Module-I GRAPH THEORY

CO'S	Course Outcomes
CO1	Outline the graph terminologies, graph representation, and relate them to practical examples
CO2	Build efficient graph routing algorithms for various optimization problems on graphs
CO3	Use effective techniques from graph theory to solve problems in networking and telecommunication.
CO4	Interpret the fundamental concepts of polynomials, roots of equations and solve corresponding problems using computer programs.
CO5	Apply the knowledge of numerical methods to solve algebraic and transcendental equations arising in real-life situations.
CO6	Solve numerical integrals and ordinary differential equations to simulate discrete time algorithms.

Module-I GRAPH THEORY

- Graph terminology
- Digraphs
- Weighted graphs
- Complete graphs
- Graph complements
- Bipartite graphs
- Graph combinations
- Isomorphisms
- Matrix representations of graphs
- Incidence and Adjacency matrices
- Degree sequence

Module-I GRAPH THEORY

➢ Graph terminology

A graph is collection of points called **vertices** & collection of lines called **edges** each of which joins either a pair of points or single points to itself.

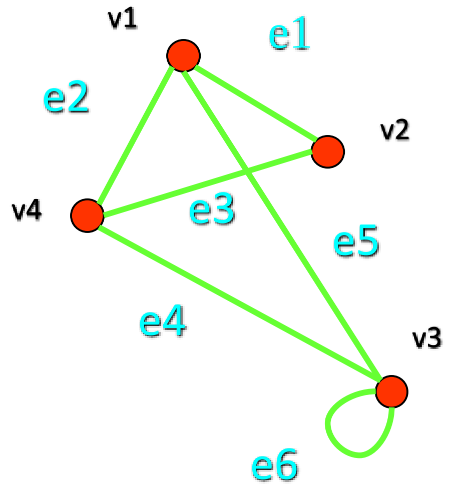
Mathematically graph G is an ordered pair of (V, E)

Each edge e_{ij} is associated with an ordered pair of vertices (V_i,V_j).

In Fig. G has graph 4 vertices namely v1, v2, v3, v4& 7 edges

Namely e1, e2, e3, e4, e5, e6, e7 Then e1=(v1, v2)

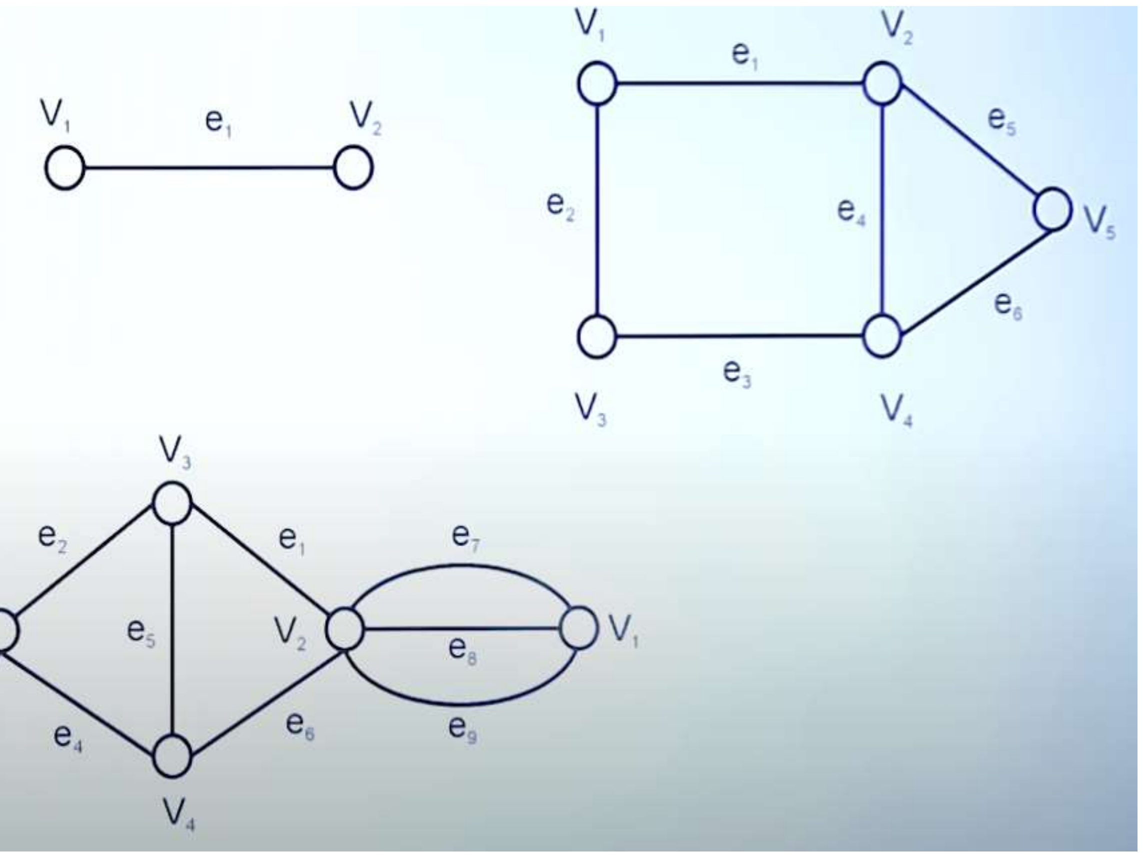
Similarly for other edges.



In short, we can represent G=(V,E) where V=(v1, v2, v3, v4) & E=(e1, e2, e3, e4, e5, e6,e7)

Module-I GRAPH THEORY

Example :

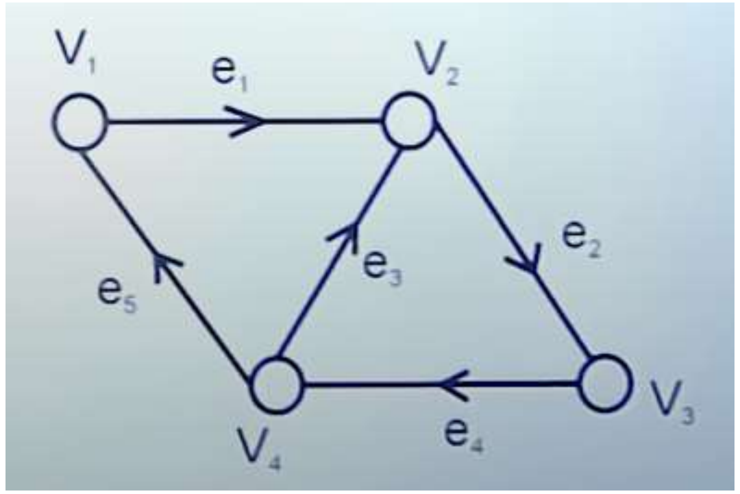


Module-I GRAPH THEORY

Trivial graph: A graph consisting only one vertex and no edge.
Example:

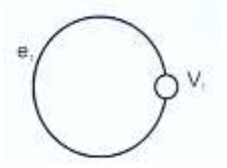
Null Graph: A Graph consisting n vertices and no edge.
Example:

Directed Graph: A Graph consist the direction of edges then this called directed graph.
Example:



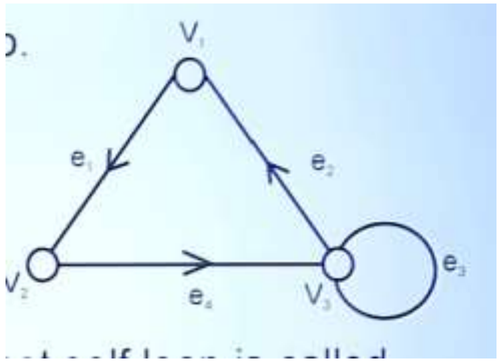
Module-I GRAPH THEORY

Undirected Graph: A Graph which is not directed then it is called undirected graph.
Example:

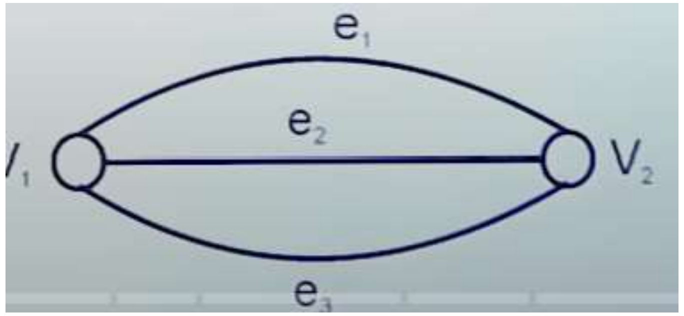


Self loop in a graph: If edge having the same vertex as both its end vertices is called self loop.

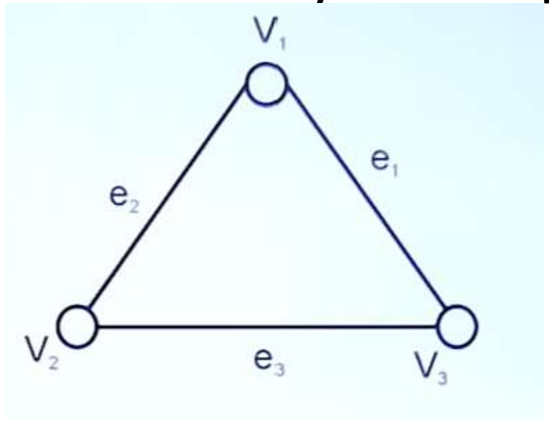
Proper edge: An edge which is not self loop is called proper edge.
Example:



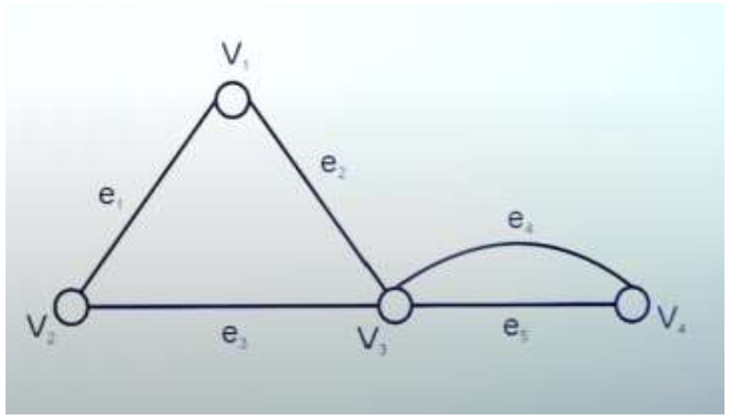
Multi edge : A collection of two or more edges having identically end points.
Example:



Simple graph: A Graph does not contain any self loop and multi edge.
Example:

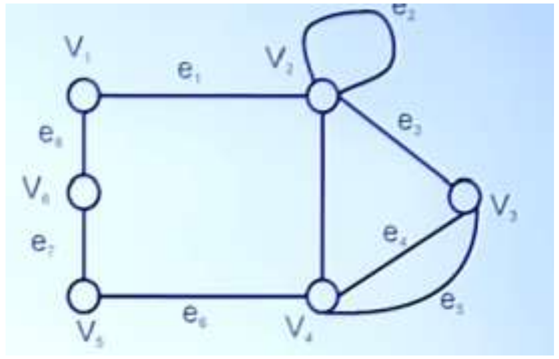


Multigraph: A graph does not contain any self loop but contain multiedge is called multigraph.



Module-I GRAPH THEORY

Pseudo Graph: A Graph contain both self loop and multiedge is called pseudo graph.



Incidence and adjacency: Let ek be an edge joining two vertices Vi and Vj then ek is said to be incident of Vi and Vj.Two vertices are said to be adjacent if there exsist an edge joining this vertices.

Example:

Here e1 is incident of V1 & V2 and V1 & V2 are adjacent but for e3,V1 & V2 are not adjacent.

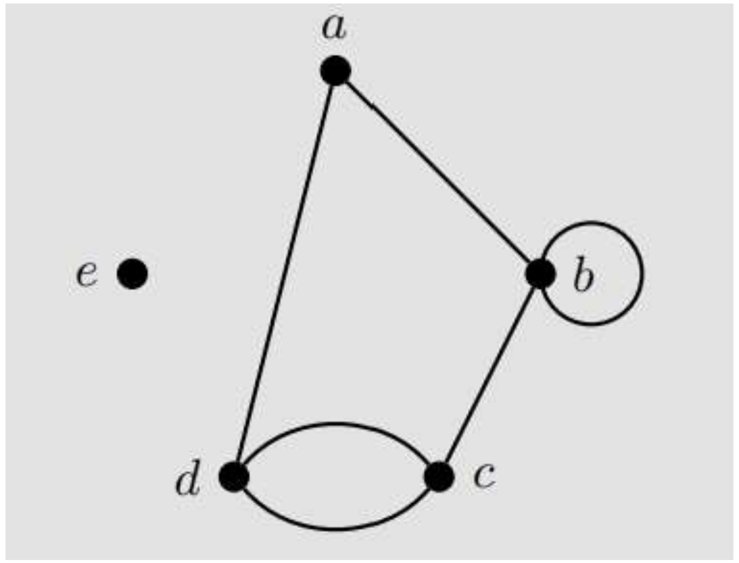
Module-I GRAPH THEORY

Degree of Vertex:The degree of vertex V in a graph G written as d(V) is equal to number of edges which are incident on V with self loop counted twice.

Example:

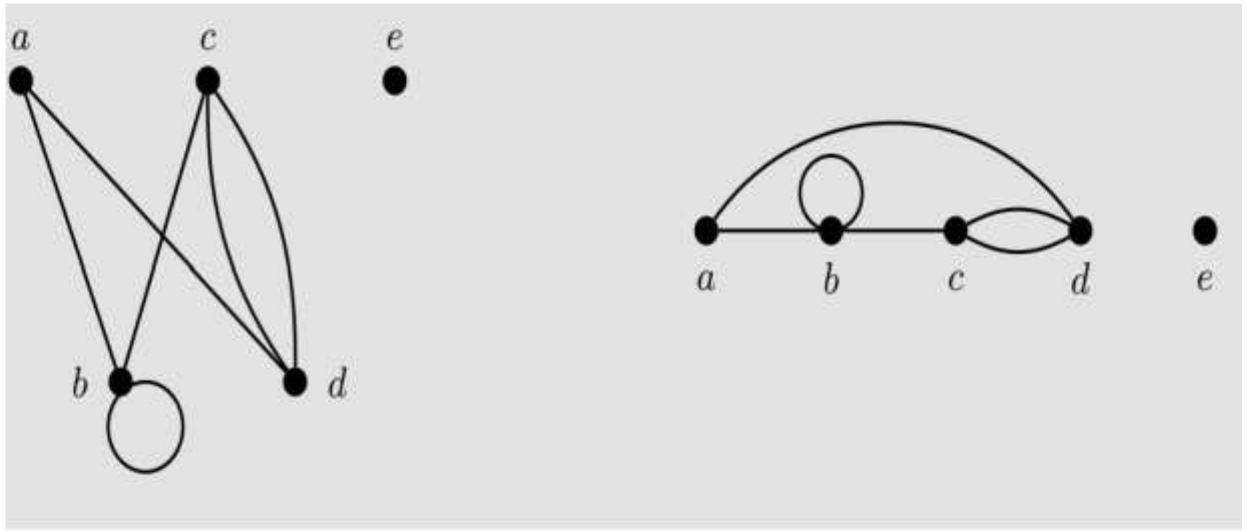
Here d(V1)=2, d(V2)=4, d(V3)=2, d(V4)=3, d(V5)=2, d(V6)=2, d(V7)=1, d(V8)=0

Definition 1.1 A graph G consists of two sets: V (G), called the vertex set, and E(G), called the edge set. An edge, denoted xy, is an unordered pair of vertices. We will often use G or G = (V, E) as short-hand.



Definition 1.2 The number of vertices in a graph G is denoted |V (G)|, or more simply |G|.

The number of edges is denoted |E(G)| or ||G||.



Definition 1.3 Let G be a graph.

• If xy is an edge, then x and y are the endpoints for that edge. We say x is incident to edge e if x is an endpoint of e.

• If two vertices are incident to the same edge, we say the vertices are adjacent, denoted $x \sim y$.
Similarly, if two edges share an endpoint, we say they are adjacent. If two vertices are adjacent, we say they are neighbors and the set of all neighbors of a vertex x is denoted N(x).

– ab and ad are adjacent edges in G4 since they share an endpoint, namely vertex a.
– $a \sim b$, that is a and b are adjacent vertices as ab is an edge of G4
– N(d) = {a, c} and N(b) = {a, b, c}

Module-I GRAPH THEORY

- If two vertices (or edges) are not adjacent then we call them *independent*.
- If a vertex is not incident to any edge, we call it an *isolated vertex*. **–** *e* is an

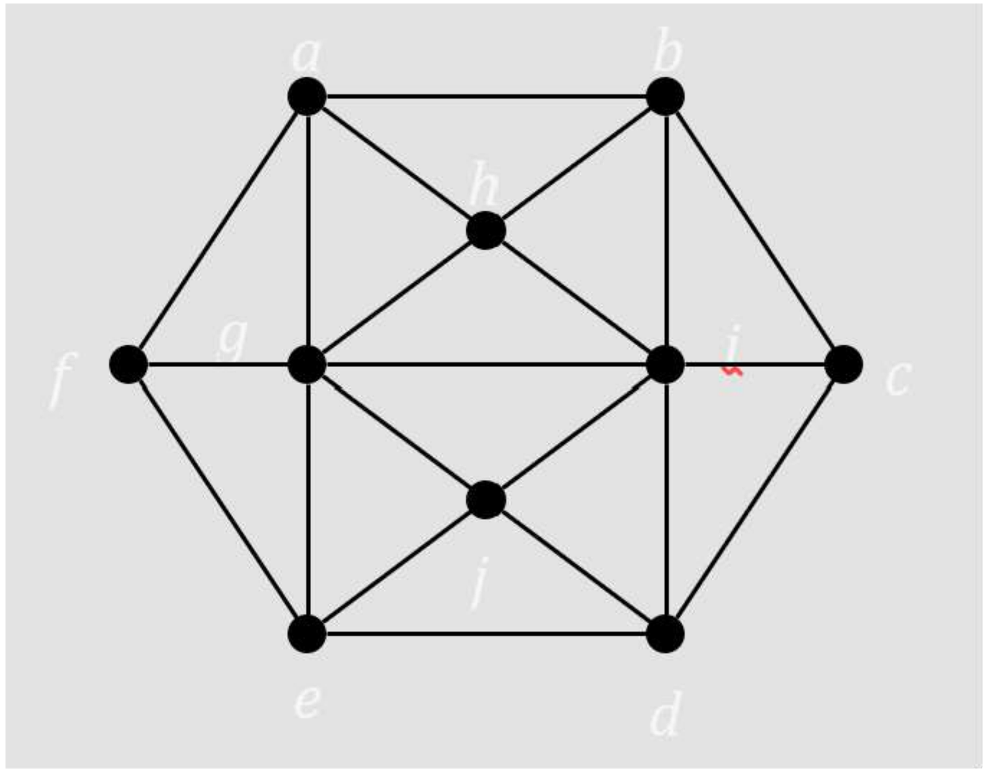
  isolated vertex of G_4
- If both endpoints of an edge are the same vertex, then we say the edge is a *loop*.
  - *bb* is a loop in G_4
- If there is more than one edge with the same endpoints, we call these *multi-edges*.
  - *cd* is a multi-edge of G_4

Module-I GRAPH THEORY

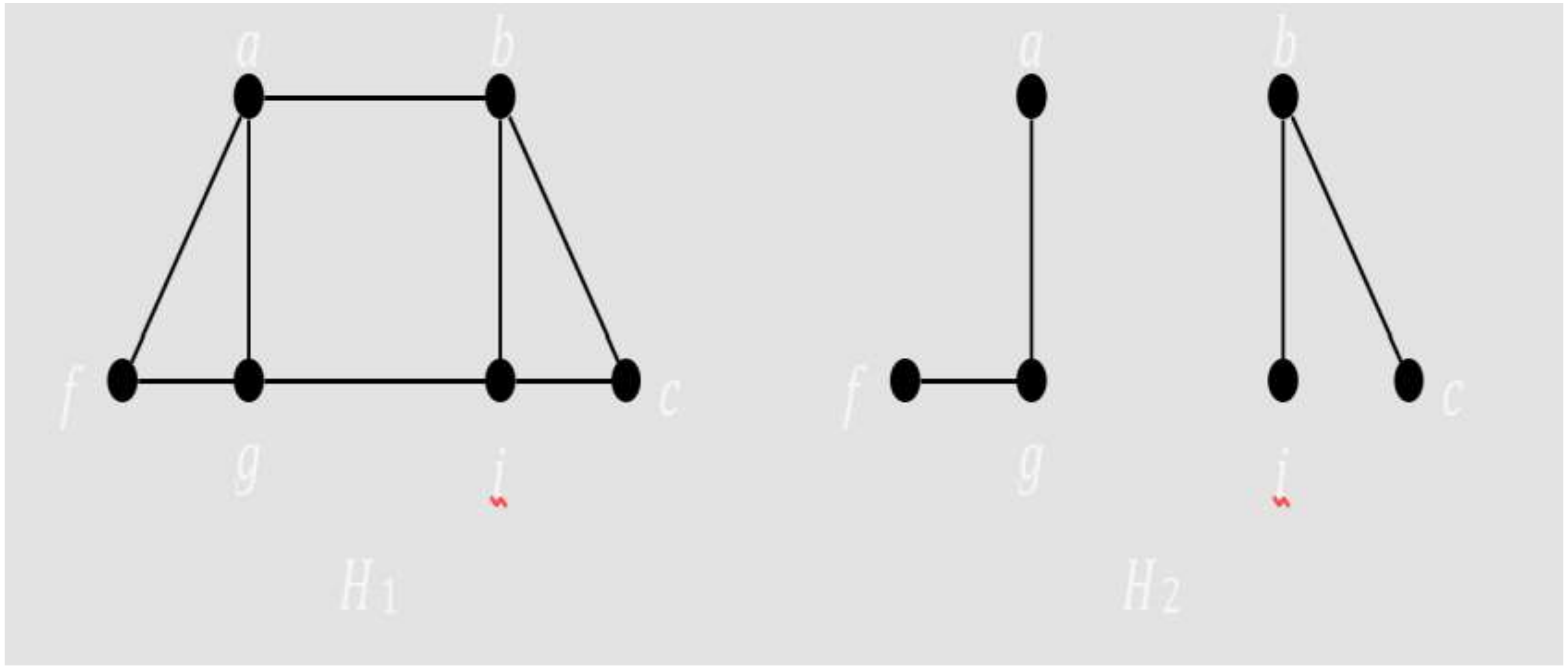
- If a graph has no multi-edges or loops, we call it *simple*.
- The *degree* of a vertex *v*, denoted deg(*v*), is the number of edges incident to *v*, with a loop adding two to the degree. If the degree is even, the vertex is called *even*; if the degree is odd, then the vertex is *odd*.
  - deg(*a*) = 2, deg(*b*) = 4, deg(*c*) = 3, deg(*d*) = 3, deg(*e*) = 0

If all vertices in a graph *G* have the same degree *k*, then *G* is called a ***k*-regular** graph. When *k* = 3, we call the graph **cubic**

Definition 1.4 A *subgraph* H of a graph G is a graph where H contains some of the edges and vertices of G; that is, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

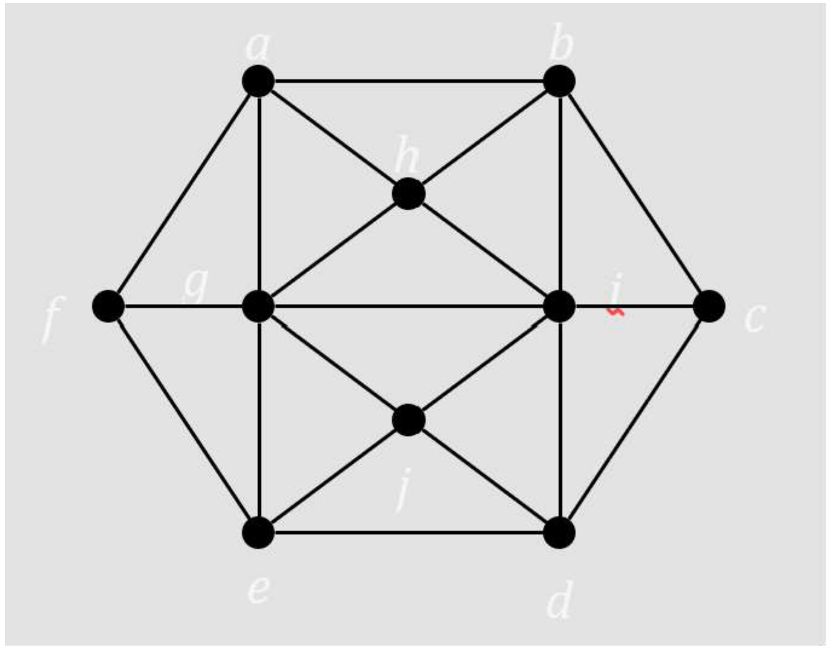


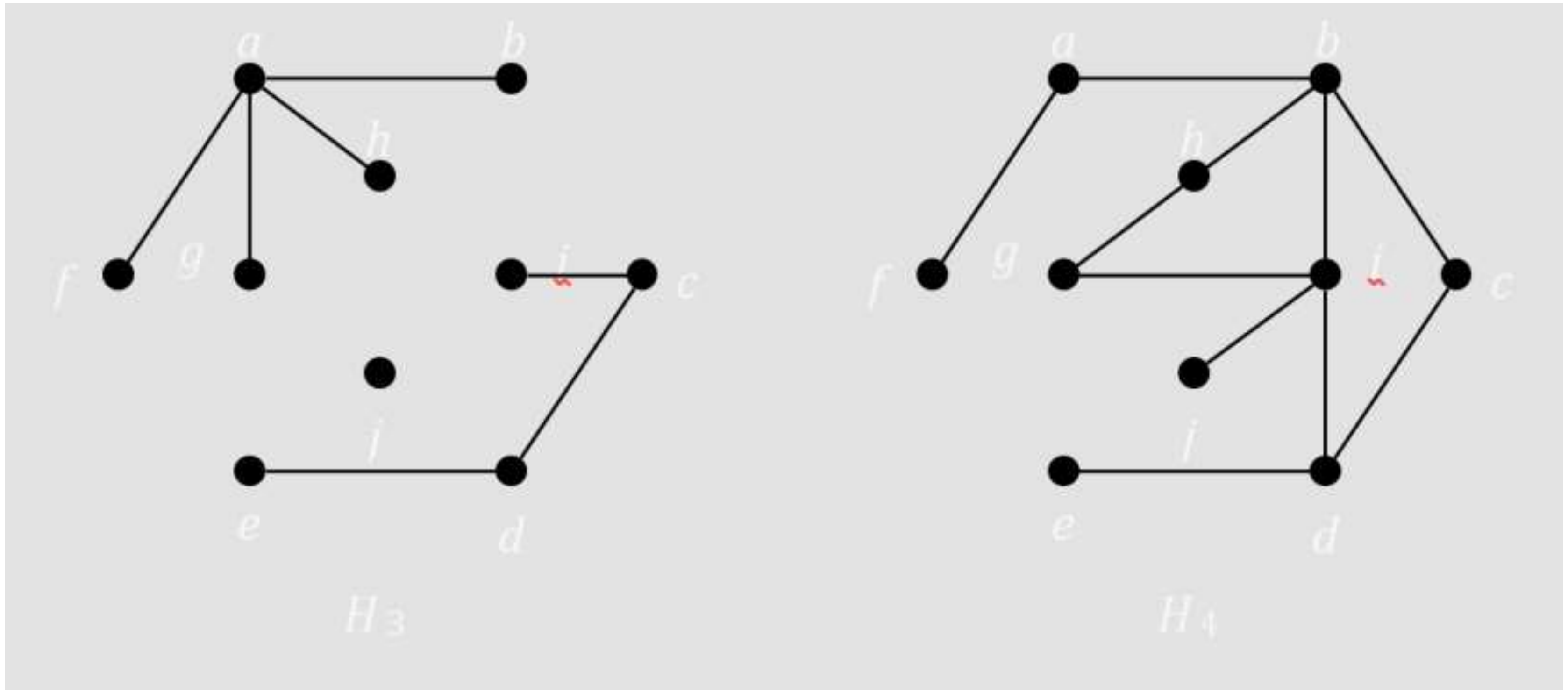
Module-I GRAPH THEORY



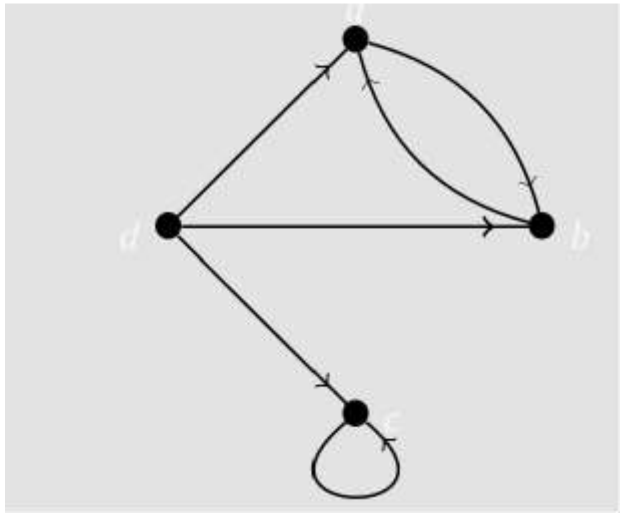
Definition 1.5 Given a graph $G = (V,E)$, an *induced subgraph* is a subgraph $G[V^0]$ where $V^0 \subseteq V$ and every available edge from G between the vertices in V^0 is included.

We say H is a *spanning subgraph* if it contains all the vertices but not necessarily all the edges of G; that is, $V(H) = V(G)$ and $E(H) \subseteq E(G)$.




a
b
h
f
g
i
c
j
e
d
H3
a
b
h
f
g
i
c
j
e
d
H4

Definition 1.6 A *directed graph*, or *digraph*, is a graph $G = (V,A)$ that consists of a vertex set $V(G)$ and an *arc set* $A(G)$. An *arc* is an ordered pair of vertices



Digraphs have many similar properties to (undirected) graphs. Looking at the digraph above, we can see that the number of wins is modeled as the number of arcs coming from a team's vertex, and the number of losses is the number of arcs entering the vertex.

Definition 1.7 Let $G = (V,A)$ be a digraph.

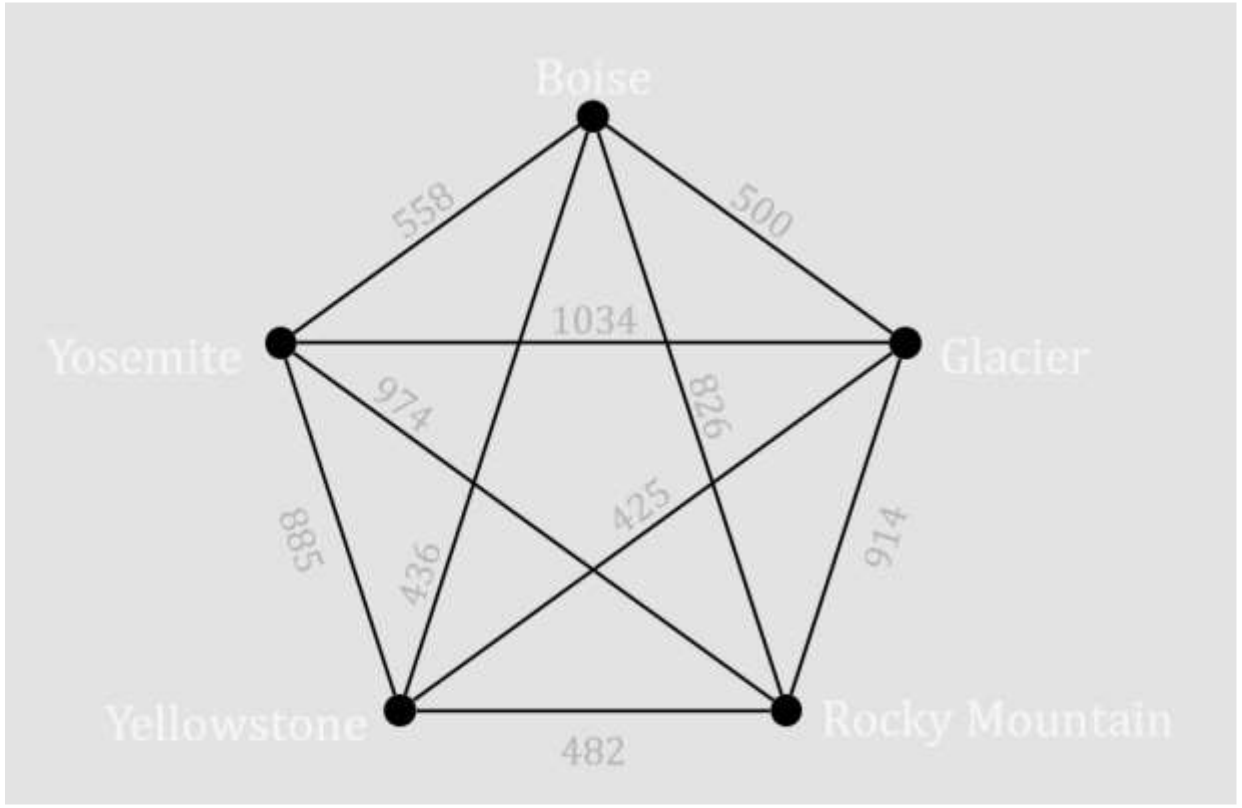
- Given an arc xy, the *head* is the starting vertex x and the *tail* is the ending vertex y.
  - a is the head of arc ab and the tail of arcs da and ba from G_5
- Given a vertex x, the *in-degree* of x is the number of arcs for which x is a tail, denoted $\deg^-(x)$. The *out-degree* of x is the number of arcs for which x is the head, denoted $\deg^+(x)$.
  - $\deg^-(a) = 2$, $\deg^-(b) = 2$, $\deg^-(c) = 2$, $\deg^-(d) = 0$
  - $\deg^+(a) = 1$, $\deg^+(b) = 1$, $\deg^+(c) = 1$, $\deg^+(d) = 3$

The *underlying graph* for a digraph is the graph $G^0 = (V,E)$ which is formed by removing the direction from each arc to form an edge.

Module-I GRAPH THEORY

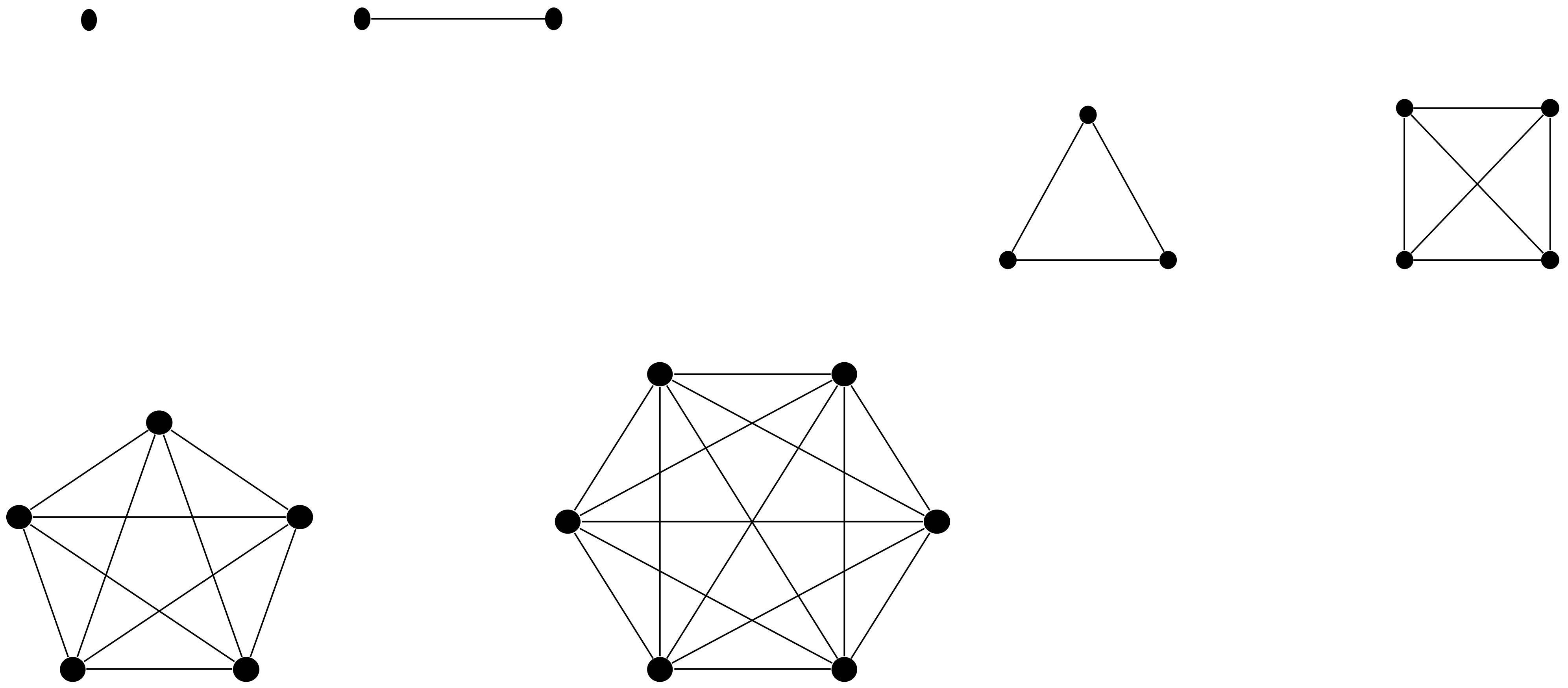
Module-I GRAPH THEORY

Definition 1.8 A *weighted graph G* = (*V,E,w*) is a graph where each of the edges has a real number associated with it. This number is referred to as the *weight* and denoted *w*(*xy*) for the edge *xy*.



Module-I GRAPH THEORY

Definition 1.9 A simple graph G is *complete* if every pair of distinct vertices is adjacent. The complete graph on n vertices is denoted K_n.



$\frac{n(n-1)}{2}$

Properties of K_n

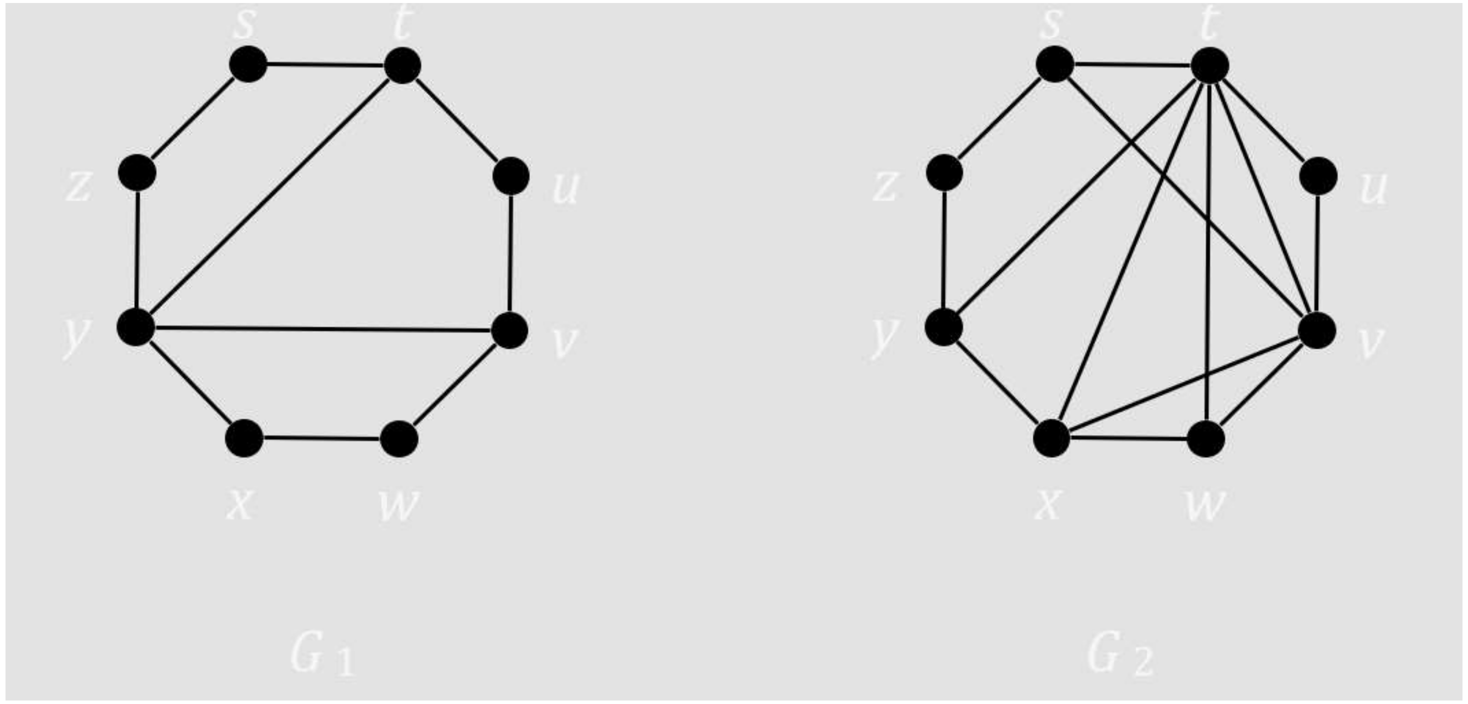
(1) Each vertex in K_n has degree $n - 1$.

(2) K_n has n(n-1)/2 edges

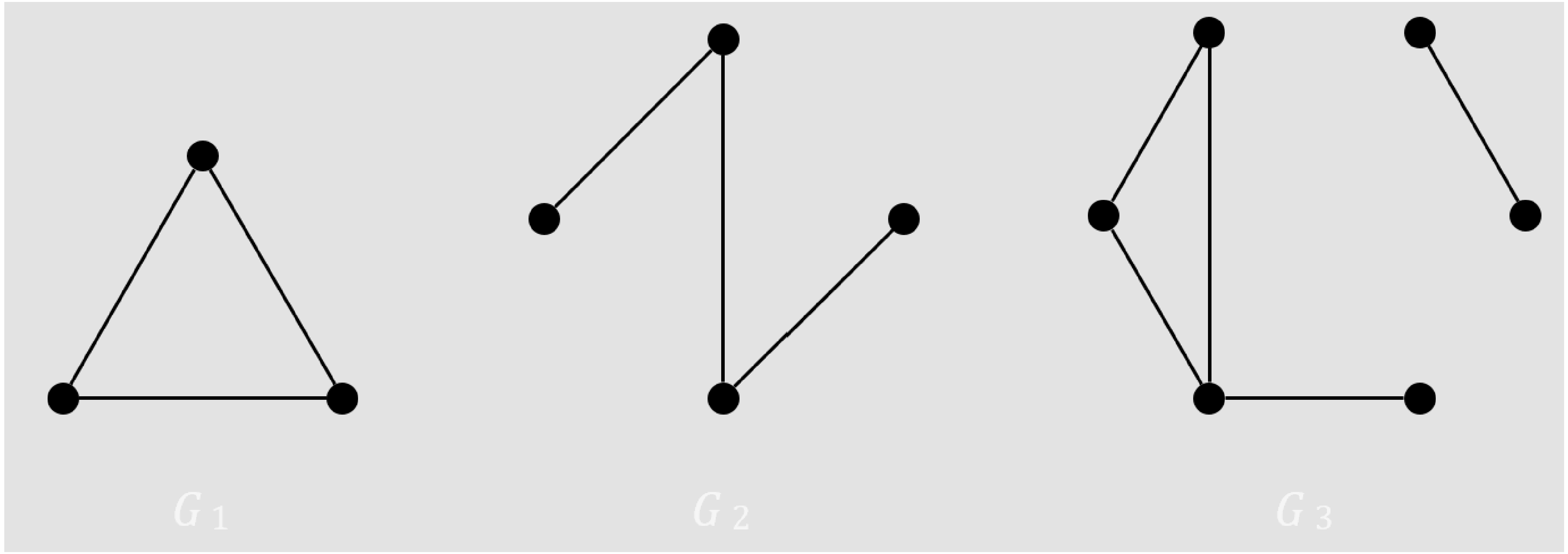
(3) K_n contains the most edges out of all simple graphs on n vertices.

Definition 1.10 The *clique-size* of a graph, $\omega(G)$, is the largest integer n such that K_n is a subgraph of G but K_{n+1} is not.

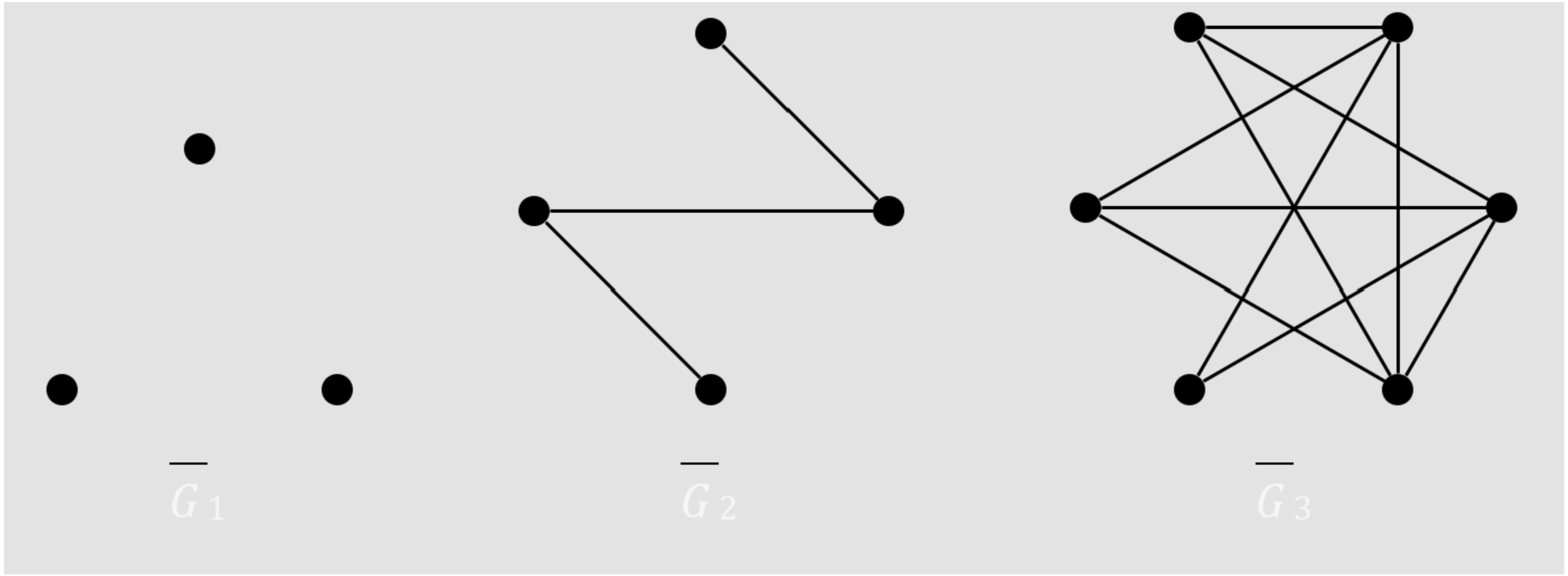
The largest complete graph that appears as a subgraph. This is called the *clique-size* of a graph.



Definition 1.11 Given a simple graph *G*=(*V,E*), define the *complement* of *G* as the graph *G* = (*V,E*), where an edge *xy* ∈ *E* if and only if *xy* /∈ *E*.

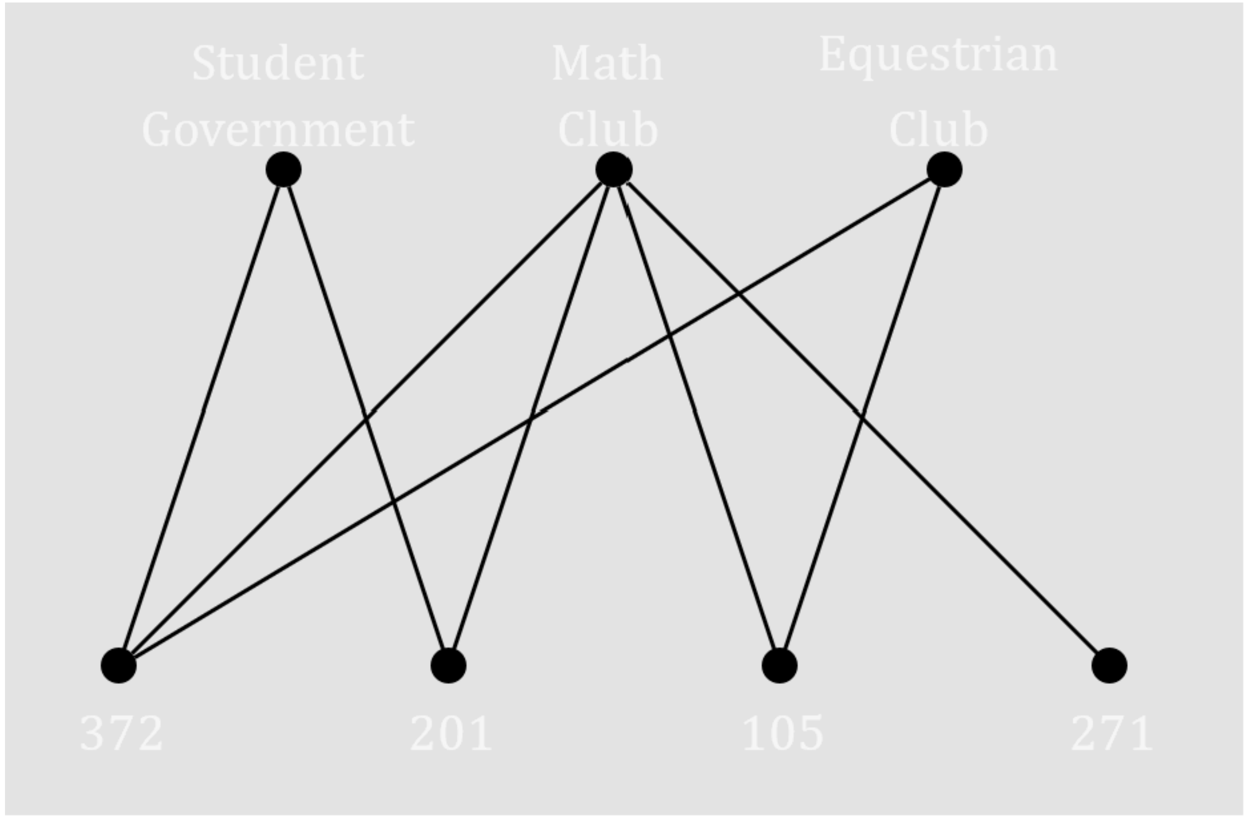


Module-I GRAPH THEORY

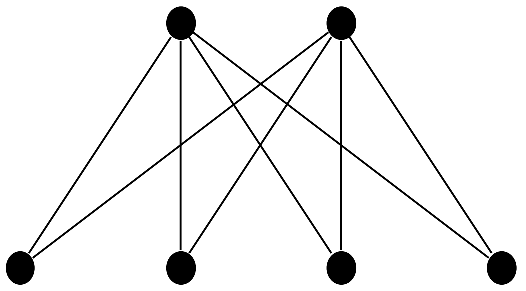
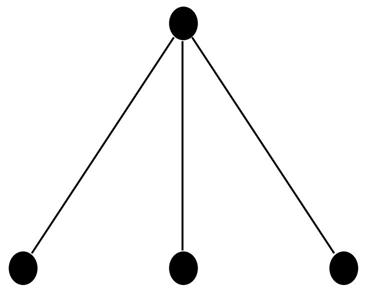
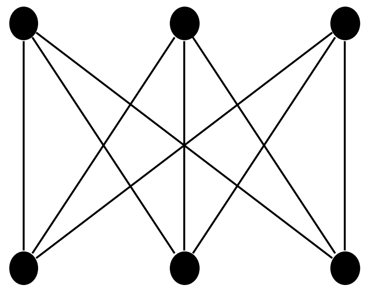
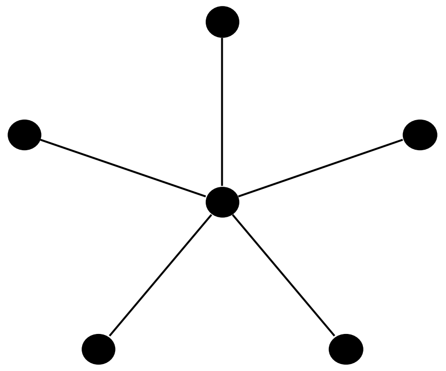
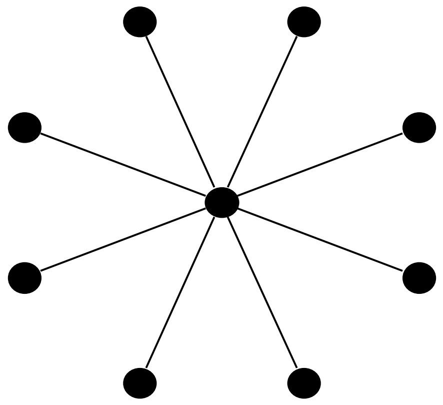


If we want to display the relationship between different types of objects, we would use a *bipartite graph*.

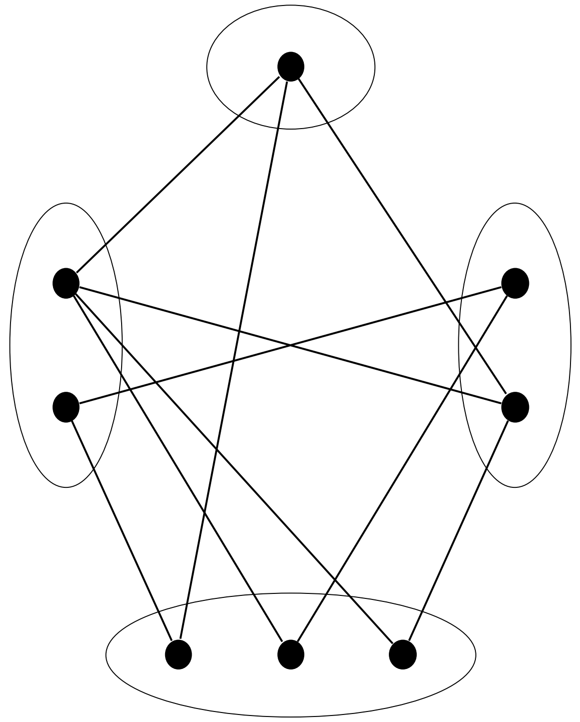
Definition 1.12 A graph *G* is *bipartite* if the vertices can be partitioned into two sets *X* and *Y* so that every edge has one endpoint in *X* and the other in *Y* .

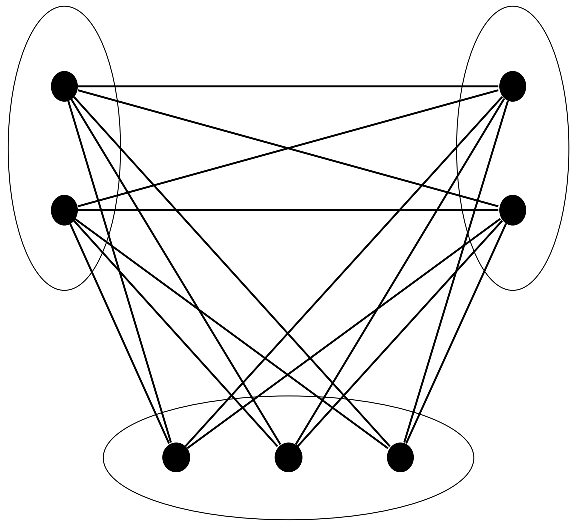


Definition 1.13 $K_{m,n}$ is the *complete bipartite graph* where $|X| = m$ and $|Y| = n$ and every vertex in X is adjacent to every vertex in Y.

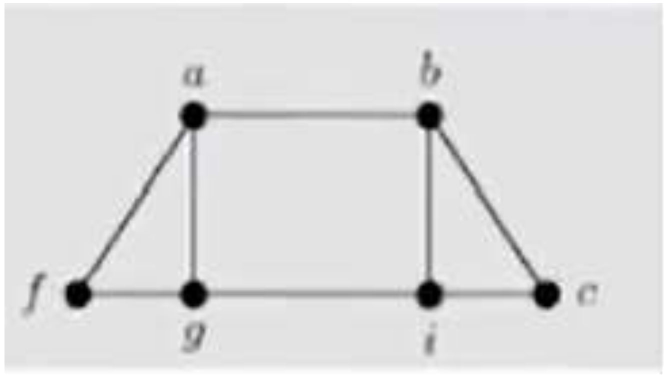






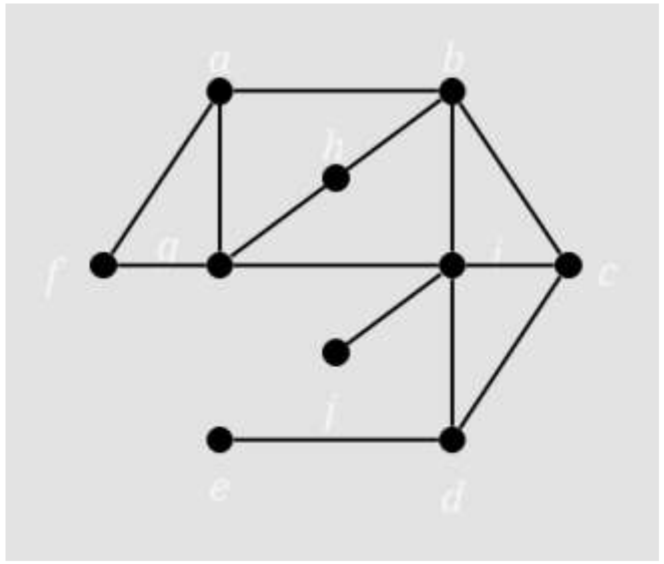
Definition 1.14 A graph G is *k-partite* if the vertices can be partitioned into k sets $X_1, X_2, \ldots X_k$ so that every edge has one endpoint in X_i and the other in X_j where i 6= j.



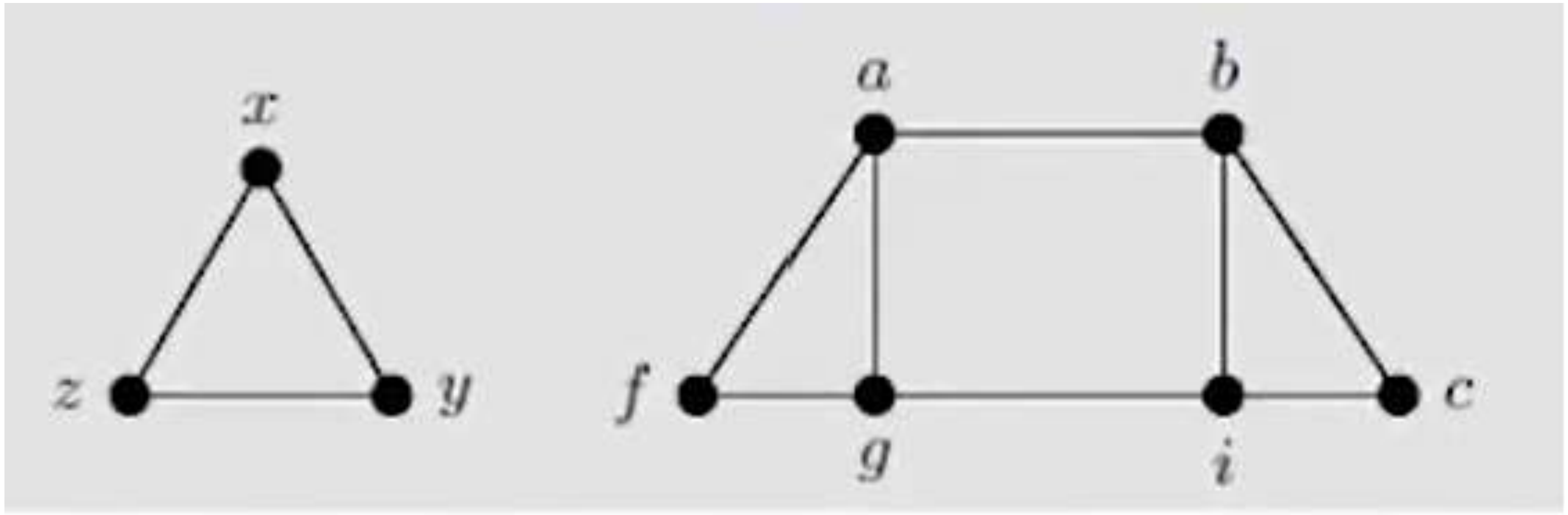


Graph combination: Definition 1.15 Given two graphs G and H the *union* $G \cup H$ is the graph with vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H)$.





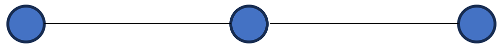
- If the vertex-sets are disjoint (that is $V(G) \cap V(H) = \emptyset$) then we call the disjoint union the *sum*, denoted $G + H$.

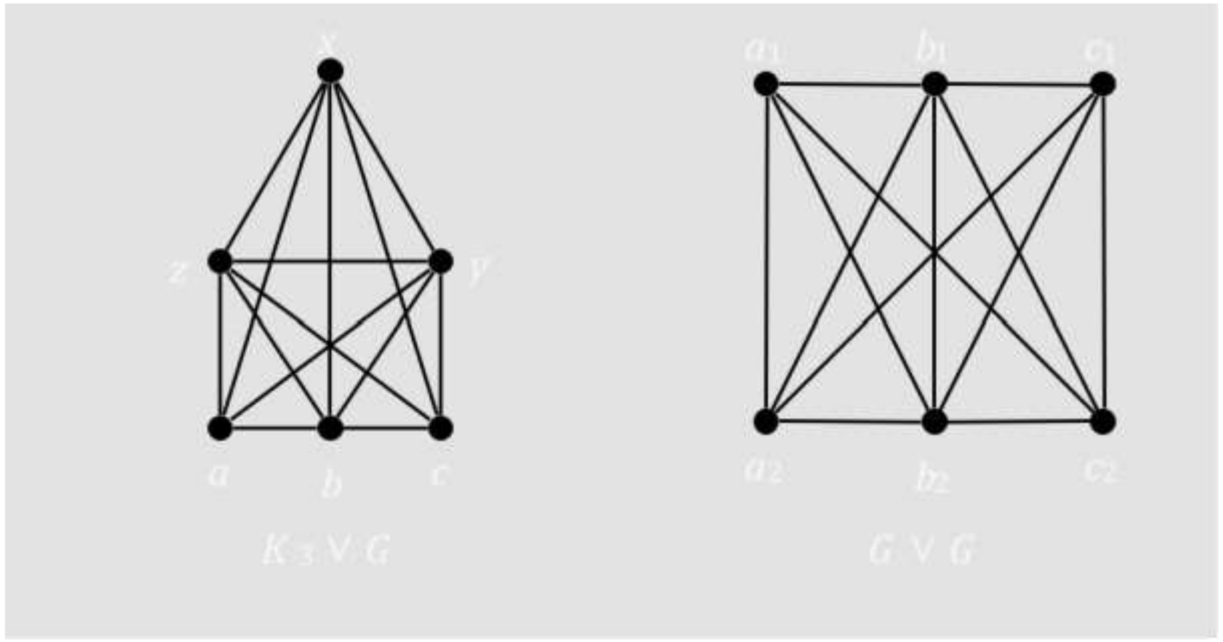


Module-I GRAPH THEORY

Definition 1.16 The *join* of two graphs G and H, denoted $G \vee H$, is the sum $G + H$ together with all edges of the form xy where $x \in V(G)$ and $y \in V(H)$.

Example: Find the join of k3 and the graph G below consisting of three vertices and two edges as well as the join G V G.



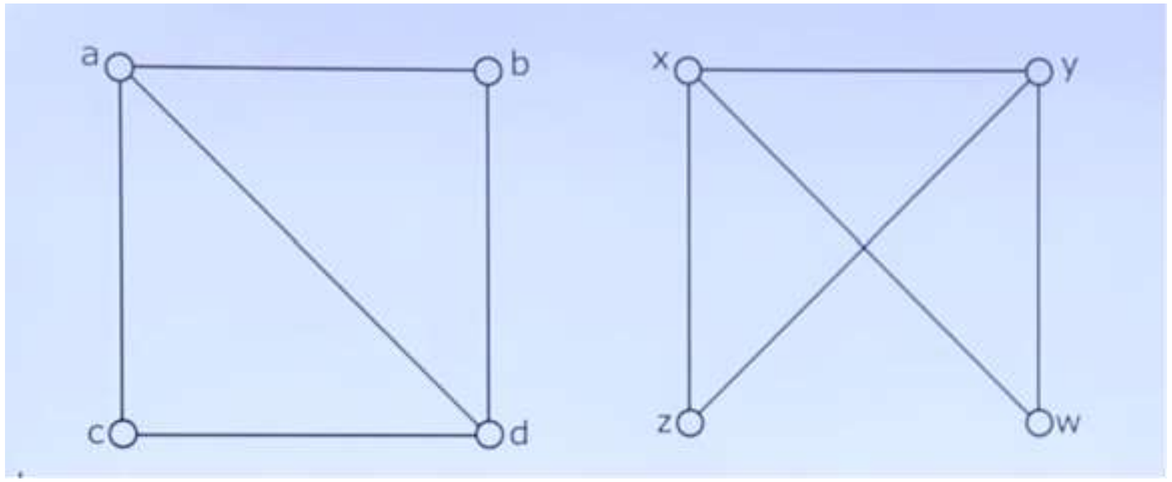


Isomorphisms: Definition 1.17 Two graphs G_1 and G_2 are *isomorphic*, denoted $G_1 \tilde{=} G_2$, if there exists a bijection $f : V(G_1) \rightarrow V(G_2)$ so that $xy \in E(G_1)$ if and only if $f(x)f(y) \in E(G_2)$.

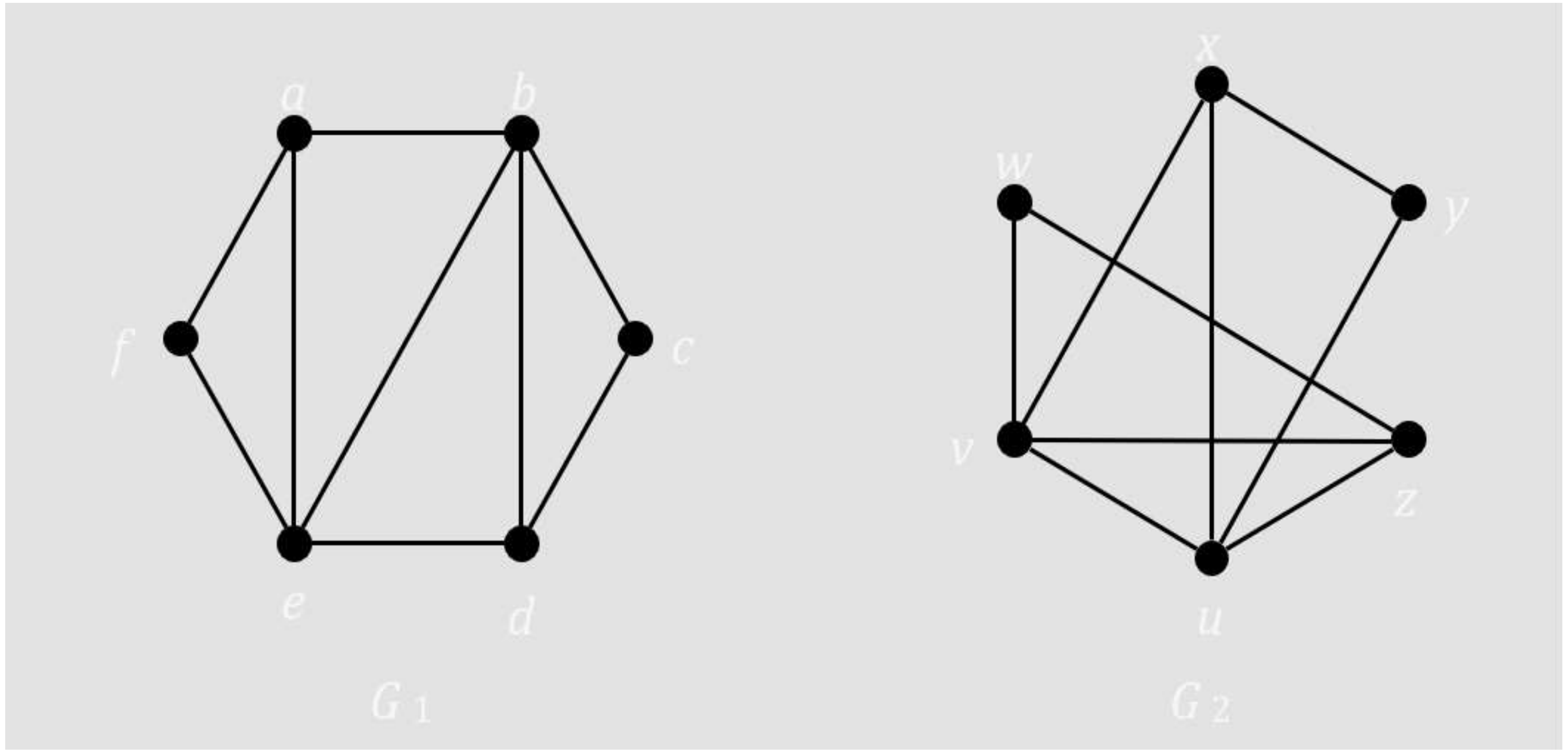
The definition of isomorphic uses a special function, called a bijection, between the vertices of G_1 and G_2; the common properties that must be maintained with isomorphic graphs, called graph invariants.

- Number of vertices
- Number of edges
- Vertex degrees
- Mapping vertices

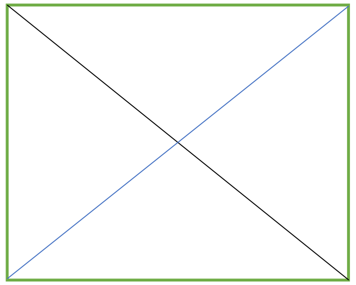
Module-I GRAPH THEORY

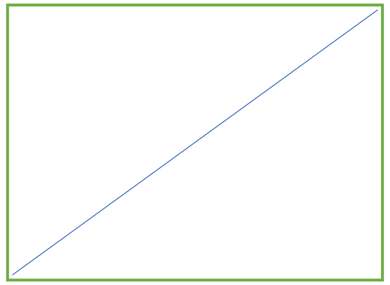


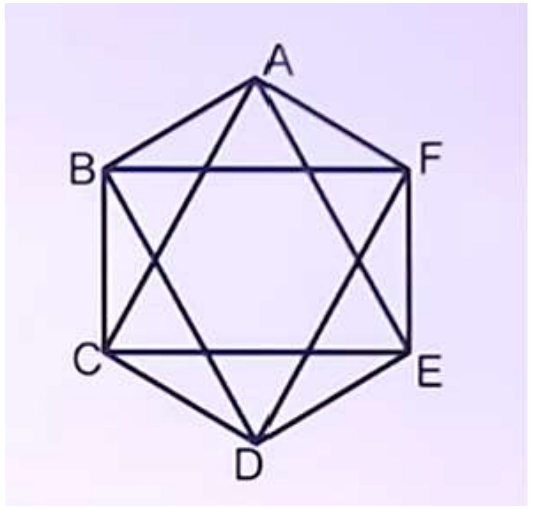
Example 1.14 Determine if the following pair of graphs are isomorphic. If so, give the vertex pairings; if not, explain what property is different among the graphs.



Planare graph: A graph which can be drawn in the plane so that its edges do not cross is called planare.







Module-I GRAPH THEORY

Module-I GRAPH THEORY

Module-I GRAPH THEORY

Module-I GRAPH THEORY

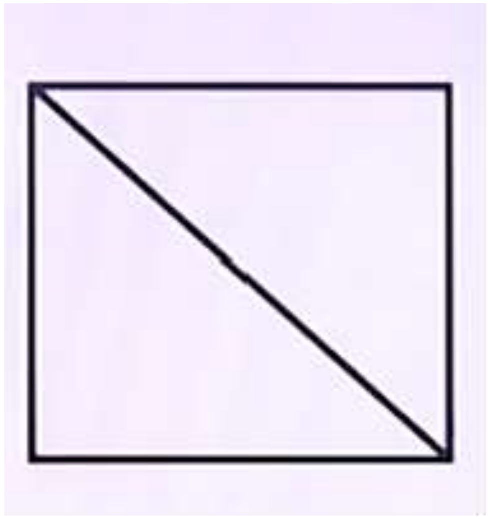
Module-I GRAPH THEORY

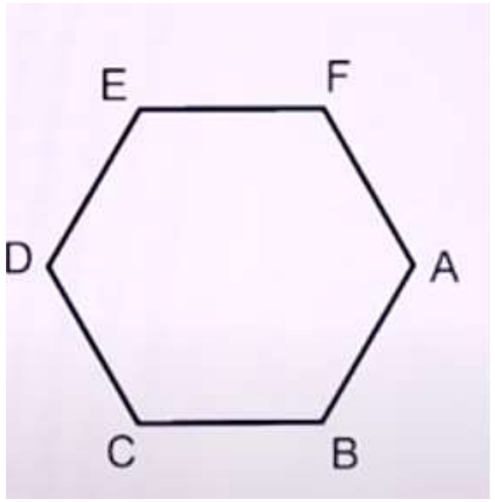
Check for bipartite graph

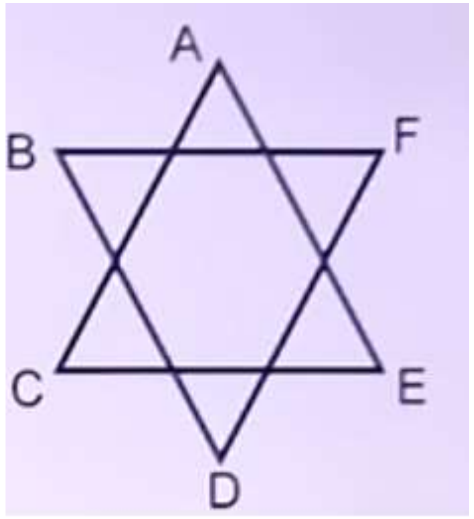
Module-I GRAPH THEORY

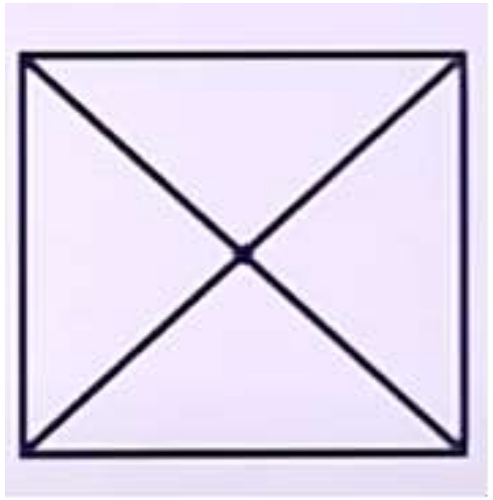
6. Draw $K_{3,4}$ bipartite graph

7. Which of the following ia regular graph

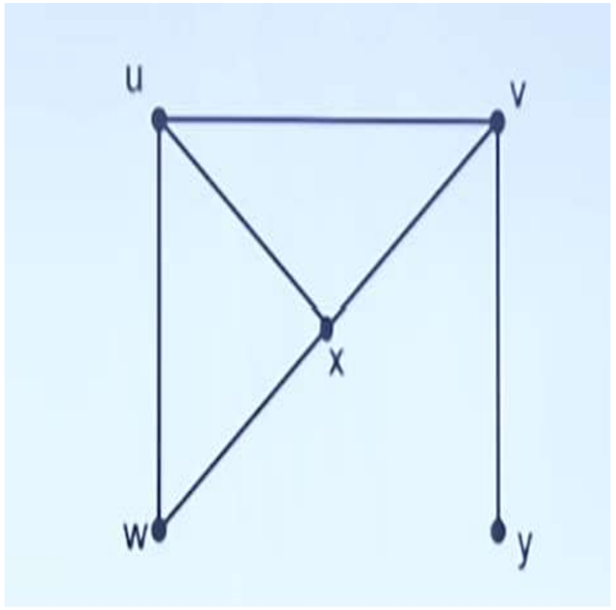


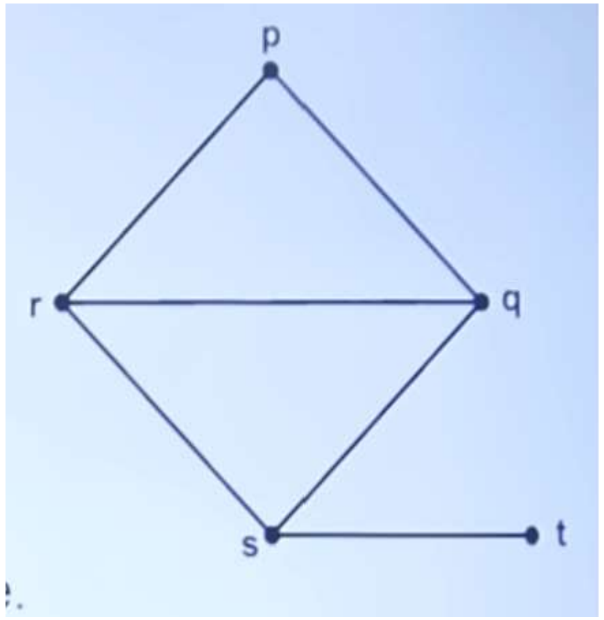




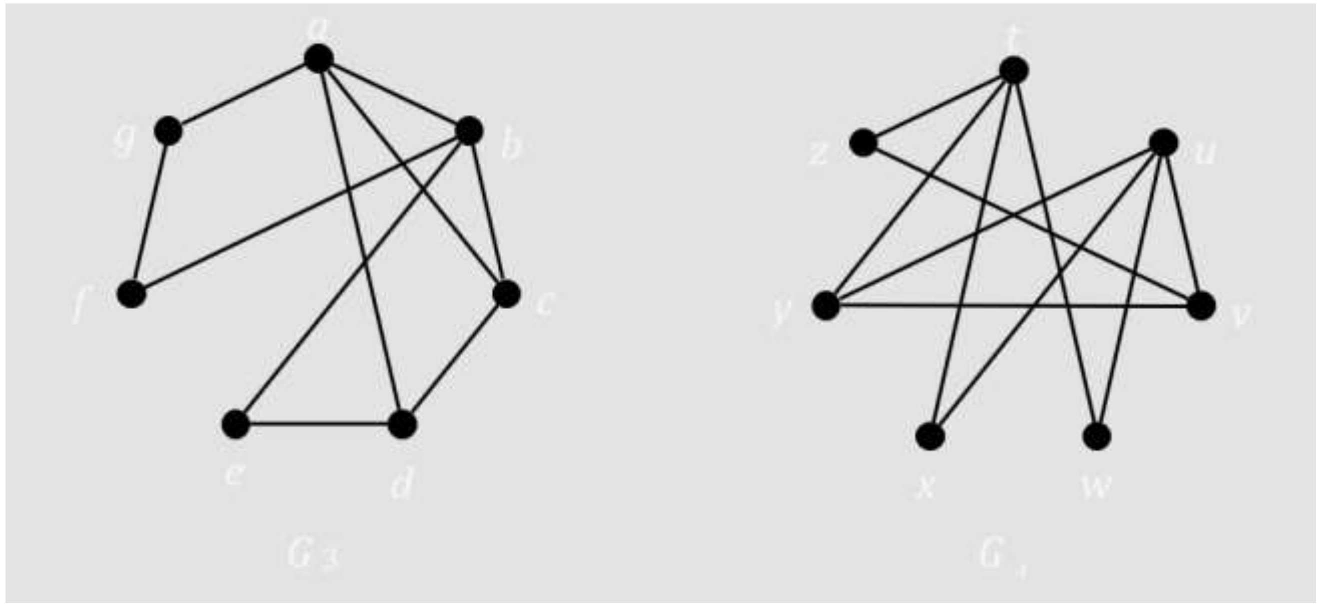


8. Check for isomorphisms

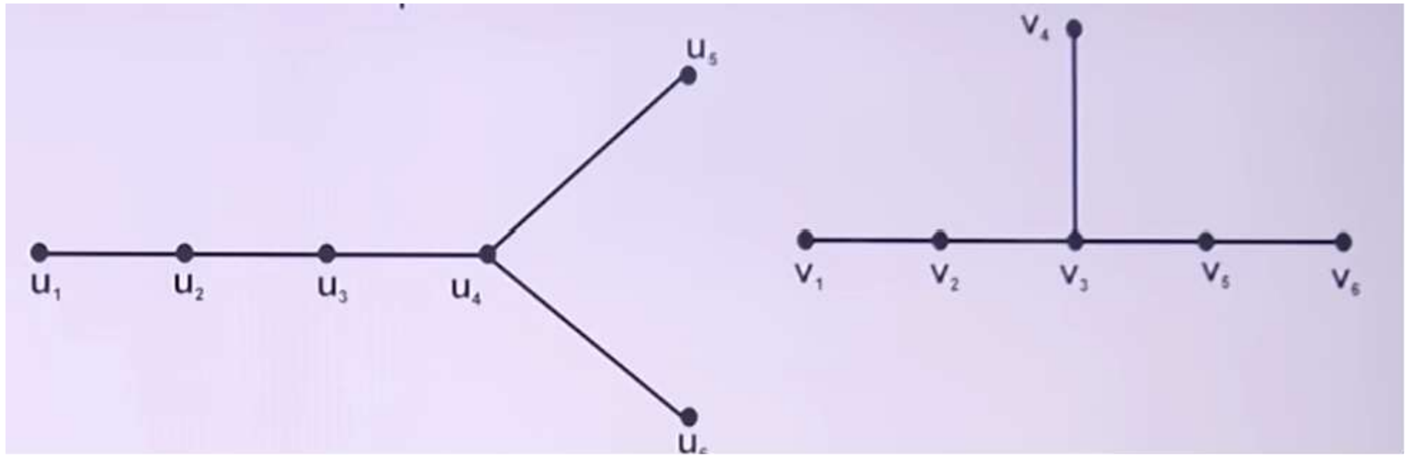


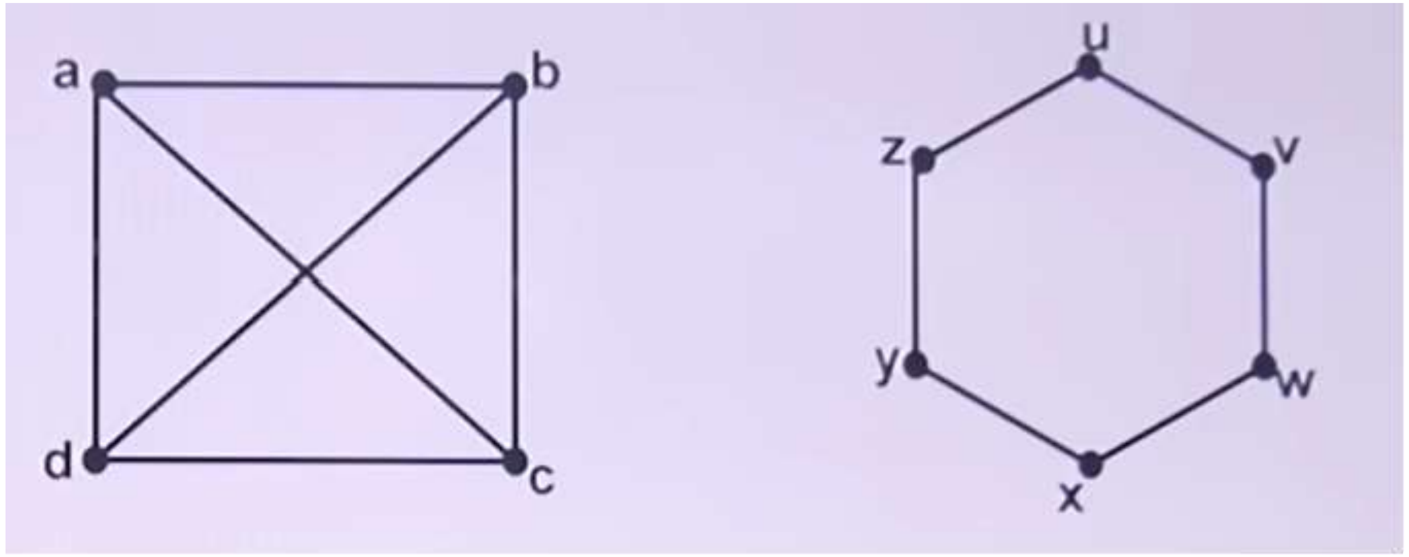


Module-I GRAPH THEORY



Module-I GRAPH THEORY



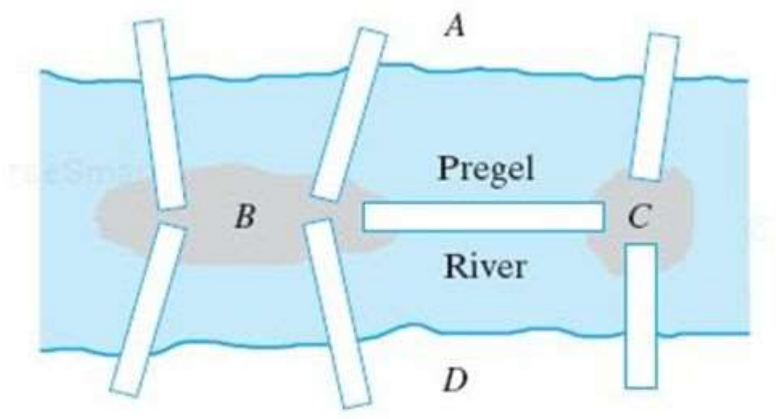


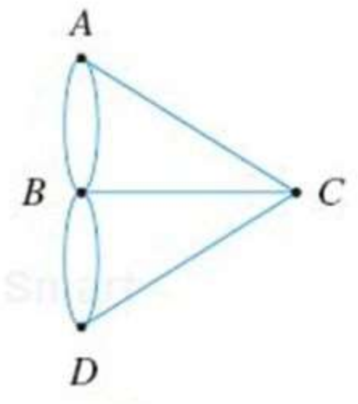
Module-I GRAPH THEORY

Module-I GRAPH THEORY

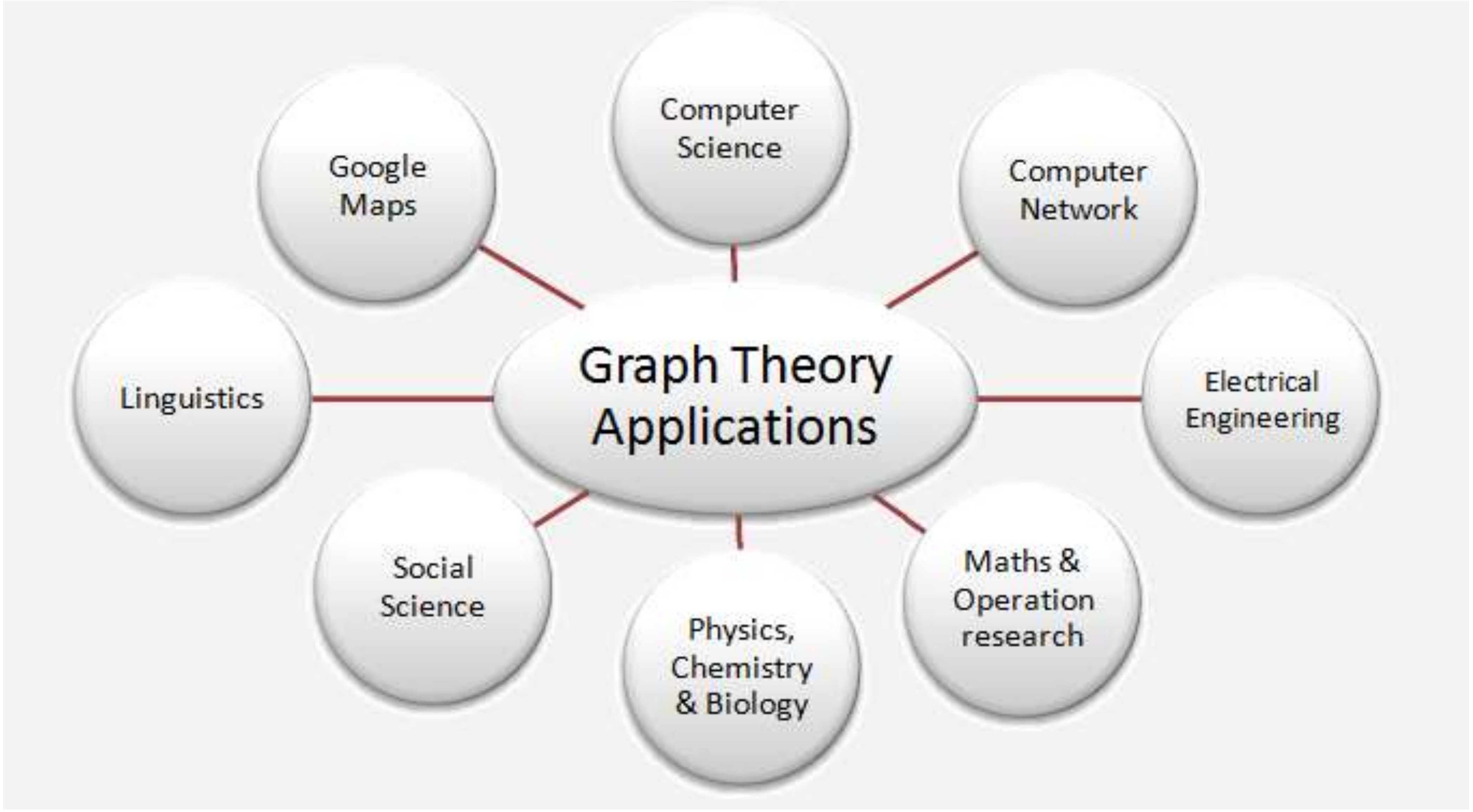
The Königsberg bridge problem

- Edges represent bridges and each vertex represents a region.





Module-I GRAPH THEORY



Algorithm for Modern Real-life Application

Google Maps

- For construction and transport systems.
- The intersection of two (or more) roads is considered a vertex, and the road connecting two vertices is considered an edge.
- Their navigation system then uses the algorithm to calculate the shortest path between two vertices.
- In GPS we also use different shortest path algorithms such as DFS (Depth first search) and BFS (Breath first search) algorithm.
- By the Dijkstra algorithm, one can find the shortest route between a given node (source node) and all other nodes (destination node) in a graph.
- This algorithm uses edge weights to find a way to reduce the total distance (weight) between the source node and all other nodes.

Facebook and LinkedIn

- Users as a graph in which each vertex is a user profile. The edge between two persons is the fact that they are friends among themselves or follow one another. Facebook is one example of an undirected graph.

World Wide Web

- On the World Wide Web, web pages are considered vertices. There is an edge between page ‘u’ and another page ‘v’ if there is a link from page ‘v’ to page ‘u’. That’s an example of a directed graph. That is the basic concept behind Google Page Rank Algorithm.

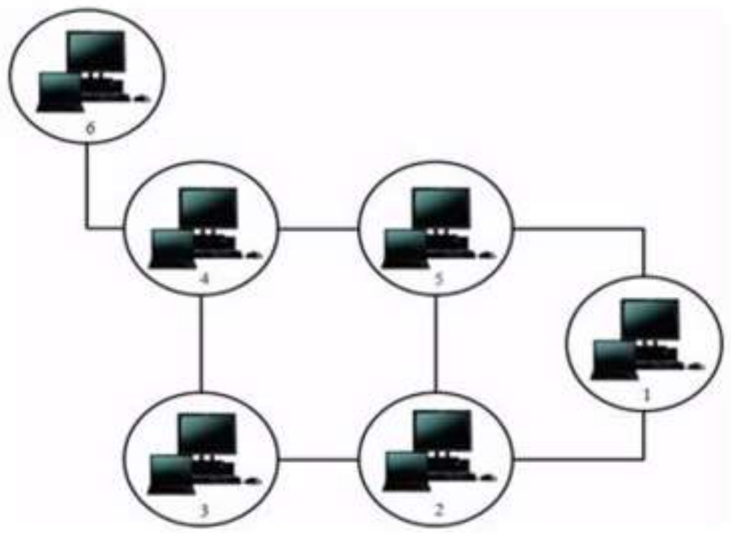
Social Network:

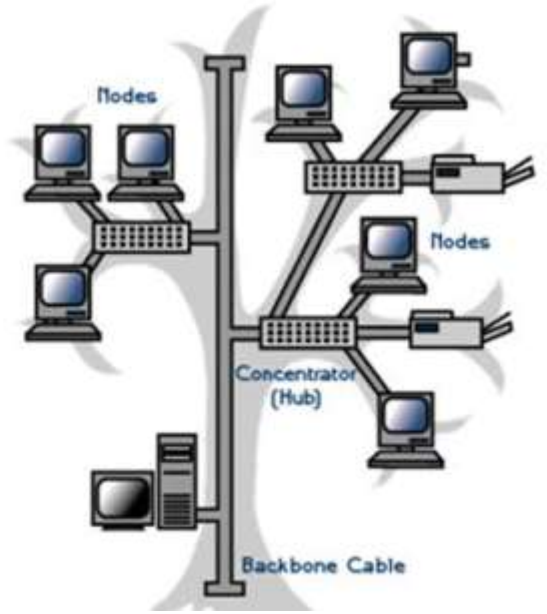
- Graphs to track user information. i.e.showing preferred post suggestions, recommendations, etc.

OTT

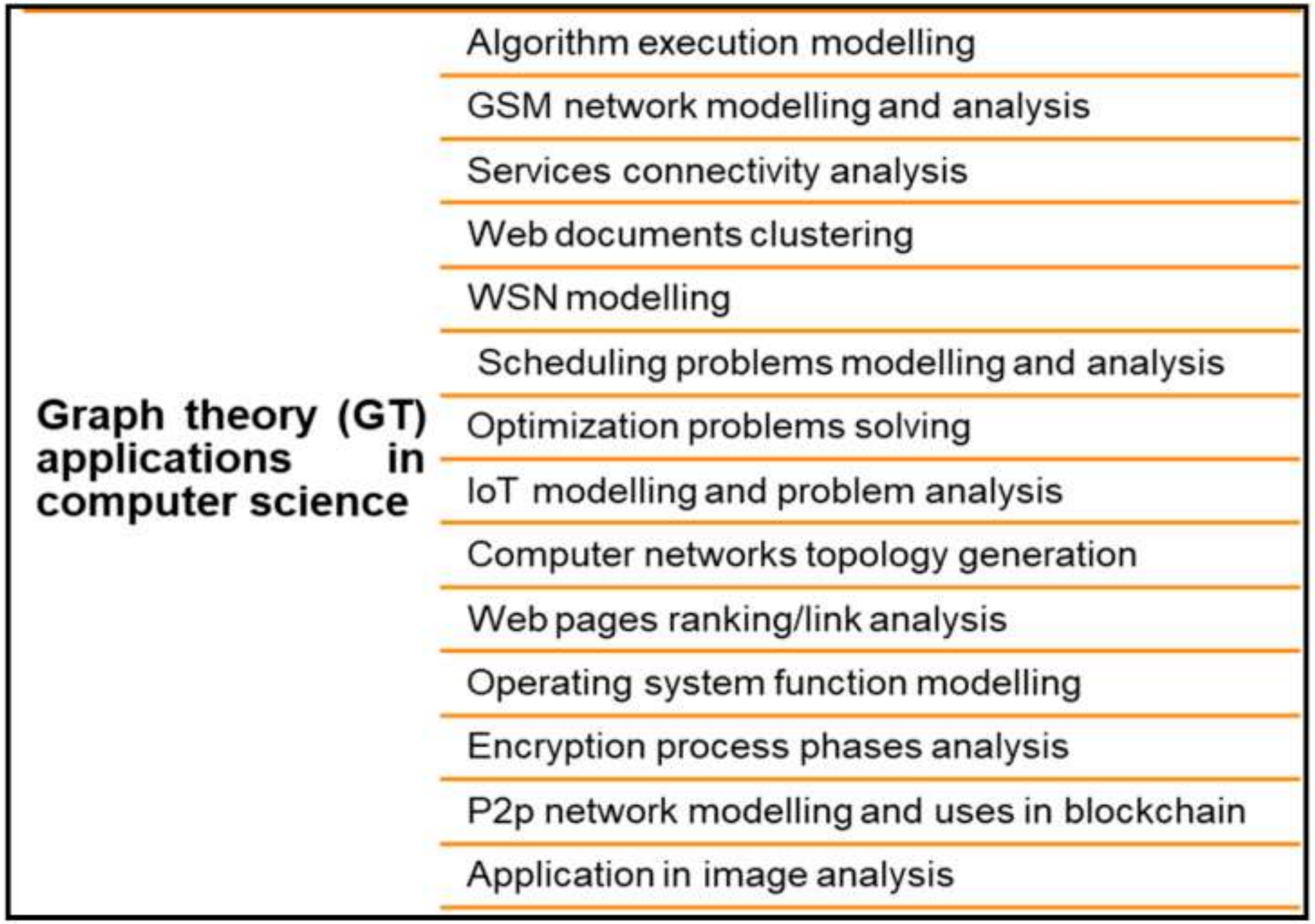
- Graph theory is used in Netflix and other OTT platforms to enhance recommendation systems.
- Users and content as nodes and their relationships as edges.
- It helps identify patterns and connections.
- It enables personalized recommendations by analyzing the user’s viewing history, ratings, and similar preferences of other users with similar viewing patterns.

Networks: Graph theory can be used in computer networks, for security purpose or to schematize network topologies, for example.





Module-I GRAPH THEORY



Graph theory (GT) applications in computer science	Algorithm execution modelling
	GSM network modelling and analysis
	Services connectivity analysis
	Web documents clustering
	WSN modelling
	Scheduling problems modelling and analysis
	Optimization problems solving
	IoT modelling and problem analysis
	Computer networks topology generation
	Web pages ranking/link analysis
	Operating system function modelling
	Encryption process phases analysis
	P2p network modelling and uses in blockchain
	Application in image analysis

Cliques can also be used in machine learning algorithms to find groups of similar data points.