Methods OF MOMENTS

$X_1, X_2, \ldots, X_n \sim f(x \mid \theta_1, \theta_2, \ldots \theta_k)$

Method of moments estimators are found by equating the first k sample moments to the corresponding k population moments, and solving the resulting system of simultaneous eqn. More precisely, define

$$m_1 = \frac{1}{n} \sum_{i=1}^{n} X_i^1 \quad , \quad \mu_1' = E[X^1]$$

$$m_2 = \frac{1}{n} \sum_{i=1}^{n} X_i^2 \quad , \quad \mu_2' = \mathbb{E}[X^2]$$

$$\vdots$$

$$m_k = \frac{1}{n} \sum_{i=1}^{n} X_i^k \quad , \quad \mu_k' = \mathbb{E}[X^k]$$

each μ_j' is a function of $\theta_1, \theta_2, \ldots \theta_k$

$\mu_j'(\theta_1, \theta_2, \ldots \theta_k)$

Solving the following system of eqn for $(\theta_1, \theta_2, \ldots \theta_k)$ in terms of $(m_1, m_2, \ldots m_k)$:

$$\Rightarrow \quad m_1 = \mu_1'(\theta_1, \ldots, \theta_k)$$
$$m_2 = \mu_2'(\theta_1, \theta_2, \ldots, \theta_k)$$
$$\vdots$$
$$m_k = \mu_k'(\theta_1, \theta_2, \ldots, \theta_k)$$

Example 7.2.1 (Normal method of moments)

$X_1, X_2, \ldots, X_n \overset{iid}{\sim} n(\theta, \sigma^2)$

$$\Rightarrow \quad \theta_1 = \mu, \quad \theta_2 = \sigma^2$$

we have $m_1 = \bar{x}$, $m_2 = \frac{1}{n}\sum x_i^2$

$\mu_1' = \theta$ $\quad \mu_2' = \mathbb{E}[x^2]$

$= var(x) + E[x]^2$

$= \sigma^2 + \theta^2$

$\Rightarrow \quad \bar{x} = \theta$

$\frac{1}{n}\sum x_i^2 = \theta^2 + \sigma^2$

Solve for θ & σ^2

$\Rightarrow \quad \hat{\theta} = \bar{x}$

$\hat{\sigma}^2 = \frac{1}{n}\sum x_i^2 - \bar{x}^2$

$= \frac{1}{n}\left(\sum x_i^2 - n\bar{x}^2\right)$

$= \frac{1}{n}\left(\sum (x_i - \bar{x})^2\right)$

Example 7.2.2 (Binomial method of moments)

$X_1, X_2, \ldots, X_n$ iid binomial (k, p) that

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$$\mathbb{P}(X_i \mid k, p) = \binom{k}{x} p^x (1-p)^{k-x}$$

$x = 0, 1, 2, \ldots k$

we have

$$m_1 = \frac{1}{n} \sum X_i$$

$$m_2 = \frac{1}{n} \sum X_i^2$$

$$\mu_1' = \mathbb{E}[X]$$

$$= kp$$

$$\mu_2' = \mathbb{E}[X^2] = kp(1-p) + k^2p^2$$

therefore

$$\left.\begin{aligned} \bar{X} &= kp \\ \frac{1}{n}\sum X_i^2 &= kp(1-p) + k^2p^2 \end{aligned}\right\} \Rightarrow \text{solve for } \hat{k}, \hat{p}$$

$$\hat{p} = \frac{\bar{X}}{\hat{k}}$$

$$\frac{1}{n}\sum x_i^2 = \bar{x}\left(1-\frac{\bar{x}}{\hat{k}}\right)+\bar{x}^2$$

$$\Rightarrow \frac{\bar{X}^2}{\hat{k}} = \bar{x}+\bar{x}^2-\frac{1}{n}\sum x_i^2$$

$$\Rightarrow \frac{\bar{X}^2}{\hat{k}} = \bar{x}^2-\frac{1}{n}\left(\sum x_i^2-n\bar{x}\right)$$

$$\Rightarrow \frac{\bar{X}^2}{\hat{k}} = \bar{x}^2-\frac{1}{n}\left(\sum(x_i-\bar{x})^2\right)$$

$$\Rightarrow \frac{\hat{k}}{\bar{X}^2} = \frac{1}{\frac{1}{n}\bar{x}^2-\sum(x_i-\bar{x})^2}$$

$$\boxed{\Rightarrow \hat{k} = \frac{\bar{x}^2}{\frac{1}{n}\bar{x}^2-\sum_{i=1}^{n}(x_i-\bar{x})^2} \qquad \hat{p} = \frac{\bar{X}}{\hat{k}}}$$