Then taking the inverse Laplace transform, we have as required

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \frac{P(\alpha_1)}{Q'(\alpha_1)}e^{\alpha_1 t} + \cdots + \frac{P(\alpha_k)}{Q'(\alpha_k)}e^{\alpha_k t} + \cdots + \frac{P(\alpha_n)}{Q'(\alpha_n)}e^{\alpha_n t} = \sum_{k=1}^{n}\frac{P(\alpha_k)}{Q'(\alpha_k)}e^{\alpha_k t}$$

30. Find $\mathcal{L}^{-1}\left\{\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$.

We have $P(s) = 2s^2-4$, $Q(s) = (s+1)(s-2)(s-3) = s^3-4s^2+s+6$, $Q'(s) = 3s^2-8s+1$, $\alpha_1=-1$, $\alpha_2=2$, $\alpha_3=3$. Then the required inverse is by Problem 29,

$$\frac{P(-1)}{Q'(-1)}e^{-t} + \frac{P(2)}{Q'(2)}e^{2t} + \frac{P(3)}{Q'(3)}e^{3t} = \frac{-2}{12}e^{-t} + \frac{4}{-3}e^{2t} + \frac{14}{4}e^{3t} = -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$$

Compare with Problem 25.

31. Find $\mathcal{L}^{-1}\left\{\dfrac{3s+1}{(s-1)(s^2+1)}\right\}$.

We have $P(s) = 3s+1$, $Q(s) = (s-1)(s^2+1) = s^3-s^2+s-1$, $Q'(s) = 3s^2-2s+1$, $\alpha_1=1$, $\alpha_2=i$, $\alpha_3=-i$ since $s^2+1=(s-i)(s+i)$. Then by the Heaviside expansion formula the required inverse is

$$\frac{P(1)}{Q'(1)}e^{t} + \frac{P(i)}{Q'(i)}e^{it} + \frac{P(-i)}{Q'(-i)}e^{-it} \tag{1}$$

$$= \frac{4}{2}e^{t} + \frac{3i+1}{-2-2i}e^{it} + \frac{-3i+1}{-2+2i}e^{-it}$$

$$= 2e^t + (-1-\tfrac{1}{2}i)(\cos t + i\sin t) + (-1+\tfrac{1}{2}i)(\cos t - i\sin t)$$

$$= 2e^t - \cos t + \tfrac{1}{2}\sin t - \cos t + \tfrac{1}{2}\sin t$$

$$= 2e^t - 2\cos t + \sin t$$

Compare with Problem 27.

Note that some labor can be saved by observing that the last two terms in (1) are complex conjugates of each other.

THE BETA FUNCTION

32. Prove that $B(m,n) = \displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ where $m>0, n>0$.

Consider

$$G(t) = \int_0^t x^{m-1}(t-x)^{n-1}\,dx$$

Then by the convolution theorem, we have

$$\mathcal{L}\{G(t)\} = \mathcal{L}\{t^{m-1}\}\,\mathcal{L}\{t^{n-1}\} = \frac{\Gamma(m)}{s^m}\cdot\frac{\Gamma(n)}{s^n} = \frac{\Gamma(m)\Gamma(n)}{s^{m+n}}$$

Thus

$$G(t) = \mathcal{L}^{-1}\left\{\frac{\Gamma(m)\Gamma(n)}{s^{m+n}}\right\} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}t^{m+n-1}$$

or

$$\int_0^t x^{m-1}(t-x)^{n-1}\,dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}t^{m+n-1}$$

Letting $t=1$, we obtain the required result.

Letting $u = tv$ this becomes

$$J_0(t) = \frac{1}{\pi}\int_0^1 e^{it(1-2v)}\, v^{-1/2}(1-v)^{-1/2}\, dv$$

or if $1 - 2v = w$,

$$J_0(t) = \frac{1}{\pi}\int_{-1}^1 e^{itw}(1-w^2)^{-1/2}\, dw$$

46. Prove that $J_0(t) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos(t\cos\theta)\, d\theta$.

Let $w = \cos\theta$ in the result of Problem 45. Then

$$J_0(t) = \frac{1}{\pi}\int_0^\pi e^{it\cos\theta}\, d\theta = \frac{1}{\pi}\int_0^\pi \cos(t\cos\theta)\, d\theta + \frac{i}{\pi}\int_0^\pi \sin(t\cos\theta)\, d\theta$$

Equating real and imaginary parts or by showing directly that the last integral is zero, we have as required

$$J_0(t) = \frac{1}{\pi}\int_0^\pi \cos(t\cos\theta)\, d\theta$$

Another method.

Let $G(t) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos(t\cos\theta)\, d\theta = \dfrac{2}{\pi}\displaystyle\int_0^{\pi/2} \cos(t\cos\theta)\, d\theta$. Then taking Laplace transforms,

$$\mathcal{L}\{G(t)\} = \frac{2}{\pi}\int_0^{\pi/2} \frac{s}{s^2+\cos^2\theta}\, d\theta = \frac{2}{\pi}\int_0^{\pi/2} \frac{s\sec^2\theta}{s^2\tan^2\theta + s^2 + 1}\, d\theta$$

$$= \frac{2}{\pi}\frac{1}{\sqrt{s^2+1}}\tan^{-1}\left(\frac{s\tan\theta}{\sqrt{s^2+1}}\right)\Bigg|_0^{\pi/2} = \frac{1}{\sqrt{s^2+1}}$$

Thus $G(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{\sqrt{s^2+1}}\right\} = J_0(t)$, as required.

Supplementary Problems

INVERSE LAPLACE TRANSFORMS

47. Determine each of the following:

(a) $\mathcal{L}^{-1}\left\{\dfrac{3}{s+4}\right\}$ (c) $\mathcal{L}^{-1}\left\{\dfrac{8s}{s^2+16}\right\}$ (e) $\mathcal{L}^{-1}\left\{\dfrac{3s-12}{s^2+8}\right\}$ (g) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^5}\right\}$ (i) $\mathcal{L}^{-1}\left\{\dfrac{12}{4-3s}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{1}{2s-5}\right\}$ (d) $\mathcal{L}^{-1}\left\{\dfrac{6}{s^2+4}\right\}$ (f) $\mathcal{L}^{-1}\left\{\dfrac{2s-5}{s^2-9}\right\}$ (h) $\mathcal{L}^{-1}\left\{\dfrac{1}{s^{7/2}}\right\}$ (j) $\mathcal{L}^{-1}\left\{\dfrac{s+1}{s^{4/3}}\right\}$

Ans. (a) $3e^{-4t}$ (e) $3\cos 2\sqrt{2}\,t - 3\sqrt{2}\sin 2\sqrt{2}\,t$ (i) $-4e^{4t/3}$

(b) $\frac{1}{2}e^{5t/2}$ (f) $2\cosh 3t - \frac{5}{3}\sinh 3t$ (j) $(t^{-2/3} + 3t^{1/3})/\Gamma(\frac{1}{3})$

(c) $8\cos 4t$ (g) $t^4/24$

(d) $3\sin 2t$ (h) $8t^{5/2}/15\sqrt{\pi}$

Solved Problems

ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

1. Solve $Y'' + Y = t$, $Y(0) = 1$, $Y'(0) = -2$.

Taking the Laplace transform of both sides of the differential equation and using the given conditions, we have

$$\mathcal{L}\{Y''\} + \mathcal{L}\{Y\} = \mathcal{L}\{t\}, \qquad s^2y - sY(0) - Y'(0) + y = \frac{1}{s^2}$$

$$s^2y - s + 2 + y = \frac{1}{s^2}$$

Then
$$y = \mathcal{L}\{Y\} = \frac{1}{s^2(s^2+1)} + \frac{s-2}{s^2+1}$$
$$= \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{s}{s^2+1} - \frac{2}{s^2+1}$$
$$= \frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}$$

and
$$Y = \mathcal{L}^{-1}\left\{\frac{1}{s^2} + \frac{s}{s^2+1} - \frac{3}{s^2+1}\right\} = t + \cos t - 3\sin t$$

Check: $Y = t + \cos t - 3\sin t$, $Y' = 1 - \sin t - 3\cos t$, $Y'' = -\cos t + 3\sin t$. Then $Y'' + Y = t$, $Y(0) = 1$, $Y'(0) = -2$ and the function obtained is the required solution.

For another method, using the convolution integral, see Problem 7 and let $a = 1$, $F(t) = t$.

2. Solve $Y'' - 3Y' + 2Y = 4e^{2t}$, $Y(0) = -3$, $Y'(0) = 5$.

We have
$$\mathcal{L}\{Y''\} - 3\mathcal{L}\{Y'\} + 2\mathcal{L}\{Y\} = 4\mathcal{L}\{e^{2t}\}$$

$$\{s^2y - sY(0) - Y'(0)\} - 3\{sy - Y(0)\} + 2y = \frac{4}{s-2}$$

$$\{s^2y + 3s - 5\} - 3\{sy + 3\} + 2y = \frac{4}{s-2}$$

$$(s^2 - 3s + 2)y + 3s - 14 = \frac{4}{s-2}$$

$$y = \frac{4}{(s^2-3s+2)(s-2)} + \frac{14-3s}{s^2-3s+2}$$
$$= \frac{-3s^2+20s-24}{(s-1)(s-2)^2}$$
$$= \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

Thus
$$Y = \mathcal{L}^{-1}\left\{\frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}\right\} = -7e^t + 4e^{2t} + 4te^{2t}$$

which can be verified as the solution.

To determine c, note that by series expansion,

$$y = \frac{1}{s} + \frac{2}{s^2} + \frac{c}{s^2}\left(1 - \tfrac{1}{2}s^2 + \tfrac{1}{8}s^4 - \cdots\right)$$

$$= \frac{1}{s} + \frac{c+2}{s^2} - c\left(\tfrac{1}{2} - \tfrac{1}{8}s^2 + \cdots\right)$$

Then since $\mathcal{L}^{-1}\{s^k\} = 0,\ k = 0, 1, 2, \ldots,$ we obtain on inverting,

$$Y = 1 + (c+2)t$$

But $Y'(0) = 2$, so that $c = 0$ and we have the required solution

$$Y = 1 + 2t$$

SIMULTANEOUS ORDINARY DIFFERENTIAL EQUATIONS

12. Solve $\begin{cases} \dfrac{dX}{dt} = 2X - 3Y \\[6pt] \dfrac{dY}{dt} = Y - 2X \end{cases}$ subject to $X(0) = 8,\ Y(0) = 3.$

Taking the Laplace transform, we have, if $\mathcal{L}\{X\} = x,\ \mathcal{L}\{Y\} = y,$

$$sx - 8 = 2x - 3y \quad \text{or} \quad (1)\quad (s-2)x + 3y = 8$$

$$sy - 3 = y - 2x \quad \text{or} \quad (2)\quad 2x + (s-1)y = 3$$

Solving (1) and (2) simultaneously,

$$x = \frac{\begin{vmatrix} 8 & 3 \\ 3 & s-1 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{8s-17}{s^2-3s-4} = \frac{8s-17}{(s+1)(s-4)} = \frac{5}{s+1} + \frac{3}{s-4}$$

$$y = \frac{\begin{vmatrix} s-2 & 8 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} s-2 & 3 \\ 2 & s-1 \end{vmatrix}} = \frac{3s-22}{s^2-3s-4} = \frac{3s-22}{(s+1)(s-4)} = \frac{5}{s+1} - \frac{2}{s-4}$$

Then

$$X = \mathcal{L}^{-1}\{x\} = 5e^{-t} + 3e^{4t}$$

$$Y = \mathcal{L}^{-1}\{y\} = 5e^{-t} - 2e^{4t}$$

13. Solve $\begin{cases} X'' + Y' + 3X = 15e^{-t} \\ Y'' - 4X' + 3Y = 15\sin 2t \end{cases}$ subject to $X(0) = 35,\ X'(0) = -48,\ Y(0) = 27$

$Y'(0) = -55.$

Taking the Laplace transform, we have

$$s^2x - s(35) - (-48) + sy - 27 + 3x = \frac{15}{s+1}$$

$$s^2y - s(27) - (-55) - 4\{sx - 35\} + 3y = \frac{30}{s^2+4}$$

(c) If $E = 100 \sin 3t$, then (2) becomes

$$\frac{d^2Q}{dt^2} + 8\frac{dQ}{dt} + 25Q = 50 \sin 3t$$

Taking the Laplace transform, we find

$$(s^2 + 8s + 25)q = \frac{150}{s^2 + 9}$$

and

$$q = \frac{150}{(s^2+9)(s^2+8s+25)}$$

$$= \frac{75}{26}\frac{1}{s^2+9} - \frac{75}{52}\frac{s}{s^2+9} + \frac{75}{26}\frac{1}{(s+4)^2+9} + \frac{75}{52}\frac{s+4}{(s+4)^2+9}$$

Thus

$$Q = \frac{25}{26}\sin 3t - \frac{75}{52}\sin 3t + \frac{25}{26}e^{-4t}\sin 3t + \frac{75}{52}e^{-4t}\cos 3t$$

$$= \frac{25}{52}(2 \sin 3t - 3 \cos 3t) + \frac{25}{52}e^{-4t}(3 \cos 3t + 2 \sin 3t)$$

and

$$I = \frac{dQ}{dt} = \frac{75}{52}(2 \cos 3t + 3 \sin 3t) - \frac{25}{52}e^{-4t}(17 \sin 3t + 6 \cos 3t)$$

For large t, those terms of Q or I which involve e^{-4t} are negligible and these are called the *transient terms* or *transient part* of the solution. The other terms are called the *steady-state terms* or *steady-state part* of the solution.

17. Given the electric network of Fig. 3-12, determine the currents in the various branches if the initial currents are zero.

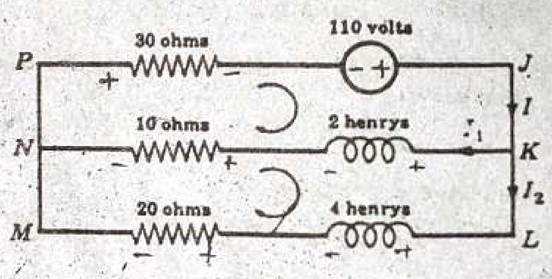

Fig. 3-12

Kirchhoff's second law [see Page 80] states that the algebraic sum of the voltage or potential drops around a closed loop is zero. Let us traverse loops $KLMNK$ and $JKNPJ$ in a clockwise fashion as shown. In traversing these loops we shall consider voltage drops as positive when we travel against the current. A voltage rise is considered as the negative of a voltage drop.

Let I be the current in $NPJK$. This current divides at the junction point K into I_1 and I_2 so that $I = I_1 + I_2$. This is equivalent to Kirchhoff's first law [see Page 80].

Applying Kirchhoff's second law to loops $KLMNK$ and $JKNPJ$, we then have respectively

$$\left.\begin{aligned} -10I_1 - 2\frac{dI_1}{dt} + 4\frac{dI_2}{dt} + 20I_2 &= 0 \\ 30I - 110 + 2\frac{dI_1}{dt} + 10I_1 &= 0 \end{aligned}\right\}$$

or

$$\left.\begin{aligned} -5I_1 - \frac{dI_1}{dt} + 2\frac{dI_2}{dt} + 10I_2 &= 0 \\ \frac{dI_1}{dt} + 20I_1 + 15I_2 &= 55 \end{aligned}\right\}$$

subject to the conditions $I_1(0) = I_2(0) = 0$.

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PIECEWISE CONTINUOUS FUNCTIONS

A function $f(x)$ is said to be *piecewise continuous* in an interval if (i) the interval can be divided into a finite number of subintervals in each of which $f(x)$ is continuous and (ii) the limits of $f(x)$ as x approaches the endpoints of each subinterval are finite. Another way of stating this is to say that a piecewise continuous function is one that has at most a finite number of finite discontinuities. An example of a piecewise continuous function is shown in Fig. 2-2. The functions of Fig. 2-1(a) and (c) are piecewise continuous. The function of Fig. 2-1(b) is continuous.

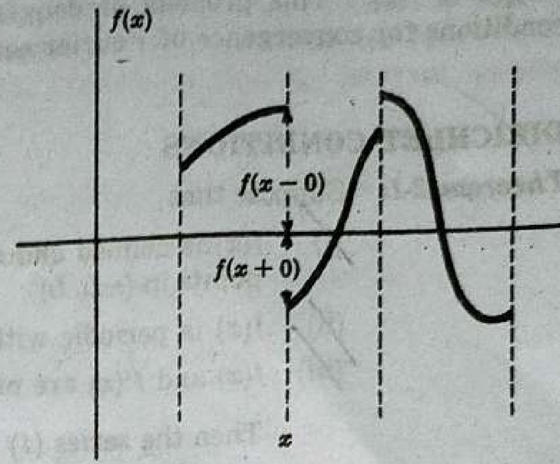

Fig. 2-2

The *limit of* $f(x)$ *from the right* or the *right-hand limit* of $f(x)$ is often denoted by $\lim_{\epsilon \to 0} f(x+\epsilon) = f(x+0)$, where $\epsilon > 0$. Similarly, the *limit of* $f(x)$ *from the left* or the *left-hand limit* of $f(x)$ is denoted by $\lim_{\epsilon \to 0} f(x-\epsilon) = f(x-0)$, where $\epsilon > 0$. The values $f(x+0)$ and $f(x-0)$ at the point x in Fig. 2-2 are as indicated. The fact that $\epsilon \to 0$ and $\epsilon > 0$ is sometimes indicated briefly by $\epsilon \to 0+$. Thus, for example, $\lim_{\epsilon \to 0+} f(x+\epsilon) = f(x+0)$, $\lim_{\epsilon \to 0+} f(x-\epsilon) = f(x-0)$.

DEFINITION OF FOURIER SERIES

Let $f(x)$ be defined in the interval $(-L, L)$ and determined outside of this interval by $f(x+2L) = f(x)$, i.e. assume that $f(x)$ has the period $2L$. The *Fourier series* or *Fourier expansion* corresponding to $f(x)$ is defined to be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \tag{1}$$

where the *Fourier coefficients* a_n and b_n are

$$\begin{cases} a_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx \\ b_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx \end{cases} \qquad n = 0, 1, 2, \ldots \tag{2}$$

Motivation for this definition is supplied in Problem 2.4.

If $f(x)$ has the period $2L$, the coefficients a_n and b_n can be determined equivalently from

$$\begin{cases} a_n = \dfrac{1}{L} \displaystyle\int_{c}^{c+2L} f(x) \cos \frac{n\pi x}{L}\, dx \\ b_n = \dfrac{1}{L} \displaystyle\int_{c}^{c+2L} f(x) \sin \frac{n\pi x}{L}\, dx \end{cases} \qquad n = 0, 1, 2, \ldots \tag{3}$$

where c is any real number. In the special case $c = -L$, (3) becomes (2). Note that the constant term in (1) is equal to $\frac{a_0}{2} = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx$, which is the *mean* of $f(x)$ over a period.

If $L = \pi$, the series (1) and the coefficients (2) or (3) are particularly simple. The function in this case has the period 2π.

It should be emphasized that the series (1) is only the series which *corresponds* to $f(x)$. We do not know whether this series converges or even, if it does converge, whether it converges to $f(x)$. This problem of convergence was examined by *Dirichlet,* who developed conditions for convergence of Fourier series which we now consider.

DIRICHLET CONDITIONS

Theorem 2-1: Suppose that

(i) $f(x)$ is defined and single-valued except possibly at a finite number of points in $(-L, L)$

(ii) $f(x)$ is periodic with period $2L$

(iii) $f(x)$ and $f'(x)$ are piecewise continuous in $(-L, L)$

Then the series (*1*) with coefficients (*2*) or (*3*) converges to

(*a*) $f(x)$ if x is a point of continuity

(*b*) $\dfrac{f(x+0) + f(x-0)}{2}$ if x is a point of discontinuity

For a proof see Problems 2.18–2.23.

According to this result we can write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{4}$$

at any point of continuity x. However, if x is a point of discontinuity, then the left side is replaced by $\frac{1}{2}[f(x+0) + f(x-0)]$, so that the series converges to the mean value of $f(x+0)$ and $f(x-0)$.

The conditions (i), (ii) and (iii) imposed on $f(x)$ are *sufficient* but not *necessary,* i.e. if the conditions are satisfied the convergence is guaranteed. However, if they are not satisfied the series may or may not converge. The conditions above are generally satisfied in cases which arise in science or engineering.

There are at present no known necessary and sufficient conditions for convergence of Fourier series. It is of interest that continuity of $f(x)$ does not *alone* insure convergence of a Fourier series.

ODD AND EVEN FUNCTIONS

A function $f(x)$ is called *odd* if $f(-x) = -f(x)$. Thus x^3, $x^5 - 3x^3 + 2x$, $\sin x$, $\tan 3x$ are odd functions.

A function $f(x)$ is called *even* if $f(-x) = f(x)$. Thus x^4, $2x^6 - 4x^2 + 5$, $\cos x$, $e^x + e^{-x}$ are even functions.

The functions portrayed graphically in Fig. 2-1(*a*) and 2-1(*b*) are odd and even respectively, but that of Fig. 2-1(*c*) is neither odd nor even.

In the Fourier series corresponding to an odd function, only sine terms can be present. In the Fourier series corresponding to an even function, only cosine terms (and possibly a constant, which we shall consider to be a cosine term) can be present.

HALF-RANGE FOURIER SINE OR COSINE SERIES

A half-range Fourier sine or cosine series is a series in which only sine terms or only cosine terms are present, respectively. When a half-range series corresponding to a given

Similar results can be obtained for cosine series or for series having both sines and cosines.

These ideas can be generalized to *triple Fourier series*, etc.

APPLICATIONS OF FOURIER SERIES

There are numerous applications of Fourier series to solutions of boundary value problems. For example:

1. **Heat flow.** See Problems 2.25–2.29.
2. **Laplace's equation.** See Problems 2.30, 2.31.
3. **Vibrating systems.** See Problems 2.32, 2.33.

Solved Problems

FOURIER SERIES

2.1. Graph each of the following functions.

(a) $f(x) = \begin{cases} 3 & 0 < x < 5 \\ -3 & -5 < x < 0 \end{cases}$ Period $= 10$

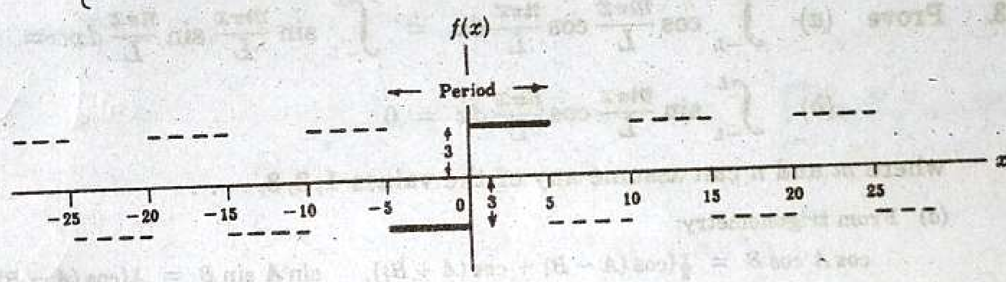

Fig. 2-3

Since the period is 10, that portion of the graph in $-5 < x < 5$ (indicated heavy in Fig. 2-3 above) is extended periodically outside this range (indicated dashed). Note that $f(x)$ is not defined at $x = 0, 5, -5, 10, -10, 15, -15$, etc. These values are the *discontinuities* of $f(x)$.

(b) $f(x) = \begin{cases} \sin x & 0 \leqq x \leqq \pi \\ 0 & \pi < x < 2\pi \end{cases}$ Period $= 2\pi$

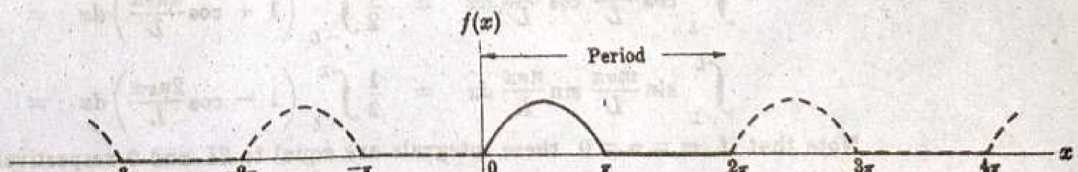

Fig. 2-4

Refer to Fig. 2-4 above. Note that $f(x)$ is defined for all x and is continuous everywhere.

If $m = n$, $$\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx \;=\; \frac{1}{2}\int_{-L}^{L}\sin\frac{2n\pi x}{L}\,dx \;=\; 0$$

The results of parts (*a*) and (*b*) remain valid when the limits of integration $-L, L$ are replaced by $c, c+2L$ respectively.

2.4. If the series $A + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$ converges uniformly to $f(x)$ in $(-L, L)$, show that for $n = 1, 2, 3, \ldots$,

(*a*) $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$, (*b*) $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$, (*c*) $A = \frac{a_0}{2}$.

(*a*) Multiplying $$f(x) \;=\; A + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) \tag{1}$$

by $\cos\frac{m\pi x}{L}$ and integrating from $-L$ to L, using Problem 2.3, we have

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx \;&=\; A\int_{-L}^{L}\cos\frac{m\pi x}{L}\,dx\\ &\quad+ \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx\right\}\\ &=\; a_mL \quad \text{if } m \neq 0\end{aligned} \tag{2}$$

Thus $$a_m \;=\; \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx \qquad \text{if } m = 1, 2, 3, \ldots$$

(*b*) Multiplying (*1*) by $\sin\frac{m\pi x}{L}$ and integrating from $-L$ to L, using Problem 2.3, we have

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx \;&=\; A\int_{-L}^{L}\sin\frac{m\pi x}{L}\,dx\\ &\quad+ \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx\right\}\\ &=\; b_mL\end{aligned}$$

Thus $$b_m \;=\; \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx \qquad \text{if } m = 1, 2, 3, \ldots$$

(*c*) Integration of (*1*) from $-L$ to L, using Problem 2.2, gives

$$\int_{-L}^{L} f(x)\,dx \;=\; 2AL \qquad \text{or} \qquad A \;=\; \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

Putting $m = 0$ in the result of part (*a*), we find $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$ and so $A = \frac{a_0}{2}$.

The above results also hold when the integration limits $-L, L$ are replaced by $c, c+2L$.

Note that in all parts above, interchange of summation and integration is valid because the series is *assumed* to converge uniformly to $f(x)$ in $(-L, L)$. Even when this assumption is not warranted, the coefficients a_m and b_m as obtained above are called *Fourier coefficients* corresponding to $f(x)$, and the corresponding series with these values of a_m and b_m is called the *Fourier series* corresponding to $f(x)$. An important problem in this case is to investigate conditions under which this series actually converges to $f(x)$. Sufficient conditions for this convergence are the *Dirichlet conditions* established below in Problems 2.18–2.23.

2.5. (*a*) Find the Fourier coefficients corresponding to the function

$$f(x) \;=\; \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases} \qquad \text{Period} = 10$$

(b) Write the corresponding Fourier series.

(c) How should $f(x)$ be defined at $x = -5$, $x = 0$ and $x = 5$ in order that the Fourier series will converge to $f(x)$ for $-5 \leqq x \leqq 5$?

The graph of $f(x)$ is shown in Fig. 2-6 below.

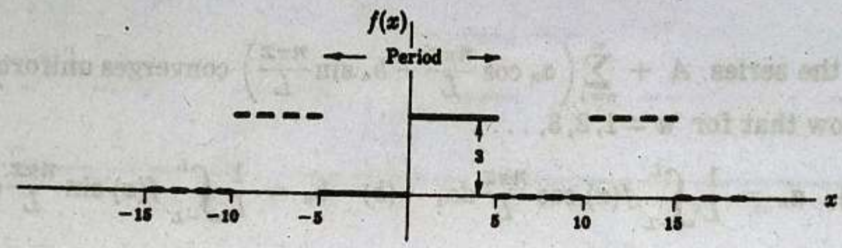

Fig. 2-6

(a) Period $= 2L = 10$ and $L = 5$. Choose the interval c to $c + 2L$ as -5 to 5, so that $c = -5$. Then

$$\begin{aligned} a_n &= \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi x}{5}\,dx \\ &= \frac{1}{5}\left\{\int_{-5}^{0}(0)\cos\frac{n\pi x}{5}\,dx + \int_0^5 (3)\cos\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_0^5 \cos\frac{n\pi x}{5}\,dx \\ &= \frac{3}{5}\left(\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right)\Big|_0^5 = 0 \quad \text{if } n \neq 0 \end{aligned}$$

If $n = 0$, $a_n = a_0 = \frac{3}{5}\int_0^5 \cos\frac{0\pi x}{5}\,dx = \frac{3}{5}\int_0^5 dx = 3.$

$$\begin{aligned} b_n &= \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi x}{5}\,dx \\ &= \frac{1}{5}\left\{\int_{-5}^{0}(0)\sin\frac{n\pi x}{5}\,dx + \int_0^5 (3)\sin\frac{n\pi x}{5}\,dx\right\} = \frac{3}{5}\int_0^5 \sin\frac{n\pi x}{5}\,dx \\ &= \frac{3}{5}\left(-\frac{5}{n\pi}\cos\frac{n\pi x}{5}\right)\Big|_0^5 = \frac{3(1-\cos n\pi)}{n\pi} \end{aligned}$$

(b) The corresponding Fourier series is

$$\begin{aligned} \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right) &= \frac{3}{2} + \sum_{n=1}^{\infty}\frac{3(1-\cos n\pi)}{n\pi}\sin\frac{n\pi x}{5} \\ &= \frac{3}{2} + \frac{6}{\pi}\left(\sin\frac{\pi x}{5} + \frac{1}{3}\sin\frac{3\pi x}{5} + \frac{1}{5}\sin\frac{5\pi x}{5} + \cdots\right) \end{aligned}$$

(c) Since $f(x)$ satisfies the Dirichlet conditions, we can say that the series converges to $f(x)$ at all points of continuity and to $\dfrac{f(x+0)+f(x-0)}{2}$ at points of discontinuity. At $x = -5$, 0 and 5, which are points of discontinuity, the series converges to $(3+0)/2 = 3/2$, as seen from the graph. The series will converge to $f(x)$ for $-5 \leqq x \leqq 5$ if we redefine $f(x)$ as follows:

$$f(x) = \begin{cases} 3/2 & x = -5 \\ 0 & -5 < x < 0 \\ 3/2 & x = 0 \\ 3 & 0 < x < 5 \\ 3/2 & x = 5 \end{cases} \qquad \text{Period} = 10$$

2.6. **Expand $f(x) = x^2$, $0 < x < 2\pi$, in a Fourier series if the period is 2π.**

The graph of $f(x)$ with period 2π is shown in Fig. 2-7.

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MATHEMATICAL METHODS

MATHEMATICAL PHYSICS

VOLUME TWO

PROFESSOR MD. ABDUR RAHAMAN

INTEGRAL TRANSFORMS AND INITIAL & BOUNDARY VALUE PROBLEMS WITH APPLICATIONS

$$L\ \{F(t)\}=\int_0^\infty \bar{e}^{st}F(t)\,dt,\ t>0$$

$$-y\ = f\ (x), y(0)=0, y(1)=0$$

$$y=\int_0^1\ G(x,s)f\ (s)ds$$

$$\text{where}\quad G(x,s)=\begin{cases} s(1-x), & 0\leq s\leq x \\ x(1-s), & x\leq s\leq 1. \end{cases}$$

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WORKED OUT EXAMPLES

Example 1. The function x^2 is periodic with period $2l$ on the interval $[-l, l]$. Find its Fourier series.

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Solution : $f(x) = x^2$. $f(-x) = (-x)^2 = x^2 = f(x)$ so $f(x)$ is an even function and hence sine terms will vanish. i.e $b_n = 0$.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where $a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx \ (n = 0)$

and $a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l} dx \ (n = 1, 2, 3 \ ....)$

Since $f(x)$ is even.

$$a_0 = \frac{1}{l}\int_0^l f(v)dv. \ a_n = \frac{2}{l}\int_0^l f(v) \cos\frac{n\pi v}{l}dv$$

$$a_0 = \frac{1}{l}\int_0^l x^2 dx = \frac{1}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{1}{3}l^2.$$

$$a_n = \frac{2}{l}\int_0^l x^2 \cos\frac{n\pi x}{l} dx. \text{ (Integrating by parts)}$$

$$= \frac{2}{l}\left[x^2. \frac{l}{n\pi}\sin\frac{n\pi x}{l}\right]_0^l - \frac{2}{l}. 2\int_0^l x. \frac{l}{n\pi}\sin\frac{n\pi x}{l}dx$$

$$= 0 - \frac{4}{n\pi}\int_0^l x \sin\frac{n\pi x}{l} dx$$

$$= \frac{4}{n\pi}. \frac{l}{n\pi}\left[x \cos\frac{n\pi x}{l}\right]_0^l - \frac{4}{n\pi}. \frac{l}{n\pi}\int_0^l \cos\frac{n\pi x}{l} dx$$

$$= \frac{4l}{n^2\pi^2}[l \cos n\pi - 0] - \frac{4l^2}{n^3\pi^3}\left[\sin\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{4l^2}{n^2\pi^2}(-1)^n - 0 = \frac{4l^2}{n^2\pi^2}.(-1)^n. \therefore a_n = \frac{4l^2}{n^2\pi^2}(-1)^n.$$

Therefore, $f(x) = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2}(-1)^n \cos\frac{n\pi x}{l}$

$$= \frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2}(-1)^n. \cos\frac{n\pi x}{l}$$

$$= \frac{l^2}{3} + \frac{4l^2}{\pi^2}\left[-\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{2^2}\cos\frac{2\pi x}{l} - \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{4^2}\cos\frac{4\pi x}{l} \cdots\right]$$

$$= \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} - \frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} - \frac{1}{4^2}\cos\frac{4\pi x}{l} + \cdots\right]$$

Example 2. Obtain the Fourier series of the function

$$f(x) = \begin{cases} 0, -\pi \le x \le 0 \\ 1, 0 \le x \le \pi \end{cases}$$

and verify the result by assuming the complex form of Fourier series. **D. U. M. SC (F) 1991**

Solution : Def^n : The **complex form of the Fourier series** can be written as.

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{c}}, -c < x < c$$

where $C_n = \frac{1}{2c}\int_{-c}^{c} f(x)\, e^{-\frac{in\pi x}{c}} dx$

and $n = 0, \pm 1, \pm 2, \ldots\ldots$

Ist portion

By def^n, we have $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ (1)

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$