MODULE-V : NUMERICAL METHODS-2

* Numerical Integration:

(a) Trapezoidal Rule

(b) Simpson's 1/3 Rule

(c) Simpson's 3/8 Rule

* Ordinary Differential equations:

(a) Taylor's series

(b) Euler's Method

(c) Runge-Kutta Method

1) Trapezoidal Rule:

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots)\right]$$

where, $h=\frac{b-a}{n}$, h = length of subintervals,

example-1

Evaluate $\int_0^1 x^3 dx$ with 5 sub-intervals by Trapezoidal rule.

sol: $\int_a^b f(x)dx = \int_0^1 x^3 dx$

$y=f(x)=x^3$, $h=\frac{b-a}{n}=\frac{1-0}{5}=0.2$

x	0	0.2	0.4	0.6	0.8	1
y	0	0.008	0.064	0.216	0.512	1

$$\int_0^1 x^3\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots)\right]$$

$$= \frac{h}{2}\left[(y_0+y_5)+2(y_1+y_2+y_3+y_4)\right]$$

$$= \frac{0.2}{2}\left[(0+1)+2(0.008+0.064+0.216+0.512)\right]$$

$$= 0.26 \,//.$$

② evaluate $\int_0^1 (1+x^3)^{1/2}\,dx$ taking $h=0.1$ using trapezoidal rule. Ans = 1.11226

Sol:- $\int_a^b f(x) = \int_0^1 (1+x^3)^{1/2}$

$y = f(x) = (1+x^3)^{1/2}$ $\quad h = 0.1$

$h = \frac{b-a}{n}$ $\quad n = \frac{b-a}{h} = \frac{1-0}{0.1}$

$= \frac{1}{0.1}$

$n = 10$

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
y	1	1.0004	1.0039	1.013	1.031	1.06	1.102	1.158	1.229	1.314	1.414

Now.

$$\int_0^1 (1+x^3)^{1/2} = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots)\right]$$

$$= \frac{0.1}{2}\left[(1+1.414)+2(1.0004+1.0039+1.013+1.031+1.06+1.102+1.158+1.229+1.314)\right]$$

$$= \frac{0.1}{2}(22.2366)$$

$$= 1.1118 \,//.$$

③ Evaluate $\int_0^{\pi/2} e^{\sin x}\,dx$ using trapezoidal rule.

Sol:- let $n=6$

$$h=\frac{b-a}{n}$$

$$h=\frac{\pi/2}{6}$$

$$h=\frac{\pi}{12}=0.2617$$

$$\int_0^{\pi/2} e^{\sin x}\,dx \Rightarrow f(x)=y=e^{\sin x}$$

x	0	0.2617 $(\pi/12)$	0.5234 $(\pi/6)$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
y	1	1.2954	1.6487	2.0281	2.3774	2.6272	2.7183

$$\int_a^b f(x)\,dx=\frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots)\right]$$

$$=\frac{\pi}{24}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$=\frac{\pi}{24}\left[(1+2.7183)+2(1.2954+1.6487+2.0281+2.3774+2.6272+2.7183)\right]$$

$$=3.0986$$

*Simpson $\frac{1}{3}$ Rule:-

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\cdots)+2(y_2+y_4+y_6+\cdots)\right]$$

where, $h=\frac{b-a}{n}$, h = length of subintervals

$y_1+y_3+y_5+\cdots$ = sum of odd terms

$y_2+y_4+y_6+\cdots$ = sum of even terms

Example-①:- Find the solution's 1/3 Rule:- $\int_0^6 \frac{1}{1+x^2}\,dx$ with 6-subintervals

Sol:-

X	0	1	2	3	4	5	6
Y	1	0.5	0.2	0.1	0.588	0.0385	0.027

Given
$n=6$

$h=\frac{b-a}{n}$

$=\frac{6-0}{6}$ $\quad y=\frac{1}{1+x^2}$

$=1$

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\cdots)+2(y_2+y_4+\cdots)\right]$$

$$\int_0^6 \frac{1}{1+x^2} = \frac{1}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{1}{3}\left[(1+0.027)+4(0.5+0.1+0.0385)+2(0.2+0.588)\right]$$

$= 1.719$ //.

*Taylor's Series:

It is an expansion of some function into an infinite sum of terms.

consider the first order equation

$$\frac{dy}{dx} = f(x,y).$$

diff ①, we have

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \quad \text{i.e } y'' = f_x + f_y f'.$$

Hence, the taylor's series

$$y = y_0 + (x - x_0)(y')_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \cdots$$

problem-①:

using Taylor's method $dy/dx = x^2+y^2$ for $x=0.4$ given that $y=0$, when $x=0$.

Sol: Given

$y(0)=0, \quad y(x_0)=y_0$

comparing $x_0=0$ and $y_0=0$

$$y(x) = y(x_0) + (x-x_0) * y'(x_0) + (x-x_0)^2/2! * y''(x_0) + (x-x_0)^3/3! * y'''(x_0)$$

$$y' = (x^2+y^2) \Rightarrow y'(x_0) = (x_0)^2 + (y_0)^2 = 0$$

$$y'' = (2.x + 2yy') \Rightarrow y''(x_0) = 2.x_0 + 2y_0y'$$

$$= 2*0 + 2*0*0 = 0$$

$$y''' = (2 + 2(yy'' + (y')^2)) \Rightarrow y'''(x_0) = 2 + 2[y_0y'' + (y')^2]$$

$$= 2 + 2(0*0+0]$$

$$= 2$$

$y(0)=0;\ y(x_0)=y_0$ comparing $x_0=0$ and $y_0=0$

$$y(x)=y(x_0)+(x-x_0)*y'(x_0)+(x-x_0)^2/2!*y''(x_0)+(x-x_0)^3/3!*y'''(x_0)$$

$$y(x)=0+0+0+x^3/3$$

$$y(x)=x^3/3$$

at $x=0.4$

$$y(0.4)=(0.4)^3/3=0.0213.$$

2

(5) Use Taylor's series method to find y' at $x=0.1, 0.2, 0.3$ considering terms upto the third degree given $dy/dx=x^2+y^2$ and $y(0)=1$.

Sol: Given

$y(0)=1;\ y(x_0)=y_0$ comparing $x_0=0$ and $y_0=1$.

$$y(x)=y(x_0)+(x-x_0)*y'(x_0)+(x-x_0)^2/2!*y''(x_0)+\frac{(x-x_0)^3}{3!}*y'''(x_0)$$

$$y'=(x^2+y^2)\Rightarrow y'(x_0)=(x_0)^2+(y_0)^2=1$$

$$y''=(2.x+2yy')\Rightarrow y''(x_0)=2x_0+2y_0y'=2*0+2*1*1=2$$

$$y'''=(2+2(yy''+(y')^2))\Rightarrow y'''(x_0)=(2+2[y_0y''+(y')^2])=2+2[1*2+1^2]=8.$$

$y(0)=0;\ y(x_0)=y_0$ comparing $x_0=0$ and $y_0=1$

$$y(x)=y(x_0)+(x-x_0)*y'(x_0)+(x-x_0)^2/2!*y''(x_0)+(x-x_0)^3/3!*y'''(x_0)$$

$$y(x)=\cancel{y(x_0)+}\ 1+x+\frac{2x^2}{2!}+\frac{8x^3}{3!}$$

$$y(x)=1+x+x^2+4x^3/3$$

at $x=0.1, 0.2, 0.3$

$$y(0.1) = 1+0.1+(0.1)^2+\frac{4(0.1)^3}{3} = 1.1113$$

$$y(0.2) = 1+0.2+(0.2)^2+\frac{4(0.2)^3}{3} = 1.2507$$

$$y(0.3) = 1+0.3+(0.3)^2+\frac{4(0.3)^3}{3} = 1.426$$

③ use Taylor's method to find the solution of $dy/dx = 1+y^2$, $y(0)=0$. At $x=0.1$ taking $\boxed{h=0.1}$ correct upto 3 decimal places.

ans: $y(0.1) = 0.10033$

sol: $y(0)=0$ $\qquad h = x - x_0$

$y_0 = 0, \; x_0 = 0$

$y' = 1+y^2$

$y'(x_0) = 1+(y_0)^2 = 1+0 = 1$

$y'' = 2yy' \Rightarrow y''(x_0) = 2*0*1 = 0$

$y''' = 2(yy'' + (y')^2) = 2(0+1)$

$= 2$

$$y(x) = y(x_0) + (x-x_0)*y'(x_0) + \frac{(x-x_0)^2 * y''(x_0)}{2!} + \frac{(x-x_0)^3 * y'''(x_0)}{3!}$$

$$= 0 + (0.1)(1) + \frac{(0.1)^2 \cdot 0}{2!} + \frac{(0.1)^3(2)}{3\times 2\times 1}$$

$y(0.1) = 0.10033$

② Find $y(0.2)$ for $y' = x^2y - 1$, $y(0) = 1$ with step length 0.1 using taylor series method.

Sol: Given

$$y' = x^2y - 1, \quad y(0) = 1, \quad h = 0.1, \quad y(0.2) = ?$$

Here, $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$$y' = x^2y - 1 \Rightarrow y_0' = x_0^2 y_0 - 1 = -1$$

$$y'' = 2xy + x^2y' \Rightarrow y_0'' = 2x_0y_0 + 2x_0y_0' = 0$$

$$y''' = 2y + 4xy' + x^2y'' \Rightarrow y_0''' = 2y_0 + 4x_0y_0' + x_0^2y_0'' = 2$$

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0'''' + \cdots$$

$$= 1 + 0.1\cdot(-1) + \frac{(0.1)^2}{2!}\cdot(0) + \frac{(0.1)^3}{3!}\cdot(2) + \frac{(0.1)^4}{4!}\cdot(-6) + \cdots$$

$$= 1 - 0.1 + 0 + 0.00033 + 0 + \cdots$$

$$= 0.90031$$

$$\therefore y(0.1) = 0.90031$$

* Euler's Method:

$$\frac{dy}{dx} = f(x, y)$$

$y_1 = y_0 + hf(x_0, y_0)$

$y_2 = y_1 + hf(x_1, y_1)$

⋮

$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

Example-①:

If $dy/dx = x+y$; $y(0)=1$ then find $y(0.3)$ by taking step size as 0.1 using Euler's method.

Sol: Given,

$y(0)=1$, $y(x_0)=y_0$ then and

$y_0=1$, $x_0=0$, $h=0.1$

$$\frac{dy}{dx} = x+y$$

$y' = x+y$

$f(x, y) = x+y$

acc. to Euler's method, $y_{n+1} = y_n + h * f(x_n, y_n)$

$y_0 = 1$, $x_0 = 0$, and $h = 0.1$

Substitute $n=0$ to get y_1 term

$y_1 = y_0 + h \cdot f(x_0, y_0)$

$y_1 = 1 + 0.1 * f(0, 1)$

$y_1 = 1 + 0.1 * (0+1)$

$y_1 = 1.1$

$y_1 = 1.1$, $x_1 = x_0 + h = 0 + 0.1$
$= 0.1$

substitute $n=1$ to get y_2 term

$y_2 = y_1 + h \cdot f(x_1, y_1)$

$y_2 = 1.1 + 0.1 * f(0.1, 1.1)$

$y_2 = 1.1 + 0.1 * (0.1 + 1.1)$

$y_2 = 1.22$

$y_2 = 1.22,\ x_2 = x_1 + h = 0.1 + 0.1$
$= 0.2$

Sub $n=2$ to get y_3 term

$y_3 = y_2 + h.f(x_2, y_2)$

$y_3 = 1.22 + 0.1 * f(0.2, 1.22)$

$y_3 = 1.22 + 0.1 * (0.2 + 1.22)$

$y_3 = 1.362$

$\therefore y(0.3) = 1.362.$

(2) use euler's method with $h=0.1$ to find approximate values for the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x},\ y(0) = 1$$

at $x = 0.1, 0.2, 0.3.$

Sol:-

$y_1 = 0.8$

$y_2 = 0.64$

$y_3 = 0.51$

(3) use euler's method to solve:-

(a) $\frac{dy}{dx} = \frac{3}{5}x^3 y,\ y(0) = 1$ $h = 0.1$

$y_1 = 1$

$y_2 = 1.6$

$y_3 = 9.28$

$y_4 = 159.616$

(b) $\frac{dy}{dx} = 1 + y^2,\ y(0) = 0$

$y_1 = 1$

$y_2 = 3$

$y_3 = 13$

$y_4 = 183$

* Runge- Kutta Method:-

→ 1st order Runge-Kutta method:-

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= y_0 + hy_0' \quad \{\text{Since } y' = f(x,y)\}$$

This formula is same as the Euler's method

→ 2nd order Runge-Kutta Method:-

$$y_1 = y_0 + \left(\frac{1}{2}\right)(K_1 + K_2)$$

Here,

$$K_1 = hf(x_0, y_0)$$

$$K_2 = hf(x_0 + h, y_0 + K_1)$$

→ 3rd order Runge-Kutta Method:-

$$y_1 = y_0 + (1/6)(K_1 + 4K_2 + K_3)$$

Here,

$$K_1 = hf(x_0, y_0)$$

$$K_2 = hf\left[x_0 + \left(\frac{1}{2}\right)h, y_0 + \left(\frac{1}{2}\right)K_1\right]$$

$$K_3 = hf(x_0 + h, y_0 + K') \text{ such that } K' = hf(x_0 + h, y_0 + K_1)$$

→ 4th order Runge-Kutta Method:-

$$y_{n+1} = y_n + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

where,

$$K_1 = hf(x_n, y_n)$$

$$K_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}\right)$$

$$K_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{K_2}{2}\right)$$

$$K_4 = hf(x_n + h, y_n + K_3)$$

problem-①:-

consider an ordinary differential equation $\frac{dy}{dx} = x^2+y^2$, $y(1)=1.2$. Find $y(1.05)$ using the fourth order Runge-Kutta method.

$\to x_0$ $\to x$

sol:- Given, $\frac{dy}{dx} = x^2+y^2$, $y(1)=1.2$ $\qquad h = x - x_0$

$f(x,y) = x^2+y^2$

$x_0 = 1$ and $y_0 = 1.2$

also, $h = 0.05$.

$y_1 = y_0 + \left(\frac{h}{6}\right)(K_1+2K_2+2K_3+K_4)$

$y_1 = 1.2 + \left(\frac{h}{6}\right)(K_1+2K_2+2K_3+K_4)$

$K_1 = hf(x_0, y_0)$

$= (0.05)\left[(1)^2+(1.2)^2\right]$

$K_1 = 0.122$

$K_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}\right)$

$= (0.05)f\left(1+\frac{0.05}{2}, 1.2+\frac{0.122}{2}\right)$

$= (0.05)f(1.025, 1.261)$

$= (0.05)\left[(1.025)^2+(1.261)^2\right]$

$K_2 = 0.1320$

$K_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}\right)$

$= hf\left(1+\frac{0.05}{2}, 1.2+\frac{0.1320}{2}\right)$

$= hf(1.025, 1.266)$

$= (0.05)\left[(1.025)^2+(1.266)^2\right]$

$K_3 = 0.1326$

$K_4 = hf(x_0+h, y_0+K_3)$

$= (0.05)f(1+0.05, 1.2+0.1326)$

$= (0.05)f(1.05, 1.3326)$

$= (0.05)\left[(1.05)^2+(1.3326)^2\right]$

$= 0.1439$

$y_1 = 1.2 + \left(\frac{0.05}{6}\right)(0.122+2(0.1320)+2(0.1326)+0.1439)$

$y_1 = 1.2066$

$$y_2 = y_1 + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1.2066 + \frac{0.05}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

But given to find $y(1.05)$

$\therefore y_1 = y(1.05)$

$y_1 = 1.2066$

② Find the value of K_1 by Runge-Kutta method of fourth order if $\frac{dy}{dx} = 2x + 3y^2$ and $y(0.1) = 1.1165$ $h = 0.1$.

Sol:-

$$f(x,y) = 2x + 3y^2$$

$x_0 = 0.1 \quad y_0 = 1.1165$

and $h = 0.1$

now $K_1 = hf(x_n, y_n)$

$$= 0.1 f(x_0, y_0)$$

$$= (0.1) f(0.1, 1.1165)$$

$$K_1 = (0.1)(2(0.1) + 3(1.1165)^2)$$

$$K_1 = 0.3939$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

$$= (0.1) f\left(0.1 + \frac{0.1}{2}, 1.1165 + \frac{0.3939}{2}\right)$$

$$= (0.1) f(0.15, 1.31345)$$

$$= (0.1)\,(2(0.15) + 3(1.31345)^2)$$

$$K_2 = 0.5475$$