IOE 611/ Math 663
Lecture 2: Convex sets (2)

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Recap

- **affine combination** of x_1 and x_2: any point x of the form

$$x = \theta x_1 + (1-\theta) x_2 \qquad \text{where } \theta \in \mathbf{R}$$

  affine set: contains the line through any two distinct points in the set (i.e., a set that's closed under affine combinations)
  example: solution set of linear equations $\{x \mid Ax = b\}$

- **convex combination** of x_1 and x_2: any point x of the form

$$x = \theta x_1 + (1-\theta) x_2 \text{ where } 0 \leq \theta \leq 1$$

  convex set: contains line segment between any two points in the set (i.e., closed under convex combinations)

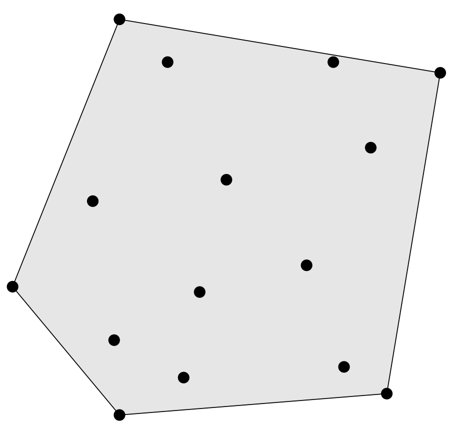
Convex combination and convex hull

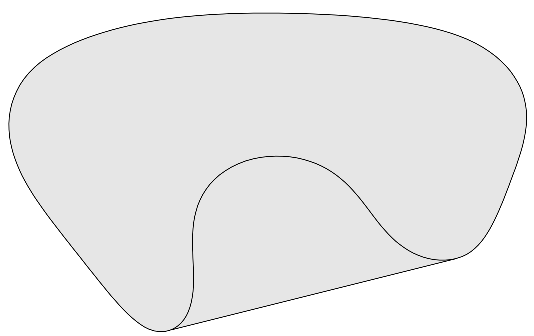
convex combination of $x_1, \ldots, x_k$: any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \geq 0$

convex hull conv S: set of all convex combinations of points in S





Convex hull of S is the smallest convex set that contain's S.

Proof:

$$\left.\begin{array}{l} S \subseteq D \\ D \text{ convex} \end{array}\right\} \Rightarrow \operatorname{conv} S \subseteq D$$

Contradiction:

Suppose this is not the case $\operatorname{conv} S \nsubseteq D$

$\Rightarrow$ $\exists\, x \in \operatorname{conv} S$ such that $x \notin D$

$\Rightarrow$ we know $x \in \operatorname{conv} S$, so by definition of convex hull $\exists\, x_1, x_2, \dots, x_k \in S$

& $\theta_1 + \theta_2 + \dots + \theta_k = 1$, $\theta_i \geq 0\ \forall i$

with $x = \theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k$

$\Rightarrow$ Since $S \subseteq D \Rightarrow x_1, x_2, \dots x_k \in D$

$\Rightarrow$ Since D is convex then the convex combination of $x_1, x_2, \dots, x_k \in D$

$\Rightarrow$ $x \in D$ or D is not convex if $x \notin D$

$\Rightarrow$ contradicts the convexity of D

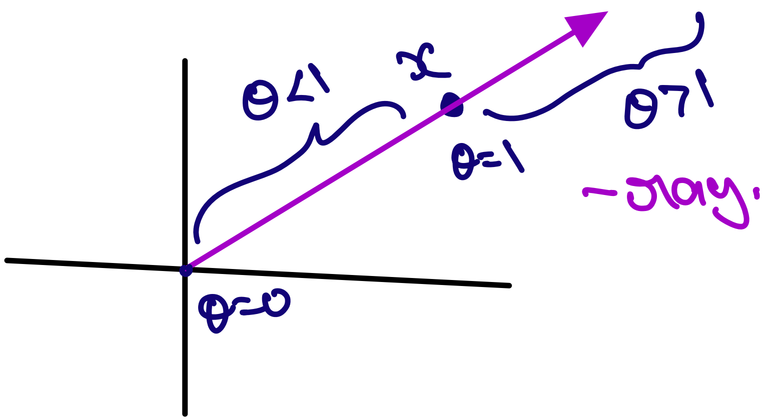
Convex cones

x can be scalar, vector, matrix etc

$x \in \mathbb{R}^n$, $\mathbb{R}^{n \times m}$, $\mathbb{R}$ etc.

cone: a set C such that for any $x \in C$ and $\theta \geq 0$, $\theta x \in C$

* intutively we can think of a cone as set of points that the ray passing through that point from origin also contain's in set.



is cone convex? (Not conic combination)

NO

② Are Affine sets Cones ? NO

for them to be Cone, they need to Pass through Origin

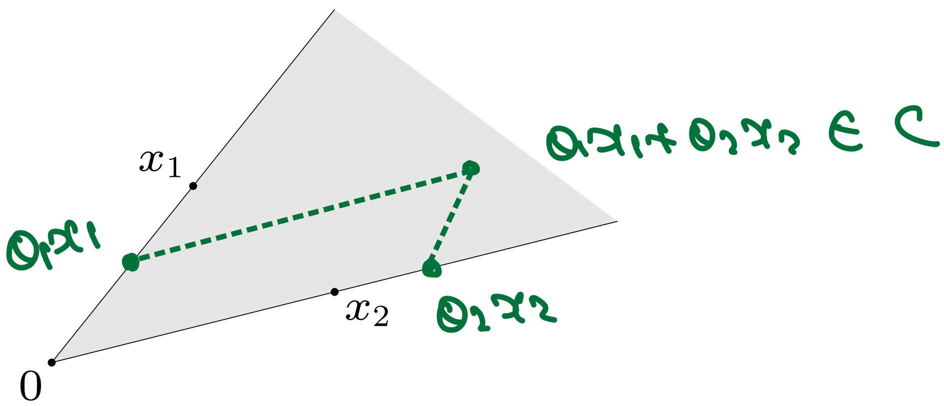
③ Convex set's are Cones? NO

Convex cones

cone: a set C such that for any $x \in C$ and $\theta \geq 0$, $\theta x \in C$

conic (nonnegative) combination of x_1 and x_2: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2 \text{ where } \theta_1, \theta_2 \geq 0$$



Are conic combination convex? yes

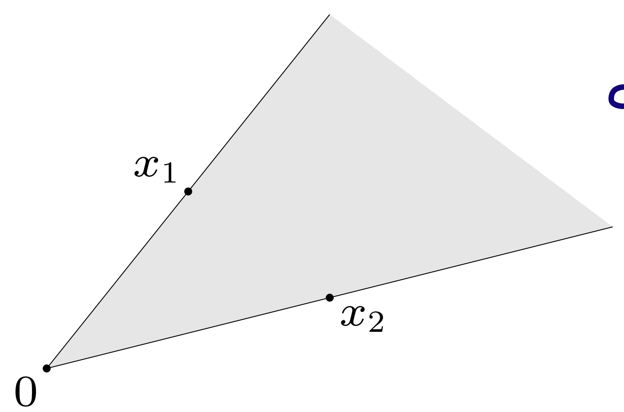
$x = \theta_1 x_1 + \theta_2 x_2 \in C$ where $\theta_1, \theta_2 \geq 0$ $\Longrightarrow$ $x = \theta_1 x_1 + \theta_2 x_2 \in C$ where $\theta_1, \theta_2 \geq 0$, $\theta_1 + \theta_2 = 1$ (convex)

Convex cones

cone: a set C such that for any $x \in C$ and $\theta \geq 0$, $\theta x \in C$

conic (nonnegative) combination of x_1 and x_2: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2 \text{ where } \theta_1, \theta_2 \geq 0$$



if a cone containts all of its conic combination then it is convex. Why?

convex cone: a cone that is convex, *i.e.*, set that contains all conic combinations

$$\{\theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k \mid \theta_i \geq 0\}$$

of points in the set (i.e., closed under conic combinations)

Examples

set that contain's only one element

- Singleton $C = \{x_0\}$

only singleton that is cone is $C = \{0\}$

Affine ✓ Convex ✓ Cone ✗

- Line $C = \{x : a^T x = b\}$

(only if passing through 0)

Affine ✓ Convex ✓ Cone ✗

- Line segment

Affine ✗ Convex ✓ Cone ✗

- A ray

Affine ✗ Convex ✓ Cone ✓

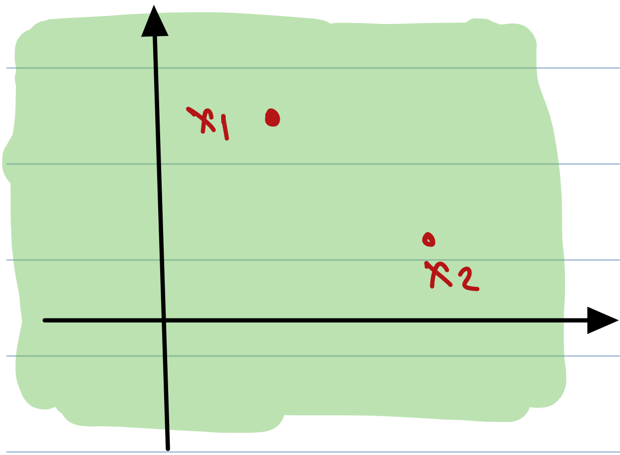
- A subspace

Affine ✓ Convex ✓ Cone ✓

SUMMARY

Let $x_1, x_2 \in \mathbb{R}^2$ be two independent points

Linear combination



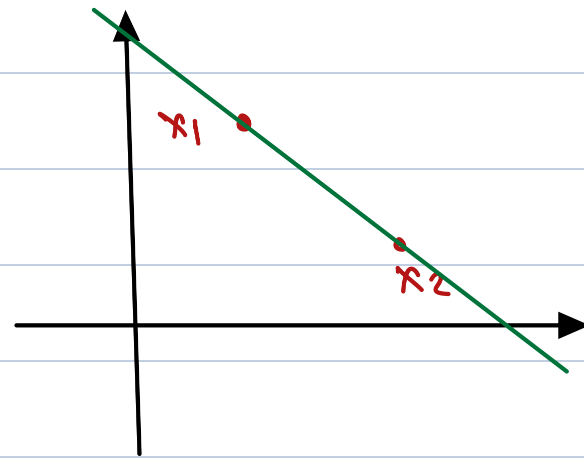
Linear combination:

$x = \theta_1 x_1 + \theta_2 x_2$

$\theta_1, \theta_2 \in \mathbb{R}$

Entire $\mathbb{R}^2$

Affine combination



Affine combination

$x = \theta_1 x_1 + \theta_2 x_2$

$\theta_1 + \theta_2 = 1$

Line passing through x_1, x_2

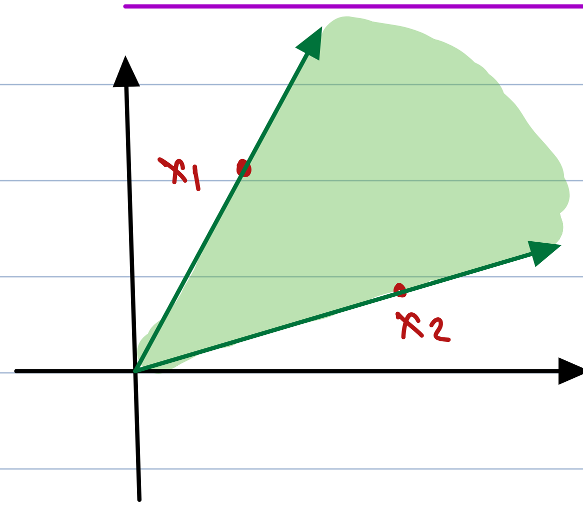
Convex combination

$x = \theta_1 x_1 + \theta_2 x_2$

$\theta_1 + \theta_2 = 1$; $\theta_1, \theta_2 > 0$

Line segment

Conic combination



$x = \theta_1 x_1 + \theta_2 x_2$

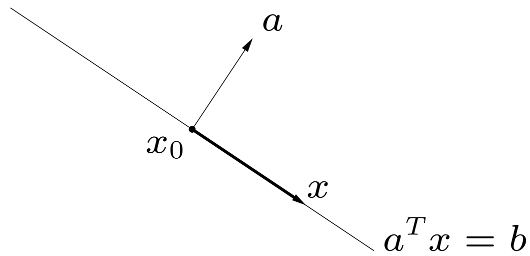
$\theta_1, \theta_2 \geq 0$

Hyperplanes and halfspaces

$x \in \mathbb{R}^n,\ a \in \mathbb{R}^n,\ b \in \mathbb{R}$

hyperplane: set of the form $\{x \mid a^T x = b\}$ $(a \neq 0)$

$\|a\| = 1$



- a is the **normal vector** of the hyperplane

Because any vector in this hyperplane is $\perp^r$ to Normal vector

Another interpretation: for any fixed $x_0 \in C$, we have $a^T(x - x_0) = 0 \quad \forall x \in C$

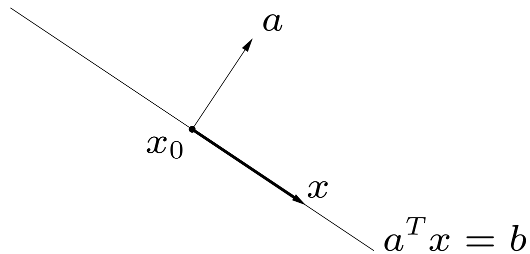
Hyperplanes and halfspaces

hyperplane: set of the form $\{x \mid a^T x = b\}$ $(a \neq 0)$

Are hyperplane's

Convex ? ✓

Affine ? ✓

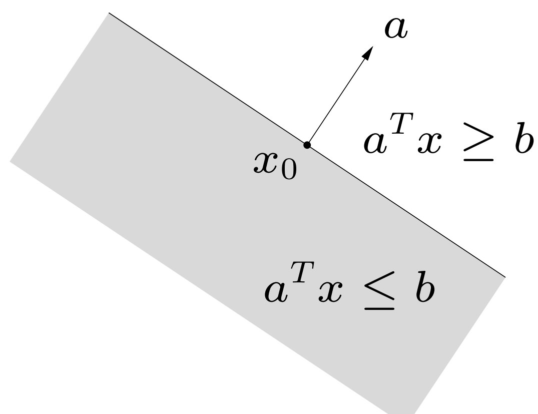


- a is the **normal vector** of the hyperplane

halfspace: set of the form $\{x \mid a^T x \leq b\}$ $(a \neq 0)$

Convex ✓

Affine ✗



Hyperplane:

$$\{x \mid a^T x = b\}$$

let $a^T x_0 = b$ for $x_0 \in \mathbb{R}^n$ (x_0 = origin of hyperplane)

$$\Rightarrow \{x \mid a^T x = a^T x_0\} = \{x \mid a^T(x - x_0) = 0\}$$

we can also write as

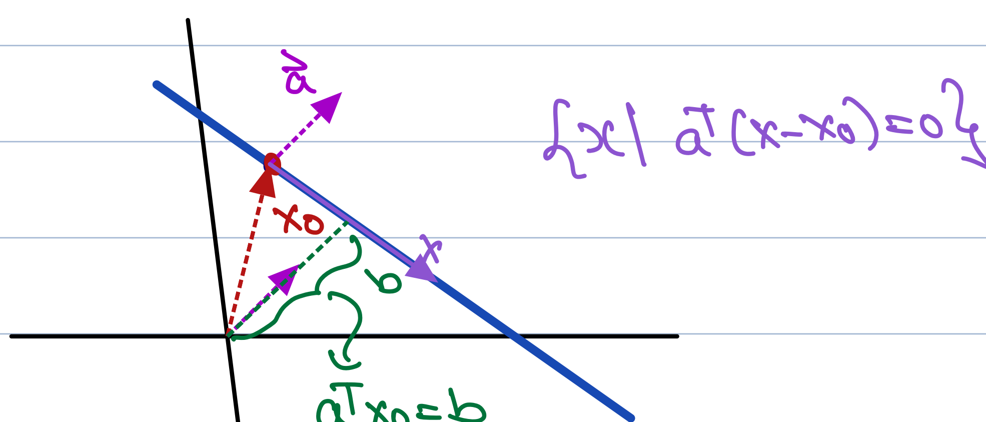
$$\{x \mid a^T(x - x_0) = 0\} = x_0 + a^\perp$$

Where $a^\perp$ denotes the orthogonal complement of a i.e the set of all vector's orthogonal to it.

$$a^\perp = \{v \mid a^T v = 0\}$$

This show's that the hyperplane consists of an offset x_0, + all the vector's orthogonal to the (normal) vector a.

let $\|a\|_2 = 1$



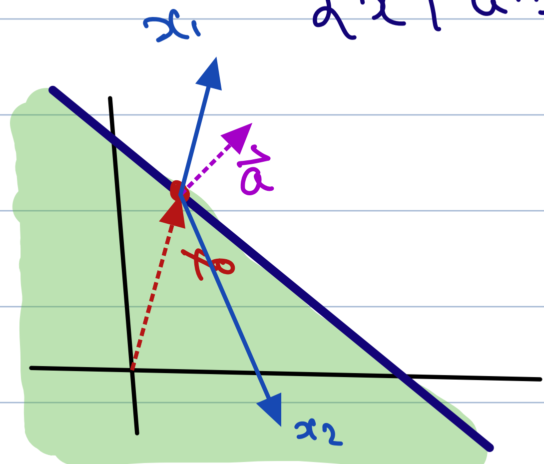
if $\|a\|_2 = 1$, $b = \perp^r$ distance of plane from origin

Half-space:

A hyperplane divides $\mathbb{R}^n$ into two halfspaces.

A closed half space is a set of the form

$$\{x \mid a^T x \le b\}$$ let $\|a\|_2 = 1$ (without loss of generality)



$a^T x_0 = b$

then

$$a^T (x - x_0) \le 0$$

Half space: $\{x \mid a^T(x - x_0) \le 0\}$

$(x_1 - x_0)$ make's acute angle with $\vec{a}$

$\Rightarrow$ $a^T (x_1 - x_0)$ the dot product is +ve

because $\theta < 90^\circ \Rightarrow \cos\theta > 0$

$(x_2 - x_0)$ make's obtuse angle with $\vec{a}$

$\Rightarrow$ $a^T (x_2 - x_0)$ the dot product is -ve because

$\theta > 90^\circ \Rightarrow \cos\theta < 0$

therefore $x_2 \in$ Half-space, $x_1 \notin$ Half-space

The Boundary of Half space $\{x \mid a^T x = b\}$ is hyperplane.

$\{x \mid a^T x < b\}$ open-Half-space.

Polyhedron

A solution set of finitely many linear inequalities and equalities

$$Ax \preceq b, \qquad Cx = d$$

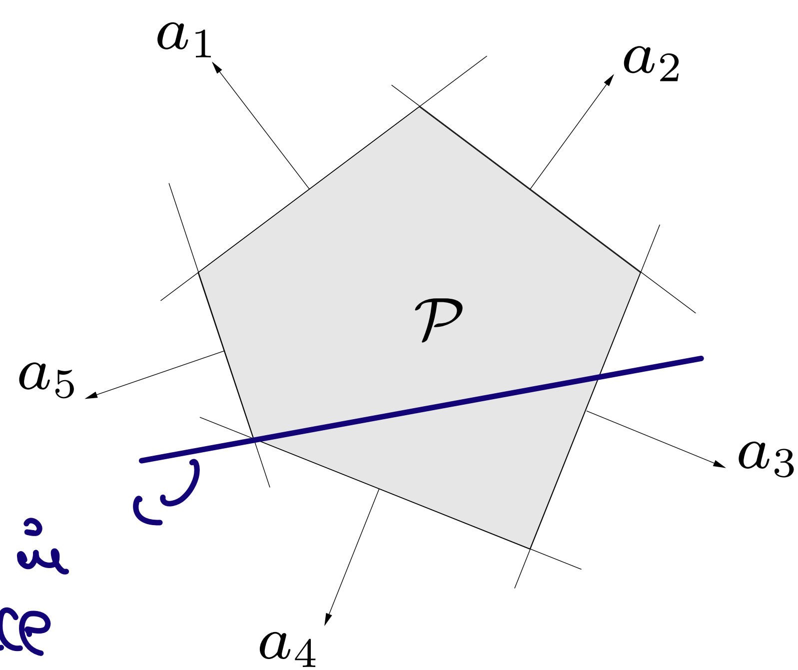
$x, y \in \mathbb{R}^n$, $x \preceq y \Rightarrow x_i \leq y_i \ \forall i$

($A \in \mathbf{R}^{m\times n}$, $C \in \mathbf{R}^{p\times n}$, $\preceq$ is component-wise inequality)

<u>Polyhedron:</u>

Convex ✓

Affine ✗



intersection of 5 Halfspaces

$a_1^T x \leq b_1$

$a_2^T x \leq b_2$

$a_3^T x \leq b_3$

$a_4^T x \leq b_4$

$a_5^T x \leq b_5$

is there a half space $c_1^T x = d_1$

a polyhedron is an intersection of a finite number of halfspaces and hyperplanes $\mathcal{P} = \{x \mid Ax \preceq b,\ Cx = d\}$

Positive semidefinite cone

notation:

- $\mathbf{S}^n$ is set of symmetric $n \times n$ matrices
- $\mathbf{S}^n_+ = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: positive semidefinite $n \times n$ matrices

$$X \in \mathbf{S}^n_+ \iff z^T X z \geq 0 \text{ for all } z$$

X is PSD $\iff$ $x \in S^n$, $z^T x z \geq 0$ $\forall z \in \mathbb{R}^n$

why a cone? $x \in S^n_+$, $z^T x z \geq 0$ $\forall z \in \mathbb{R}^n$

for $\theta > 0$ $\quad \theta z^T x z \geq 0$ $\forall z \in \mathbb{R}^n$

$z^T(\theta x) z \geq 0$ $\forall z \in \mathbb{R}^n$

$\Rightarrow \theta x \in S^n_+$

$\Rightarrow S^n_+$ is cone

is PSD cones convex? i.e S^n_+ convex?

<u>Yes</u>

$$x_1 \in S^n_+ \iff z^T x_1 z \geq 0 \quad \forall z \in \mathbb{R}^n$$

$$x_2 \in S^n_+ \iff z^T x_2 z \geq 0 \quad \forall z \in \mathbb{R}^n$$

$1 \geq \theta \geq 0$ then $\theta z^T x_1 z \geq 0$

$$(1-\theta) z^T x_2 z \geq 0$$

$$\Rightarrow \theta z^T x_1 z + (1-\theta) z^T x_2 z \geq 0$$

$$\Rightarrow z^T (\theta x_1 + (1-\theta) x_2) z \geq 0 \quad \forall z$$

$$\Rightarrow \theta x_1 + (1-\theta) x_2 \in S^n_+$$

$\Rightarrow$ The set S^n_+ is convex, cone

is the set S^n_+ Affine?

Positive semidefinite cone

notation:

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$$X \in \mathbf{S}^n_+ \quad \Longleftrightarrow \quad z^T X z \geq 0 \text{ for all } z$$

- $\mathbf{S}^n_{++} = \{X \in \mathbf{S}^n \mid X \succ 0\}$: positive definite $n \times n$ matrices

$$X \in S^n_{++} \Longleftrightarrow z^T X z > 0 \quad \forall z \neq 0$$

Positive semidefinite cone

notation:

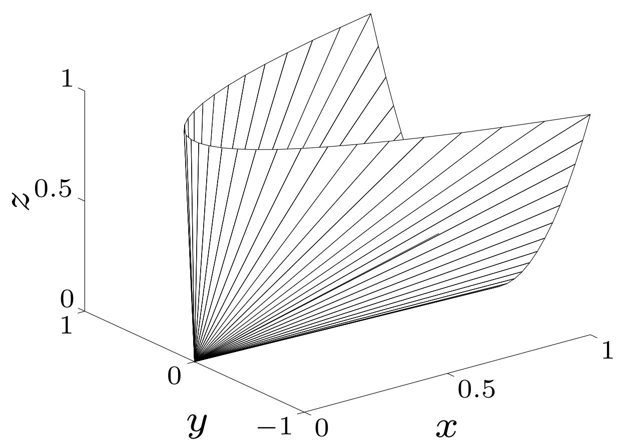
- $\mathbf{S}^n$ is set of symmetric $n \times n$ matrices
- $\mathbf{S}^n_+ = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: positive semidefinite $n \times n$ matrices

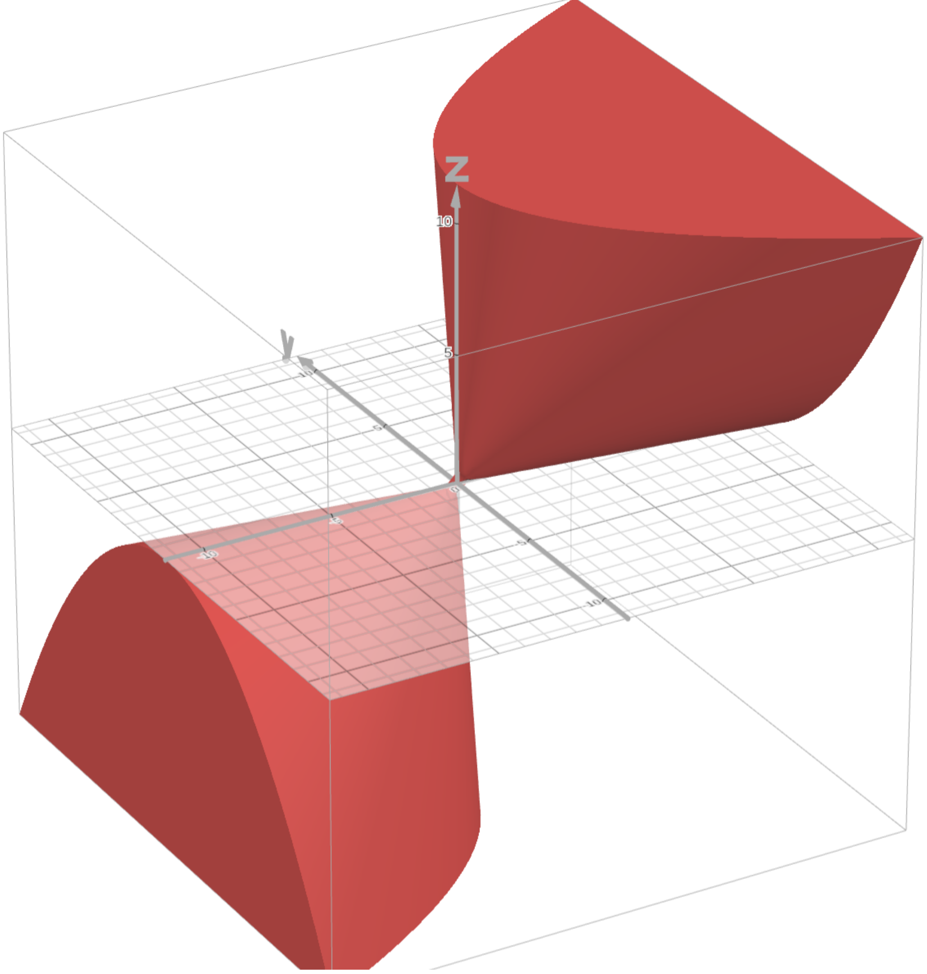
$$X \in \mathbf{S}^n_+ \quad \Longleftrightarrow \quad z^T X z \geq 0 \text{ for all } z$$

- $\mathbf{S}^n_{++} = \{X \in \mathbf{S}^n \mid X \succ 0\}$: positive definite $n \times n$ matrices

$\mathbf{S}^n_+$ is a convex cone

example: $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}^2_+$





The set
$xz - y^2 \geq 0$

A symmetric matrix X is PSD $\iff$ its eigenvalue are non-negative

A symmetric matric X is PD $\iff$ if its eigenvalue are +ve.

Operations that preserve convexity

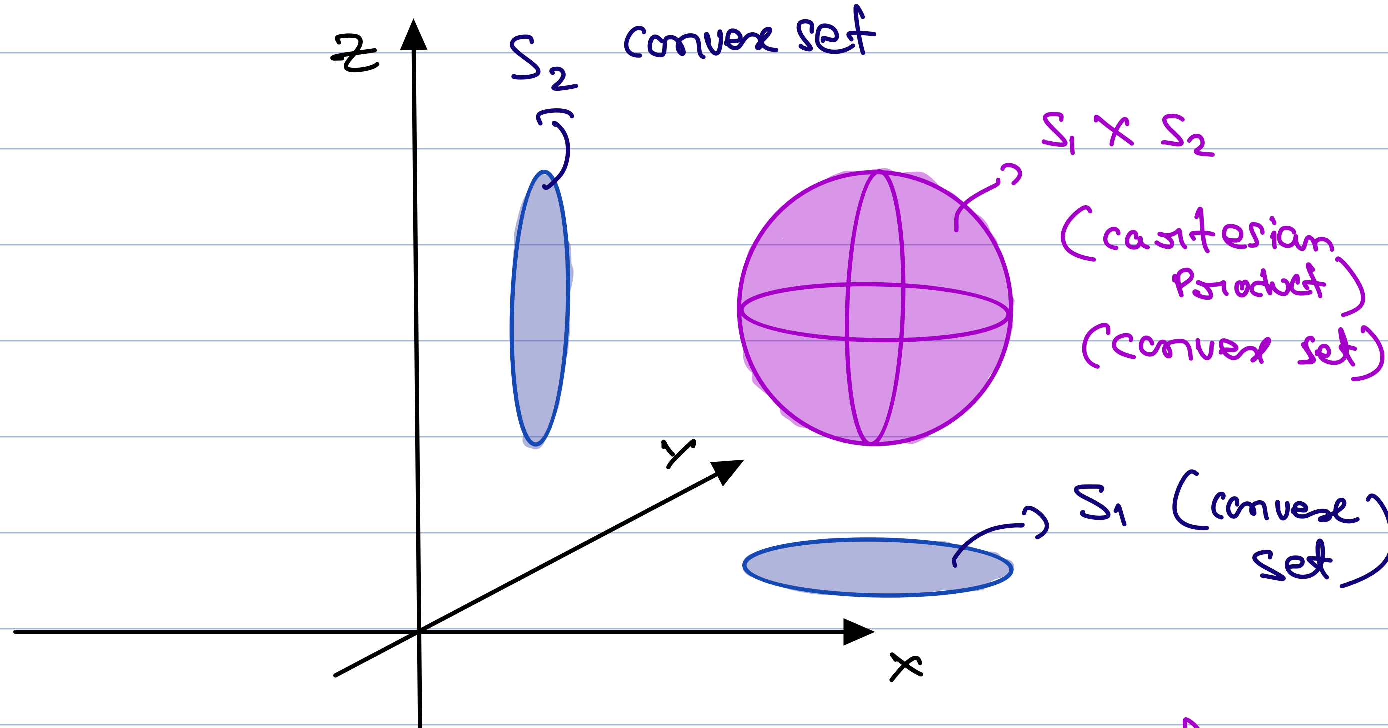
How to show the convexity of C?

1. apply definition

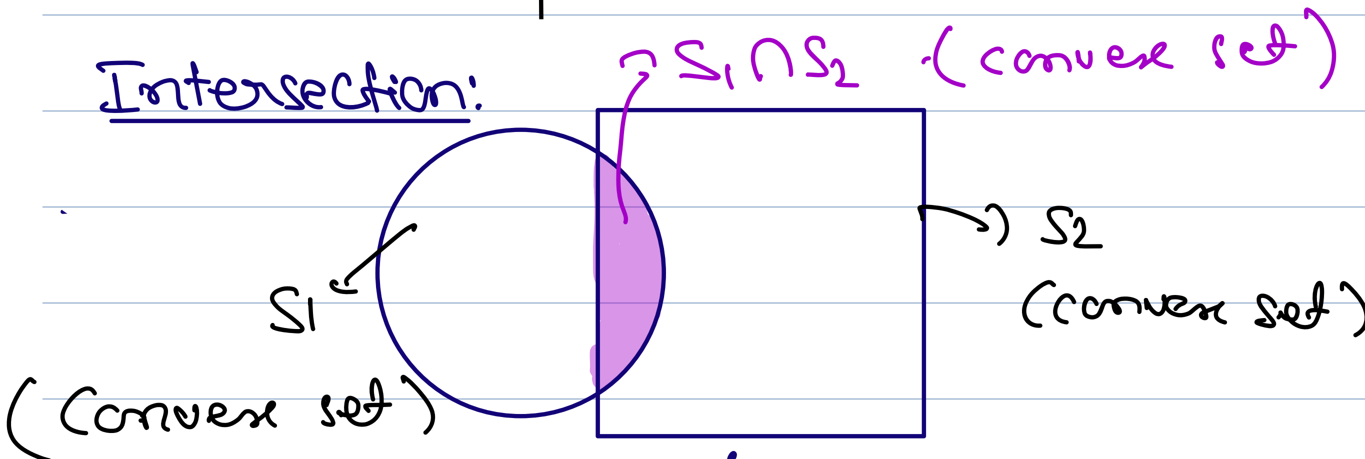
$$x_1, x_2 \in C, \quad 0 \le \theta \le 1 \quad \Longrightarrow \quad \theta x_1 + (1-\theta)x_2 \in C$$

2. show that C is obtained from simple convex sets (hyperplanes, halfspaces, ...) by operations that preserve convexity
   - Cartesian product
     ($S_1 \times S_2 = \{(x_1, x_2) \mid x_1 \in S_1,\ x_2 \in S_2\}$; if S_1 and S_2 are convex, so is $S_1 \times S_2$)
   - intersection
   - affine functions
   - perspective function
   - linear-fractional functions

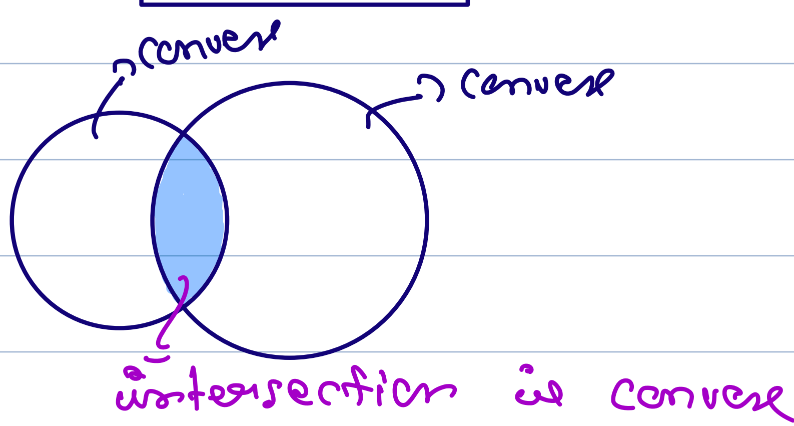
Cartesian Product:

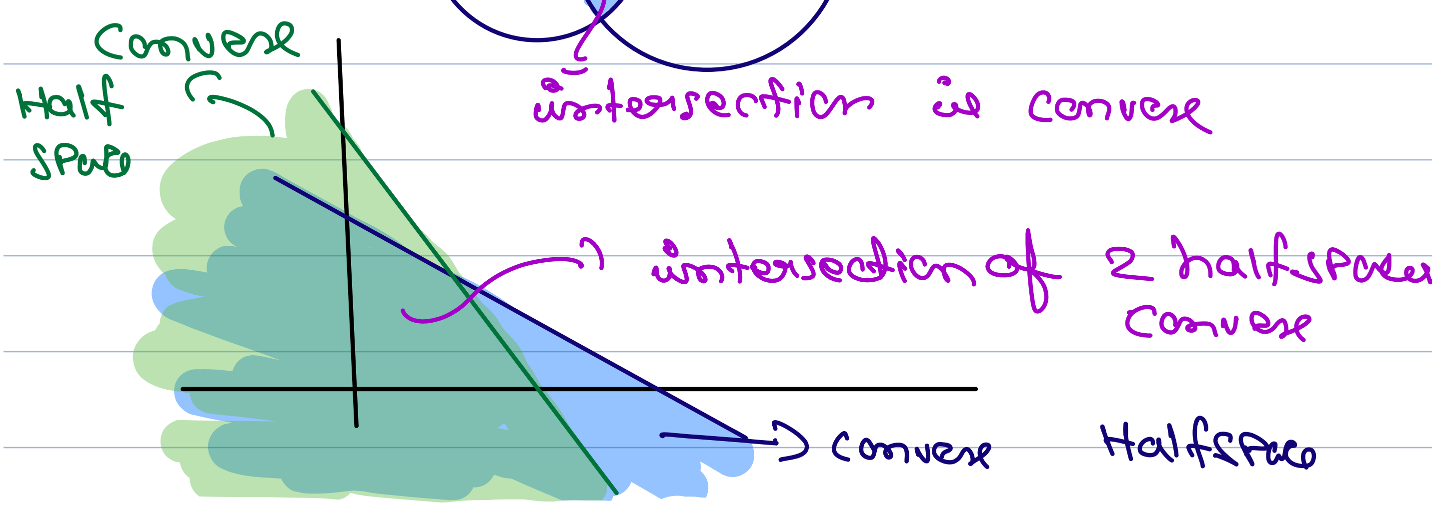


Intersection:



Ex:





Intersection

If two sets S_1 and S_2 are convex, so is their intersection $S_1 \cap S_2$.

suppose $x_1 \in S_1 \cap S_2$ & $x_2 \in S_1 \cap S_2$ our goal is to show : $\theta x_1 + (1-\theta) x_2 \in S_1 \cap S_2$

$$x_1, x_2 \in S_1 \cap S_2 \Longrightarrow \begin{array}{l} \theta_1 x_1 + (1-\theta) x_2 \in S_1 \\ \theta_2 x_2 + (1-\theta) x_2 \in S_2 \end{array}$$

$$\Downarrow$$

$$\theta_1 x_1 + (1-\theta) x_2 \in S_1 \cap S_2$$

$\Longrightarrow$ we can extend these properties to k sets.

$S_1, S_2, \ldots S_k$ are convex $\Longrightarrow$ $S_1 \cap S_2 \cap S_3 \cap \ldots \cap S_k$ is also convex.

PSD cone

$$\mathbf{S}_+^n = \{X : X \text{ is symmetric}, z^\top X z \geq 0, \forall z\}$$

Prove convexity of PSD cone using Intersection.

Let $C = \left\{ X \,;\, X \in S^n \,,\, \underset{z}{z^T X z \geq 0 \text{ for some}} \right\}$

$z^T X z = \sum z_i z_j x_{ij} \Rightarrow$ This is a Linear function of x (z is fixed)

Ex: $z_1 z_2 x_{12} + z_2 z_1 x_{21} + z_1 z_1 x_{11} + z_2 z_2 x_{22}$

$$\Rightarrow \begin{bmatrix} z_1 z_2 & z_2 z_1 & z_1 z_1 & z_2 z_2 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{21} \\ x_{11} \\ x_{22} \end{bmatrix}$$

$\Rightarrow z^T x$

we know that $z^T x z \geq 0$ (Half space) for fixed z.

* So for each $z \in \mathbb{R}^n$ (fixed) $z^T x z \geq 0$ is a half space.

* we are looking for set of all x's such that $z^T x z \geq 0 \quad \forall z$

$\Rightarrow$ set S^n_+ is the intersection of halfspaces

$\Rightarrow$ Like $z^T x z \geq 0$ (for a fixed z) with a infinitely many z.

$\Rightarrow$ Half space is convex, therefore intersection of convex set's is convex.

$\Rightarrow$ S^n_+ is convex (through intersection of infinitely many half spaces)

$$S^n_+ = \left\{ x \;\middle|\; \begin{array}{l} z_1^T x z_1 \geq 0,\ z_2^T x z_2 \geq 0,\ z_3^T x z_3 \geq 0, \ldots \\ \underbrace{x_{ij} = x_{ji}}_{x_{ij} - x_{ji} = 0 \text{ (HyperPlane)}} \quad \forall i,j \end{array} \right\}$$

$\Rightarrow$ $x_{ij} - x_{ji} = 0 \quad \forall i,j$ (intersection of finite many HyperPlane)

$\Rightarrow$ S_n^+ = intersection of infinitely many Half spaces and finitely many HyperPlanes)

Affine function

suppose function $f : \mathbf{R}^n \to \mathbf{R}^m$ is affine ($f(x) = Ax + b$ with $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$)

Affine function

suppose function $f : \mathbf{R}^n \to \mathbf{R}^m$ is affine ($f(x) = Ax + b$ with $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$)

- the image of a convex set under f is convex

$$S \subseteq \mathbf{R}^n \text{ convex} \quad \Longrightarrow \quad f(S) = \{f(x) \mid x \in S\} \subseteq \mathbf{R}^m \text{ convex}$$

- **examples:** scaling, translation, projection, sum of sets

S is convex

scaling : $\alpha S = \{\alpha x : x \in S\}$

$\Rightarrow \quad f(x) = \underbrace{\alpha I}_{A} x$ (Affine function)

Translation:

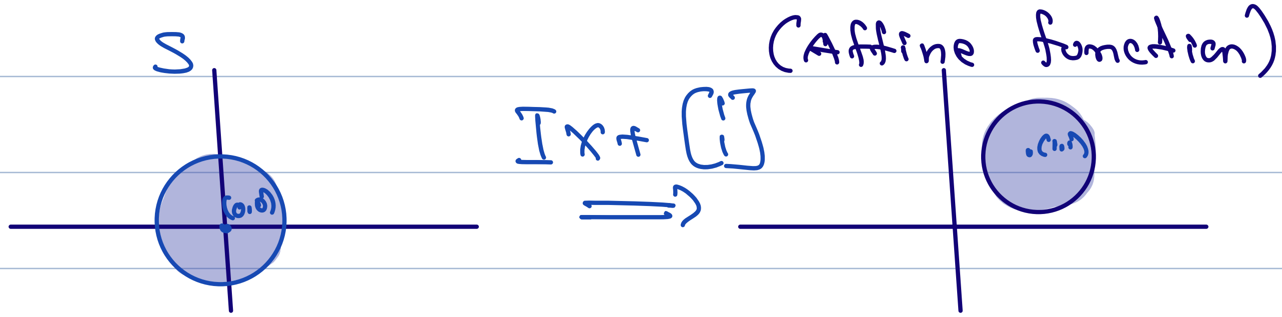
$$S + a = \{x + a : x \in S\}$$

$$\Downarrow$$

$$f(x) = Ix + a$$

(Affine function)

Ex:

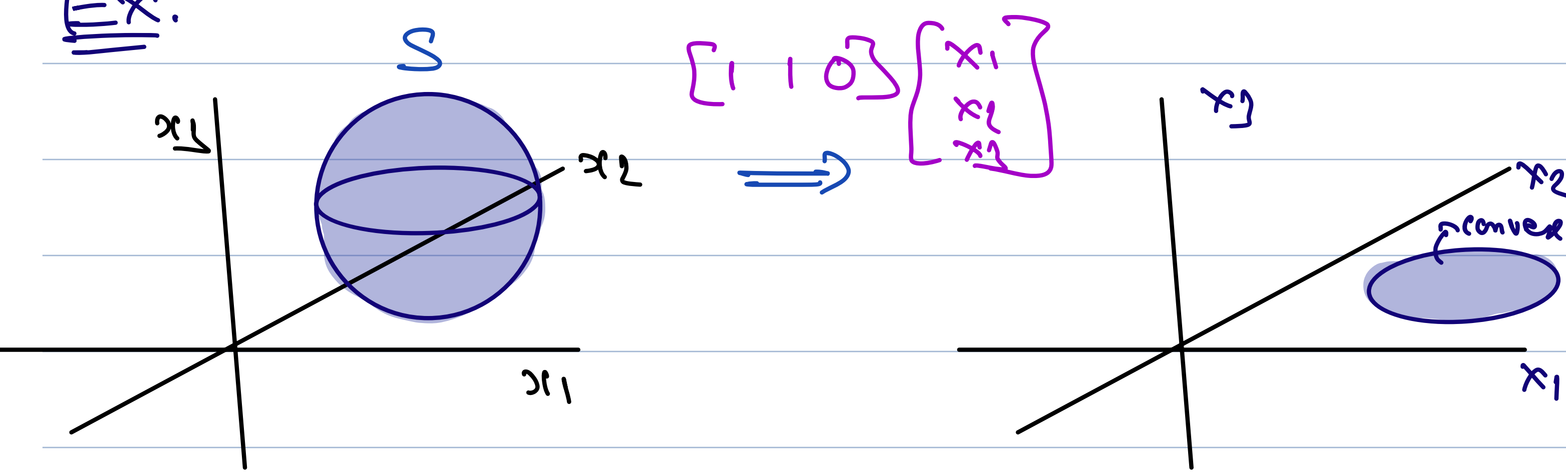


Projection:

$$T = \{x_1 : (x_1, x_2) \in S\}$$

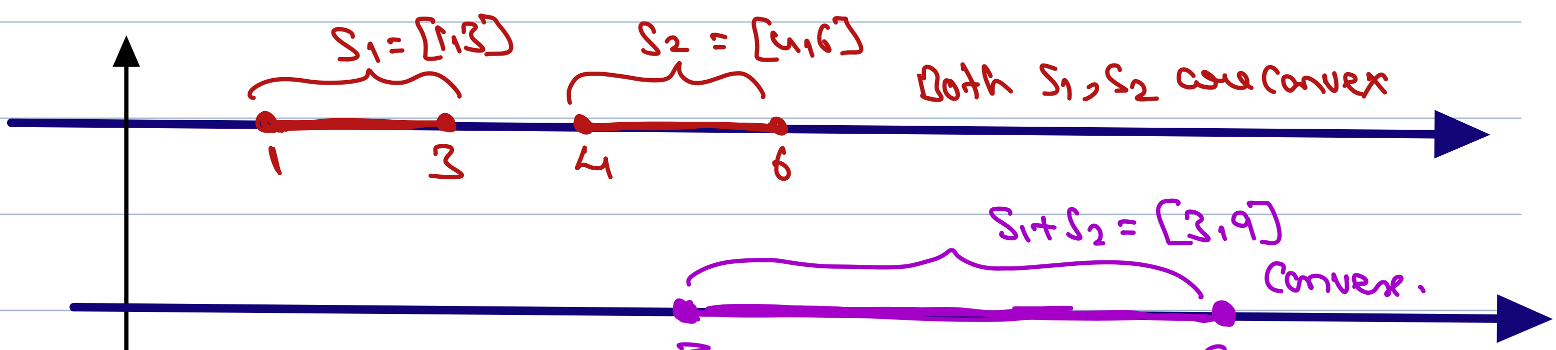
$$f(x) = \underbrace{\begin{bmatrix} I & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Ex:



* Sum's of sets

: $S_1 + S_2 = \{x_1 + x_2 : x_1 \in S_1, x_2 \in S_2\}$



Affine function

suppose function $f : \mathbf{R}^n \to \mathbf{R}^m$ is affine ($f(x) = Ax + b$ with $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$)

- the image of a convex set under f is convex

$$S \subseteq \mathbf{R}^n \text{ convex} \quad \Longrightarrow \quad f(S) = \{f(x) \mid x \in S\} \subseteq \mathbf{R}^m \text{ convex}$$

  - **examples:** scaling, translation, projection, sum of sets
- the inverse image $f^{-1}(C)$:

$$f^{-1}(C) = \{x \in \mathbf{R}^n \mid f(x) \in C\}$$

  the inverse image of a convex set under f is convex

$$C \subseteq \mathbf{R}^m \text{ convex} \quad \Longrightarrow \quad f^{-1}(C) \text{ convex}$$

Examples

- **example:** solution set of linear matrix inequality $\{x \mid x_1 A_1 + \cdots + x_m A_m \preceq B\}$ (with $A_i, B \in \mathbf{S}^n$)

If S is convex and f is Affine function then we want to prove $f(S)$ The Image of convex set is convex.

$$f(x) = Ax + b$$

let $f(S) = \{ f(x) : x \in S \}$

let $f(x_1) \in f(S)$ then
$f(x_2) \in f(S)$

$$\theta f(x_1) + (1-\theta) f(x_2)$$

$$= \theta (Ax+b) + (1-\theta)(Ax+b)$$

$$= A(\theta x + (1-\theta)x) + \theta b + (1-\theta) b$$

$$= A(\underbrace{\theta x + (1-\theta) x}_{\in S}) + b \Rightarrow \in f(S)$$

So if $f(x_1), f(x_2) \in f(S)$ then $\theta f(x_1) + (1-\theta) f(x_2)$ is also $\in f(S) \Rightarrow f(S)$ convex