Where is Jupiter
May 2021 Revision (original)
Saied Seghatoleslami

As I am drifting into retirement, I thought I would take up ham radio again. My license had expired, so I started over again. The new rule is that if your Extra Class license has expired, you can take the Technician exam and present a copy of your old license and you can get your Extra ticket back. I did that yesterday and I am waiting for my name to show up in the FCC database.

In our homeowner's association, large outdoor antennas are not permitted and beside that, the Ten Tec Omni D is not working. So, until I figure out how to get the old rig fixed and build a stealthy flag-pole antenna, I have to do something else. VHF/UHF are good candidates but putting up a J-pole antenna and getting on the repeaters, as interesting as that might be, does not feel like it would be fully satisfying. Of course, I will do that once my ticket arrives, but I am going to explore amateur satellites. That sounds like a lot more fun.

My plan is to build a satellite tracking antenna using stepper motors driven by a Raspberry Pi 4 programmed in Go. The details of the design are to be worked out, but this is the high-level idea. I am starting with Don Koks' paper from the Australian Department of Defense Science and Technology Department titled "Changing Coordinates in the Context of Orbital Mechanics". In the process, I have also worked out some of the math where he "leaves the details for the student to work out". I will include them here at some future revision.

For the rest of this note, I am assuming you have read the paper and understand it. I will be using his notation with one exception that I will note. My objective is to do 3 things:

1. Repeat his calculations for Adelaide to make sure that I have fully understood the concepts.
2. Do the same calculations for Cranbury, NJ where I live, to confirm my understanding of the concepts.
3. Reframe the model for earth orbiting satellites (he suggests in his paper that it should not be hard, we will see).

Independent of the frame of reference, the following is true:

$$r_{JA} = r_{JS} + r_{SE} + r_{EA} = r_{JS} - r_{ES} - r_{AE}$$

Where:

- J is Jupiter
- A is Adelaide
- S is the Sun
- E is Earth
- The two letters (e.g., SE) represent the direction of the vector, in the case of SE from Sun to Earth and ES from the Earth to Sun.

We are interested in the azimuth and elevation of Jupiter in the Adelaide sky at some specific time, so eventually, everything has to be translated to Adelaide ENU (East North Up) coordinates. Will start with Jupiter's location in the Jupiter orbital coordinates and work our way back to the Adelaide ENU coordinates.

$$[r_{JA}]_{ENU} = [r_{JS} - r_{ES} - r_{AE}]_{ENU}$$

Or:

$$[r_{JA}]_{ENU} = \mu_{ENU}^{OPJ}[r_{JS}]_{OPJ} - \mu_{ENU}^{OPE}[r_{ES}]_{OPE} - \mu_{ENU}^{ECEF}[r_{AE}]_{ECEF}$$

Where μ_{XYZ}^{ABC} means transformation of the vector coordinates from the ABC coordinate system to the XYZ coordinate system.

Let's start with Jupiter in its own orbital plane.

$$x_{op} = a\cos E - ae \qquad y_{op} = b\sin E$$

Where:

- "a" is the semi-major axis
- "b" is the semi-minor axis
- "E" the plants *eccentric anomaly*
- "e" is the eccentricity of the orbit

From geometry we know that: $b = a\sqrt{1-e^2}$

Now for calculating E we need to solve:

$$E - e\sin E = M$$

Where M is the mean anomaly and given by:

$$M = M_0 + 2\pi(t - t_0)/T$$

Where M_0 is mean anomaly at some epoch t_0 and T is the period of the plan that we will calculate next.

$$T^2 = \frac{4\pi^2}{G(MM + m)}a^3$$

Where MM is the mass of the sun (this is the one place I have deviated from and m is the mass of Jupiter. I have used MM instead of as not to confuse it with mean anomaly.

It is important to note that time is measured in Julian Days or JD. I recommend building a spreadsheet for easy conversion between the Georgian Calendar and JD. That is what I did.

Now we get the data. This is where I got my data (in addition to some data from the paper) and that is also where I think the author got his:

http://www.met.rdg.ac.uk/~ross/Astronomy/Planets.html

m	Mass of Jupiter	1.8986×10^{27} kg
a	Jupiter semi-major axis	5.203 363 01 AU
e	Jupiter orbit eccentricity	0.048 392 66
i	Orbital inclination	1.305 30 degrees
Ω	Longitude of the ascending node	100.556 15 degrees
	Longitude of the perihelion	14.753 85 degrees
	Mean longitude	34.404 38 degrees
t_0	Epoch J2000	Noon Jan. 1, 2000
ω	Argument of the perifocus	-85.80 degrees
M_0	Mean anomaly at epoch	19.65 degrees
AU	Astronomical Unit	$1.495\ 978\ 707 \times 10^{11}$ meters
MM	Mass of the Sun	1.989×10^{30} kg
G	Gravitational constant	$6.673\ 84 \times 10^{-11}$ SI Units
Now	The time of interest to the author	9:00 pm DST, March 22, 2014
$t - t_0$	Difference to J2000	5193.9375 days[1]

Author points out that two of the traditionally given orbital parameters, longitude of perihelion (also known as the longitude of the perifocus) and mean longitude make no mathematical sense. The first is the sum of two angles that are in two different planes. One is the longitude of the ascending node which lies in the reference plane (in this case in the XY Sun Centered Inertial frame) and ω, argument of the perifocus, which lies in the orbital plane. So, we will disaggregate these numbers per author's advice.

$$\omega = longitude\ of\ perifocus - \Omega$$

Mean magnitude is defined as:

$$mean\ longitude = \ \Omega + \omega + M_0$$

And it suffers from the same problems and more, so it will also be disaggregated:

$$M_0 = mean\ longitude - \ longitude\ of\ perifocus$$

So, we can add these quantities to the above table and proceed with the calculations.

[1] Note to self: I had to put the minutes to 30 and hour to 10 to get the same number as the author. Either my understanding of JD is incorrect or there is a bug in the spreadsheet.

$$T^2 = \frac{4\pi^2}{6.67384 \times 10^{-11}(1.989 \times 10^{30} + 1.8986 \times 10^{27})}(5.203\,363\,01 \times 1.495978707 \times 10^{11})^3$$

Which yields:

$$T = 4332.8\ days$$

Now we evolve the Jupiter's mean anomaly at the epoch to the time of interest:

$$M = 0.343 + 2\pi \times \frac{5193.9375}{4332.8} = \ 7.87\ radians = 91.2\ mod\ 360\ degrees$$

Note that 19.65 degrees needs to be converted to radians and the results converted back to degrees (modulo 360) to match the author's number as he points out in the paper.

Now we iteratively solve for E by using (note radians):

$$E = 0.04839266 \sin E + 7.87$$

Which is an easy task on a spreadsheet and mod 360 gives:

$$E = 93.97\ degrees$$

Now we can write the coordinates of Jupiter in the Jupiter orbit relative to the Sun at the time of interest:

$$[r_{JS}]_{OPJ} = \begin{bmatrix} a(\cos E - e) \\ b \sin E \\ 0 \end{bmatrix} = \begin{bmatrix} -0.91 \\ 7.76 \\ 0 \end{bmatrix} \times 10^{11}\ meters$$

These are the next steps:

1. Transform the coordinates in Jupiter orbit to the sun centered inertial orbit
2. Then transform to Earth centered inertial orbit
3. Then to Earth Centered, Earth Fixed orbit
4. Then to Adelaide ENU orbit

There are two steps to the transformation. One is translating from the Jupiter orbital plane to the Sun centered inertial frame and the second step is to translate from the sun centered intertrial frame, to the Adelaide ENU. Note that this transformation is independent of any specific planet once its location has been translated to the Sun Centered Inertial frame.

$$\mu_{ENU}^{OPX} = \mu_{ENU}^{SCI}\ \mu_{SCI}^{OPJ}$$

Taking up the invariant transformation:

$$\mu_{ENU}^{SCI} = \mu_{ENU}^{ECEF}\ \mu_{ECEF}^{ECI}\ \mu_{ECI}^{SCI}$$

$$\mu_{ENU}^{SCI} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} E_2^{\lambda}\ E_3^{-\phi}\ E_3^{-\gamma}\ E_1^{\tau}\ E_3^{p}$$

And:

$$\mu_{SCI}^{OPJ} = E_3^{\Omega}\, E_1^{i}\, E_3^{\omega}$$

Where:

λ	Adelaide latitude	-34.9 degrees
ϕ	Adelaide longitude	138.60 degrees
γ	Greenwich Sideral angle	280.46 + 56.71 degrees
τ	Earth's tilt	23.439 degrees
p	Earth's precession	0.20 degrees

And Es are the Euler transformation matrices where subscripts 1, 2 and 3 represent rotation about x, y or z axis.

Greenwich Sideral angle and Earth's precession require additional calculations.

Earth's Sideral angle was measured at J2000 to a great degree of precision to be 280.46062 degrees. For $t - t_0 = 5193.9375$ as calculated before:

$$\frac{360\ degrees}{23h\ 56m\ 4.09890s\ (Sideral\ Day)} \times 5193.9375\ \times 24\ h \cong 56.71\ degrees$$

The Earth precession period is 25,770 years:

$$p = \frac{360^{\circ} \times 5193.9375\ days}{25770\ years\ \times 365.25\ days/yer} = 0.2^{\circ}\ Earth's\ precession$$

$$\mu_{ENU}^{SCI} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(34.9) & 0 & -\sin 34.9 \\ 0 & 1 & 0 \\ \sin 34.9 & 0 & \cos 34.9 \end{bmatrix} \begin{bmatrix} \cos 130.68 & \sin 130.68 & 0 \\ -\sin 130.68 & \cos 130.68 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 337.17 & -\sin 337.17 & 0 \\ \sin 337.17 & \cos 337.17 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 23.439 & -\sin 23.439 \\ 0 & \sin 23.439 & \cos 23.439 \end{bmatrix} \begin{bmatrix} \cos 0.2 & -\sin 0.2 & 0 \\ \sin 0.2 & \cos 0.2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that cosine is an even function and sine is an odd function.

So, if you have not made any mistakes and have been diligent in working spreadsheet (Go program later), you should get the following rounded to two decimal points (I will continue the calculations with the spreadsheet precision):

$$\mu_{ENU}^{SCI} = \begin{bmatrix} -0.90 & -0.40 & 0.17 \\ -0.25 & 0.80 & 0.55 \\ -0.35 & 0.45 & -0.82 \end{bmatrix}$$

And we work the same with the other transformation:

$$\mu_{SCI}^{OPJ} = \begin{bmatrix} \cos 100.55615 & -\sin 100.55615 & 0 \\ \sin 100.55615 & \cos 100.55615 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 1.30530 & -\sin 1.30530 \\ 0 & \sin 1.30530 & \cos 1.30530 \end{bmatrix} \begin{bmatrix} \cos(-85.80) & -\sin(-85.80) & 0 \\ \sin(-85.80) & \cos(-85.80) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And again, assuming no mistakes, you should get:

$$\mu_{SCI}^{OPJ} = \begin{bmatrix} 0.9668 & -0.2547 & .0224 \\ 0.2547 & 0.9670 & 0.0042 \\ -0.0227 & 0.0017 & 0.9997 \end{bmatrix}$$

Now we can calculate the location of Jupiter in the Sun Centered Inertial frame:

$$[r_{JS}]_{SCI} = \mu_{SCI}^{OPJ} [r_{JS}]_{OPJ} = \begin{bmatrix} -2.8562 \\ 7.2723 \\ 0.0336 \end{bmatrix} \times 10^{11}$$

Now we will transform from the SCI to the ENU coordinates:

$$[r_{JS}]_{ENU} = \mu_{ENU}^{SCI} [r_{JS}]_{SCI} = \begin{bmatrix} -0.90 & -0.40 & 0.17 \\ -0.25 & 0.80 & 0.55 \\ -0.35 & 0.45 & -0.82 \end{bmatrix} \begin{bmatrix} -2.8562 \\ 7.2723 \\ 0.0336 \end{bmatrix} \times 10^{11}$$

Or:

$$[r_{JS}]_{ENU} = \begin{bmatrix} -0.2960 \\ 6.5375 \\ 4.2684 \end{bmatrix} \times 10^{11}$$

Recall that:

$$[r_{JA}]_{ENU} = \mu_{ENU}^{OPJ}[r_{JS}]_{OPJ} - \mu_{ENU}^{OPE}[r_{ES}]_{OPE} - \mu_{ENU}^{ECEF}[r_{AE}]_{ECEF}$$

Now that you are done with Jupiter, we need to work on Earth. Here is the Earth data (I have repeated some of the data for convenience:

m	Mass of Earth	5.9736x10^{24} kg
a	Earth semi-major axis	1.000 000 11 AU
e	Earth orbit eccentricity	0.01671022 392 66
b	Earth semi-minor axis	2.12x10^{11} meters
i	Orbital inclination	0.000 05 degrees
Ω	Longitude of the ascending node	11.26064 degrees
	Longitude of the perihelion	102.947 19 degrees
	Mean longitude	100.464 35 degrees
t_0	Epoch J2000	Noon Jan. 1, 2000
ω	Argument of the perifocus	91.680 790 degrees
M_0	Mean anomaly at epoch	-2.48284 degrees
AU	Astronomical Unit	1.495 978 707 x 10^{11} meters
MM	Mass of the Sun	1.989x10^{30} kg
G	Gravitational constant	6.673 84x10^{-11} SI Units
Now	The time of interest to the author	9:00 pm DST, March 22, 2014
t - t_0	Difference to J2000	5193.9375 days

And the same calculations:

$$\omega = longitude\ of\ perifocus - \Omega = 102.94719 - 11.26064 = 91.680790\ degrees$$

$$M_0 = mean\ longitude - \ longitude\ of\ perifocus = 100.64635 - 102.94719 = \ -2.48284\ degrees$$

$$T^2 = \frac{4\pi^2}{6.67384 \times 10^{-11}(1.989 \times 10^{30} + 5.9736 \times 10^{24})}(1.000\,005 \times 1.495978707 \times 10^{11})^3$$

$$T = 365.2175\ days$$

$$M = M_0 + 2\pi(t - t_0)/T$$

$$M = -0.0433 + 2\pi \times \frac{5193.9375}{365.2175} = \ 89.3129\ radians = 77.254\ mod\ 360\ degrees$$

$$E - e\sin E = M$$

$$E = 0.000005\sin E + 89.3129 = 79.96\ degrees$$

$$[r_{ES}]_{OPE} = \begin{bmatrix} a(\cos E - e) \\ b\sin E \\ 0 \end{bmatrix} = \begin{bmatrix} 0.281 \\ 1.46 \\ 0 \end{bmatrix} \times 10^{11}\ meters$$

Now we transform to the Sun Centered Inertial frame:

$$\mu_{SCI}^{OPE} = E_3^{\Omega}\, E_1^{i}\, E_3^{\omega} = \begin{bmatrix} \cos 11.26 & -\sin 11.26 & 0 \\ \sin 11.26 & \cos 11.26 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 0.00005 & -\sin 0.00005 \\ 0 & \sin 0.00005 & \cos 0.00005 \end{bmatrix} \begin{bmatrix} \cos 91.68 & -\sin 91.68 & 0 \\ \sin 91.68 & \cos 91.68 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And combining it with the SCI to ENU transform matrix (recall, it is independent of plants).

$$\mu_{ENU}^{OPE} = \begin{bmatrix} -0.1837 & 0.9676 & 0.1729 \\ 0.8346 & 0.0605 & 0.5475 \\ 0.5193 & 0.2449 & -0.8187 \end{bmatrix}$$

$$[r_{ES}]_{ENU} = \mu_{ENU}^{OPE}[r_{ES}]_{OPE} = \begin{bmatrix} 1.36115 \\ 0.3230 \\ 0.5305 \end{bmatrix} \times 10^{11}$$

Recall that this is what we are solving for:

$$[r_{JA}]_{ENU} = [r_{JS} - r_{ES} - r_{AE}]_{ENU}$$

We have solved for the first two terms, now it’s time for the third term:

For the Earth:

$$equotorial\!: a = 6{,}378{,}137\ m \qquad polar\!: b = 6{,}356{,}752.3142\ m$$

And:

$$k \equiv (a^2(\cos\lambda)^2 + b^2(\sin\lambda)^2)^{1/2}$$

$$[r_{AE}]_{ECEF} = \begin{bmatrix} \left(\frac{a^2}{k} + h\right)\cos\lambda\cos\phi \\ \left(\frac{a^2}{k} + h\right)\cos\lambda\sin\phi \\ \left(\frac{b^2}{k} + h\right)\sin\lambda \end{bmatrix}$$

$$[r_{AE}]_{ECEF} = \begin{bmatrix} -3.93 \\ 3.46 \\ -3.63 \end{bmatrix} \times 10^6\ m$$

And:

$$\mu_{ENU}^{ECED} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} E_2^{\lambda} E_3^{-\phi}$$

Or:

$$\mu_{ENU}^{ECEF} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos 34.9 & 0 & \sin 34.9 \\ 0 & 1 & 0 \\ -\sin 34.9 & 0 & \cos 34.9 \end{bmatrix} \begin{bmatrix} \cos 130.68 & -\sin 130.68 & 0 \\ \sin 130.68 & \cos 130.68 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.66 & 0.75 & 0 \\ -0.43 & 0.38 & 0.82 \\ -0.62 & 0.54 & -0.57 \end{bmatrix}$$

After applying to the position vector:

$$[r_{AE}]_{ENU} = \begin{bmatrix} 0.00357 \\ 0.01865 \\ 6.37127 \end{bmatrix} \times 10^6\ m$$

Now we have arrived at the final point:

$$\left[r_{JA}\right]_{ENU} = \left[r_{JS} - r_{ES} - r_{AE}\right]_{ENU}$$

$$\left[r_{JA}\right]_{ENU} = \begin{bmatrix} -0.2602 \\ 6.5375 \\ 4.2684 \end{bmatrix} \times 10^{11} - \begin{bmatrix} 1.3612 \\ 0.3230 \\ 0.5035 \end{bmatrix} \times 10^{11} - \begin{bmatrix} 0.00357 \\ 0.01865 \\ 6.37127 \end{bmatrix} \times 10^6\ m$$

$$\left[r_{JA}\right]_{ENU} = \begin{bmatrix} -1.6570 \\ 6.2145 \\ 3.7648 \end{bmatrix} \times 10^{11}\ m$$

$$D = (x_{ENU}^2 + y_{ENU}^2)^{1/2}$$

$$\sin\beta = \frac{x_{ENU}}{D} \qquad \cos\beta = \frac{y_{ENU}}{D} \qquad \tan\epsilon = \frac{z_{ENU}}{D}$$

Where β is the bearing and ϵ the elevation.

$$D = 6.4317 \times 10^{11} \qquad \beta = 345.1\ degrees \qquad \epsilon = 30.34\ degrees$$

Bearing is the same as Azimuth, at least based on a quick Google search.

Final statement of phase 1: I have gained real respect for the astronomers of old without Excel spreadsheet, 11C calculator and linear algebra (not to mention Newtonian mechanics).