_i}{p} - \frac{n - \sum x_i}{1-p}$$

$$\frac{\partial^2 l}{\partial p^2} = -\frac{\sum x_i}{p^2} - \frac{n - \sum x_i}{(1-p)^2}$$

$$\Rightarrow I(\theta) = -E\left[\frac{\partial^2 l}{\partial p^2}\right] = \mathbb{E}\left[\frac{\sum x_i}{p^2} + \frac{n - \sum x_i}{(1-p)^2}\right]$$

$$= \frac{np}{p^2} + \frac{n - np}{(1-p)^2}$$

$$= \frac{n}{p} + \frac{n}{1-p} = \frac{n}{p(1-p)}$$

$$CRLB = \frac{1}{I(\theta)} = \frac{p(1-p)}{n} = Var(\hat{p}_{MLE})$$

(b)

for $n \geq 4$, show that the product $x_1 x_2 x_3 x_4$ is an unbiased estimator of p^4, and use the fact to find the best unbiased estimator of p^4

Sol^n $\quad X_1 X_2 X_3 X_4$

$$\Rightarrow \mathbb{E}[X_1 X_2 X_3 X_4] = E[X_1]E[X_2]E[X_3]E[X_4]$$
$$= p^4$$

$\Rightarrow$ $X_1 X_2 X_3 X_4$ is an unbaised Estimator of p^4

Apply Rao-Blackwell theorem

$$\phi(T) = \mathbb{E}[W \mid T]$$
$$= \mathbb{E}[X_1 X_2 X_3 X_4 \mid \bar{X}]$$

$$f(x_1 x_2 x_3 x_4 \mid \bar{x})$$
$$= \frac{f(x_1 x_2 x_3 x_4, \bar{x})}{f(\bar{x})}$$

$$\mathbb{E}[X_1 X_2 X_3 X_4 \mid \bar{X}] = \frac{\mathbb{P}(X_1=1, X_2=1, X_3=1, X_4=1, \Sigma x = t \geq 4)}{P(\Sigma x = t)}$$

$$= \frac{p^4 \binom{n-4}{t-4} p^{t-4} (1-p)^{n-4}}{\binom{n}{t} p^t (1-p)^{n-t}}$$

$$E[x_1 x_2 x_3 x_4 | \bar{x}] = \frac{\binom{n-4}{t-4}}{\binom{n}{t}}$$

This should be the best unbaised estimator for p^4, according to Rao-Blackwell theorem.

7.49 $x_1, x_2, \ldots x_n \sim \exp(\lambda)$

(a) find UE of λ based only on $Y = \min\{x_1, \ldots x_n\}$

soln

$$f_x(x|\lambda) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \quad 0 < x < \infty$$

$$f_n(x|\lambda) = \frac{1}{\lambda^n} e^{-\frac{\sum x_i}{\lambda}}$$

$T(x) = \sum x_i$ is sufficient statistic.

we want an unbiased estimator based on $X_{(1)}$

$$f(x|\lambda) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}}$$

$$F(x|\lambda) = \int_0^x \frac{1}{\lambda} e^{-\frac{x}{\lambda}} dx$$

$\frac{x}{\lambda} = y$

$\frac{dx}{\lambda} = dy$

$$= -e^{-\frac{x}{\lambda}} \Big]_0^x$$

$$F(x|\lambda) = 1 - e^{-\frac{x}{\lambda}} \qquad x > 0$$

$$f_{X_{(1)}}(x) = \frac{n!}{(n-1)!} f_x(x) \left(1 - F_x(x)\right)^{n-1}$$

$$= n \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \left(e^{-\frac{x}{\lambda}}\right)^{n-1}$$

$$f_{X_{(1)}}(x) = \frac{n}{\lambda} e^{-\frac{nx}{\lambda}} \qquad x>0$$

$$\sim \exp\left(\frac{\lambda}{n}\right)$$

$$\sim \text{gamma}\left(1, \frac{\lambda}{n}\right)$$

$$\mathbb{E}[X_{(1)} \mid \lambda] = \frac{\lambda}{n}$$

$\Rightarrow$ unbiased estimator of λ

using $Y = \min\{x_1, x_2, \dots x_n\}$

$$\Rightarrow \hat{\lambda} = nY$$

(b) Find a better estimator than the one in part (a). Prove it is better

sol^n we know : $\sum x_i$ is complete sufficient statistic so the best unbiased Estimator for λ

should be function of $\sum x_i$

$$\mathbb{E}[\sum x_i] = \sum \mathbb{E}[x_i] = n\mathbb{E}[x]$$

$$= n\lambda$$

$\Rightarrow$ $\hat{\lambda} = \overline{X}$ is the Best

unbiased estimator of λ.

(c) Data x: 50.1, 70.1, 137.0, 166.9,

170.5, 152.8, 80.5, 123.5, 112.6, 148.5

160.0, 125.4

Soln

$$\hat{\lambda}_1 = n\, X_{(1)}$$

$$= 12 \cdot 50.1 = 601.2$$

$$\hat{\lambda}_2 = \overline{X} = 124.825$$