THÈSE DE DOCTORAT
DE L'UNIVERSITÉ THE TOP UNIVERSITY

Spécialité : Subject of the Thesis
École doctorale : Doctoral School Name

Réalisée au

LABORATORY NAME

présentée par

A Good STUDENT

pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ THE TOP UNIVERSITY

Sujet de la thèse :

Un Titre Interessant

Soutenue le 8 octobre 2019 devant le jury composé de :

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Un soir, j'ai assis la Beauté sur mes genoux. – Et je l'ai trouvée amère – Et je l'ai injuriée.

— Arthur Rimbaud

Dedication
goes here.
Dedicated to
the right person.

Remerciements

Une thèse c'est un travail long et complexe, et donc on peut comprendre qu'il soit impossible de le complêter sans l'aide incommensurable de beaucoup de personnes. Peut-être que j'oublierai quelqu'un, mais si leurs noms ne sont pas dans ma tête, ils sont dans mon coeur.

Tout d'abord, je tiens à remercier vivement ma directrice de thèse, Michela, pour avoir accepté de me prendre en thèse, quand je ne savais presque rien de la théorie des cordes. J'ai beaucoup appris sous sa tutelle, en physique et en general d'un point de vue humain. Je la remercie pour toutes les discussions sans fin et pour les réponses à mes questions souvent bêtes. Je souhaite lui exprimer ma gratitude pour ça et pour avoir été un point de référence et un exemple incroyable de rigueur scientifique et personnelle.

L'amitié avec mes collègues thésards a été précieuse. Merci vraiment à tous. En particulier, sans les pauses café avec Enrico, Ruben, Sophie, Thomas, Charles, Johannés, Alessandro et Constantin, je n'aurais jamais survécu à la derniere année. Sans l'aide de ces pauvres diables que sont Hugo, Matthieu et Thomas je n'aurai jamais appris le français et jamais eu l'opportunité de devenir si fort à SportsHead. Enfin, merci à Enrico pour sa force tranquille et à Ruben pour sa folle sagesse.

Merci à tous les membres du LPTHE pour leur accueil chaleureux et pour avoir été une communauté de laquelle je me suis senti toujours partie. En particulier, je suis vraiment reconnaissant envers Isabelle et Françoise pour leur aide avec la bureaucratie qui a été pour moi un obstacle parfois plus grand que la théorie des cordes.

Je veux remercier aussi Dan Waldram, qui m'a introduit à la geometrie généralisée, et m'a donné confiance quand je n'étais rien que un étudiant effrayé, et Charles Strickland-Constable pour les discussions précieuses sur la théorie des groups et des représentations. Une mention speciale à Davide Cassani, qui n'a pas seulement été une source des profondes connaissances de la théorie des cordes, mais aussi un support pendant mes moments de difficulté.

Ma vie à Paris n'aurais pas été la même sans l'amitié de Andrea, Davide, Lara, et tous les autres "monas", merci de tout mon coeur. Merci à mes amis loins, qui sont proches dans l'esprit, parce que quand on se voit c'est comme si on avait toujours été ensemble.

Je souhaite aussi souligner que je ne serais pas ici aujourd'hui sans ma famille. Merci à ma mère et à mon père pour être toujours là, sans être intrusifs, pour m'avoir enseigné à penser de façon indépendante et libre. Merci à ma soeur Sara, pour me donner toujours une comparaison et pour me soutenir quand je me sens perdu. Merci à ma soeur Camilla, pour être joyeuse et joviale, sans jamais être superficielle, pour être sage et gentille avec tous.

Finalement, je remercie chaleureusement les membres du jury pour avoir accepté d'être présents à la soutenance de cette thèse. Merci à Anna Ceresole, Mariana Graña, Henning Samtleben, Jan Troost et Alberto Zaffaroni.

An Interesting Title

Abstract

English Abstract.

Un Titre Interessant

Résumé

Resumée en Français.

Contents

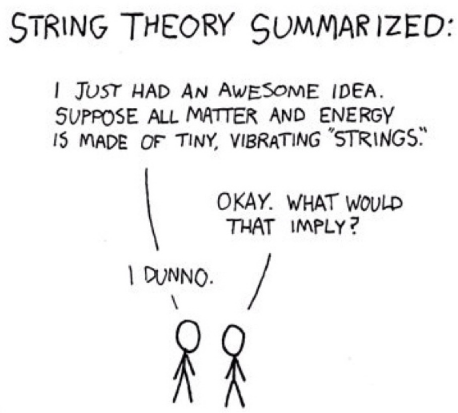
Introduction

The major success of theoretical physics in the last half century is certainly the unification of the electromagnetic, strong and weak forces in the framework of the Standard Model. This is is based on *Quantum Field Theory* – a framework where local excitations interact according to the laws of quantum mechanics and special relativity – and is, as far as we know, the most complete, experimentally verifiable theory of fundamental interactions.

What still remains an open question is how to include gravity in this picture, and this is due to the lack of renormalizability of the theory. Usually a quantum field theory has ultra-violet divergencies that are cured by adding to the action a finite number of terms canceling the divergencies and such that the physical quantities do not depend on them. This procedure does not hold for Einstein gravity and, at present, there is no satisfactory way to quantize gravity.

Among the candidates for a quantum theory of gravity, string theory is perhaps the most promising one since, at least in principle, it also leads to unification of gravity and the other forces.

String theory is based on the simple but revolutionary idea of replacing, at the fundamental scale, point-like particles with one-dimensional extended objects – the strings.



The fundamental constituents of the universe are extremely tiny vibrating strings moving in spacetime. Strings can have the topology of a segment – open strings – or that of a circle – closed strings.

The quantised string has a discrete spectrum of vibration modes, which at large distances (much larger than the characteristic string length $\ell_s = \sqrt{\alpha'}$) can be effectively interpreted as different point particles. The spectrum contains a finite number of massless states and infinite tower of massive states with masses of order $1/\sqrt{\alpha'}$. The theory was invented in another context (to describe strong interactions), but it came back to glory when it was realised that in the spectrum of the closed string there is always a massless spin 2 mode. One can identify it with the graviton, and therefore string theory automatically incorporates gravity. Since, among the massless states there can also be gauge bosons, string theory provides a framework for the unification of all fundamental forces, that reproduces, as low energy limit, Einstein theory and gauge theories.

The short distance singularities are avoided due to the extended nature of the string. A string sweeps a two-dimensional surface – the world-sheet – that is a smooth manifold and the interaction vertices are given by diagrams as in figure 1. This provides an

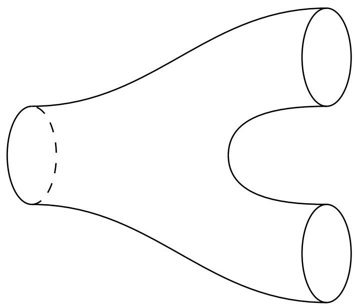


FIGURE 1: Feynman diagram of a string interaction vertex. Imposing the finite dimension of the fundamental objects, we lose the *locality* of the interactions, but we can cure the short-distance divergences. Now, Feynman diagrams are smooth 2-dimensional surfaces and the interaction vertices have been "smoothed out".

ultraviolet regularisation of the graviton scattering amplitudes, whose divergence was due to the point-wise nature of the interaction.

Generically the string spectrum contains tachyons, which can be interpreted as instabilities of the space-time. These can be avoided by imposing that the spectrum is supersymmetry and gives rise to *superstring theory.*
The evolution of the string is described by a two-dimensional conformal field theory defined on its world-sheet. To keep conformal invariance at the quantum level constrains the space-time where the string leaves to be ten-dimensional.
Taking into account all the constraints and consistency conditions, it turns out that there are only five possible superstring theories: type I, type IIA, type IIB and the two heterotic theories SO(32) and $E_8 \times E_8$, which have different field contents. In all these cases, it is possible to derive an effective quantum field theory for the massless states: these are called supergravity theories since they contain gravity and are supersymmetric. Supergravity theories have been discovered independently from string theory in the mid seventies, and only later stage it was realised that they corresponded to the low energy limit of string theory. Notice also that all supergravity theories are non renormalizable, but they make sense as effective theories of the string.
As we will discuss in detail later, a common feature of all ten-dimensional supergravity theories is the presence of higher-rank gauge fields – the NS and RR fields – which played a major role in all recent developments of string theory.

In mid 1990s the discovery of string dualities allowed to show that superstring theories are actually different formulations of the same theory, which are valid in different corners of the parameter space and are related to more fundamental theory, which is conjectured to live in eleven spacetime dimensions, and has been given the name of M-theory.[1].

[1]We know the 11-dimensional supergravity, so the corresponding high energy fundamental theory

The net of dualities involve other fundamental dynamical objects, besides strings, called branes. A Dp-brane is a solitonic entended[2] object that is charged under one of the NS and RR potentials, generalising the coupling of charged particle to the electric field. Crucially for many applications, in string theory, a D-brane also has a perturbative description as dynamical hyperplanes on which open strings can end.[3]

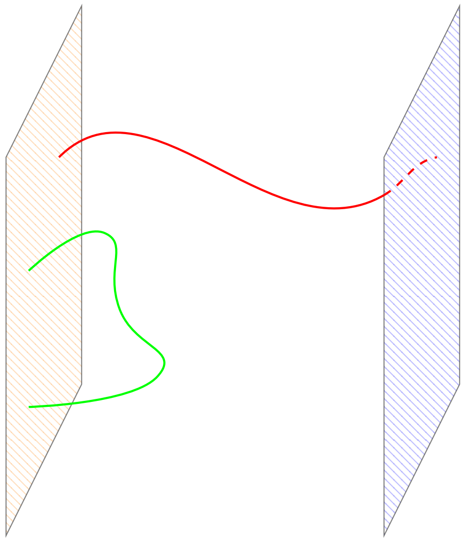


FIGURE 2: Branes are hypersurface where open strings (or other branes) can end.

Because of the different spectra, each theory has its own stable branes. In M-theory there are M2 and M5 branes. Type IIA has D_p branes with p even, while Type IIB has D_p branes with p odd. Further, both type II string theories share a brane that couples electrically with the 2-form Kalb-Ramond potential B, a 5-brane usually called NS5 brane. Branes are dynamical objects whose dynamic is governed by the fields that leaves on their world-volume. This is a very important property that lead to the discovery of the AdS/CFT correspondence and all its developments.

The introduction one-dimensional elementary objects seems to solve the conflict between general relativity and quantum field theory and to provide a framework for the unification of all fundamental forces. However, we pay a price. Space-time has extra-dimensions. This is one of the most striking predictions of string theory, and a model based on this framework with some hope to describe nature has to cope with this issue.

One option to face this concern is called *compacification*. It consists in assuming the original spacetime to have four large non-compact spacetime directions – the ones we have experience of – while the remaining ones are instead wrapped on themselves to form a very tiny compact space. The characteristic dimension of this space is very small, such to explain why we do not have access to it (there are actually some bounds on the maximal length these dimensions can have. These bounds are being updated constantly due to on going measures at LHC, see for example [1, 2]). More formally,

is identified with M-theory. We do not have a complete formulation of M-theory yet, but we know the degrees of freedom of the theory. This tells us that the fundamental dynamical ingredients of this theory are not strings, but higher dimensional objects: branes. We are going back to branes in the following.

[2]It has a (p+1)-dimensional world volume.

[3]More precisely a Dp-brane is an open string with $p+1$ *Dirichelet* boundary conditions.

we are interested in string solutions with a topology like,

$$\mathcal{M}_{10} = \mathcal{X}_4 \times M_6 \,,$$

where $\mathcal{X}$ is the non-compact *external spacetime*, and M_d is the compact *internal* manifold. Because of the small size of the extra-dimensions, the motion and excitation of the strings will look to us essentially four-dimensional. However the external dimensions are not invisible: the features of the effective four-dimensional theory – particle content, symmetries, masses, etc. – depend on the geometry of the internal space.

Supersymmetry comes naturally in string theory, A major question is what amount of supersymmetry has the low-energy effective theory. From a phenomenological point of view, the most desirable case is preserving a minimal amount of supersymmetry – $\mathcal{N} = 1$ – since this allows for chiral fermions and it is compatible with the minimal extensions of the Standard Model.[4] A residual supersymmetry is welcome also for technical purposes, since it guarantees the stability of given string solutions and also allows for simpler ways of finding string vacua.

What is very important for this thesis is that the conditions for a given string background to be supersymmetric translate into topological and differential conditions on the internal manifold and strongly constrains its geometry. The best known example is the case purely geometric compactifications of string theory to four-dimensions, where the internal manifold is constrained to be a Calabi-Yau three-fold, namely a 6-dimensional manifold with SU(3) holonomy [3]. The geometry of Calbi-Yau manifolds is well known and this makes it possible to explicitly compute the low-energy four-dimensional effective actions on these compactifications. However the theories obtained this way are not good for phenomenology since they contain many massless scalar fields – the moduli – that are not constrained by any potential.

A possible way to solve the moduli problem is to find a mechanism to generate a potential for them in the lower-dimensional action. This would have the effect of stabilise the moduli (giving them a mass and a fixed vev). A great number of results in this direction have been reached in the last twenty years, realising that it is possible to generate a non-trivial scalar potential considering more general string back-grounds where some of the RR and NS field take non-trivial expectation values. *fluxes* [4–6]. One can find some nice reviews of the subject in [7–10]. However such fields back-react on the geometry and new tools are required to analyse and study the internal compactification manifolds, which are not Calabi-Yau. The methods described in this thesis are promising tools in this direction.

Usually, a dimensionally reduced theory has an infinite number of states with higher dimensional origin. We are interested in constructing effective actions with a finite number of degrees of freedom, hence we have to give a prescription to *truncate* out some of the modes. We call *truncation ansatz* this prescription. Among all the possible truncation ansatze, a particular class is made by the ones having the property of being *consistent*. A *consistent truncation* is a choice of a finite set of modes, where the omitted ones are not sourced by the subset chosen. This is equivalent to say that the set of truncated modes has a dynamics which is not affected by the others. We are interested in such truncations since any solution of the lower dimensional theory can be uplifted to a solution of the higher dimensional one.

[4]The scale of susy breaking can tested in present day accelators. It is highly possible that the scale of susy breaking is much higher than what expected until now and this must be taken into account in string compactification models.

Reasons to study compactifications are not only phenomenological. There are important formal motivations. Many supergravity theories in various dimensions are connected by compactifications. Historically, since the birth of supergravity, dimensional reductions have been used to build lower dimensional models from the higher dimensional ones. A first milestone example is the derivation of the four-dimensional maximally supersymmetric supergravity theory from the eleven-dimensional supergravity, due to Cremmer and Julia [11].

Outline of the thesis

Extension of differential geometry turned out to be powerful tools to study generic string compactifications. One of these is Generalised Geometry and the main goal of this thesis is to study its applications to various contexts in supergravity, principally consistent truncations and brane calibrations. The thesis is organised as follows.

In the first two chapters we introduce the mathematical tools which are needed in the following. In particular, in chapter 1 are given the main definitions and examples of G-structures. The concept of torsion of a structure is analysed and we show how the torsionless conditions for some structure are equivalent to reformulate the supersymmetry conditions on the manifold. Finally, we expose some facts about the special holonomy of a manifold.

Chapter 3 is needed to give the physical environment we are moving in. We briefly review the main features of the theories of supergravity we analyse in this thesis. In addition we show how the G-structures can be fruitfully used to describe fluxless compactifications, but how they fail to capture all the informations in the presence of fluxes.

Chapter 4 is based on the exposition of the main aspect of Generalised Geometry, both complex (useful as introduction) and exceptional. We will face the generalisation of the G-structures in this context and we will build an example of how flux compactifications are elegantly encoded in this formalism. In particular, an appealing feature of this approach is that one can predicts the lower-dimensional supergravity independently of many explicit computations. We will focus on maximally supersymmetric truncations, dealing with generalised parallelisation, that we will briefly review in 4.3.5.

The core of the thesis are chapters **??** and 2. The **??** is about flux compactifications and consistent truncations. In particular, we define generalised Scherk-Schwarz reductions, and we build a concrete example of truncations of massive IIA, building also an appropriate version of Exceptional Generalised Geometry adapted to it [12].

On the other hand, the chapter 2 is centered on branes and their calibrations. We study brane probes in AdS backgrounds (both in M-theory and type IIB) and we look for the supersymmetric configurations of these probes. It is a notorious fact that these corrispond to probes wrapping (generalised) calibrated submanifolds. Using the formalism of generalised geometry and G-structures we show how the integrability conditions on generalised HV structures [13] are equivalent to have supersymmetric branes in AdS backgrounds. In other words, one can state that HV structures provide generalised calibrations of branes in such backgrounds.

The thesis is completed by two appendices collecting conventions and technical remarks in exceptional generalised geometry.

The core of this thesis is built on the following works

[12] D. Cassani, O. de Felice, M. Petrini, C. Strickland-Constable, and D. Waldram, *Exceptional generalised geometry for massive IIA and consistent reductions*, JHEP, 08 (2016), p. 074

[14] O. de Felice and J. Geipel, *Generalised Calibrations in AdS backgrounds from Exceptional Sasaki-Einstein Structures*, 1704.05949, (2017)

Other projects still in progress are

[15] O. de Felice, P. Ntokos, and M. Petrini, *Leigh-Strassler exactly marginal deformations from Exceptional Structures*, to appear, (2017)

[16] A. Ashmore, O. de Felice, and P. Ntokos, *Exactly marginal deformations of massive IIA Chern-Simons dual*, to appear, (2017)

[17] D. Cassani, O. de Felice, M. Petrini, C. Strickland-Constable, and D. Waldram, *Systematics of consistent truncations from generalised geometry*, to appear, (2017)

1

An Important Chapter

1.1 Introduction and Motivations

2

Generalised Calibrations in AdS backgrounds

2.1 Introduction and Motivations

In this chapter, based mainly on [14], we are interested in investigating the relation between the Exceptional Sasaki-Einstein structures [13] presented in section 4.3.6 and generalised brane calibrations, in $\mathrm{AdS}_5 \times M_5$ backgrounds in type IIB and in $\mathrm{AdS}_5 \times M_6$ and $\mathrm{AdS}_4 \times M_7$ compactifications in M-theory.

Also in this case it is useful to analyse the problem through the lens of G-structures. We have mentioned how requiring the AdS background to be supersymmetric is equivalent to put integrability conditions on HV structures, that in that case take the name of *Exceptional Sasaki-Einstein structures* [13], or of *Exceptional Calabi-Yau spaces (ECY)* for compactifications to Minkowski spacetimes [18].

G-structures also appear naturally in defining calibration forms on the compactification manifolds.

A p-form ϕ on a d-dimensional manifold M $(d > p)$ is a *calibration* if and only if it is closed, *i.e.* $\mathrm{d}\phi = 0$, and its pull-back to any tangent p-plane $\mathcal{S}$ satisfies the inequality

$$P_{\mathcal{S}}[\phi] \leq \mathrm{vol}_{\mathcal{S}}\,, \tag{2.1.1}$$

where $\mathrm{vol}_{\mathcal{S}}$ is the volume form on the plane $\mathcal{S}$ induced from the metric on M [19].

The ordering relation in (2.1.1) has to be read as $P_{\mathcal{S}}[\phi] = \alpha\,\mathrm{vol}_{\mathcal{S}}$, with $\alpha \in \mathbb{R}^+$ and $\alpha \leq 1$. For the supersymmetric backgrounds relevant in string and M-theory the calibration forms can be written as bilinears in the supersymmetry Killing spinors. For instance, on Calabi-Yau manifolds there are two types of calibration forms, which corresponds to products of the Kähler form and to the real part of the holomorphic form on the Calabi-Yau.

A p-dimensional submanifold Σ_p is called *calibrated* if it saturates the condition (2.1.1) at each point: $P_{\Sigma_p}[\phi] = \mathrm{vol}_{\Sigma_p}$. One can show that a calibrated submanifold minimises the volume in its homology class. Indeed, given another submanifold Σ', such that

$\Sigma - \Sigma' = \partial B$ is the boundary of a $p+1$-dimensional manifold B, one has (see e.g. [19])

$$\mathrm{Vol}(\Sigma') = \int_{\Sigma'} \mathrm{vol}_{\Sigma'} \geq \int_{\Sigma} P_{\Sigma}[\phi] + \int_{\partial B} P_{\partial B}[\phi] = \int_{\Sigma} \mathrm{vol}_{\Sigma} + \int_{B} \mathrm{d}P_B[\phi] = \mathrm{Vol}(\Sigma)\,,$$

where we used the definition of calibration form, Stokes theorem and the fact that Σ saturates the inequality (2.1.1). For a nice review on these arguments, one can refer, for instance, to [20].

Calibrations are useful tools in string theory because they provide a classification of supersymmetric branes in a given background. In a purely geometric background (no fluxes) supersymmetric branes wrap calibrated submanifolds, so that they minimise their volume [21–24]. The calibration form is constructed as a bilinear in the Killing spinors of the background geometry, and its closure follows from the Killing spinor equations of the background.

In the more general case of a background with non-trivial fluxes supersymmetric branes are associated with *generalised calibrations*. Since the branes couple with the background fluxes, they do not correspond to minimal volume submanifolds but to configurations that minimise the energy. As in the fluxless case the generalised calibration form is related to the Killing spinors of the background [25–34]. Also in these cases the calibration forms can be written as bilinears in the Killing spinors and the closure of the generalised calibration form can then be deduced from the Killing spinor equations of the supersymmetric background [26, 31, 32, 35].

In section 4.3.6 we have seen how the exceptional HV structure contains a generalised vector K that generalises the Reeb vector field and the contact structure of usual Sasakian geometry. For this reason, it is believed to encode information on brane configurations and conformal dimensions of chiral operators, as the contact structure does in [36]. In particular, in [13] the form parts of the generalised vector K were conjectured to describe generalised calibrations for brane configurations dual to barionic operators in the dual gauge theory. The aim of this chapter is to prove this conjecture and to show that the vector structure is indeed associated to generalised calibrations.

In the following, we focus on the calibrations forms associated to branes wrapping cycles in the internal manifolds and that are point-like in the AdS space. We show that for these configurations the general expression for the calibration forms that can be constructed using κ-symmetry can be expressed in terms of the generalised Killing vector K defining the Exceptional Sasaki-Einstein structure and that the closure of the calibration forms is given by the integrability (more precisely the L_K condition) of the ESE structure. Our results proves the conjecture appeared in [13], that the generalised Killing vector is a generalised calibration. We also partially discuss other brane configurations that are calibrated by the vector K.

The analysis in this chapter is far from being complete. For instance we did not fully study the calibration forms for branes with world-volumes spanning different directions in the AdS space. These should be related to components of the hypermultiplet structures and their closure to the moment map conditions. It would also be interesting to perform a similar analysis for compactifications to Minkowski space where the relevant structures are Generalised Calabi-Yau's [18]. We leave this analysis for future work.

Conventions for Clifford algebras and bilinears of spinor notations are relegated to appendix A.

2.2 Generalised calibrations in M-theory

The aim of this section is to study calibrations for supersymmetric brane configurations of M-theory on AdS backgrounds of the form

$$\mathrm{d}s^2 = e^{2\Delta}\mathrm{d}s^2(X_D) + \mathrm{d}s^2(M_d)\,,$$

in terms of exceptional geometry. AdS calibrations have been thoroughly discussed in the literature [25, 26, 31, 37] and led to the notion of generalised calibration. In this section we will interpret these calibrations in terms of the Exceptional Sasaki-Einstein structures describing the AdS background.

Supersymmetry static M-theory backgrounds have been studied in [38]. As we have seen in **??**, a supersymmetric background admits a Majorana Killing spinor ε satisfying,

$$\nabla_M\varepsilon + \frac{1}{288}\Big[\Gamma_M{}^{NPQR} - 8\delta_M{}^N\Gamma^{PQR}\Big]G_{NPQR}\,\varepsilon = 0\,, \tag{2.2.1}$$

where $M, N, \ldots = 0, 1, \ldots, 10$, $G = \mathrm{d}A$ is the four-form field strength and the Gamma matrices are the Clifford algebra elements in 11 dimensions. The four-form G and the metric g satisfy the relative equations of motion

$$R_{MN} - \frac{1}{12}\left(G_{MPQR}G_N{}^{PQR} - \frac{1}{12}g_{MN}G^2\right) = 0\,, \tag{2.2.2}$$

$$\mathrm{d}\star G + \frac{1}{2}G\wedge G = 0\,. \tag{2.2.3}$$

The Killing spinor can then be used to build one-, two- and five-forms

$$\mathcal{K}_M = \bar\varepsilon\Gamma_M\varepsilon\,, \tag{2.2.4a}$$

$$\omega_{MN} = \bar\varepsilon\Gamma_{MN}\varepsilon\,, \tag{2.2.4b}$$

$$\Sigma_{MNPQR} = \bar\varepsilon\Gamma_{MNPQR}\varepsilon\,, \tag{2.2.4c}$$

and the supersymmetry conditions imply that

$$\mathrm{d}\mathcal{K} = \tfrac{2}{3}\iota_\omega G + \tfrac{1}{3}\iota_\Sigma\star G\,, \tag{2.2.5}$$

$$\mathrm{d}\omega = \iota_{\mathcal{K}}G\,, \tag{2.2.6}$$

$$\mathrm{d}\Sigma = \iota_{\mathcal{K}}\star G - \omega\wedge G\,. \tag{2.2.7}$$

Supersymmetry also implies that the vector $\hat{\mathcal{K}}^M$ dual to the one-form (2.2.4a) is a Killing vector, *i.e.*

$$\mathcal{L}_{\hat{\mathcal{K}}}g = 0\,, \qquad \mathcal{L}_{\hat{\mathcal{K}}}G = 0\,. \tag{2.2.8}$$

The vector $\hat{\mathcal{K}}^M$ can be either null or time-like, and for the backgrounds of interest here it is time-like[1].

Now let us focus on AdS backgrounds,

$$\mathrm{d}s^2 = e^{2\Delta}\mathrm{d}s^2(\mathrm{AdS}) + \mathrm{d}s^2(M)\,, \tag{2.2.9}$$

where Δ is a real function on M, the warp factor.

[1] In this case the forms $\mathcal{K}$, ω and Σ define an $SU(5)$ structure in 11 dimensions.

As usual, to construct the generalised calibrations for M-branes we can make use of κ-symmetry. A supersymmetric brane satisfies the bound

$$\hat{\Gamma}\varepsilon = \varepsilon\,, \tag{2.2.10}$$

where ε is the background Killing spinor and the κ-symmetry operator $\hat{\Gamma}$ depends on the type of brane. For an M5-brane this is defined as [31, 39],

$$\hat{\Gamma} = \frac{1}{L_{DBI}}\Gamma_0\left[\frac{1}{4}\Gamma^{\alpha}(\tilde{H}\lrcorner H)_{\alpha} + \frac{1}{2}\Gamma^{\alpha\beta}\tilde{H}_{\alpha\beta} + \frac{1}{5!}\Gamma^{\alpha_1\ldots\alpha_5}\epsilon_{\alpha_1\ldots\alpha_5}\right], \tag{2.2.11}$$

where $H = dB + P[A]$ is the world-volume three-form, $\tilde{H}$ is its world-space dual [40–42][2]and L_{DBI} is the Dirac-Born-Infeld Lagrangian for the M5 brane,

$$L_{DBI} = -\sqrt{-\det(P[g] + \tilde{H})}\,. \tag{2.2.13}$$

Per usual, $P[\bullet]$ denotes the pull-back on the M5 world-volume and we defined

$$\Gamma_{\alpha_1\ldots\alpha_s} = \Gamma_{M_1\ldots M_s}\partial_{\alpha_1}X^{M_1}\ldots\partial_{\alpha_s}X^{M_s}\,. \tag{2.2.14}$$

As discussed in [31, 39], the κ-symmetry condition (2.2.10) can be used to derive the following bound [43],

$$\|\varepsilon\|^2 L_{DBI}\ \ \mathrm{vol}_5 \geq \left[\frac{1}{2}P[\iota_{\hat{\mathcal{K}}}H]\wedge H + P[\omega]\wedge H + P[\Sigma]\right], \tag{2.2.15}$$

where K, Σ and ω are defined in (2.2.4). To satisfy the bound one has to take into account that the space is Anti-de Sitter. As discussed in [39], the norm $\varepsilon^{\dagger}\varepsilon$ depends on the AdS coordinates and the bound is saturated when the M5 brane sits at the center of AdS. Explicitly, the metric of AdS$_n$ in global coordinates can be written as,

$$\mathrm{d}s^2 = R^2\left(-\cosh^2\varrho\,\mathrm{d}t^2 + \mathrm{d}\varrho^2 + \sinh^2\varrho\,\mathrm{d}\Omega_{n-2}\right), \tag{2.2.16}$$

and $\varepsilon^{\dagger}\varepsilon \propto \cosh\varrho$, thus, the (2.2.15) can be saturated only for $\varrho = 0$, *i.e.* in the center of AdS.

Further, the bound (2.2.15) can be used to derive a bound on the energy of the M5 brane [31, 39]. The energy of the an M5-brane is given by

$$E_{\mathrm{M5}} = -\int_{\mathcal{S}}\mathrm{d}^5\sigma\ g(\hat{P},\hat{\mathcal{K}})\,, \tag{2.2.17}$$

where $\mathcal{S}$ denotes the 5-dimensional world-space of the brane, $\hat{P}_M$ is the conjugate momentum[3] [42],

$$\begin{aligned}\hat{P}_M =& \frac{\partial L_{M5}}{\partial(\partial_\tau X^M)} = P_M + \frac{1}{4}\frac{1}{5!}\epsilon^{\tau\alpha_1\ldots\alpha_5}H_{\alpha_1\alpha_2\alpha_3}H_{\alpha_4\alpha_5\alpha_6}\partial^{\alpha_1}X_M\\ &- \frac{\tau_5}{5!}\epsilon^{\tau\alpha_1\ldots\alpha_5}\left[\iota_M\tilde{A} - \tfrac{1}{2}\iota_M A\wedge(A-2H)\right]_{\alpha_1\ldots\alpha_5},\end{aligned} \tag{2.2.18}$$

[2]The field $\tilde{H}$ is defined in terms of an auxiliary scalar field a, which is needed to ensure the Lorentz covariance of the world-volume Lagrangian [40, 41],

$$\tilde{H}_{\mu\nu} = \frac{1}{\sqrt{|\partial a|^2}}(\star H)_{\mu\nu\alpha}\partial^{\alpha}a(\sigma)\,. \tag{2.2.12}$$

The scalar a is subject to a gauge transformation and one usually fixes it by going to the *temporal gauge*, *i.e.* $a = \sigma^0 = t$. This gauge fixing procedure breaks the Lorentz invariance $SO(1,5)$ down to $SO(5)$ and sets $\tilde{H}$ equal to the world-space dual of H.

[3]To write this expression, we have chosen again the static gauge $X^M = (t, \sigma^{\alpha})$.

where X^M are the embedding coordinates of the brane. The quantity $g(\hat{P}, \hat{\mathcal{K}}) = \hat{P}^M \hat{\mathcal{K}}^N g_{MN}$ can be interpreted as a Noether charge density of the symmetry generated by $\hat{\mathcal{K}}$ [44]. Then the inequality (2.2.15) gives a bound on the energy of the brane,

$$E_{\mathrm{M5}} \geq E_{\mathrm{M5}}^{BPS} \,, \tag{2.2.19}$$

where

$$E_{\mathrm{M5}}^{BPS} = \int_{\mathcal{S}} P[\Sigma] + P[\iota_{\hat{\mathcal{K}}} \tilde{A}] + P[\omega] \wedge H + \tfrac{1}{2} P[\iota_{\hat{\mathcal{K}}} H] \wedge (A - 2H) \,. \tag{2.2.20}$$

As shown in [39], the form

$$\Phi_{\mathrm{M5}} = \Sigma + \iota_{\hat{\mathcal{K}}} \tilde{A} + \omega \wedge H + \tfrac{1}{2} \iota_{\hat{\mathcal{K}}} H \wedge (A - 2H) \,, \tag{2.2.21}$$

is a generalised calibration, namely is closed by supersymmetry and it minimise the energy in its homology class being a topological quantity [31].

The discussion for M2 works analogously, and the calibration form is

$$\Phi_{\mathrm{M2}} = \omega + \iota_{\hat{\mathcal{K}}} H \,. \tag{2.2.22}$$

As final comment, we want just to point out that the construction above can also be derived by the supersymmetry algebra. The same calibration forms emerge in the supersymmetry algebra with the central extensions due to the presence of BPS extended objects, and one can prove their closure by using the Killing spinor equations [26, 32, 33].

2.2.1 Calibrations on $\mathrm{AdS}_5 \times M_6$

Even if the formalism described in the previous section is completely general, in what follows we will focus on static M-branes in backgrounds of the type (2.2.9) and we will show how the calibration forms (2.2.21) and (2.2.22) are naturally encoded in the generalised Sasaki-Einstein structure.

We consider first the case of compactifications to 5-dimensional AdS. The supersymmetry conditions for backgrounds of this type are give in [45] while the corresponding exceptional generalised geometry is given in [13,46], and we briefly review it below. We refer to these works also for notation and conventions, and we collect, for convenience, again the relevant conventions used here in **??**.

The metric takes the form

$$\mathrm{d}s^2 = e^{2\Delta} \mathrm{d}s^2_{\mathrm{AdS}_5} + \mathrm{d}s^2_{M_6} \,, \tag{2.2.23}$$

where we denote the inverse AdS radius as m. As shown in [45], supersymmetry constrains the geometry of the six-dimensional internal manifold: M_6 has a local $SU(2)$ structure and topologically is a two-sphere bundle over a four-dimensional base[4].

There is a non-trivial four-form field strength $\mathcal{F}$ with non-zero components along the internal manifold M_6,

$$F_{m_1 \ldots m_4} = (\mathcal{F})_{m_1 \ldots m_4} \,, \tag{2.2.24}$$

[4]The four dimensional base can be either a Kähler-Einstein manifold with positive curvature or a product of two constant curvature Riemannian surfaces. The latter case in non-Einstein [45].

while the external components are set to zero, $\mathcal{F}_{\mu_1\dots\mu_4} = 0$.

The internal flux F satisfies the equations of motion and the Bianchi identity

$$\mathrm{d}F = 0\,, \qquad \mathrm{d}(e^{\Delta} \star_6 F) = 0\,, \tag{2.2.25}$$

with $\star_6$ the Hodge star on M_6, while the dual form $\tilde{F}_{m_1\dots m_7} = (\star_{11}\mathcal{F})_{m_1\dots m_7}$ identically vanishes on the six-dimensional internal manifold M_6.

The Clifford algebra Cliff$(1,10)$ decomposes in Cliff$(1,4)$ and Cliff$(0,6)$:

$$\hat{\Gamma}^{\mu} = e^{\Delta}\,\rho^{\mu} \otimes \gamma_7\,, \qquad \hat{\Gamma}^{m+4} = \mathbb{1}_4 \otimes \gamma^m\,, \tag{2.2.26}$$

with $\gamma_7 = -i\gamma^1 \dots \gamma^6$ the chiral operator in 6 dimensions, and ρ and γ matrices satisfying

$$\{\rho_\alpha, \rho_\beta\} = 2\eta_{\alpha\beta}\mathbb{1}\,, \qquad \{\gamma_a, \gamma_b\} = 2\delta_{ab}\mathbb{1}\,, \tag{2.2.27}$$

in terms of the frame indices $\alpha, \beta = 0, \dots, 4$ on AdS$_5$ and $a, b = 1, \dots, 6$ on M_6. We collect further conventions about spinors and Clifford algebras in **??**.

To have an $\mathcal{N} = 2$ supersymmetric background we decompose the 11-dimensional spinor as

$$\varepsilon = \psi \otimes \chi + \psi^c \otimes \chi^c\,, \tag{2.2.28}$$

where ψ is an element of Cliff$(1,4)$. Notice that, in order to have an AdS backgrounds, the internal spinor χ cannot be a chirality eigenstate [45]. Hence, it can be written as,

$$\chi = \sqrt{2}(\cos\alpha\chi_1 + \sin\alpha\chi_2^*)\,, \tag{2.2.29}$$

where α is a parameter chosen to get the unit norm for the spinor, as in [45].

The exceptional geometry for these backgrounds is given in [13, 46]. The exceptional bundles to consider are again (4.3.1) and (4.3.11). The vector structure $K \in E$ and the hypermultiplet structure $J_a \in \mathrm{ad}F$ can be expressed in terms of the $SU(2)$ structure of [45].

In this discussion we are mostly concerned with the generalised Killing vector K. Its untwisted version is given by

$$\tilde{K} = \xi - e^{\Delta}Y' + e^{\Delta}Z \equiv \xi + \tilde{\omega} + \tilde{\sigma} \quad \in \tilde{E} \tag{2.2.30}$$

with the vector ξ, the two-form Y' and five-form Z defined in terms of spinor bilinears as in [13],

$$\xi = -i\big(\bar{\chi}_1 + \chi_2^T\big)\gamma^{(1)}(\chi_1 - \chi_2^*)\,, \tag{2.2.31}$$

$$Y' = -i\big(\bar{\chi}_1 + \chi_2^T\big)\gamma_{(2)}(\chi_1 - \chi_2^*)\,, \tag{2.2.32}$$

$$Z = -i\big(\bar{\chi}_1 + \chi_2^T\big)\gamma_{(5)}(\chi_1 - \chi_2^*)\,. \tag{2.2.33}$$

The twisted version of the V structure is obtained by the (exponentiated) adjoint action

$$K = e^{A+\tilde{A}}\tilde{K}\,, \tag{2.2.34}$$

where A is the three-form and $\tilde{A}$ the six-form potential of M-theory. Using the expressions for the commutator and the adjoint action from [18], one obtains [13]

$$K = \xi + \big(\iota_\xi A - e^{\Delta}Y'\big) + \big(e^{\Delta}Z - e^{\Delta}A \wedge Y' + \tfrac{1}{2}\iota_\xi A \wedge A\big)\,. \tag{2.2.35}$$

As discussed above, the tensor $\tilde{R}$ must vanish for the generalised Lie derivative to reduce to the usual one, and this is equivalent to [13]

$$\mathrm{d}\tilde{\omega} = \iota_\xi F \,, \qquad \mathrm{d}\tilde{\sigma} = \iota_\xi \tilde{F} - \tilde{\omega} \wedge F \,. \tag{2.2.36}$$

On M_6, this yields the differential conditions

$$\mathrm{d}\big(e^{\Delta} Y'\big) = -\iota_\xi F \,, \tag{2.2.37a}$$

$$\mathrm{d}\big(e^{\Delta} Z\big) = e^{\Delta} Y' \wedge F \,, \tag{2.2.37b}$$

which we refer to as L_K conditions in the language of exceptional generalised geometry. Supersymmetry gives also the Killing vector condition

$$\mathcal{L}_\xi F = \mathcal{L}_\xi \Delta = \mathcal{L}_\xi g = 0 \,. \tag{2.2.38}$$

We can now discuss how the generalised Killing vector K is related to the calibration forms for supersymmetric branes. The general calibrations for M5 and M2 branes are given by the (2.2.21) and (2.2.22). With an appropriate choice of the AdS$_5$ gamma matrices (see **??**), the 11-dimensional Killing vector $\mathcal{K}_M$ in (2.2.4a) has only two non-zero components,

$$\mathcal{K}_0 = \bar{\psi} \rho_0 \psi \tag{2.2.39a}$$

$$\mathcal{K}_m = -i\big(\bar{\chi}_1 + \chi_2^T\big)\gamma_m(\chi_1 - \chi_2^*) = \xi_m \,, \tag{2.2.39b}$$

where we fixed the norms of the spinors to $\bar{\chi}\chi = 1$ and $(\bar{\psi}\psi) = i/2$ and ξ_m is the Reeb vector on M_6. Consistently, we also fixed the value of the angular parameter to $\alpha = \pi/4$ in (2.2.29).

The specific expression of the calibration forms Φ_{M5} and Φ_{M2} in (2.2.21) and (2.2.22) depends on how many AdS directions are spanned by the world-volume of the branes.

Consider an M5 wrapping a 5-cycle in M_6. We choose again the static gauge for the brane embedding and we set to zero the world-volume gauge field (so $H = A$). The the relevant components of the Σ and ω in (2.2.4b) and (2.2.4c) are the internal ones,

$$\omega_{m_1 m_2} = e^{\Delta}\, \bar{\chi}\gamma_7\gamma_{m_1 m_2}\chi = e^{\Delta} Y' \,, \tag{2.2.40a}$$

$$\Sigma_{m_1 \dots m_5} = e^{\Delta}\, \bar{\chi}\gamma_7\gamma_{m_1 \dots m_5}\chi = e^{\Delta} Z \,, \tag{2.2.40b}$$

and the calibration form in (2.2.21) reads (recall that the pull-back of $\tilde{A}$ is zero),

$$\Phi_{\mathrm{M5}} = e^{\Delta} Z - e^{\Delta} A \wedge Y' + \tfrac{1}{2}\iota_\xi A \wedge A \,. \tag{2.2.41}$$

Note that this is exactly the pull-back on the brane of the twisted generalised vector K in (2.2.35). A similar computation for an M2-brane wrapping a 2-cycle in M_6 gives

$$\Phi_{\mathrm{M2}} = e^{\Delta} Y' - \iota_\xi A \,, \tag{2.2.42}$$

which is again the pull-back on the M2-brane of the twisted generalised vector K. Using the L_K conditions (2.2.37) and choosing the a gauge for A such that $\mathcal{L}_\xi A = 0$, it is straightforward to check that Φ_{M5} and Φ_{M2} are closed. Explicitly, for instance for M5, one has,

$$\begin{aligned} \mathrm{d}\Phi_{\mathrm{M5}} &= \mathrm{d}(e^{\Delta} Z) - \mathrm{d}(e^{\Delta} A \wedge Y') + \tfrac{1}{2}\mathrm{d}(\iota_\xi A \wedge A) = \\ &= e^{\Delta} Y' \wedge F - F \wedge e^{\Delta} Y' + \iota_\xi F \wedge A + \tfrac{1}{2}\mathrm{d}(\iota_\xi A) \wedge A + \tfrac{1}{2}\iota_\xi A \wedge F = \\ &= \iota_\xi F \wedge A + \tfrac{1}{2}\mathcal{L}_\xi A \wedge A - \tfrac{1}{2}\iota_\xi F \wedge A - \tfrac{1}{2} A \wedge \iota_\xi F = 0 \,. \end{aligned} \tag{2.2.43}$$

Analogously, one can verify that $\Phi_{\rm M2}$ is also closed, showing that the purely internal configuration of the membrane is supersymmetric.

The generalised vector K is also related to the calibration forms for other types of brane probes. Here we focus on branes with one one leg in the external space-time, that is a string moving in AdS. We leave the study of other membrane configurations to future work. The calibration forms for M2 and M5-branes of this kind are given by (2.2.21) and (2.2.22) in this case, take the following form

$$\Phi_{\rm M2} = e^{2\Delta}\tilde{\zeta}_1 \tag{2.2.44}$$
$$\Phi_{\rm M5} = e^{2\Delta} \star Y' + e^{2\Delta}\tilde{\zeta}_1 \wedge A \tag{2.2.45}$$

where $Z = \star\tilde{\zeta}_1$. The two calibrations are components of the (poly)-form

$$\Phi = e^{2\Delta}\tilde{\zeta}_1 + e^{2\Delta} \star_6 Y' + e^{2\Delta}\tilde{\zeta}_1 \wedge A\,. \tag{2.2.46}$$

which is the Hodge dual of the vector structure (2.2.35) . We want now to study it's closure and its relation to the integrability conditions. In this case, the closure follows from the moment map condition $\mu_3 \equiv 0$, rather than from the L_K condition. In [13], it is shown that this moment map condition requires

$$\mathrm{d}\big(e^{2\Delta}\tilde{\zeta}_1\big) = 0\,, \tag{2.2.47}$$

so that this form calibrates a M2-brane. Again, combining the two conditions, we get

$$\mathrm{d}\big(e^{3\Delta}\sin\Theta\big) = 2me^{2\Delta}\tilde{\zeta}_1 \quad \text{and} \quad \mathrm{d}\big(e^{3\Delta}V\big) = e^{3\Delta}\sin\Theta F + 2me^{2\Delta}\star Y'\,. \tag{2.2.48}$$

From the vanishing of μ_3 in [13], it is easy to verify that the form $e^{2\Delta}\star_6 Y' + e^{2\Delta}\tilde{\zeta}_1 \wedge A$ is closed (for non-vanishing m).

2.2.2 Calibrations in $\mathrm{AdS}_4 \times M_7$

In this section, we discuss M-theory calibrations on AdS_4 backgrounds. Again, we first review the exceptional generalised geometry [13] and then relate it to generalised calibrations. Conventions for the spinor bilinears and the supersymmetry equations for the internal forms can be found in [39] and the relevant ones for this work are collected in appendix **??**.

The background metric has the following form

$$\mathrm{d}s^2 = e^{2\Delta}\mathrm{d}s^2_{\mathrm{AdS}_4} + \mathrm{d}s^2_{M_7}\,. \tag{2.2.49}$$

We set the inverse AdS_4 radius to $m = 2$. In addition, there is a non trivial four-form flux

$$G = m\,\mathrm{vol}_4 + F\,, \tag{2.2.50}$$

where $F = \mathrm{d}A$ is the flux component on M_7 and it satisfies the following Bianchi identity and equations of motion

$$\mathrm{d}F = 0\,, \qquad \mathrm{d}(e^{2\Delta}\star_7 F) = -mF\,, \tag{2.2.51}$$

with $\star_7$ the Hodge star on M_7. We will also need its dual $\tilde{F} = \mathrm{d}\tilde{A} - \frac{1}{2}A \wedge F$.

The 11-dimensional gamma matrices split as

$$\Gamma_\mu = e^{\Delta}\rho_\mu \otimes \mathbb{1} \quad \text{and} \quad \Gamma_m = e^{\Delta}\rho_5 \otimes \gamma_m\,, \tag{2.2.52}$$

with $\{\rho_\mu, \rho_\nu\} = 2g_{\mu\nu}$ and $\{\gamma_m, \gamma_n\} = g_{mn}$. The matrix $\rho_5 = i\rho_{0123}$ is the chirality operator in four dimensions, and $\gamma_{1\dots7} = i\mathbb{1}$. For further details about Clifford algebra conventions we refer to the **??**.

The spinor ansatz preserving eight supercharges reads [39, 47]

$$\begin{aligned}\varepsilon &= \sum_{i=1,2} \psi_i \otimes e^{\Delta/2}\chi_i + \psi_i^c \otimes e^{\Delta/2}\chi_i^c \\ &= e^{\Delta/2}\psi_+ \otimes \chi_- + e^{\Delta/2}\psi_- \otimes \chi_+ + \text{c.c.}\end{aligned} \tag{2.2.53}$$

where $\chi_\pm := \frac{1}{\sqrt{2}}(\chi_1 \pm i\chi_2)$ and $\psi_\pm := \frac{1}{\sqrt{2}}(\psi_1 \pm \psi_2)$. In addition, we take the AdS$_4$ spinors ψ_i to have positive chirality, *i.e.* $\rho_5\psi_i = \psi_i$.

Combining the supersymmetry conditions and equations of motion for the fluxes one can express the internal fluxes in terms of spinor bilinears by [39],

$$F = \frac{3m}{\tilde{f}}\, \mathrm{d}(e^{6\Delta} i(\bar{\chi}_+^c \gamma_{(3)}\chi_-))\,, \tag{2.2.54}$$

$$\tilde{F} = -\tilde{f}\ \mathrm{vol}_7\,. \tag{2.2.55}$$

The features of the solutions depend on the *electric* charge m. When $m = 0$ the solutions correspond to near horizon geometries of M5-branes wrapped on internal cycles (no M2 charge). The geometries with $m \neq 0$ correspond to the presence of a non-vanishing M2 charge. For $m \neq 0$ the internal manifolds always admit a canonical contact structure, as shown in [39].

The generalised geometry relevant for backgrounds of this kind is discussed in [13]. The HV structure is given by a generalised vector X in the fundamental of $E_{7(7)}$ and a triplet J_a in the adjoint representation. The untwisted vector reads

$$\tilde{X} = \xi + e^{3\Delta}Y + e^{6\Delta}Z - ie^{9\Delta}\tau\,, \tag{2.2.56}$$

where the forms are bilinears in the internal background spinors

$$\sigma = i\bar{\chi}_+^c\gamma_{(1)}\chi_-\,, \qquad Y = i\bar{\chi}_+^c\gamma_{(2)}\chi_-\,, \qquad Z = \star_7 Y\,, \qquad \tau = \sigma \otimes \mathrm{vol}_7\,, \tag{2.2.57}$$

and ξ is the vector dual to the one-form σ. Notice that the vector structure has the same form in both cases of a Sasaki-Einsten internal manifold and of a generic flux background [13]. Indeed the seven-dimensional manifolds giving $\mathcal{N} = 2$ supersymmetry always admit a local $SU(2)$ structure. Moreover, the Killing vector constructed by spinor bilinears in (2.2.57) (or equivalently its dual one-form σ) defines a contact structure. This allows us to write the M_7 metric as a Reeb foliation, analogously to the case of a Sasaki-Einstein manifold [39]. As a consequence, the volume form can be written making use of the contact structure,

$$\frac{1}{3!}\sigma \wedge \mathrm{d}\sigma \wedge \mathrm{d}\sigma \wedge \mathrm{d}\sigma = \left(\frac{3m^2}{\tilde{f}}\right)^3 e^{9\Delta}\,\mathrm{vol}_7 = 2\left(\frac{3m^2}{\tilde{f}}\right)^3 \sqrt{q(K)}\,, \tag{2.2.58}$$

where $q(K)$ is the $E_{7(7)}$ invariant and K is the real part of the twisted vector structure X

$$K = \xi - \frac{1}{2}\sigma \wedge \omega \wedge \omega + \iota_\xi \tilde{A}\,. \tag{2.2.59}$$

As already mentioned in section 4.3.6, supersymmetry implies that X is a generalised vector and its vector part, ξ, is a Killing vector. Through AdS/CFT, ξ is the dual of the R-symmetry of the conformal $\mathcal{N} = 2$ gauge theory in three dimensions. Then, as discussed in section 4.3.6, the generalised Lie derivative along X must reduce to $\mathcal{L}_\xi$, which implies the vanishing of the tensor $\tilde{R}$, or more explicitly

$$\begin{aligned} \mathrm{d}(e^{3\Delta}Y) &= \iota_\xi F\,,\\ \mathrm{d}(e^{6\Delta}Z) &= \iota_\xi \tilde{F} - e^{3\Delta}Y \wedge F\,. \end{aligned} \tag{2.2.60}$$

As expected, these reproduce part of the supersymmetry equations in [39].

One can choose the gamma matrices and spinors in such a way that the Killing vector $\mathcal{K}$ has components [39]

$$\begin{aligned} \mathcal{K}_0 &= \sum_i \bar{\psi}_i \rho_0 \psi_i\,,\\ \mathcal{K}_m &= -\tfrac{i}{2} e^{2\Delta}\, \bar{\chi}^c_+ \gamma_m \chi_-\,. \end{aligned} \tag{2.2.61}$$

As in the previous section, the form of the generalised calibrations for M5 and M2 branes, (2.2.21) and (2.2.22), depends on the direction spanned by the branes. Again, we considered first an M5 wrapping a 5-cycle in M_7, with a zero world-volume gauge field ($H = A$) and in the static gauge.

In this case, the relevant components of the forms (2.2.4b) and (2.2.4c) are

$$\begin{aligned} \omega &= \tfrac{i}{2} e^{3\Delta} \bar{\chi}^c_+ \gamma_{(2)} \chi_- = e^{3\Delta} Y\,,\\ \Sigma &= -e^{6\Delta}(\bar{\chi}_+ \gamma_{(5)} \chi^c_+ + \bar{\chi}^c_- \gamma_{(5)} \chi_-) = e^{6\Delta} Z\,, \end{aligned}$$

and the calibration Φ_{M5} gives

$$\Phi_{\mathrm{M5}} = (e^{6\Delta} Z + A \wedge e^{3\Delta} Y + \tfrac{1}{2}\iota_\xi A \wedge A + \iota_\xi \tilde{A})\,. \tag{2.2.62}$$

One can also add an M2 completely arranged along the internal directions. The corresponding calibration form is given by,

$$\Phi_{\mathrm{M2}} = (e^{3\Delta} Y + \iota_\xi A)\,, \tag{2.2.63}$$

which together with Φ_{M5} gives rise to a poly-form,

$$\Phi = (e^{3\Delta} Y + \iota_\xi A) + (e^{6\Delta} Z + A \wedge e^{3\Delta} Y + \tfrac{1}{2}\iota_\xi A \wedge A + \iota_\xi \tilde{A})\,, \tag{2.2.64}$$

and this, again, corresponds to the vector structure. The closure of Φ, follows from supersymmetry[5], more precisely, from the L_K conditions (2.2.60),

$$\begin{aligned} \mathrm{d}\Phi &= \mathrm{d}(e^{3\Delta} Y + \iota_\xi A) + \mathrm{d}(e^{6\Delta} Z + A \wedge e^{3\Delta} Y + \tfrac{1}{2}\iota_\xi A \wedge A + \iota_\xi \tilde{A})\\ &= \iota_\xi F + \mathrm{d}(\iota_\xi A) + \iota_\xi \tilde{F} - e^{3\Delta} Y \wedge F + F \wedge e^{3\Delta} Y - A \wedge \iota_\xi F\\ &\quad + \tfrac{1}{2}\mathrm{d}(\iota_\xi A) \wedge A + \tfrac{1}{2}\iota_\xi A \wedge F + \mathrm{d}(\iota_\xi \tilde{A})\\ &= \mathcal{L}_\xi A + \mathcal{L}_\xi \tilde{A} + \tfrac{1}{2}\mathcal{L}_\xi A \wedge F + \tfrac{1}{2} A \wedge \iota_\xi F - A \wedge \iota_\xi F + \tfrac{1}{2}\iota_\xi A \wedge F = 0\,, \end{aligned} \tag{2.2.65}$$

[5] Note that L_K conditions imply that Φ_{M5} and Φ_{M2} are separately closed.

where in the last line we made a gauge choice, such that,

$$\mathcal{L}_\xi A = 0\,, \qquad\qquad \mathcal{L}_\xi \tilde{A} = 0\,. \tag{2.2.66}$$

One can consider not only branes wrapping internal cycles. For instance, for M5-brane spanning two spatial directions, we can show that – also in this case – the related calibration form comes from the vector K. The relevant form is given by

$$\Phi = (e^{3\Delta} Y + \iota_\xi A)\,. \tag{2.2.67}$$

The closure of this form comes directly from (2.2.60).

It is also easy to see that, in this case, branes with one leg aligned with an external space direction are not supersymmetric, as indeed already discussed in [48].

Indeed, for instance in the case of an M2 wrapping an internal cycle and with one external leg, the candidate calibration form is proportional to σ in (2.2.57), which is not closed – *i.e.* $\mathrm{d}\sigma \sim \omega$. This condition, in the language of Exceptional generalised geometry, is a part of $L_K J_a = \epsilon_{abc}\ \lambda_b J_c$. It is interesting to note that, on the other hand, this configuration is supersymmetric in the case of a Minkowski background, since the analogous condition reads $L_K J_a = 0$, [18].

To conclude the analysis, let us focus on space filling brane configurations. This case corresponds to the J_a components of the Exceptional Sasaki-Einstein structure. For instance, for an M5-brane we find

$$\Phi = -e^{4\Delta} V_-\,, \tag{2.2.68}$$

where V_- is the two-form defined from spinor bilinears as follows,

$$V_\pm \coloneqq \frac{1}{2i}\big(\bar{\chi}_+ \gamma_{(2)} \chi_+ \pm \bar{\chi}_- \gamma_{(2)} \chi_-\big)\,, \tag{2.2.69}$$

which gives the $TM \otimes T^*M$-component of J_a by raising one index. In particular, in the limit of a Sasaki-Einstein manifold (the only one for which the expression of J_a is given explicitly in [13]), the calibration form (2.2.68) corresponds to J_3, and we have good reasons to trust this result also for the cases where generic fluxes are turned on. We leave the complete discussion of these cases for future work.

2.3 Supersymmetric branes in type IIB

In this section, we want to discuss the analogous conditions to have supersymmetric extended objects in a type IIB supergravity AdS background and their connections to Exceptional Sasaki-Einstein structures defining such backgrounds.

We are interested in backgrounds with non trivial fluxes. The NS three-form is $H = \mathrm{d}B$ and the RR fields are

$$F_1 = \mathrm{d}C_0 \qquad F_3 = \mathrm{d}C_2 \qquad F_5 = \mathrm{d}C_4 - \frac{1}{2} H \wedge C_2 + \frac{1}{2} F_3 \wedge B \tag{2.3.1}$$

The field strengths F satisfy the duality condition

$$F_p = (-1)^{\left[\frac{p}{2}\right]} \star F_{10-p}\,, \tag{2.3.2}$$

while the S-duality of type IIB is reflected in the fact that B with C_2 form an $SL(2)$ doublet. It could be useful to define a complexified version of the three-form flux [49],

$$G = F_3 + iH_3 \,. \tag{2.3.3}$$

The Bianchi identities are written as

$$\mathrm{d}F_5 = \frac{1}{8}\,\mathbb{I}\mathrm{m}\; G \wedge G^* \,, \qquad \mathrm{d}G = 0 \,. \tag{2.3.4}$$

The generalised calibrations for backgrounds with non-trivial NS-NS flux have been constructed in [44] (see also [32, 34] for an equivalent derivation in terms of the supersymmetry algebra). The two Majorana-Weyl supersymmetry parameters ε_1 and ε_2 can be used to construct the following bilinears [50],

$$\mathcal{K} = \frac{1}{2}(\bar\varepsilon_1\Gamma_M\varepsilon_1 + \bar\varepsilon_2\Gamma_M\varepsilon_2)\;\mathrm{d}x^M \,, \tag{2.3.5}$$

$$\omega = \frac{1}{2}(\bar\varepsilon_1\Gamma_M\varepsilon_1 - \bar\varepsilon_2\Gamma_M\varepsilon_2)\;\mathrm{d}x^M \,, \tag{2.3.6}$$

$$\Psi = \sum_{k=0}^{2} \frac{1}{(2k+1)!}\bar\varepsilon_1\Gamma_{M_1\dots M_{2k+1}}\varepsilon_2\;\mathrm{d}x^{M_1}\wedge\dots\wedge\mathrm{d}x^{M_{2k+1}} \,. \tag{2.3.7}$$

Using the Killing spinor equations for type IIB, one can show that the vector $\hat{\mathcal{K}}$ dual to the one form $\mathcal{K}$ is a Killing vector [51],

$$\mathcal{L}_{\hat{\mathcal{K}}}\, g = 0 \,, \qquad \mathcal{L}_{\hat{\mathcal{K}}}\, F = 0 \,. \tag{2.3.8}$$

Notice that also the spinor bilinears (2.3.5)–(2.3.7) are invariant under the transformation generated by $\hat{K}$

$$\mathcal{L}_{\hat{\mathcal{K}}}\,\omega = 0 \,, \qquad \mathcal{L}_{\hat{\mathcal{K}}}\,\Psi = 0 \,. \tag{2.3.9}$$

As discussed in [44], we may write the κ-symmetry condition to have a supersymmetric Dp-brane

$$\hat\Gamma_{\mathrm{D}p}\;\varepsilon_2 = \varepsilon_1 \,, \tag{2.3.10}$$

where, the κ-symmetry operator is defined as [52, 53]

$$\hat\Gamma_{\mathrm{D}p} = \frac{1}{\sqrt{-\det(P[G]+\mathcal{F})}} \sum_{2l+s=p+1} \frac{\epsilon^{\alpha_1\dots\alpha_{2l}\beta_1\dots\beta_s}}{l!s!2^l}\mathcal{F}_{\alpha_1\alpha_2}\dots\mathcal{F}_{\alpha_{2l-1}\alpha_{2l}}\Gamma_{\beta_1\dots\beta_s} \,, \tag{2.3.11}$$

and $P[\bullet]$ denotes the pullback to the $(p+1)$-dimensional brane world-volume and $\mathcal{F} = F + P[B]$, with B the NS two-form and F the world-volume gauge field-strength. The energy of the brane (the charge associated to the transformation generated by $\hat{\mathcal{K}}$) is

$$E = -\int_{\mathcal{S}} \mathrm{d}^p\sigma\;\hat P_M \hat{\mathcal{K}}^M \,, \tag{2.3.12}$$

where $\mathcal{S}$ is the brane world-space and $\hat P = \frac{\partial L_{\mathrm{D}p}}{\partial(\partial_\tau X^M)}$ takes the form

$$\hat P_M = -\mu_{\mathrm{D}p}e^{-\phi}\sqrt{-\det\mathcal{M}}(\mathcal{M}^{-1})^{(\alpha\tau)}B_{MN}\partial_\alpha X^N + \frac{\mu_{\mathrm{D}p}}{p!}\epsilon^{\tau\alpha_1\dots\alpha_p}\left[\iota_M(C\wedge e^{\mathcal{F}})\right]_{\alpha_1\dots\alpha_p} \,, \tag{2.3.13}$$

where we denoted $\mathcal{M} = P[g] + \mathcal{F}$. Note that again we are in the temporal gauge in adapted coordinates, such that the world-volume of the brane is $\mathbb{R} \times \mathcal{S}$. One has the usual BPS bound,

$$E \geq E_{BPS}\,, \tag{2.3.14}$$

with

$$\begin{aligned} E_{BPS} = \mu_{\mathrm{D}p} \int_{\mathcal{S}} \mathrm{d}^p\sigma\, P\Big[e^{-\phi}\Psi - \iota_{\hat{\mathcal{K}}}C - \omega \wedge C\Big] \wedge e^{\mathcal{F}} \\ + \mu_{\mathrm{D}p} \int_{\mathcal{S}} \mathrm{d}^p\sigma\, P\big[\omega - \iota_{\hat{\mathcal{K}}}B\big] \wedge \big(C \wedge e^{\mathcal{F}}\big)\big|_{p-1}\,. \end{aligned} \tag{2.3.15}$$

Thus, one can read the generalised calibration form from the last expression,

$$\Phi_{\mathrm{D}p} = e^{-\phi}\Psi - \iota_{\hat{\mathcal{K}}}C - \omega \wedge C \wedge e^{\mathcal{F}} + \omega - \iota_{\hat{\mathcal{K}}}B \wedge \big(C \wedge e^{\mathcal{F}}\big)\big|_{p-1}\,. \tag{2.3.16}$$

One can show [44, 54] that this is a topological quantity. In addition, one can also show that this form is closed, making use of potential configurations preserving the symmetry generated by $\hat{\mathcal{K}}$, *i.e.*

$$\mathcal{L}_{\hat{\mathcal{K}}}B = 0\,, \qquad\qquad \mathcal{L}_{\hat{\mathcal{K}}}C = 0\,, \tag{2.3.17}$$

analogously to what has been done in the previous section for M-theory. As a final observation, we would like to point out that the same conclusions about calibration forms can be obtained by supertranslation algebra, as done for example in [26, 32, 34].

Let us now focus on type IIB compactifications to AdS_5-backgrounds. As for the discussion of supersymmetric extended objects in M-theory above, we now apply the supersymmetry conditions and the aforementioned approach to branes in type IIB string theory on

$$\mathrm{d}s^2 = e^{2\Delta}\mathrm{d}s^2_{\mathrm{AdS}_5} + \mathrm{d}s^2_{M_5}\,, \tag{2.3.18}$$

and relate the calibration forms to the geometric description by the vector and hypermultiplet structures. The exceptional geometry of this setup is discussed in [13, 46], based on the geometric description in [55]. For $\mathcal{N} = 2$ backgrounds of the form (2.3.18) with generic fluxes, the internal manifold m_5 admits a (local) identity structure [55–57].

The two ten-dimensional Majorana-Weyl spinors of the same chirality which describe a IIB background of the form (2.3.18) can be decomposed as in [46][6],

$$\varepsilon_i = \psi \otimes \chi_i \otimes u + \psi^c \otimes \chi_i^c \otimes u\,. \tag{2.3.19}$$

Here ψ denotes the external Spin(4, 1) spinor, χ_i are the internal Spin(5) spinors and u a two-component spinor. It might be convenient to define the complex spinors $\zeta_1 = \chi_1 + i\chi_2$ and $\zeta_2^c = \chi_1^c + i\chi_2^c$. As for the previous cases, one can construct the relevant bilinears defining a local identity structure on the internal manifold [55]. One introduces the vectors

$$\begin{aligned} K_0^m &:= \bar{\zeta}_1^c\gamma^m\zeta_2\,, \\ K_3^m &:= \bar{\zeta}_2\gamma^m\zeta_1\,, \\ K_4^m &:= \tfrac{1}{2}\big(\bar{\zeta}_1\gamma^m\zeta_1 - \bar{\zeta}_2\gamma^m\zeta_2\big)\,, \\ K_5^m &:= \tfrac{1}{2}\big(\bar{\zeta}_1\gamma^m\zeta_1 + \bar{\zeta}_2\gamma^m\zeta_2\big)\,, \end{aligned} \tag{2.3.20}$$

[6] We follow the conventions given in the appendix of [46]. We collect them in **??**.

that are not all linear independent. Relations between these forms comes from supersymmetry, as shown in [55]. Then, one can use the following scalars to parametrise the norms of the spinors

$$\begin{aligned} A &:= \tfrac{1}{2}\big(\bar{\zeta}_1\zeta_1 + \bar{\zeta}_2\zeta_2\big)\,, \\ A\sin\Theta &:= \tfrac{1}{2}\big(\bar{\zeta}_1\zeta_1 - \bar{\zeta}_2\zeta_2\big)\,, \\ S &:= \bar{\zeta}_2^c\zeta_1\,, \\ Z &:= \bar{\zeta}_2\zeta_1\,. \end{aligned} \tag{2.3.21}$$

Finally, one considers the two-forms

$$\begin{aligned} U_{mn} &:= -\tfrac{i}{2}\big(\bar{\zeta}_1\gamma_{mn}\zeta_1 + \bar{\zeta}_2\gamma_{mn}\zeta_2\big)\,, \\ V_{mn} &:= -\tfrac{i}{2}\big(\bar{\zeta}_1\gamma_{mn}\zeta_1 - \bar{\zeta}_2\gamma_{mn}\zeta_2\big)\,, \\ W_{mn} &:= -\bar{\zeta}_2\gamma_{mn}\zeta_1\,. \end{aligned} \tag{2.3.22}$$

The HV structure for these backgrounds can be found in [13, 46]. The untwisted generalised vector structure $K \in \Gamma(\tilde{E})$ in (4.3.72) is given in terms of the identity structure above by

$$\tilde{K} = \tilde{\xi} + \tilde{\lambda}^i + \tilde{\rho} + \tilde{\sigma}^i = K_5^\sharp + e^{2\Delta - \frac{\phi}{2}}\begin{pmatrix}\mathbb{R}\mathrm{e}\,K_3 \\ \mathbb{I}\mathrm{m}\,K_3\end{pmatrix} - e^{4\Delta-\phi}\star V\,, \tag{2.3.23}$$

where ϕ is the dilaton and Δ the warp factor. Notice that for these backgrounds the five-forms vanish $\sigma^i = 0$. The twisted vector structure is obtained by acting on $\tilde{K}$ with the adjoint element as in appendix E of [18]

$$K = \xi + \lambda^i + \rho\,, \tag{2.3.24}$$

where the twisted quantities are [13, 18]

$$\xi = \tilde{\xi}\,, \tag{2.3.25a}$$

$$\lambda^i = \tilde{\lambda}^i + \iota_\xi B^i\,, \tag{2.3.25b}$$

$$\rho = \tilde{\rho} + \iota_\xi C + \epsilon_{ij}\tilde{\lambda}^i \wedge B^j + \tfrac{1}{2}\epsilon_{ij}\big(\iota_\xi B^i\big)\wedge B^j\,. \tag{2.3.25c}$$

and we defined $B^1 = B$, $B^2 = C_2$, $C = C_4$, $F^1 = H$, $F^2 = F_3$ and $F = F_5$.

As already discussed the condition that the generalised Lie derivative L along the Reeb vector K has to reduce to the conventional one, $\mathcal{L}_\xi$ implies some differential equations on the elements of the vector structure that reproduce some of the supersymmetry conditions on the identity structure derived in [55],

$$\begin{aligned} \mathrm{d}\tilde{\lambda}^i &= \iota_\xi F^i\,, \\ \mathrm{d}\tilde{\rho} &= \iota_\xi F + \epsilon_{ij}\tilde{\lambda}^i \wedge F^j\,. \end{aligned} \tag{2.3.26}$$

Analogously to the M-theory case, we want now to express the calibration conditions for a Dp probe in these backgrounds in terms of the generalised structure and check that their closure in implied by differential conditions on Exceptional Sasaki-Einstein structures. To this purpose we have to specialise the calibrations (2.3.5), (2.3.6) and (2.3.7) to the various brane configurations. The AdS$_5$ geometry and, in particular the products of external spinors, is the same as in the previous section. Thus, in our conventions, the Killing vector $\mathcal{K}$ has the following components,

$$\begin{aligned} \mathcal{K}_0 &= e^{-2\Delta}\bar{\psi}\rho_0\psi \otimes A\,, \\ \mathcal{K}_m &= \tfrac{1}{2}\big(\bar{\zeta}_1\gamma^m\zeta_1 + \bar{\zeta}_2\gamma^m\zeta_2\big) = \xi_m\,, \end{aligned} \tag{2.3.27}$$

where ξ is the Reeb vector. We also fix the norm of the internal spinors such that $A = 1$.

As in the previous sections, we focus on the cases of point-like AdS particles and space-filling branes where the calibrations are related to the generalised vector K. Consider first a D1 wrapping an internal one-cycle. The relevant terms is (2.3.16) are

$$\begin{aligned}
\omega &= -e^{2\Delta-\phi/2}\tfrac{1}{2}\left(\bar{\zeta}_2\gamma_m\zeta_1 + \zeta_2^T\gamma_m\zeta_1^*\right) = -e^{2\Delta-\phi/2}\,\mathbb{R}\mathrm{e}\,K_3\,,\\
\Psi &= -e^{2\Delta+\phi/2}\tfrac{1}{2i}\left(\bar{\zeta}_2\gamma_m\zeta_1 - \zeta_2^T\gamma_m\zeta_1^*\right) = -e^{2\Delta+\phi/2}\,\mathbb{I}\mathrm{m}\,K_3\,,\\
\iota_{\hat{\mathcal{K}}}B &= \iota_\xi B^1 =: \iota_\xi B\,,\\
\iota_{\hat{\mathcal{K}}}C &= \iota_\xi B^2\,.
\end{aligned} \tag{2.3.28}$$

and it is immediate to see that the calibration form is given by the generalised vector K

$$\Phi_{\mathrm{D1}} = -\tilde{\lambda}^2 - \iota_\xi B^2 = -e^{2\Delta-\phi/2}\,\mathbb{I}\mathrm{m}\,K_3 - \iota_\xi B^2\,. \tag{2.3.29}$$

Using equation (2.3.26) one can show that Φ_{D1} is closed. Using again the properties of the AdS_5 spinors, it is easy to show that a space-filling D5-brane is also calibrated by the same form,

$$\Phi_{\mathrm{D5}} = (-\tilde{\lambda}^2 - \iota_\xi B^2)\otimes\star 1 = (-\tilde{\lambda}^2 - \iota_\xi B^2)\otimes \mathrm{vol}_{\mathrm{AdS}_5}\,. \tag{2.3.30}$$

Similarly, one can find the calibration form for a D3-brane wrapping purely internal cycles,

$$\Phi_{\mathrm{D3}} = \tilde{\rho} + \iota_\xi C + \epsilon_{ij}\tilde{\lambda}^i\wedge B^j + \tfrac{1}{2}\epsilon_{ij}\iota_\xi B^i\wedge B^j\,. \tag{2.3.31}$$

Its closure follows from the L_K conditions under the gauge choice[7] that the potential are invariant under K. Again, this form provides also the calibration for a space-filling D7-brane.

In the particular case where the only non-trivial background flux is the five-form, the generalised Sasaki-Einstein structure reduces to the standard one and the internal manifold is Sasaki-Einstein. In this case the spinor ansatz (2.3.19) simplifies since the two internal spinors are proportional to each other, *i.e.* $\chi_2 = i\chi_1$, and, consequently, the one-form part vanishes. The twisteed vector (2.3.24) simplifies to

$$K = \xi - \sigma\wedge\omega + \iota_\xi C\,, \tag{2.3.32}$$

and the (untwisted) hypermultiplet structure is [13]

$$\tilde{J}_+ = \tfrac{1}{2}\kappa u^i\Omega - \tfrac{i}{2}\kappa u^i\Omega^\sharp\,, \tag{2.3.33}$$

$$\tilde{J}_3 = \tfrac{1}{2}\kappa I + \tfrac{1}{2}\kappa\hat{\tau}^i{}_j + \tfrac{1}{8}\kappa\Omega^\sharp\wedge\bar{\Omega}^\sharp - \tfrac{1}{8}\kappa\Omega\wedge\bar{\Omega}\,, \tag{2.3.34}$$

where $u^i = (-i, 1)^i$ and I, ω and Ω are the complex structure, the symplectic and the holomorphic two-forms on the Kähler-Einstein basis of M_5.

Note that in the vector structure, the three-form is $\sigma\wedge\omega$ and by the structure equation (2.3.26) one immediately sees that adding the potential part $\iota_\xi C$ yields to a closed form (up to a gauge choice),

$$\mathrm{d}(\tilde{\rho} + \iota_\xi C) = \iota_\xi\mathrm{d}C + \mathcal{L}_\xi C - \iota_\xi\mathrm{d}C = \mathcal{L}_\xi C = 0\,. \tag{2.3.35}$$

[7] We also need to use $\tilde{\rho}\propto\star V$.

Since in this case, the form of the hypermultiplet structure is simple, we can also study calibrations that are not associated to the vector K. We will do it in the simplest Sasaki-Einstein background, namely $\mathrm{AdS}_5 \times S^5$, where S^5 is the five-dimensional sphere. The background is given by

$$\begin{aligned} \mathrm{d}s^2 &= \frac{R^2}{r^2}\mathrm{d}r^2 + \frac{r^2}{R^2}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}s^2(S^5)\,, \\ C_4 &= \left(\frac{r^4}{R^4} - 1\right)\mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^3\,. \end{aligned} \tag{2.3.36}$$

while all other fluxes, dilaton and warp factors vanish. The S^5 can be written as a $U(1)$ fibration over $\mathbb{CP}^2$,

$$\mathrm{d}s^2(S^5) = \mathrm{d}\Sigma_4^2 + \sigma \otimes \sigma\,, \tag{2.3.37}$$

where $\mathrm{d}\Sigma_4^2$ is the *Fubini-Study metric* over $\mathbb{CP}^2$. The form σ is given by $\sigma = \mathrm{d}\psi + A$, where A is a connection such that $\mathcal{F} = \mathrm{d}A = 2\omega$, and ψ is the periodic coordinate on the circle $U(1)$ with period 6π.

Explicitly, the sphere S^5 takes the form [58]

$$\begin{aligned} \mathrm{d}s^2(S^5) = \mathrm{d}\alpha^2 &+ \frac{1}{4}\sin^2\alpha(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) + \frac{1}{4}\cos^2\alpha \sin^2\alpha(\mathrm{d}\beta + \cos\theta \mathrm{d}\phi)^2 \\ &+ \frac{1}{9}\left[\mathrm{d}\psi - \frac{3}{2}\sin^2\alpha(\mathrm{d}\beta + \cos\theta \mathrm{d}\phi)\right]^2\,, \end{aligned} \tag{2.3.38}$$

with $\psi \in [0, 6\pi]$, $\beta \in [0, 4\pi]$, $\alpha \in [0, \pi/2]$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. In these coordinates the holomorphic form has the following expression,

$$\Omega = -\frac{1+i}{\cos\theta}\,\mathrm{d}\beta \wedge \mathrm{d}\theta + \frac{1+i}{8}\cos\sigma \cos\theta \sin\theta \sin^3\sigma\;\mathrm{d}\sigma \wedge \mathrm{d}\phi + (1+i)\;\mathrm{d}\theta \wedge \mathrm{d}\phi\,. \tag{2.3.39}$$

First, consider a D5-brane spanning the directions $0, 1, 2, r$ in AdS_5. The world-volume of the brane is $\mathrm{AdS}_4 \times S^2$, where S^2 is the sphere parametrized by the angles (θ, ϕ). Then the expression (2.3.16) reduces to

$$\begin{aligned} \Phi_{\mathrm{D5}} &= -\;\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^4 \wedge \tfrac{(1-i)}{2}e^{4\Delta}\Omega \\ &= -\;\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}r \wedge \mathrm{vol}_{S^2}\,, \end{aligned} \tag{2.3.40}$$

and we see that it corresponds to the two-form part of $\tilde{J}_+$ in (2.3.33). Modulo choice of coordinates,[8] it agrees with the analogous form in [32].

We can also consider a D3-brane probe spanning the directions $0, 1, r$ of AdS_5. The world-volume is now $\mathrm{AdS}_3 \times S^1$ and the calibration is given by the Hodge dual of the 4-form part of $\tilde{J}_3$,

$$\Phi_{\mathrm{D3}} = \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}r \wedge \tfrac{1}{8}e^{4\Delta} \star (\Omega \wedge \bar{\Omega})\,. \tag{2.3.41}$$

One can prove the closure of this form by the $L_K J$ relations. In particular,

$$\mathrm{d}(e^{4\Delta}\;\star(\Omega \wedge \bar{\Omega})) = -m\;\iota_\xi\,\mathrm{vol}_5 = 0\,, \tag{2.3.42}$$

where the first equality comes from the conditions to have a vanishing $\tilde{R}$-tensor [13,18]. In other words, it is the rewriting of the (2.3.26) in the Sasaki-Einstein case.

[8] Here $\mathrm{d}x^4 \propto \sin\theta\;\mathrm{d}r + \mathrm{d}\sigma + \mathrm{d}\beta$.

3
Flux Compactifications

3.1 Introduction and motivations

This thesis is devoted to the study of supersymmetric compactifications with non-trivial fluxes. We will see in the first part of this chapter how requiring some amount of supersymmetry on the lower dimensional theory constrains the geometry of the internal manifold M, such that it must admit geometrical structures like the ones we described in chapter 1. For the well-known case of fluxless compactification of a 10-dimensional type II supergravity to a minimal supergravity in 4 dimensions, the constraints on the internal manifold requires it to be a Calabi-Yau three-fold [3]. When we allow fluxes to be turned on, the supersymmetry conditions can be cast in a compact and elegant form using *Generalised Geometry* and generalised structures we will introduce in chapter 4.

3.2 Supergravity theories

Supergravity theories are theories combining general relativity with supersymmetry (making this a local symmetry). These can be seen as low-energy effective theories of the different string theories. There exists also an eleven-dimensional maximally supersymmetric supergravity, which is not connected (as low-energy limit) to any string theory. This has been interpreted to have its higher dimensional origin in M-theory.

The aim of this section is to describe the main feature of type II and eleven-dimensional supergravity theories, with their effective actions and the gauge symmetries of their potentials.

3.2.1 Eleven-dimensional supergravity

This section is devoted to the description of eleven-dimensional supergravity, *i.e.* the low energy effective theory of M-theory. This is not meant to be an exhaustive treatment and we refer to [59, 60] for further details.

The bosonic degrees of freedom of eleven dimensions supergavity consist of the metric g, a three-form potential A and its dual. Although it does not transport independent degrees of freedom, one often introduces the dual seven-form $\tilde{G}$, whose six-form potential is conventionally denoted by $\tilde{A}$.

The theory is invariant under diffeomorphisms and the gauge transformations

$$A_3 \longrightarrow A_3 + \mathrm{d}\Lambda_2\,, \tag{3.2.1}$$

where Λ_2 is a two-form. The gauge invariant field strength is $G_4 = \mathrm{d}A_3$.

The bosonic action of the eleven-dimensional supergravity is

$$\begin{aligned} S_{11} =& \frac{1}{2\kappa^2}\int \mathrm{d}^{11}x\,\sqrt{g}\Big[\Big(R - \frac{1}{2}|G|^2\Big)\Big] - \\ & \underbrace{-\frac{1}{6}G_4 \wedge G_4 \wedge A}_{S_{CS}}\,, \end{aligned} \tag{3.2.2}$$

The equation of motion and Bianchi identity can be written (in a sourceless case) as

$$\begin{aligned} &\mathrm{d}\star G + \frac{1}{2}G \wedge G = 0\,, \\ &\mathrm{d}G = 0\,. \end{aligned} \tag{3.2.3}$$

The theory is supersymmetric, with $\mathcal{N} = 1$ supersymmetry. Notice that this is the maximal possible supersymmetry in eleven dimensions. The fermionic degrees of freedom are completely captured by the gravitino Ψ.

In addition, the equation of motion for the metric g, *i.e.* the Einstein equation, can be written as follows,

$$R_{MN} - \frac{1}{12}\Big(G_{MPQR}G_N{}^{PQR} - \frac{1}{12}g_{MN}G^2\Big) = 0\,. \tag{3.2.4}$$

3.2.2 Type II theories

Type II supergravities are the ten-dimensional effective theories for massless fields type II string theories. There are two such theories that differ in the chirality of the fermionic fields and the rank of the form potentials.

The bosonic sector consists of two sets of fields: the Neveu-Schwarz Neveu-Schwarz (NSNS) and the Ramond-Ramond (RR).

As one can see from table 3.1, the NSNS sector is the same for both type II theories. It contains the metric g, the dilaton ϕ and the NSNS two-form B. The latter is a U(1) gauge potential with field strength $H = \mathrm{d}B$.

The RR sector depends on the theory. Type IIA contains odd forms, while for type IIB has even ones. These are also U(1) gauge potentials. For applications to generalised geometry is it convenient to use the *democratic formulation* [61], of supergravity. This formulation considers RR potentials of all ranks C_p, with $p = 1, 3, \ldots, 9$ for type IIA and $p = 0, 2, \ldots, 8$ for IIB. These are not all independent since their field strengths

	g	metric (graviton)
NSNS	B	Kalb-Ramond 2-form
	ϕ	dilaton
RR	A_p	p odd for type IIA p even for type IIB
	$\psi_M^{\alpha,+}$, $\psi_M^{\alpha,\mp}$	Gravitinos
	λ_α^-, $\lambda_\alpha^\pm$	Dilatinos

TABLE 3.1: Type II supergravities spectrum in ten dimensions. The different chiralities of spinors define the two theories. Upper signs refer to type IIA, while lower ones to type IIB.

have to satisfy duality relations with respect to the Hodge dual. The field strength[1] are defined by,

$$F_p = \mathrm{d}C_{p-1} + H \wedge C_{p-3} + e^B F_0 \,, \tag{3.2.5}$$

where $F_0 = m$ is the Romans mass, which can be added only in type IIA [62], and the duality relations,

$$F_p = (-1)^{\left[\frac{p+3}{2}\right]} \star F_{10-p} \,. \tag{3.2.6}$$

The fermionic sector of the two theories consists of two Majorana-Weyl spinors of spin 3/2, the gravitinos ψ_M^α, and two Majorana-Weyl spin 1/2 spinors λ^α, the dilatinos. Gravitinos and dilatinos have opposite chirality. In type IIA the gravitinos have opposite chirality, while in type IIB they have the same chirality (chosen positive by convention). As a consequence, type IIB dilatinos will both have negative chirality. This is the difference between type IIA and IIB, the former is a non-chiral theory, while the latter is chiral. Nevertheless, they are both maximal supersymmetry in ten dimensions, *i.e.* they are $\mathcal{N} = 2$. An important fact is that type IIA supergravity can be obtained by the eleven-dimensional one by a compactifiation on a circle. We will analyse this reduction in a while.

The string frame[2] action for the bosonic fields of type IIA is (we follow the conventions of [61])

$$S_{IIA} = \frac{1}{2\kappa^2}\int \mathrm{d}^{10}x \; \underbrace{\sqrt{g}\left[e^{-2\phi}\left(R + 4\nabla\phi^2 - \frac{1}{2}|H|^2\right)\right]}_{S_{NS}} - \underbrace{\frac{\sqrt{g}}{2}\sum_{k=0}^{2}|F_{2k}|^2}_{S_R}$$
$$\underbrace{-\frac{1}{2}B\wedge\mathcal{F}_4\wedge\mathcal{F}_4}_{S_{CS}} \,,$$

where $\mathcal{F}_4 = \mathrm{d}C_3$, while F_p are the field strength defined above in (3.2.5).

[1] There exists another common choice for the RR potential, the so-called A-basis, which is related to the C-basis we use as $A = e^{-B} \wedge C$. In this basis the field strength (3.2.5) reads $F = e^B \wedge (\mathrm{d}A + m)$.

[2] Einstein frame and string frame metric are related by a dilaton rescaling, *i.e.*

$$g = e^{\phi/2} g^E \,.$$

The bosonic action for type IIB reads

$$S_{IIB} = \frac{1}{2\kappa^2}\int \mathrm{d}^{10}x \; \underbrace{\sqrt{g}\left[e^{-2\phi}\left(R + 4\nabla\phi^2 - \frac{1}{2}|H|^2\right)\right]}_{S_{NS}} - \underbrace{\frac{\sqrt{g}}{2}\sum_{k=0}^{2}\frac{1}{k!}|F_{2k+1}|^2}_{S_R}$$
$$- \underbrace{\frac{1}{2}C_4 \wedge H_3 \wedge \mathcal{F}_3}_{S_{CS}} \,.$$

Analogously to the type IIA case, we introduced $\mathcal{F}_n = \mathrm{d}C_{n-1}$. This action has a constant shift symmetry $C_0 \to C_0 + c$, where c is a constant. Hence, it is referred to as an *axion* [60, 63]. Furthermore, the five-form field strength F_5 satisfies the self-duality condition

$$F_5 = \star F_5 \,, \tag{3.2.7}$$

which has to be imposed as a further constraint together with the equations of motion.

It is be useful to collect all the RR field strengths and potentials into a single polyform,

$$C = \textstyle\sum_p C_p \quad p \text{ odd/even for type IIA/IIB}\,,$$
$$F = \textstyle\sum_p F_p \quad p \text{ even/odd for type IIA/IIB}\,.$$

In this notation, the (3.2.5) and (3.2.6) take the following form,

$$F = \mathrm{d}_H C + e^B F_0 \,,$$
$$F = \star s(F) \,,$$

where we introduced the differential operator $\mathrm{d}_h := \mathrm{d} - H\wedge$ acting on polyforms, called *H-twisted exterior derivative*, and the *index reversal operator s*,

$$s(A_p) = (-1)^{[p/2]} A_p \,. \tag{3.2.8}$$

The field strengths defined above are invariant under gauge transformations of potentials,

$$\begin{aligned} \delta B &= -\mathrm{d}\lambda \,, \\ \delta C &= -e^B \wedge (\mathrm{d}\omega - m\lambda) \,, \end{aligned} \tag{3.2.9}$$

where λ is a one-form, ω is a polyform made of even/odd forms for type IIA/IIB and the term proportional to the Romans mass m is there only in the type IIA case.

The RR field strengths have the following equations of motions and Bianchi identities (when there are no sources, like D_p branes). Bosonic fields equations for type II appear as,

$$(\mathrm{d}e^{-2\phi} \star H) \pm \frac{1}{2} F \wedge \star F = 0 \,, (\mathrm{d} + H) \star F = 0 \,, \tag{3.2.10}$$

where $\pm$ sign is referred to type IIA/B respectively, and

$$\mathrm{d}F = H \wedge F \,. \tag{3.2.11}$$

Notice that for type IIA the (3.2.10),

$$\mathrm{d}(e^{-2\phi} \star H) + \frac{1}{2}[F \wedge \star F]_8 = 0 \,, \tag{3.2.12}$$

can be interpreted as the Bianchi identity for the dual seven-form field strength,

$$\tilde{H} = e^{-2\phi} \star H \,. \tag{3.2.13}$$

We denoted by $[\ldots]_k$ the rank k form of the polyform in the bracket.

Making use of the self-duality relation for F (3.2.6), we can rewrite the (3.2.12) as,

$$\mathrm{d}\Big(\tilde{H} + \frac{1}{2}\big[s(F) \wedge C + m e^{-B} \wedge C\big]_7\Big) = 0 \,, \tag{3.2.14}$$

which is solved by,

$$\tilde{H} = \mathrm{d}\tilde{B} - \frac{1}{2}\big[s(F) \wedge C + m e^{-B} \wedge C\big]_7 \,. \tag{3.2.15}$$

Thus we introduce a new potential $\tilde{B}$ [64,65], whose (linearised) gauge transformations are fixed by requiring the invariance of its field strength,

$$\delta\tilde{B} = -(\mathrm{d}\sigma + m\omega_6) - \frac{1}{2}\big[e^B \wedge (\mathrm{d}\omega - m\lambda) \wedge s(C)\big]_6 \,, \tag{3.2.16}$$

where σ is a five-form, while ω and λ are the parameters of the gauge transformations (3.2.9).

An interesting point to notice about the massive IIA theory [62] is that one can obtain it from the non-massive one by shifting the gauge parameters as,

$$\begin{aligned} \mathrm{d}\omega_0 &\longrightarrow \mathrm{d}\omega_0 - m\lambda \,, \\ \mathrm{d}\sigma &\longrightarrow \mathrm{d}\sigma + m\omega_6 \,. \end{aligned} \tag{3.2.17}$$

These relations will be the key of the construction of the exceptional generalised geometry for massive type IIA [12].

Type IIB theory exhibits a non-compact global symmetry $\mathrm{SL}(2,\mathbb{R})$. This is not evident in the formulation we gave above, so we want to make it explicit. The two two-form potentials B and C_2 can be organised into a doublet of $\mathrm{SL}(2,\mathbb{R})$,

$$B^i := \begin{pmatrix} B \\ C_2 \end{pmatrix}^i \,. \tag{3.2.18}$$

Similarly, we introduce $F^i = \mathrm{d}B^i$. Under an $\mathrm{SL}(2,\mathbb{R})$ transformations the B fields transform linearly,

$$B^i \longrightarrow \Lambda_{ij} B^j \,, \qquad \Lambda_{ij} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \,. \tag{3.2.19}$$

One can also define a complex scalar field τ which is the complex combination of the axion and the dilaton field, for this reason this is called *axion-dilaton field.* This is useful since it transforms nicely under $\mathrm{SL}(2,\mathbb{R})$,

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d} \,. \tag{3.2.20}$$

Then, type IIB action S_{IIB} can be re-written in terms of $\mathrm{SL}(2,\mathbb{R})$ representations, like the symmetric matrix h,

$$h_{ij} = \begin{pmatrix} |\tau|^2 & -C_0 \\ -C_0 & 1 \end{pmatrix}_{ij} \,, \tag{3.2.21}$$

transforming under $\mathrm{SL}(2,\mathbb{R})$ as

$$h_{ij} \longrightarrow \Lambda^{ik} h_{kl} \Lambda^{lj} \,. \tag{3.2.22}$$

Then the action S_{IIB} in terms of $\mathrm{SL}(2,\mathbb{R})$ covariant objects can be recast as,

$$\begin{aligned} S_{IIB} = \frac{1}{2\kappa^2} \int \mathrm{d}^{10}x \, \sqrt{g} \Big[e^{-2\phi} \Big(R - \frac{1}{12} F^i h_{ij} F^j + \frac{1}{4} \partial h_{ij} \partial h^{ji} \Big) \Big] \\ - \frac{1}{8\kappa^2} \int \mathrm{d}^{10}x \, \left[\sqrt{g} |F_5|^2 - \epsilon_{ij} C_4 \wedge F^i \wedge F^j \right] . \end{aligned} \tag{3.2.23}$$

The self duality condition on the 5-form field strength (3.2.7) (which is a constraint in this formalism) is also $\mathrm{SL}(2,\mathbb{R})$ invariant. Moreover, one can re-write its definition in an $\mathrm{SL}(2,\mathbb{R})$ invariant form,

$$F_5 = \mathrm{d}C_4 + \frac{1}{2} \epsilon_{ij} B^i \wedge H^j \,. \tag{3.2.24}$$

3.3 Supersymmetric backgrounds and compactifications

After describing the actions and the equations of motion of eleven- and ten-dimensional supergravities, we are interested in solutions. Since we want to study compactifications, we look for solutions that are warped products

$$\mathcal{M}_{10} = \mathcal{X} \times M_d \,,$$

of a maximally symmetric external spacetime (Minkowski, Anti-de Sitter, de Sitter) and an internal space M_d. In order to preserve Poincaré invariance in the external spacetime, we must set all fermionic fields to zero so the background is purely bosonic.

Then, the metric ansatz reads

$$\mathrm{d}s^2_{\mathcal{M}} = e^{2A} \mathrm{d}s^2_{\mathcal{X}} + \mathrm{d}s^2_M \,, \tag{3.3.1}$$

where A is a real function of the coordinates on M_d, the *warp factor*.

We look for supersymmetric solutions. A background is supersymmetric if all the supergravity fields (and hence the solutions) are invariant under supersymmetry transformations. Choosing ϵ as the quantity parameterising supersymmetry variations, one is allowed to write (schematically)

$$\delta(\text{boson}) = \epsilon(\text{fermion}) \,, \qquad \delta(\text{fermion}) = \epsilon(\text{boson}) \,. \tag{3.3.2}$$

The variations of the bosonic fields always contain a fermionic field, and since we have set these to zero the variations automatically vanish. On the other hand, we get non-trivial conditions from the variations of the fermionic fields. Then, supersymmetry of the background is equivalent to the existence of a non-vanishing spinor ε for which the supersymmetry variations vanish. These can be recast into differential and algebraic equations, known as *Killing spinor equations.* The spinor ϵ solving these is then called *Killing spinor.* A background is supersymmetric if it admits Killing spinors. The Killing spinor equation for M-theory is

$$\nabla_M \epsilon + \frac{1}{288} \Big[\Gamma_M{}^{NPQR} - 8 \delta_M{}^N \Gamma^{PQR} \Big] G_{NPQR} \, \epsilon = 0 \,, \tag{3.3.3}$$

where $M, N, \ldots = 0, 1, \ldots, 10$, ϵ is a Majorana spinor and the Gamma matrices are the Clifford algebra elements in 11 dimensions.

Type IIA Killing spinor equation can be derived by the previous one by a compactification, so we do not give it explicitly. Finally, for type IIB the Killing spinor equations, given in terms of ten-dimensional Gamma matrices are

$$\nabla_M \epsilon - \frac{1}{96}\Big[\Gamma_M{}^{PQR} - 9\Gamma^{PQ}\Big] G_{MPQ}\, \epsilon^c + \frac{1}{192}\Gamma^{PQRS} F_{MPQRS}\epsilon = 0\,, \tag{3.3.4a}$$

$$i\Gamma^M P_M \epsilon^c + \frac{i}{24}\Gamma^{PQR} G_{PQR}\epsilon = 0\,. \tag{3.3.4b}$$

where, following [55], we defined $P = \frac{i}{2}e^{\phi}\mathrm{d}C_0 + \frac{1}{2}\mathrm{d}\phi$ and $G = ie^{\phi/2}(\tau \mathrm{d}B - \mathrm{d}C_2)$, and here $M, N = 0, 1, \ldots, 9$.

The Killing spinor equations are central in the study of supersymmetric string backgrounds. This is due to the fact that for backgrounds of the (3.3.1) one can show that the supersymmetry variations plus the Bianchi identities for the NS and RR fields imply all other equations of motion. Thus a solution of the Killing spinor equations is automatically a solution of the supergravity equations of motions.

On backgrounds of the type (3.3.1) the supersymmetry parameters factorise accordingly

$$\epsilon = \sum_{i=N} \varepsilon_i \otimes \chi\,, \tag{3.3.5}$$

where ε_i are anticommuting spinors on the external space and χ is a generic commuting spinor on the internal manifold. The number N of spinors ε_i detemine the number of supersymmetries preserved by the background. This splitting induces also a splitting of the Killing spinor equations into distinct conditions for ε and for χ. The existence of Killing spinors (and the differential conditions they have to satisfy) on the internal manifold M puts several constraints on the geometry of the manifold. Investigating how this happens and how this allows compactifications with fluxes is the goal of next sections.

3.3.1 Calabi-Yau backgrounds in type II

Let us start by a famous example of compactifications to four dimensions. We consider a purely geometric solutions where the only non-trivial field is the metric. We are going to see how supersymmetry conditions constrain the internal geometry to be Calabi-Yau.

As discussed above, to find solutions it is enough to solve the supersymmetry variations for the spinors (3.3.2). In type II theory, in absence of fluxes they reduce to

$$\delta\lambda_1 = \partial_M \phi \Gamma^M \epsilon_1 = 0\,, \qquad \delta\lambda_2 = \partial_M \phi \Gamma^M \epsilon_2 = 0\,, \tag{3.3.6}$$

for the dilatino variations, while the gravitino variations reduce to the requirement that the supersymmetry parameters must be covariantly constant

$$\delta\psi^1_M = \nabla_M \epsilon_1 = 0\,, \qquad \delta\psi^2_M = \nabla_M \epsilon_2 = 0\,. \tag{3.3.7}$$

The supersymmetry parameters decompose as

$$\epsilon_1 = \zeta_1 \otimes \eta_1 + \text{c.c.}\,, \tag{3.3.8}$$

$$\epsilon_2 = \zeta_2 \otimes \eta_2 + \text{c.c.}\,. \tag{3.3.9}$$

Here ζ is a four-dimensional chiral spinor ($\gamma_5\zeta = \zeta$) and $\eta_{1,2}$ are six-dimensional chiral spinors, of opposite chirality in IIA and same chirality in IIB

$$\gamma_7\eta_1 = \eta_1 \qquad \gamma_7\eta_2 = \mp\eta_2 \qquad \text{in IIA/IIB}\,. \tag{3.3.10}$$

Using the decomposition ansatz for the metric (3.3.1), and supersymmetry parameters (3.3.8), we obtain the six-dimensonal equations

$$\partial\!\!\!/\phi\,\eta_{1,2} = 0\ , \tag{3.3.11}$$

where $\partial\!\!\!/\phi = \gamma^m\partial_m\phi$. This implies that the dilaton must be constant, $\partial_m\phi = 0$, since $||\partial\!\!\!/\phi\,\eta_{1,2}||^2 = (\partial\phi)^2||\eta_{1,2}||^2$.

The gravitino variations reduce to

$$\nabla_\mu \to \nabla_\mu\otimes\mathbb{1} + \frac{1}{2}e^A(\gamma_\mu\gamma_5\otimes\partial\!\!\!/A)\ , \tag{3.3.12}$$
$$\nabla_m \to \mathbb{1}\otimes\nabla_m\ , \tag{3.3.13}$$

where, on the right hand side, ∇_μ and ∇_m are the covariant derivatives with respect to the external four-dimensional unwarped metric and the internal six-dimensional metric, respectively. Using again (3.3.8), we can decompose (3.3.7) into an external (four-dimensional) and an internal (six-dimensional) part as

$$\nabla_\mu\zeta_1\otimes\eta_1 - \frac{1}{2}e^A(\gamma_\mu\zeta^*\otimes\partial\!\!\!/A\,\eta_1^*) + \text{c.c.} = 0\,, \tag{3.3.14}$$
$$\zeta_1\otimes\nabla_m\eta_1 + \text{c.c.} = 0\ . \tag{3.3.15}$$

An identical equation holds for η_2. The external gravitino equations (3.3.14) imply that $\partial\!\!\!/A\,\eta_{1,2}^*$ should be proportional to $\eta_{1,2}$, which is impossible since $\eta_{1,2}^\dagger\gamma_m\eta_{1,2} = 0$. It follows that the warping must be constant. Taking this into account, (3.3.14) further reduces to

$$\nabla_\mu\zeta_{1,2} = 0\,. \tag{3.3.16}$$

The commutator of two external covariant derivatives gives

$$[\nabla_\mu,\nabla_\nu] = \frac{1}{4}R_{\mu\nu\rho\sigma}\gamma^{\rho\sigma} = \frac{\Lambda}{6}\gamma_{\mu\nu}\,, \tag{3.3.17}$$

where we used the expression for the curvature tensor for a maximally symmetric (unwarped) four-dimensional metric: $R_{\mu\nu\rho\sigma} = \frac{1}{3}\Lambda(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$. From (3.3.16) and (3.3.17), it then follows $\Lambda\gamma_{\mu\nu}\zeta = 0$ which implies the vanishing of the cosmological constant

$$\Lambda = 0\,. \tag{3.3.18}$$

Then, the external gravitino equations require that the warp factor A must be constant, and the four-dimensional space must be Minkowski ($\Lambda = 0$).

Let us now turn to the internal gravitino equations, (3.3.15), which reduce to

$$\nabla_m\eta_1 = 0\,, \qquad \nabla_m\eta_2 = 0\ . \tag{3.3.19}$$

Applying the same argument as below (3.3.17), we see that the internal metric must be Ricci flat.

Moreover, a covariantly constant spinor implies a reduction of the holonomy group of a Riemannian manifold. From (3.3.19) it follows that the internal metric must have at most holonomy SU(3). A Ricci flat manifold of SU(3) holonomy is a Calabi-Yau.

If the internal metric has strict SU(3)-holonomy, then η^1_+ and η^2_+ must be proportional. Without loss of generality we can set $\eta^1_+ = \eta^2_+ = \eta$ with $\eta^\dagger\eta = 1$. We can express

We can construct the forms ω and Ω as spinors bilinears of the covariantly constant spinor η. Then we can rewrite the Calabi-Yau condition $\nabla_m\eta = 0$ in the alternative form

$$\mathrm{d}\omega = 0\,, \qquad \mathrm{d}\Omega = 0\,. \tag{3.3.20}$$

Thus, we got an important result: in absence of fluxes, looking for a supersymmetric vacuum requires to consider a Calabi-Yau three-fold as internal manifold.

The properties of Calabi-Yau's manifolds are such that one can explicitely derive the four-dimensional effective action (see [66, 67] for details). This is an $\mathcal{N} = 2$ supergravity theory in four-dimensions, that is characterised by the presence of massless uncostrained scalars, the moduli. In supersymmetric theories massless scalar fields are not a problem, the trouble is if some of them stay massless after SUSY breaking: massless scalar fields would provide long range interactions that are not observed in nature. One solutions to the moduli problem is to find ways to generate potential terms for some or all such scalars. One way is to consider compactifications admitting non-trivial fluxes.

3.3.2 Backgrounds with fluxes in type II

We now turn to the study of more general solutions of type II supergravity where some of the fluxes have non-zero values. The presence of fluxes drastically changes the properties of the solutions. This can be seen both from the equations of motion and the supersymmetry variations. Indeed, from the Einstein equation, which reads schematically

$$R_{MN} \sim H_{MPQ}H_N{}^{PQ} + \sum_p F_{MQ_1\ldots Q_p}F_N{}^{Q_1\ldots Q_p}\,, \tag{3.3.21}$$

we see that the fluxes back-react on the metric, which generically cannot be Ricci-flat (and thus Calabi-Yau) anymore. Another generic feature is a non trivial warp factor in the ten-dimensional metric.

The supersymmetry variations are also modified. For example, from (3.3.2) one can see that in the presence of RR fluxes the supersymmetry conditions relate ϵ^1 and ϵ^2 so that the four-dimensional components $\zeta_{1,2}$ cannot be chosen independently anymore, as in (3.3.8). Therefore, in the presence of RR-fluxes one generically obtains $\mathcal{N} = 1$ in four dimensions.

Repeating the strategy used in the fluxless case, we decompose the supersymmetry conditions according to the compactification ansatz. We do not give all details here, but one can easily see that the internal gravitino variations become

$$\begin{aligned}(\nabla_m + \frac{1}{4}\not{H}_m)\eta_1 + \frac{1}{8}e^\phi \not{F}\gamma_m\gamma_7\eta_2 = 0\ ,\\ (\nabla_m - \frac{1}{4}\not{H}_m)\eta_2 - \frac{1}{8}e^\phi \not{F}^\dagger\gamma_m\gamma_7\eta_1 = 0\ ,\end{aligned} \tag{3.3.22}$$

from which we see that, generically, the internal manifold is no longer Ricci flat and hence no longer Calabi-Yau, since

$$[\nabla_m, \nabla_n]\eta_{1,2} = \frac{1}{4}R_{mn}{}^{pq}\gamma_{pq}\eta_{1,2} \neq 0\,. \tag{3.3.23}$$

It is therefore natural to wonder whether it is still possible to say something about the geometry of the internal manifold. Generalized Complex Geometry provides a general framework to describe flux backgrounds in string theory.

4

Generalising the geometry

4.1 Introduction and motivations

The aim of this chapter is to introduce Generalised Geometry, both complex (its simpler version) and Exceptional.

Generalised complex geometry, as originally proposed by Hitchin [68,69], geometrises the NSNS sector of type II supergravity. As described in the previous chapter, Hitchin's generalised tangent bundle is isomorphic to the sum $TM \oplus T^*M$ of the tangent and cotangent bundle to the d-dimensional compactification manifold M_d, and is patched by $\mathrm{GL}(d,\mathbb{R})$ transformations and gauge shifts of the NSNS two-form B. The structure group of this extended bundle is $\mathrm{O}(d,d)$, *i.e.* the T-duality group of the compactification on a d-dimensional torus. From a string theory perspective, T and T^* parameterise the quantum number of the string, that is momentum and winding charge. Extending this construction to include the RR potentials in type II supergravity [70–72], or adapting it to M-theory compactifications [70,73,74], leads to exceptional generalised geometry. In this case the structure group of the generalised tangent bundle is the U-duality group, and the bundle parameterises all the charges of the theory under study, that is momenta and winding, as well as NS- and D-brane (or M-brane) charges.

4.2 Generalised complex geometry

Generalised complex geometry was introduced by Hitchin [68] and Gualtieri [69] to find a structure interpolating between complex and symplectic geometries.

The main idea of generalised geometry is to geometrise the gauge transformations of a two form. This is done by introducing the generalised tangent bundle. Given a d-dimensional manifold M generalised tangent bundle is the extension of the tangent space by the cotangent space

$$\begin{tikzcd} 0 \arrow[r] & T^*M \arrow[r, "i"] & E \arrow[r, "\pi"] & TM \arrow[l, bend left, "B"] \arrow[r] & 0 \end{tikzcd}\ , \tag{4.2.1}$$

where π is the so-called *anchor* map, that is a projection $\pi : E \to TM$, to not be confused with the usual projection map on a bundle. Its action on sections is simply projecting out the form part.

At any point $p \in M$, E is isomorphic to the sum of the tangent and the cotangent bundle

$$E \cong \tilde{E} = TM \oplus T^*M \,. \tag{4.2.2}$$

The sections of E are called *generalised vectors*. On patch U_α they can be written as,

$$V_{(\alpha)} = v_{(\alpha)} + \mu_{(\alpha)} \in TM \oplus T^*M \,, \tag{4.2.3}$$

and at the intersection $U_\alpha \cap U_\beta$ the patch non-trivially

$$V_{(\alpha)} = v_{(\alpha)} + \mu_{(\alpha)} = A_{(\alpha\beta)} v_{(\beta)} + A^{-T}_{(\alpha\beta)} \mu_{(\beta)} - \iota_{A_{(\alpha\beta)} v_{(\beta)}} \mathrm{d}\Lambda_{(\alpha\beta)} \,, \tag{4.2.4}$$

where $A_{(\alpha\beta)}$ is a $\mathrm{GL}(d, \mathbb{R})$ transition function and Λ is a one-form gauge parameter satisfying the co-cycle condition on the triple overlap $U_\alpha \cap U_\beta \cap U_\gamma$,

$$\Lambda_{(\alpha\beta)} + \Lambda_{(\beta\gamma)} + \Lambda_{(\gamma\alpha)} = -i g^{-1}_{(\alpha\beta\gamma)} \mathrm{d} g_{(\alpha\beta\gamma)} \,, \tag{4.2.5}$$

with g is an element of $\mathrm{U}(1)$, satisfying the condition for transition functions

$$g_{(\beta\gamma\delta)} g^{-1}_{(\alpha\gamma\delta)} g_{(\alpha\beta\delta)} g^{-1}_{(\alpha\beta\gamma)} = 1 \,, \qquad \text{on } U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta \,. \tag{4.2.6}$$

This construction allows to introduce a two-form B, transforming as

$$B_{(\alpha\beta)} := B_{(\alpha)} - B_{(\beta)} = \mathrm{d}\Lambda_{(\alpha\beta)} \,. \tag{4.2.7}$$

which is a generalisation of the standard $\mathrm{U}(1)$ connection. B is a *connection on a gerbe* [75], that is a higher rank form generalisation of a connection over a fiber bundle [76]. We can see that the map $B_{(\alpha\beta)}$ gives the non-trivial fibration of the cotangent bundle on the tangent one. In other words, the vector part of the generalised vector V is a well-defined vector, meaning it patches correctly as a vector over the manifold. On the other hand, as one can observe from (4.2.4) the form part does not patch as one can expect from a one-form, but it has an extra-part parametrised by $\mathrm{d}\Lambda_{(\alpha\beta)}$. This is what we mean when we say that the isomorphism (formally called *splitting* [77]) $E \cong TM \oplus T^*M$ is not canonical, but it depends on the choice of the map B. The transition functions of the generalised tangent bundle are $\mathrm{GL}(d, \mathbb{R}) \times \Lambda^2 T^*M$. Notice that generally the B-field is only defined locally, however its field strength $H = \mathrm{d}B$ is globally defined. In applications to string theory, B is identified with the NS-NS two-form potential and the patching rules (4.2.19) and (4.2.7) will correctly reproduce the gauge transformations prescribed by supergravity.

The generalised tangent bundle – because of the duality between TM and T^*M as linear spaces – is equipped with a natural $\mathrm{O}(d,d)$ symmetric bilinear form η, *i.e.* a metric

$$\eta(V, W) = \eta(v + \mu, w + \lambda) = \frac{1}{2}(\mu(w) + \lambda(v)) \,, \tag{4.2.8}$$

where $\mu(w) = \iota_w \mu$ denote the contraction of the vector w with the one-form μ. One can also write the relation above as a matrix equation $\eta(V, W) = V^T \eta W$, where,

$$V = \begin{pmatrix} v \\ \mu \end{pmatrix} , \qquad W = \begin{pmatrix} w \\ \lambda \end{pmatrix} , \qquad \eta = \frac{1}{2} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} . \tag{4.2.9}$$

One can diagonalise the matrix η and make the signature (d,d) explicit

$$\tilde{\eta} = \frac{1}{2}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} . \tag{4.2.10}$$

As discussed in chapter 1, defining a metric is equivalent to define a G-structure. In this case, the metric η defines $\mathrm{O}(d,d)$ on $TM \oplus T^*M$.

In addition to the metric, the generalised tangent bundle $\tilde{E}$ has an orientation too [69]. It can be defined by the η metric through the Levi-Civita symbol,

$$\mathrm{vol}_\eta = \frac{1}{(d!)^2}\epsilon^{m_1\ldots m_d}\epsilon_{n_1\ldots n_d}\partial_{m_1}\wedge\ldots\wedge\partial_{m_d}\wedge \mathrm{d}x^{n_1}\wedge\ldots\wedge \mathrm{d}x^{n_d} , \tag{4.2.11}$$

where ∂_m and $\mathrm{d}x^n$ denote a basis on TM and T^*M.

The structure group preserving both the metric and the volume form is $\mathrm{SO}(d,d)$. The Lie algebra of $\mathrm{SO}(TM \oplus T^*M) \cong \mathrm{SO}(d,d)$ is given by

$$\mathfrak{so}(TM\oplus T^*M) = \{T \mid \eta(TV,W) + \eta(V,TW) = 0\}, \tag{4.2.12}$$

i.e. generators are antisymmetric. This algebra decomposes [78] in

$$\mathrm{End}(TM) \oplus \Lambda^2 TM \oplus \Lambda^2 T^*M , \tag{4.2.13}$$

or, equivalently, a generic element $T \in \mathfrak{so}(TM \oplus T^*M)$ can be written as

$$T = \begin{pmatrix} A & \beta \\ B & -A^T \end{pmatrix} , \tag{4.2.14}$$

where $A \in \mathrm{End}(TM)$, $B \in \Lambda^2 T^*M$, $\beta \in \Lambda^2 TM$, and hence

$$\begin{aligned} A &: TM \longrightarrow TM \\ B &: TM \longrightarrow T^*M \\ \beta &: T^*M \longrightarrow TM \ . \end{aligned}$$

The action of the three subgroups can be done explicitly. For $\mathrm{GL}(d,\mathbb{R})$ part,

$$e^A \cdot V = Av + A^{-T}\mu , \tag{4.2.15}$$

for the so-called B-transformation,

$$e^B \cdot V = v + \mu - \iota_v B , \tag{4.2.16}$$

and, finally for the β-transformation,

$$e^\beta \cdot V = v - \beta\lrcorner\mu + \mu . \tag{4.2.17}$$

Here $\cdot$ denotes the adjoint action of the $\mathfrak{so}(d,d)$ algebra. A noteworthy fact is that since both B and β are not invariant under $\mathrm{GL}(d,\mathbb{R})$, their actions do not commute with the $\mathrm{GL}(d,\mathbb{R})$ one. Another observation one can make is that, since both B and β are antisymmetric tensors, the symmetric product defined by the metric η is invariant under B and β transformations,

$$\eta(e^{B+\beta}V, e^{B+\beta}W) = \eta(V,W) . \tag{4.2.18}$$

The patching conditions can be rewritten as

$$V_{(\alpha)} = e^{A_{(\alpha\beta)} + \mathrm{d}\Lambda_{(\alpha\beta)}} \cdot V_{(\beta)} \tag{4.2.19}$$

The bundle $\tilde{E}$ is some-time called the untwisted generalised and its sections $\tilde{V} \in \Gamma(TM \oplus T^*M)$ are called untwisted generalised vectors. They are related to the sections of E by a B-transformation

$$V = e^B\tilde{V} = e^B(v+\mu) = v + \mu - \iota_v B . \tag{4.2.20}$$

4.2.1 The generalised frame bundle

The definition of generalised frame bundle is a straightforward generalisation of that of frame bundle. Given a frame on E, that we call $\{\hat{E}_A\}$, satisfying the orthonormality condition with respect to the natural inner product

$$\eta\left(\hat{E}_A, \hat{E}_B\right) = \eta_{AB} = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}_{AB}, \tag{4.2.21}$$

we can define the *generalised frame bundle* as follows. The *frame bundle* is the bundle associated to these basis vectors. Points on the fibre (frames) are connected by $\mathrm{O}(d,d)$ transformations. Conversely, all frames connected by $\mathrm{O}(d,d)$ transformations to a frame that satisfies (4.2.21) will satisfy it too. In other words, these frames form an $\mathrm{O}(d,d)$-bundle, that we call *generalised frame bundle*,

$$F := \bigsqcup_{p\in M}\left\{\left(p,\, \hat{E}_A\right) \mid p \in M,\ \eta\left(\hat{E}_A, \hat{E}_B\right) = \eta_{AB}\right\}. \tag{4.2.22}$$

Given the frame $\{\hat{e}_a\}$ for TM and the coframe $\{e^a\}$ for the cotangent bundle T^*M, we can make a particular choice of frame, the *split frame*, such that we can keep track of the vector and form part of our generalised sections [79,80]. Explicitly we can choose,

$$\hat{E}_A := \begin{cases} \begin{pmatrix} \hat{e}_a \\ -i_{\hat{e}_a}B \end{pmatrix}, & A = a\,, \\ \begin{pmatrix} e^a \\ 0 \end{pmatrix}, & A = a + d\,. \end{cases} \tag{4.2.23}$$

Note that the B-shift is present in our definition of the split frame. This is because one can obtain the split frame by a twist of a generic basis of $TM \oplus T^*M$.

We can then define a *generalised G-structure*, as a sub-bundle of the principal generalised frame bundle associated to E. In other words, a generalised G-structure is a set of generalised tensors that are invariant under the action of a subgroup $G \subset \mathrm{O}(d,d)$.

4.2.2 Generalised metric

Proceeding in with the ordinary structures, we want to describe the analogue of a Riemannian metric on the generalised tangent bundle.

One can define a *generalised metric* on a generalised tangent bundle E, as a positive definite sub-bundle of rank d, such that the restriction of the natural metric η is positive definite [81, def. 4.1.1]. In terms of generalised G-structures, we say that a generalised metric is an $\mathrm{O}(d) \times \mathrm{O}(d)$-structure over M.

The presence of such a structure splits the generalised tangent bundle E into two sub-bundles,

$$E \cong C_+ \oplus C_-\,, \tag{4.2.24}$$

corresponding to spaces where the inner product η has a definite sign [69,82]. This allows us to define a generalised metric [81]

$$\mathcal{G}(\cdot,\cdot) = \eta(\cdot,\cdot)\big|_{C_+} - \eta(\cdot,\cdot)\big|_{C_-}\,. \tag{4.2.25}$$

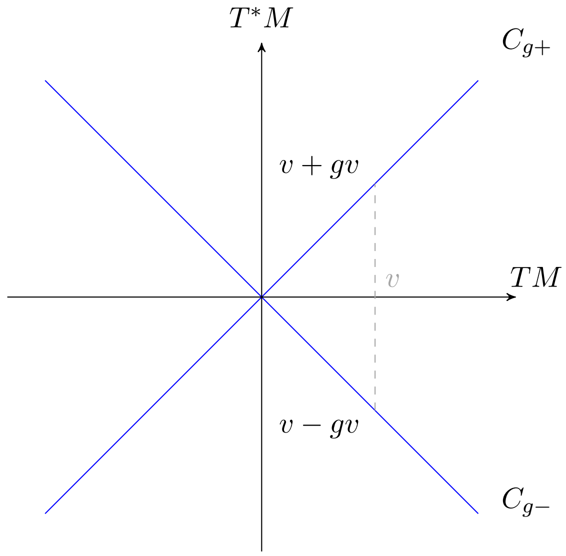


FIGURE 4.1: We can represent the splitting of E into the sub-bundles $C_+ \oplus C_-$ by the graph of a linear map $h : TM \longrightarrow T^*M$. Here is shown the particular case of a zero B field transformation.

Since any generalised section which is made only by a vector field or only a form has a zero norm with respect to the metric η, we can state for instance $T^*M \cap C_\pm = \{0\}$. (Analogously intersections between TM and $C_\pm$ are just made by the zero section, as represented in figure 4.1.) Thus, we can define a map $h : TM \longrightarrow T^*M$ such that C_+ is the graph of h, and C_- its orthogonal complement, and explicitly

$$C_+ = \{v + hv \mid v \in \Gamma(TM)\}. \tag{4.2.26}$$

The map h provides an isomorphism between TM and C_+. One can see h as an element of $T^*M \otimes T^*M$, *i.e.* a 2-tensor, and hence can be written as a sum of a symmetric and an antisymmetric part: $h = g + B$, exploiting the decomposition $T^*M \otimes T^*M \cong Sym^2 T^*M \oplus \Lambda^2 T^*M$, where $g \in Sym^2 T^*M$ and $B \in \Lambda^2 T^*M$. Thus one can write a general element $V_+ \in C_+$ as $V_+ = v + (B + g)v$, where we denote with Bv the contraction $\iota_v B$. The orthogonality condition between C_+ and C_- force us to write $V_- \in C_-$ as $V_- = v + (B - g)v$ and so

$$C_- = \{v + (B - g)v \mid v \in \Gamma(TM)\}\,. \tag{4.2.27}$$

Thanks of the symmetry of the inner product induced by η under B shifts, we can identify g as an ordinary Riemannian metric on M. Indeed, given any $V_+, W_+ \in C_+$ and their inner product

$$\begin{aligned}\eta(V_+, W_+) &= \eta(v + \iota_v B + gv, w + \iota_w B + gw) \\ &= \eta(v + gv, w + gw) \\ &= \frac{1}{2}(\iota_w gv + i_v gw) = g(v, w)\,.\end{aligned}$$

By construction, g is a positive definite Riemannian metric on M.

One can find an explicit form for the generalised metric $\mathcal{G}$ (4.2.25) by studying its action on $C_\pm$. The construction below will closely follow the one in [81], which we refer to for further details. Given a generalised vector $V \in E$, one can write it as $V = V_+ + V_-$, where $V_\pm \in C_\pm$. Thus, we have the map

$$\begin{aligned} \mathcal{G}:\ E &\longrightarrow E^* \cong E \\ V &\longmapsto \mathcal{G}(V) = \mathcal{G}(V,\cdot) \end{aligned} \tag{4.2.28}$$

where we denoted by $\mathcal{G}(V)$ the *generalised one-form* $\mathcal{G}(V,\cdot)$, but since $(TM \oplus T^*M)^* \cong TM \oplus T^*M$ it can be thought as a generalised vector and then decomposed in $C_\pm$ components,

$$\mathcal{G}(V) = \mathcal{G}(V,\cdot) = \eta(V_+ + V_-,\cdot)\big|_{C_+} - \eta(V_+ + V_-,\cdot)\big|_{C_-} = V_+ - V_-\,. \tag{4.2.29}$$

From the expression above, one can state $\mathcal{G}^2 = \mathbb{1}$, and $C_\pm$ are the eigenspaces relative to the eigenvalues ± 1 of $\mathcal{G}$. Consider the usual Riemannian metric g over M, this induces the splits of $TM \oplus T^*M$ into

$$C_{g\pm} = \{v \pm gv \mid v \in TM\}\,. \tag{4.2.30}$$

A generic vector in $C_{g\pm}$ can be written as $V_{g\pm} = v \pm gv$. In this particular case $2x = V_{g+} + V_{g-}$, thus, we are now allowed to write

$$2\mathcal{G}(v) = V_{g+} - V_{g-} = 2g(v),$$

and since $\mathcal{G}^2 = \mathbb{1}$, it holds

$$2\mathcal{G}^2(v) = 2\mathcal{G}(g(v)) = V_{g+} + V_{g-} = 2v\,.$$

We look for an explicit form of $\mathcal{G}$ in terms of the two quantities g and B. The simplest form the matrix $\mathcal{G}$ can take (compatible with the two conditions above) is with $B = 0$,

$$\mathcal{G}_g = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}.$$

Now we want to reintroduce the B field we have ignored so far. Recall that

$$e^B V_{g\pm} = (v \pm gv + Bv) = V_\pm$$

and also that

$$\mathcal{G}(V_\pm) = \pm V_\pm. \tag{4.2.31}$$

Using the previous relations and applying the B transformation to $V_{g\pm}$, we can obtain a matrix representation for $\mathcal{G}$ as follows,

$$\begin{aligned} V_\pm = e^B V_{g\pm} &= \pm e^B \mathcal{G}_g V_{g\pm} = \\ &= \pm e^B \mathcal{G}_g e^{-B} e^B V_{g\pm} = \\ &= \pm e^B \mathcal{G}_g e^{-B} V_\pm\,. \end{aligned}$$

This is true if and only if

$$\mathcal{G} = e^{-B}\mathcal{G}_g e^B = \begin{pmatrix} g^{-1}B & g^{-1} \\ g - Bg^{-1}B & -Bg^{-1} \end{pmatrix}. \tag{4.2.32}$$

The generalised metric is an element of the coset space $\mathrm{O}(d,d)/O(d) \times O(d)$ and encodes both the metric and the B-field [82]. One can also introduce *generalised* *vielbeins*, parametrising the coset, the local flat symmetry in ordinary geometry is here replaced by $O(d) \times O(d)$. We also require that the set of local vielbeins $\{\hat{E}_A\}$ makes the generalised metric take the following form,

$$\eta = \hat{E}^T \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \hat{E}\,, \qquad \mathcal{G} = \hat{E}^T \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \hat{E}\,, \tag{4.2.33}$$

An explicit form is given by,

$$\hat{E} = \frac{1}{\sqrt{2}} \begin{pmatrix} e_+ - \hat{e}_+^T B & \hat{e}_+^T \\ -e_- - \hat{e}_-^T B & \hat{e}_-^T \end{pmatrix}, \tag{4.2.34}$$

where $e_\pm$ are two sets of vielbeins and $\hat{e}_\pm$ their inverse. They have to satisfy

$$g = e_\pm^T e_\pm\,, \qquad g^{-1} = \hat{e}_\pm \hat{e}_\pm^T\,, \tag{4.2.35}$$

which are the canonical conditions on ordinary vielbeins.

4.2.3 Generalised almost complex structure

A *generalised almost complex structure* is a map

$$\mathcal{I} : E \longrightarrow E\,, \tag{4.2.36}$$

compatible with the bundle structure, *i.e.* $\pi(\mathcal{I}V) = \pi(V)$ and with the property analogous to (**??**),

$$\mathcal{I}^2 = -\mathbb{1}\,. \tag{4.2.37}$$

In addition, compatibility with the $\mathrm{O}(d,d)$ metric is required,

$$\mathcal{I}^T \eta \mathcal{I} = \eta\,, \tag{4.2.38}$$

or equivalently

$$\eta(\mathcal{I}V, \mathcal{I}W) = \eta(V,W)\,, \qquad \forall V, W\,. \tag{4.2.39}$$

An ordinary complex structure reduces the structure group to $\mathrm{U}(d/2)$. Here something analogous happens, the generalised almost complex structure implies M is even-dimensional, and the structure group of the exceptional tangent bundle $\mathrm{O}(d,d)$ is reduced to $\mathrm{U}(d/2, d/2)$.

Also in this case, one can adopt a "matrix" notation and describe the generalised almost complex structure as a block matrix acting on a generalised vector. From (4.2.38), it follows that the generalised almost complex structure takes the form,

$$\mathcal{I} = \begin{pmatrix} -I & R \\ L & I^T \end{pmatrix}, \tag{4.2.40}$$

where R and L are an antisymmetric two-vector and a two-form respectively. The requirement for $\mathcal{I}$ to square to $-\mathbb{1}$ imples,

$$I^2 + RL = -\mathbb{1}_d\,, \tag{4.2.41}$$

$$-IR + RI^T = 0\,, \tag{4.2.42}$$

$$-LI + I^T L = 0\,. \tag{4.2.43}$$

An important feature of the generalised almost complex structure is that it contains both the ordinary almost complex and the almost symplectic structures,

$$\mathcal{I}_I = \begin{pmatrix} -I & 0 \\ 0 & I^T \end{pmatrix}, \qquad \mathcal{I}_\omega = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}. \tag{4.2.44}$$

We can think at the general case (4.2.40) as a set of continuous intermediate structures, interpolating between the two extrema of complex and symplectic geometry, by varying the tensors R and L. Indeed, the original purpose of Hitchin and Guatieri [69, 75] was to find a way to unify the symplectic and complex geometry.

As in the ordinary case, also the generalised almost complex structure generates two *generalised distributions*, that is the two eigenbundles associated to the eigenvectors $\pm i$,

$$E \otimes \mathbb{C} = L_{\mathcal{I}} \oplus \overline{L}_{\mathcal{I}}\,. \tag{4.2.45}$$

This is somehow analogous to the split seen previously induced by the generalised metric.

$L_{\mathcal{I}}$ and $\overline{L}_{\mathcal{I}}$ are *maximally isotropic sub-bundles* of $E \otimes \mathbb{C}$. Recall that a generalised sub-bundle L is *isotropic* if and only if it is *null* with respect to the natural $\mathrm{O}(d,d)$ metric,

$$\eta(V,W) = 0 \qquad \forall V, W \in L\,. \tag{4.2.46}$$

In addition, it is maximally isotropic if its rank is half of the rank of E.

We can show that $L_{\mathcal{I}}$ is actually maximally isotropic. Given two vectors $V, W \in L_{\mathcal{I}}$, their inner product is,

$$\eta(V,W) = V^T \eta W = V^T \mathcal{I}^T \eta \mathcal{I} W = (iV)^T \eta (iW) = -V^T \eta W = -\eta(V,W) = 0\,,$$

where we used the (4.2.38) in the second equality and the fact that both V and W are elements of $L_{\mathcal{I}}$ in the third one. Moreover, since $L_{\mathcal{I}}$ and $\bar{L}_{\mathcal{I}}$ have the same rank, we have that both have complex dimension d, such that $L_{\mathcal{I}}$ is maximally isotropic.

4.2.4 Dorfman derivative and Courant bracket

One can define a differential operator generalising the action of the Lie derivative in the ordinary case. We will also see how to generalised the concept of infinitesimal diffeomorphism for the generalised tangent bundle.

Given two sections of E, for instance $V = v + \mu$ and $W = w + \lambda$, where $v, w \in \Gamma(TM)$ and $\mu, \lambda \in \Gamma(T^*M)$, we define the *Dorfman derivative* or *generalised Lie derivative* [68, 73, 79] as

$$L_V W := \mathcal{L}_v w + \mathcal{L}_v \lambda - \iota_w \mathrm{d}\mu\,. \tag{4.2.47}$$

The Dorfman derivative enjoys the Leibnitz rule, *i.e.*

$$L_V(L_W Z) = L_{L_V W} Z + L_W(L_V Z)\,. \tag{4.2.48}$$

This gives the generalised tangent bundle, endowed with the generalised Lie derivative the structure of a *Leibnitz algebroid* [83].

Note that we can give a definition that makes explicit the action of the $\mathrm{O}(d,d)$ group. This comes from the observation that the Lie derivative between two ordinary vectors v and v' of $\Gamma(TM)$ can be written in components using the $\mathfrak{gl}(d,\mathbb{R})$ action,

$$(\mathcal{L}_v v')^m = v^n\, \partial_n v'^m - (\partial \times v)^m{}_n\, v'^n\,.$$

Thus, by analogy one can use the $\mathrm{O}(d,d)$ action to write the Dorfman derivative as

$$\left(L_V V'\right)^M = V^N \partial_N V'^M - (\partial \times_{\mathrm{ad}} V)^M{}_N V'^N \,, \tag{4.2.49}$$

where $\times_{\mathrm{ad}}$ denotes the projection onto the adjoint bundle,

$$\times_{\mathrm{ad}} : E^* \times E \longrightarrow \mathrm{ad} \,. \tag{4.2.50}$$

This definition is useful not only because allows to define the action of the L_V operator also to representations of $\mathrm{O}(d,d)$ other than the fundamental one (like the adjoint, the second rank symmetric, etc.), but also because it is easily extendable to the exceptional case.

The Dorfman derivative is not antisymmetric. Its antisymmetrisation defines the *Courant bracket* [79,84],

$$[\![V,W]\!] := \frac{1}{2}(L_V W - L_W V) = [v,w] + \mathcal{L}_v\lambda - \mathcal{L}_w\mu - \frac{1}{2}\mathrm{d}(\iota_v\lambda - \iota_w\mu) \,, \tag{4.2.51}$$

which makes explicit the fact that in the $\mathrm{O}(d,d)$ generalised geometry we have a *Courant algebroid* [84,85]. A nice historical review on the subject can be found in [86].

The Courant bracket educes to the ordinary Lie bracket when restricted to vectors, while it vanishes on one-forms. It is invariant under diffeomorphisms, and under B-shifts parametrised by a closed 2-form b,

$$[\![e^b \cdot V, e^b \cdot W]\!] = e^b [\![V,W]\!] + \iota_v\iota_w \mathrm{d}b \,. \tag{4.2.52}$$

The relation above suggests the introduction of a *twisted* Courant bracket by a 3-form H

$$[\![V,W]\!]_H = [\![V,W]\!] + \iota_v\iota_w H \,. \tag{4.2.53}$$

One can see that the twisted Courant bracket is the right differential operator acting on untwisted vectors, while the (4.2.51) is the one used in the twisted picture, where the B-shift is already encoded in generalised vectors.

Integrability

Recall that for an ordinary complex structure, integrability can be expressed in terms of the involutivity of its $\pm i$ eigenbundles with respect to the Lie bracket. Here, we can give a definition of integrability in terms of Courant bracket and the involutivity of the generalised subbundle $L_{\mathcal{I}}$.

Given a generalised almost complex structure $\mathcal{I}$ on E, we say it is *integrable* if and only if its i-eigenspace $L_{\mathcal{I}}$ is closed under the Courant bracket,

$$[\![V,W]\!] \in \Gamma(L_{\mathcal{I}}) \,, \qquad \forall V,W \in \Gamma(L_{\mathcal{I}}) \,. \tag{4.2.54}$$

A manifold admitting such a structure (called *generalised complex structure*) is said *generalised complex manifold.*

4.2.5 Generalised geometry and compactifications

Generalised geometry allows to treat on the same ground diffeomorphisms and gauge transformations of the NS sector of type II supergravities. It is also a powerful tool to classify and study flux vacua. Let us consider again the case of $\mathcal{N} = 1$ flux compactifications to four dimensions.

The idea is that one can define a pair of bispinors built out the supersymmetry parameters, η_1 and η_2

$$\begin{aligned} \Phi^+ &= e^{-\phi}e^{-B}\big(\eta_1^+ \otimes \overline{\eta}_2^+\big) \in \Gamma(\Lambda^{\text{even}}T^*M)\,, \\ \Phi^- &= e^{-\phi}e^{-B}\big(\eta_1^+ \otimes \overline{\eta}_2^-\big) \in \Gamma(\Lambda^{\text{odd}}T^*M)\,, \end{aligned} \tag{4.2.55}$$

which are globally defined. The two polyforms $\Phi^{\pm}$ can be seen as sections of the positive and negative helicity $\mathrm{Spin}(6,6)$ spinor bundles associated to E through the Clifford map. And are associated to an almost generalised complex structure each. Each of them is stabilised by a different $\mathrm{SU}(3,3)$ subrgroup of $\mathrm{Spin}(6,6)$. Hence, each of them defines a different $\mathrm{SU}(3,3)$ generalised structure. The compatibility condition implies that the group leaving both invariant has to be the intersection of the two $\mathrm{SU}(3,3)$ subgroups, then $\mathrm{SU}(3) \times \mathrm{SU}(3)$. Thus, we see that all $\mathcal{N} = 1$ flux backagrounds must have $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure [87, 88].

On can show [87, 89], that Killing spinor equations can be rewritten as differential conditions on such spinors

$$\mathrm{d}_H(e^{3A}\Phi_1) = 0\,, \tag{4.2.56}$$

$$\mathrm{d}_H(e^{2A}\,\mathbb{I}\mathrm{m}\,\Phi_2) = 0\,, \tag{4.2.57}$$

$$\mathrm{d}_H(e^{4A}\,\mathbb{R}\mathrm{e}\,\Phi_2) = e^{4A} \star \lambda(F)\,, \tag{4.2.58}$$

where d_H is the H-twisted derivative and Φ_1 and Φ_2 correspond to Φ_+ and Φ_- in type IIA and vice-versa for IIB. It is possible to show that such conditions correspond to the integrability of the generalised complex structure associated to Φ_1. The supersymmetry conditions are also equivalent to the existence of a torsion-free generalised connection and structure-compatible.

The ordinary Calabi-Yau case can be retrieved as a particular choice of the $\Phi^{\pm}$,

$$\Phi^+ = e^{-\phi}e^{-B}e^{i\omega}\,, \qquad \Phi^- = ie^{-\phi}e^{-B}\Omega\,, \tag{4.2.59}$$

with B closed (eventually it can be made zero by a gauge transformation) and ϕ constant. From this particular case we can see that Φ^+ is a generalisation of the symplectic structure, while Φ^- captures the generalisation of the complex structure.

4.3 Exceptional generalised geometry

Generalised complex geometry has a natural application to string theory, since it allows to treat in a geometric way diffeomorphisms and gauge transformations of the NS sector of supergravity theories, This motivated the introduction of Exceptional generalised geometry where the generalised tangent bundle admits the action of larger structure groups, $\mathrm{E}_{d(d)} \times \mathbb{R}^+$ [70, 73, 90][1], thus allowing to encode also the RR degrees of freedom of the various supergravities.

[1] $\mathrm{E}_{d(p)}$ is a non-compact version of the exceptional group E_d, meaning the group having as Lie algebra the exceptional one $\mathfrak{e}_d$, with a number p of non-compact generators. $\mathrm{E}_{d(d)}$ is the maximal non-compact form.

The general definitions of frame bundle, generalised G-structures and generalised Lie derivative hold also here. An important difference between $\mathrm{O}(d,d)$ generalised geometry and the exceptional one is that in the $\mathrm{O}(d,d)$ case the same structure of the generalised tangent bundle allows to describe both type IIA and IIB and furthermore it does not depend on the dimension of the manifold M_d, as we will see in **??**. On the other hand, the exceptional tangent bundle takes a different form depending on whether one works in M-theory, type IIA or type IIB supergravity, and depending on the dimension of M_d, its fibres transform in different representations of the structure group.

Thus, we are going to analyse the various cases separately, describing the suitable generalised geometry to describe the theories we will focus on.

4.3.1 M-theory

Here we review from [90] the construction of the exceptional geometry for M-theory. The idea is to construct a generalised tangent bundle whose transition functions contain the three- and six-form potentials of M-theory.

Given a d-dimensional manifold M with $d \leq 7$. The $\mathrm{O}(d,d)$ group of generalised geometry is replaced by $\mathrm{E}_{d(d)}$. We define the generalised tangent bundle as isomorphic to a sum of tensor bundles [70,73], corresponding to the different $\mathrm{GL}(d,\mathbb{R})$n irreducible representations

$$E \cong TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \oplus (T^*M \otimes \Lambda^7 T^*M) \,. \tag{4.3.1}$$

where for $d < 7$ some of these terms will not be present.

A generic section of E is written as,

$$V = v + \omega + \sigma + \tau \,, \tag{4.3.2}$$

where $v \in \Gamma(TM)$ is a vector, $\omega \in \Gamma(\Lambda^2 T^*M)$, so is a two-form, etc.

The bundle is defined together with patching rules. These are such that, given a chart U_α of an atlas covering M, we have

$$\begin{aligned} V_{(\alpha)} &= v_{(\alpha)} + \omega_{(\alpha)} + \sigma_{(\alpha)} + \tau_{(\alpha)} \\ &\quad \in \Gamma\big(TU_{(\alpha)} \oplus \Lambda^2 T^*U_{(\alpha)} \oplus \Lambda^5 T^*U_{(\alpha)} \oplus (T^*U_{(\alpha)} \otimes \Lambda^7 T^*U_{(\alpha)})\big) \,, \end{aligned} \tag{4.3.3}$$

for a local section. Then these sections are patched through the whole E as

$$V_{(\alpha)} = e^{\mathrm{d}\Lambda_{(\alpha\beta)} + \mathrm{d}\tilde{\Lambda}_{(\alpha\beta)}} \cdot V_{(\beta)} \,, \qquad \text{on } U_\alpha \cap U_\beta \,. \tag{4.3.4}$$

The two quantities $\Lambda_{(\alpha\beta)}$ and $\tilde{\Lambda}_{(\alpha\beta)}$ are respectively a two- and a five-form, and $\cdot$ denotes the adjoint action of $\mathrm{E}_{d(d)}$. In components this reads

$$v_{(\alpha)} = v_{(\beta)} \,, \tag{4.3.5a}$$

$$\omega_{(\alpha)} = \omega_{(\beta)} + \iota_{v_{(\beta)}} \mathrm{d}\Lambda_{(\alpha\beta)} \,, \tag{4.3.5b}$$

$$\sigma_{(\alpha)} = \sigma_{(\beta)} + \mathrm{d}\Lambda_{(\alpha\beta)} \wedge \omega_{(\beta)} + \tfrac{1}{2} \mathrm{d}\Lambda_{(\alpha\beta)} \wedge \iota_{v_{(\beta)}} \mathrm{d}\Lambda_{(\alpha\beta)} + \iota_{v_{(\beta)}} \mathrm{d}\tilde{\Lambda}_{(\alpha\beta)} \,, \tag{4.3.5c}$$

$$\begin{aligned} \tau_{(\alpha)} &= \tau_{(\beta)} + j\mathrm{d}\Lambda_{(\alpha\beta)} \wedge \sigma_{(\beta)} - j\mathrm{d}\tilde{\Lambda}_{(\alpha\beta)} \wedge \omega_{(\beta)} + j\mathrm{d}\Lambda_{(\alpha\beta)} \wedge \iota_{v_{(\beta)}} \mathrm{d}\tilde{\Lambda}_{(\alpha\beta)} \\ &\quad + \tfrac{1}{2} j\mathrm{d}\Lambda_{(\alpha\beta)} \wedge \mathrm{d}\Lambda_{(\alpha\beta)} \wedge \omega_{(\beta)} + \tfrac{1}{6} j\mathrm{d}\Lambda_{(\alpha\beta)} \wedge \mathrm{d}\Lambda_{(\alpha\beta)} \wedge \iota_{v_{(\beta)}} \mathrm{d}\Lambda_{(\alpha\beta)} \,, \end{aligned} \tag{4.3.5d}$$

where j denotes the operator

$$(j\lambda \wedge \mu)_{m,m_1\dots m_d} = \frac{d!}{(p-1)!(d-p+1)!}\,\lambda_{m[m_1\dots m_{p-1}}\mu_{m_p\dots m_d]}\,, \tag{4.3.6}$$

for $\lambda \in \Lambda^p T^*$ and $\mu \in \Lambda^{d-p+1}T^*$.

The collection of $\Lambda_{(\alpha\beta)}$ defines this connective structure on the gerbe, satisfying the series of relations analogous to the (4.2.5),

$$\begin{aligned}
&\Lambda_{(\alpha\beta)} + \Lambda_{(\beta\gamma)} + \Lambda_{(\gamma\alpha)} = \mathrm{d}\Lambda_{(\alpha\beta\gamma)} && \text{on } U_\alpha \cap U_\beta \cap U_\gamma\,,\\
&\Lambda_{(\beta\gamma\delta)} - \Lambda_{(\alpha\gamma\delta)} + \Lambda_{(\alpha\beta\delta)} - \Lambda_{(\alpha\beta\gamma)} = \mathrm{d}\Lambda_{(\alpha\beta\gamma\delta)} && \text{on } U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta\,.
\end{aligned} \tag{4.3.7}$$

Similar relations hold for $\tilde{\Lambda}$ [72],

$$\begin{aligned}
\tilde{\Lambda}_{(\alpha\beta)} - \tilde{\Lambda}_{(\beta\gamma)} + \tilde{\Lambda}_{(\gamma\alpha)} =& \mathrm{d}\tilde{\Lambda}_{(\alpha\beta\gamma)}\\
&+ \tfrac{1}{2(3!)}\big(\Lambda_{(\alpha\beta)} \wedge \mathrm{d}\Lambda_{(\beta\gamma)} + \text{antisymm. in } [\alpha\beta\gamma]\big)\,,\\
\tilde{\Lambda}_{(\alpha\beta\gamma)} - \tilde{\Lambda}_{(\alpha\beta\delta)} + \tilde{\Lambda}_{(\alpha\gamma\delta)} - \tilde{\Lambda}_{(\beta\gamma\delta)} =& \mathrm{d}\tilde{\Lambda}_{(\alpha\beta\gamma\delta)}\\
&+ \tfrac{1}{2(4!)}\big(\Lambda_{(\beta\gamma\delta)} \wedge \mathrm{d}\Lambda_{(\gamma\delta)} + \text{antisymm. in } [\alpha\beta\gamma\delta]\big)\,.
\end{aligned}$$

In this case one can notice that the connective structure for $\tilde{\Lambda}$ depends on Λ, this further generalises the previous gerbe construction [75], but it generates the correct patching rules that will be reinterpreted as gauge transformations of supergravity potentials in the next chapter.

Technically the patching rules (4.3.4) defines the generalised tangent bundle as a series of extensions,

$$\begin{aligned}
&0 \longrightarrow \Lambda^2 T^*M \longrightarrow E'' \longrightarrow TM \longrightarrow 0\,,\\
&0 \longrightarrow \Lambda^5 T^*M \longrightarrow E' \longrightarrow E'' \longrightarrow 0\,,\\
&0 \longrightarrow T^*M \otimes \Lambda^7 T^*M \longrightarrow E \longrightarrow E' \longrightarrow 0\,.
\end{aligned} \tag{4.3.8}$$

Analogously to what we have seen in (4.2.1), these extensions are splitted [77] into the isomorphism (4.3.1) by the choice of some potentials, formally some connections on a gerbe [75].

As for generalised geometry, we can define a frame bundle. Let us define $\{\hat{E}_A\}$ as a basis of a fibre of the exceptional tangent bundle (A runs over the dimension of the bundle). As in (4.2.22), the frame bundle is a principal bundle by construction. An exceptional G-structure is defined as a principal sub-bundle of F, such that its structure group is reduced to G.

Take into account a point $p \in M$ and the exceptional fibre in that point E_p. Let $\{\hat{e}_a\}$ be a basis for T_pM and $\{e^a\}$ a basis for T_p^*M. Following [72], an explicit basis of E_p can be constructed as

$$\{\hat{E}_A\} = \{\hat{e}_a\} \cup \{e^{ab}\} \cup \{e^{a_1\dots a_5}\} \cup \{e^{a.a_1\dots a_6}\}\,. \tag{4.3.9}$$

Thus, the definition for the exceptional frame bundle (analogous to (4.2.22)) reads

$$F = \Big\{\Big(x, \{\hat{E}_A\}\Big) \mid x \in M\,,\ \text{and } \{\hat{E}_A\} \text{ basis of } E\Big\}\,. \tag{4.3.10}$$

By construction, this is a principal bundle with fibre $\mathrm{E}_{d(d)} \times \mathbb{R}^+$. It might be useful to consider the decomposition under $\mathrm{GL}(d,\mathbb{R})$ of the bundle transforming in the adjoint representation,

$$\mathrm{ad} \cong \mathbb{R} \oplus (TM \otimes T^*M) \oplus \Lambda^3 T^*M \oplus \Lambda^6 T^*M \oplus \Lambda^3 TM \oplus \Lambda^6 TM\,, \tag{4.3.11}$$

then one can see it contains a scalar l, a $\mathfrak{gl}(d,\mathbb{R})$ element r, three- and a six-form A and $\tilde{A}$ and a three- and a six-vector α and $\tilde{\alpha}$. Since we are interested in describing the degrees of freedom of eleven-dimensional supergravity, we will interpret A and $\tilde{A}$ as the supergravity potentials. In exceptional generalised geometry these are gerbe connections patched on an overlap $U_\alpha \cap U_\beta$ as,

$$\begin{aligned} A_{(\alpha)} &= A_{(\beta)} + \mathrm{d}\Lambda_{(\alpha\beta)}\,, \\ \tilde{A}_{(\alpha)} &= \tilde{A}_{(\beta)} + \mathrm{d}\tilde{\Lambda}_{(\alpha\beta)} - \frac{1}{2}\mathrm{d}\Lambda_{(\alpha\beta)} \wedge A_{(\beta)}\,. \end{aligned} \tag{4.3.12}$$

We will see how these reproduces the gauge transformations for potentials in eleven-dimensional supergravity [70, 72, 73]. Indeed the invariant field strengths

$$\begin{aligned} F &= \mathrm{d}A\,, \\ \tilde{F} &= \mathrm{d}\tilde{A} - \frac{1}{2} A \wedge F\,, \end{aligned} \tag{4.3.13}$$

reproduce the supergravity ones.

Also in the exceptional case we can define untwisted vectors $\tilde{V}$ as

$$V = e^{A+\tilde{A}} \cdot \tilde{V}\,, \tag{4.3.14}$$

where $\cdot$ denotes the adjoint action of the $\mathfrak{e}_7 \oplus \mathbb{R}^+$ algebra (given explicitly in **??**) [73]. The sections of E are called *twisted* vectors, while the $\tilde{V}$ take the name of untwisted generalised sections.

Also here the Dorfman derivative is constructed as a generalisation of the Lie derivative. In particular, it holds also the (4.2.49), and similarly indicating by V^M the components of V in a standard coordinate basis, and embedding the standard derivative operator as a section of the dual generalised tangent bundle E^*, one can define the Dorfman derivative as

$$\left(L_V V'\right)^M = V^N \partial_N V'^M - (\partial \times_{\mathrm{ad}} V)^M{}_N V'^N\,, \tag{4.3.15}$$

where again $\times_{\mathrm{ad}}$ is the projection onto the adjoint bundle,

$$\times_{\mathrm{ad}} : E^* \times E \to \mathrm{ad}\,. \tag{4.3.16}$$

In terms of $\mathfrak{gl}(d,\mathbb{R})$ components the generalised Lie derivative can be expressed as

$$\begin{aligned} L_V V' = \mathcal{L}_v v' &+ \left(\mathcal{L}_v \omega' - \iota_{v'} \mathrm{d}\omega\right) + \left(\mathcal{L}_v \sigma' - \iota_{v'} \mathrm{d}\sigma - \omega' \wedge \mathrm{d}\omega\right) \\ &+ \left(\mathcal{L}_v \tau' - j\sigma' \wedge \mathrm{d}\omega - j\omega' \wedge \mathrm{d}\sigma\right)\,. \end{aligned} \tag{4.3.17}$$

The version of Dorfman derivative being able to act on untwisted objects is called *twisted Dorfman derivative* and denoted by $\mathbb{L}_{\tilde{V}}$,

$$\mathbb{L}_{\tilde{V}} \tilde{\mathcal{A}} = e^{-A-\tilde{A}} L_{e^{A+\tilde{A}}\tilde{V}} (e^{A+\tilde{A}} \tilde{\mathcal{A}})\,, \tag{4.3.18}$$

where $\tilde{\mathcal{A}}$ is a generic generalised tensor.

The explicit form of the twisted derivative is the same as the untwisted one, modulo the following replacements,

$$\begin{aligned} \mathrm{d}\omega &\to \mathrm{d}\tilde{\omega} - \iota_{\tilde{v}} F\,, \\ \mathrm{d}\sigma &\to \mathrm{d}\tilde{\sigma} - \iota_{\tilde{v}} \tilde{F} + \tilde{\omega} \wedge F\,. \end{aligned} \tag{4.3.19}$$

We collect in the **??** the other relevant objects and representation bundles for exceptional generalised geometry.

4.3.2 Type IIA

The relevant generalised geometry for type IIA theories has been constructed in [12]. The structure group for a d-dimensional manifold is $\mathrm{E}_{d+1(d+1)} \times \mathbb{R}^+$. The generalised geometry for IIA can obtained by a reduction o f the M-theory one. We give this construction explicitly in **??**, here we just collect the most important results and definitions.

The generalised tangent bundle iE is isomorphic to

$$E \cong TM \oplus T^*M \oplus \Lambda^5 T^*M \oplus \Lambda^{\mathrm{even}} T^*M \oplus (T^*M \otimes \Lambda^6 T^*M)\,, \tag{4.3.20}$$

and a generic section can be decomposed according to a $\mathrm{GL}(d, \mathbb{R})$ as

$$V = v + \lambda + \sigma + \omega + \tau\,, \tag{4.3.21}$$

where v is a vector, λ a one-form, σ a five-form, ω a polyform in $\Gamma(\Lambda^{\mathrm{even}} T^*M)$ and $\tau \in \Gamma(T^*M \otimes \Lambda^6 T^*M)$.

As in M-theory E s defined by a series of extensions,

$$\begin{array}{l} 0 \longrightarrow T^*M \longrightarrow E''' \longrightarrow TM \longrightarrow 0\,, \\ 0 \longrightarrow \Lambda^{\mathrm{even}} T^*M \longrightarrow E'' \longrightarrow E''' \longrightarrow 0\,, \\ 0 \longrightarrow \Lambda^5 T^*M \longrightarrow E' \longrightarrow E'' \longrightarrow 0\,. \\ 0 \longrightarrow T^*M \otimes \Lambda^6 T^*M \longrightarrow E \longrightarrow E' \longrightarrow 0\,. \end{array} \tag{4.3.22}$$

In order to define the parching rules for potentials and generalised vectors, first we define the *untwisted* section $\tilde{V} = \tilde{v} + \tilde{\lambda} + \tilde{\sigma} + \tilde{\omega} + \tilde{\tau}$ as

$$V = e^{\tilde{B}} e^{-B} e^{C} \cdot \tilde{V}\,, \tag{4.3.23}$$

where, as usual, $\cdot$ denotes the adjoint action of the structure group algebra. This twist concretely specifies the isomorphism (4.3.20). An explicit expansion of (4.3.23) can be written as,

$$\begin{aligned} v &= \tilde{v}\,, \\ \lambda &= \tilde{\lambda} + \iota_{\tilde{v}} B\,, \\ \sigma &= \tilde{\sigma} + \iota_{\tilde{v}} \tilde{B} - \Big[s(C) \wedge \big(\tilde{\omega} + \tfrac{1}{2} \iota_{\tilde{v}} C + \tfrac{1}{2} \tilde{\lambda} \wedge C \big) \Big]_5\,, \\ \omega &= e^{-B} \wedge \big(\tilde{\omega} + \iota_{\tilde{v}} C + \tilde{\lambda} \wedge C \big)\,, \\ \tau &= \tilde{\tau} + jB \wedge \Big[\tilde{\sigma} - s(C) \wedge \big(\tilde{\omega} + \tfrac{1}{2} \iota_{\tilde{v}} C + \tfrac{1}{2} \tilde{\lambda} \wedge C \big) \Big]_5 + j\tilde{B} \wedge (\tilde{\lambda} + \iota_{\tilde{v}} B) \\ &\quad - js(C) \wedge \big(\tilde{\omega} + \tfrac{1}{2} \iota_{\tilde{v}} C + \tfrac{1}{2} \tilde{\lambda} \wedge C \big)\,, \end{aligned} \tag{4.3.24}$$

where $[\cdot]_k$ denotes the degree k-form of a polyform.

The patching rules for the generalised vector V on the intersection $U_\alpha \cap U_\beta$ of two charts U_α and U_β, reads

$$V_{(\alpha)} = e^{\mathrm{d}\tilde{\Lambda}_{(\alpha\beta)}} e^{\mathrm{d}\Omega_{(\alpha\beta)}} e^{\mathrm{d}\Lambda_{(\alpha\beta)}} V_{(\beta)} \,, \tag{4.3.25}$$

where $\Lambda_{(\alpha\beta)}$ is a one-form, $\tilde{\Lambda}_{(\alpha\beta)}$ a five-form, and $\Omega_{(\alpha\beta)}$ a poly-form of even degree, all defined on $U_\alpha \cap U_\beta$. In the (4.3.25) we have dropped the transformations due to the $\mathrm{GL}(d,\mathbb{R})$ action. Plugging (4.3.23) into (4.3.25) and reorganising the exponentials on the right hand side, one obtains the patching conditions for the adjoint fields (corresponding to gauge transformations of supergravity potentials),

$$\begin{aligned} B_{(\alpha)} &= B_{(\beta)} + \mathrm{d}\Lambda_{(\alpha\beta)} \,, \\ C_{(\alpha)} &= C_{(\beta)} + e^{B_{(\beta)}+\mathrm{d}\Lambda_{(\alpha\beta)}} \wedge \mathrm{d}\Omega_{(\alpha\beta)} \,, \\ \tilde{B}_{(\alpha)} &= \tilde{B}_{(\beta)} + \mathrm{d}\tilde{\Lambda}_{(\alpha\beta)} + \tfrac{1}{2}\Big[\mathrm{d}\Omega_{(\alpha\beta)} \wedge e^{B_{(\beta)}+\mathrm{d}\Lambda_{(\alpha\beta)}} \wedge s(C_{(\beta)})\Big]_6 \,. \end{aligned} \tag{4.3.26}$$

As we clarify in **??**, these do indeed correspond to the finite supergravity gauge transformations between patches (here given for vanishing Romans mass, $m = 0$). As in the previous case, this construction generalises the standard definition of a gerbe connection.

It is also useful to consider the decomposition under $\mathrm{GL}(d,\mathbb{R})$ of the adjoint bundle $\mathrm{ad} \subset E \otimes E^*$,

$$\begin{aligned} \mathrm{ad} = \mathbb{R}_\Delta \oplus \mathbb{R}_\phi &\oplus (TM \otimes T^*M) \oplus \Lambda^2 TM \oplus \Lambda^2 T^*M \oplus \Lambda^6 TM \oplus \Lambda^6 T^*M \\ &\oplus \Lambda^{\mathrm{odd}} TM \oplus \Lambda^{\mathrm{odd}} T^*M \,. \end{aligned} \tag{4.3.27}$$

Its sections R can be written as

$$R = l + \varphi + r + \beta + B + \tilde{\beta} + \tilde{B} + \Gamma + C \,, \tag{4.3.28}$$

where $r \in \mathrm{End}(T)$ will correspond to the $GL(d,\mathbb{R})$ action, while the scalars l and φ will be related to the shifts of the warp factor and dilaton, respectively. The forms B, $\tilde{B}$ and $C = C_1 + C_3 + C_5$ will encode the internal components of the NSNS two-form, of its dual and of the RR potentials. The other elements are poly-vectors obtained by raising the indices of the forms, and do not have an immediate supergravity counterpart.

To conclude this discussion, we want to raise an observation about the relation between the $\mathrm{E}_{d(d)} \times \mathbb{R}^+$ generalised geometry and the $\mathrm{O}(d,d)$ one. Indeed, one can additionally view E as an extension of Hitchin's generalised tangent space E' [68,69] by $\mathrm{O}(d,d) \times \mathbb{R}^+$ tensor bundles, as we describe in **??**.

The Dorfman derivative for the massless type IIA can be obtained by the (4.3.17) by a dimensional reduction, or by analogy from (4.3.15). Using an index M to denote the components of a generalised vector V in a standard coordinate basis,

$$V^M = \{v^m, \lambda_m, \sigma_{m_1\dots m_5}, \tau_{m,m_1\dots m_6}, \omega, \omega_{m_1 m_2}, \omega_{m_1\dots m_4}, \omega_{m_1\dots m_6}\} \,, \tag{4.3.29}$$

and embedding the standard derivative operator as a section of the dual generalised tangent bundle E^*, $\partial_M = (\partial_m, 0, \dots, 0)$, the Dorfman derivative is defined as [72]

$$(L_V V')^M = V^N \partial_N V'^M - (\partial \times_{\mathrm{ad}} V)^M{}_N V'^N \,, \tag{4.3.30}$$

where $\times_{\text{ad}}$ is again the projection onto the adjoint bundle. In this case, this explicitly gives

$$\partial \times_{\text{ad}} V = \partial \times v - \mathrm{d}\lambda + \mathrm{d}\sigma + \mathrm{d}\omega \ . \tag{4.3.31}$$

We recall this operator satisfies the Leibniz property (4.2.48) and is not antisymmetric. Hence, the generalised Lie derivative for massless type IIA in $\mathrm{GL}(d,\mathbb{R})$ decomposition reads

$$\begin{aligned} L_V V' =& \mathrm{d}\mathcal{L}_v v' + \left(\mathcal{L}_v\lambda' - \iota_{v'}\mathrm{d}\lambda\right) + \left(\mathcal{L}_v\sigma' - \iota_{v'}\mathrm{d}\sigma + [s(\omega')\wedge \mathrm{d}\omega]_5\right) \\ &+ \left(\mathcal{L}_v\tau' + j\sigma'\wedge\mathrm{d}\lambda + \lambda'\otimes\mathrm{d}\sigma + js(\omega')\wedge\mathrm{d}\omega\right) \\ &+ \left(\mathcal{L}_v\omega' + \mathrm{d}\lambda\wedge\omega' - (\iota_{v'} + \lambda'\wedge)\mathrm{d}\omega\right) \ . \end{aligned} \tag{4.3.32}$$

This can also be written in terms of natural derivative operators in $\mathrm{O}(d,d)$ generalised geometry, see **??**.

The action of the generalised Lie derivative on the untwisted bundle can also by defined. Let us denote $\mathbb{L}$ the *twisted Dorfman derivative*[2], defined as follows,

$$\mathbb{L}_{\tilde{V}}\tilde{V}' = e^{-C}e^{B}e^{-\tilde{B}} \cdot L_V V' \ . \tag{4.3.33}$$

This is completely analogous to the *twisted Courant bracket* defined in (4.2.53). Operationally it is useful to get the twisted derivative by the untwisted one by replacing in the (4.3.32),

$$\begin{aligned} \mathrm{d}\tilde{\lambda} &\longrightarrow \mathrm{d}\tilde{\lambda} - \iota_{\tilde{v}}H \ , \\ \mathrm{d}\tilde{\sigma} &\longrightarrow \mathrm{d}\tilde{\sigma} - [s(\tilde{\omega})\wedge F]_6 \ , \\ \mathrm{d}\tilde{\omega} &\longrightarrow \mathrm{d}_H\tilde{\omega} - (\iota_{\tilde{v}} + \tilde{\lambda}\wedge)F \ . \end{aligned} \tag{4.3.34}$$

where H is the three-form $H = \mathrm{d}B$ on M, $F = F_2 + F_4 + F_6$ is a polyform made out of field strengths of the potentials C in (4.3.28), and d_H is the twisted exterior derivative defined by $\mathrm{d}_H = \mathrm{d} - H\wedge$. These field strength forms transform in a generalised bundle which is a the $\mathbf{912}_{-1}$ irreducible representation of $\mathrm{E}_{7(7)}\times\mathbb{R}^+$ [71].

In view of the application of this formalism to supergravity theories, it is useful to stress again that the Dorfman derivative generates the infinitesimal generalised diffeomorphisms on the internal manifold M. Interpreting a generalised vector V as a gauge parameter, the infinitesimal gauge transformation of any field is given by

$$\delta_V = L_V \ .$$

The Leibniz property (4.2.48) then just expresses the gauge algebra $[\delta_V, \delta_{V'}] = \delta_{L_V V'}$.

We want now to see how to include in the formalism the Romans mass. In a string theory perspective this corresponds to a D8 brane filling the ten-dimensional spacetime. One of the main results of my work is the description of a generalised geometry for type IIA accommodating also the Romans mass.

We have seen in the previous chapter how exceptional generalised geometry can accomodate all the fluxes of type II supergravity and M-theory by twisting the exceptional tangent bundle by their potentials. The difficulty in incorporating the mass m in the generalised geometry formalism is that the zero-form flux $m = F_0$ is not expressible as the derivative of a potential. This means that it is not possible to introduce the mass term as an additional twist of the generalised bundle E, as for the other fluxes.

[2]When the generalised tangent bundle is untwisted, the Dorfman derivative is twisted, and vice-versa.

The key point in solving this problem is to look at the way the gauge transformations of the NSNS and RR potentials are realised in exceptional generalised geometry.

We saw in (3.2.17) how the mass affects the gauge transformations of type IIA supergravity. Since the gauge transformations of the supergravity potentials are encoded in the way the twisted generalised vectors patch, the introduction of the Romans mass requires a modification of the patching conditions (4.3.25) and (4.3.26). Following a reasoning that schematically derives the patching conditions from gauge transformations (see **??** for a more detailed discussion), we find the new patching conditions of the form

$$V_{(\alpha)} = e^{\mathrm{d}\tilde{\Lambda}_{(\alpha\beta)}} e^{\mathrm{d}\Omega_{(\alpha\beta)} + m\Omega_{6(\alpha\beta)}} e^{-\mathrm{d}\Lambda_{(\alpha\beta)} - m\,\Lambda_{(\alpha\beta)}} \cdot V_{(\beta)} \,. \tag{4.3.35}$$

This condition reproduces the massive supergravity gauge transformations on overlapping patches $U_\alpha \cap U_\beta$. A first-principles derivation of this is also given in appendix **??**.

Although the structure of the exact sequences (4.3.22) is left intact by this deformation, the precise details of the twisting (4.3.35) do change[3]. An important feature of massive type IIA is that by virtue of the Bianchi identity we have (globally)

$$H_3 = \tfrac{1}{m}\,\mathrm{d}F_2 \,, \tag{4.3.36}$$

so that for $m \neq 0$, H_3 is trivial in cohomology. Thus, the first extension in (4.3.22) is naturally equivalent to the trivial one.

Also, a pure NSNS gauge transformation no longer acts in the $\mathrm{O}(d,d)$ subgroup of $E_{d+1(d+1)} \times \mathbb{R}^+$, simply because it also generates a C_1 RR potential. As such, there is no massive version of Hitchin's $\mathrm{O}(d,d)$ generalised geometry[4].

The modification (4.3.35) of the patching condition also requires a deformation of the Dorfman derivative. Recall that the latter generates the infinitesimal gauge transformations, and that these are affected by the Romans mass via the shifts (3.2.17). It follows that the massive form of the Dorfman derivative is obtained implementing the same shift in the massless expression (4.3.32)

$$\begin{aligned} L_V V' =& \mathcal{L}_v v' + \big(\mathcal{L}_v \lambda' - \iota_{v'} \mathrm{d}\lambda\big) \\ &+ \big(\mathcal{L}_v \sigma' - \iota_{v'}(\mathrm{d}\sigma + m\omega_6) + [s(\omega') \wedge (\mathrm{d}\omega - m\lambda)]_5\big) \\ &+ \big(\mathcal{L}_v \tau' + j\sigma' \wedge \mathrm{d}\lambda + \lambda' \otimes (\mathrm{d}\sigma + m\omega_6) + js(\omega') \wedge (\mathrm{d}\omega - m\lambda)\big) \\ &+ \big(\mathcal{L}_v \omega' + \mathrm{d}\lambda \wedge \omega' - (\iota_{v'} + \lambda'\wedge)(\mathrm{d}\omega - m\lambda)\big) \,, \end{aligned} \tag{4.3.37}$$

which contains the mass as a deformation parameter.

More formally, (4.3.37) is related to the massless Dorfman derivative (here denoted by $L^{(m=0)}$) as

$$L_V V' = L_V^{(m=0)} V' + \underline{m}(V) \cdot V' \,, \tag{4.3.38}$$

where, given a generalised vector V, we define the map $\underline{m}$ such that

$$\underline{m}(V) = m\lambda - m\omega_6 \tag{4.3.39}$$

[3] A consequence of this is the following. In massless IIA we can project a generalised vector onto its vector and zero-form parts $v + \omega_0$, giving a well-defined section of a bundle with seven-dimensional fibre. This is the dimensional reduction of the M-theory tangent bundle TM_7. However, with the massive IIA patching rules (4.3.35), this projection would no longer give a section of a bundle with seven-dimensional fibre. Hence, the massive patching rules do not arise from a seven-dimensional geometry.

[4] Though see [91] for a double field theory approach to this, where the F_0 flux is generated by introducing a linear dependence on the additional non-geometric coordinates dual to the winding modes of the string.

is an object that acts in the adjoint of $\mathrm{E}_{7(7)}$ (see (**??**)) as

$$\underline{m}(V) \cdot V' = m\big(-\iota_{v'}\omega_6 - \lambda \wedge \omega_4' + \lambda' \otimes \omega_6 - \lambda \otimes \omega_6' + \iota_{v'}\lambda + \lambda' \wedge \lambda\big) . \tag{4.3.40}$$

It is a tedious but straightforward computation to verify that (4.3.37) satisfies the Leibniz property (4.2.48)[5].

To justify further our definition, we rewrite the massive Dorfman derivative in the untwisted picture. Using (4.3.33) we find

$$\begin{aligned}
\mathbb{L}_{\tilde{V}}\tilde{V}' =& \mathcal{L}_{\tilde{v}}\tilde{v}' + (\mathcal{L}_{\tilde{v}}\tilde{\lambda}' - \iota_{\tilde{v}'}\mathrm{d}\tilde{\lambda} + \iota_{\tilde{v}'}\iota_{\tilde{v}}H) \\
& + \mathcal{L}_{\tilde{v}}\tilde{\sigma}' - \iota_{\tilde{v}'}\mathrm{d}\tilde{\sigma} \\
& \qquad + \Big[\iota_{\tilde{v}'}(s(\tilde{\omega}) \wedge F) + s(\tilde{\omega}') \wedge \big(\mathrm{d}\tilde{\omega} - H \wedge \tilde{\omega} - (\iota_{\tilde{v}} + \tilde{\lambda}\wedge)F\big)\Big]_5 \\
& + \mathcal{L}_{\tilde{v}}\tilde{\tau}' + j\tilde{\sigma}' \wedge (\mathrm{d}\tilde{\lambda} - \iota_{\tilde{v}}H) + \tilde{\lambda}' \otimes \big(\mathrm{d}\tilde{\sigma} - [s(\tilde{\omega}) \wedge F]_6\big) \\
& \qquad + js(\tilde{\omega}') \wedge \big(\mathrm{d}\tilde{\omega} - H \wedge \tilde{\omega} - (\iota_{\tilde{v}} + \tilde{\lambda}\wedge)F\big) \\
& + \mathcal{L}_{\tilde{v}}\tilde{\omega}' + (\mathrm{d}\tilde{\lambda} - \iota_{\tilde{v}}H) \wedge \tilde{\omega}' \\
& \qquad - (\iota_{\tilde{v}'} + \tilde{\lambda}'\wedge)\big(\mathrm{d}\tilde{\omega} - H \wedge \tilde{\omega} - (\iota_{\tilde{v}} + \tilde{\lambda}\wedge)F\big) \ ,
\end{aligned} \tag{4.3.41}$$

where $F = F_0 + F_2 + F_4 + F_6$ is now the complete $O(6,6)$ spinor with $m \neq 0$. So the twisted version of the massive Dorfman derivative produces precisely the expected gauge transformations with flux terms including the Romans mass. Again, these are given by the action of (3.2.9), now with $m \neq 0$.

Note that of all the flux terms in (4.3.41), the mass term is the only one which is diffeomorphism-invariant. It is also the only true deformation of the generalised Lie derivative, since it cannot be removed by twisting the generalised tangent bundle.

4.3.3 Type IIB

For the review of this part we will closely follow [18]. For type IIB the structure group of the principal frame bundle is the same as the type IIA case, $\mathrm{E}_{d+1(d+1)} \times \mathbb{R}^+$ for a d-dimensional manifold M [73, 92].

On a d-dimensional manifold M, the generalised tangent bundle is

$$\begin{aligned}
E &\cong TM \oplus T^*M \oplus (T^*M \oplus \Lambda^3T^*M \oplus \Lambda^5T^*M) \oplus \Lambda^5T^*M \\
&\qquad \oplus (T^*M \otimes \Lambda^6T^*M) \\
&\cong TM \oplus (T^*M \otimes S) \oplus \Lambda^3T^*M \oplus (\Lambda^5T^*M \otimes S) \oplus (T^*M \otimes \Lambda^6T^*M) \,,
\end{aligned} \tag{4.3.42}$$

where, as usual, E is defined formally by an extension and it is isomorphic to the sum of spaces in (4.3.42) by choosing the potential maps, *i.e.* the connective structures on the gerbe. In the (4.3.42) the S transforms as a doublet of $\mathrm{SL}(2,\mathbb{R})$. We write sections of this bundle as

$$V = v + \lambda^i + \rho + \sigma^i + \tau \,, \tag{4.3.43}$$

where $v \in \Gamma(TM)$, $\lambda^i \in \Gamma(T^*M \otimes S)$, $\rho \in \Gamma(\Lambda^3T^*M)$, $\sigma \in \Gamma(\Lambda^5T^*M \otimes S)$ and $\tau \in \Gamma(T^*M \otimes \Lambda^6T^*M)$, the index $i = 1, 2$ is the one labelling the fundamental representation of $\mathrm{SL}(2,\mathbb{R})$.

[5] A very subtle point is that neither of the terms on the RHS of (4.3.38) transforms correctly as a generalised vector under (4.3.35), and as a consequence $\underline{m}(V)$ does not transform as a section of the adjoint bundle. However, overall $L_V V'$ defines a good section of E.

As before, the patching conditions reproduce the type IIB supergravity gauge transformations. Given a cover $\{U_\alpha\}$ of M on can define the generalised section $V_{(\alpha)}$ on $U_\alpha \cap U_\beta$ by the $V_{(\beta)}$ as follows,

$$V_{(\alpha)} = e^{\mathrm{d}\Lambda^i_{(\alpha\beta)} + \mathrm{d}\Omega_{(\alpha\beta)}} \cdot V_{(\beta)} \,, \tag{4.3.44}$$

which $\cdot$ denoting the adjoint action and $\Lambda^{(i)}$ and Ω are locally a pair of one-forms and a three-form respectively. By defining the untwisted vector

$$V = e^{-B^i - C}\tilde{V} \,. \tag{4.3.45}$$

and comparing the two actions, we find

$$\begin{aligned} B^i_{(\alpha)} &= B^i_{(\beta)} + \mathrm{d}\Lambda^{(i)}_{(\alpha\beta)} \,, \\ C_{(\alpha)} &= C_{(\beta)} + \mathrm{d}\Omega_{(\alpha\beta)} + \frac{1}{2}\epsilon_{ij}\mathrm{d}\Lambda^i_{(\alpha\beta)} \wedge B^i_{(\beta)} \,. \end{aligned} \tag{4.3.46}$$

The adjoint bundle is

$$\begin{aligned} \mathrm{ad}\,\tilde{F} =& \mathbb{R} \oplus (TM \otimes T^*M) \oplus (S \otimes S^*)_0 \oplus (S \otimes \Lambda^2 TM) \oplus (S \otimes \Lambda^2 T^*M) \\ & \oplus \Lambda^4 TM \oplus \Lambda^4 T^*M \oplus (S \otimes \Lambda^6 TM) \oplus (S \otimes \Lambda^6 T^*M) \,, \end{aligned} \tag{4.3.47}$$

where the subscript on $(S \otimes S^*)_0$ denotes the traceless part. We write sections of the adjoint bundle as

$$R = l + r + a + \beta^i + B^i + \gamma + C + \tilde{\alpha}^i + \tilde{a}^i \,, \tag{4.3.48}$$

where $l \in \mathbb{R}$, $r \in \Gamma(\mathrm{End}(TM))$, etc.

The action of the Dorfman derivative on a generalised vector in type IIB is

$$\begin{aligned} L_V V' =& \mathcal{L}_v v' + (\mathcal{L}_v \lambda'^i - \imath_{v'}\mathrm{d}\lambda^i) + (\mathcal{L}_v \rho' - \imath_{v'}\mathrm{d}\rho + \epsilon_{ij}\mathrm{d}\lambda^i \wedge \lambda'^j) \\ & + (\mathcal{L}_v \sigma'^i - \imath_{v'}\mathrm{d}\sigma^i + \mathrm{d}\rho \wedge \lambda'^i - \mathrm{d}\lambda^i \wedge \rho') \\ & + (\mathcal{L}_v \tau' - \epsilon_{ij} j\lambda'^i \wedge \mathrm{d}\sigma^j + j\rho' \wedge \mathrm{d}\rho + \epsilon_{ij} j\sigma'^i \wedge \mathrm{d}\lambda^j) \,. \end{aligned} \tag{4.3.49}$$

The expression for Dorfman derivative acting on a section of the adjoint bundle is given in the appendix **??**.

Also here one can give the *twisted* Dorfman derivative $\mathbb{L}_V$ of an untwisted generalised tensor $\tilde{\mathcal{A}}$, defined by

$$\mathbb{L}_{\tilde{V}}\tilde{\mathcal{A}} = e^{-B^i - C} L_{e^{B^i + C}\tilde{V}}(e^{B^i + C}\tilde{\mathcal{A}}). \tag{4.3.50}$$

The twisted Dorfman derivative $\mathbb{L}_V$ is given by the same expression as the usual Dorfman derivative but with the substitutions

$$\begin{aligned} \mathrm{d}\lambda^i &\to \mathrm{d}\tilde{\lambda}^i - \imath_{\tilde{v}} F^i \,, \\ \mathrm{d}\rho &\to \mathrm{d}\tilde{\rho} - \imath_{\tilde{v}} F - \epsilon_{ij}\tilde{\lambda}^i \wedge F^j \,, \\ \mathrm{d}\sigma^i &\to \mathrm{d}\tilde{\sigma}^i + \tilde{\lambda}^i \wedge F - \tilde{\rho} \wedge F^i \,. \end{aligned} \tag{4.3.51}$$

We collect further details about the exceptional generalised geometry in **??**.

4.3.4 Generalised metric

In this section we briefly review some examples of generalised structures. For concreteness we restrict ourself to type IIA, however we refer to [18, 72] for a detailed discussions of the other cases.

In the same way as the ordinary metric on a manifold M can be seen as an $\mathrm{O}(d)$ structure on TM parameterising the coset $\mathrm{GL}(d,\mathbb{R})/\mathrm{O}(d)$, the generalised metric can be seen as an $\mathrm{SU}(8)/\mathbb{Z}_2$ structure on the generalised tangent bundle, and for a six-dimensional manifold it parameterises the coset $\mathrm{E}_{7(7)}/(\mathrm{SU}(8)/\mathbb{Z}_2)$. The construction of the generalised metric is a natural extension of the one we have described for the $\mathrm{O}(d,d)$ case in section 4.2.2.

The generalised metric $\mathcal{G}$ can be defined by its action on two generalised vectors V and V' as

$$\begin{aligned}\mathcal{G}(V,V') &= \tilde{v}\lrcorner\tilde{v}' + \tilde{\lambda}\lrcorner\tilde{\lambda}' + \tilde{\sigma}\lrcorner\tilde{\sigma}' + \tilde{\tau}\lrcorner\tilde{\tau}' + \sum_{k=0}^{3}\tilde{\omega}_{2k}\lrcorner\tilde{\omega}'_{2k} \\ &= \tilde{v}^m\tilde{v}'_m + \tilde{\lambda}^m\tilde{\lambda}'_m + \tfrac{1}{5!}\tilde{\sigma}^{m_1\dots m_5}\tilde{\sigma}'_{m_1\dots m_5} + \tfrac{1}{6!}\tilde{\tau}^{m,m_1\dots m_6}\tilde{\tau}'_{m,m_1\dots m_6} \\ &\quad + \sum_{k=0}^{3}\tfrac{1}{(2k)!}\,\tilde{\omega}^{m_1\dots m_{2k}}\tilde{\omega}'_{m_1\dots m_{2k}}\,,\end{aligned} \tag{4.3.52}$$

where the indices are lowered/raised using the ordinary metric g_{mn} and its inverse g^{mn}.

One can also define a generalised frame $\{\hat{E}_A\}$ on E and then construct the inverse generalised metric as the tensor product of two such frames

$$\mathcal{G}^{-1} = \delta^{AB}\hat{E}_A\otimes\hat{E}_B\,. \tag{4.3.53}$$

We will give below a precise definition of this product. To construct the generalised frame, we first consider the *untwisted* generalised tangent bundle $\tilde{E}$, in an analogous way to the construction in $\mathrm{O}(d,d)$ geometry. Let $\hat{e}_a$, with $a = 1,\dots,6$, be an ordinary frame, namely a basis for the tangent space at a point of M_6, and let e^a be the dual basis for the cotangent space[6]. Then we can define a frame $\tilde{\hat{E}}_A$ for the untwisted generalised tangent space as the collection of bases for the subspaces that compose it

$$\begin{aligned}\{\tilde{\hat{E}}_A\} = \{\hat{e}_a\}\cup\{e^a\}\cup\{e^{a_1\dots a_5}\}\cup\{e^{a,a_1\dots a_6}\}\cup\{1\} \\ \cup\{e^{a_1a_2}\}\cup\{e^{a_1\dots a_4}\}\cup\{e^{a_1\dots a_6}\}\,,\end{aligned} \tag{4.3.54}$$

where $e^{a_1\dots a_p} = e^{a_1}\wedge\dots\wedge e^{a_p}$ and $e^{a,a_1\dots a_6} = e^a\otimes e^{a_1\dots a_6}$. A frame for the *twisted* generalised tangent space is obtained by twisting (4.3.54) by the local $\mathrm{E}_{7(7)}\times\mathbb{R}^+$ transformation

$$\hat{E}_A = e^{\tilde{B}}e^{-B}e^{C}e^{\Delta}e^{\phi}\cdot\tilde{\hat{E}}_A\,, \tag{4.3.55}$$

where in addition to the twist (4.3.23) we also include a rescaling by the dilaton ϕ and warp factor Δ, acting as specified in (??). Because of the rescaling by Δ the frame (4.3.55) was called *conformal split frame* in [74]. Note that (4.3.55) is just a particular choice of frame, not the most general one. Any other frame can be obtained from (4.3.55) acting with an $\mathrm{E}_{7(7)}\times\mathbb{R}^+$ transformation.

[6] We are using the hat symbol to distinguish frame vectors, $\hat{e}_a$, from co-frame one-forms, e^a. Similarly, the hat on $\hat{E}_A$ indicates that this is a generalised frame vector.

We denote the components of $\hat{E}_A$ carrying different flat indices as

$$\{\hat{E}_A\} = \{\hat{\mathcal{E}}_a\} \cup \{\mathcal{E}^a\} \cup \{\mathcal{E}^{a_1\dots a_5}\} \cup \{\mathcal{E}^{a,a_1\dots a_6}\} \cup \{\mathcal{E}\} \cup \{\mathcal{E}^{ab}\} \cup \{\mathcal{E}^{a_1\dots a_4}\} \cup \{\mathcal{E}^{a_1\dots a_6}\} .$$

Explicit expressions for each of these terms are given in appendix **??**.

Once we have the generalised frame, we can derive the expression for the inverse generalised metric $\mathcal{G}^{-1}$. Expanded in GL(6) components, the product (4.3.53) becomes

$$\begin{aligned}\mathcal{G}^{-1} =& \delta^{aa'}\hat{\mathcal{E}}_a \otimes \hat{\mathcal{E}}_{a'} + \delta_{aa'}\mathcal{E}^a \otimes \mathcal{E}^{a'} + \mathcal{E}\otimes\mathcal{E} + \tfrac{1}{2}\delta_{a_1a_1'}\delta_{a_2a_2'}\mathcal{E}^{a_1a_2}\otimes\mathcal{E}^{a_1'a_2'} \\ &+ \tfrac{1}{4!}\delta_{a_1a_1'}\cdots\delta_{a_4a_4'}\mathcal{E}^{a_1\dots a_4}\otimes\mathcal{E}^{a_1'\dots a_4'} + \tfrac{1}{5!}\delta_{a_1a_1'}\cdots\delta_{a_5a_5'}\mathcal{E}^{a_1\dots a_5}\otimes\mathcal{E}^{a_1'\dots a_5'} \\ &+ \tfrac{1}{6!}\delta_{a_1a_1'}\cdots\delta_{a_6a_6'}\mathcal{E}^{a_1\dots a_6}\otimes\mathcal{E}^{a_1'\dots a_6'} + \tfrac{1}{6!}\delta_{aa'}\delta_{a_1a_1'}\cdots\delta_{a_6a_6'}\mathcal{E}^{a,a_1\dots a_6}\otimes\mathcal{E}^{a',a_1'\dots a_6'} .\end{aligned}$$

which is nothing else than the inverse of (4.3.52) calculated on frames. The full expression for $\mathcal{G}^{-1}$ is long and ugly, so we only give the terms that will be relevant for the next discussion. Arranging them according to their curved index structure, we have

$$\begin{aligned}(\mathcal{G}^{-1})^{mn} &= e^{2\Delta}g^{mn} , \\ (\mathcal{G}^{-1})^{m} &= e^{2\Delta}g^{mn}C_n , \\ (\mathcal{G}^{-1})^{m}{}_{n} &= -e^{2\Delta}g^{mp}B_{pn} , \\ (\mathcal{G}^{-1})^{m}{}_{np} &= e^{2\Delta}g^{mq}(C_{qnp} - C_qB_{np}) , \\ (\mathcal{G}^{-1})^{m}{}_{npqr} &= e^{2\Delta}g^{ms}\big(C_{snpqr} - C_{s[np}B_{qr]} + \tfrac{1}{2}C_sB_{[np}B_{qr]}\big) , \\ (\mathcal{G}^{-1}) &= e^{2\Delta}\Big(e^{-2\phi} + g^{mn}C_mC_n\Big) .\end{aligned} \tag{4.3.56}$$

These terms will be sufficient to read off all the supergravity physical fields from the generalised metric (we are omitting the formula determining $\tilde{B}_{m_1\dots m_6}$). Some other components of $\mathcal{G}^{-1}$ are

$$\begin{aligned}(\mathcal{G}^{-1})_{m} &= e^{2\Delta}g^{np}C_nB_{pm} , \\ (\mathcal{G}^{-1})_{(mn)} &= e^{2\Delta}(g_{mn} + g^{pq}B_{pm}B_{qn}) , \\ (\mathcal{G}^{-1})_{[mn]} &= -e^{2\Delta}\Big(e^{-2\phi}B_{mn} - g^{pq}C_q(C_{pmn} - C_pB_{mn})\Big) , \\ (\mathcal{G}^{-1})_{m,np} &= -e^{2\Delta}\big(g_{m[n}C_{p]} + g^{qr}B_{qm}(C_{rnp} - C_rB_{np})\big) .\end{aligned} \tag{4.3.57}$$

There is also a density associated to the generalised metric which trivialises the $\mathbb{R}^+$ factor of the $\mathrm{E}_{d+1(d+1)} \times \mathbb{R}^+$ structure group. In terms of the field content of type IIA it is given by

$$\Phi = (\det\mathcal{G})^{-(9-d)/(\dim E)} = g^{1/2}\,e^{-2\phi}e^{(8-d)\Delta} , \tag{4.3.58}$$

as can be seen by decomposing the corresponding M-theory density [74]. This equation provides an easy way to solve relations such as (4.3.56) explicitly for the supergravity fields. For example, to solve the first, second and last of equations in (4.3.56), one can begin by setting

$$(M^{-1})^{mn} := (\mathcal{G}^{-1})^{mn} = e^{2\Delta}g^{mn} . \tag{4.3.59}$$

The second of equations (4.3.56) then becomes

$$C_m = M_{mn}(\mathcal{G}^{-1})^{n} , \tag{4.3.60}$$

which can be substituted into the last equation in (4.3.56) to give

$$e^{2\Delta}e^{-2\phi} = (\mathcal{G}^{-1}) - M_{mn}(\mathcal{G}^{-1})^m(\mathcal{G}^{-1})^n := Q\,. \tag{4.3.61}$$

One then easily obtains the expressions for g_{mn}, C_m, e^{Δ} and $e^{-2\phi}$ as

$$\begin{aligned} e^{\Delta} &= \left(\frac{\Phi}{Q\sqrt{\det M}}\right)^{1/6}\,, & e^{-2\phi} &= \left(\frac{Q^4\sqrt{\det M}}{\Phi}\right)^{1/3}\,,\\ g_{mn} &= M_{mn}\left(\frac{\Phi}{Q\sqrt{\det M}}\right)^{1/3}\,, & C_m &= M_{mn}(\mathcal{G}^{-1})^n\,, \end{aligned} \tag{4.3.62}$$

where M_{mn}, Q and Φ are given in terms of the generalised metric as above. In particular, we have expressions for e^{Δ} and g_{mn}, so that solving the remaining relations in (4.3.56) becomes straightforward.

The above method to compute the warp factor from an arbitrary generalised metric involves evaluating $\det \mathcal{G}$, which is in general a slightly difficult computation. A simpler way to attain the same result is to evaluate the determinant of a subset of the components of the generalised metric, denoted $\mathcal{H}$, corresponding to the degrees of freedom in the coset

$$\mathcal{H} \in \frac{\mathrm{SO}(d,d)\times\mathbb{R}^+}{\mathrm{SO}(d)\times\mathrm{SO}(d)}\,. \tag{4.3.63}$$

Explicitly, we construct $\mathcal{H}^{-1}$ in components via

$$\mathcal{H}^{-1} = \begin{pmatrix} (\mathcal{G}^{-1})^{mn} & (\mathcal{G}^{-1})^m{}_n \\ (\mathcal{G}^{-1})_m{}^n & (\mathcal{G}^{-1})_{mn} \end{pmatrix} = e^{2\Delta}\begin{pmatrix} g^{mn} & -(g^{-1}B)^m{}_n \\ (Bg^{-1})_m{}^n & (g-Bg^{-1}B)_{mn} \end{pmatrix} \tag{4.3.64}$$

where in the second equality we have used (4.3.56) and (4.3.57). We recognise the last matrix as the components of (the inverse of) the $O(d,d)$ generalised metric (4.2.32), which has unit determinant [69]. Therefore we can immediately write

$$e^{\Delta} = (\det\mathcal{H})^{-1/4d}\,. \tag{4.3.65}$$

We comment on the appearance of the $\mathrm{O}(d,d)$ generalised metric in appendix **??**.

4.3.5 Generalised parallelisation

The goal of this section is to extend the definition of identity structure given in the previous chapter to the exceptional case. This was firstly defined in [92], and extended in [12].

Namely, a generalised parallelisation $\{\hat{E}_A\}$ ($A = 1,\dots,N$, where N is the dimension of the generalised bundle) is a globally defined frame, or a set of N globally defined vector fields defining a basis of $E|_p$ at each point p of M. The latter is a topological condition. In addition one can add a differential constraint on the frame $\{\hat{E}_A\}$,

$$L_{\hat{E}_A}\hat{E}_B = X_{AB}{}^C\hat{E}_C\,, \tag{4.3.66}$$

with constant coefficients $X_{AB}{}^C$. Following [92], we call this a *generalised Leibniz parallelisation*. The name comes from the fact that since the Dorfman derivative is not antisymmetric, the frame algebra (4.3.66) is not a Lie algebra, but in general a Leibniz one.

In [82] using $\mathrm{O}(d,d)$ generalised geometry, it is proven that a necessary condition to admit a generalised Leibniz parallelisation is to be an homogeneous space, that is a coset space in the form G/H. This result is extendable to the full $\mathrm{E}_{d(d)}$ bundle.

4.3.6 Generalised HV structures

Here we review briefly the results of [13], in order to describe the so-called HV structures as generalised G-structures on the exceptional frame bundle. These will describe AdS backgrounds of various supergravity theories.

As for ordinary G-structures, the existence of globally defined generalised tensors reduces the structure group of E and defines generalised G-structures in exceptional geometry.

As shown in [13,18,46], a supergravity solution with eight supercharges is characterised by the existence of the so-called *hyper-* and *vector-multiplet*structures, defining the relative generalised G-structure.

A *hypermultiplet structure* [18], or *H structure* for short, is a triplet of sections of the weighted adjoint bundle (one can see **??**) for details)

$$J_a \in \Gamma(\mathrm{ad}\tilde{F} \otimes (\det T^*M)^{1/2}) \,, \tag{4.3.67}$$

such that

$$[J_a, J_b] = 2\kappa\epsilon_{abc}J_c \qquad \text{and} \qquad \mathrm{tr}(J_aJ_b) = -\kappa^2\delta_{ab} \,. \tag{4.3.68}$$

In the cases we are going to be interested in, *i.e* $d = 6, 7$, the triplet J_a realises a $\mathrm{Spin}^*(12) \subset \mathrm{E}_{7(7)} \times \mathbb{R}^+$ structure and an $\mathrm{SU}^*(6) \subset \mathrm{E}_{6(6)} \times \mathbb{R}^+$ respectively.

A *vector structure*, or *V structure*, is given by a generalised vector

$$K \in \Gamma(E) \tag{4.3.69}$$

that has positive norm

$$q(K) > 0 \quad \text{or} \quad c(K) > 0 \,, \tag{4.3.70}$$

where $q(K)$ denotes the $\mathrm{E}_{7(7)}$ quartic and $c(K)$ is the $\mathrm{E}_{6(6)}$ cubic invariant. The generalised vector K defines an $\mathrm{E}_{6(2)}$ and $\mathrm{F}_{4(4)}$ structure in $D = 4$ and $D = 5$, respectively.

One can impose the following compatibility conditions on J_a and K

$$J_a \cdot K = 0 \quad \text{and} \quad \mathrm{tr}(J_aJ_b) = \begin{cases} -2\sqrt{q(K)}\delta_{ab} & D = 4 \\ -c(K)\delta_{ab} & D = 5 \end{cases} \tag{4.3.71}$$

The pair (J_a, K) is then called an *HV structure* and defines an $\mathrm{SU}(6) = \mathrm{Spin}^*(12) \cap \mathrm{E}_{6(2)}$ structure and a $\mathrm{USp}(6) = \mathrm{SU}^*(6) \cap F_{4(4)}$ structure in $D = 4$ and $D = 5$, respectively (see [18]). The explicit form of the *HV* structure depends on the theory and the dimension of the compactification manifold. For instance, the generalised vector K is

$$K = \begin{cases} \xi + \omega + \sigma + \tau & \text{M theory} \\ \xi + \lambda^i + \rho + \sigma^i + \tau & \text{type IIB} \end{cases} . \tag{4.3.72}$$

As discussed in [13, 18], and analogously to other G-structures, the integrability of these structures is achieved by imposing a set of integrability conditions on (J_a, K),

$$\mu_a(V) = \lambda_a\gamma(V) \qquad \forall V \in \Gamma(E) \,, \tag{4.3.73a}$$
$$L_K K = 0 \,, \tag{4.3.73b}$$
$$L_K J_a = \epsilon_{abc}\lambda_b J_c \,, \qquad L_{\tilde{K}} J_a = 0 \,, \tag{4.3.73c}$$

where the second condition in (4.3.73c) only applies for $D = 4$.

When $\lambda_a \neq 0$, this structure is called *exceptional Sasaki-Einstein* [13], while in the case of $\lambda_a = 0$ it takes the name of *exceptional Calabi-Yau* [18].

On can show that these are equivalent to the Killing spinor equations for backgrounds preserving $\mathcal{N} = 2$ supersymmetry.

The functions $\mu_a(V)$ are a triplet of moment maps for the action of the generalised diffeomorphisms,

$$\mu_a(V) = -\frac{1}{2}\epsilon_{abc}\int_M \mathrm{tr}(J_b L_V J_c)\,. \tag{4.3.74}$$

We will see how the constants λ_a depend on the theory: they are zero for Minkowski backgrounds, while for AdS are related to the inverse of the AdS radius $|\lambda| = 2m$ for $D = 4$ and $|\lambda| = 3m$ for $D = 5$, where $|\lambda|^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$. Finally, the function γ is defined as

$$\begin{aligned} &\gamma(V) = 2\textstyle\int_M q(K)^{-1/2} q(V,K,K,K) && D = 4\,,\\ &\gamma(V) = \textstyle\int_M c(V,K,K) && D = 5\,. \end{aligned} \tag{4.3.75}$$

The integrability conditions have important consequence, that we are going to explore further in the next chapters. For instance, the generalised vector K is a generalised Killing vector, that is

$$L_K \mathcal{G} = 0\,, \tag{4.3.76}$$

where the generalised metric $\mathcal{G}$ in (4.3.52). The generalised Killing vector condition for M-theory is equivalent to

$$\mathcal{L}_\xi g = 0\,, \qquad \mathcal{L}_\xi A - \mathrm{d}\omega = 0\,, \qquad \mathcal{L}_\xi \tilde{A} - \mathrm{d}\sigma + \tfrac{1}{2}\mathrm{d}\omega \wedge A = 0\,, \tag{4.3.77}$$

while in type IIB one has

$$\begin{aligned} &\mathcal{L}_\xi g = 0\,, && \mathcal{L}_\xi C = \mathrm{d}\rho - \tfrac{1}{2}\epsilon_{ij}\mathrm{d}\lambda^i \wedge B^j\,,\\ &\mathcal{L}_\xi B^i - \mathrm{d}\lambda^i = 0\,, && \mathcal{L}_\xi \tilde{B}^i = \mathrm{d}\sigma^i + \tfrac{1}{2}\mathrm{d}\lambda^i \wedge C - \tfrac{1}{2}\mathrm{d}\rho \wedge B^i + \tfrac{1}{12}\epsilon_{kl}B^i \wedge B^k \wedge \mathrm{d}\lambda^l\,. \end{aligned} \tag{4.3.78}$$

The generalised Killing vector condition on K means that the action of the generalised Lie derivative on the untwisted objects reduces to the usual one,

$$\hat{L}_K \cdot = \mathcal{L}_\xi \cdot\,, \tag{4.3.79}$$

where ξ denotes the (necessarily non-vanishing) vector component of K. By virtue of (**??**) this is equivalent to the vanishing of the tensor $R_{\mathbb{L}_{\tilde{V}}}$ in (**??**). We will refer to (4.3.73a) as *moment map condition* while to (4.3.73b), (4.3.73c) and the vanishing of $R_{\mathbb{L}_{\tilde{V}}}$ in (**??**) as L_K *condition.* K is called *generalised Reeb vector* because it naturally generalises the isometry described by the usual Reeb vector in Sasakian geometry.

We will analyse further these structures in the chapters **??** and 2.

Conclusions

The major aim of this thesis was to study flux backgrounds. In the first part we focused on the problem of seeking for a systematic way to find consistent truncations in the presence of fluxes. In the second part, we were more concerned in reformulating the supersymmetry conditions for brane probes in AdS backgrounds, in terms of integrability of exceptional structures in exceptional geometry.

This thesis contains the construction of a generalised geometric description of type IIA flux backgrounds.

We made a large use of the formalism of G-structures and generalised geometry. As in conventional geometry, integrability is defined as the existence of a generalised torsion-free connection that is compatible with the structure, or equivalently as the vanishing of the generalised intrinsic torsion. In the case analysed, *i.e.* truncations of massive type IIA, the integrability conditions correspond to the Leibnitz algebra for parallelisations. These enquires the trucation preserves maximal supersymmetry in lower dimension [92]. Moreover, the notion of *generalised Leibniz parallelisation* is the key ingredient to construct a consistent truncation ansatz. We showed in chapter **??** how to build consistent truncation ansatze, by the so-called *Generalised Scherk-Scwharz reductions*, making use of generalised Leibniz parallelisations. We constructed various examples of massive type IIA spheres truncations. In the case of the truncation on a six-dimensional sphere, we obtained a generalised parallelisation on S^6 satisfying the ISO(7) algebra, and spelled out the corresponding truncation ansatz as obtained from the generalised Scherk–Schwarz prescription. As recently described in [93, 94], the Romans mass introduces a magnetic gauging of the ISO(7) translations in the truncated four-dimensional theory, yielding a symplectic deformation [95] of the type first found in [96] for the SO(8) gauging. We found the same phenomenon for type IIA supergravity on the six-dimensional hyperboloids $H^{p,7-p}$: on these spaces one can define a consistent truncation down to $\mathrm{ISO}(p, 7-p)$ supergravity in four-dimensions; switching the Romans mass on leads to the symplectically-deformed $\mathrm{ISO}(p, 7-p)$ gauging described in [95]. We also obtained generalised Leibniz parallelisations on S^4, S^3 and S^2 for vanishing Romans mass, reproducing the Leibniz algebra of known consistent truncations of massless type IIA supergravity on these manifolds. When the Romans mass is switched on, these parallelisations no more satisfy a Leibniz algebra. We offered an explanation of why this is the case by showing that the frame lies in the stabiliser group of the Romans mass only for the parallelisation on S^6. For massive type IIA on S^3 we presented a no-go result indicating that a consistent truncation including the SO(4) algebra does not exist. It would be interesting to see whether similar no-go theorems can be proved for the S^4 and S^2 cases.

As said, in order to construct apply the generalised Scherk-Schwarz prescription, we built the adapted version of generalised geometry for massive type IIA. The principal issue in this construction is accommodating the flux due to the Romans mass. This is achieved by deforming the generalised Lie derivative such that the deformed one generates the gauge transformations of type IIA supergravity with a non-zero m-flux.

An interesting fact is the existence of an alternative massive type IIA [97]. This can be obtained from eleven-dimensional supergravity by gauging a combination of the GL(1) global symmetry and the trombone symmetry of the equations of motion. It is not known a description of such a theory through a Lagrangian. One may wonder whether other massive extensions of type IIA can exist. However, in [98], making use of superspace arguments, it was discussed how this and the Romans mass are the only possible extensions. It is natural to ask how this deformation appears in our formalism. We want deformation parameters to be diffeomorphism invariant then we require them to appear as GL(6) singlets with zero $\mathbb{R}^+$ weight. There are precisely two such singlets in the $\mathbf{912}_{-1}$ representation of $E_{7(7)} \times \mathbb{R}^+$, one of which we have already identified as the Romans mass deformation. There is also a singlet in the $\mathbf{56}_{-1}$ representation, which is another part of the generalised torsion [74], and which could also be used to deform the Dorfman derivative. When performing generalised Scherk-Schwarz reductions, this additional $\mathbf{56}_{-1}$ part of the embedding tensor is generated by gauging the trombone symmetry [99], and the resulting theory does not have an action. It is natural to conjecture that deforming the Dorfman derivative by switching on a combination of the second singlet in $\mathbf{912}_{-1}$ and the singlet in $\mathbf{56}_{-1}$ would give the relevant gauge algebra for the theory described in [97]. The result of [98] can be verified in this case, since, by considering the closure of the gauge algebra, one can argue that there are no others deformations, as singlets of GL(6) in the torsion representation bundle.

In the last chapter of the thesis, we focused on brane calibrations. The tools used are again generalised G-structures. In particular, we concentrated our attention on AdS backgrounds with eight supercharges. These have an elegant description in generalised geometry in terms of exceptional Sasaki-Einstein structures [13, 46]. In chapter 2, we studied the relation between the Exceptional Sasaki-Einstein structures and generalised brane calibrations in $\mathrm{AdS}_5 \times M_5$ backgrounds in type IIB and in $\mathrm{AdS}_5 \times M_6$ and $\mathrm{AdS}_4 \times M_7$ compactifications in M-theory. We focussed on the calibrations forms associated to branes wrapping cycles in the internal manifolds and that are point-like in the AdS space. We showed that for these configurations the general expression for the calibration forms that can be constructed using κ-symmetry can be expressed in terms of the generalised Killing vector K defining the Exceptional Sasaki-Einstein structure and that the closure of the calibration forms is given by the integrability (more precisely the L_K condition) of the ESE structure. The results of the chapters prove the conjecture appeared in [13] that the (form part of the) generalised Killing vector is a generalised calibration. The motivation of this conjecture can be found in holography [100]. One can observe that the generalised killing vector K generates the global R-symmetry of the filed theory dual to the AdS background in supergravity. It is made by a combination of the vector part (generating diffeomorphism) and p-forms (parametrising gauge transformations), under which the generalised metric is invariant. Thus there is a sign that K is an object related to R-symmetry in a non-trivial way, since it encodes informations about the gauge transformations of flux potentials. In AdS/CFT correspondence, BPS branes have volume associated to the conformal dimension of chiral operator in the dual SCFT, thus finding a calibration (that is a supersymmetric circle on which branes wrap) is equivalent to find the conformal dimensions of the dual operators.

We have seen how to construct these calibrations in generalised geometry and how

BPS conditions correspond to the integrability of the structures.

We also partially discussed other brane configurations that are calibrated by the vector K. However we did not perform a complete analysis, leaving the discussion for a future work.

Future works

There are many other directions for future study.

For intance, so far, there is not a description of supergravity flux backgrounds in terms of generalised geometry for any amount of supersymmetry. As stated, supersymmetric background preserving $\mathcal{N}$ supersymmetries are given by integrable G-structures, where G is the stabiliser group of the $\mathcal{N}$ Killing spinors [101]. Maximally supersymmetric backgrounds, as seen are described by parallelisations [92, 102]. Half-maximal supergravity truncations have been recently described in exceptional field theory [103], but a generalised geometry formulation is not known at the moment. The $\mathcal{N} = 2$ backgrounds, as seen in [13, 18, 71, 104], is a rich field for generalised G-structure applications. An $\mathcal{N} = 1$ formalism to describe vacua is known in $\mathrm{O}(d, d)$ generalised geometry [87–89] and recently an exceptional picture has been found in [105]. However, the analysis is far to be complete. Hence, obvious extension is to consider backgrounds with different amounts of supersymmetry, which will be described by new geometric structures within generalised geometry. In [17], there is a work in progress development of the structures to describe supersymmetric backgrounds with sixteen supercharges. Furthermore, the hope is not just to find a new descriptions of such backgrounds, but that this would lead to discover new examples of truncations. In addition, an ambitious project would be a complete classification of such structures analysing the mathematical constraints on the internal geometries and finding a coherent structure describing them.

Moreover, all these techniques are may also be applied in different contexts from consistent truncations, like for example, studies of holography effects, marginal deformations of the dual field theories, etc.

More specifically, the formalism developed in the first part of this thesis about massive type IIA may also be applied to investigate marginal deformations of conformal field theories holographically dual to (massive) type IIA AdS backgrounds. Recently this theories have been identified with a class of Chern-Simons-matter theories [93], and in [16], we aim to describe exactly marginal deformations of such conformal theories by constructing (and studying their deformations) the exceptional Sasaki-Einstein structures for massive type IIA, describing the AdS background in [106] preserving eight supercharges. The work of [107] is the first example in this sense. There, the authors study the exceptional Sasaki-Einstein structures describing AdS_5 background in both type IIB and eleven-dimensional supergravity to analyse marginal deformations of the dual $\mathcal{N} = 1$ CFT in four dimensions. A famous result in gauge theory [108] states that marginal deformations are determined by imposing F-term conditions on operators of conformal dimension three and then quotienting by the complexified global symmetry group. In [107], it was shown that this result has a geometrical interpretation: the marginal deformations are obtained as solutions of moment maps for the generalised diffeomorphism group that have the correct charge under the Reeb vector. Indeed, the Reeb generates the $U(1)_R$ symmetry group. In the case this is the only symmetry of the background, then all the marginal deformations are exactly marginal. When there are other global symmetries, the field theory result predicts one has to quotient out these symmetries. On the supergravity side, this can be read as fixed points of the moment maps, being an obstruction for marginal deformations to be exactly marginal.

This analysis holds for any kind of internal geometry in an AdS background, however, so far the examples to which this has been applied are all Sasaki-Einstein geometries. So, it would be interesting to apply it to one of the few examples of non-Sasaki–Einstein backgrounds, such as the Pilch–Warner solution [109]. This would give the marginal deformations of the dual SCFT, the so-calle Leigh-Strassler theory [110]. We aim to study this example in [15].

To conclude, generalised geometry gives tools to better analyse several aspects of supergravity and string theory. It has also applications in holography and field theory. Furthermore, it is interesting also in pure mathematics, since it is related to various areas beyond differential geometry, like algebraic topology, algebraic geometry and group theory. Thus it provides an example of a topic that lies at the frontier between mathematics and physics, on the one hand, receiving deep insights from both fields, but on the other hand it could also give some useful tools to understand and answer questions in both areas. For these reasons, generalised geometry seems to be worth of further efforts and studies, since its U-duality covariant approach may reveal something hidden so far and perhaps help us to understand the geometrical nature of dualities in string theory.

A
Very important appendix

A.1 Very important section

B

An even more important appendix

B.1 An even more important section

C
The most important appendix

C.1 The most important section

C.1.1 The most important subsection

...

C.2 The second most important section

C.2.1 The second most important subsection

...

C.2.2 The almost important subsection

...

C.2.3 The not so important subsection

...

C.2.4 Another subsection

...

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List of Figures

Résumé

Resumée en Français.

Abstract

English Abstract.