Factor Graphs

Ruediger Urbanke, EPFL

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~ 2-3 hours

Distributive Law

$$ab + ac = a(b + c)$$

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$$\sum_{i,j} a_i b_j \qquad\qquad \left(\sum_i a_i\right)\left(\sum_j b_j\right)$$

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$$f(x_1, x_2, x_3, x_4, x_5, x_6) = f_1(x_1, x_2, x_3) f_2(x_1, x_4, x_6) f_3(x_4) f_4(x_4, x_5)$$

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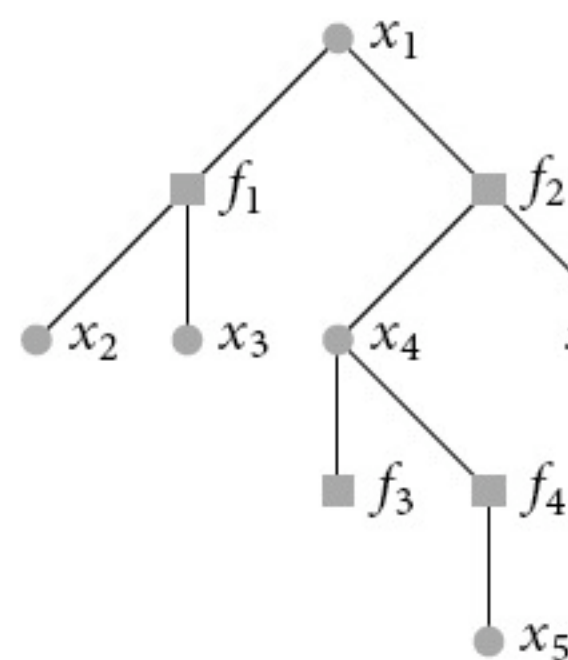

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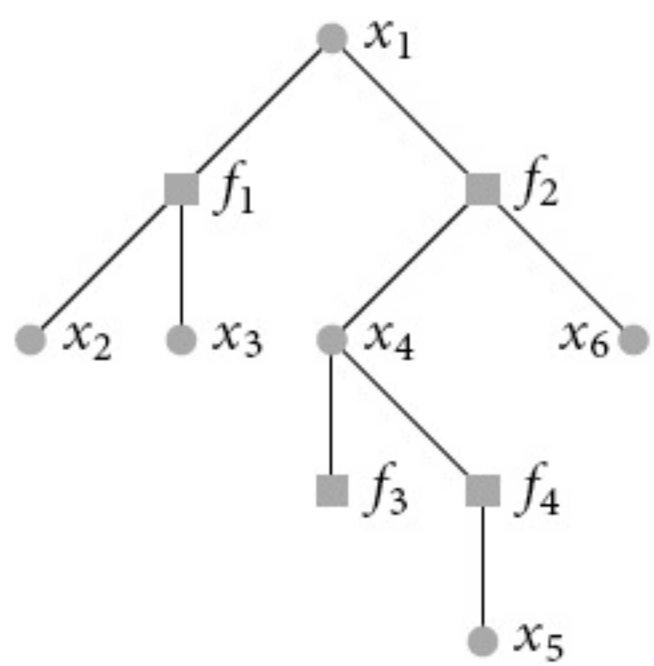

$$f(x_1) = \sum_{x_2, x_3, x_4, x_5, x_6} f(x_1, x_2, x_3, x_4, x_5, x_6) = \sum_{\sim x_1} f(x_1, x_2, x_3, x_4, x_5, x_6)$$

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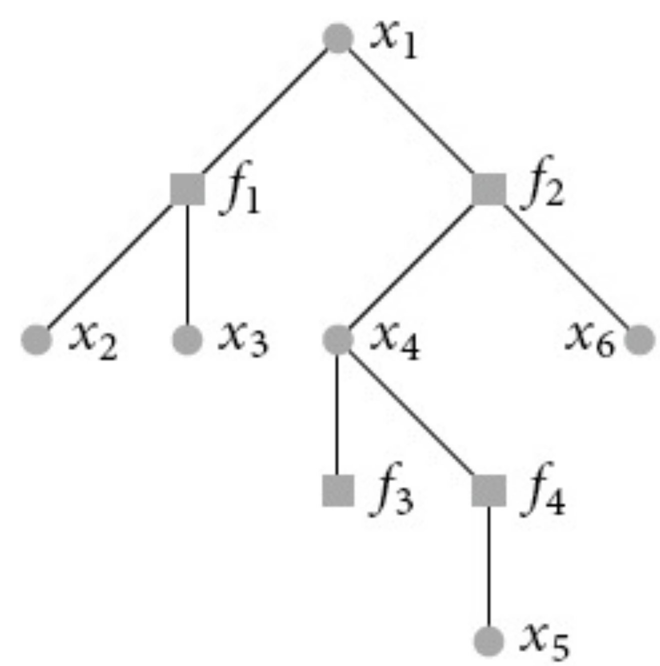

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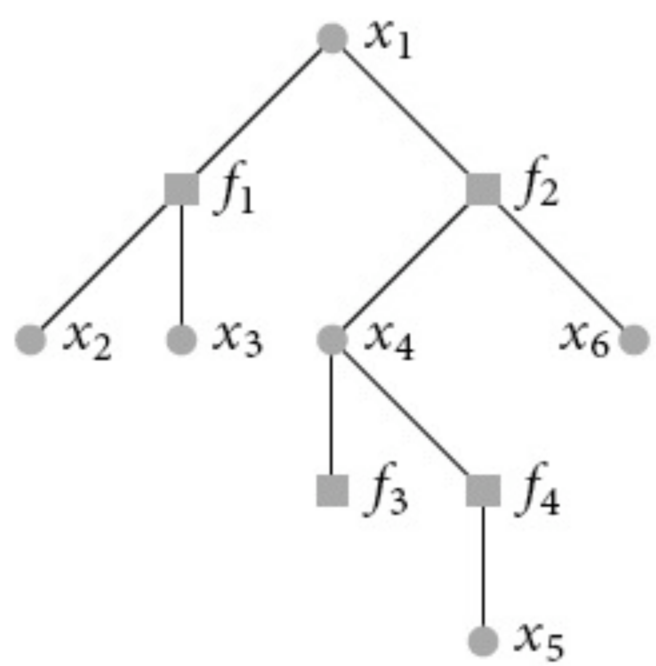

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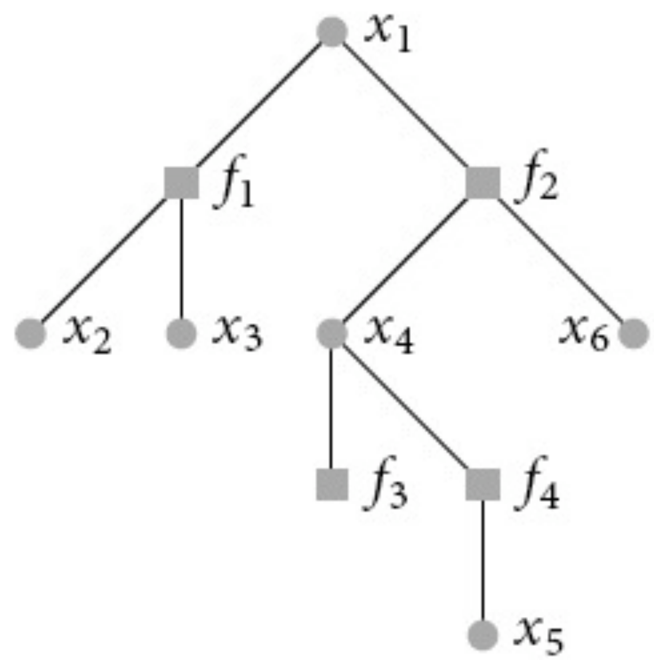

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$\Theta\left(|\mathcal{X}|^6\right)$ brute force complexity

Example

$$f(x_1) = \Big[\sum_{x_2,x_3} f_1(x_1,x_2,x_3)\Big]\Big[\sum_{x_4} f_3(x_4)\Big(\sum_{x_6} f_2(x_1,x_4,x_6)\Big)\Big(\sum_{x_5} f_4(x_4,x_5)\Big)\Big]$$

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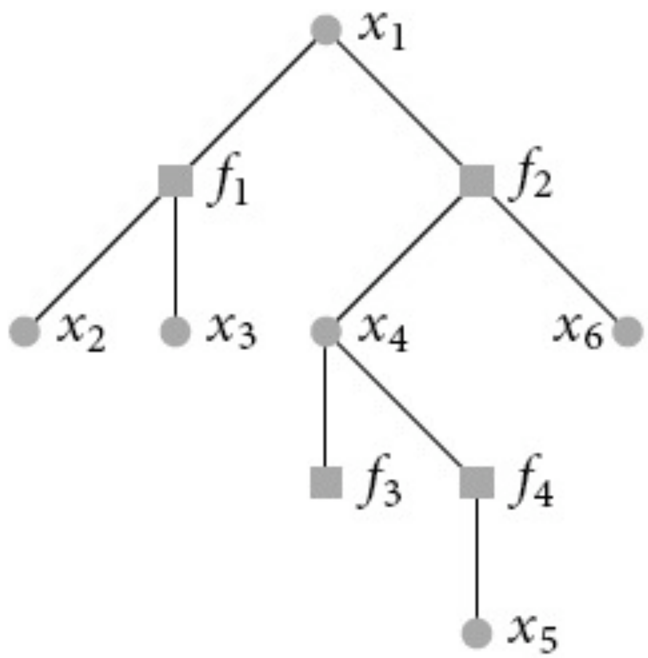

Does there exist a systematic way to find this low complexity scheme using the structure of the graph?

Marginalization via Message Passing for Trees

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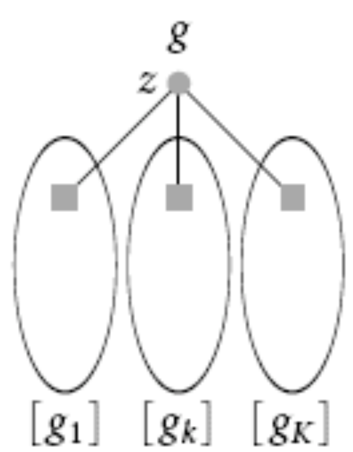

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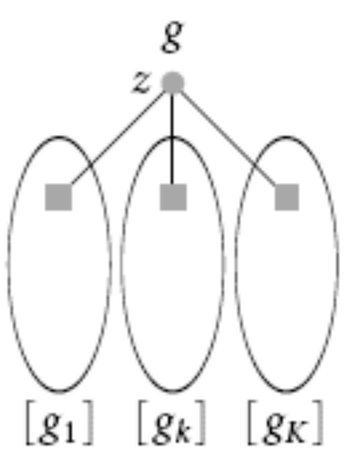

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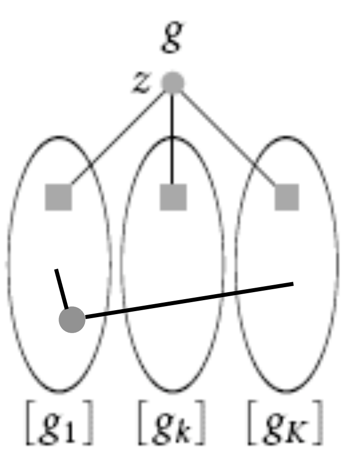

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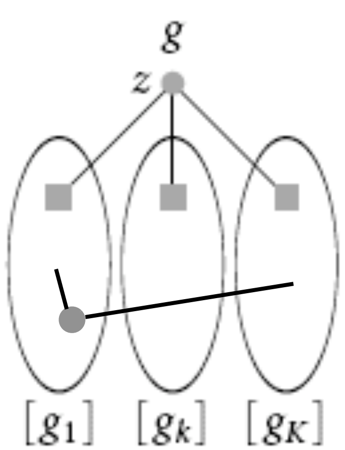

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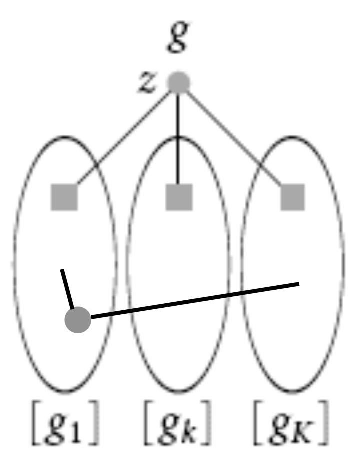

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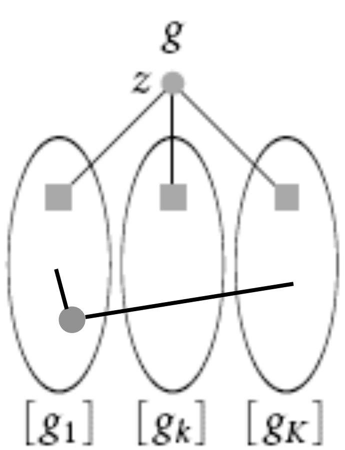

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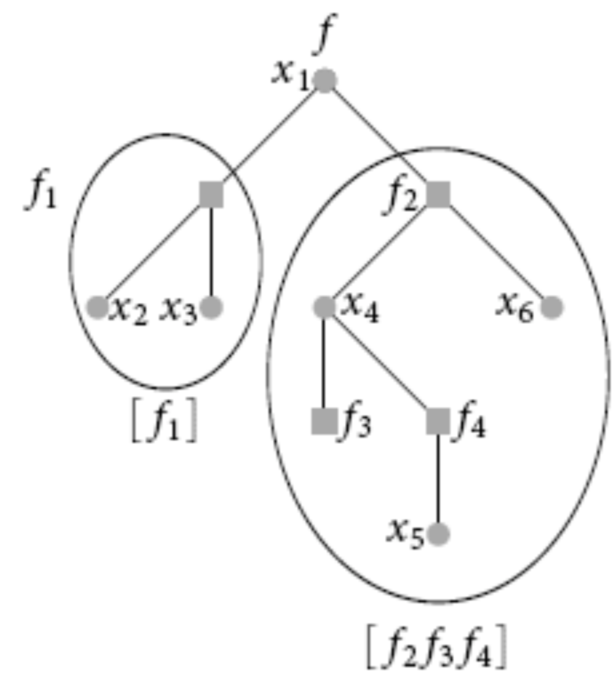

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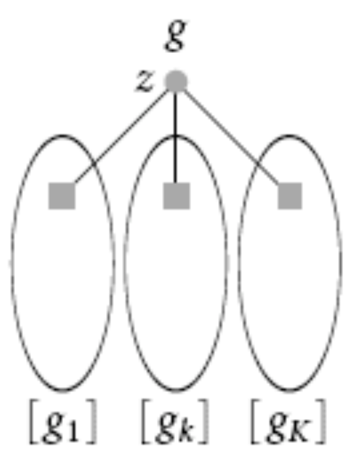

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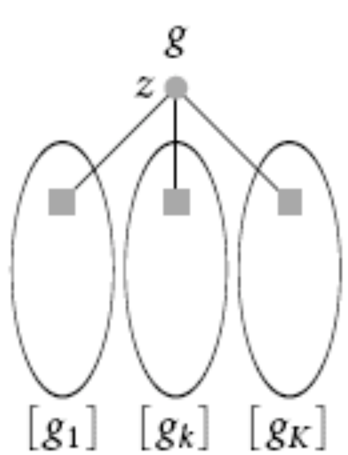

marginal $\sum_{\sim z} g(z, \ldots)$ is the product of the individual marginals

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recall that g(z) is a function, taking a distinct value for each value of z

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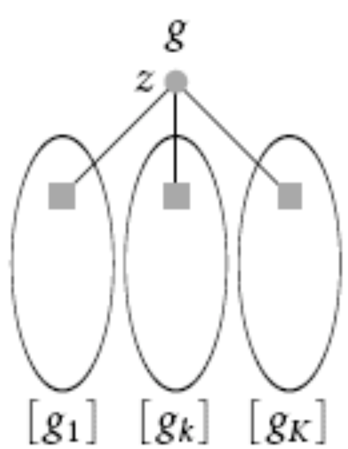

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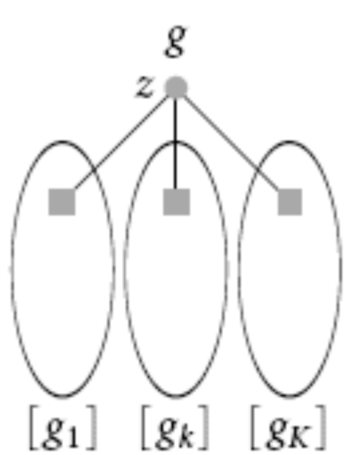

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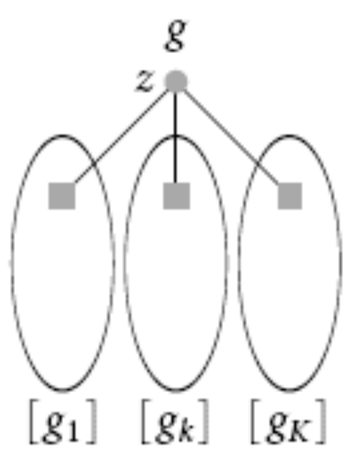

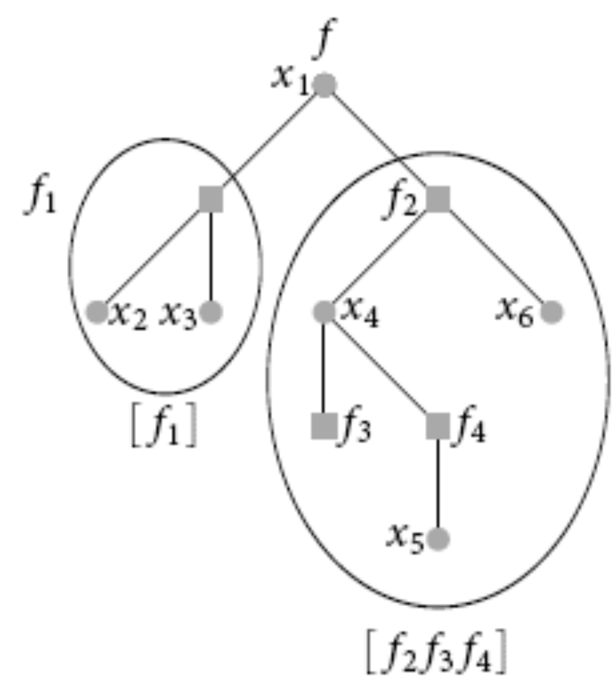

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$$g_k(z,\dots) = \underbrace{h(z,z_1,\dots,z_J)}_{\text{kernel}} \prod_{j=1}^{J} \underbrace{\big[h_j(z_j,\dots)\big]}_{\text{factors}}$$

"kernel" $h(z,z_1,\dots,z_J)$

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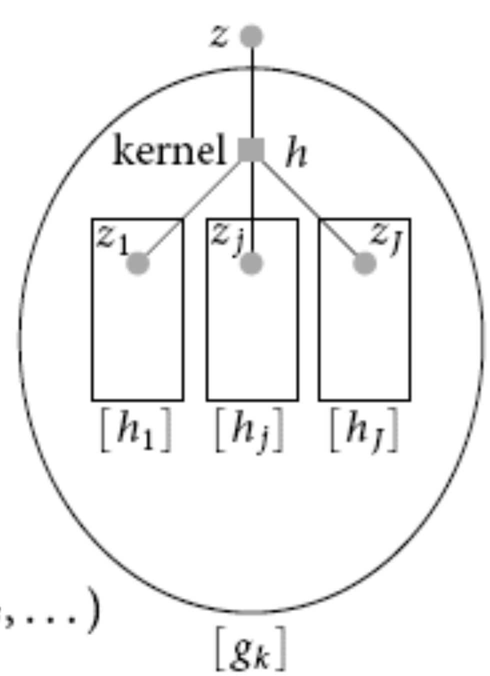

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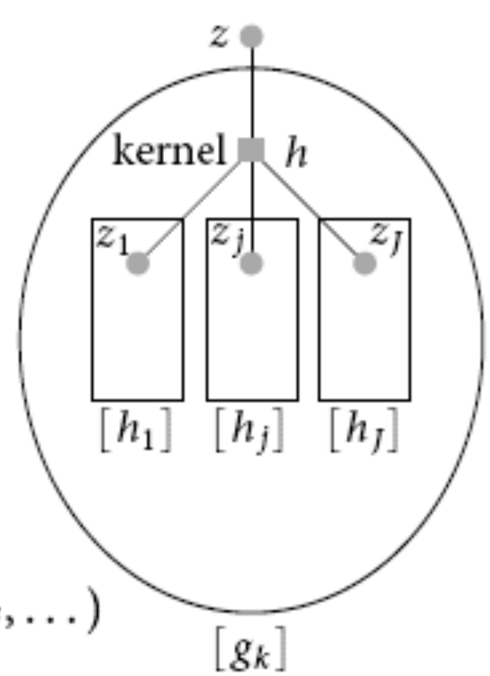

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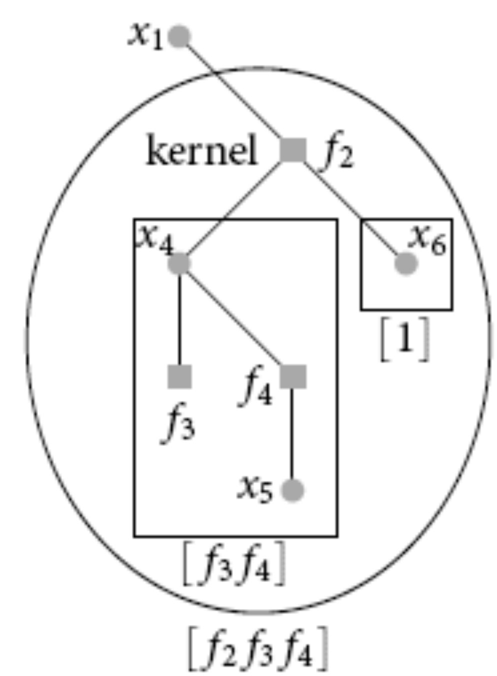

$$f_2(x_1,x_4,x_6)\, f_3(x_4)\, f_4(x_4,x_5) = \underbrace{f_2(x_1,x_4,x_6)}_{\text{kernel}} \underbrace{[f_3(x_4) f_4(x_4,x_5)]}_{x_4} \underbrace{[1]}_{x_6}.$$

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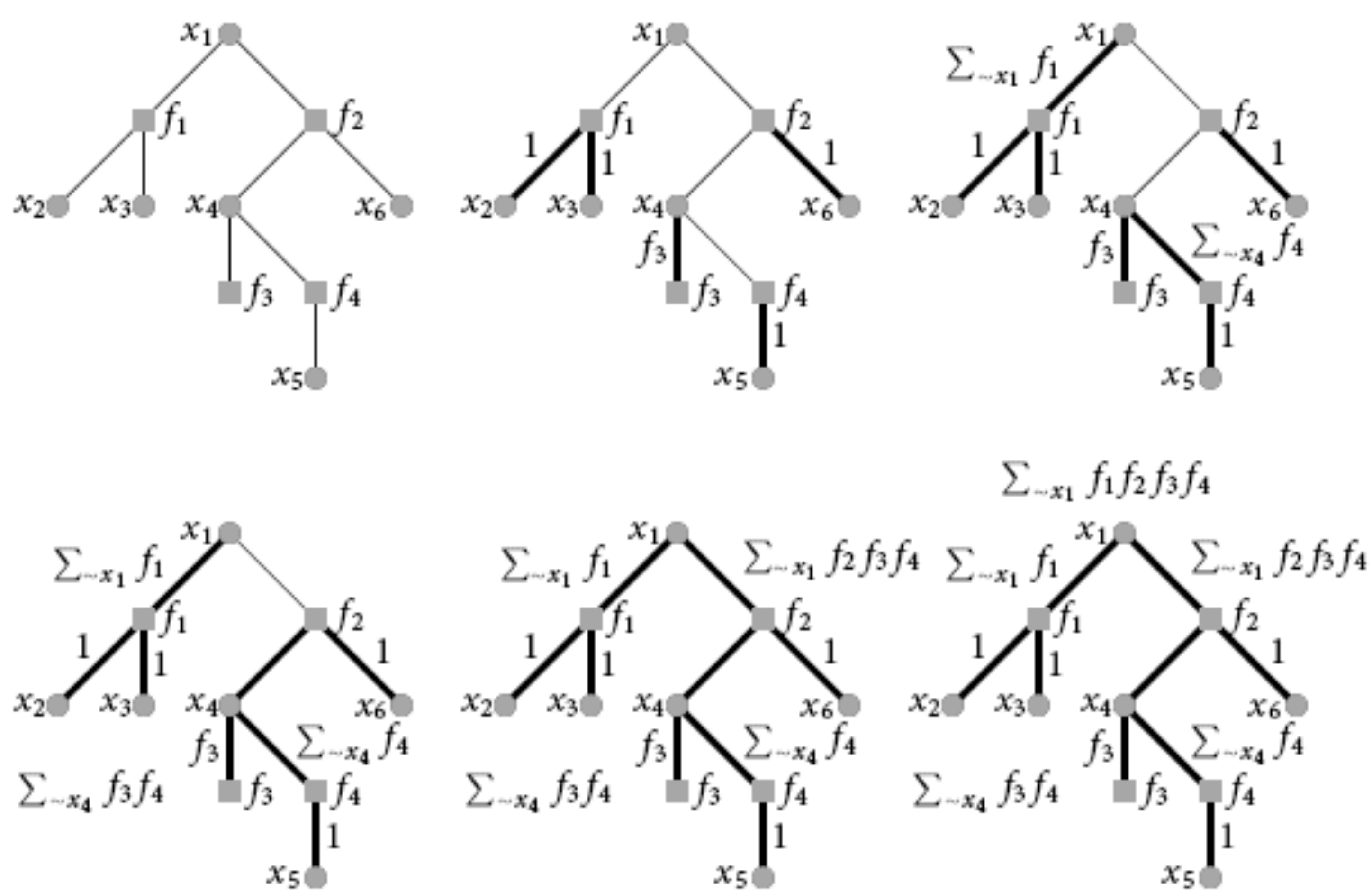

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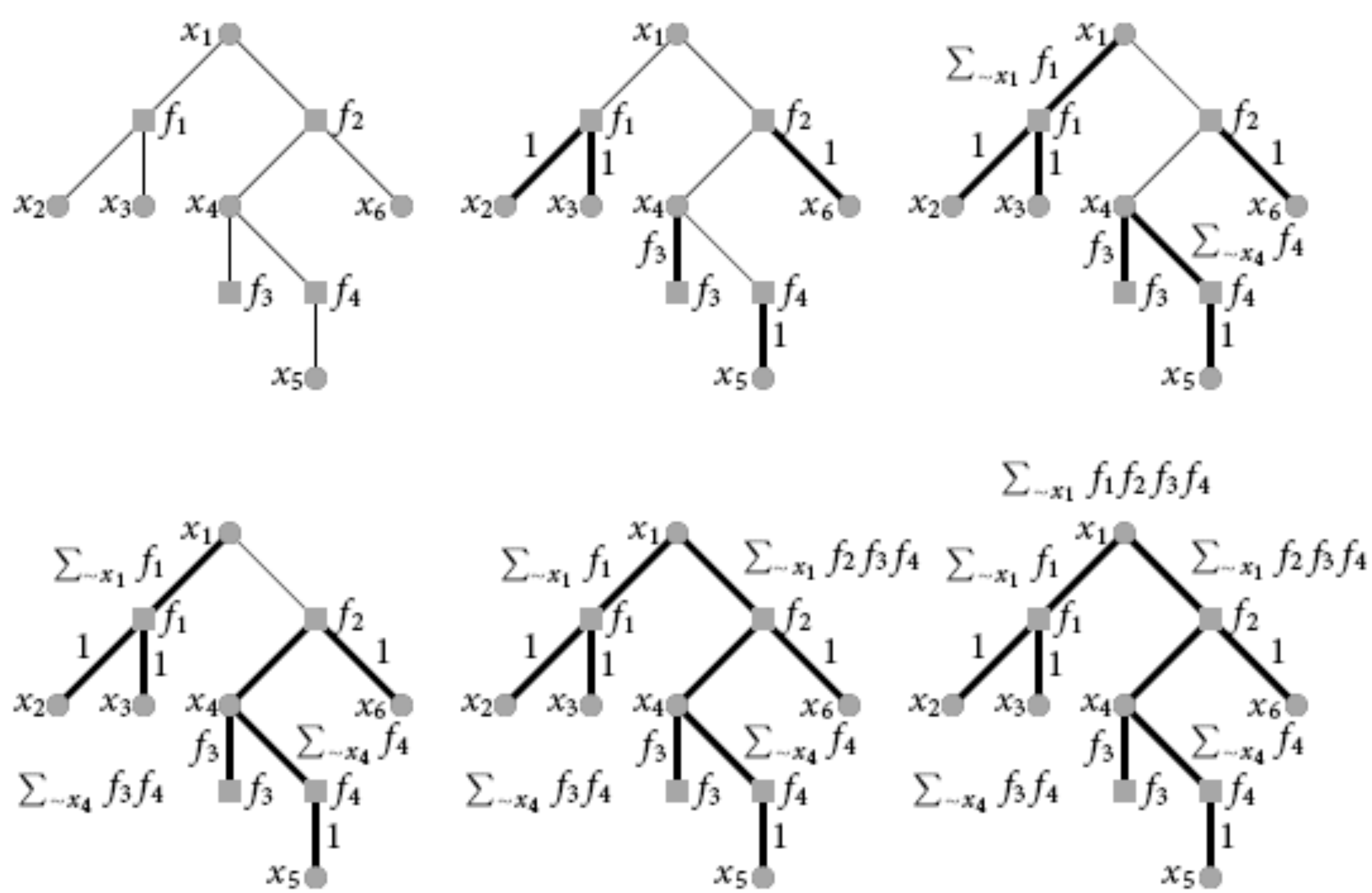
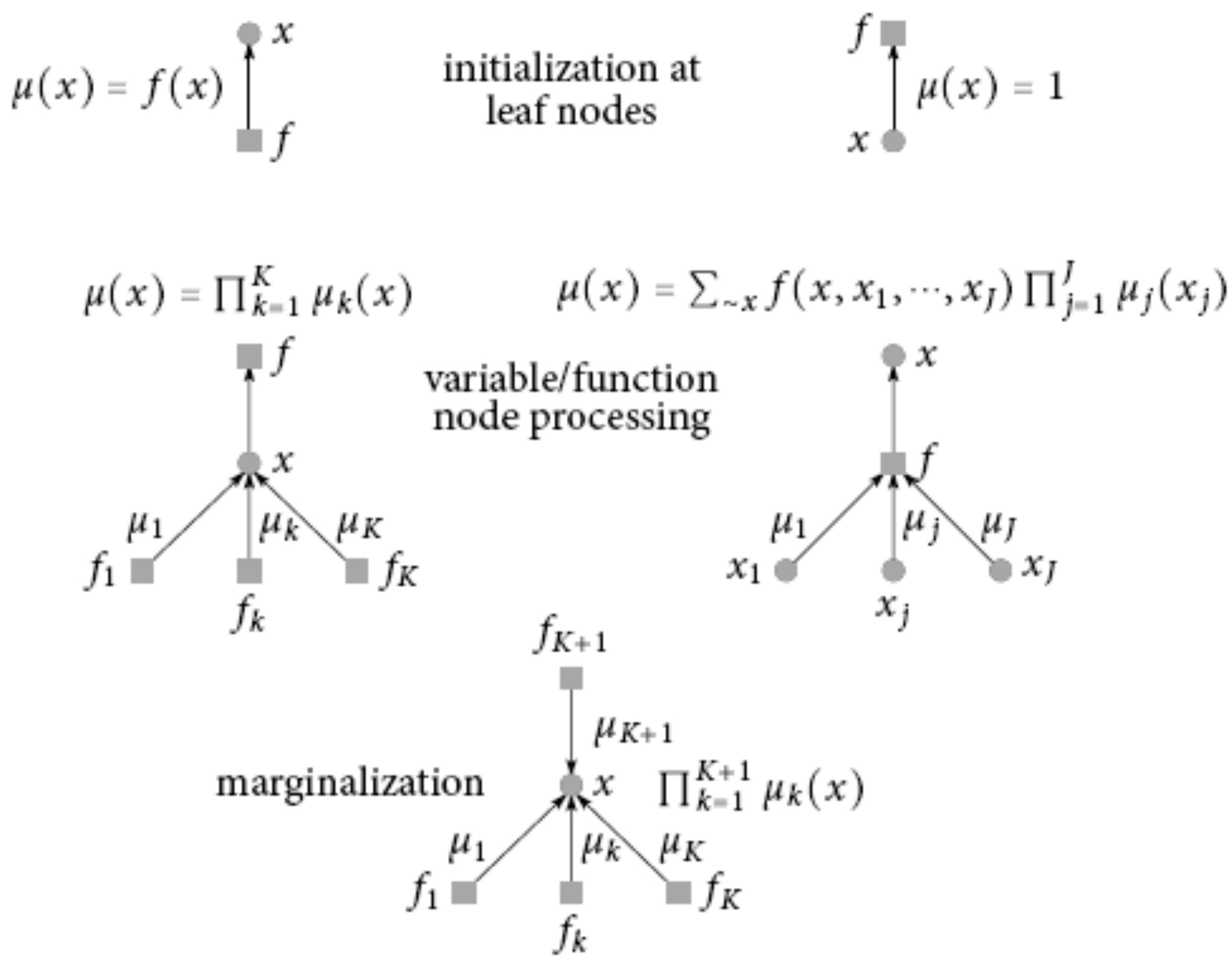


complexity proportional to highest degree

Message Passing Rules



Summary and Limitations

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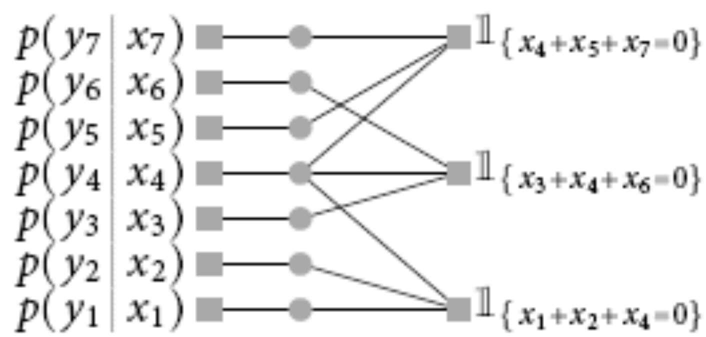

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