MŰEGYETEM 1782

CONTROL ENGINEERING

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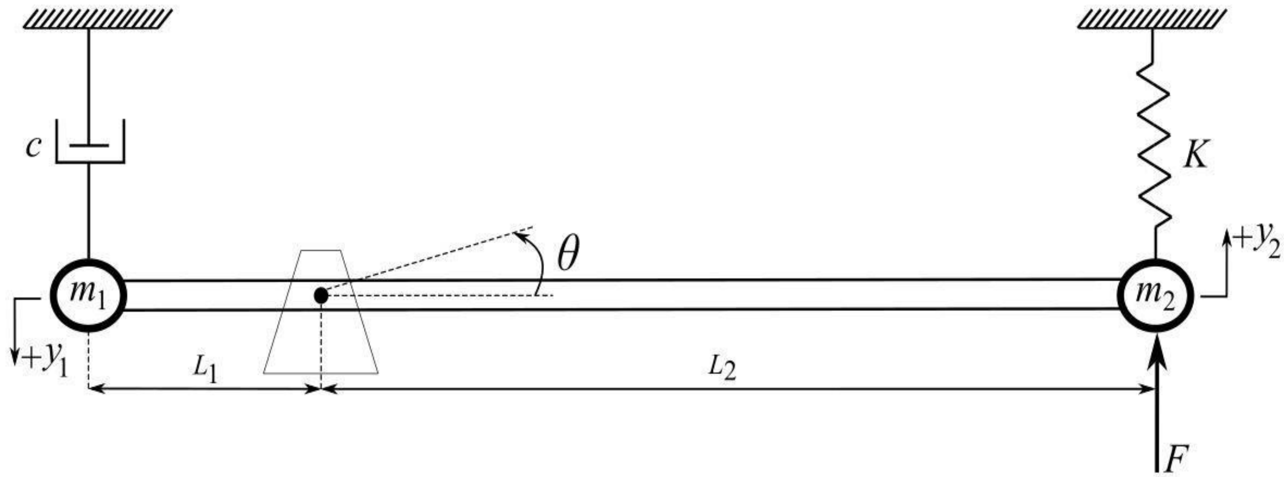
- # Task Figure:



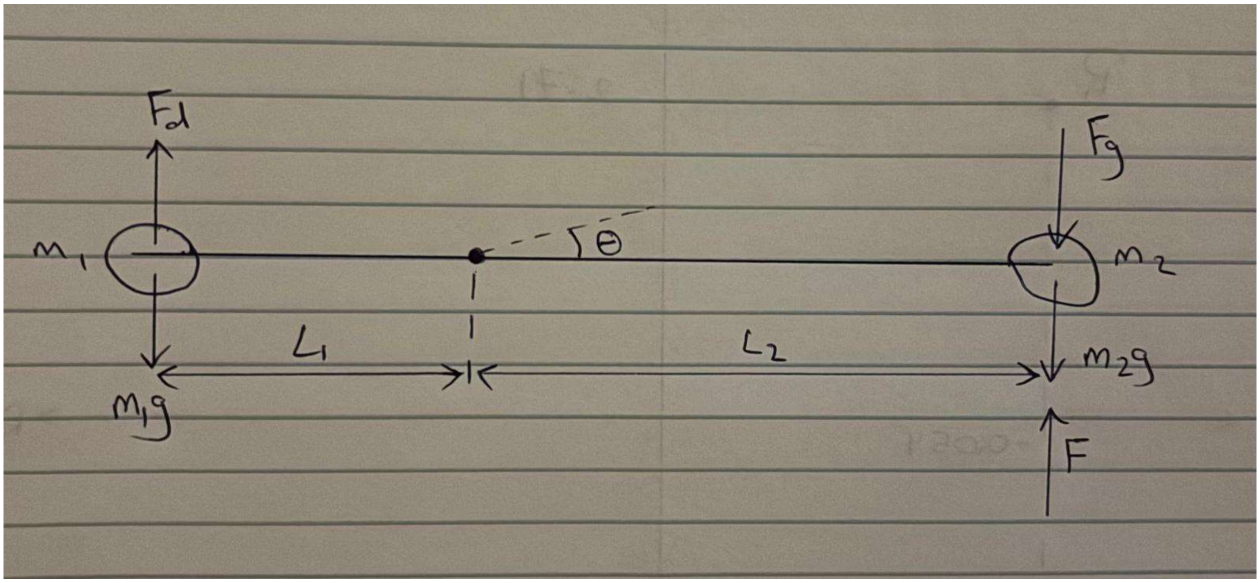
- # Given Data:

m_1= 3.45 [Kg]
m_2= 2.04 [Kg]
L_1= 0.94 [m]
L_2= 2.13 [m]
w= 2.04 [rad/s]
K= 20 [N/m]
c= 1.5 [N*s/m]
g= 9.81 [m/s^2]

- # SOLUTIONS:

- ## TASK 1 – Linearization:

1.1 Free Body Diagram and a Nonlinear Model of the System



Damping force , $F_d(t) = c\dot{y}_1(t)$

Spring force, $F_s = ky_2(t)$

$y_1(t) = L_1\sin\theta(t)$

$y'_1(t) = L_1\theta'(t)\cos\theta(t)$

$y_2(t) = L_2\sin\theta(t)$

Moment of Inertia,

$I = m_1(L_1)^2 + m_2(L_2)^2 = 11.64\ [Kgm^2]$

Sum of the moments,

$M = FL_2 - F_sL_2 - m_2gL_2 - F_dL_1 + m_1gL_1 = 2.13F - 2.13F_s - 42.63 - 0.94F_d + 31.81$

Let us assume,

$x_1 = \theta$

$x'_1 = x_2 = \theta'$

$x'_2 = \theta''$

Euler Equation of Moments,

$M = I\theta''$

$$\theta'' = \frac{M}{I} = \frac{FL_2 - FsL_2 - m2gL_2 - F_dL_1 + m1gL_1}{I}$$

$$= (FL_2 - KL_2^2\sin\theta - m_2gL_2 - cL_1^2\theta'\cos\theta + m_1gL_1) / I$$

1.2 Trim Input $\bar{u}$.

$x'_1 = 0 \qquad x'_2 = 0$

$\bar{x_1} = 0 \qquad \bar{x_2} = 0$

$F = \frac{42.63-31.81}{2.13} = 5.08 = \bar{u}$

1.3 Linearization

$x'_2 = 0.17F - 0.17F_S - 3.47 - 0.08F_d + 2.59$

$= 0.17u - 3.46y_2 - 0.11y'_1 - 0.88$

$= 0.17u - 7.37\sin(x_1) - 0.10x_2\cos(x_1) - 0.88$

General Form, $\{ \delta'_x = \nabla f_x \ (\bar{x}, \bar{u})\delta_x + \nabla f_u(\bar{x}, \bar{u})\delta_u$
$\delta'y = \nabla h_x \ (\bar{x}, \bar{u})\delta_x + \nabla h_u(\bar{x}, \bar{u})\delta_u$

$$\nabla f_x(\bar{x},\bar{u}) = \begin{bmatrix} \left.\frac{\partial f_1}{\partial x_1}\right| \bar{x},\bar{u} & \left.\frac{\partial f_1}{\partial x_2}\right| \bar{x},\bar{u} \\ \left.\frac{\partial f_2}{\partial x_1}\right| \bar{x},\bar{u} & \left.\frac{\partial f_2}{\partial x_2}\right| \bar{x},\bar{u} \end{bmatrix}$$

$x'_1 = x_2 \rightarrow f_1$
$x'_2 = 0.17F - 7.37\sin(x_1) - 0.10x_2\cos(x_1) - 0.88 \rightarrow f_2$
$y = x_1 \rightarrow h$

$\frac{\partial f1}{\partial x1} = 0, \frac{\partial f1}{\partial x2} = 1,$
$\frac{\partial f2}{\partial x1} = -7.37\cos(x_1) + 0.10x_2\sin(x_1);\ \frac{\partial f1}{\partial x1}|\bar{x},\bar{u} = -7.37,$
$\frac{\partial f2}{\partial x2} = -0.10\cos(x_1);\ \frac{\partial f}{\partial x2}|\bar{x},\bar{u} = -0.10$
$\nabla fx(\bar{x}, \bar{u}) = \begin{vmatrix} 0 & 1 \\ -7.37 & -0.10 \end{vmatrix}$

$$\nabla\ f_x\ (\bar{x},\bar{u}) = \begin{bmatrix} \left.\frac{\partial f_1}{\partial u}\right| \bar{x},\bar{u} \\ \left.\frac{\partial f_2}{\partial u}\right| \bar{x},\bar{u} \end{bmatrix}$$

$\frac{\partial f1}{\partial u} = 0, \frac{\partial f2}{\partial u} = 0.17$

$\nabla fu(\bar{x}, \bar{u}) = \begin{vmatrix} 0 \\ 0.17 \end{vmatrix}$

$$\nabla h_x(\overline{x}, \overline{u}) = \left(\frac{\partial h}{\partial x_1}(\overline{x}, \overline{u}) \quad \frac{\partial h}{\partial x_2}(\overline{x}, \overline{u})\right)$$

$\frac{\partial h}{\partial x1}(\bar{x}, \bar{u}) = 1$

$\frac{\partial \mathrm{h}}{\partial \mathrm{x}}(\bar{\mathrm{x}}, \bar{\mathrm{u}}) = 0$

$\nabla hx(\bar{x}, \bar{u}) = |1 \quad 0|$

$\nabla\ hx\ (\bar{x}, \bar{u}) = \frac{\partial \mathrm{h}}{\partial \mathrm{u}}(\bar{\mathrm{x}}, \bar{\mathrm{u}}) = 0$

$\delta' x = \begin{vmatrix} 0 & 1 \\ -7.37 & -0.10 \end{vmatrix} \delta x + \begin{vmatrix} 0 \\ 0.17 \end{vmatrix} \delta u$

$\delta' \mathrm{y} = |1 \quad 0|\ \delta x$

- **TASK 2 – System Solution:**

2.1 ODE from LTI System

$$A = \begin{vmatrix} 0 & 1 \\ -7.37 & -0.10 \end{vmatrix}$$

Eigenvalues of A

$$\begin{vmatrix} 0-\lambda & 1 \\ -7.37 & -0.10-\lambda \end{vmatrix} = 0$$

$\lambda^2 + 0.10\lambda + 7.37 = 0$

ODE: $x'' + 0.10x' + 7.37x = 0.17\,u$

2.2 ODE Solution

From the characteristics Equation, $x'' + 2wn\zeta x' + w_n^2 x = bu$

$w_n^2 = 7.37$

$2wn\zeta = 0.10$

Natural Frequency, $w_n = 2.71$ [rad/s]

Damping Factor, $\zeta = 0.02$ [-] < 1 (Underdamped System)

Damped Natural Frequency, $w_d = w_n \sqrt{\mathbf{1-\zeta^\wedge 2}} = 2.71$ [rad/s]

Initial Conditions: $x(0) = 0$ [rad]

$x'(0) = 0$ [rad/s]

Homogenous Solution:

$x_H = \boldsymbol{e^{-\zeta\omega nt}}[A\cos(w_d t)+B\sin(w_{dt})]$

$x_H = e^{-0.054t}[A\cos(2.71t)+B\sin(2.71t)]$

Particular Solution:

$x_P = \frac{\mathbf{b}}{\mathbf{wn^\wedge 2}} = 0.023$

General Solution:

$x(t) = e^{-0.054t}[A\cos(2.71t)+B\sin(2.71t)] + 0.023$

$x(0) = A + 0.023 = 0$ ***A= -0.023***

$x'(t) = -0.054e^{-0.054t}[A\cos(2.71t)+B\sin(2.71t)] + e^{-0.054t}[-2.71A\sin(2.71t) + 2.71B\cos(2.71t)]$

$x'(0) = -0.054*-0.023 + 2.71B$ ***B= -4.58*10⁻⁴***

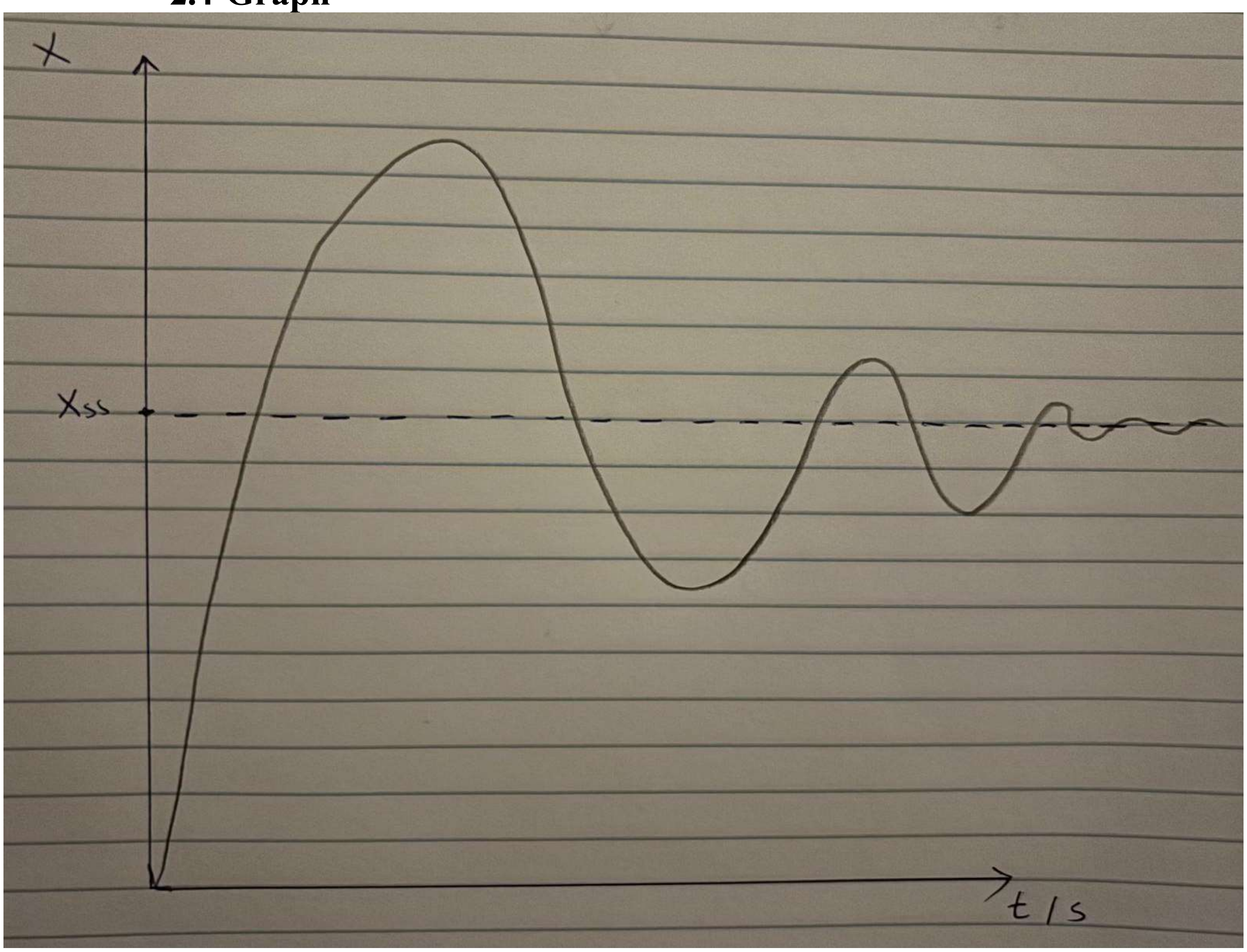
Inhomogeneous Solution:

$x(t) = e^{-0.054t}[-0.023\cos(2.71t) - 4.58*10^{-4}\sin(2.71t)] + 0.023$

2.3 Steady State Value & Transient Response

Steady State Response, $x_{ss} = x(\infty) = b/w_n^2 = 0.023$
Transient Response, $x_H = e^{-0.054t}[-0.023\cos(2.71t) - 4.58*10^{-4}\sin(2.71t)]$

2.4 Graph



- ## TASK 3 - Time Domain Performance Specifications:

3.1 Calculations

From **2.2**,

Natural Frequency, w_n = 2.71 [rad/s]

Damping Factor, ζ = 0.02 [-]

Damped Natural Frequency, w_d = 2.71 [rad/s]

Now,

Settling Time, t_s (5%) = 3 * $\tau = 55.35$ [s]

Peak Time, $t_p = \frac{\pi}{wd}$ = 1.16 [s]

Rise Time, $t_r = \frac{\pi}{2*wd}$ = 0.58 [s]

100% Rise Time, t_r (100%) = $(1/\omega d) * arctg(\omega d / \xi \omega n)$ = 0.57 [s]

Delay Time, $t_d = \frac{\pi}{4*wd}$ = 0.29 [s]

Maximum Overshoot, $M_p = e^{(-\pi\zeta / \sqrt{1-\zeta^2})} * 100$ = 93.91%

Decay Parameter, $\lambda = w_n \zeta = 0.054$

Number of Oscillation = $t_s / (2\pi/w_d)$ = 23.87

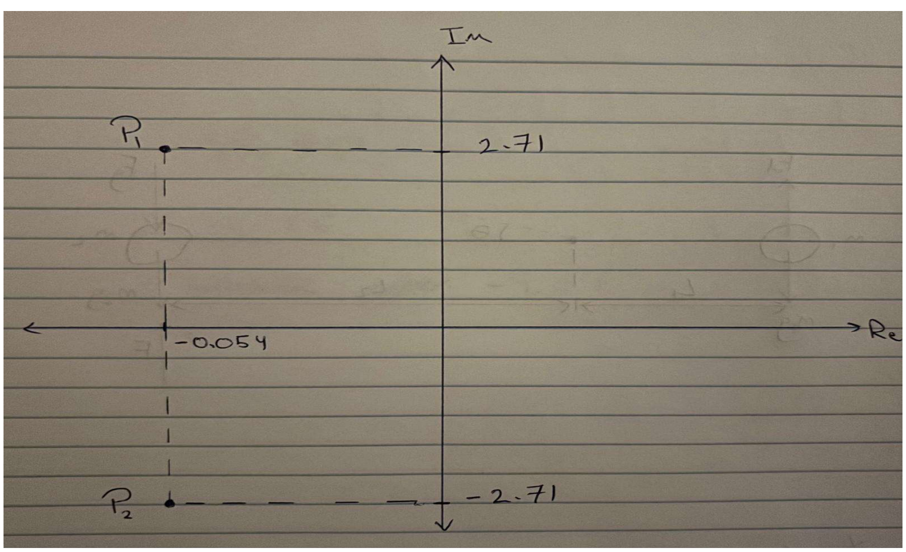
Time Constant, $\tau = 1 / w_n \zeta$ = 18.45 [s]

3.2 S Plane

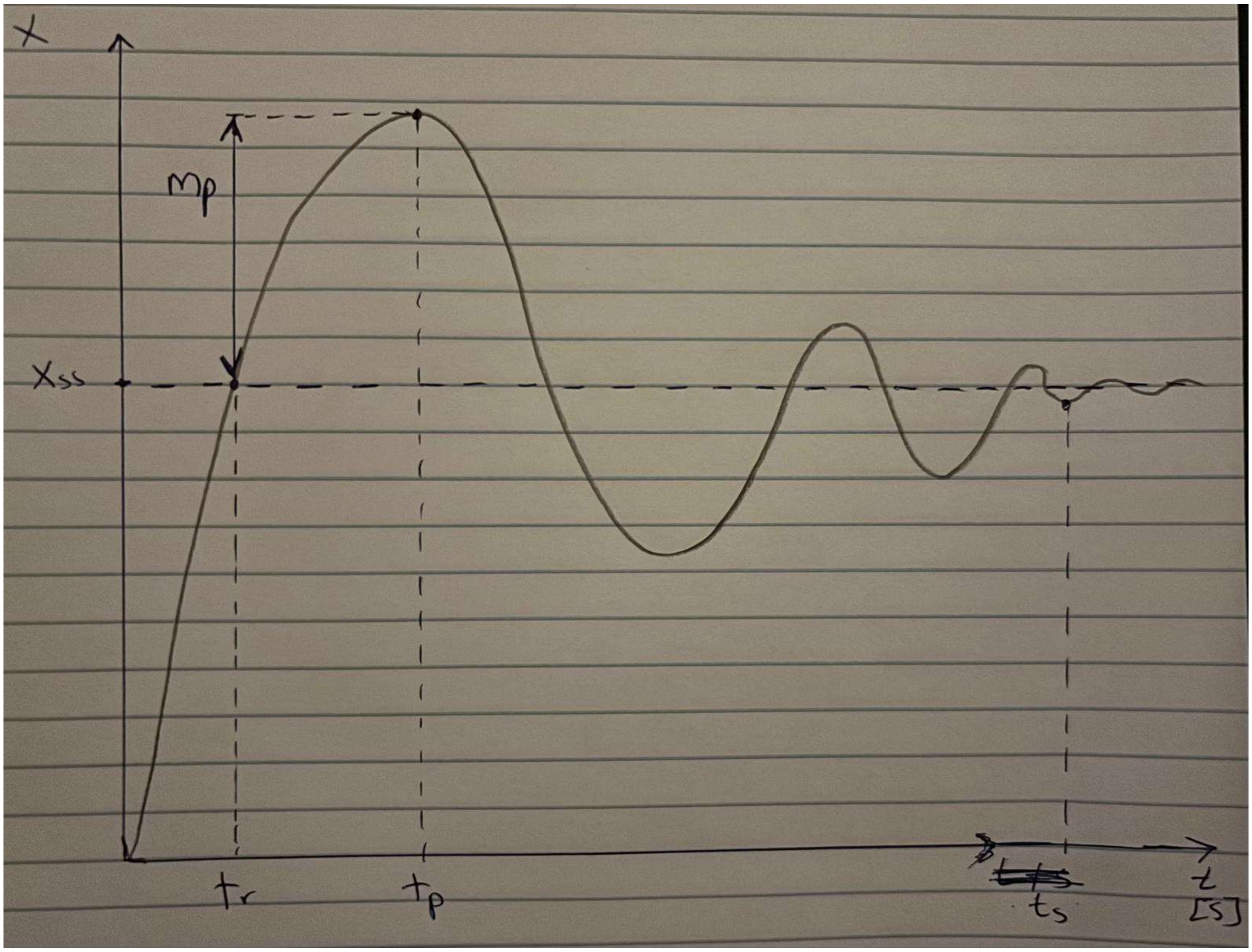
$r^2 + 2w_n \zeta r + w_n^2 = 0$

$r^2 + 0.1084r + 7.37 = 0$

$r_{1,2} = -0.054 \pm 2.71i$



3.3 Response Plot



▪ TASK 4 – Controller Design:

4.1 PID Design

$$\boldsymbol{u(t) = K_p e(t) + K_d \frac{de\ (t)}{dt} + K_i \int_0^t e(\tau) d\tau}$$

$$\boldsymbol{e(t) = r(t) - x(t)}$$

$x'' + 0.10x' + 7.37x = 0.17\ (u+d)$

$= 0.17\ [\ K_p(r-x) + K_d(r'-x') + K_i\int_0^t (r-x)d\tau + d\ ]$

$x''' + 0.10x'' + 7.37\ x' = 0.17K_p(r'-x') + 0.17K_d(r''-x'') + 0.17K_i(r-x) + 0.17d'$

$$r(t) = \begin{cases} 0 \; for \; t \; 0 \\ 1 \; for \; t \geq 0 \end{cases}$$

$$d(t) = \begin{cases} 0 \; for \; t \; 0 \\ -20 \; for \; t \geq 0 \end{cases}$$

r''=r'=d'=0

Closed Loop Model

$x''' + (0.10+0.17K_d)x'' + (7.37+0.17K_p)x' + 0.17K_i x = 0.17K_i$

For stability the conditions are as follows:

$$r = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \qquad\qquad r = -p < 0$$

$$(r^2 + 2\omega_n\zeta r + \omega_n^2)(r+p) = 0$$

$$r^3 + (p + 2\omega_n\zeta)r^2 + (2\omega_n\zeta p + \omega_n^2)r + \omega_n^2 p = 0$$

Comparing with the closed loop model

$0.10+0.17K_d = p + 2w_n\zeta$
$7.37+0.17K_p = 2w_n\zeta p + w_n^2$
$0.17K_i = w_n^2 p$

Settling Time, $t_s = 3/ w_n\zeta = 0.10$ [s]
We assume that $p=w_n$

Set 1: $\begin{cases} \zeta' = 1 \\ \omega_n' = 30 \end{cases}$ Set 2: $\begin{cases} \zeta' = \frac{\sqrt{2}}{2} \\ \omega_n' = 42.9 \end{cases}$

Initial Data				Gain Factors		
ζ	w_n	p	t_s	K_p	K_d	K_i
1	30	30	0.1	15839.00	528.82	158823.53
$\frac{\sqrt{2}}{2}$	42.9	42.9	0.1	26092.78	608.65	464432.88

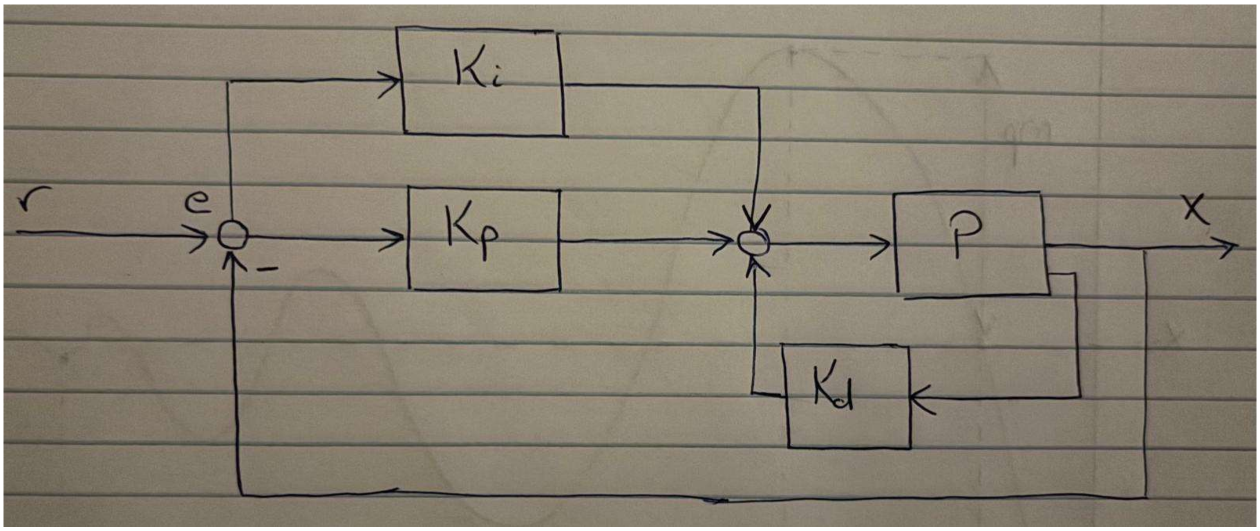
As we have a PID, there's no steady state error and the disturbance has no effect.

Since Set 1 is critically damped, there isn't any overshoot

But for Set 2,

$$M_p = exp\,(-\pi\zeta\,/\,\sqrt{1-\ \zeta^\wedge 2}\,)*100 = 4.32\%$$

4.2 Block Diagram



- **TASK 5 – Laplace Domain:**

5.1 Transfer Function

x''+ 0.10x' + 7.37x = 0.17 u

x'' − βx' − ax = γu

Transfer Function, $G(s) = \frac{\gamma}{s^2 - \beta s - a} = \frac{0.17}{s^2 + 0.10s + 7.37}$

5.2 Zero(s) & Time Constant

Comparing with, $G(s) = \frac{b}{s^2 + 2wn\zeta s + wn^\wedge 2}$

$s_{1,2}$ = = - 0.054± 2.71i

Time Constant, $\tau = 1\,/\,w_n\,\zeta = 1\,/\,(-\beta/2) = 20$ [s]

5.3 Steady State Value

$$F(t) = 150 \sin \omega t = u(t)$$

$$F(s) = \frac{150\omega}{s^2 + \omega^2}$$

$$x_{ss} = \lim_{s \to 0} s\ G(s)\, F(s)$$

$$x_{ss} = \lim_{s \to 0} \frac{0.17}{s^2 + 0.10s + 7.37} * \frac{150w}{s^2 + w^2} = 0$$

- **TASK 6 - Frequency Domain:**

6.1 Bode Plot

$$F(t) = 150 \sin \omega t = u(t)$$

$$\ddot{x} + 2\xi\omega_n\dot{x} + {\omega_n}^2 x = bu$$

$$H = \frac{150b}{\left(\frac{s^2}{{\omega_n}^2} + 2\xi\frac{s}{\omega_n} + 1\right){\omega_n}^2}$$

$$K = 150 * \frac{b}{{\omega_n}^2}$$

K = 150*0.023 = 3.45

Magnitude

$$|H(j\omega)| = \frac{K}{\sqrt{\left(1 - \frac{\omega^2}{{\omega_n}^2}\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

$$\emptyset(H(j\omega)) = \arctan\left(\frac{Im}{Re}\right)$$

Phase Angle

$$\emptyset\left(H(j\omega)\right) = -\arctan\left(\frac{Im}{Re}\right) = -\arctan\left(\frac{2\xi\frac{\omega}{\omega_n}}{1-\frac{\omega^2}{{\omega_n}^2}}\right)$$

$$|\emptyset\left(H(j\omega)\right)|_{dB} = 20\log\left(|G(j\omega)|\right)$$

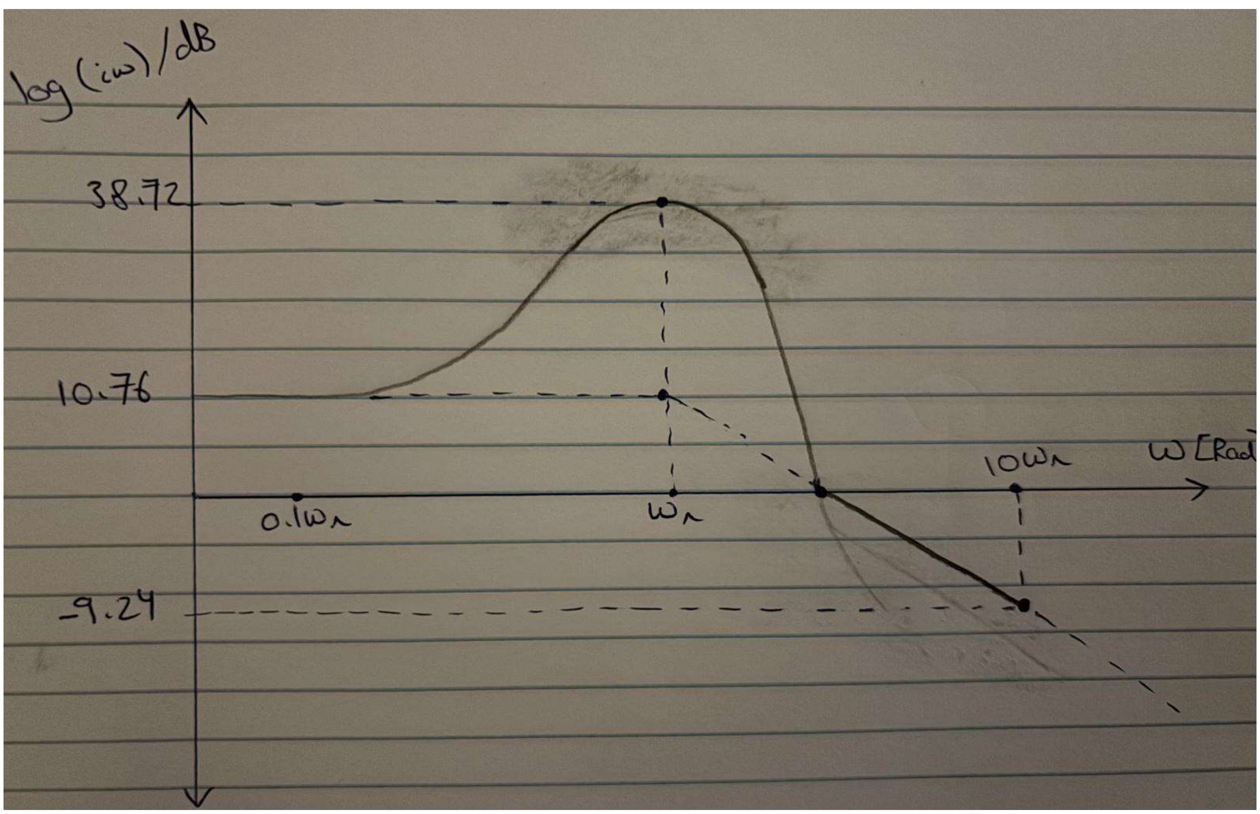
$$|\emptyset\left(H(j\omega)\right)|_{dB} = 20\log(K) - 10(log(1-\frac{\omega^2}{{\omega_n}^2})^2 + (2\xi\frac{\omega}{\omega_n})^2)$$

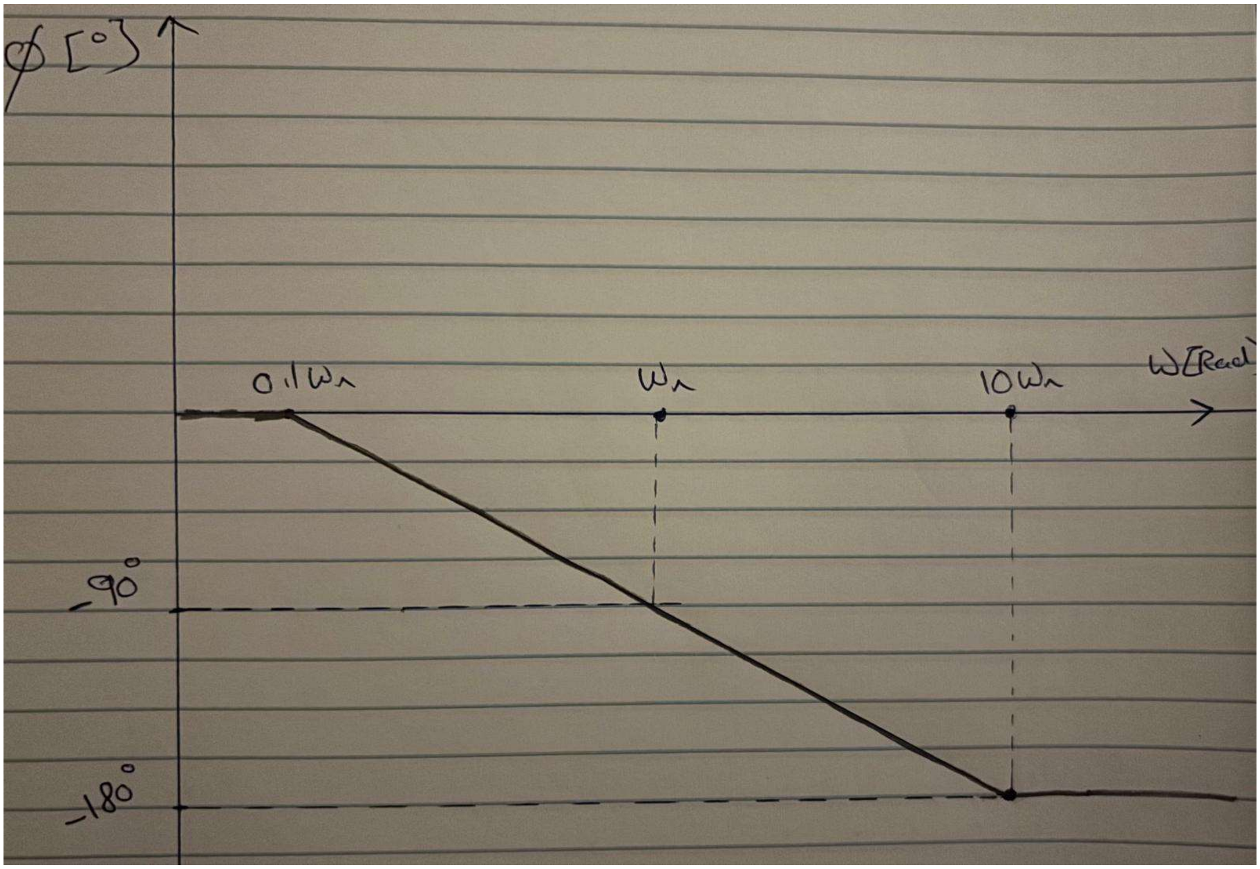
20 log (3.45) = 10.76

Frequency Peak

$$20\log\left(\frac{K}{2\xi\sqrt{1-\xi^2}}\right)$$

= 38.72





6.2 Gain Margin

$$G(r) = \frac{150\gamma}{r^2 - \beta r - \alpha} \Rightarrow G(j\omega) = \frac{150\gamma}{-\omega^2 - j\beta\omega - \alpha}$$

$$= \frac{\mathbf{150*0.17}}{\mathbf{-2.04^2 - i(-0.10)(2.04) - (-7.37)}}$$

$$= \frac{\mathbf{25.5}}{\mathbf{3.2084 + i0.204}}$$

$$= 7.89 - i\,0.507$$

6.3 Phase Margin

$$20\log G(j\omega)\mathrm{I}\omega^{*} = 0db$$

$$G(j\omega^{*}) = 1 \Rightarrow \frac{150\gamma}{\sqrt{(\alpha+\omega^{2})^{2}+\beta^{2}\omega^{2}}} = 1$$

$$\omega^{4} + (2\alpha+\beta^{2})\omega^{2} + (\alpha^{2} - 150^{2}\gamma^{2}) = 0$$

w* = 4.26 [rad/s]

$$ø\big(H(j\omega^{*})\big) = -atg\left(\frac{2\,\zeta\frac{\omega^{*}}{\omega_{n}}}{1-\left(\frac{\omega^{*}}{\omega_{n}}\right)^{2}}\right)$$

= 0. 043 [rad]

= 4.93˚