Chapter 1:

Partial Derivatives

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Recall: Differentiation

Basic Properties

1. $\frac{d}{dx}(c) = 0$
2. $\frac{d}{dx}(x) =1$
3. $\frac{d}{dx}(cf) = cf'$
4. $\frac{d}{dx}(x^n) = nx^{n-1}$
5. $\frac{d}{dx}(f \pm g) = f' \pm g'$
6. $\frac{d}{dx}(fg) = fg' + gf'$
7. $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{gf'-fg'}{g^2}$

Trigonometric Functions

1. $\frac{d}{dx}(\sin x) = \cos x$
2. $\frac{d}{dx}(\cos x) = -\sin x$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$
4. $\frac{d}{dx}(\sec x) = \sec x \tan x$
5. $\frac{d}{dx}(\csc x) = -\csc x \cot x$
6. $\frac{d}{dx}(\cot x) = -\csc^2 x$

Common Derivatives

1. $\frac{d}{dx}(a^x) = a^x \ln a$
2. $\frac{d}{dx}(e^x) = e^x$
3. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
4. $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

Basic Chain Rule

1. $\frac{d}{dx}([f(x)]^n) = n[f(x)]^{n-1}f'$
2. $\frac{d}{dx}(e^{f(x)}) = f'e^{f(x)}$
3. $\frac{d}{dx}(\ln f(x)) = \frac{1}{f(x)}f'$
4. $\frac{d}{dx}(\sin(f(x))) = \cos(f(x)) \cdot f'$
5. $\frac{d}{dx}\left(\cos(f(x))\right) = -\sin(f(x)) \cdot f'$
6. $\frac{d}{dx}(\tan(f(x)) = \sec^2(f(x)) \cdot f'$

Partial Derivatives

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First Partial

For $z = f(x, y, z)$,

$$f_x = \frac{\partial f}{\partial x},\ f_y = \frac{\partial f}{\partial y},\ f_z = \frac{\partial f}{\partial z}$$

Second Partial

For $z = f(x, y)$,

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Clairaut's Theorem

$$f_{xy}(a, b) = f_{yx}(a, b)$$

The same concept also applies for multivariable functions

$$f_{xyz} = f_{xzy} = f_{yxz} = f_{zyx} = \cdots \text{ etc}$$

Implicit Differentiation

If y can be expressed completely in terms of x, then y is called an **explicit function** of x.
If it is **not possible,** then y is called an **implicit function** of x.

Case 1 Given an implicit function $y = f(x)$ in an equation of the form $F(x, y) = 0$

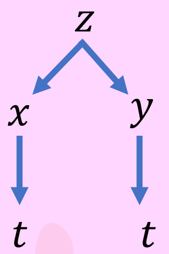
$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Case 2 For $z = f(x, y)$ in an equation of the form $F(x, y, z) = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Chain Rule

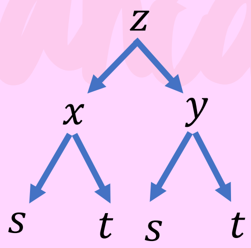
Case 1



If $z = f(x, y)$, $x = g(t)$ and $y = h(t)$,
Then z is a composite function of t

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Case 2

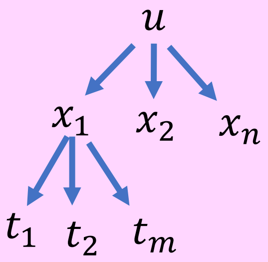


If $z = f(x, y)$, $x = g(s, t)$ and $y = h(s, t)$,
Then z is a composite function of s and t

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Case 3



If $u = f(x_1, x_2, \dots, x_n)$,
and $x_i = h(t_1, t_2, \dots, t_m)$,
Then z is a composite function of $t_1, t_2, \dots, t_m$

$$\frac{du}{dt_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

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Partial Derivatives

Critical Points

- Critical point of f is when
  $f_x(x,y) = 0$ and $f_y(x,y) = 0$
- 3 types –
  local min, local max or saddle point

Second Derivative Test (Max Min)

$$D(x,y) = f_{xx}f_{yy} - \left(f_{xy}\right)^2$$

$f_{xx}(a,b)$	$D(a,b)$	Point type
$f_{xx} > 0$	$D > 0$	Local min
$f_{xx} < 0$	$D > 0$	Local max
	$D < 0$	Saddle point

Vector

- Has magnitude (length) and direction
- Can be written in the form:
  $\bar{a} = \mathbf{a} = \vec{a} = a_1\bar{\imath} + a_2\bar{\jmath} = \langle a_1, a_2 \rangle$

Unit Vector

- Vector with length 1
- To convert any vector $\bar{a}$ into $\bar{u}$, divide with its length

$$\bar{u} = \frac{\bar{a}}{|\bar{a}|} = \frac{a_1\bar{\imath} + a_2\bar{\jmath} + a_3\bar{k}}{\sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}}$$

- If given an angle θ,
  $\bar{u} = \cos\theta\,\bar{\imath} + \sin\theta\,\bar{\jmath}$
- If given 2 points ; P and Q
  $\overline{PQ} = \overline{PO} + \overline{OQ}$

$$\bar{u} = \frac{\overline{PQ}}{|\overline{PQ}|}$$

- If no vector given, follow direction of ∇f

$$\bar{u} = \frac{\nabla f}{|\nabla f|}$$

Directional Derivatives

Gradient Vector, ∇f

Direction of maximum rate of change

$$\text{Grad } f = \nabla f = \frac{\partial f}{\partial x}\bar{\imath} + \frac{\partial f}{\partial y}\bar{\jmath} + \frac{\partial f}{\partial z}\bar{k}$$

To get $\nabla f(a,b,c)$
substitute a, b, c into x, y, z

Maximum Rate of Change, $|\nabla f|$

Maximum rate of change at (a,b,c)

$$|\nabla f(a,b,c)| = \sqrt{f_x(a,b,c)^2 + f_y(a,b,c)^2 + f(a,b,c)_z{}^2}$$

Directional Derivative, $D_u f$

- The rate of change at (a,b,c)
  $D_u f(a,b,c) = \nabla f(a,b,c) \cdot \bar{u}$