Functions.

A ***function*** $y = f(x)$ is a **relation** between x (the **independent variable**) and y (the **dependent variable,** or **function value**) where y depends on x in such a way **that for each x there is no more than one y-value**. For example, $y=3x^2$ is a function but $y^2=x$ is not (as for $x = 4$ there are two y-values, $y=\pm 2$).

The set of all x-values is called the ***domain*** and denoted $D(f)$. The set of all y-values is called ***range***, $R(f)$.

The ***graph*** of a function consists of all those points whose coordinates (x, y) satisfy the equation $y = f(x)$.

The number $c\in D(f)$ is called ***a zero*** (or a ***root***) of a function if $\underline{f(c)=0}$. In the graph, it is an **x-intercept**.

A function is said to be ***increasing*** on some segment I of the domain if for *any two* x-values x_1 and x_2 from this segment $\underline{(x_1<x_2)\Rightarrow(f(x_1)<f(x_2))}$, and ***decreasing*** if $\underline{(x_1<x_2)\Rightarrow(f(x_1)>f(x_2))}$.
Increasing and decreasing functions are called ***monotoneous***.

A function is called ***odd*** if for all $x\in D(f)$, $(-x)\in D(f)$, and $\underline{f(-x)=-f(x)}$. The graph is symmetrical with respect to the origin of coordinates.

A function is called ***even*** if for all $x\in D(f)$, $(-x)\in D(f)$, and $\underline{f(-x)=f(x)}$. The graph is symmetrical with respect to the y–axis.

A function is called ***periodic*** if there is such a positive number T that for all $x\in D(f)$, $(x+T)\in D(f)$, and $\underline{f(x+T)=f(x)}$. The least of such numbers T is called the period of the function.

If $y=f(u)$ and $u=g(x)$, then the function $\underline{y=f\left(g(x)\right)}$ is called the ***composite*** of f and g; and denoted $y=f\left(g(x)\right)$. The function g is called the inner function and f is the outer function.

Let the function $y=f(x)$ have the domain $D(f)=X$ and range $R(f)=Y$. The function $y=g(x)$ is called the ***inverse function*** of f if it has the domain Y and range X and $\underline{g\left(f(x)\right)=f\left(g(x)\right)=x}$. The inverse of $y=f(x)$ is denoted by $y=f^{-1}(x)$.

The ***graphs*** of the inverse function and the given function are ***symmetrical*** with respect to the line $y=x$ (that is, the bisector of the 1st and 3rd quadrant). Note that the function $y=f(x)$ must be monotoneous in order for the inverse function to exist.

The function is called ***explicit*** if it is written in the form $y=f(x)$ and ***implicit*** if it is written as $F(x,y)=0$.

Examples.

1. To find the domain of $y=\sqrt{4-x^2}+\dfrac{1}{\lg(x+1)}$, solve

$$\begin{cases}4-x^2\ge 0\\ x+1>0\\ \lg(x+1)\ne 0\end{cases} : \quad \begin{cases}x^2\le 4\\ x>-1\\ x+1\ne 1\end{cases} \quad \begin{cases}-2\le x\le 2\\ x>-1\\ x\ne 0\end{cases} \quad x\in(-1;0)\cup(0;2]$$

2. The function $f(x)=|3x|\cdot\sqrt{x^4+1}$ is even because $f(-x)=|3(-x)|\cdot\sqrt{(-x)^4+1}=|3x|\cdot\sqrt{x^4+1}=f(x)$.

3. The function $f(x)=x^3+8$ is neither odd nor even because $\begin{aligned}&f(-x)=(-x)^3+8=-x^3+8\ne f(x);\\ &f(-x)=-(x^3-8)\ne -f(x).\end{aligned}$

4. If $f(x)=\sqrt{x}$ and $g(x)=3x^2+1$, the composite function $f\left(g(x)\right)=\sqrt{g(x)}=\sqrt{3x^2+1}$.

5. Finding the inverse function of $y=2+\lg\dfrac{x-1}{3}$:

$$x=2+\lg\frac{y-1}{3}\Rightarrow \lg\frac{y-1}{3}=x-2\Rightarrow \frac{y-1}{3}=10^{x-2}\Rightarrow y=3\cdot 10^{x-2}+1.$$

The **basic elementary functions** are:

(1) **Power functions** $y = x^{\alpha}$ where $\alpha \in \mathbb{R}$

(2) **Exponential functions** $y = a^x$ where $a \in \mathbb{R}, a > 0, a \neq 1$

(3) **Logarithmic functions** $y = \log_a x$ where $a \in \mathbb{R}, a > 0, a \neq 1$

(4) **Trigonometric functions** $y = \sin x,\ y = \cos x,\quad y = \tan x$ (or tg x), $y = \cot x$ (or ctg x)

(5) **Inverse trigonometric functions** (cyclometric functions)
$y = \arcsin x,\ y = \arccos x,\ y = \arctan x,\ y = \operatorname{arccot} x$

By combining these functions using addition, subtraction, multiplication, division and composition, all other **elementary functions** are obtained.

Exercises.

1. (a) Given $f(x) = \dfrac{x+2}{x^2-3}$, find $f(1),\ f(a+2),\ f(a)+2,\ f\left(\dfrac{1}{b}\right), \dfrac{1}{f(b)},\ f(3c), 3f(c)$.

   (b) Given $f(x) = \dfrac{x^2-1}{2+x}$, find $f(2),\ f\left(\dfrac{1}{3}\right),\ f\left(\dfrac{1}{a}\right) f(2a),\ f(a+1),\ f(a)+1$.

2. Find the values of functions:

   (a) $\varphi(-1), \varphi(-0,001), \varphi(100)$ if $\varphi(x) = \lg\left(x^2\right)$

   (b) $F(1), F(2), F(3), F(6)$ if $F(y) = \sin\dfrac{\pi}{y} + \cos\pi y$

   (c) $\varphi(1) + f(3),\ f(9) + \varphi(0)$ if $f(t) = \log_{\frac{1}{3}} t,\ \varphi(t) = \log_3(2t+1)$

   (d) $F(0), F\left(\frac{1}{2}\right), F\left(\frac{1}{4}\right)$ if $F(s) = \arccos(2s-1)$

3. Write the composite functions :

   (a) $\varphi(\psi(x))$ and $\psi(\varphi(x))$ if $\varphi(x) = x^2$ and $\psi(x) = 3^x$

   (b) $F(x) = f(\varphi(x-1))$ if $\varphi(x) = \lg(x+1)$ and $f(x) = \arcsin x$; find $F(10), F(1), F(0.1)$.

3. The function $2x^3 - \arctan y + 1 = 0$ is given implicitly. Write it as an explicit function.

4. For each if the following equations, write at least one explicit function that satisfies it:
   (a) $x^2 - \arcsin y = 2$ (b) $x^2 + y^2 = 4$ (c) $y^2 - 4xy - x = 0$ (d) $\lg x + \lg(y+1) = 4$

5. Find the domain of the function:

(a) $y = \lg\left(x^2 - 3x - 4\right)$ (b) $y = \cos x + \dfrac{3}{x^2-4}$ (c) $y = \dfrac{3}{\sin x}$ (d) $y = \sqrt{\lg x}$ (e) $y = \arccos\dfrac{x}{2}$

(f) $y = \sqrt{9-x^2} + \dfrac{1}{x}$ (g) $y = \sqrt{x+3} + \dfrac{1}{\lg(2+x)}$ (h) $y = \arcsin\dfrac{3-2x}{5} + \sqrt{3-x}$

(i) $y = \arcsin\left(\log_2\dfrac{x}{8}\right)$ (j) $y = \log_3\left(\log_2(x+2)\right)$

6. State whether each function is odd, even, or neither odd nor even.

(a) $f(x) = x^3\sin^2 x + \dfrac{\cos x}{x}$ (b) $F(x) = \sin 2x + x^2\sqrt{x}$ (c) $y = |x| + 3^{x^2-5}$ (d) $f(x) = \lg\dfrac{3-x}{3+x}$

(e) $\varphi(t) = \dfrac{2^t + 2^{-t}}{3t} + t\cdot 2^{-t^2}$ (f) $f(x) = 2a^x + x\sin x$ $(a > 0, \text{const})$ (g) $f(s) = \sqrt{1-s^2} + \sqrt{(1-s)^2}$

7. Find the inverse function : (a) $f(x) = \log_3(3x-9)$ (b) $f(x) = 10^{2x-5}$ (c) $f(x) = \dfrac{4x+5}{3x-7}$

8. Find the inverse function and its domain and range:

   (a) $f(x) = x^2 + 2x - 3$ for $x \geq -1$ (b) $y = \arccos\dfrac{x}{3}$

   (c) $y = \sin(3x-1)$ if $x \in \left[-\dfrac{\pi}{6} + \dfrac{1}{3}; \dfrac{\pi}{6} + \dfrac{1}{3}\right]$ (d) $y = \cos^2 x - \sin^2 x$ if $x \in \left[0; \dfrac{\pi}{2}\right]$

ANSWERS

1. (a)

$$f(1)=-1.5,\ f(a+2)=\frac{a+4}{a^2+4a+1},\ f(a)+2=\frac{2a^2+a-4}{a^2-3},\ f\left(\frac{1}{b}\right)=\frac{b+2b^2}{1-3b^2},\ \frac{1}{f(b)}=\frac{b^2-3}{b+2},$$

$$f(3c)=\frac{3c+2}{9c^2-3},\ 3f(c)=\frac{3c+6}{c^2-3}$$

(b)

$$f(2)=0.75,\ f\left(\frac{1}{3}\right)=-\frac{8}{21},\ f\left(\frac{1}{a}\right)=\frac{1-a^2}{2a^2+a};\ f(2a)=\frac{4a^2-1}{2+2a},\ f(a+1)=\frac{a^2+2a}{3+a},\ f(a)+1=\frac{a^2+a+1}{2+a}$$

2.

(a) $\varphi(-1)=0,\ \varphi(-0,001)=6,\ \varphi(100)=4$

(b) $F(1)=-1,\ F(2)=2,\ F(3)=\dfrac{\sqrt{3}-2}{2},\ F(6)=1\tfrac{1}{2}$

(c) $\varphi(1)+f(3)=0,\ f(9)+\varphi(0)=-2$

(d) $F(0)=\pi,\ F\left(\tfrac{1}{2}\right)=\tfrac{\pi}{2},\ F\left(\tfrac{1}{4}\right)=\tfrac{2\pi}{3}$

3.

(a) $\varphi(\psi(x))=\left(3^x\right)^2=3^{2x}=9^x$ and $\psi(\varphi(x))=3^{x^2}$

(b) $F(x)=f\left(\varphi(x-1)\right)=\arcsin\left(\lg x\right)$; $F(10)=\tfrac{\pi}{2},\ F(1)=0,\ F(0.1)=-\tfrac{\pi}{2}$.

3. $y=\tan\left(2x^3+1\right)$

4. (a) $y=\sin\left(x^2-2\right)$ (b) $y=\sqrt{4-x^2}$ or $y=-\sqrt{4-x^2}$

(c) $y=2x+\sqrt{4x^2+x}$ or $y=2x-\sqrt{4x^2+x}$ (d) $y=\dfrac{10000-x}{x}$

5. (a) $(-\infty;-1)\cup(4;+\infty)$ (b) all x except ± 2 (c) all x except πn , $n\in\mathbb{Z}$ (d) $[1;+\infty)$ (e) $[-2;2]$
(f) $[-3;0)\cup(0;3]$ (g) $(-2;-1)\cup(-1;+\infty)$ (h) $[-1;3]$ (i) $[4;16]$ (j) $(-1;+\infty)$

6. (a) odd (b) neither odd nor even (c) even (d) even
(e) odd (f) neither odd nor even (g) neither odd nor even

7.

(a) $f^{-1}(x)=3^{x-1}+3$ (b) $f^{-1}(x)=\dfrac{\lg x+5}{2}$ (c) $f(x)=\dfrac{7x+5}{3x-4}$

8. (a) $f(x)=\sqrt{x+4}-1,\ \ x\ge -4,\ y\ge -1$ (b) $y=3\cos x,\ \ x\in\mathbb{R},\ y\in[-3;3]$

(c) $y=\dfrac{\arcsin x+1}{3},\ x\in[-1;1],\ y\in\left[-\dfrac{\pi}{6}+\dfrac{1}{3};\dfrac{\pi}{6}+\dfrac{1}{3}\right]$ (d) $y=\tfrac{1}{2}\arccos x,\ x\in[-1;1],\ y\in\left[0;\dfrac{\pi}{2}\right]$