Law of iterated expectation, law of total variance

from chapter 4 that if X & Y are any two random variable's, then

$$\mathbb{E}_X[X] = \mathbb{E}_Y\left[E_{X|Y}[X|Y]\right]$$ (Law of iterated expectation)

$$\mathrm{Var}(X) = \mathrm{Var}\left(E[X|Y]\right) + \mathbb{E}\left[\mathrm{Var}(X|Y)\right]$$ (Law of total variance)

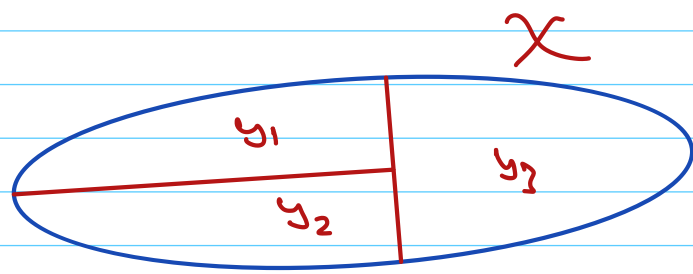
Intution + Proof:

These Law's Assuming that we are "Breaking up" the sample space of X based on the value's of some other r.v Y.

* each relization assume's that, First we

draw Y, from its unconditional distribution $f_Y(y)$, then sample X from its conditional distribution $f_{X|Y}(x|y=y)$

Ex:

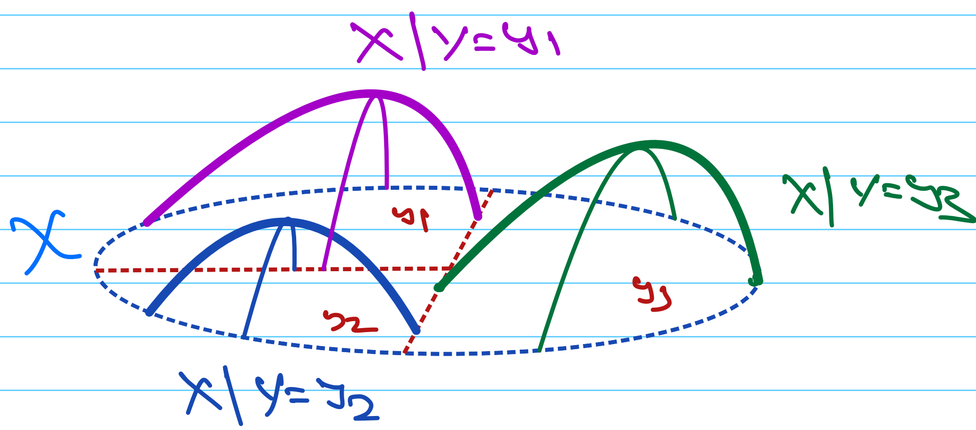


Breaking up sample space X into 3 disjoint set's and defining a r.v Y over them

$P(Y=y_1) = p_1$

$P(Y=y_2) = p_2$ $\qquad p_1+p_2+p_3=1$

$P(Y=y_3) = p_3$

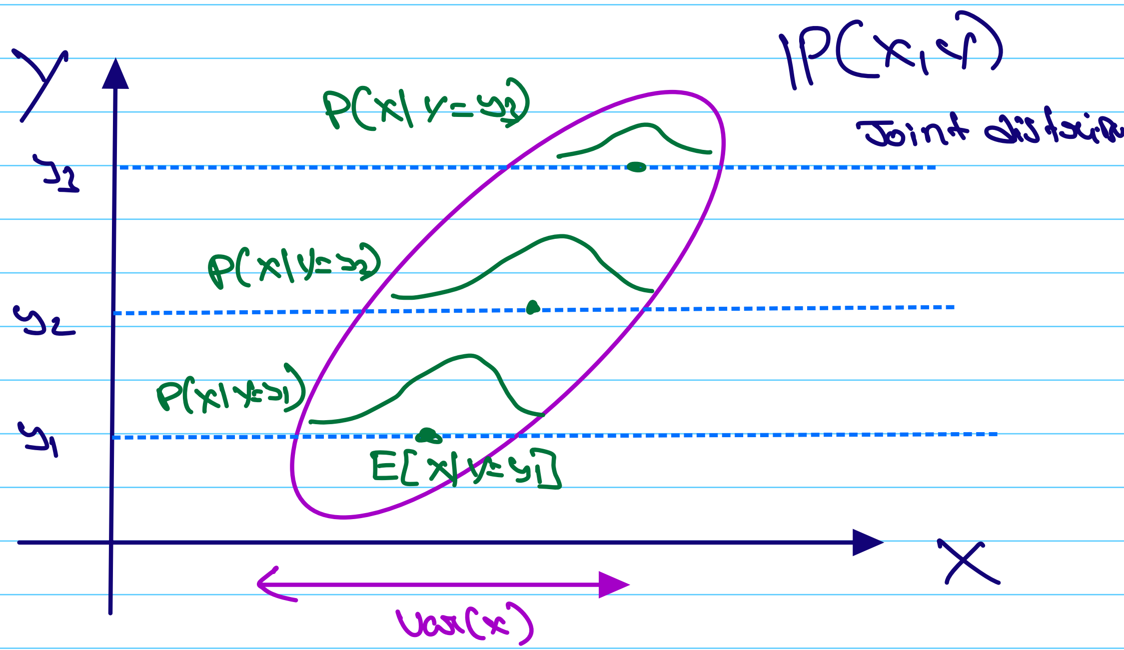


$$\mathbb{E}[X] = \mathbb{E}[X|y=y_1]\,\mathbb{P}(y=y_1)$$

$$+ \mathbb{E}[X|Y=y_2]\,\mathbb{P}[Y=y_2]$$

$$+ \mathbb{E}[X|Y=y_3]\,\mathbb{P}(Y=y_3)$$

Ex2



$f(x,Y)$ is a Joint distribution of X,Y

$$\mathbb{E}[X] = \int\int x\, f(x,y)\, dx\, dy$$

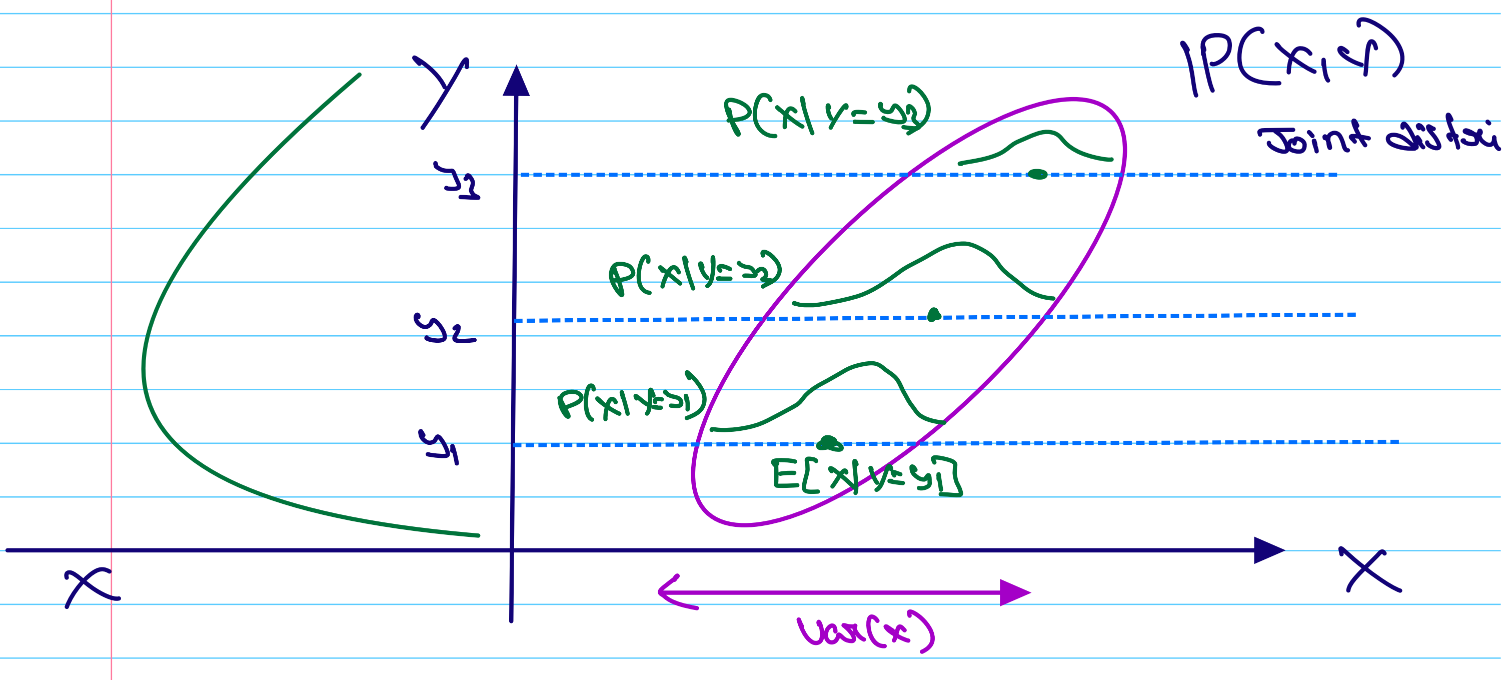
$$= \int\int x\, f(x|y)\, f(y)\, dx\, dy$$

$$= \int \left[\int x \, f(x|y) \, dx \right] f(y) \, dy$$

$$= \int E_{X|Y}[X \mid Y=y] \, f(y) \, dy$$

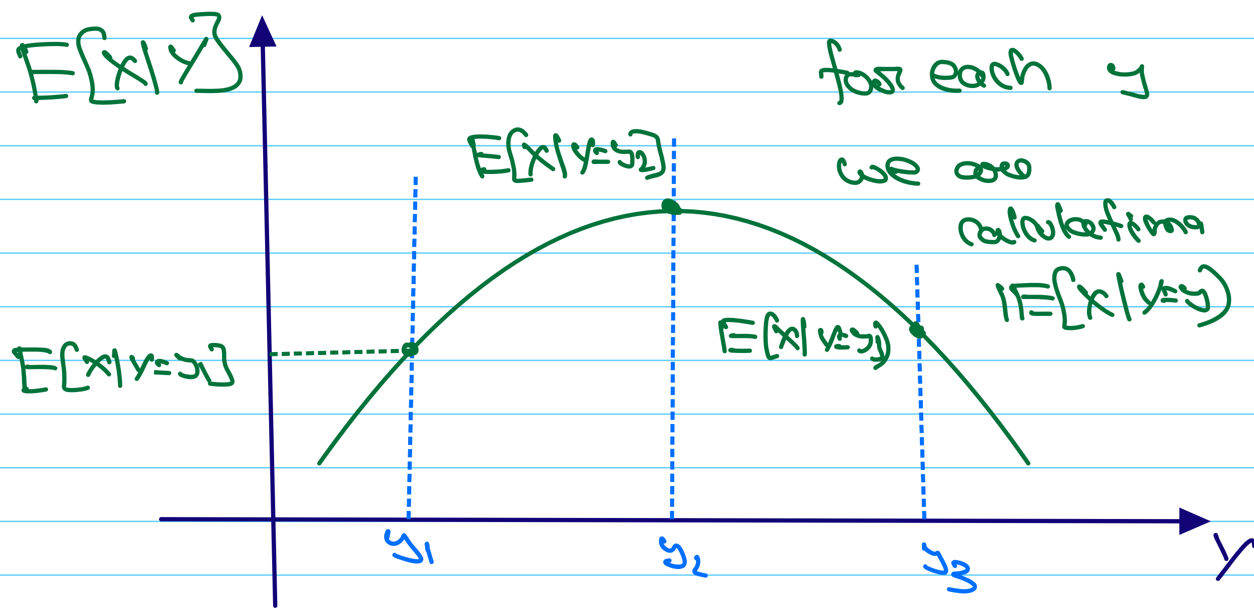
$$= \mathbb{E}_Y \left[\mathbb{E}_{X|Y}[X|Y] \right]$$

* $\mathbb{E}[X|Y]$ is a random variable



$\mathbb{E}[X|Y]$ is a random variable

where the sample space of this r.v is Y, and the value's of r.v lives in X



(This is not a probability distribution)
(Just a Transformation)

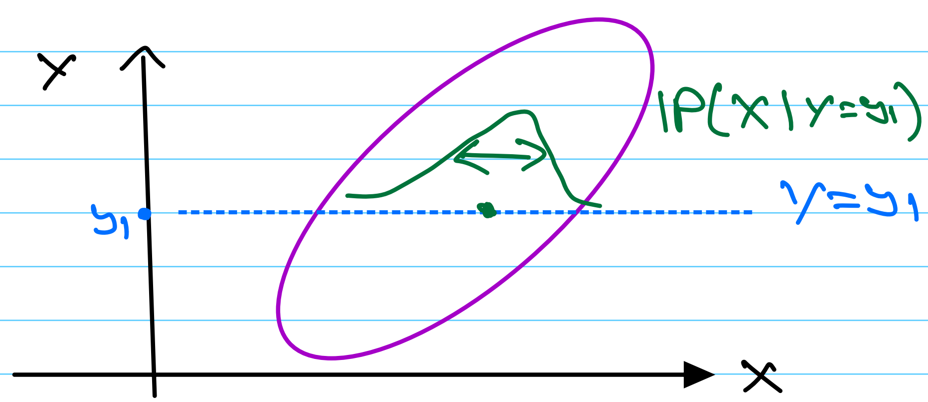
② Law of total Variance :

$$\mathrm{Var}(X) = \mathrm{Var}(\mathbb{E}[X|Y]) + \mathbb{E}[\mathrm{Var}(X|Y)]$$

Both $\mathrm{Var}(X|Y)$ & $\mathbb{E}[X|Y]$ are random variable's

$\mathrm{Var}(X|Y) \Rightarrow$ at each $Y=y$ we are calculating variance of the distribution $X|Y=y$ i.e

$$\mathrm{Var}(X|Y=y) = \mathbb{E}\left[\left(X - \mathbb{E}[X|Y=y]\right)^2\right]$$



$\mathbb{E}[X|Y]$ is a r.v

at each $Y=y_1$ we are calculating Expectation of the $X|Y=y_1$ distribution & forming a r.v

$$var(X) = \mathbb{E}[Var(X|Y)] + Var(E[X|Y])$$

* The 1^{st} term say's that we want the expected variance of X as we average over all value of Y. However, remember that the $var(X|Y=y)$ is taken w.r.t Conditional mean $\mathbb{E}[X|Y=y]$. Therefore, this does not take into account the movement's of the mean itself, Just the variation about each possible varying mean.

* This is where the second term comes in, it does not care about the variability about $\mathbb{E}[Y|X=x]$, just the variability of $\mathbb{E}[Y|X]$ itself.