Drop the first eqn

$$\frac{dS}{dt} = \frac{\beta S}{T}(1-S-\varphi-A) - \frac{\rho S A}{T} + \sigma\varphi S - \mu S$$

$$\frac{d\varphi}{dt} = \frac{\rho S A}{T} - \sigma\varphi S - \gamma\varphi - \mu\varphi$$

$$\frac{dA}{dt} = \mu\rho T - \mu A + \gamma\varphi$$

Define the following:

$$\phi = \mu T(\gamma+\mu)$$
$$\omega = \mu T\sigma - \gamma\rho$$
$$h = (\gamma+\mu)(\rho\mu T^2)$$
$$k = \rho\mu T^2\sigma \quad \text{— (A)}$$

Total population: T

Let's express A as a function of φ:

(Use $\frac{dA}{dt}$)
$$\boxed{A(\varphi) = \rho T + \frac{\gamma}{\mu}\varphi} \quad \text{— (1)}$$

Let's express φ as a function of S and A,

(Use $\frac{d\varphi}{dt}$)
$$\boxed{\varphi(S,A) = \frac{\rho S A}{T(\sigma S+\gamma+\mu)}} \quad \text{— (2)}$$

Using this in (1) to get A in terms of S,

$$\Rightarrow A = \rho T + \frac{\gamma\rho S A}{\mu T(\sigma S+\gamma+\mu)}$$

$$\Rightarrow A\left[1 - \frac{\gamma\rho S}{\mu T(\sigma S+\gamma+\mu)}\right] = \rho T$$

$$\Rightarrow A(S) = \frac{\rho\mu T^2(\sigma S+\gamma+\mu)}{\mu T(\sigma S+\gamma+\mu) - \gamma\rho S}$$

$$\Rightarrow \boxed{A(S) = \frac{\rho\mu T^2(\sigma S+\gamma+\mu)}{S(\mu T\sigma-\gamma\rho) + \mu T(\gamma+\mu)}} \quad \text{— (3)}$$

Using (A) we get,

$$\boxed{A(S) = \frac{kS+h}{S\omega+\phi}}$$

Using $A(s)$ in ⑦ in ②,

$$\Phi(s) = \frac{ps(p\mu T^2)\,\cancel{(\sigma s+\gamma+\mu)}}{\cancel{T(\sigma s+\gamma+\mu)}\,[s\omega+\phi]}$$

$$\Rightarrow \boxed{\Phi(s) = \frac{p^2 \mu s T}{s\omega+\phi}}$$

Using initial set of eqⁿs

$[S, \Phi, A]$

So, we have, $A(s)$ and $\Phi(s)$ as functions of s.

$$\Rightarrow \boxed{A(s) = \frac{ks+h}{s\omega+\phi}, \quad \Phi(s) = \frac{p^2\mu s T}{s\omega+\phi}} \quad ④$$

We can use ④ in the eqⁿ for $\frac{dS}{dt}$.

$$\frac{B}{T}(1-S-\Phi-A) - \frac{pA}{T} + \sigma\Phi = \mu$$

$$\Rightarrow B\left[1-S-\Phi(s)-A[s]\right] - pA = T(\mu-\sigma\Phi)$$

$$\Rightarrow B\left[1-S-\frac{p^2\mu s T}{s\omega+\phi} - \frac{(ks+h)}{(s\omega+\phi)}\right] - \frac{p(ks+h)}{s\omega+\phi}$$

$$= T\left(\mu - \frac{p^2\sigma\mu s T}{s\omega+\phi}\right)$$

Multiplying both sides by $s\omega+\phi$, we get,

$$\mathcal{B}\left[(s\omega+\phi)-s(s\omega+\phi)-p^2\mu sT-(Ks+h)\right]$$
$$-p(Ks+h)=T\left[\mu(s\omega+\phi)-p^2\sigma\mu sT\right]$$

$$\Rightarrow \cancel{\mathcal{B}s\omega}+\mathcal{B}\phi-\cancel{s^2\mathcal{B}\omega}-\cancel{\mathcal{B}\phi s}-\cancel{p^2\mu sT}-\cancel{Ks}-h$$
$$-\cancel{pKs}-ph=\cancel{sT\mu\omega}+T\mu\phi-\cancel{Tp^2\sigma\mu sT}$$

$$\Rightarrow -s^2(\mathcal{B}\omega)+s\big(\mathcal{B}\omega-\mathcal{B}\phi-p^2\mu T-K-pK$$
$$-T\mu\omega+Tp^2\sigma\mu T\big)$$
$$+\mathcal{B}\phi-ph-T\mu\phi=0$$

$$\Rightarrow s^2(\mathcal{B}\omega)-s\big(\mathcal{B}\omega-\mathcal{B}\phi-p^2\mu T-K-pK$$
$$-T\mu\omega+T^2p^2\sigma\mu\big)$$
$$-\mathcal{B}\phi+ph+T\mu\phi=0$$

We have a quadratic in s,

$$\boxed{\begin{aligned} a&=\mathcal{B}\omega\\ b&=\mathcal{B}\omega-\mathcal{B}\phi-p^2\mu T-K-pK-T\mu\omega+T^2p^2\sigma\mu\\ c&=ph+\phi(T\mu-\mathcal{B})\end{aligned}}$$

Now, $a>0 \Rightarrow \mathcal{B}\omega>0$

and, $\mu T\sigma-\gamma p>0 \Rightarrow \mu T\sigma>\gamma p$

$$\Rightarrow \boxed{\frac{\gamma p}{\mu T\sigma}<1}$$

Upon ~~symbolic computation~~, ~~the expressions~~ for a, b, c are:

$$a = B(T\mu\sigma - \gamma p)$$

$$b = -T^2\mu p\sigma + T(-B\mu(\gamma+\mu) - \mu p^2 - \mu(T\mu\sigma - \gamma p)) + a$$

$$c = T\mu(T\gamma\mu + T\gamma p^2 + T\mu^2 + T\mu p^2 - B\gamma - B\mu)$$

If $c > 0$,

$$\cancel{T\mu}\left[T\gamma(\mu+p^2) + T\mu(\mu+p^2) - B(\gamma+\mu)\right] > 0$$

$$\Rightarrow T\cancel{(\gamma+\mu)}(\mu+p^2) > B\cancel{(\gamma+\mu)}$$

$$\Rightarrow \boxed{\frac{T(\mu+p^2)}{B} > 1}$$

$$\Rightarrow \boxed{\frac{B}{T(\mu+p^2)} < 1}$$