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INSTITUTE OF AERONAUTICAL ENGINEERING

(Autonomous)

DUNDIGAL-500043, HYDERABAD

EDUCATION FOR LIBERATION

SEMESTER END EXAMINATIONS QUESTION PAPER – CHECK LIST ON OBE LEARNING DOMAINS

Name of the examiner: P. Shantan Kumar

Designation: Asst. Prof.

Academic Year: 2023-24

Month / Year of exam: Feb-2024

Programme: B.Tech / M.Tech / MBA

Regulation: BT23

Year & Semester: I - I (Regular)

Branch: Common to all

Course Name: MATRICES AND CALCULUS

Course Code: AHSB02

Q.No	Marks	Course Outcomes (Put Tick)						Bloom's Thinking Levels (Put Tick)						Program Outcomes (PO) (Put Tick √ on all correlating PO's)											
		1	2	3	4	5	6	1		2		3		1	2	3	4	5	6	7	8	9	10	11	12
		CO-1	CO-2	CO-3	CO-4	CO-5	CO-6	Remember	Understand	Apply	Analyze	Evaluate	Create	Engineering Knowledge	Problem Analysis	Design & Development	Analysis, Design, Research	Modern Tool Usage	Society & Culture	Environment & Sustainability	Ethics	Individuals &Team Work	Communication	Project &Team Work	Life Long Learning
1(a)	6	✓								✓															
1(b)	6	✓								✓															
2(a)	6		✓							✓															
2(b)	6		✓							✓															
3(a)	6			✓						✓															
3(b)	6			✓						✓															
4(a)	6				✓					✓															
4(b)	6				✓					✓															
5(a)	6					✓				✓															
5(b)	6					✓				✓															
6(a)	6					✓				✓															
6(b)	6					✓				✓															
7(a)	6						✓			✓															
7(b)	6						✓			✓															
8(a)	6						✓			✓															
8(b)	6						✓			✓															
9(a)	6																								
9(b)	6																								
10(a)	6																								
10(b)	6																								
Mark Distribution across Bloom's Level:									%	100 %		%													

13/02/24.

CHIEF EXAMINER **HOD** **Dean LET**

Hall Ticket No | | | | | | | | | | | Question Paper Code: AHSD02

INSTITUTE OF AERONAUTICAL ENGINEERING

(Autonomous)

Dundigal-500043, Hyderabad

B.Tech I SEMESTER END EXAMINATIONS (REGULAR) - FEBRUARY 2024

Regulation: BT23

MATRICES AND CALCULUS

(COMMON TO ALL BRANCHES)

Time: 3 Hours   Max Marks: 60

Answer ALL questions in Module I and II

Answer ONE out of two questions in Modules III, IV and V

All Questions Carry Equal Marks

All parts of the question must be answered in one place only

MODULE – I

1. (a) Reduce the matrix in echelon form and find its rank $\begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 0 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$

[BL: Apply| CO: 1|Marks: 6]

(b) Investigate for what value of λ and μ the equations x + y + z =6, x + 2y + 3z =10, $x+2y+\lambda z = \mu$ have

i) No solution

ii) Unique solution

iii) Infinite solution

[BL: Apply| CO: 1|Marks: 6]

MODULE – II

2. (a) Find the eigen values and eigen vectors of the following matrix $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

[BL: Apply| CO: 2|Marks: 6]

(b) Diagonalize the matrix $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ [BL: Apply| CO: 2|Marks: 6]

MODULE – III

3. (a) Discuss the maxima and minima of $f(x,y) = x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10$

[BL: Apply| CO: 3|Marks: 6]

(b) Determine the value of c using Lagrange's mean value theorem for $f(x) = x(x-1)(x-2)$ in $(0, \frac{1}{2})$ [BL: Apply| CO: 3|Marks: 6]

4. (a) If x+y+z=u, y+z=uv, z=uvw then show that $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$ [BL: Apply| CO: 4|Marks: 6]

(b) Examine the functional dependence or independence of $u = \dfrac{x-y}{x+y}$ and $v = \dfrac{x+y}{x}$. If dependent, find the relation between them.

[BL: Apply| CO: 4|Marks: 6]

MODULE – IV

5. (a) Obtain the Fourier series expansion for the function $f(x) = x(2\pi - x)$ in $0 \leq x \leq 2\pi$

[BL: Apply| CO: 5|Marks: 6]

(b) Find the half range sine series for $f(x) = x^2$ in $(0, 2\pi)$ [BL: Apply| CO: 5|Marks: 6].

6. (a) Find the Fourier series expansion for $f(x) = \pi - x$ in $[0, 2\pi]$ with period 2π. Hence find the sum of the series $1 - \dfrac{1}{3} + \dfrac{1}{5} \ldots.$. [BL: Apply| CO: 5|Marks: 6]

(b) Obtain the Fourier series of $f(x) = x^3$ in $[-\pi, \pi]$ [BL: Apply| CO: 5|Marks: 6]

MODULE – V

7. (a) Evaluate $\int\int y\,dx\,dy$ over the part of the curves bounded by the line $y = x$ and the parabola $y = 4x - x^2$ [BL: Apply| CO: 6|Marks: 6]

(b) Compute the value of integral $\int_0^{\pi/2} \int_0^{sin\theta} r\,dr\,d\theta$ [BL: Apply| CO: 6|Marks: 6]

8. (a) Change the order of integration and evaluate $\int_0^a \int_{y^2/a}^a \dfrac{y\,dx\,dy}{(a-x)\sqrt{ax - y^2}}$

[BL: Apply| CO: 6|Marks: 6]

(b) Evaluate $\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z\,dz\,dy\,dx$ [BL: Apply| CO: 6|Marks: 6]

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Scheme of Evaluation $\longrightarrow$ AHSD02

B-Tech-I-Semester End Exam (Regular) - Feb - 2024.

Regulation - BT23

MATRICES AND CALCULUS

(Common to all Branches)

MODULE-I

(I) a) Reduce the matrix in Echelon form and find the rank $\begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 0 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$

$$\Rightarrow \quad A = \begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 0 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$$

$$\xrightarrow{R_4-2R_1} \begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 0 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & -2 & -6 & 5 \end{bmatrix}$$

$$\xrightarrow[R_4-R_3]{R_2 \leftrightarrow R_3} \begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & -2 & 1 & -4 & 2 \\ 0 & 0 & -1 & -9 & 4 \end{bmatrix} \quad \text{— (2M)}$$

$$\xrightarrow{R_3+2R_2} \begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 3 & 2 & 4 \\ 0 & 0 & -1 & -9 & 4 \end{bmatrix}$$

$$\xrightarrow{3R_4+R_3} \begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 3 & 2 & 4 \\ 0 & 0 & 0 & -25 & 16 \end{bmatrix} \quad \text{— (2M)}$$

$\therefore$ which is in Echelon form.

$\therefore$ Number of non-zero rows $= 4$

$\therefore \rho(A) = 4$. — (2M)

b) $x+y+z=6,\ x+2y+3z=10,\ x+2y+\lambda z=\mu$

i) No solⁿ. ii) Unique solⁿ. iii) Infinite solⁿ.

$$[A\ \ B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}, \quad n=3\ (x,y,z)$$ (1M)

$$\xrightarrow{R_2-R_1,\ R_3-R_1} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix}$$

$$\xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$ (2M)

i) For no solution:- $\rho(A) \neq \rho(A\ \ B)$

iff $\lambda-3=0,\ \mu-10\neq 0$.

$\Rightarrow \lambda=3,\ \mu\neq 10$. (1M)

ii) For Unique solution: $\rho(A)=\rho(A\ \ B)=n$

$\Rightarrow \rho(A)=\rho(A\ \ B)=3 \quad \because n=3$.

iff $\lambda-3\neq 0,\ \forall \mu$

$\Rightarrow \lambda\neq 3,\ \forall \mu$. (1M)

iii) For Infinite solution: $\rho(A)=\rho(A\ \ B)<n$

$\Rightarrow \rho(A)=\rho(A\ \ B)<3 \quad \because n=3$

iff $\lambda-3=0,\ \mu-10=0$.

$\Rightarrow \lambda=3,\ \mu=10$. (1M)

MODULE - II.

2 a) Find the Eigen values and Eigen vectors. $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

$\Rightarrow$ The characteristic eqn of $A \Rightarrow |A - \lambda I| = 0$

$$|A-\lambda I| = \begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$ — (1M)

$$\Rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$
$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$
$$\Rightarrow \lambda = 1, 2, 3.$$

(OR) — (2M)

$$|A-\lambda I| = \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$
$$S_1 = 6, \quad S_2 = 11, \quad S_3 = 6.$$

$$\Rightarrow \lambda = 1, 2, 3.$$

Case-I:- If $\lambda = 1 \Rightarrow [A - \lambda I] X = 0$

$$\Rightarrow \begin{bmatrix} 7 & -8 & -2 \\ 4 & -4 & -2 \\ 3 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left.\begin{aligned} 7x_1 - 8x_2 + 2x_3 &= 0 \\ 4x_1 - 4x_2 - 2x_3 &= 0 \\ 3x_1 - 4x_2 + 0.x_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{array}{cccc} x_1 & x_2 & x_3 & x_1 \\ 7 & -8 & -2 & 7 \\ 4 & -4 & -2 & 4 \end{array}$$

$$\frac{x_1}{16-8} = \frac{x_2}{-8+14} = \frac{x_3}{-28+32} = K \text{ (let)}$$

$$\Rightarrow \frac{x_1}{8} = \frac{x_2}{6} = \frac{x_3}{4} = K$$

$$\Rightarrow X_1 = \begin{bmatrix} 8K \\ 6K \\ 4K \end{bmatrix} = \begin{bmatrix} 4K \\ 3K \\ 2K \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \text{ or } \begin{bmatrix} 2 \\ 1.5 \\ 1 \end{bmatrix}$$ — (1M)

Case-II: If $\lambda = 2 \Rightarrow [A-\lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 6 & -8 & -2 \\ 4 & -5 & -2 \\ 3 & -4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left.\begin{aligned} 6x_1 - 8x_2 - 2x_3 &= 0 \\ 4x_1 - 5x_2 - 2x_3 &= 0 \\ 3x_1 - 4x_2 - x_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{matrix} x_1 & x_2 & x_3 & x_1 \\ 6 & -8 & -2 & 6 \\ 4 & -5 & -2 & 4 \end{matrix}$$

$$\frac{x_1}{16-10} = \frac{x_2}{-8+12} = \frac{x_3}{-30+32} = k \text{ (let)}$$

$$\Rightarrow \frac{x_1}{6} = \frac{x_2}{4} = \frac{x_3}{2} = k$$

$$\Rightarrow X_2 = \begin{bmatrix} 6k \\ 4k \\ 2k \end{bmatrix} = \begin{bmatrix} 3k \\ 2k \\ k \end{bmatrix} \text{ or } \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$ ——— (1M)

Case-III: If $\lambda = 3 \Rightarrow [A-\lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 5 & -8 & -2 \\ 4 & -6 & -2 \\ 3 & -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left.\begin{aligned} 5x_1 - 8x_2 - 2x_3 &= 0 \\ 4x_1 - 6x_2 - 2x_3 &= 0 \\ 3x_1 - 4x_2 - 2x_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{matrix} x_1 & x_2 & x_3 & x_1 \\ 5 & -8 & -2 & 5 \\ 4 & -6 & -2 & 4 \end{matrix}$$

$$\frac{x_1}{16-12} = \frac{x_2}{-8+10} = \frac{x_3}{-30+32} = k$$

$$\Rightarrow \frac{x_1}{4} = \frac{x_2}{2} = \frac{x_3}{2} = k$$

$$\Rightarrow X_3 = \begin{bmatrix} 4k \\ 2k \\ 2k \end{bmatrix} = \begin{bmatrix} 2k \\ k \\ k \end{bmatrix} \text{ or } \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$ ——— (1M)

Note:- Calculate any other method.

b):- Diagonalize the matrix $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

$\Rightarrow$ The characteristic eqn of $A \Rightarrow |A-\lambda I| = 0$

$$\Rightarrow |A-\lambda I| = \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$\Rightarrow \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0 \Rightarrow (\lambda-2)(\lambda-3)(\lambda-6) = 0$$

$$\Rightarrow \lambda = 2, 3, 6.$$

— (1M)

Case→I:- If $\lambda = 2 \Rightarrow [A-\lambda I]X = 0 \Rightarrow [A-2I]X = 0$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} = 0$$

By solving, we get $\begin{bmatrix} K \\ 0 \\ -K \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ — (1M)

Case→II:- If $\lambda = 3 \Rightarrow [A-\lambda I]X = 0 \Rightarrow [A-3I]X = 0$

$$\Rightarrow \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

By solving, we get $\begin{bmatrix} K \\ K \\ K \end{bmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ — (1M)

Case→III:- If $\lambda = 6 \Rightarrow [A-\lambda I]X = 0 \Rightarrow [A-6I]X = 0$

$$\Rightarrow \begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

By solving, we get $\begin{bmatrix} K \\ -2K \\ K \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ — (1M)

$\therefore$ Modal matrix $P = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$

$\therefore$ Diagonalize :- $P^{-1}AP = D$.

where, $P^{-1} = \left(\frac{-1}{6}\right)\begin{bmatrix} 3 & 0 & -3 \\ -2 & -2 & -2 \\ -1 & 2 & -1 \end{bmatrix}$

$\therefore P^{-1}AP = \left(-\frac{1}{6}\right)\begin{bmatrix} 3 & 0 & -3 \\ -2 & -2 & -2 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}$

$= \left(-\frac{1}{6}\right)\begin{bmatrix} -12 & 0 & 0 \\ 0 & -18 & 0 \\ 0 & 0 & -36 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ (2M)

$= D$.

$\therefore$ Diagonalise the matrix.

[signature]

MODULE-III

(3) a):- Discuss the maxima & minima of $f(x,y) = x^3 + xy^2 - 12x^2 - 2y^2 + 21x + 10$

$\Rightarrow$ For stationary points,

$$p = \frac{\partial f}{\partial x} = 0 \quad , q = \frac{\partial f}{\partial y} = 0$$

$$\therefore p = \frac{\partial f}{\partial x} = 3x^2 + y^2 - 24x + 21 = 0 \quad \text{——— (i)}$$

$$q = \frac{\partial f}{\partial y} = 2xy - 4y = 0 \Rightarrow 2y(x-2) = 0 \quad \text{——— (ii)}$$

— (2M)

from (ii), $y = 0$, $x = 2$

If $y = 0 \overset{(i)}{\Rightarrow} 3x^2 - 24x + 21 = 0 \Rightarrow x^2 - 8x + 7 = 0$

$\Rightarrow x = 1, 7.$

$\therefore$ stationary points $\Rightarrow (1,0), (7,0).$

If $x = 2 \overset{(i)}{\Rightarrow} 12 + y^2 - 48 + 21 = 0 \Rightarrow y^2 - 15 = 0$

$\Rightarrow y = \sqrt{15}, -\sqrt{15}.$

$\therefore$ stationary points $\Rightarrow (2, \sqrt{15}), (2, -\sqrt{15})$

$\therefore$ Total stationary points, $(1,0), (7,0), (2,\sqrt{15}), (2,-\sqrt{15})$

— (2M)

$\therefore$ For maxima & minima,

$$l = \frac{\partial^2 f}{\partial x^2} = 6x - 24$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 2y$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2x - 4$$

Here $(2,\sqrt{15}), (2,-\sqrt{15})$ there is no extreme values.

at $(1,0) \Rightarrow$ maximum $\Rightarrow f(1,0) = 1 + 0 - \cancel{12} 0 + 21 + 10 = 30$ maximum

at $(7,0) \Rightarrow$ minimum $\Rightarrow f(7,0) = 7^3 - 12(7^2) + 21(7) = -98$ minimum.

— (2M)

b):- Find c, By using Lagrange's mean value theorem.

$$f(x) = x(x-1)(x-2) \text{ in } (0, \tfrac{1}{2})$$

$\Rightarrow$ $f(x) = x(x-1)(x-2)$.

i) Given function is continuous in $[0, 1/2]$

ii) Given function is derivable in $(0, 1/2)$

then, $\exists\ c \in (a,b)$. such that

$$f'(c) = \frac{f(b) - f(a)}{b-a} \quad \text{———} (I)$$

— (2M)

$\therefore f(x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x$ ~~$\Rightarrow f'(c) = 3$~~

$f'(x) = 3x^2 - 6x + 2 \Rightarrow f'(c) = 3c^2 - 6c + 2$

$f(a) = f(0) = 0$ $\because$ from $f(x)$

$f(b) = f(1/2) = \frac{3}{8}$ $\because$ from $f(x)$

— (2M)

$\therefore$ from (I) $\Rightarrow$ ~~$x^3 - 3x^2 + 2x$~~ =

$$3c^2 - 6c + 2 = \frac{\frac{3}{8} - 0}{\frac{1}{2} - 0}$$

$\Rightarrow 3c^2 - 6c + 2 = \frac{3}{4}$

$\Rightarrow 12c^2 - 24c + 5 = 0$

$\Rightarrow c = \frac{24 \pm \sqrt{336}}{24} = \frac{6 \pm \sqrt{21}}{6}$

$\Rightarrow c = 1 - \frac{\sqrt{21}}{6} \in (0, 1/2)$.

— (2M)

$\therefore$ verified Lagrange's mean value theorem.

4 a) If $x+y+z=u$, $y+z=uv$, $z=uvw$, show that $\frac{\partial(x,y,z)}{\partial(u,v,w)}=u^2v$

$\Rightarrow$ $x+y+z=u$ — (i)

$y+z=uv$ — (ii)

$z=uvw$. — (iii)

$\therefore z=uvw$

from eqn. (ii) $\Rightarrow y=uv-z=uv-uvw$ — (iv)

from eqn. (i) $\Rightarrow x=u-(y+z)=u-uv$ — (v)

— (2M)

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$ — (2M)

$$=\begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}=u^2v.$$ — (2M)

(OR)

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=(-1)^3\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}=(-1)\cdot\frac{\begin{vmatrix} 1 & 0 & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix}}{\begin{vmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix}}=u^2v.$$

$\because f_1=u-x-y-z$

$f_2=uv-y-z$

$f_3=uvw-z$

b):- Examine $u = \frac{x-y}{x+y}$, $v = \frac{x+y}{x}$ is F.D. / F.I.

Find relation?

$\Rightarrow$ Given, $u = \frac{x+y}{x-y} \Rightarrow \frac{\partial u}{\partial x} = \frac{(x+y)(1)-(x-y)(1)}{(x+y)^2} = \frac{2y}{(x+y)^2}$

$$\frac{\partial u}{\partial y} = \frac{(x+y)(1)-(x-y)(-1)}{(x+y)^2} = \frac{2x}{(x+y)^2}$$ ← (2M)

$$v = \frac{x+y}{x} \Rightarrow \frac{\partial v}{\partial x} = -\frac{y}{x^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{x}$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{2y}{(x+y)^2} & \frac{2x}{(x+y)^2} \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix}$$ ← (3M)

$$\Rightarrow \frac{2y}{x(x+y)^2} - \frac{2xy}{x^2(x+y)^2} = 0$$

$\therefore u, v$ are functionally dependent.

Now, To find relation :-

$u = \frac{x-y}{x+y}$, $v = \frac{x+y}{x} = 1 + \frac{y}{x}$

$\Rightarrow \frac{y}{x} = v - 1$ ← (1M)

$$\Rightarrow u = \frac{1 - y/x}{1 + y/x}$$

$$= \frac{1-(v-1)}{1+v-1} = \frac{2-v}{v}$$

$\Rightarrow uv = 2 - v \Rightarrow uv + v = 2$

$\Rightarrow v(1+u) = 2.$

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MODULE - IV

(5) a) :- $f(x) = x(2\pi - x)$, $(0, 2\pi)$.

Required Fourier series, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]$ — (I)

where, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}(2\pi x - x^2)\,dx = \frac{4\pi^2}{3}$. — (1M)

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}(2\pi x - x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\underbrace{(2\pi x - x^2)\left(\frac{\sin nx}{n}\right)}_{0} - (2\pi - 2x)\left(-\frac{\cos nx}{n^2}\right) + \underbrace{(0-2)\left(-\frac{\sin nx}{n^3}\right)}_{0}\right]_0^{2\pi}$$

$$= \frac{2}{\pi}\left[(\pi - x)\frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi}{n^2} - \frac{\pi}{n^2}\right] = -\frac{4}{n^2}.$$

— (2M)

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}(2\pi x - x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(2\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - \underbrace{(2\pi - 2x)\left(-\frac{\sin nx}{n^2}\right)}_{0} + (0-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= 0.$$

— (2M)

$\therefore$ from eqn (I),

$$x(2\pi - x) = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$$ — (1M)

b) :- Find half range sine series for $f(x) = x^2$, $(0, 2\pi)$

$\Rightarrow$ For half range sine series, Required limits are $(0, \pi)$

Required series,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \qquad \text{I} \qquad \because a_0 = a_n = 0$$ (1M)

$$= b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\,dx$$ (1M)

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - \underbrace{2x\left(-\frac{\sin nx}{n^2}\right)}_{0} + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi^2\frac{\cos n\pi}{n} + \frac{2\cos n\pi}{n^3} + 0 - \frac{2\cos 0}{n^3}\right]$$

$$= \frac{2}{\pi}\left[-\frac{(-1)^n\pi^2}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right]$$ (2M)

$\therefore$ from eq^n. (I),

$$x^2 = \sum_{n=1}^{\infty} \frac{2}{\pi}\left(-\frac{(-1)^n\pi^2}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right)\sin nx.$$ (2M)

(OR) [Note].

If ~~any~~ student will attempt ~~this~~ problem, will get 6 marks

6 a):- Find the F.S. $f(x) = \pi - x$, $[0, 2\pi]$, find the sum of $1 - \frac{1}{3} + \frac{1}{5} - \cdots$

$\Rightarrow$ $f(x) = \pi - x$, $[0, 2\pi]$.

Required Fourier series, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$ — (I) (1M)

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)\,dx$$

$$= \frac{1}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi} = \frac{1}{\pi}\left(2\pi^2 - \frac{4\pi^2}{2}\right) = 0. \quad \text{(1M)}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\underbrace{(\pi - x)\left(\frac{\sin nx}{n}\right)}_{0} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi} = -\frac{1}{\pi}\left(\frac{\cos nx}{n^2}\right)_0^{2\pi} = 0. \quad \text{(1M)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} (\pi - x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - \underbrace{(-1)\left(-\frac{\sin nx}{n^2}\right)}_{0}\right]_0^{2\pi} = -\frac{1}{\pi}\left[(\pi - x)\frac{\cos nx}{n}\right]_0^{2\pi} = \frac{2}{n}. \quad \text{(1M)}$$

$\therefore$ from eqn (I),

$$\pi - x = \sum_{n=1}^{\infty} \frac{2}{n}\sin nx. \quad \text{(1M)}$$

put $x = \frac{\pi}{2} \Rightarrow \frac{\pi}{2} = 2\sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n}$

$$\Rightarrow \frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{n}$$

$$\Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}. \quad \text{(1M)}$$

b):- Obtain the Fourier series $f(x) = x^3$, $[-\pi, \pi]$.

$\Rightarrow$ $f(x) = x^3$, $[-\pi, \pi]$.

Given function in $(-\pi, \pi)$ is odd function.

$\therefore$ To find b_n. $(\because a_0 = a_n = 0)$

$\therefore$ Required series, $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$. —— I —— (1M)

(2M)

$$\therefore b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx \, dx$$

$$= \frac{2}{\pi}\left[x^3\left(-\frac{\cos nx}{n}\right) - \underbrace{3x^2\left(-\frac{\sin nx}{n^2}\right)}_{0} + 6x\left(\frac{\cos nx}{n^3}\right) - \underbrace{6\left(\frac{\sin nx}{n^4}\right)}_{0}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{x^3 \cos nx}{n} + 6x\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{-\pi^3(-1)^n}{n} + \frac{6\pi(-1)^n}{n^3}\right] \qquad \because f(0) = 0.$$

$$= \frac{(-1)^n 2}{n}\left[-\pi^2 + \frac{6}{n^2}\right]$$

$$= 2(-1)^n\left[\frac{-\pi^2}{n} + \frac{6}{n^3}\right]$$

(2M)

$\therefore$ from eqn (I), $x^3 = \sum_{n=1}^{\infty} 2(-1)^n\left[\frac{-\pi^2}{n} + \frac{6}{n^3}\right]\sin nx$.

$$\Rightarrow x^3 = 2\sum_{n=1}^{\infty}(-1)^n\left[\frac{-\pi^2}{n} + \frac{6}{n^3}\right]\sin nx. \quad \text{(1M)}$$

MODULE - V.

15

7 a):- Evaluate $\iint y\,dx\,dy$, $y = x$, $y = 4x - x^2$

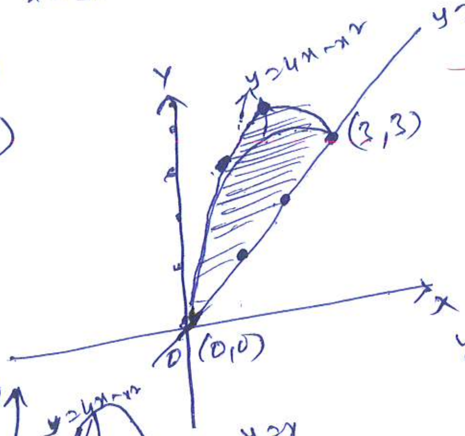
$\Rightarrow$ Given curves are, $y = x$, $y = 4x - x^2$

$\Rightarrow x = 4x - x^2$

$\Rightarrow x^2 - 3x = 0 \Rightarrow x = 0, 3$ — (2M)

If $x = 0 \Rightarrow y = 0 \Rightarrow (0,0)$

$x = 3 \Rightarrow y = 3 \Rightarrow (3,3)$

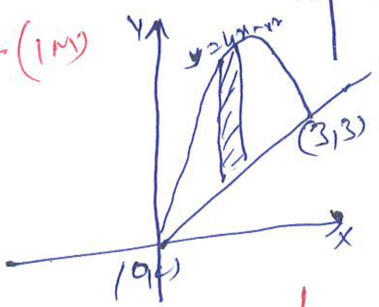


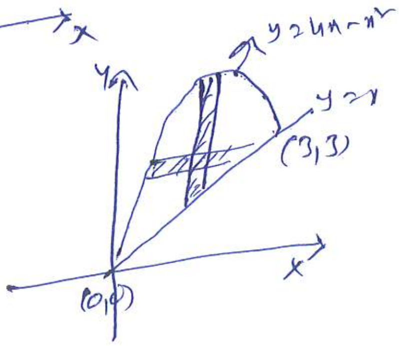
$y = 4x - x^2$

x	0	1	2
y	0	3	4

$\Rightarrow \iint y\,dx\,dy$

$$= \int_{x=0}^{3} \int_{y=x}^{4x-x^2} y\,dy\,dx \quad \text{— (1M)}$$





Graph — (1M)

$$= \int_{x=0}^{3} \left(\frac{y^2}{2}\right)_x^{4x-x^2} dx$$

$$= \frac{1}{2}\int_{x=0}^{3} \left(y^2\right)_x^{4x-x^2} dx$$

$$= \frac{1}{2}\int_{x=0}^{3} \left[(4x-x^2)^2 - x^2\right] dx$$

$$= \frac{1}{2}\left[\int_{x=0}^{3} (16x^2 - x^4 - 8x^3 - x^2)\,dx\right]$$

$$= \frac{1}{2}\int_{x=0}^{3} (-8x^3 + 15x^2 - x^4)\,dx$$

$$= \frac{1}{2}\left[-\frac{8x^4}{4} + 15\cdot\frac{x^3}{3} - \frac{x^5}{5}\right]_0^3$$

$$= \frac{1}{2}\left[-162 + 135 - \frac{243}{5}\right] = \frac{-678}{2} = -336$$

— (2M)

b):- Compute the value of $\int_0^{\pi/2}\int_0^{\sin\theta} r\,dr\,d\theta$

$\Rightarrow \int_{\theta=0}^{\pi/2}\int_{r=0}^{\sin\theta} r\,dr\,d\theta = \int_{\theta=0}^{\pi/2}\left(\frac{r^2}{2}\right)_0^{\sin\theta} d\theta$ (2M)

$= \frac{1}{2}\int_{\theta=0}^{\pi/2}\sin^2\theta\,d\theta$ (1M)

$= \frac{1}{2}\int_0^{\pi/2}\frac{1-\cos 2\theta}{2}\,d\theta$

$= \frac{1}{4}\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta$

$= \frac{1}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$ (3M)

$= \frac{1}{4}\left[\frac{\pi}{2} - 0 - 0 - 0\right]$

$= \frac{\pi}{8}$.

(OR)

(OR)

$= \frac{1}{2}\int_{\theta=0}^{\pi/2}\sin^2\theta\,d\theta$

$= \frac{1}{2}\cdot\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{2+0+2}{2}\right)}$

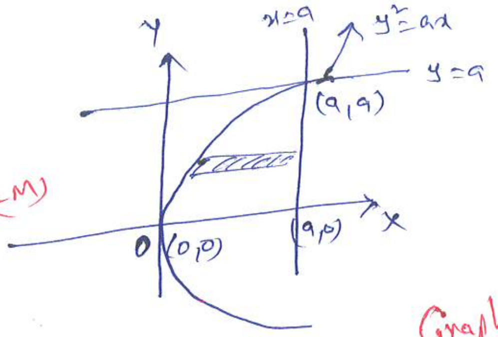
$= \frac{1}{2}\cdot\frac{\Gamma(3/2)\,\Gamma(1/2)}{2\Gamma(2)}$

$= \frac{1}{2}\cdot\frac{\frac{1}{2}\sqrt{\pi}\cdot\sqrt{\pi}}{2(1)}$

$= \frac{\pi}{8}$.

(8) a):- Change the order of integration & evaluate $\int_0^a \int_{y^2/a}^a \frac{y\,dx\,dy}{(a-x)\sqrt{ax-y^2}}$

$$\Rightarrow \int_0^a \int_{y^2/a}^a \frac{y}{(a-x)\sqrt{ax-y^2}}\,dx\,dy$$



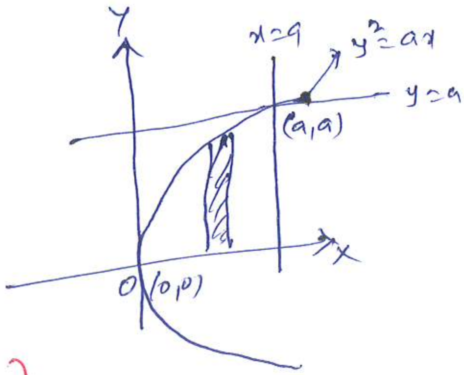
i.e. $x = \frac{y^2}{a} \Rightarrow y^2 = ax$

$x = a \Rightarrow x = a.$

By solving, we get $(0,0), (a,a)$ — (2M)

Graph(1M)

By applying, changing the order of integration



$$\Rightarrow \int_{x=0}^a \int_{y=0}^{\sqrt{ax}} \frac{y}{(a-x)\sqrt{ax-y^2}}\,dy\,dx. \quad \text{— (1M)}$$

$$\Rightarrow -\frac{1}{2}\int_{x=0}^a \int_{y=0}^{\sqrt{ax}} \frac{-2y}{(a-x)\sqrt{ax-y^2}}\,dy\,dx$$

$$\Rightarrow -\int_{x=0}^a \frac{1}{a-x}\left[\int_{y=0}^{\sqrt{ax}} d.\sqrt{ax-y^2}\,.dy\right]dx.$$

$$\Rightarrow -\int_{x=0}^a \frac{1}{a-x}\left(\sqrt{ax-y^2}\right)_0^{\sqrt{ax}}\,dx$$

$$\Rightarrow -\int_{x=0}^a \frac{1}{a-x}.\left[\sqrt{ax-ax} - \sqrt{ax-0}\right]dx$$

$$\Rightarrow -\int_{x=0}^a \frac{1}{a-x}.(-\sqrt{ax})\,dx$$

$$\Rightarrow \sqrt{a}.\int_{x=0}^a x^{1/2}(a-x)^{-1}\,dx$$

— (2M)

& Solve.

b):- Evaluate $\int_0^1 \int_0^{1-x} \int_0^{x+y} e^x \cdot dz\,dy\,dx$

$\Rightarrow \int_0^1 \int_0^{1-x} \int_0^{x+y} e^x \,dz\,dy\,dx = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{x+y} e^x \,dz\,dy\,dx$ — (1M)

$= \int_{x=0}^{1} \int_{y=0}^{1-x} e^x \cdot (z)_0^{x+y} \,dy\,dx$

$= \int_{x=0}^{1} \int_{y=0}^{1-x} e^x \cdot (x+y) \,dy\,dx$ — (1M)

$= \int_{x=0}^{1} e^x \left[xy + \frac{y^2}{2}\right]_0^{1-x} dx$

$= \int_{x=0}^{1} e^x \left(x(1-x) + \frac{(1-x)^2}{2}\right] dx$ — (2M)

$= \int_{x=0}^{1} e^x \left[x - x^2 + \frac{1+x^2-2x}{2}\right] dx$

$= \int_{x=0}^{1} e^x \left(\frac{2x-2x^2+1+x^2-2x}{2}\right) dx$

$= \frac{1}{2}\int_{x=0}^{1} e^x (-x^2+1)\,dx$

$= \frac{1}{2}\left[e^x \left(\frac{(-x^2+1)}{1} - \frac{(-2x)}{1^2} + \frac{(-2)}{1^3}\right)\right]_0^1$

$= \frac{1}{2}\left(e\left[\frac{1-1}{1} + \frac{2}{1} - \frac{2}{1}\right] - e^0\left[1+0-2\right]\right)$ — (2M)

$= \frac{1}{2}\left[1\right]$

$= \frac{1}{2}.$