

Modelling and Analysis of Traffic Flows using Three-Dimensional Non-Linear Dynamical System of Ordinary Differential Equations

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<u>Abstract:</u>

Many researchers have described the nature of traffic flows using ***partial differential equations (PDE's)*** and ***ordinary differential equations (ODE's)***. Most of them use the concept of the ***conservation laws in fluid mechanics***. For example, the model introduced by Lighthill-Whitman-Richards ***(LWR Model)*** is formulated as a nonlinear partial differential equation, derived by using the conservation of vehicles on a single road. In this paper we have described the nature of traffic flow using a ***three-dimensional nonlinear dynamical system*** called the ***FBDF model***. We have checked the ***well posedness*** of the model. The ***local stability analysis*** of the equilibrium points has been carried out using the ***retardation number*** while the ***global stability analysis*** has been performed using ***LaSalle's invariance principle.*** The numerical simulations were carried out using ***Python***. Further we have presented the ***sensitivity analysis*** of the parameters via two methods: ***a derivative-based local method*** and ***algorithmic differentiation***. Some important results and observations have been drawn and have been presented in the text of this paper.

<u>Key Terms:</u> Nonlinear dynamical system, equilibrium points, local stability analysis, global stability analysis, retardation number, sensitivity analysis, derivative based local method, algorithmic differentiation, next-generation matrices, spectral radius, Routh-Hurwitz criterion, LaSalle's invariance principle, Lyapunov function.

1. <u>INTRODUCTION:</u>

Since the beginning of the twentieth century, researchers have been developing mathematical models that could describe the behavior of traffic flow. As a result, two classes of models came into existence viz., ***microscopic*** and ***macroscopic*** models. Microscopic models simulate ***single vehicle driver units***. The dynamic variables of the models represent microscopic properties like the ***position*** and the ***velocity*** of single vehicles. However, microscopic models result in complications when the road has a large number of vehicles. On the other hand, a macroscopic model formulates the relationships among traffic flow characteristics like ***density, flow, mean speed of a traffic stream***, etc. The method of modeling traffic flow at a macroscopic level originated under an assumption that traffic streams as a whole are comparable to ***fluid streams***. An example of a macroscopic model is the LWR model. It was first introduced by Lighthill and Whitham in 1955 and independently by Richards in 1956. Another class of models called the ***mesoscopic*** models have come into picture in order to fill the gap between the microscopic and macroscopic families of models.

Traditional mesoscopic models describe vehicle flow in ***aggregate terms*** such as in ***probability distributions***. However, behavioral rules are defined for individual vehicles.

In the model that we have presented in this paper, we are using the concept of a ***three-dimensional nonlinear dynamical system*** to illustrate the flow of vehicles and blocking effects.

Why ***three***? Because the vehicles running on a road have been classified into three categories: ***Free(F)***, ***Blocked(B)*** and ***Discharged(D)*** vehicles.

Why ***nonlinear***? As we will see later on in this paper, the differential equations (2) - (4) contain a nonlinear term ***(FB).*** This will become clearer once we go through the construction of the model.

Let's briefly look at the variables we are going to consider in this paper.

A) **Free vehicles (F(t)):** Free vehicles are the normal vehicles that are moving at a speed as per the driver's wish. These are the ***potential*** vehicles that are likely to get blocked.
B) **Blocked vehicles (B(t)):** Blocked vehicles are those which have been ***completely blocked*** by other vehicles and they are capable of ***propagating*** the blockage to other vehicles.
C) **Discharged vehicles (D(t)):** Discharged vehicles are those vehicles which have already ***been blocked*** by other vehicles and have ***gotten discharged*** via some means, say the blocking vehicle moved away.

At this stage we would like to point out the ***dependent*** and ***independent*** variables.

The set of the variables defined above i.e.: F(t), B(t), D(t) are the dependent ones and they ***count the number of vehicles in each of the groups*** as a function of time, making time the independent variable.

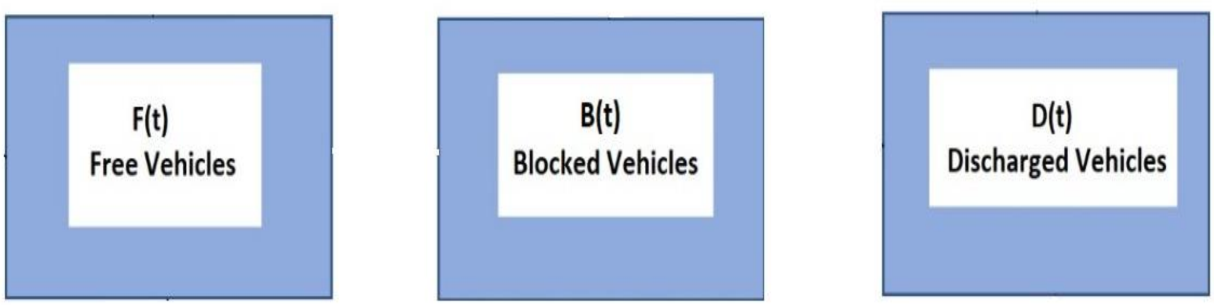
2. ASSUMPTIONS:

We formulated the equations (2) - (4) while keeping in mind a set of assumptions. We will walk you through each one of them and by the end of this section we will be looking at the block diagram of our ***FBDF model.***

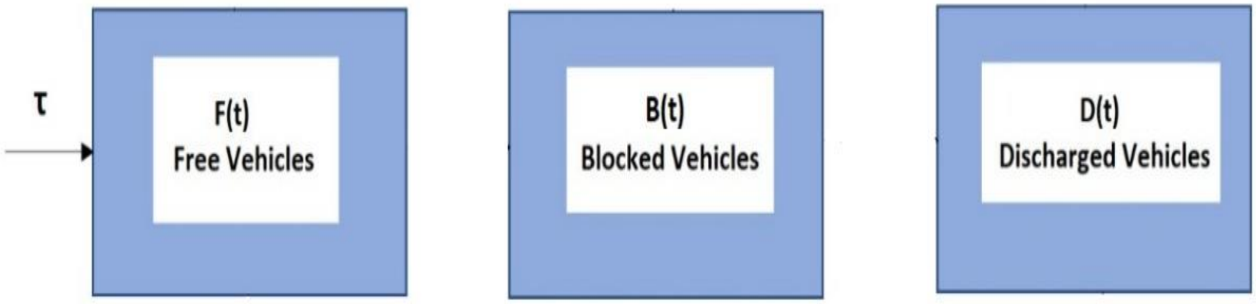
1. We have assumed that, at any time, the vehicle population is ***homogenously mixed***, that is, we assume that the blocked vehicles and the free vehicles are

always ***randomly distributed*** over the ***single*** road under consideration. Thus, every free vehicle is ***equally likely*** to get blocked, every blocked vehicle is equally likely to get discharged and every discharged vehicle is equally likely to get free. This assumption is necessary because otherwise we will have to use a ***network-based approach*** which will ***explicitly model*** the ***heterogeneity*** of the free-blocked vehicles contact pattern.

2. Rate of blocking is proportional to B i.e.: the number of blocked vehicles. This assumption is the reason why the non-linearity comes into picture. More on this in point 8.
3. Vehicle population is large enough to ***ignore random fluctuations*** between individual vehicles. This takes care of the fact that we do not end up with a ***stochastic*** model. Here we would like to point out that the ***FBDF*** model is ***deterministic***.
4. The blocking is ***propagated*** only by other blocked vehicles. We are not considering blockage due to other reasons say, a tree is obstructing the road or some construction work is going on. This means that in our initial conditions, B(0) has to be ***at least one*** for the ***blocking steady state*** to exist. This initial source of blocking could be something like, say a flat tire. But the rest of the vehicles will be blocked due to the other blocked vehicles only.
5. There are three compartments, F(t), B(t), D(t).



6. We assumed that new vehicles will be joining the road at a ***constant rate*** τ. This is essentially the ***growth rate*** of the free vehicles on the road. Also, it is assumed that the new vehicles joining the road will be free. They will not get blocked immediately. So, we are only considering the ***growth rate of free vehicles*** and ***not*** the growth rate of blocked vehicles.

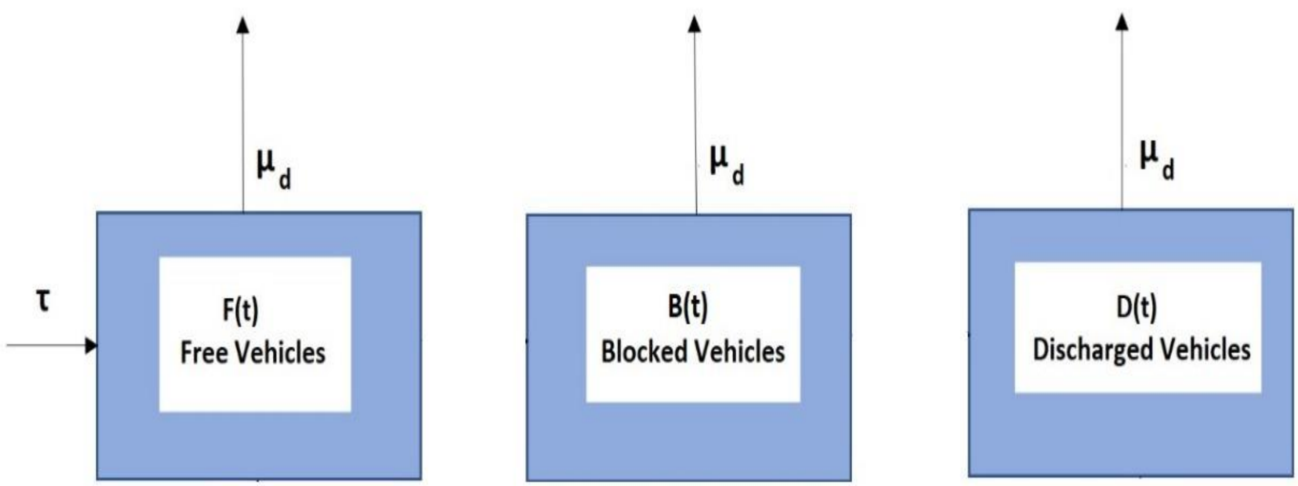


7. Vehicles may leave the present road to follow another route. The reasons could involve cutting down on time, etc. To take this possibility into account we introduced a parameter $\mu_{d.}$ This is the rate at which ***vehicles go off the road***. It is assumed that this parameter remains the same for all the three compartments. Or in other words we assumed that the rate at which free/blocked/discharged vehicles may leave the road is proportional to the number of free/blocked/discharged vehicles respectively and μ_d is the positive constant of proportionality that comes into picture. Thus, we can write,

{rate of free vehicles leaving the road} = $-\mu_d F$.

{rate of blocked vehicles leaving the road} = $-\mu_d B$.

{rate of discharged vehicles leaving the road} = $-\mu_d D$.

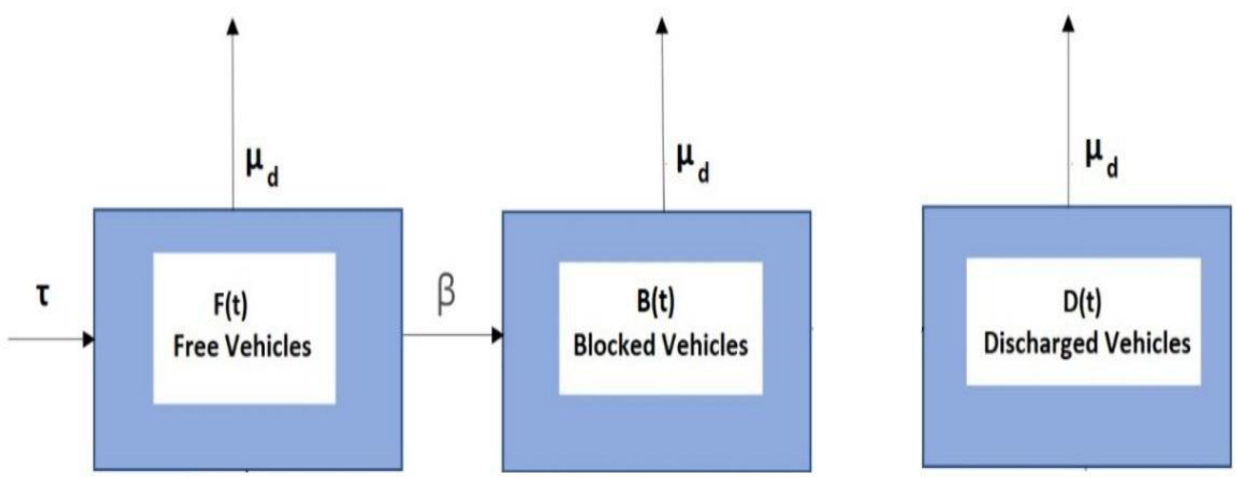


8. To model the total rate of free vehicles getting blocked, we will consider the ***free vehicles blocked by a single blocked vehicle***. It is evident that the greater number of free vehicles, the greater the increase in the number of blocked vehicles. Thus, the rate of free vehicles blocked by a single blocked vehicle will be an ***increasing function*** of the number of free vehicles. For simplicity, we assumed that this rate is directly proportional to the number of free vehicles. The number of free vehicles at time t is denoted by $F(t)$. Then the rate of free vehicles getting blocked is $\lambda(t)F(t)$. However, it is not reasonable to assume that λ is a constant, since the more blocked vehicles there are, the higher the possibility that a single free vehicle will get blocked. Thus, we can write,

{rate of free vehicles getting blocked} **=** $\lambda(t)F(t)$.

λ is the ***per-capita rate*** at which free vehicles become newly blocked. We are calling $\lambda(t)$ as the ***force of blocking***. It has the units of $time^{-1}$. We can also interpret it as the ***instantaneous probability per unit of time*** of a free vehicle becoming blocked, and thus, for very small-time intervals, $\lambda(t)\Delta(t)$ is the probability of a free vehicle getting blocked. $\lambda(t)$ depends on the current number of blocked vehicles $B(t)$ and increases as the proportion of blocked vehicles in the population increases. One model for $\lambda(t)$ could be $\lambda(t) = \beta B(t)$. Therefore, we can write,

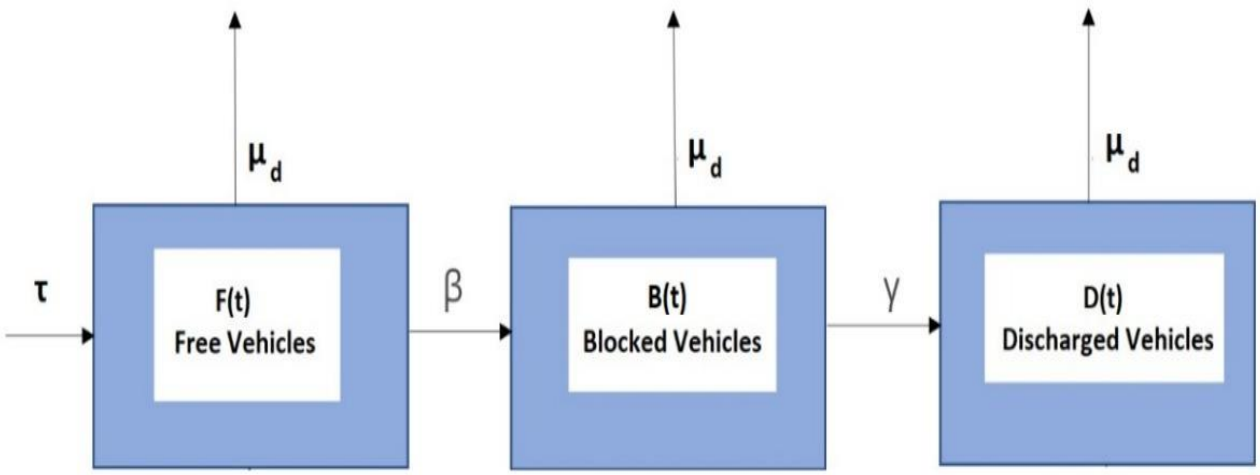
{rate of free vehicles getting blocked} **=** $\beta B(t)F(t)$.



9. The number of blocked vehicles removed from the compartment should not depend in any way upon the number of free vehicles, but only on the number of blocked vehicles. We assume that the rate at which blocked vehicles become discharged is directly proportional to the number of blocked vehicles and write,

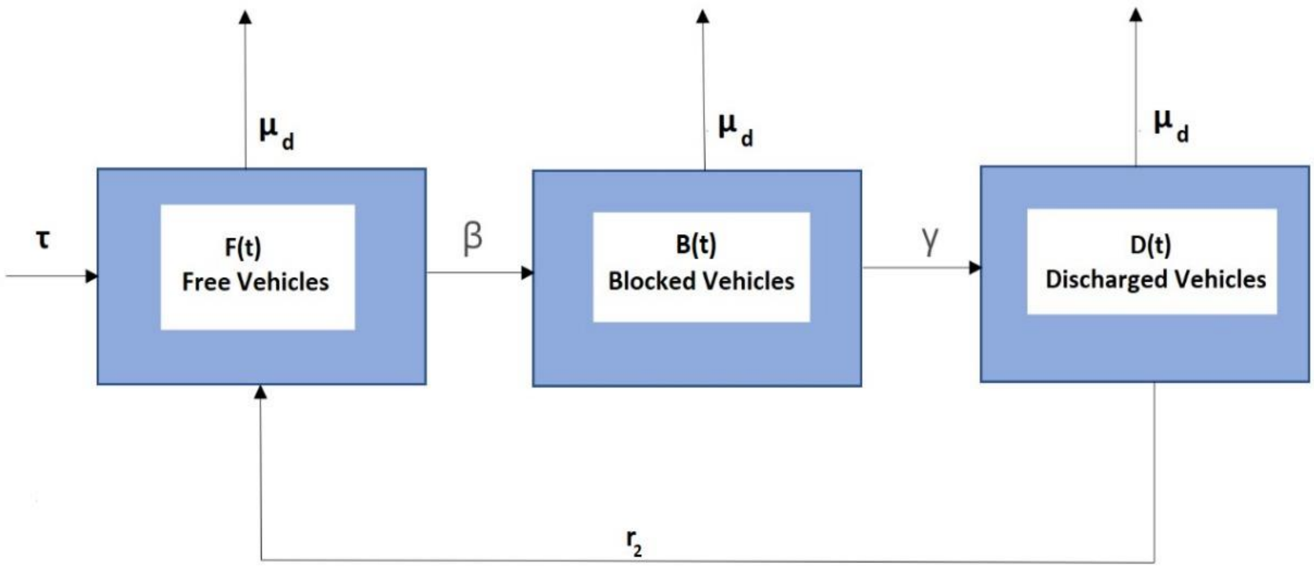
{rate of blocked vehicles getting discharged} **=** $-\gamma B(t)$.

This γ is a positive constant of proportionality, which we are calling as the ***discharge rate***, or more generally, the ***removal rate***. The rate γ is a per-capita rate. It's reciprocal,γ^{-1}, can be identified with the ***residence time in the blocked compartment***, that is, the mean time a vehicle is blocked.



10. The discharged vehicles become free vehicles at a rate r_2. We here consider the fact that the driver may take some time to realize that he is free from any kind of blockings and thus some time would lapse before the vehicle is moving freely at a speed as per the driver's wish. We assume that the rate at which discharged vehicles become free is directly proportional to the number of discharged vehicles and write,

{rate of discharged vehicles getting free} **=** $-r_2 D(t)$.



This is the complete ***block diagram*** of the ***FBDF*** model.

3. MODEL FORMULATION:

Here we introduce a new variable N(t), which denotes the ***total population size*** of the vehicles under consideration.

So, we have,

$$N(t) = F(t) + B(t) + D(t) \qquad (1)$$

PARAMETER	DESCRIPTION
τ	Rate of new vehicles joining the road. Alternatively, one could also say that it's the growth rate of free vehicles.
μ_d	Rate of vehicles leaving the present road to follow other routes. They may do this to save time, or maybe they have reached their destinations.

β	Rate at which free vehicles get blocked because of other blocked vehicles.
r_2	Rate at which discharged vehicles become free vehicles.
γ	Rate at which blocked vehicles become discharged vehicles.

Table 1: Description of Model parameters

Based on the assumptions, block diagrams and word equations given in section 2, the mathematical model describing the dynamics of population sizes of various vehicles pertaining to a traffic flow on a road can be expressed as a ***system of nonlinear differential equations as***:

$$\frac{dF}{dt} = \tau - \beta FB + r_2 D - \mu_d F \qquad (2)$$

$$\frac{dB}{dt} = \beta FB - \gamma B - \mu_d B \qquad (3)$$

$$\frac{dD}{dt} = \gamma B - \mu_d D - r_2 D \qquad (4)$$

With the initial conditions, $F(0)$, $B(0)$, and $D(0)$ being non-negative. Of course, initially, F(0) > 0, B(0) has to be ***at least 1*** and D(0) = 0. B(0) has to be at least 1 because without any blocked vehicle there is no possibility of further blockings (point 4 in section 2).

4. ANALYSIS OF THE MODEL:

In this section we have carried out the mathematical analysis of the model (2) – (4). The analysis covers the following features:

(i) Non-negativity of the solutions.

(ii) Boundedness of the solutions.

(iii) Existence and uniqueness of the solutions.

(iv) Equilibrium Points and Stability analysis.

4.1 Non-negativity of the solutions:

If the equations (2) - (4) are to have a valid physical meaning, then it is necessary that the ***state variables are non-negative***. We state below a result that ensures the non-negativeness of F(t), B(t), D(t).

Result 1: If the initial conditions $F(0)$, $B(0)$ and $D(0)$ are non-negative then the solution region $R = \{F(t), B(t), D(t)\}$ of the system of equations (2) – (4) is non-negative.

Proof:

To show that the solution region R is non-negative we have taken each equation of the dynamical system separately and have proven that each of one them has a non-negative solution.

A) Non-negativity of F(t): Consider equation (2). Without any loss of generality, we can ignore the positive terms $(\tau + r_2 D)$.

$$\Rightarrow \frac{dF}{dt} \geq -(\beta B + \mu_d)F$$

$$\Rightarrow \frac{dF}{dt} + (\beta B + \mu_d)F \geq 0$$

This ***differential inequality*** is ***first order and linear***. Thus, the integrating factor will be $e^{-\aleph}$ where, $\aleph = -\int_0^t (\beta B + \mu_d)dt$. (It will become clear later on as to why we defined $\aleph$ to be this way).

$\Rightarrow \frac{d}{dt}[F(t)e^{-\aleph}] \geq 0$ (Since the integrating factor is exponential, multiplying it on both the sides does not change the sign of the inequality).

Integrating both sides,

$$\Rightarrow \int_0^t \frac{d}{dt}[F(t)e^{-\aleph}]dt \geq 0$$

$$\Rightarrow F(t)e^{-\aleph} - F(0) \geq 0$$

$\Longrightarrow\ F(t) \geq F(0)e^{\aleph}$.

Now, F(0) is the initial population size of free vehicles and by definition is non-negative. Also, exponential functions are always are non-negative. Thus F(t) is a non-negative quantity $\forall\, t \,\epsilon\, [0,\infty)$.

B) <u>Non-negativity of B(t):</u> Consider equation (3). Without any loss of generality, we can ignore the positive term (βFB).

$$\Longrightarrow\ \frac{dB}{dt} \geq -(\gamma + \mu_d)B$$

$$\Longrightarrow\ \frac{dB}{dt} + (\gamma + \mu_d)B\ \geq 0$$

The integrating factor will be $e^{(\gamma+\mu_d)t}$. Thus,

$$\frac{d\left[B(t)e^{(\gamma+\mu_d)t}\right]}{dt} \geq 0$$

Integrating both sides,

$$\Longrightarrow\ \int_0^t \frac{d\left[B(t)e^{(\gamma+\mu_d)t}\right]}{dt} \geq 0$$

$$\Longrightarrow\ B(t)e^{(\gamma+\mu_d)t} - B(0) \geq 0$$

$$\Longrightarrow\ B(t)\ \geq\ B(0)e^{-(\gamma+\mu_d)t}$$

Now, B(0) is the initial population size of the blocked vehicles and by definition is non-negative. Also, exponential functions are always non-negative. Thus, B(t) is a non-negative quantity $\forall\, t \,\epsilon\, [0,\infty)$.

C) <u>Non-negativity of D(t):</u> Consider equation (4). Without any loss of generality, we can ignore the positive term (γB).

$$\Longrightarrow\ \frac{dD}{dt} \geq\ -(r_2 + \mu_d)D$$

$$\Longrightarrow\ \frac{dD}{dt} + (\mu_d + r_2)D\ \geq\ 0$$

The integrating factor will be $e^{(\mu_d+r_2)t}$. Thus,

$$\frac{d\left[D(t)e^{(\mu_d+r_2)t}\right]}{dt} \geq 0$$

Integrating both sides,

$$\Rightarrow \int_0^t \frac{d\left[D(t)e^{(\mu_d+r_2)t}\right]}{dt} \geq 0$$

$$\Rightarrow D(t)e^{(r_2+\mu_d)t} - D(0) \geq 0$$

$$\Rightarrow D(t) \geq D(0)e^{-(\mu_d+r_2)t}$$

Now, D(0) is the initial population size of the blocked vehicles and by definition is non-negative. Also, exponential functions are always non-negative. Thus, D(t) is a non- negative quantity $\forall\, t \in [0,\infty)$.

Thus, $R = \{F(t), B(t), D(t)$: $F(\mathrm{t}) \geq 0$, $B(t) \geq 0$, $D(t) \geq 0$, $\forall$ t $\in [0, \infty)\}$ is a set containing non-negative quantities. This essentially ensures the non-negativity of our solution region.

4.2 Boundedness of the solutions:

Result 2: The non-negative solution region $R = \{F(t), B(t), D(t)\}$ of the system of equations (2) – (4) is bounded i.e., $N(t) \leq \frac{\tau}{\mu_d}$.

Proof:

In order to make sure that the formulated model is valid and well posed, it is also necessary to show that the solution region R is bounded.

Adding (2), (3), (4), we get,

$$\frac{dF(t)}{dt} + \frac{dB(t)}{dt} + \frac{dD(t)}{dt} = -\beta FB + r_2 D + \tau - \mu_d F + \beta FB - \gamma B - \mu_d B + \gamma B - \mu_d D - r_2 D$$

After cancelling out similar terms we get,

$$\frac{d[F(t) + B(t) + D(t)]}{dt} = \tau - \mu_d[F(t) + B(t) + D(t)]$$

Using (1), this can be written as,

$$\frac{dN(t)}{dt} = \tau - \mu_d N(t)$$

$$\Longrightarrow \frac{dN(t)}{dt} + \mu_d N(t) = \tau$$

This differential equation can be solved easily. The integrating factor for this DE is:

$$e^{\mu_d t}$$

$$\Longrightarrow \frac{d[N(t)e^{\mu_d t}]}{dt} = \tau e^{\mu_d t}$$

Integrating both sides,

$$\Longrightarrow \int_0^t \frac{d[N(t)e^{\mu_d t}]}{dt} dt = \tau \int_0^t e^{\mu_d t}\, dt$$

$$\Longrightarrow N(t)e^{\mu_d t} - N(0) = \frac{\tau}{\mu_d}(e^{\mu_d t} - 1)$$

$$\Longrightarrow N(t) = \frac{\tau}{\mu_d} + \left(N(0) - \frac{\tau}{\mu_d}\right)e^{-\mu_d t}$$

Here N(0) is the initial population of all the vehicles on the road.

Now observe that, $\lim_{t\to\infty} N(t) = \frac{\tau}{\mu_d}$. This happens because the exponential factor becomes zero as t tends to infinity.

Thus, it follows that N(t) is bounded as $t \to \infty$ i.e.: $N(t) \leq \frac{\tau}{\mu_d}$, provided that the initial condition, $N(0) \leq \frac{\tau}{\mu_d}$ is satisfied. This condition ***necessitates*** that $N(t)$ is an ***increasing function of time***. How?

One can check that $\frac{dN(t)}{dt} = \left(\tau - \mu_d N(0)\right)e^{-\mu_d t}$. Applying the condition for increasing functions i.e.: $\frac{dN(t)}{dt} \geq 0$ we get that $N(0) \leq \frac{\tau}{\mu_d}$.

4.3 Existence and Uniqueness of the solutions:

Here we show that the solution for the system (2) - (4) exists and is unique. The procedure we have followed is given in Derric and Grossman (1976).

Theorem 1: Consider a system of n first order differential equations of the type $x_i' = f_i(x_1, x_2, x_3, \ldots, x_n, t)$ together with the initial conditions $x_i(t_0) = x_{i0}$ where i = 1, 2,…,n. Let R denote a region in $(n + 1)$ dimensional space among which one dimension is for t and n dimensions are for the vector x. If all the partial derivatives $\partial fi/\partial xj$ for all i, j = 1, 2,… n are continuous in $R = \{(x,t), |t - t_0| \leq a, |x - x_0| \leq b\}$ then there is exists a constant $\delta > 0$ such that there is a unique continuous vector solution x* = $[x_1(t), x_2(t), \ldots, x_n(t)]$ in the interval $|t - t_0| \leq \delta$ for the system of n equations.

Result 3: There exists a unique solution to the system of equations (2) – (4).

Proof: The system of equations (2) - (5) together with the initial conditions can be expressed as,

$$\frac{dF}{dt} = -\beta FB + r_2 D + \tau - \mu_d F \equiv f_1, \qquad F(t_0) = F_0 .$$

$$\frac{dB}{dt} = \beta FB - \gamma B - \mu_d B \equiv f_2 \qquad , \qquad B(t_0) = B_0 .$$

$$\frac{dD}{dt} = \gamma B - \mu_d D - r_2 D \equiv f3 \qquad , \qquad D(t_0) = D_0 .$$

Let $R = \{(F, B, D, t): |F - F_0| \leq a, |B - B_0| \leq b, |D - D_0| \leq c, |t - t_0| \leq d\}$

Now,

$\left\lvert\frac{\partial f_1}{\partial F}\right\rvert = \beta B + \mu_d < \infty$ $\left\lvert\frac{\partial f_1}{\partial B}\right\rvert = \beta F < \infty$ $\left\lvert\frac{\partial f_1}{\partial D}\right\rvert = r_2 < \infty$	$\left\lvert\frac{\partial f_2}{\partial F}\right\rvert = \beta B < \infty$ $\left\lvert\frac{\partial f_2}{\partial B}\right\rvert = \lvert\beta F - \gamma - \mu_d\rvert < \infty$ $\left\lvert\frac{\partial f_2}{\partial D}\right\rvert = 0 < \infty$
$\left\lvert\frac{\partial f_3}{\partial F}\right\rvert = 0 < \infty$ $\left\lvert\frac{\partial f_3}{\partial B}\right\rvert = \gamma < \infty$ $\left\lvert\frac{\partial f_3}{\partial D}\right\rvert = \mu_d + r_2 < \infty$	NOTE: Observe that all the partial derivatives are finite and continuous.

Table 2: Continuity of the partial derivatives.

In result 2, we have shown that the solutions are bounded. Therefore, by theorem1, the solution exists and is unique.

4.4 Equilibrium Points:

4.4.1 Blocking Free Equilibrium (BFE):

At the blocking free equilibrium vehicles flow freely without any kind of blockings. That is, at this equilibrium vehicles will run freely with speeds as per the wish of the drivers. $\Longrightarrow$ $B = 0.$ Under this assumption the system of equations (2) - (4) is reduced to the following form:

$$\frac{dF}{dt} = r_2 D + \tau - \mu_d F$$

$$\frac{dB}{dt} = 0$$

$$\frac{dD}{dt} = -(\mu_d + r_2)D$$

$\Longrightarrow$ $r_2 D - \mu_d F + \tau = 0$ & $-(\mu_d + r_2)D = 0$. We have to solve this system of equations to get the BFE.

The second equation is anyways equal to zero. Solving the other two using simple algebraic operations, we obtain the fixed point to be:

$$(F^{\circ}, B^{\circ}, D^{\circ}) = \left(\frac{\tau}{\mu_d}, 0, 0\right) = A_0$$

D = 0 implies that there is no car being released from blockings. And indeed, this is the case as there is no blocking here. That leaves no question of any discharge of vehicles.

4.4.2 Fixed Points (Equilibrium Points) of the System:

In order to find the fixed points of the system of equations (2) - (4), we need to set each one of them equal to 0 and solve the resulting system of equations.

$$\Longrightarrow \tau - \beta FB + r_2 D - \mu_d F = 0$$

$$\beta FB - \gamma B - \mu_d B = 0$$

$$\gamma B - \mu_d D - r_2 D = 0.$$

We have obtained one of the fixed points in the section 3.4.1. For this model we will be getting one more fixed point.

$$(F^{\infty}, B^{\infty}, D^{\infty}) = \left(\frac{\gamma+\mu_d}{\beta},\ \frac{\frac{\beta\tau}{\mu_d}-\gamma-\mu_d}{\beta(\mu_d+\gamma+r_2)},\ \frac{\gamma\left(\frac{\beta\tau}{\mu_d}-\gamma-\mu_d\right)}{\beta(\mu_d+r_2)(\mu_d+\gamma+r_2)} \right)$$

4.4.3 Local Stability Analysis:

Since now we have got our fixed points, the next step is to perform the stability analysis. Before that we will define a new quantity, ***Basic Retardation Number*** T_0.

The basic retardation number gives the ***number of secondary blockings*** resulting from a single blocked vehicle placed in a ***completely free vehicle population*** over the ***time the vehicle is blocked***.

We use T_0 to express the ***threshold*** phenomenon; assuming every vehicle in the population is initially free. The blocking takes place only if $T_0 > 1$.

In our model, in order to find T_0, we need to set $B^{\infty} > 0$. Upon performing the required the algebraic operations, one will get,

$$\frac{\beta\tau}{\mu_d(\gamma+\mu_d)} > 1$$

Here we define T_0 to be the term on the left side of the inequality.

$$\implies T_0 = \frac{\beta\tau}{\mu_d(\gamma+\mu_d)}.$$

As long as $T_0 > 1$, the blocking equilibrium will exist. Another derivation for the basic retardation number has been provided in section 4.4.4.

The second fixed point can be expressed in terms of T_0 as:

$$(F^{\infty}, B^{\infty}, D^{\infty}) = \left(\frac{\tau}{\mu_d T_0},\ \frac{(T_0-1)}{\beta\left(1+\frac{r_2}{\mu_d}\right)},\ \frac{\gamma\mu_d(T_0-1)}{\beta(\mu_d+r_2)^2} \right)$$

The Jacobian for the system of equations (2) - (4) is given by:

$$[-B\beta-\mu_d \quad -\beta F \qquad r_2]$$

$[\beta B \qquad \beta F - \gamma - \mu_d \qquad 0]$ (5)

$[0 \qquad \gamma \qquad - \mu_d - r_2]$

Let's first talk about the stability of the BFE point. In order to do so we need to evaluate the Jacobian given in (5) at the BFE and then find the eigenvalues.

The eigenvalues will be:

$$\lambda_1 = -\mu_d \, , \, \lambda_2 = \, - (\mu_d + r_2) \, , \, \lambda_3 = \frac{\beta\tau}{\mu_d} - \gamma - \mu_d \, .$$

The first two are anyways negative. Now the third one will be positive or negative depending on the parameter values. If $\lambda_3 < 0$, then we will have $\frac{\beta\tau}{\mu_d(\gamma+\mu_d)} < 1.$

Observe that the term on the left is the basic retardation number T_0.

Therefore, the BFE point is locally stable for $T_0 < 1.$

Coming to the analysis of the second fixed point, here we have taken a slightly different approach. Instead of finding the eigenvalues, we have used the ***Routh-Hurwitz criterion***. In order for the equilibria to be locally asymptotically stable, the conditions trace(A) < 0 and det(A) < 0 should be satisfied. This is true for a 3 × 3 matrix. In the case of a 2 × 2 matrix the conditions would be trace(A) < 0 and det(A) > 0. Here we have a 3 × 3 Jacobian.

Thus, after evaluating the Jacobian at the second fixed point, one can find that it's trace is:

$-\beta B^{\infty} - 2\mu_d - r_2$. Putting the value of B^{∞} into this expression we obtain,

$$-\frac{(T_0 - 1)\mu_d}{\beta(r_2 + \mu_d)} - 2\mu_d - r_2.$$

If this expression has to be less than 0 then $T_0 > 1.$

The determinant will be:

$-\beta F\big(\beta B(\mu_d + r_2)\big)$. Plugging in the values of $F^{\infty}$and B^{∞}in this expression, we get,

$\frac{\left(-\beta^2\tau(T_0-1)\,\mu_d(\mu_d+r_2)\right)}{\mu_d T_0 \beta(r_2+\mu_d)}$. After cancellation of the terms of the similar terms we are left with, $-\beta\tau\left(1-\frac{1}{T_0}\right)$.

If this expression has to be less than 0 then, $T_0 > 1$.

Therefore if $T_0 > 1$, $(F^{\infty}, B^{\infty}, D^{\infty})$ exists and is stable.

4.4.4 Derivation of Basic Retardation Number: (Using the concept of next generation matrices)

Regrouping the system of ODE's (2) - (4) into blocking class and non-blocking classes we get that (3) belongs to the blocking class and (2), (4) belong to the non-blocking class. Using only the blocking classes we can find F and ν. In our case,

$\mathrm{F} = (\beta F B)$ and $\nu = (-\gamma B - \mu_d B)$.

Say, $f_1(B) = \beta F B$ then $F = \left(\frac{\partial f_1}{\partial B}\right) = (\beta F)$. Evaluating this at the BFE, we get $F = \left(\frac{\beta\tau}{\mu_d}\right)$.

Also, $f_2(B) = -\gamma B - \mu_d B$ then $V = \left(\frac{\partial f_2}{\partial B}\right) = (-\gamma - \mu_d)$. Evaluating this at the BFE, we get the same result only as there are no terms of F, B or D. Thus $V^{-1} = \left(-\frac{1}{\gamma+\mu_d}\right)$.

As a result $FV^{-1} = -\left(\frac{\beta\tau}{\mu_d}\right)\left(\frac{1}{\gamma+\mu_d}\right) = \left(-\frac{\beta\tau}{\mu_d(\gamma+\mu d)}\right)$. The dominant eigenvalue ***(spectral radius)*** of a 1×1 matrix is the element itself and since the basic retardation number is the dominant eigenvalue of the matrix FV^{-1}, T_0 is $\frac{\beta\tau}{\mu_d(\gamma+\mu_d)}$.

It may be noted that in this section the brackets have been used to denote a 1×1 matrix.

4.4.5 Global Stability Analysis:

In order to examine the dynamics of the population one needs to perform the stability analysis. Local stability of an equilibrium point means that if you put the system somewhere nearby the point then it will itself move to the equilibrium point in some time. Global stability means that the system will come to the equilibrium point from ***any possible starting point*** (i.e.: there is no "nearby condition).

In this section we have performed the global stability analysis of the ***blocking free equilibrium point (BFE).*** As shown in section 4.2, the total population may vary with time. In the absence of the disease, the population size converges to the equilibrium $\frac{\tau}{\mu_d}$. We thus study (2), (3), (4) in the following feasible region:

$$\Omega = \left\{(F,\, B,\, D) \,\epsilon\, \mathfrak{R}_+^3 : F \geq 0, B \geq 0, D \geq 0,\, F + B + D \leq \frac{\tau}{\mu_d}\right\}.$$

The global stability of the blocking free equilibrium A_0 is easily proved by using common ***quadratic Lyapunov function*** and ***LaSalle's invariance principle.***

Result 4: If $T_0 \leq 1$, then the blocking free equilibrium A_0 is globally asymptotically stable in Ω .

Proof:

Define $V : \{(F,\, B,\, D) \,\epsilon\, \Omega\colon F \,>\, 0\} \longrightarrow \mathfrak{R}$ by $V(F,\, B,\, D) = \frac{1}{2}B^2$. Thus,

$\frac{dV}{dt} = B \cdot \frac{dB}{dt} = B^2(\beta F - \gamma - \mu_d)$. Now we know from sections 4.4.3 and 4.4.4 that $T_0 = \frac{\beta\tau}{\mu_d(\gamma+\mu_d)}$. So, $\gamma + \mu_d = \frac{\beta\tau}{\mu_d T_0}$. Using this in the above expression for $\frac{dV}{dt}$ we get that $\frac{dV}{dt} = B^2\beta\left(F - \frac{\tau}{\mu_d T_0}\right)$.

In result 2 of this paper, we stated that the solution region is bounded. Precisely speaking, the solution region is bounded by $\frac{\tau}{\mu_d}$. Since we are taking $T_0 \leq 1$, the term $\left(F - \frac{\tau}{\mu_d T_0}\right)$ will be lesser than equal to 0.

$\Longrightarrow \frac{dV}{dt} \leq 0$. Of course, here all the model parameters are positive and state variables are non-negative.

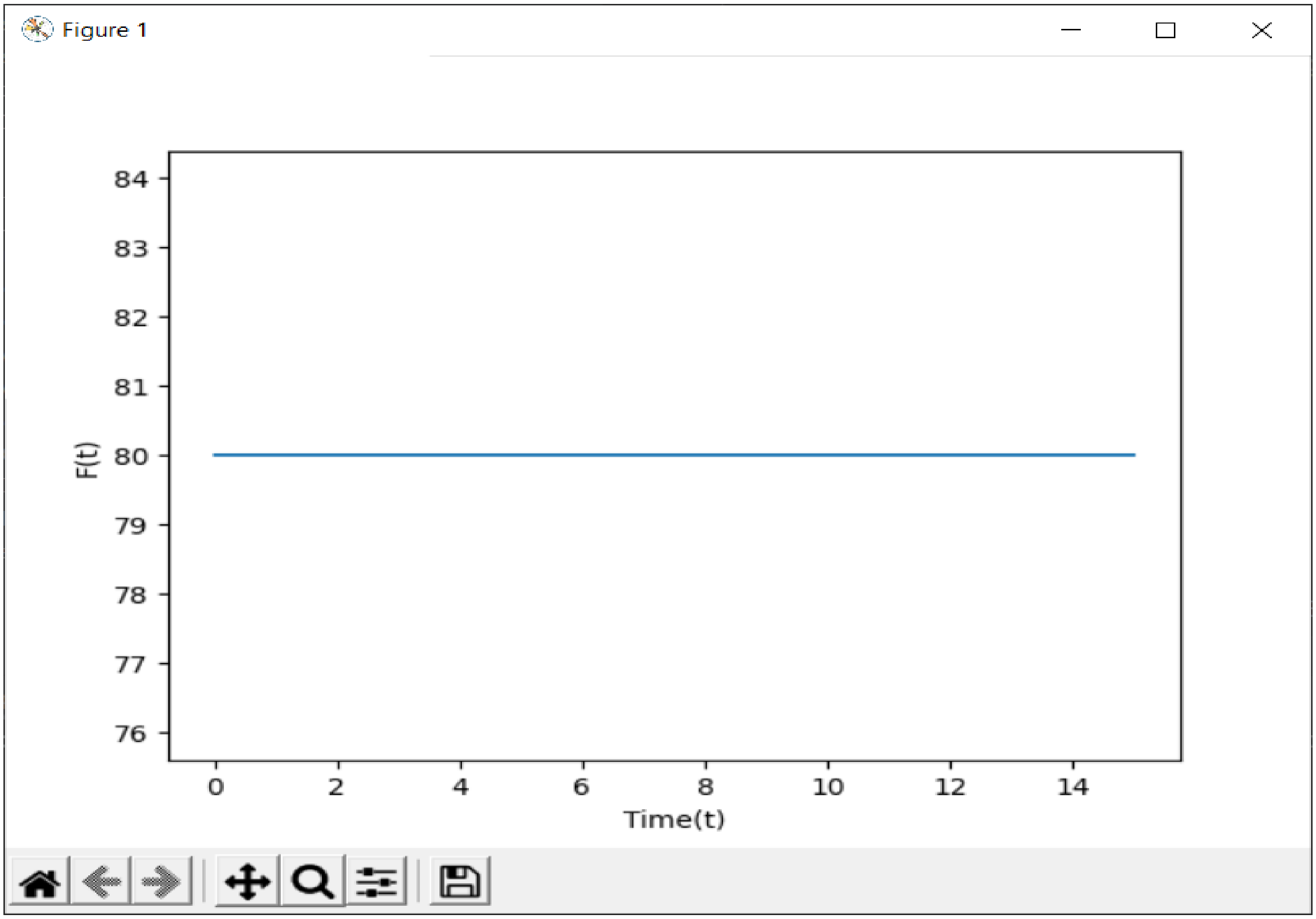
5. NUMERICAL SIMULATION:

The numerical simulation has been carried out using ***Python***. The following numerical values have been assigned to the parameters. While explaining the simulations, we have used terms like inflow, outflow and blockings. Blocking is self-explanatory while inflow refers to the parameter τ. Outflow refers to the parameter μ_d. That is, if we say that outflow is not allowed, then essentially for the sake of simulation we took $\mu_d = 0$.

Also, in all of the plots, time is shown on the x-axis and it is being measured in minutes. The y-axis labels have been provided wherever needed.

PARAMETER	**VALUE**	**SOURCE**
τ	40	Assumed
μ_d	0.1	Assumed
β	0.01	Assumed
r_2	0.5	Assumed
γ	0.7	Assumed

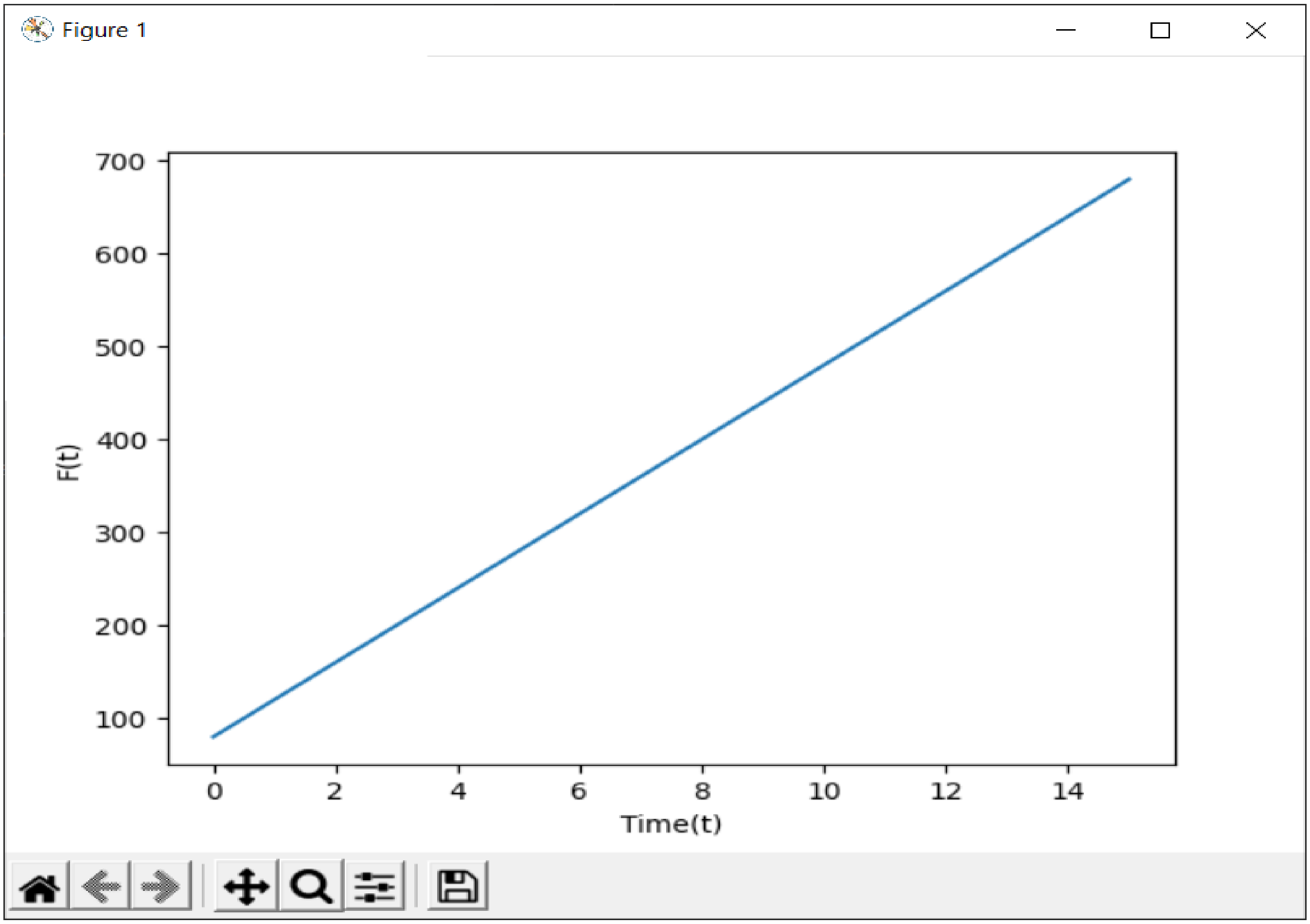
Table 3: List of values assigned to the parameters of the model.



Plot 1: F(t) vs t; Population dynamics of free vehicles in the absence of inflow, outflow and blockings.

<u>Initial Conditions:</u> (F, B, D) = (80,0,0)

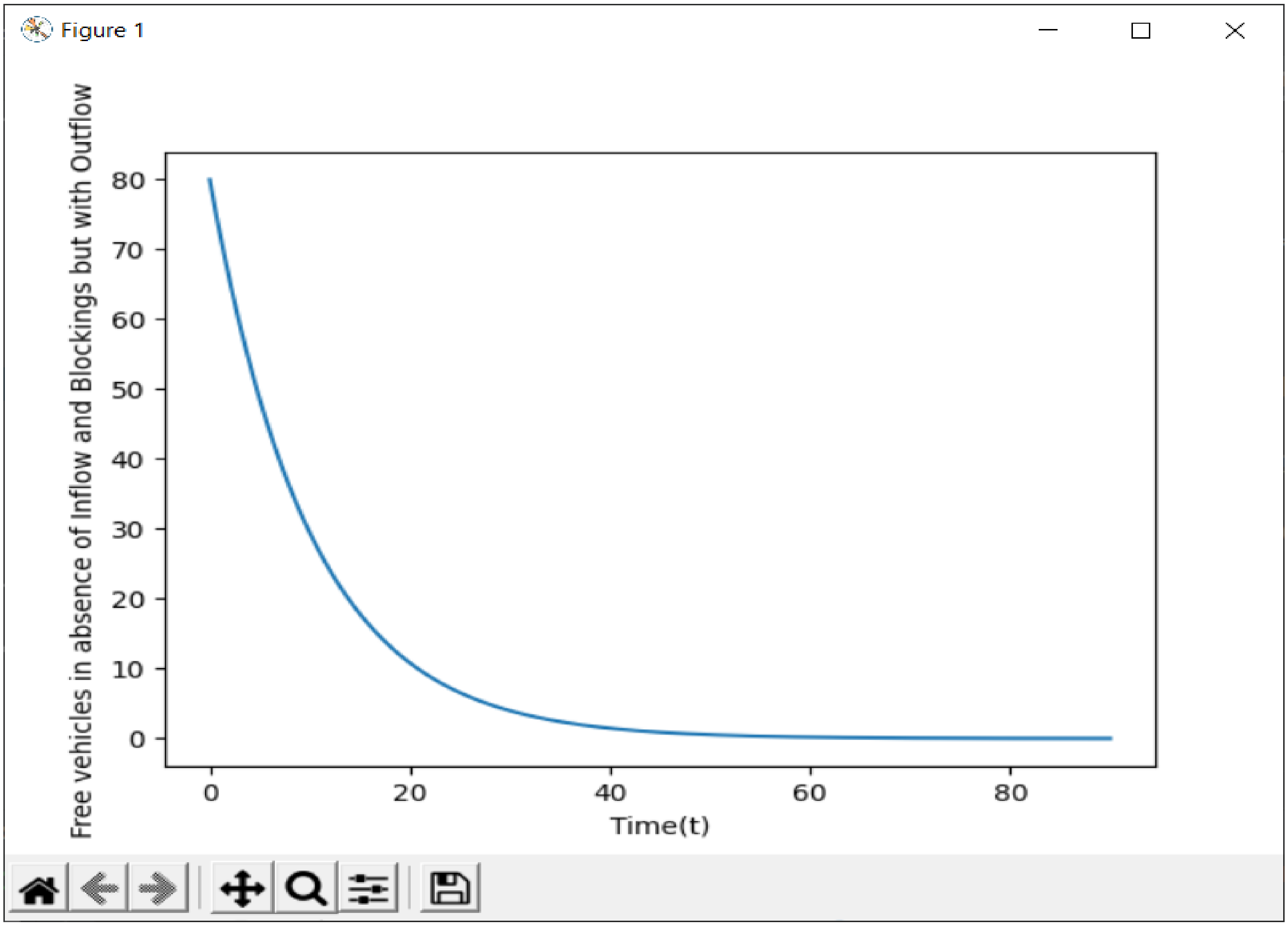
The plot shows that the number of free vehicles on the road remains the same for all future time if there is no outflow ($\mu_d = 0$), inflow ($\tau = 0$) or blocking. Leaving all the math and simulations behind for a moment, this is what any layman would also expect. If there are no vehicles joining the road and none of the initial vehicles are leaving the road then the number of vehicles on the road should remain the same. And indeed, this is what the model is predicting. Given the initial conditions, one would obtain the analytical solution to be: $F(t) = 80,\ B(t) = 0,\ D(t) = 0.$



Plot 2: F(t) vs t; Population dynamics of free vehicles with no outflow, blocking but only inflow.

Initial Conditions: (F, B, D) = (80,0,0).

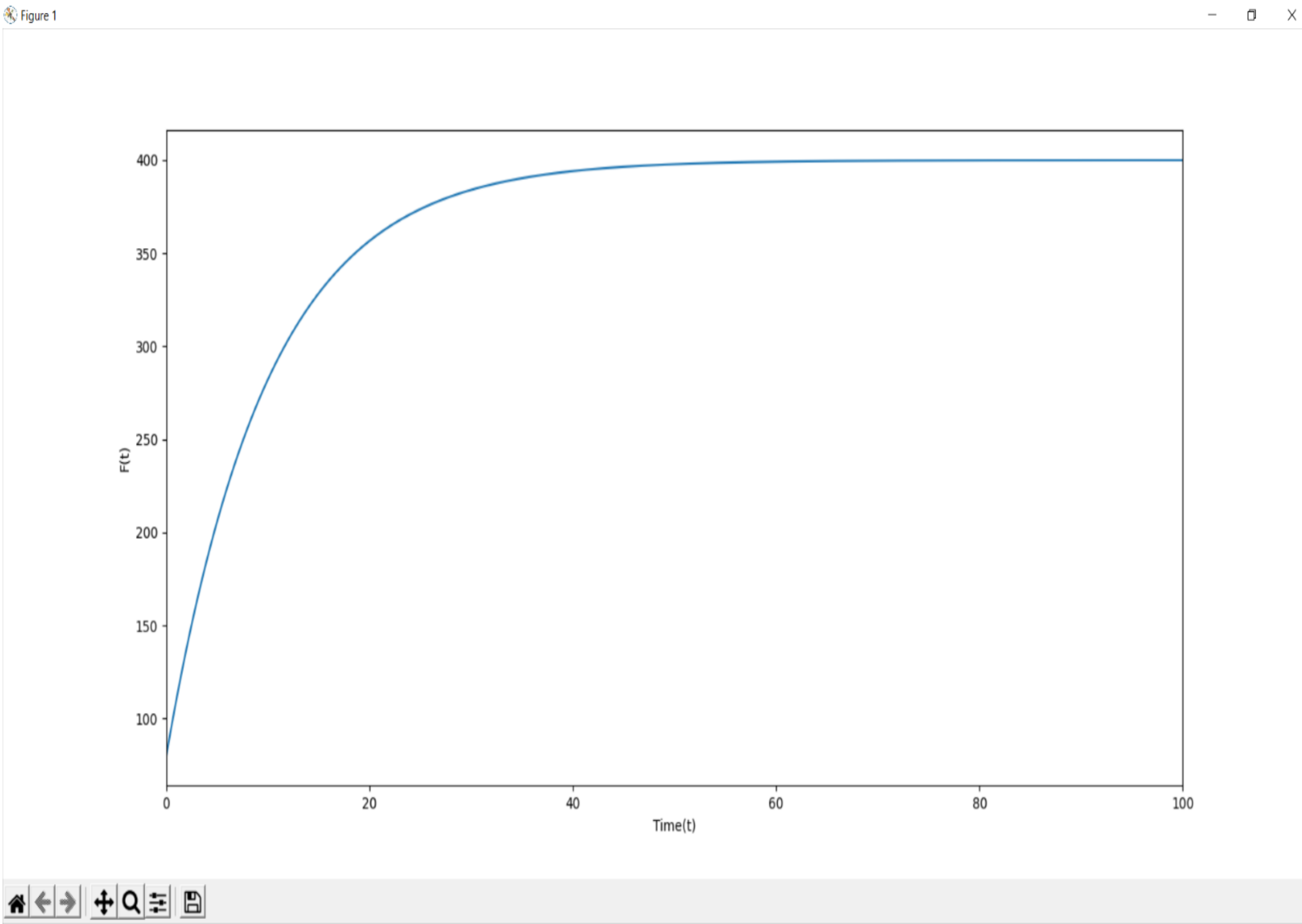
This plot shows that the population of free vehicles ***grows linearly*** from the initial size F = 80. The inflow of vehicles is allowed but there is no outflow or blocking. Thus, the number of vehicles keeps increasing as time goes on and there is no bound. This is unrealistic but considering the parameter values we took i.e.: $\tau = 40$ and $\mu_d = 0$, the upper bound will be ∞. This is a consequence of result 2 of this paper $\left(\frac{\tau}{\mu_d} \longrightarrow \infty\right)$. Also, $\tau = 40$ implies that 40 new vehicles will be joining the road every minute. With nowhere to go and no blocking as well, the number of vehicles on the road are expected to blow up as time goes on. The analytical solution given the initial conditions will be: $F(t) = 40t + 80$, $B(t) = 0$, $D(t) = 0$.



Plot 3: F(t) vs t; Population dynamics of free vehicles with no inflow no blocking but only outflow.

<u>Initial Conditions:</u> (F, B, D) = (80,0,0).

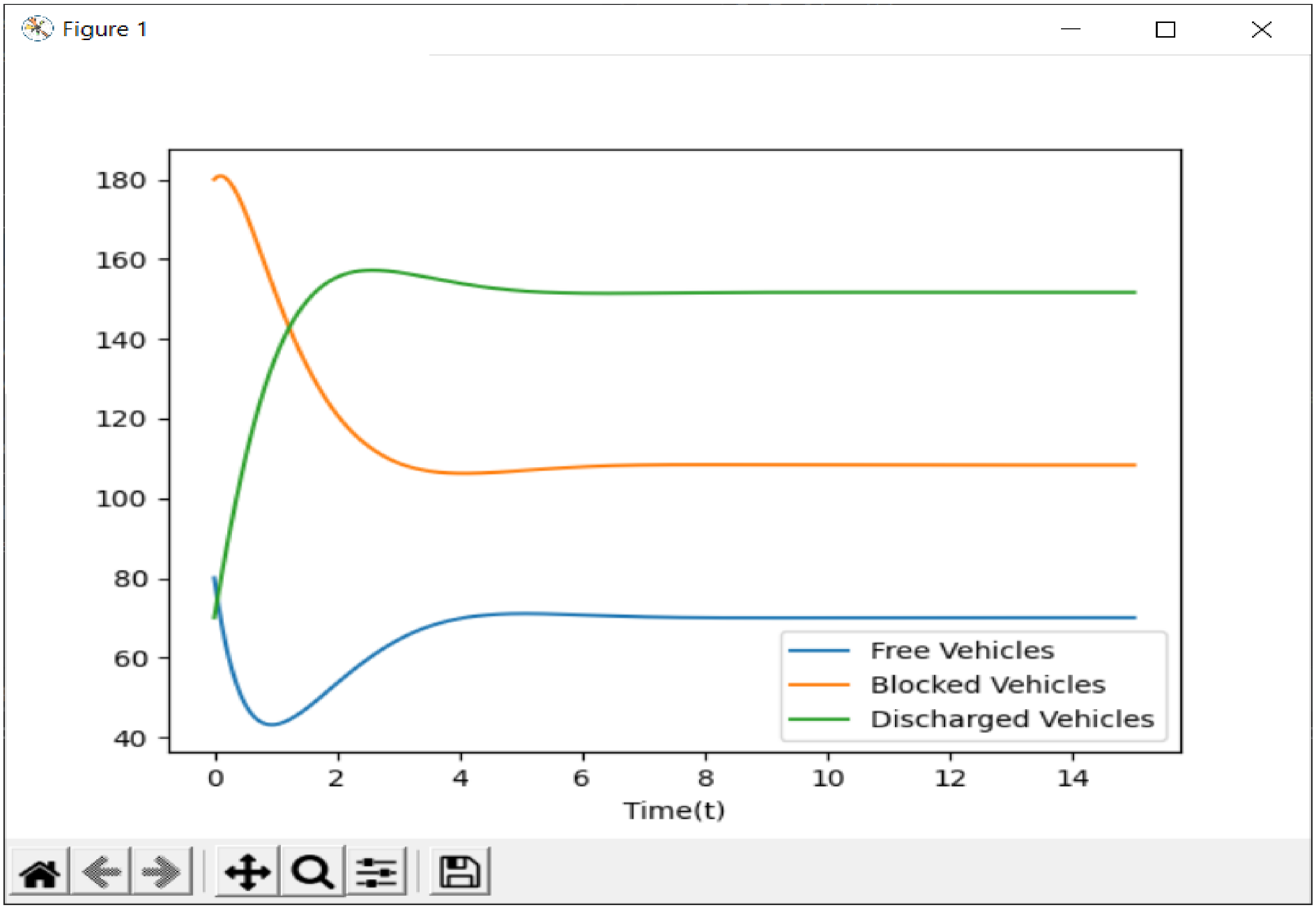
In this simulated graph we have considered only the possibility of outflow i.e.: $\mu_d = 0.1$. There is no inflow of vehicles and there is no blocking as well. In such a scenario the figure shows that the population size of the free vehicles decreases from the initial size F=80 to F=0. And this ***drop is exponential***. This means that the number of freely moving vehicles decreases exponentially till the road becomes empty of free vehicles. The analytic solution will turn out to be $F(t) = 80e^{-0.1t}$, $B(t) = 0$, $D(t) = 0$.



Plot 4: F(t) vs t; Population dynamics of free vehicles with outflow, inflow but no blocking.

Initial Conditions: (F, B, D) = (80,0,0).

In this simulated graph we have considered the possibility of outflow and inflow but no blockings. Thus $\mu_d = 0.1$ and $\tau = 40.$ In this scenario the figure shows that the number of free vehicles increases from the initial size F=80 to the upper bound F=400. The upper bound was calculated to be $\frac{\tau}{\mu_d}$ in result 2 of this paper. In this case this value is coming out to be 400. Given the initial conditions the analytic solution will turn out to be $F(t) = 400 - 320e^{-0.1t}$, $B(t) = 0$, $D(t) = 0.$

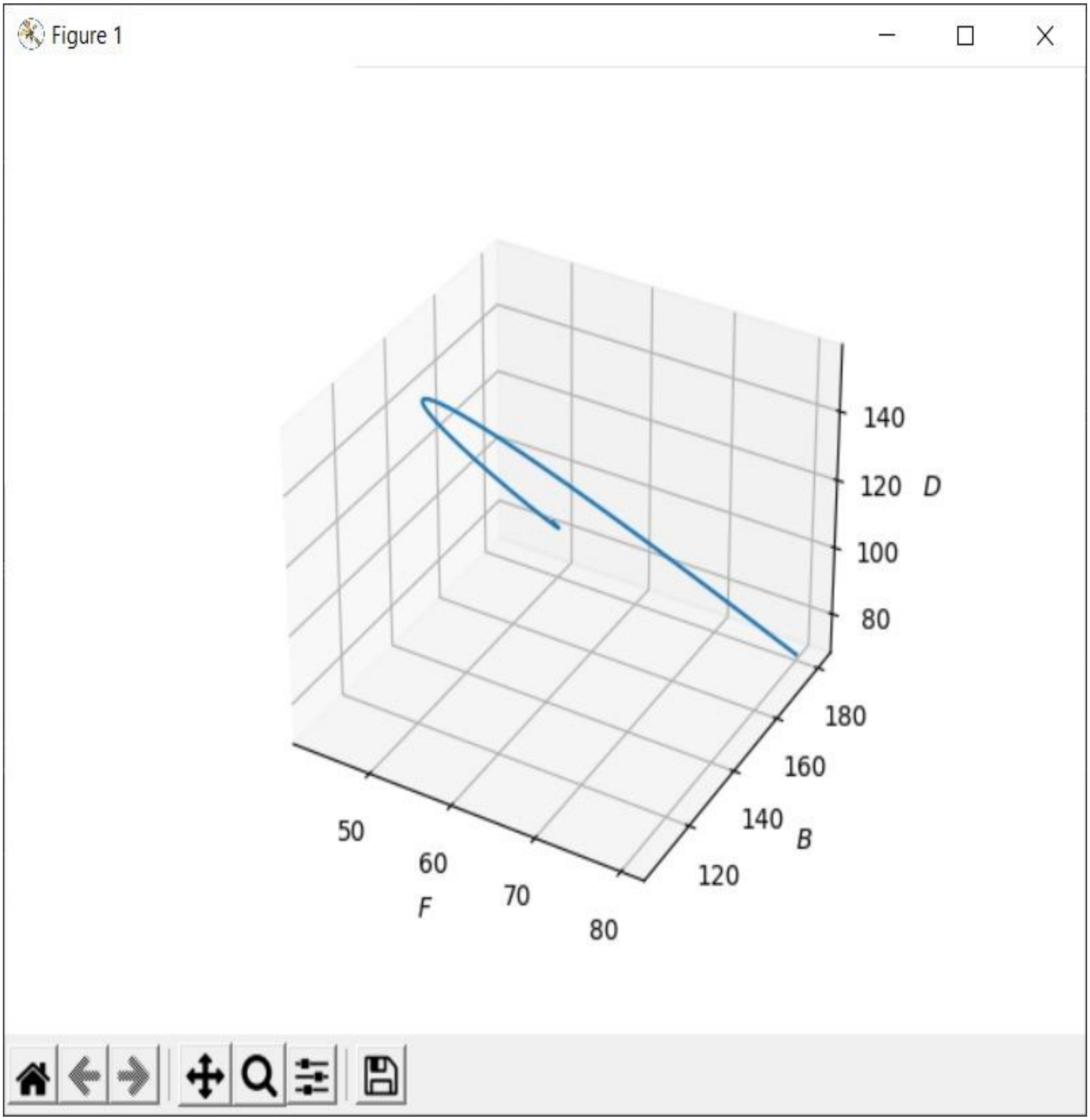


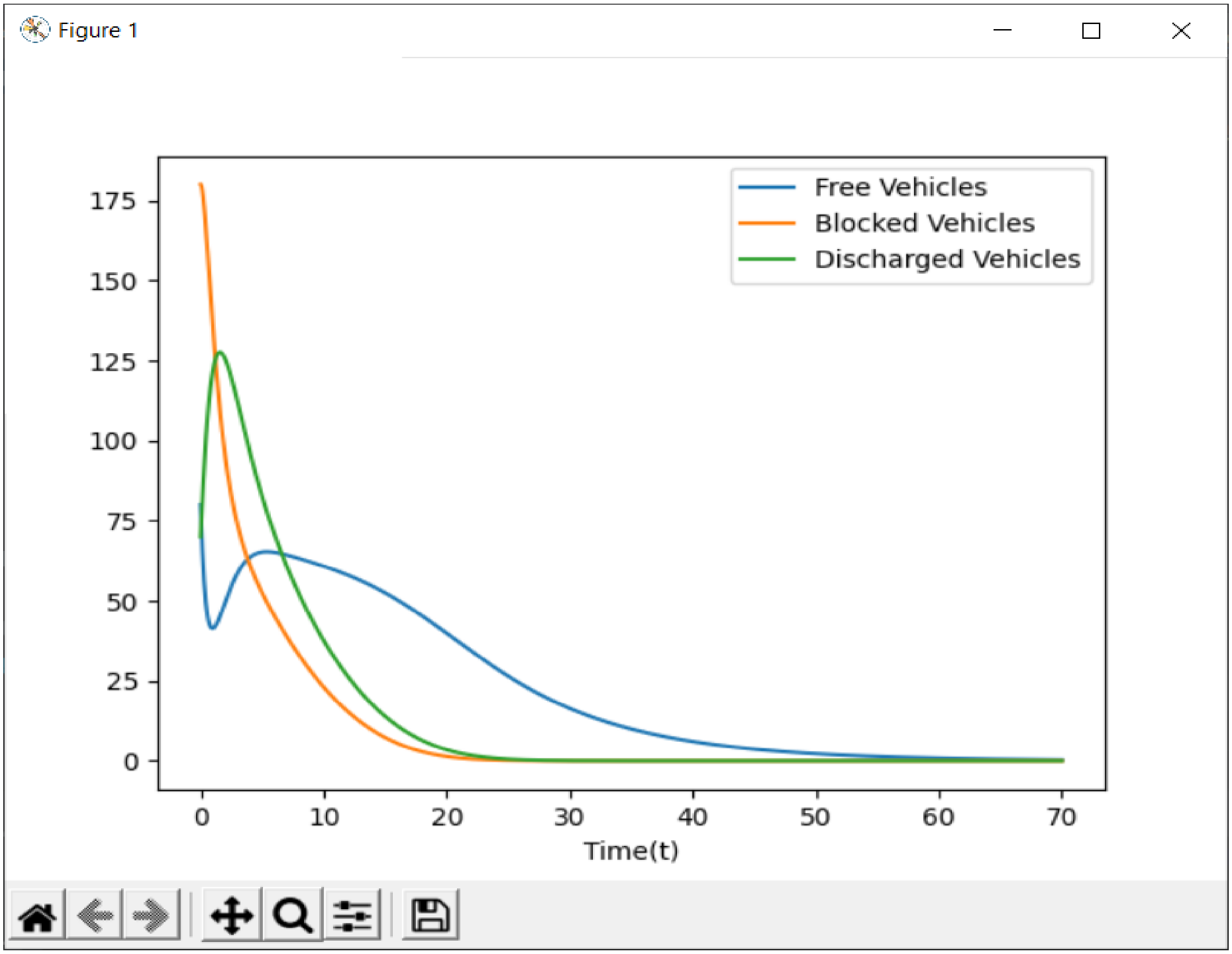
Plot 5: F(t), B(t), D(t) vs t; Population dynamics of all vehicles in the presence of blockings but no inflow or outflow.

Initial Conditions: (F, B, D) = (80,180,70).

In this simulation we can interpret the results as follows: I) The number of free vehicles initially decreases to a minimum value due to the crowdedness of the road but then starts increasing as the number of blocked vehicles gradually start to decline from their initial value of 180. II) The number of blocked vehicles start to decrease over time and as a result III) the number of discharged vehicles keeps on increasing. Finally, the number of vehicles in each category become constant. So, this road is not advisable for passengers.

Let's look at this simulation in FBD space:



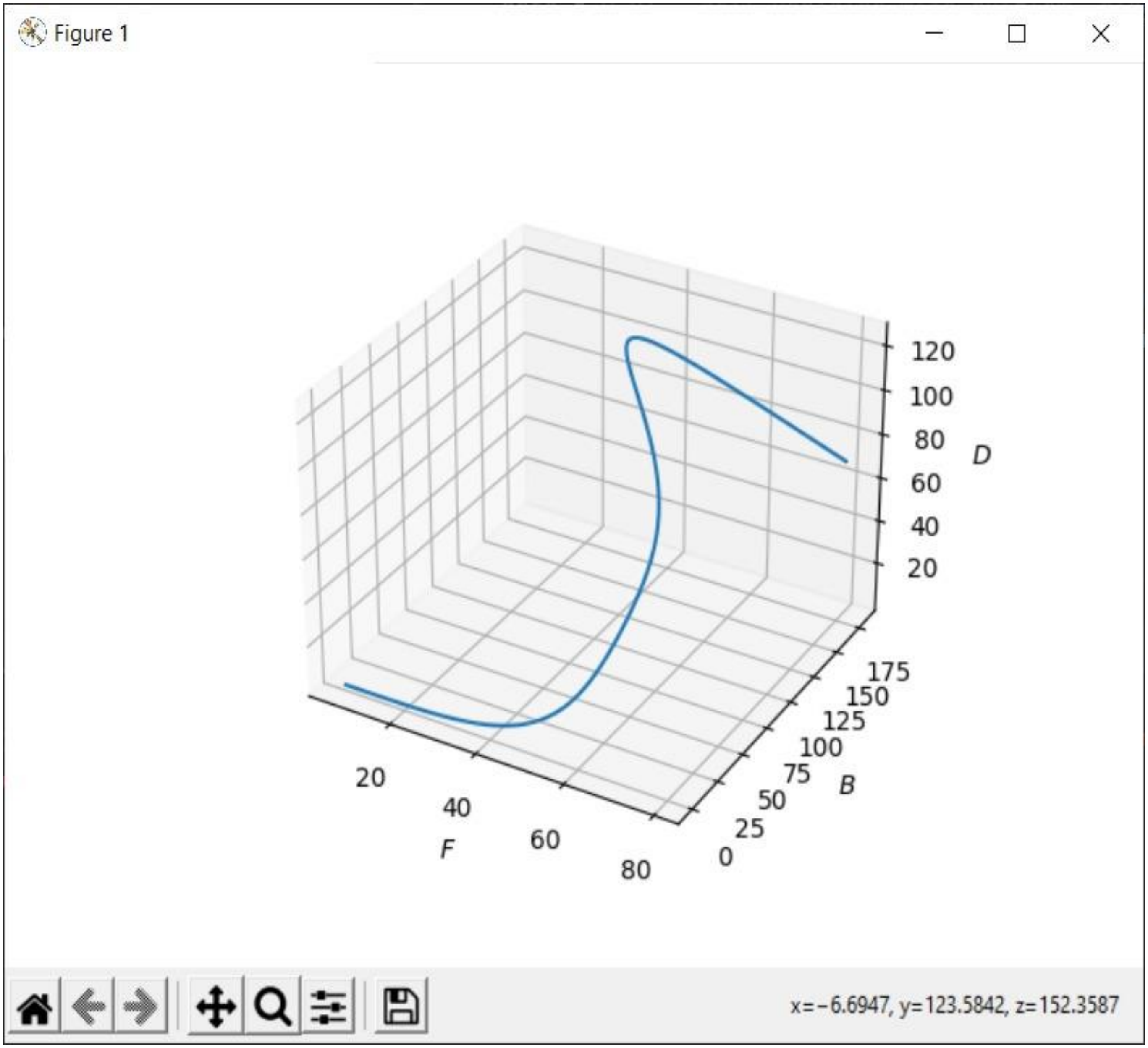


Plot 6: F(t), B(t), D(t) vs t; Population dynamics of all vehicles in the presence of outflow and blockings but no inflow.

Initial Conditions: (F, B, D) = (80,180,70).

This plot can be interpreted as follows: I) Free vehicles decrease because of the crowdedness of the road and then increase again due to increase in the number of discharged vehicles. Over time they also fall down to zero due to continuous outflow. II) Discharged vehicles increase because of a greater number of vehicles being discharged from blockings but over time decreases because of continuous outflow. III) Blocked vehicles decrease as there is discharge and outflow as well. As a result, all the vehicles move out of the road as time progresses.

Let's look at this simulation in FBD space:



6. SENSITIVITY ANALYSIS:

Sensitivity analysis is used to determine how ***"sensitive"*** a model is towards changes in the parameters of the model. It is used to discover those parameters that have a high impact on T_0. Precisely speaking, ***sensitivity indices*** allow us to measure the ***relative change*** in a variable when the parameter changes.

Here, T_0 depends on only 4 parameters. We have derived analytical expressions for it's sensitivity towards each parameter using the ***normalized forward sensitivity index as given*** by Chitnis in [1] as follows:

$$\gamma_{\beta}^{T_0} = \left(\frac{\partial T_0}{\partial \beta}\right)\left(\frac{\beta}{T_0}\right) = 1, \quad \gamma_{\tau}^{T_0} = \left(\frac{\partial T_0}{\partial \tau}\right)\left(\frac{\tau}{T_0}\right) = 1, \quad \gamma_{\mu_d}^{T_0} = \left(\frac{\partial T_0}{\partial \mu_d}\right)\left(\frac{\mu_d}{T_0}\right) = -\frac{\left(2+\frac{\gamma}{\mu_d}\right)}{\left(1+\frac{\gamma}{\mu_d}\right)},$$

$$\gamma_{\gamma}^{T_0} = \left(\frac{\partial T_0}{\partial \gamma}\right)\left(\frac{\gamma}{T_0}\right) = -\frac{\gamma}{\gamma+\mu_d} = -\frac{1}{1+\left(\frac{\mu_d}{\gamma}\right)}.$$

PARAMETER	SENSITIVITY INDEX
β	+1
τ	+1
μ_d	-1.125
γ	-0.875

Table 4: Sensitivity of T_0 against various parameters.

If the sensitivity index is negative, then the relationship between that parameter and T_0 is inversely proportional. So, a SI value of −1.125 for μ_d indicates that an increase of one unit in the value of μ_d will cause a decrease of 1.125 units in T_0. We can arrange these parameters in the decreasing order of their magnitudes as follows: μ_d , β, τ, γ. The least sensitive parameter is γ while the most sensitive parameter is μ_d.

When the problem of interest is to "***find the derivative,"*** one must be careful to distinguish which of the following two objectives one is trying to accomplish:

A) Explicitly finding a symbolic expression for the derivative, or
B) Numerically estimating the derivative by a discrete approximation, such as finite difference or a finite element method.

The previous method we used was a ***derivative based local method.*** The focus was to explicitly find the ubiquitous derivative. The method was about finding the derivative of an output variable with respect to a particular parameter, or input variable. Now, we want to perform the analysis using the methodology of ***algorithmic differentiation (AD)*** to check our results. What follows is the pseudo-code and an explanation of how this process works.

We are going to describe the main idea behind ***AD,*** in the forward mode, without discussing any of the memory allocation, coding etc., topics that are inherent to the actual execution of AD. In the jargon of AD, the standard rules of calculus would be written as the ***"tangent operations".***

Here we are going to calculate the sensitivity index of T_0 with respect to μ_d. The first step is the initialization of the parameters,

$p\_1 = 0.01; \quad // \ p1 = \beta$

$p\_2 = 40; \quad\quad // \ p2 = \tau$

$p\_3 = 0.1; \quad // \ p3 = \mu_d$

$p\_4 = 0.7; \quad // \ p4 = \gamma$

Our intention is to calculate the normalized sensitivity index with respect to the parameter μ_d, while the other parameters β, τ, γ are constant. Therefore, we initialize the derivatives as,

$dp\_1 = 0.0; \quad // \frac{\partial \beta}{\partial \mu_d} = 0$

$dp\_2 = 0.0; \quad // \frac{\partial \tau}{\partial \mu_d} = 0$

$dp\_3 = 1.0; \quad // \ \frac{\partial \mu_d}{\partial \mu_d} = 1$

$dp\_4 = 0.0; \quad // \ \frac{\partial \gamma}{\partial \mu_d} = 0$

In AD, $p1, p2, p3, p4, dp1, dp2, dp3, dp4$ are referred to as the ***input variables***. Next, we perform the forward evaluation of the intermediate variables $u1 = \beta\tau$, $u2 = \mu_d$, $u3 = \gamma + \mu_d$, $u4 = \mu_d(\gamma + \mu_d)$, along with the associated derivatives.

$$u\_1 = p\_1 \cdot p\_2 = 0.4;$$

$$du\_1 = p\_2 \cdot dp\_1 + dp\_2 \cdot p\_1 = 0.0;$$

$$u\_2 = p\_3 = 0.1;$$

$$du\_2 = dp\_3 = 1.0;$$

$$u\_3 = p\_3 + p\_4 = 0.8;$$

$$du\_3 = dp\_3 + dp\_4 = 1.0;$$

$$u\_4 = u\_2 \cdot u\_3 = 0.08;$$

$$du\_4 = u\_2 \cdot du\_3 + du\_2 \cdot u\_3 = 0.9;$$

Now forming the retardation number,

$T_0 = u5 = \frac{u1}{u4} = \frac{u1}{u2 \cdot u3} = \frac{\beta\tau}{\mu_d(\gamma+\mu_d)}$ and the derivative $\frac{\partial T_0}{\partial \mu_d}$.

$$u\_5 = \frac{u\_1}{u\_4} = 5.0;$$

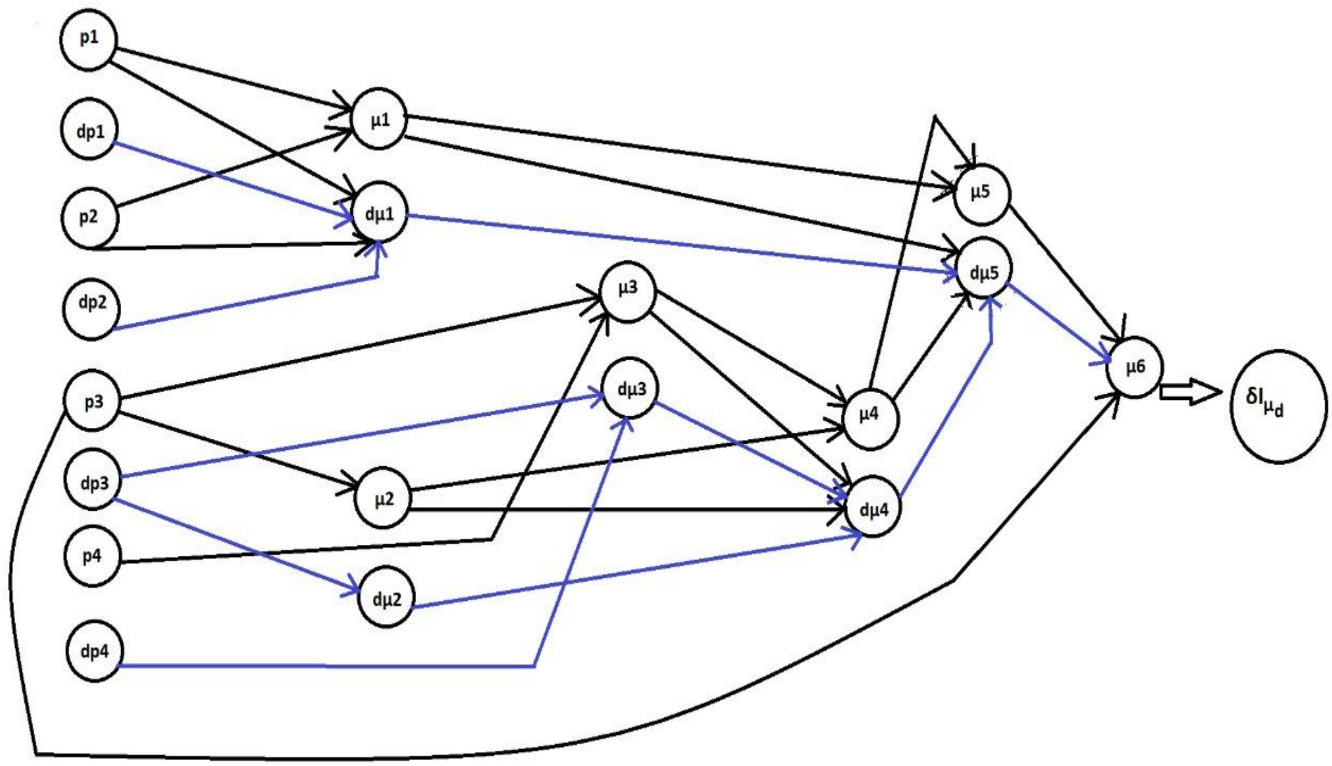
$$du\_5 = \frac{du\_1 \cdot u\_4 - u\_1 \cdot du\_4}{(u\_4)^2} = -56.25;$$

Finally, the sensitivity index is calculated, as the output variable $u6$,

$$u\_6 = \left(\frac{p\_3}{u\_5}\right) \cdot du\_5 = -1.125.$$

This is precisely what we found by direct calculations.

Evaluation of these functions can be thought of as the progression through a directed tree as shown below:



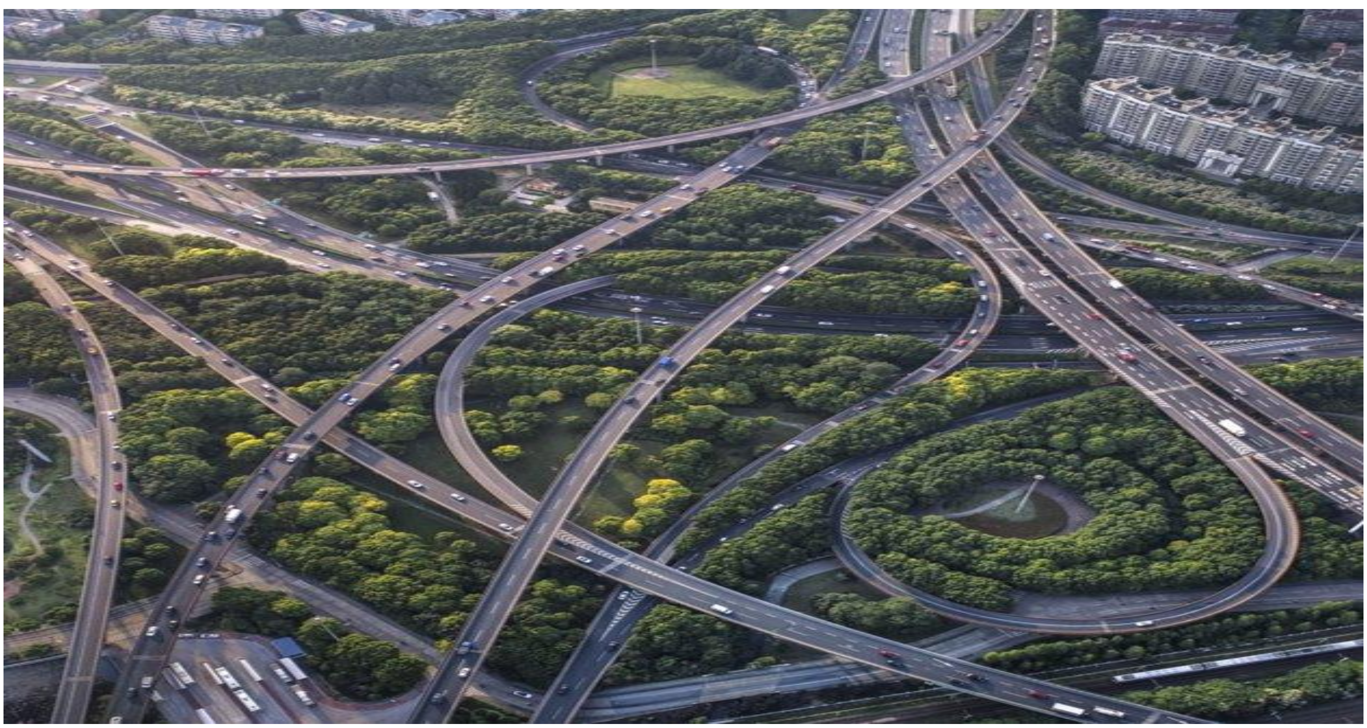
This is the ***directed computational graph*** for T_0. Black arrows denote ***forward evaluations*** and blue arrows denote ***forward derivative evaluations.*** The parameters $p1, \dots, p4$ and the derivative parameters $dp1, \dots, dp4$ are first initialized. Next the intermediate forward variables $u1, \dots, u6$ are calculated. The final step is the output variable SI_{μ_d}.

7. <u>RESULTS AND DISCUSSIONS:</u>

The focal point of the analysis carried out in this paper is that, it is for a single road. Plot 1 shows the flow of 80 vehicles for 14 minutes. In that situation there is no inflow, outflow and blocking on the road. Physically one could determine if there is going to be no inflow or outflow by analyzing the real time data for a network of roads. ***Applying*** this model to a ***number of individual roads*** can help in solving traffic flow problems.

For example, take two roads, road A and road B. The rate at which vehicles leave road A to move to road B will change the parameter τ for road B. Such intricacies will come into play when one will try applying this model to a network of roads. Also, in that case one has to ***relax the assumption*** that there will only be the growth rate of free vehicles. When considering a network of roads there could be a growth rate of blocked vehicles as well i.e.: as soon as a vehicle joins the road it will get blocked.

Plot 1 shows the flow of traffic that is quite similar to morning traffic. If the model is ***integrated*** into a system (some form of **AI,** let's call it ***the CEREBRUM***) that is ***fed*** with ***real time data*** on a ***continuous basis***, then she could advise the passenger that the road is safe to take. Similarly, for the situation in plot 3, the road becomes empty after 40 minutes. If CEREBRUM knows the initial conditions (this is where real time data comes into picture) ***with respect*** to the passenger, then she could predict and tell him/her that the road will be safe to use after some specific time. Essentially, the model can also predict how long it will take for the traffic to clear. Passengers can plan their trips according to this information. In plot 6 the road becomes empty after some 50 mins with the assumed initial conditions. That road would be safe to use for the passengers. On the other hand, in plot 5 the road will become congested in 6-8 minutes. In such a case CEREBRUM could ***notify the traffic authorities*** to intervene in order to make sure that the ***congestion is negated*** in the first place and advise all the incoming passengers to avoid that road for the next few minutes.



8. EXTENSIONS AND FUTURE WORK:

- **Bifurcations:** As of now we are thinking of using the ***Sotomayor theorem*** for bifurcations.
- **Fractional derivatives:** Apply the theory of fractional derivatives to model for different values of fractional orders. (***Caputo-Fabrizio*** fractional derivatives).
- **Parameter Estimation:** A mathematical framework to obtain estimated values of parameters and apply to real time traffic.

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