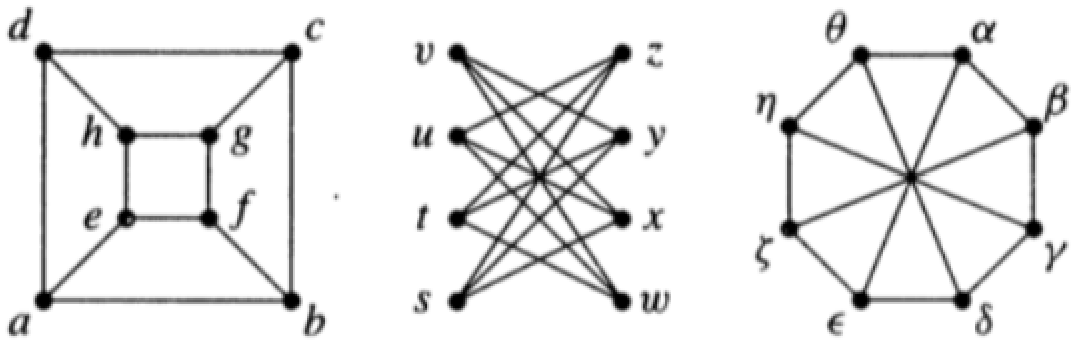
PART - C (PROBLEM SOLVING AND CRITICAL THINKING ANSWERS)

Q1) Determine which pairs of graphs below are isomorphic?

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Ans: 1.Graph A:

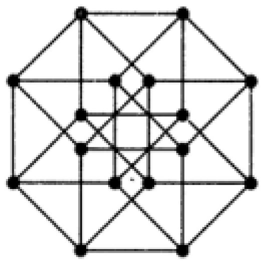
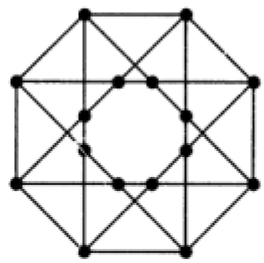
- It does not contain a **4-cycle** (a cycle with 4 vertices).
- The **Petersen graph** is isomorphic to this graph, and its automorphism group is **S5** (the symmetric group of order 5).
- Therefore, **Graph A** is **not isomorphic** to any of the other graphs.

2. Graph B:
    - It is a **regular 10-gon** (decagon) with all the long diagonals.
    - Its automorphism group contains a **10-cycle** (isomorphic to **D10**, the dihedral group of order 10).
    - The **diameter** of **Graph B** is **3**.
    - Therefore, **Graph B** is **not isomorphic** to **Graph A** or **Graph C**.
3. Graph C:
    - It contains **4-cycles**.
    - The **diameter** of **Graph C** is **2**.
    - For any two vertices in **Graph C** that are **3 units apart**, there is exactly **one path** of length **3** between them.
    - Therefore, **Graph C** is **not isomorphic** to **Graph A** or **Graph B**.

In summary:

- **Graph A** is not isomorphic to any other graph.
- **Graph B** is not isomorphic to **Graph A** or **Graph C**.
- **Graph C** is not isomorphic to **Graph A** or **Graph B**.

Q2) Determine whether the graphs below are bipartite and whether they are isomorphic.

Ans: 1.Graph G:

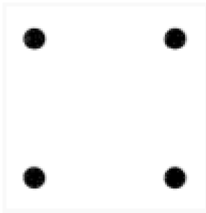
- Vertices: {a, b, c, d, g, h, i, j}
- Edges: {ab, ac, ad, ag, ah, ai, aj, bc, bd, bg, bh, bi, bj, cd, cg, ch, ci, cj, gh, gi, gj, hi, hj, ij}

2. Graph H:
    - Vertices: {1, 2, 3, 4, 5, 6, 7, 8}
    - Edges: {12, 13, 14, 15, 16, 17, 18, 23, 24, 25, 26, 27, 28, 34, 35, 36, 37, 38, 45, 46, 47, 48, 56, 57, 58, 67, 68, 78}
3. Graph G:
    - We can partition the vertices into two sets:
        i. **U**: {a, c, g, i}
        ii. **V**: {b, d, h, j}
    - All edges connect vertices from **U** to **V**, so **Graph G** is **bipartite**.
4. Graph H:
    - We can partition the vertices into two sets:
        i. **U**: {1, 3, 5, 7}
        ii. **V**: {2, 4, 6, 8}
    - All edges connect vertices from **U** to **V**, so **Graph H** is also **bipartite**.
5. Two graphs are **isomorphic** if there exists a bijection (one-to-one, onto map) between their vertex sets that preserves edges and non-edges.

    In summary, both graphs are bipartite and isomorphic.

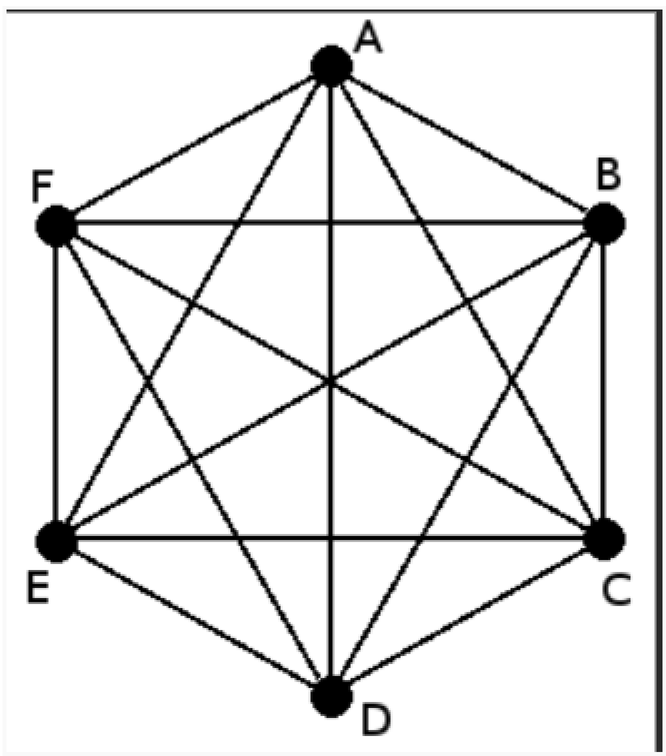
Q3) Draw the following graphs: (i) the null graph N5 (ii) the complete graph K6 (iii) the complete bipartite graph K7,4 (iv) the union of Kl,3 and

Ans: (i) Null Graph N5



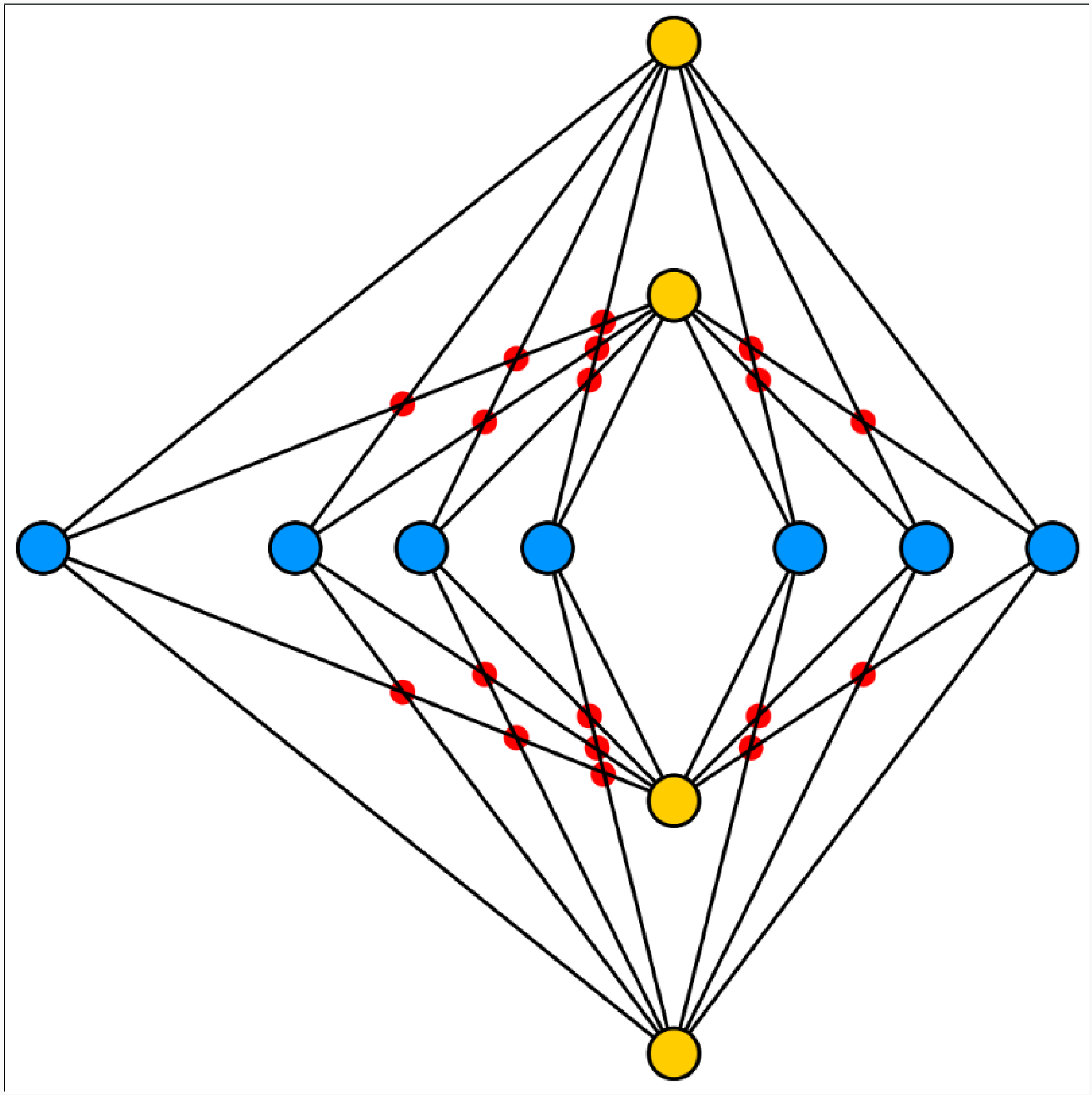
A graph whose edge-set is empty is a null graph. We denote the null graph on n vertices by Nn\ N5

(ii) Complete graph



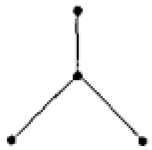

The complete graph K6 has 6 vertices, and every vertex is connected to every other vertex.it forms a regular hexagon with all the vertices connected

(iii) complete bipartite graph k7,4



A complete bipartite graph of $K_{4,7}$ showing that Turan's brick factory  with 4 storage sites (yellow spots) and 7 kilns (blue spots) requires 18 crossings (red dots).

(iv) union K1,3 and W4



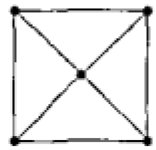


Fig  the claw k1,3 (left) and 4- wheel w4 (right)

K1,3 Graph:

- The **K1,3** graph consists of a central vertex connected to three other vertices. It forms a **claw** shape, where one vertex (the “claw”) is adjacent to the other three.

  W4 Graph:

- The **W4** graph is a **4-wheel** graph. It consists of a cycle of four vertices (forming the “wheel”) with an additional central vertex connected to all four cycle vertices.

  W4 Graph:

  The W4 graph is a 4-wheel graph. It consists of a cycle of four vertices (forming the “wheel”) with an additional central vertex connected to all four cycle vertices.

Q4) Define a directed graph or digraph? Let G5 be a

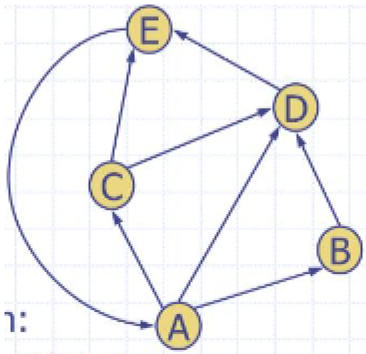
digraph where V (G5) = {a, b, c, d} andA(G5) =

{ab, ba, cc, dc, db, da}. Draw the digraph for G5?

Ans: Directed graph: A graph in which the direction of the edge is defined to a particular node is a directed graph.

Digraph:A **directed graph**, also known as a **digraph**, is a graph where the edges have a direction associated with them.

1. Graph G5:
   - Let’s create the directed graph G5 based on the given information:
     - Vertex set V(G5) = {a, b, c, d}
     - Arc set A(G5) = {ab, ba, cc, dc, db, da}



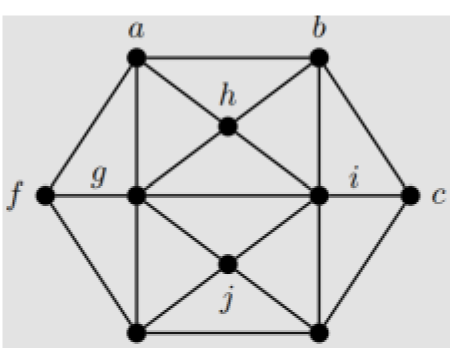

Digraph diagram

- Here's the visual representation of G5:

!G5 Digraph

- Explanation:
  - The vertices are labeled as a, b, c, and d.
  - The directed edges (arcs) are as follows:
    - ab: Arrow from a to b
    - ba: Arrow from b to a
    - cc: Self-loop on c
    - dc: Arrow from d to c
    - db: Arrow from d to b
    - da: Arrow from d to a

Q5) Consider the graph G below. Find two subgraphs of G,both of which have vertex set V` = {a, b, c, f, g, i}.




Ans: 1.)Subgraph 1:

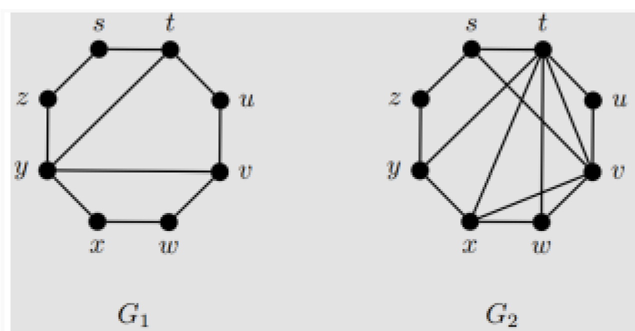
    - We can select any subset of vertices from (V’) to form a subgraph. Let’s choose the vertices ({a, b, c}).
    - The edges incident to these vertices are ({ab, ac}).
    - Therefore, the first subgraph has vertex set ({a, b, c}) and edge set ({ab, ac}).
2. Subgraph 2:
    - For the second subgraph, let’s consider the vertices ({f, g, i}).
    - The edges incident to these vertices are ({fg, fi, gi}).
    - Thus, the second subgraph has vertex set ({f, g, i}) and edge set ({fg, fi, gi}).

In summary:

- **Subgraph 1**: Vertex set ({a, b, c}), Edge set ({ab, ac})
- **Subgraph 2**: Vertex set ({f, g, i}), Edge set ({fg, fi, gi})

Remember that a subgraph is formed by selecting a subset of vertices and their corresponding edges from the original graph. These subgraphs maintain the same vertex set but have different edge sets

Q6)Find the clique-size of a graph, ω(G)for each of the graphs shown below.




Ans: Graph 1:

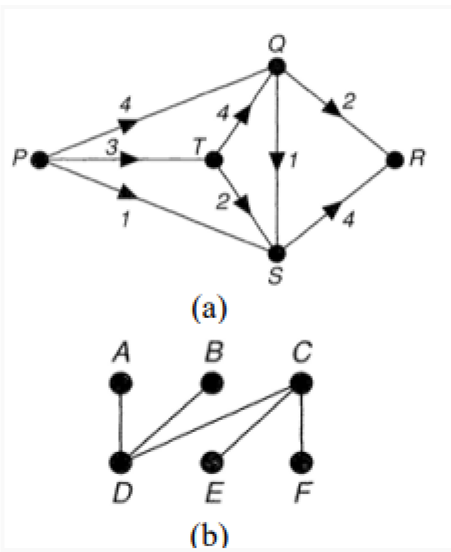
- The graph appears to have **4 vertices** and **6 edges**.
- To find the number of cliques, we can use the formula: [ \text{Number of cliques} = \frac{n \cdot (n - 1)}{2} - m + 1 ] where (n) is the number of vertices and (m) is the number of edges.
- Plugging in the values for this graph: [ \text{Number of cliques} = \frac{4 \cdot (4 - 1)}{2} - 6 + 1 = 2 ]
- Therefore, there are 2 cliques of graph.
- Graph 2:

- Unfortunately, I cannot directly analyze Graph 2 without additional information or a visual representation. If you provide more details or a description of Graph 2, I'd be happy to assist further.

Remember that a clique in an undirected graph is a complete subgraph where all vertices are connected to each other.

Part B LONG ANSWER QUESTIONS

Q1) Write down the number of vertices, the number of edges, and the degree of each vertex, in: (i) the graph in Fig. (a) (ii)the tree in Fig. (b)




(i) Graph in Fig. (a):

- The graph is not explicitly provided, but we can discuss general properties.
- To determine the number of vertices and edges, we need more information about the graph.
- However, I can share some relevant facts:
  - A graph with **n** vertices has at most **(n choose 2)** edges (where **n choose 2** represents the binomial coefficient).
  - Without specific details, we cannot compute the exact values.

2. Tree in Fig. (b):

- A tree is a special type of graph with the following properties:

   1. It is connected (there is a unique path between every pair of vertices).
   2. It has no cycles (no closed loops).
- Let's denote the number of vertices as **n**.
- Since it's a tree, there is always a vertex of degree one (a leaf).
- The total number of edges in a tree with **n** vertices is **n - 1**
- Let's analyze the degree of each vertex:
   1. The leaf vertex (degree 1) connects to exactly one other vertex.
   2. All other vertices (internal nodes) have a degree greater than one.
   3. The sum of degrees in any graph (including trees) is always even because each edge contributes to the degree of two vertices.
   4. Therefore, the internal nodes must have degrees greater than one.
   5. The maximum degree of an internal node in a tree is **n - 1** (when it connects to all other vertices).
   6. The minimum degree of an internal node is **2** (since it must connect to at least one other vertex).
- In summary:
   1. Number of vertices (**n**): Not specified.
   2. Number of edges: **n - 1**.
   3. Degree of each vertex:
      - Leaf vertex: 1 (degree one).
      - Internal nodes: Between 2 and **n - 1**.

Please provide additional details or the actual graph/tree for a more precise analysis.

Q2) Draw a digraph for the following:

(a) Snakes eat frogs and birds eat spiders; birds and spiders both eat insects; frogs eat snails, spiders and insects. Draw a digraph representing this predatory behaviour.

(b) John likes Joan, Jean and Jane; Joe likes Jane and Joan; Jean and Joan like each other. Draw a digraph illustrating these relationships between John, Joan, Jean, Jane and Joe.

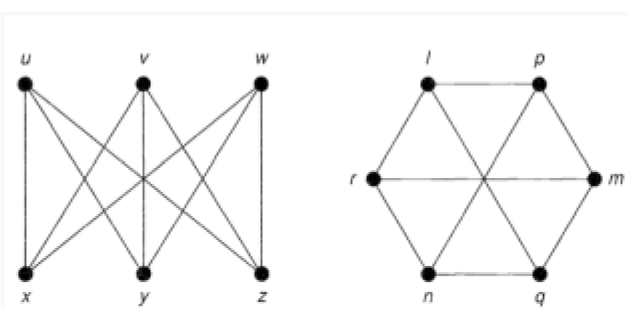
Ans: (a) Snakes eat frogs and birds eat spiders; birds and spiders both eat insects; frogs snakes spiders and insects.

```
 Snakes
  |
  v
Frogs --> Insects --> Birds
  ^          ^          |
  |          |          v
  +---------+-------> Spiders
```

(b) john likes joan,jean and jane; joe likes jane; jean and joan like each other

```
John ------> Joan <--+
 ^   ^               |
 |   |               |
 |   +-----> Jean    |
 |                   |
 +-------------------+
         |
         v
       Jane <-- Joe
```

Q3) Define isomorphism of graphs? State the two labelled graphs are isomorphic or not with reasons.



Ans: The isomorphism graph can be described as a graph in which a single graph can have more than one form.

Any two graphs will be known as isomorphism if they satisfy the following four conditions

1. There will be an equal number of vertices in the given graphs

2. There will be an equal number of edges in the given graphs.

3.There will be an equal amount of degree sequence in the given graphs

**4.Cycle Preservation**: If one graph forms a cycle of length **k** using vertices $\{v_1, v_2, v_3, \ldots, v_k\}$, the other graph must also form the same cycle of length **k** with vertices $\{v_1, v_2, v_3, \ldots, v_k\}$.

Now, let's apply these criteria to the two labeled graphs you've provided.

## Example:

Consider the following two labeled graphs:

Graph 1:

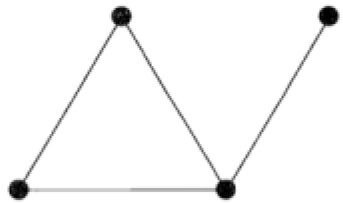
- Vertices: A, B, C, D
- Edges: AB, BC, CD, DA
- Degree sequence: Each vertex has degree 2
- Forms a cycle of length 4: ABCD

Graph 2:

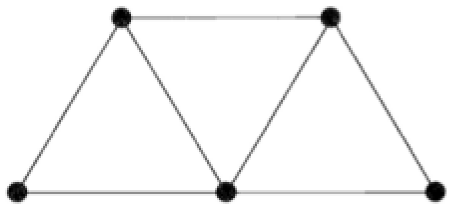
- Vertices: P, Q, R, S
- Edges: PQ, QR, RS, SP
- Degree sequence: Each vertex has degree 2
- Forms a cycle of length 4: PQRS

Based on these conditions, we can conclude that Graph 1 and Graph 2 are isomorphic.

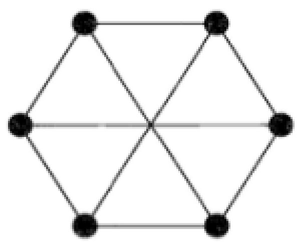
Q4)Define a subgraph in a graph? Verify the graph in (a) is a subgraph of the graph in (b), but is not a subgraph of the graph in (c).



(a)



(b)



(c)

Ans: A **subgraph** is a little replica of a larger graph. Selecting some points (vertices) from the larger graph and connecting them with lines (edges) creates it.

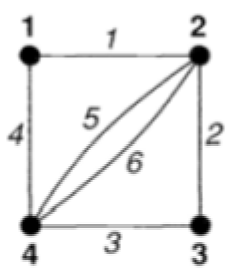
1. Formal Definition:
   - Given a graph **G(V, E)**, a graph **G1(V1, E1)** is considered a subgraph of **G** if:
     - The vertex set **V1(G)** is a subset of **V(G)**.
     - The edge set **E1(G)** is a subset of **E(G)**.
     - Each edge in **G1** has the same end vertices as in **G**.
2. Verification:
   - Let's apply this definition to the given graphs:
     - Graph **(a)**: Check if its vertices and edges are subsets of those in graph **(b)**.
     - Graph ©: Verify if it is not a subgraph of graph **(a)**.
3. Graphs for Verification:
   - Graph **(a)**:
     - Vertices: {A, B, C, D}

- Edges: {AB, BC, CD}
- Graph **(b)**:
  - Vertices: {A, B, C, D, E}
  - Edges: {AB, BC, CD, DE}
- Graph ©:
  - Vertices: {A, B, C, D, E}
  - Edges: {AB, BC, CD}

4. Verification Results:
   - Graph **(a)** is indeed a subgraph of graph **(b)** because its vertices and edges are subsets of those in **(b)**.
   - However, graph **(a)** is **not** a subgraph of graph © because it lacks the edge **DE** present in ©.

In summary, graph (a) is a subgraph of graph (b) but not of graphs

## Q5)Explain the following: (a) Adjacency matrix (b) Incidence matrix Write the adjacency and incidence matrix for the following graph given below:




Ans: (a)Adjacency Matrix:

- An adjacency matrix is a square matrix that represents the relationships between vertices in a graph.
- For a graph with *n* vertices, the adjacency matrix is an *n* × *n* matrix.
- The entry at row *i* and column *j* in the adjacency matrix indicates whether vertices *i* and *j* are adjacent (connected by an edge).
- If vertices *i* and *j* are adjacent, the entry is typically set to **1**; otherwise, it is **0**.
- Example adjacency matrix for a graph with vertices {A, B, C} and edges {(A, B), (B, C)}: [ \begin{bmatrix} 0 & 1 & 0 \ 1 & 0 & 1 \ 0 & 1 & 0 \ \end{bmatrix} ]

(b)Incidence Matrix:

- An incidence matrix represents the incidence relationships between vertices and edges in a graph.
- For a graph with *n* vertices and *m* edges, the incidence matrix is an *n × m* matrix.
- The entry at row *i* and column *j* in the incidence matrix indicates whether vertex *i* is incident to edge *j*.
- If vertex *i* is incident to edge *j*, the entry is typically set to **1**; otherwise, it is **0**.
- Example incidence matrix for the same graph: [ \begin{bmatrix} 1 & 1 \ 1 & 0 \ 0 & 1 \ \end{bmatrix} ]

In the given graph, we have the following vertices and edges:

- Vertices: {A, B, C, D}
- Edges: {(A, B), (B, C), (C, D), (D, A)}

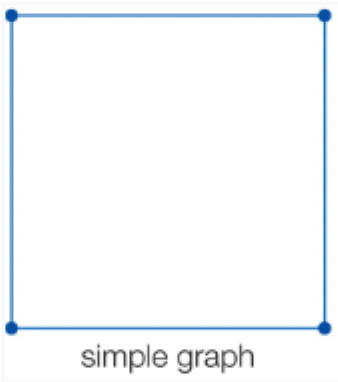
Let's construct the adjacency matrix and incidence matrix for this graph:

1. Adjacency Matrix:
   - The adjacency matrix will be a 4x4 matrix.
   - Based on the edges, we can fill in the entries: [ \begin{bmatrix} 0 & 1 & 0 & 1 \ 1 & 0 & 1 & 0 \ 0 & 1 & 0 & 1 \ 1 & 0 & 1 & 0 \ \end{bmatrix} ]
2. Incidence Matrix:
   - The incidence matrix will be a 4x4 matrix (4 vertices and 4 edges).
   - We fill in the entries as follows: [ \begin{bmatrix} 1 & 1 & 0 & 0 \ 0 & 1 & 1 & 0 \ 0 & 0 & 1 & 1 \ 1 & 0 & 0 & 1 \ \end{bmatrix} ]

Remember that the incidence matrix and adjacency matrix have a relationship, where the incidence matrix is related to the adjacency matrix through the identity matrix.

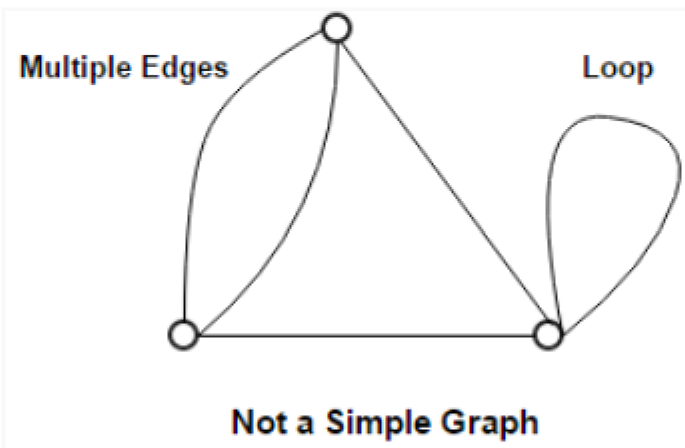
Q6) Explain and draw the following graphs (i) a simple graph, (ii) a non-simple graph with no loops, (iii)a non-simple graph with no multiple edges, each with five vertices and eight edges.

Ans: (i) a simple graph




1. Simple Graph:
   - A simple graph has no loops (edges connecting a vertex to itself) and no multiple edges (more than one edge between the same pair of vertices).
   - Here's a simple graph with five vertices and eight edges: !A simple graph with five vertices and eight edges

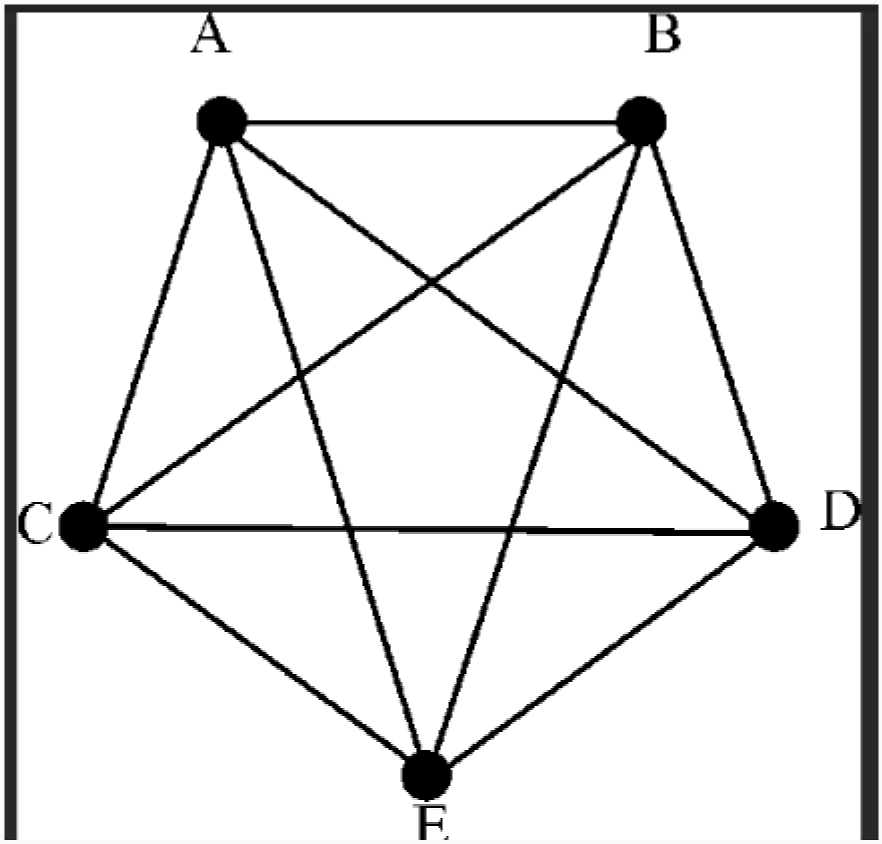
(b) non-simple graph without loops




2. a non-simple graph with no loops,

- In this graph, we'll exclude loops but allow multiple edges.
- Imagine five vertices connected by eight edges, but no vertex is connected to itself.
- The edges can repeat between the same pair of vertices.
- I won't draw this one, but you can visualize it by connecting the vertices in various ways while avoiding loops.

© a non-simple graph with no multiple edges, each with five vertices and eight edges.

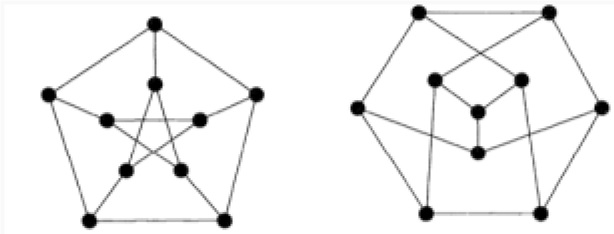



3.Non-Simple Graph without Multiple Edges:

- In this graph, we'll exclude multiple edges but allow loops.
- Again, imagine five vertices, but this time, each pair of vertices has at most one edge connecting them.
- Loops (edges connecting a vertex to itself) are allowed.
- I won't draw this one either, but you can create it by connecting the vertices with unique edges.

Remember that these are abstract representations, and you can experiment with different edge configurations to create

Q7) Show that the two graphs in Fig. (a) are isomorphic by suitably labelling the vertices, and also explain why the two graphs in Fig. (b)are not isomorphic.



(a)

Ans (a) Graphs in Fig. (a):

- Graph **G**:
  - Vertices: {A, B, C, D}
  - Edges: {AB, AC, BC}
- Graph **H**:
  - Vertices: {1, 2, 3, 4}
  - Edges: {12, 13, 23}

To show that G and H are isomorphic, we need to find a bijection (one-to-one, onto map) between their vertex sets that preserves edges. Let's define the following mapping:

- **A → 1**
- **B → 2**
- **C → 3**
- **D → 4**

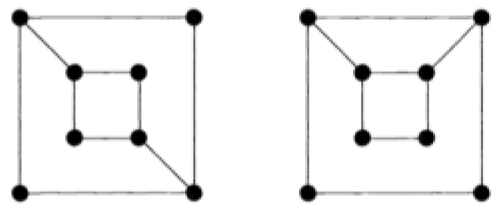
Now, let's check the edges:

- **AB** in **G** corresponds to **12** in **H**.
- **AC** in **G** corresponds to **13** in **H**.
- **BC** in **G** corresponds to **23** in **H**.

Thus, the bijection φ from G to H is:

- **φ(A) = 1**
- **φ(B) = 2**
- **φ© = 3**
- **φ(D) = 4**

Since φ preserves edges, G and H are isomorphic: G ≅ H.

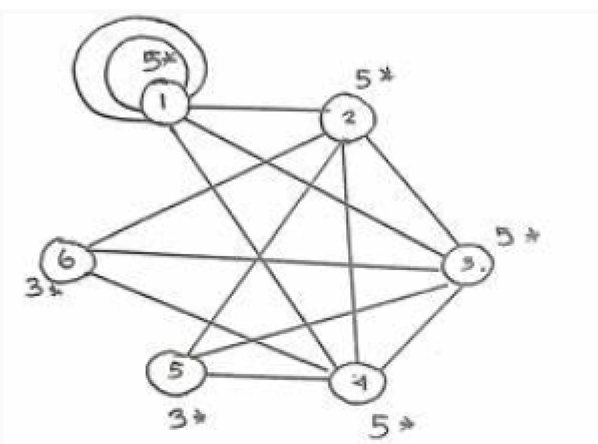

(b)

1. Graphs in Fig. (b):
    - Graph **P**:
        - Vertices: {A, B, C, D}
        - Edges: {AB, AC, BC}
    - Graph **Q**:
        - Vertices: {1, 2, 3, 4}
        - Edges: {12, 13, 24}
2. Let's examine the edges:
    - **AB** in **P** corresponds to **12** in **Q**.
    - **AC** in **P** corresponds to **13** in **Q**.
    - However, there is no edge corresponding to **BC** in **Q**.
3. Since the edge structure is not preserved, P and Q are not isomorphic: P ≠ Q.

In summary:

- **Graphs in Fig. (a)** are isomorphic (by suitable vertex labelling).
- **Graphs in Fig. (b)** are not isomorphic due to the differing edge structures.

Q8) Draw a graph on six vertices with degree sequence (3, 3, 5, 5, 5, 5); and verify does there exist a simple graph with these degrees?

Ans: 

Let’s create a graph with the given degree sequence (3, 3, 5, 5, 5, 5). We’ll start by connecting the vertices to satisfy the degree requirements.

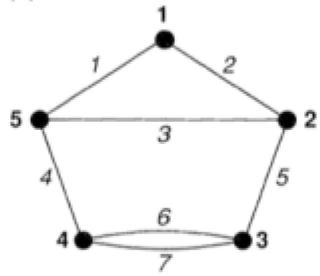
1. Begin with six vertices: $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$, $\mathbf{v_4}$, $\mathbf{v_5}$, $\mathbf{v_6}$.
2. Connect $\mathbf{v_1}$ to the other four vertices: $\mathbf{v_2}$, $\mathbf{v_3}$, $\mathbf{v_4}$, $\mathbf{v_5}$.
3. Connect $\mathbf{v_2}$ to $\mathbf{v_3}$ and $\mathbf{v_4}$.
4. Connect $\mathbf{v_3}$ to $\mathbf{v_5}$.
5. Connect $\mathbf{v_4}$ to $\mathbf{v_5}$.

The resulting graph has the following edges:

- **$(\mathbf{v_1}, \mathbf{v_2})$, $(\mathbf{v_1}, \mathbf{v_3})$, $(\mathbf{v_1}, \mathbf{v_4})$, $(\mathbf{v_1}, \mathbf{v_5})$**
- **$(\mathbf{v_2}, \mathbf{v_3})$, $(\mathbf{v_2}, \mathbf{v_4})$**
- **$(\mathbf{v_3}, \mathbf{v_5})$**
- **$(\mathbf{v_4}, \mathbf{v_5})$**

This graph satisfies the degree sequence (3, 3, 5, 5, 5, 5), and it is a simple graph. Each vertex has the correct degree, and no duplicate edges or loops exist. Therefore, a simple graph with these degrees does indeed exist!

Q9) (i) Write down the adjacency and incidence matrices of the graph in Fig. (a) (ii) Draw the graph whose adjacency matrix is given in Fig. (b) (iii) Draw the graph whose incidence matrix is given in Fig. ©




(a)

$$\begin{bmatrix} 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Ans: (i) Adjacency Matrix:

- An adjacency matrix represents the relationships between vertices in a graph. It's a square matrix where each entry indicates whether two vertices are adjacent (connected by an edge) or not.
- For the given graph in Fig. (a), let's assume it has 4 vertices labeled as A, B, C, and D. The edges are as follows:
  - Edge between **A** and **B**
  - Edge between **B** and **C**
  - Edge between **C** and **D**

(ii) An adjacency matrix is a way of representing a graph as a matrix of 0s and 1s, where each cell indicates whether there is an edge between two vertices. In this case, we'll use the adjacency matrix provided in Fig. (b).

The adjacency matrix represents the connections between vertices. Each row and column corresponds to a vertex, and the value in cell A<sub>ij</sub> indicates whether there is an edge from vertex i to vertex j. If there is an edge, A<sub>ij</sub> is 1; otherwise, it's 0

In this graph:

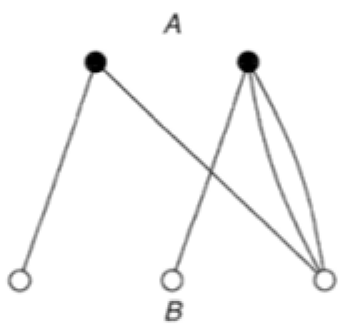
- Vertex 0 is connected to vertices 1 and 2.
- Vertex 1 is connected to vertices 0, 2, and 3.
- Vertex 2 is connected to vertices 0, 1, and 3.
- Vertex 3 is connected to vertices 1 and 2.

© The incidence matrix represents the relationship between vertices and edges in a graph. Each row corresponds to a vertex, and each column corresponds to an edge. The entry in row *i* and column *j* is 1 if vertex *i* is incident with edge *j*, and 0 otherwise.

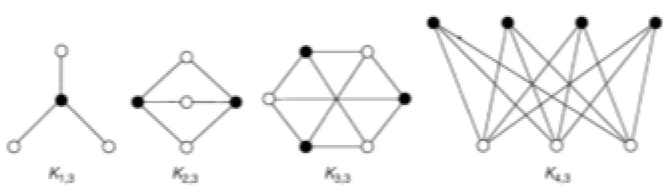
Since I don't have access to Fig. ©, I'll generate a simple example of an undirected graph using an incidence matrix. Let's assume the following incidence matrix:

[ B = \begin{bmatrix} 1 & 1 & 1 & 0 \ 1 & 0 & 0 & 0 \ 0 & 1 & 0 & 1 \ 0 & 0 & 1 & 1 \end{bmatrix} ]

Q10) Define bipartite graphs and complete bipartite graphs. Justify the graph in fig. (a) is a bipartite graph or not and also the graphs in fig. (b) are complete bipartite graphs or not.




(a)




(b)

Ans:(i) Bipartite graphs:- A graph in which the vertices can be divided into two disjoint sets such that every edge connects a vertex in one set to a vertex in the other side.

Ex: A job applicant graph where the vertices can be divided into job applicants and job openings

(ii) Complete graphs :- A graph in which each vertex is connected to every other vertex

Ex: A tournament graph where every player plays against every other player.

let's analyze the given graphs:

- Graph (a):
  - !Graph (a)
  - This graph is **bipartite** because we can partition its vertices into two disjoint subsets: **U = {A, B, C}** and **V = {D, E, F}**. All edges connect vertices from **U** to **V**, and no edges exist within the same subset.
  - Therefore, **Graph (a)** is a **bipartite graph**.
- Graph (b):
  - !Graph (b)
  - This graph is **complete bipartite** because it satisfies the conditions:
    - Vertices can be partitioned into subsets $\mathbf{V_1}$ **= {A, B, C}** and $\mathbf{V_2}$ **= {D, E, F}**.
    - All possible edges connecting vertices from $\mathbf{V_1}$ to $\mathbf{V_2}$ are present.
  - Therefore, **Graph (b)** is a **complete bipartite graph**.