CS3233
Competitive Programming

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Week 11 – (Computational) Geometry

Outline

- Mini Contest #9 + Discussion + Break + Admins
- Geometry Basics + Prepare Your Libraries
  - Points, Lines, Circles, Triangles, Polygons (Focus)
- Not discussed tonight:
  - Quadrilaterals
  - 3D Objects: Spheres
  - Other 3D Objects: Cones, Cylinders, etc
  - Plane Sweep technique
  - Intersection problems
  - Divide and Conquer in geometry problems


Jan-Apr 2009
NUSSU
Pepe Jeans
Jan-Apr 2010
FEELS
fresh
Jan-Apr 2011
ZELDA
Jan-Apr 2012

The major part of the hard copy material of a top ICPC team is usually a collection of geometric libraries…

GEOMETRY BASICS AND LIBRARIES

Some Comp Geometry Principles

- Whenever possible, we prefer test (predicates) than computing the exact numerical answers
- Tests:
  - Avoid floating point operations (division, square root, and any other operations that can produce numerical errors)
  - Preferably, all operations are done in **integers**
  - If we really need to work with floating point, we do floating point equality test this way: **fabs(a - b) < EPS** where **EPS** is a small number like **1e-9** instead of **a == b**

Geometry Basics – 0D (1)

- Point, representation + sorting feature

```
struct point_i { int x, y }; // use this whenever possible
struct point { double x, y }; // but I will use this form now

struct point { double x, y;
  point(double _x, double _y) { x = _x, y = _y; }
  bool operator < (point other) {
    if (fabs(x - other.x) > EPS) // useful for sorting
      return x < other.x; // first criteria , by x-axis
    return y < other.y;   // second criteria, by y-axis
} };
```

Geometry Basics – 0D (2)

- Comparing Points

```
bool areSame(point p1, point p2) { // floating point version
  // use EPS when testing equality of two floating points
  return fabs(p1.x - p2.x) < EPS && fabs(p1.y - p2.y) < EPS; }
```

- Euclidean Distance between two points

```
double dist(point p1, point p2) { // Euclidean distance
  // hypot(dx, dy) returns sqrt(dx * dx + dy * dy)
  return hypot(p1.x - p2.x, p1.y - p2.y); } // return double
```

Geometry Basics – 1D (1)

- Lines (ch7_01_points_lines.cpp/java)
  - Poor line equation, $y = mx + c$ (vertical line → special case)
  - Better line equation, $ax + by + c = 0$

```
struct line { double a, b, c; }; // a way to represent a line

// the answer is stored in the third parameter (pass byref)
void pointsToLine(point p1, point p2, line *l) {
  if (p1.x == p2.x) { // vertical line is handled nicely here
    l->a = 1.0;   l->b = 0.0;   l->c = -p1.x;
  } else {
    l->a = -(double)(p1.y - p2.y) / (p1.x - p2.x);
    l->b = 1.0; // fix the value of b to 1.0
    l->c = -(double)(l->a * p1.x) - (l->b * p1.y);
} }
```

Geometry Basics – 1D (2)

- Interaction between two lines

```
bool areParallel(line l1, line l2) { // check coefficient a + b
  return (fabs(l1.a-l2.a) < EPS) && (fabs(l1.b-l2.b) < EPS); }

bool areSame(line l1, line l2) { // also check coefficient c
  return areParallel(l1, l2) && (fabs(l1.c - l2.c) < EPS); }
```

Geometry Basics – 1D (3)

- Interaction between two lines – continued
  - Simple linear algebra: $a_1x + b_1y + c_1 = a_2x + b_2y + c_2$!

```
// returns true (+ intersection point) if two lines are intersect
bool areIntersect(line l1, line l2, point *p) {
  if (areSame(l1, l2)) return false; // all points intersect
  if (areParallel(l1, l2)) return false; // no intersection
  // solve system of 2 linear algebraic equations with 2 unknowns
  p->x = (double)(l2.b * l1.c - l1.b * l2.c) /
                 (l2.a * l1.b - l1.a * l2.b);
  if (fabs(l1.b) > EPS) // test for vertical line
    p->y = - (l1.a * p->x + l1.c) / l1.b; // avoid div by zero
  else // this is another special case in geometry problem...
    p->y = - (l2.a * p->x + l2.c) / l2.b;
  return true; }
```

Geometry Basics – 1D (4)

- Line segments: line with two endpoints (finite length)
- Vector: line segment with *a direction*
- We can *translate (move)* a point w.r.t a vector

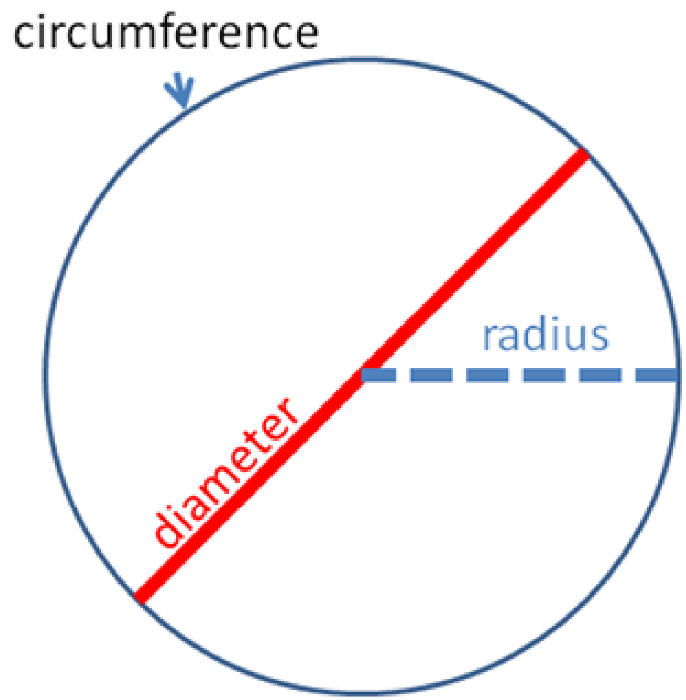
```
struct vec { double x, y; // similar to point
  vec(double _x, double _y) { x = _x, y = _y; } };
vec toVector(point p1, point p2) { // convert 2 points to vector
  return vec(p2.x - p1.x, p2.y - p1.y); }
vec scaleVector(vec v, double s) { // s = [<1 ...   1   ... >1]
  return vec(v.x * s, v.y * s); }  // shorter v  same v  longer v
point translate(point p, vec v) { // translate p according to v
  return point(p.x + v.x , p.y + v.y); }
```

Geometry Basics – 2D/Circles (1)

- Circles (ch7_02_circles.cpp/java)
  - A circle centered at (a, b) and radius r is the set of all points (x, y) such that $(x - a)^2 + (y - b)^2 = r^2$

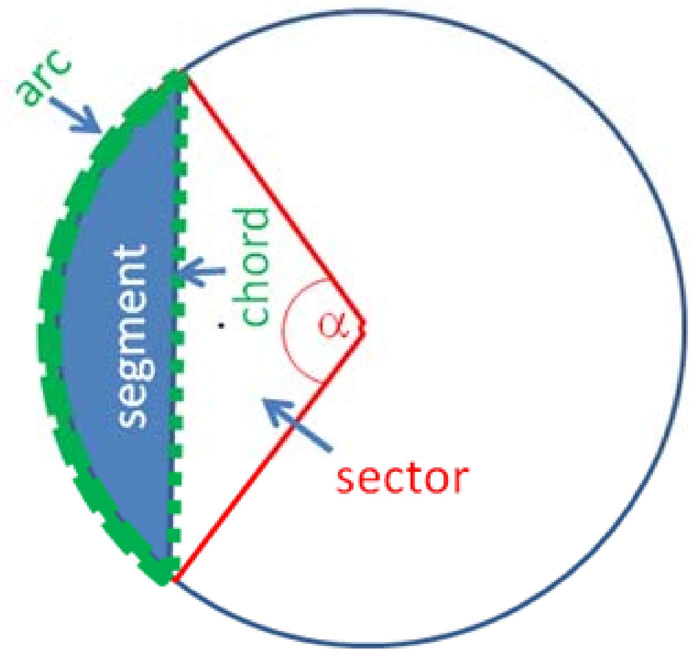
```
int in_circle(point p, point c, int r)
// 0 – inside, 1 – at border, 2 - outside
```

  - $\pi = 2 * acos(0.0)$
  - Diameter $d = 2 * r$
  - Circumference $c = \pi * d$
  - Area of circle $A = \pi * r * r$



Geometry Basics – 2D/Circles (2)

- Arc length: α / 360.0 * c
- Chord length:
  -
  -
- Sector area: α / 360.0 * A
- Segment area: sector area – isosceles t

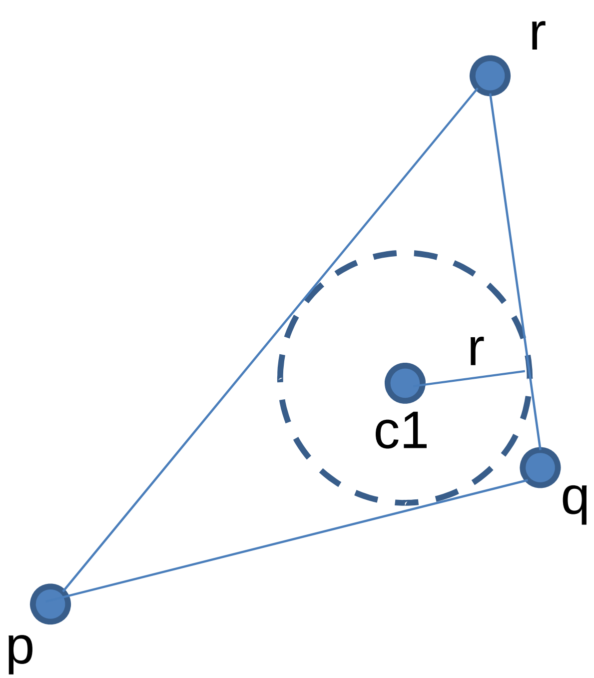


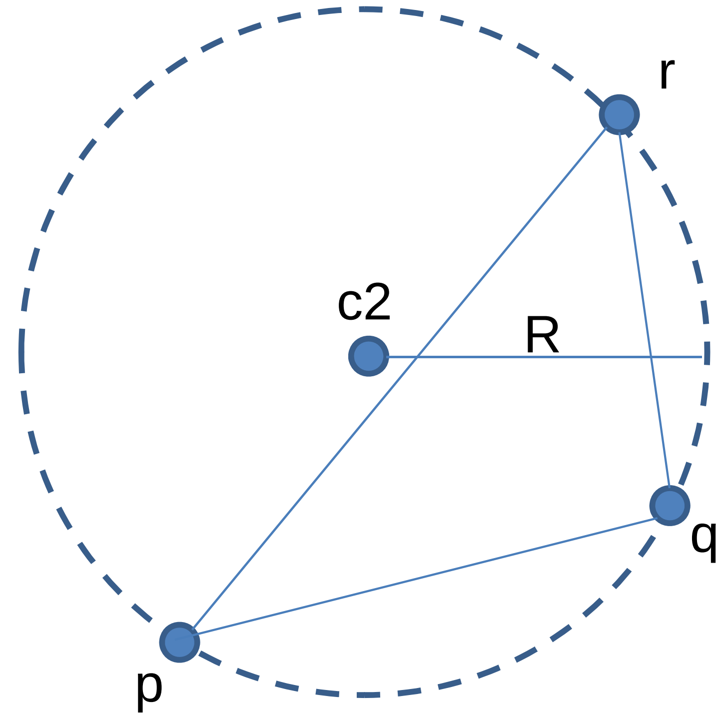
Geometry Basics – 2D/Triangles (1)

- Triangles (ch7_03_triangles.cpp/java)
  - Polygon with three vertices and three edges
  - Area of Triangle 1: A = 0.5 * b x h
  - Perimeter p = a + b + c
    - where a, b, c are the length of the 3 edges
  - Area of Triangle 2: A = sqrt(s * (s - a) * (s - b) * (s - c))
    - where semi-perimeter s = 0.5 * p
    - This is called the **Heron's formula**
    - *Safer from overflow*: A = sqrt(s) * sqrt(s - a) * sqrt(s - b) * sqrt(s - c)
      - But can be slightly more imprecise

Geometry Basics – 2D/Triangles (2)

- Given three points p, q, r
  - Determine the circumcenter c1 and radius R1 of the inner/inscribed circle/incircle and (c2, R2) of the outer/circumscribed circle/circumcircle





Geometry Basics – 2D/Triangles (3)

- Trigonometry/Law of Cosines
  - $c^2 = a^2 + b^2 - 2 * a * b * \cos(\gamma)$
- Trigonometry/Law of Sines
  - $a / \sin(\alpha) = b / \sin(\beta) = c / \sin(\gamma)$
- Trigonometry/Phytagorean Theorem
  - $c^2 = a^2 + b^2$ because cos(90.0 degrees/right angle) = 0

Geometry Basics – 2D/Others

- Quadrilaterals (no sample code)
  - Rectangles/Squares
  - Trapeziums/Parallelograms/Rhombus
  - Area
  - Perimeter
  - Etc…

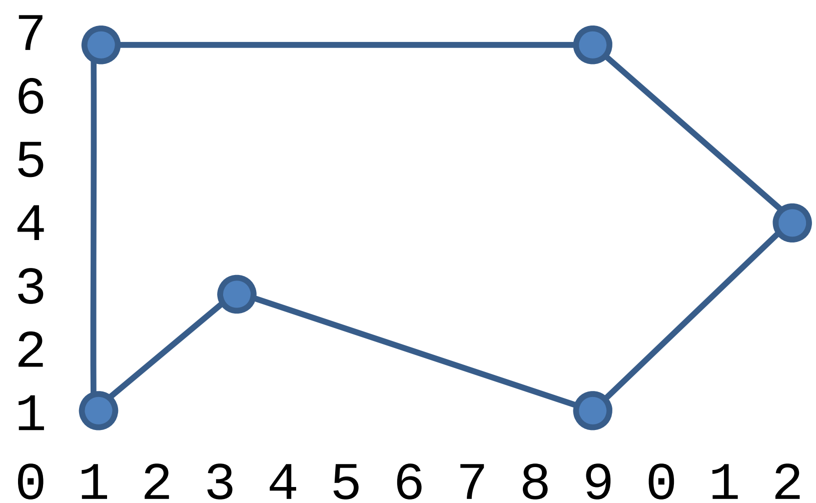
Focus for CS3233 this semester

ALGORITHMS ON POLYGON

Polygon (1)

- Sample code (ch7_05_polygon.cpp/java)
  - Plane figure that is bounded by a closed circuit composed of a **finite sequence** of **straight line segments**
  - Basic form, vertices are ordered *either* in **cw** or **ccw** order
  - **Usually the first = the last vertex**

```
vector<point> P;
P.push_back(point(1, 1));
P.push_back(point(3, 3));
P.push_back(point(9, 1));
P.push_back(point(12, 4));
P.push_back(point(9, 7));
P.push_back(point(1, 7));
P.push_back(P[0]); // loop back
```

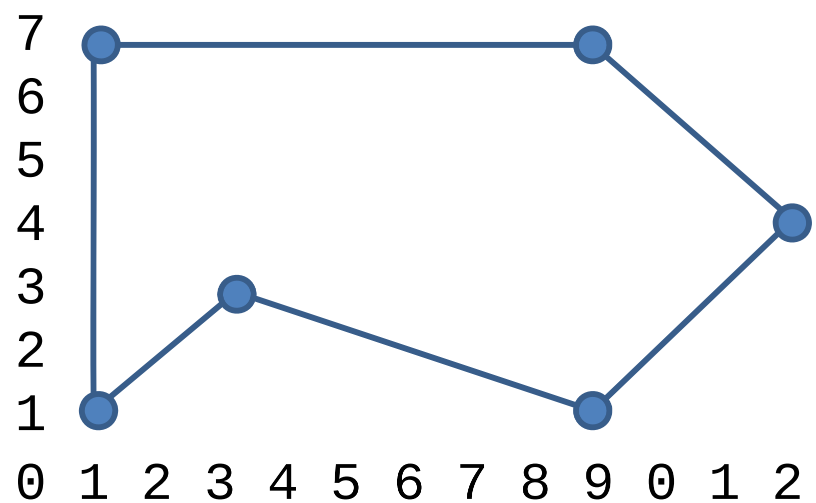


http://www.comp.nus.edu.sg/~stevenha/visualization/polygon.html

Polygon (2)

- Perimeter of polygon (trivial)

```
// returns the perimeter, which is the sum of Euclidian distances
// of consecutive line segments (polygon edges)
double perimeter(vector<point> P) {
  double result = 0.0;
  for (int i = 0; i < (int)P.size(); i++)
    result += dist(P[i], P[(i + 1) % P.size()]);
  return result; }
```



Area of a Polygon

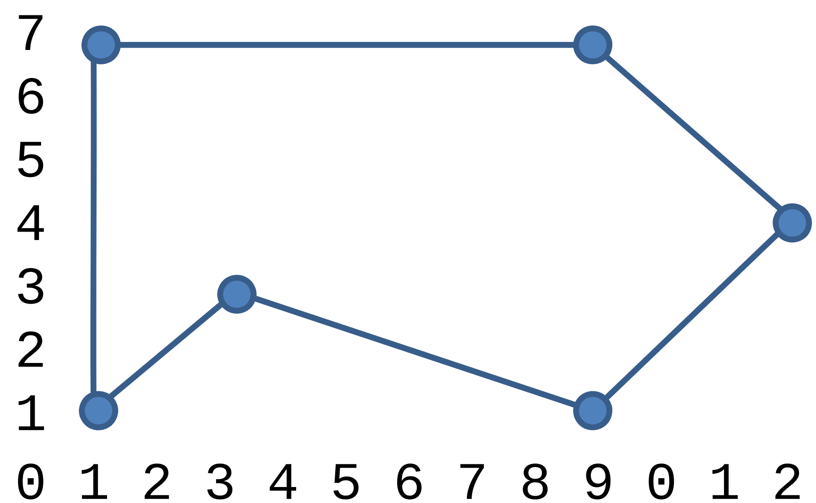
- Given the vertices of a polygon in a circular manner (cw or ccw), its area is

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ . & . \\ . & . \\ . & . \\ x_n & y_n \end{vmatrix} = \frac{1}{2}\sum_{i=1}^{n}(x_i y_{i+1 \bmod n} - x_{i+1 \bmod n} y_i)$$

Polygon (3)

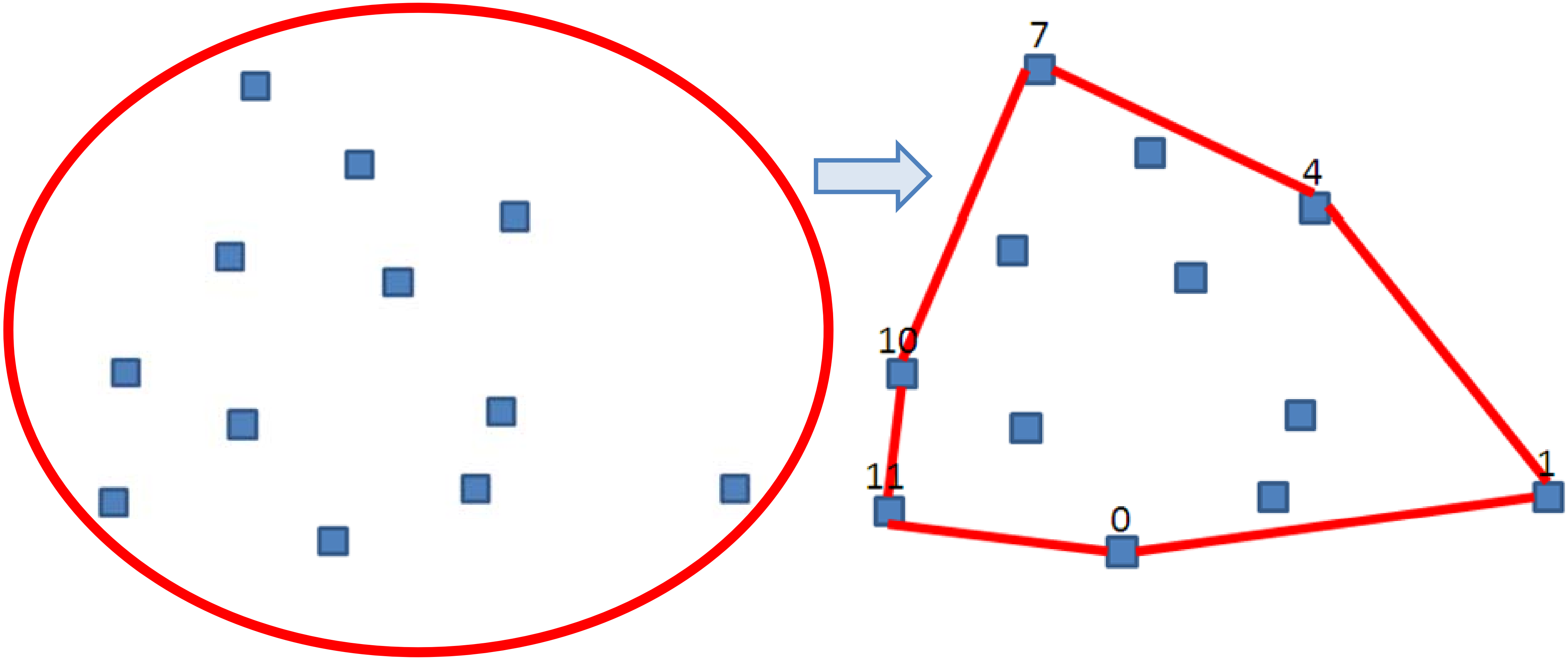
- Area of polygon

```
// returns the area, which is half the determinant
double area(vector<point> P) {
  double result = 0.0, x1, y1, x2, y2;
  for (int i = 0; i < (int)P.size(); i++) {
    x1 = P[i].x; x2 = P[(i + 1) % P.size()].x;
    y1 = P[i].y; y2 = P[(i + 1) % P.size()].y;
    result += (x1 * y2 - x2 * y1);
  }
  return fabs(result) / 2.0; }
```



Polygon/Convex Hull (1)

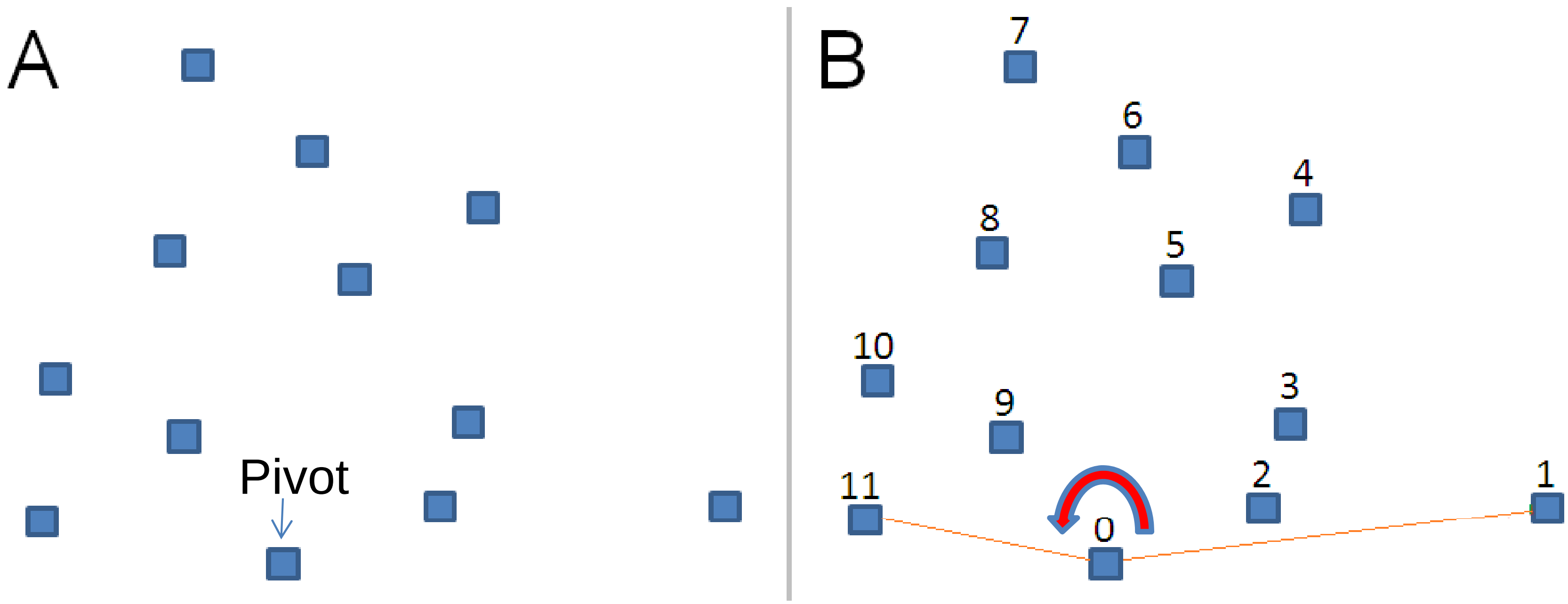
- The Convex Hull of a set of points P is the smallest convex polygon CH(P) for which each point in P is either on the boundary of CH(P) or in its interior



http://www.comp.nus.edu.sg/~stevenha/visualization/convexhull.html

Polygon/Convex Hull (2)

- Graham's Scan algorithm
  1. Find pivot (bottom most, right most point)
  2. Angular sorting w.r.t pivot (easy with library)
  3. Series of ccw tests (with help of stack)



Summary

- In this lecture, you have seen:
  - Basic geometry routines (quite substantial)
    - But still… many others routines are skipped :O
  - Focus on (some) algorithms on polygon
- But… you need to practice using them!
  - Especially, scrutinize ch7_05_polygon.cpp/java
  - Solve one UVa problem involving polygon
  - We will have a comp geo contest next week ☺

References

- CP2.9, Chapter 7
- Introduction to Algorithms, 2nd/3rd ed, Chapter 33