INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR

Control Systems lab 2

Submitted by

(Group 4)
Sai Joshitha Annareddy 21EE62R10
Prabhat kumar Satya 21EE62R12
Vidhu shree 21EE62R06

Introduction

The Rotary Inverted Pendulum (RIP) is a classical problem in the control systems area that provides a challenging platform for control study purposes. It is a highly non-linear and open loop unstable system. Moreover, the dynamic model of the RIP is very useful for the study of attitude control of space rockets, automatic aircraft landing systems or for the problem of stabilizing androids.

It has two equilibrium points: normal downward position (180 degrees) which is a stable equilibrium point and vertically upward position (0 degrees) which is an unstable equilibrium point. Basically, there are two control tasks for an inverted pendulum system: swing-up and balance control. For swing-up control, the pendulum is swing from its downward stable balance point to the upward unstable balance point. For balance control, the pendulum is in its upright vertical position and the driven arm automatically varies its angle while the controller tries to keep the system stable.

Rotary Inverted Pendulum Model

Working Principle

The rotary inverted pendulum is driven by a rotary servo motor system. The servo motor drives an independent output gear whose angular position is measured by an encoder. The rotary pendulum arm is mounted on the output gear. The pendulum is attached to a hinge instrumented with another encoder at the end of the pendulum arm. This second encoder measures the angular position of the pendulum. The system is interfaced by means of a data acquisition card and driven by Matlab / Simulink based real time software.

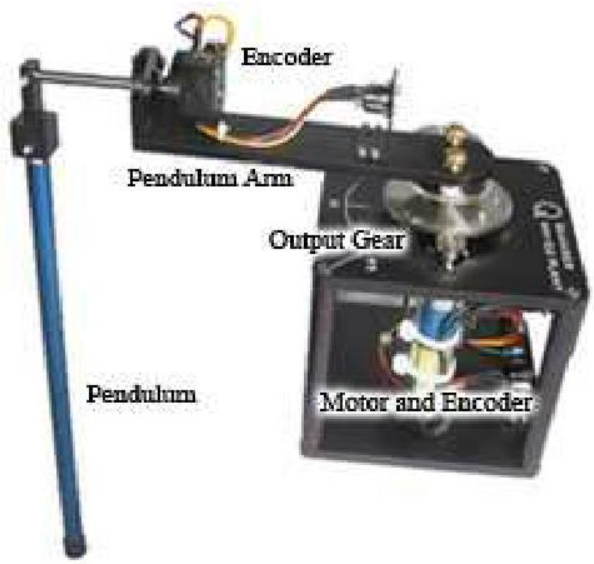


Figure 1: Rotary Inverted Pendulum

Mathematical Modelling

The following assumptions are important in modeling of the system :-

1) The system starts in a state of equilibrium meaning that the initial conditions are therefore assumed to be zero.

2) The pendulum does not move more than a few degrees away from the vertical to satisfy a linear model.

3) A small disturbance can be applied on the pendulum.

The variables used to define the model of the rotary inverted pendulum are as follows:

Symbol	Description
K_t	Motor Torque Constant
K_m	Back EMF Constant
R_m	Armature Resistance
K_g	SRV02 system gear ratio (motor->load)
η_m	Motor efficiency
η_g	Gearbox efficiency
B_{eq}	Equivalent viscous damping coefficient
J_{eq}	Equivalent moment of inertia at the load

Table 1: Description of parameters

The system model is usually determined by energy-based methods, in which differential equations from the mechanical model are obtained by using Euler Lagrange formulation. Two equations of motion obtained are :-

$\ddot{\theta} = \frac{1}{a}[(b\cos\alpha)\ddot{\alpha}-(b\sin\alpha)\dot{\alpha}^2-e\dot{\theta}+fV_m]$

$\ddot{\alpha} = \frac{1}{c}[d\sin\alpha+(b\cos\alpha)\ddot{\theta}]$

where $a = J_{eq} + mr^2$

$b = mLr$

$c = \frac{4}{3}mr^2$

$d = mgL$

$e = B_{eq}+ \frac{K_G^2 K_M K_t \eta_G \eta_M}{R_M}$

$f = \frac{K_G K_t \eta_G \eta_M}{R_M}$

For linearizing the system assume small α so that sinα=α and cosα=1

$\ddot{\theta} = \frac{1}{a}[b\ddot{\alpha}-e\dot{\theta}+fV_m]$

$\ddot{\alpha} = \frac{1}{c}[d\alpha+b\ddot{\theta}]$

To put the system in a state-space form, the state vector x for the rotary inverted pendulum system is defined such as

$x^T = [x_1\ x_2\ x_3\ x_4] = [\ \theta\ \alpha\ \dot{\theta}\ \ \dot{\alpha}\ ]$

where

Θ is the servo load gear angle (radians)

Α is the pendulum arm deflections (radians)

Input variable (u) is V_m (motor input voltage)

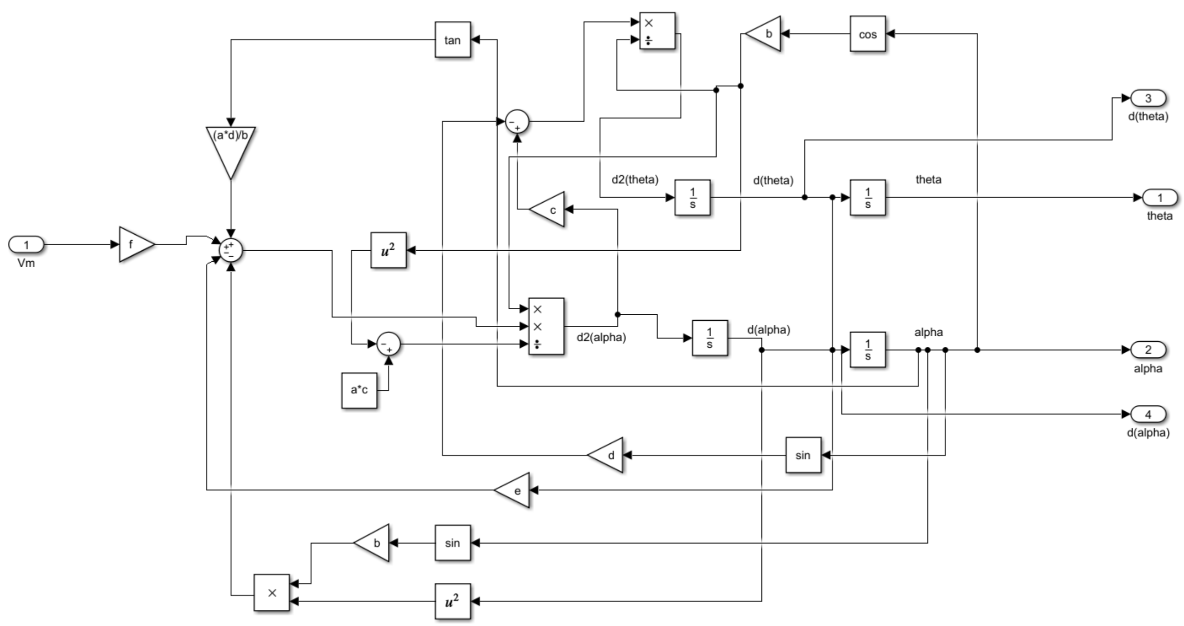
Finally the following state space representation of the complete system is obtained :-

$$\begin{bmatrix}\dot{\theta}\\ \dot{\alpha}\\ \ddot{\theta}\\ \ddot{\alpha}\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & \frac{bd}{E} & \frac{-cG}{E} & 0\\ 0 & \frac{qd}{E} & \frac{-bG}{E} & 0\end{bmatrix}\begin{bmatrix}\theta\\ \alpha\\ \dot{\theta}\\ \dot{\alpha}\end{bmatrix} + \begin{bmatrix}0\\ 0\\ c\,\frac{\eta_m \eta_g K_t K_g}{R_m E}\\ b\,\frac{\eta_m \eta_g K_t K_g}{R_m E}\end{bmatrix} V_m$$

Here, $a = J_{eq} + mr^2$, $b = mLr$, $c = 4/3mL^2$, $d = mgL$,

$$E = ac - b^2,\ \ G = \frac{\eta_m \eta_g K_t K_m K_g^2 - B_{eq} R_m)}{R_m}.$$

Simulink model of Non linear Rotary inverted pedulum



Linear Controllers : -

1. 2DOF PID Controller

The main target is to maintain pendulum angle alpha, α, as zero so that the inverted pendulum remains stable. Here the output theta θ has the responsibility to do this job. So it is necessary to design two PID compensators where one will maintain the speed and position of theta while the other controller will function based on the feedback of alpha.

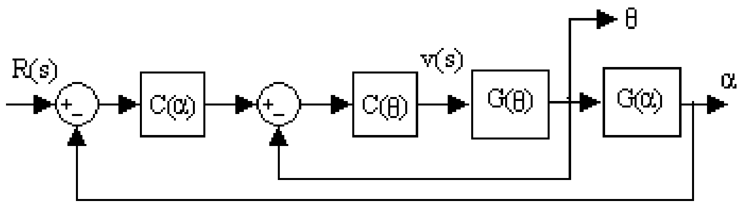


Figure 2: 2DOF PID Controller block diagram

Based on the state space equation following two transfer functions are derived :-

$$G(\theta) = \frac{\theta(s)}{V_m(s)} = \frac{fcs^2 - fd}{(ac-b^2)s^4 + ecs^3 - ads^2 - eds} = \frac{40.4091s^2 - 1775.83}{s^4 + 22.5659s^3 - 94.7608s^2 - 991.2137s}$$

$$\frac{\alpha(s)}{V_m(s)} = \frac{bfs^2}{(ac-b^2)s^4 + ecs^3 - ads^2 - eds} = \frac{38.9013s^2}{s^4 + 22.5659s^3 - 94.7608s^2 - 991.2137s}$$

From above two equations we get,

$$G(\alpha) = \frac{\alpha(s)}{\theta(s)} = \frac{bfs^2}{fcs^2 - fd} = \frac{38.9013s^2}{40.4091s^2 - 1775.83}$$

Let $C(\theta) = K_{P1} + \frac{K_{I1}}{s} + \frac{100K_{D1}s}{s+100} = \frac{(K_{P1}+100K_{D1})s^2 + (100K_{P1}+K_{I1})s + 100K_{I1}}{s^2 + 100s}$

$C(\alpha) = K_{P2} + \frac{K_{I2}}{s} + \frac{100K_{D2}s}{s+100} = \frac{(K_{P2}+100K_{D2})s^2 + (100K_{P2}+K_{I2})s + 100K_{I2}}{s^2 + 100s}$

Considering following system specifications :-

1.) Settling time < 0.5 sec

2.) Max. Peak overshoot < 10%

Let damping ratio=0.6 and natural frequency = 3 rad/s

From these we get poles at -1.8$\pm$ j2.4 and assuming other four poles at -30 we obtain gains as :-

K_{P1} = -20.79841633, K_{I1} = -39.65437772, K_{D1} = -8

K_{P2} = 56.9, K_{I2} = 2.78, K_{D2} = 8.878372

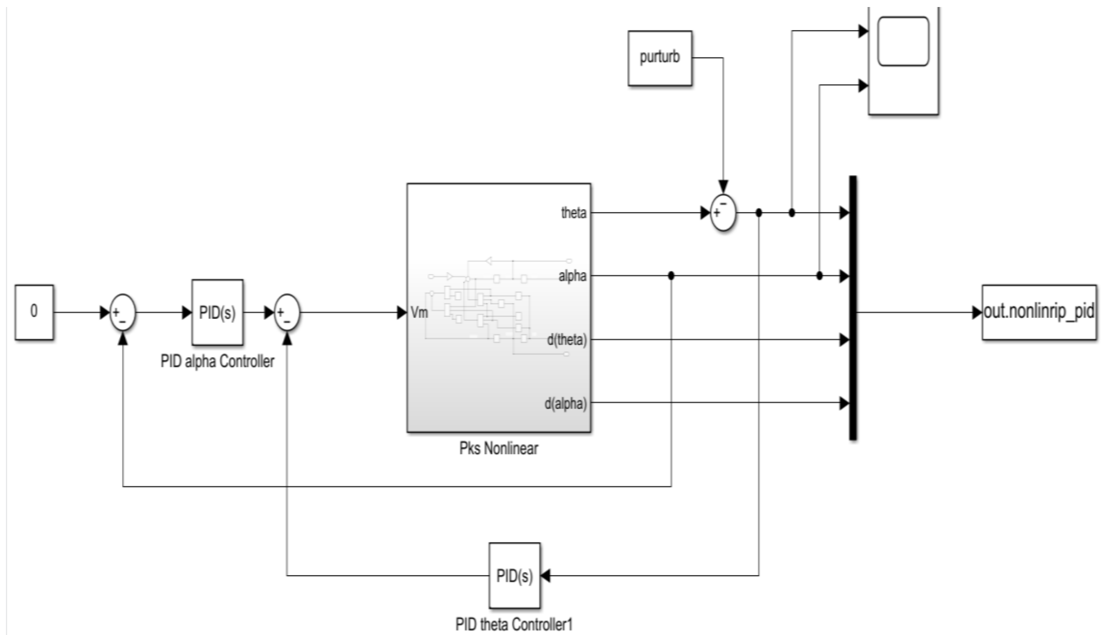
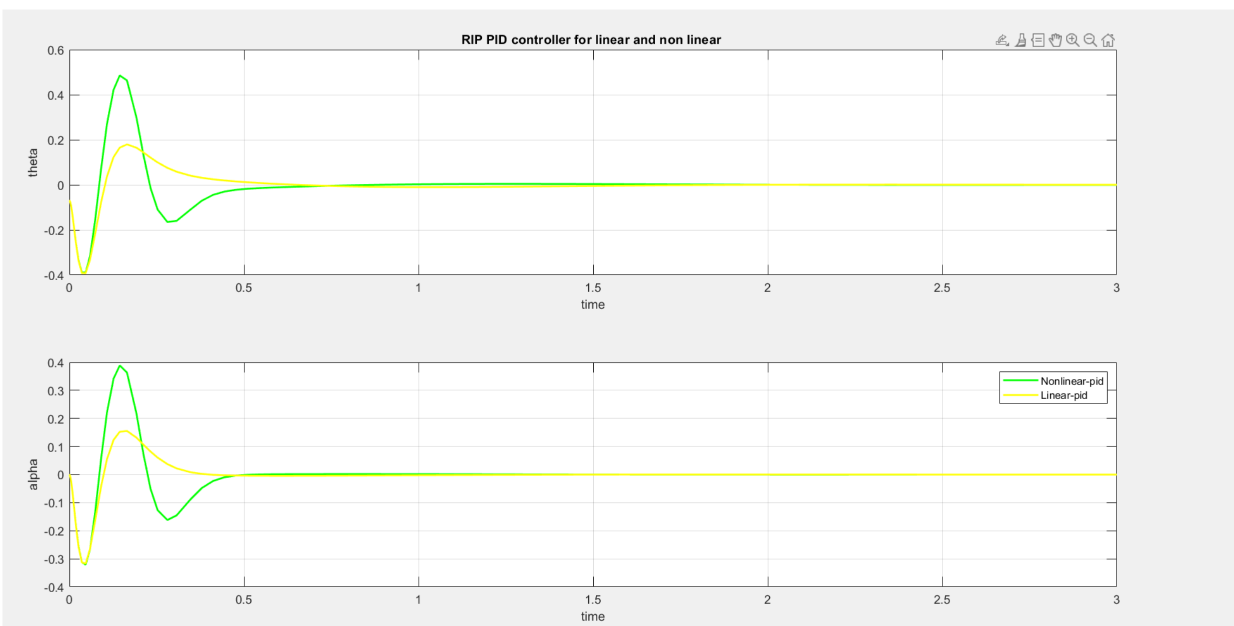


Figure 3: Simulink Model for 2DOF PID Controllers

Output graph :-



2. FSF Controller

To design a FSF controller Ackerman's formula is used which is an easy and effective method in modern control theory to design a controller via pole placement technique. Ackerman's formula is represented as,

$K = [0 \ldots\ldots.0\ 1]M_C^{-1}\varphi_d(A)$

where M_C indicates the controllability matrix and $\varphi_d(A)$ is the desired characteristic of the closed loop poles which can be evaluated as s=A.

We obtain gain K=[-5.6338 17.028 2.0795 2.1199];

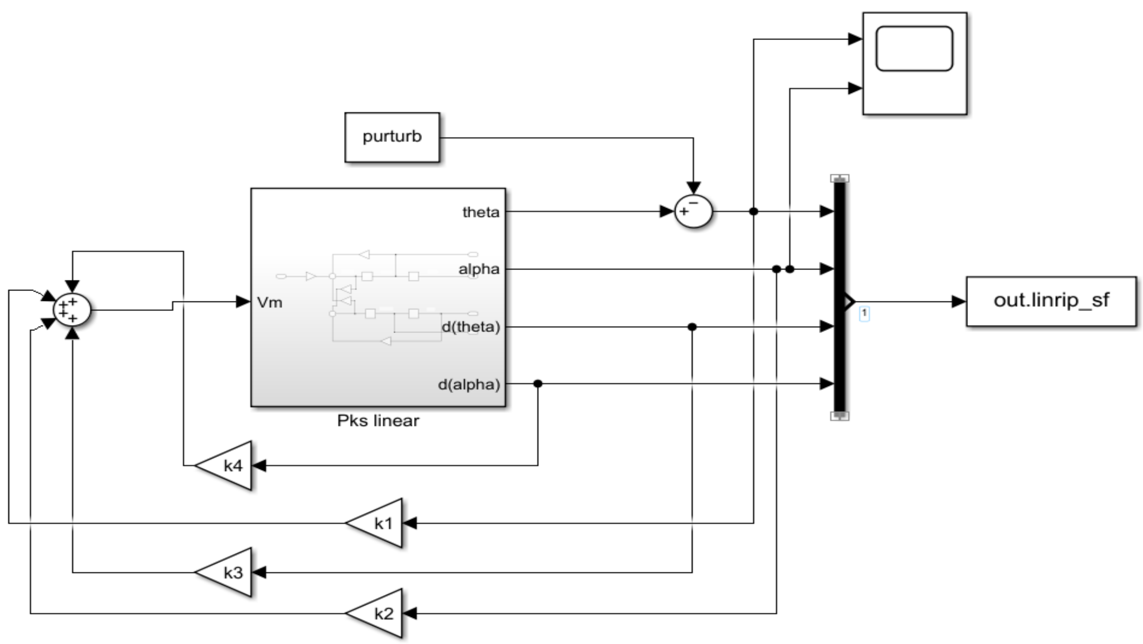
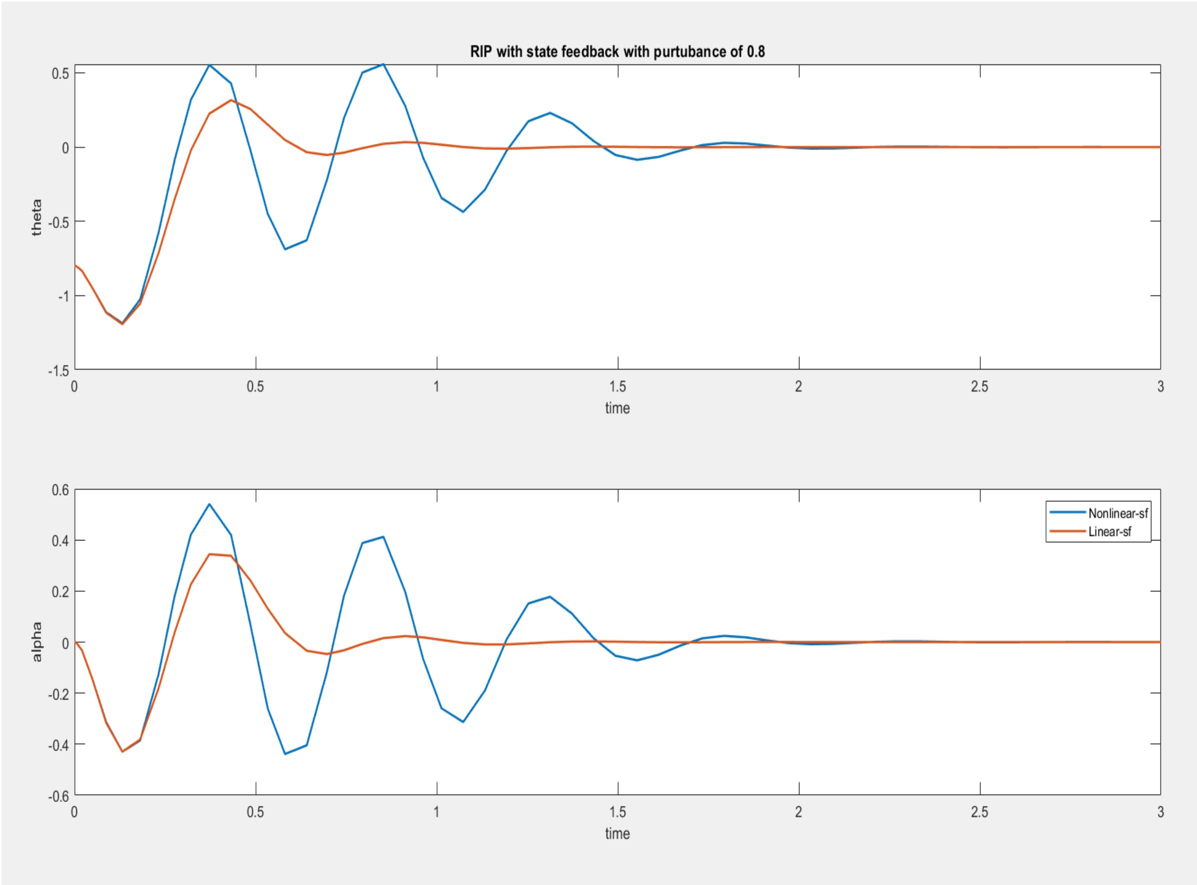


Figure 4: Simulink Model for FSF Controller

Output graph :-



3. LQR Controller

In a LQR design process, the gain matrix K, for a linear state feedback control law u = -Kx, is found by minimizing a quadratic cost function of the form as,

$$J = \int_0^{\infty} x(t)^T \mathrm{Qx(t)} + u(t)^T \mathrm{Ru(t)}\ \mathrm{dt}$$

Here Q and R are weighting parameters that penalize certain states or control inputs. In the design the weighting parameters of the optimal state feedback controller are chosen as,

$$Q = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } R=1$$

For this we obtain gain K = $[-2.4495\ \ 27.5843\ \ -2.5505\ \ 3.92]$

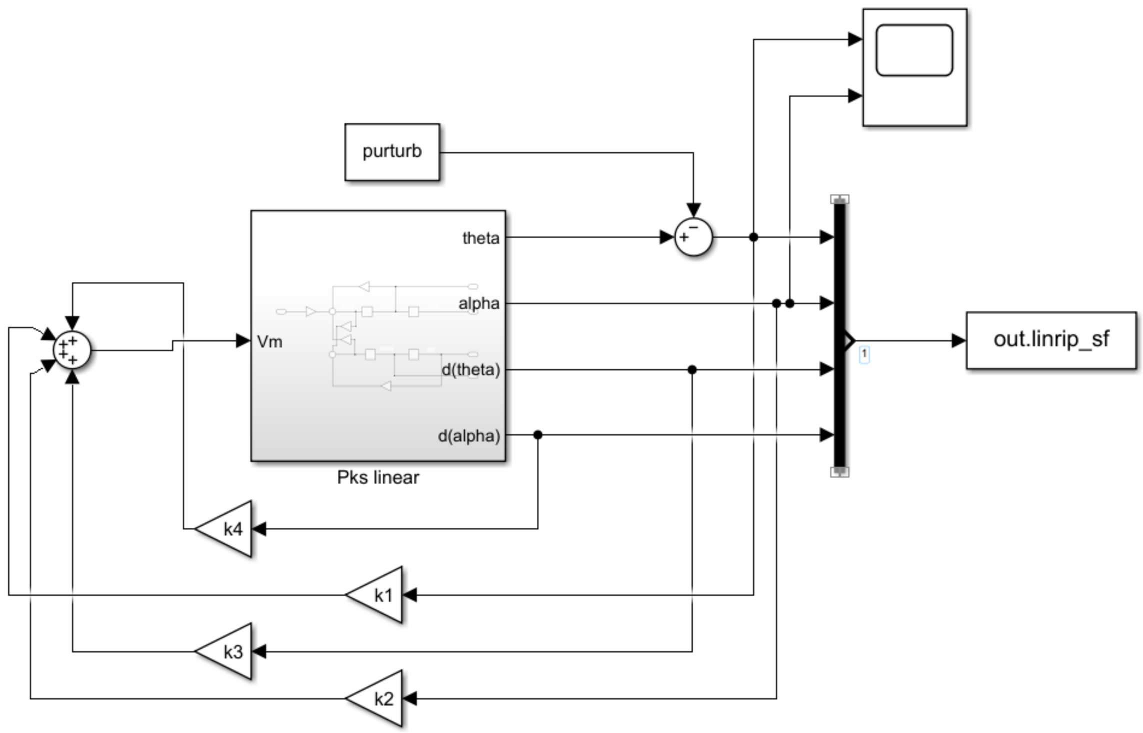
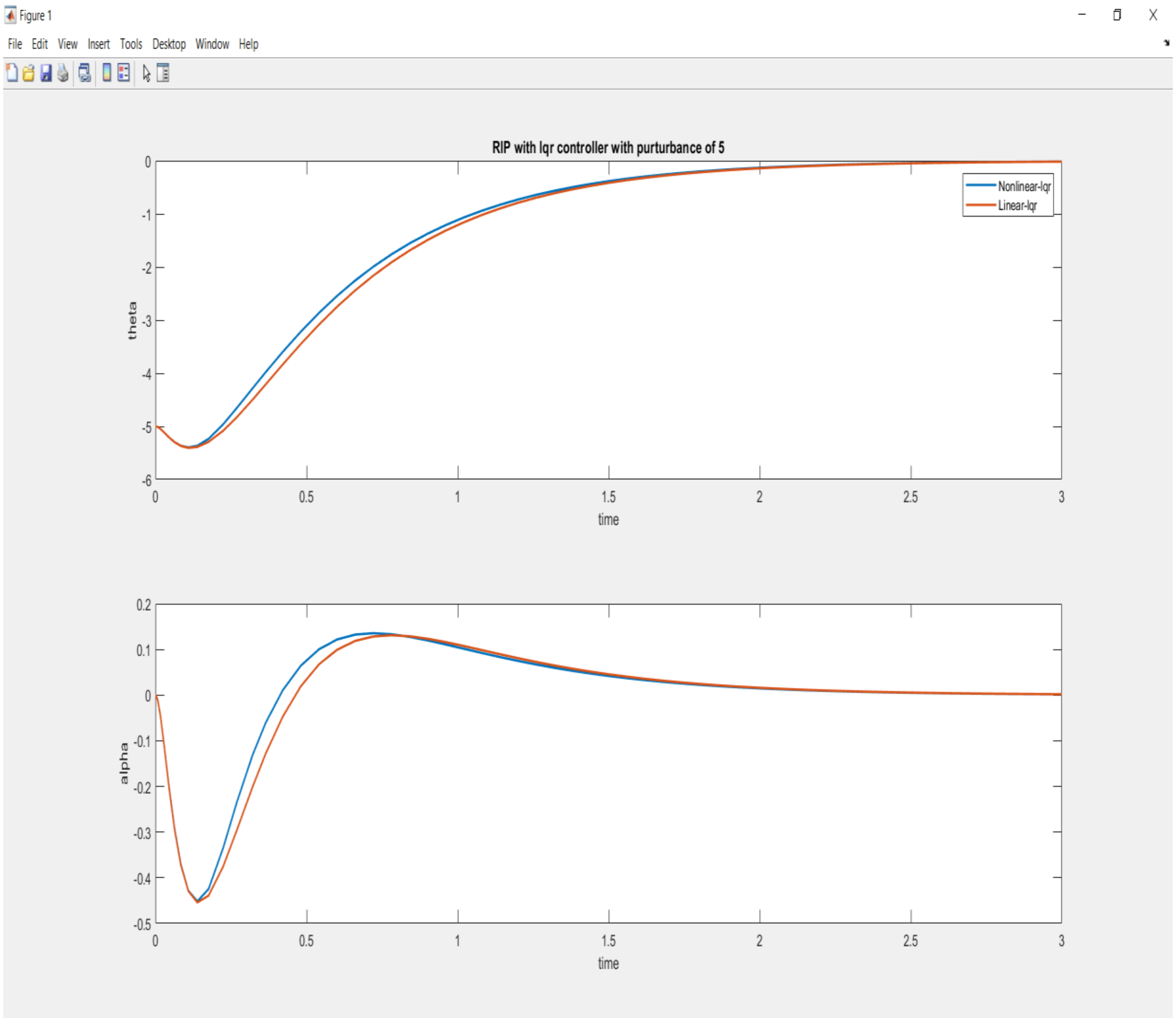


Figure 5: Simulink Model for LQR Controller

Output graph :-



Non linear Controllers:-

1. Input State feedback linearization

System equations are as below

$$a\ddot{\theta} - b\cos(\alpha)\ddot{\alpha} + b\sin(\alpha)(\alpha)^2 + G\dot{\theta} = \frac{\eta_m\eta_g K_t K_g}{R_m} V_m$$

$$c\ddot{\alpha} - b\cos(\alpha)\ddot{\theta} - d\sin(\alpha) = 0$$

Let

$$x_1 = \alpha$$

And

$$x_3 = \theta$$

So the equations are as below…

$$\begin{aligned}
\dot{x_1} &= x_2 \\
\dot{x_2} &= F_1(X) + G_1(X)u \\
\dot{x_3} &= x_4 \\
\dot{x_4} &= F_2(X) + G_2(X)u
\end{aligned}$$

$$\begin{aligned}
F_1(X) &= \frac{-b^2/2\sin(2x_1)x_2^2 - Gbx_4\cos(x_1) + ad\sin(x_1)}{ac - b^2\cos^2 x_1} \\
G_1(X) &= \frac{\eta_m\eta_g k_t k_g b}{R_m(ac - b^2\cos^2 x_1)}\cos(x_1) \\
F_2(X) &= \frac{cF_1(X) - d\sin(x_1)}{b\cos(x_1)} \\
G_2(X) &= \frac{\eta_m\eta_g k_t k_g c}{R_m(ac - b^2\cos^2 x_1)}
\end{aligned}$$

Now consider the output equation as…

$$y = h(X) = x_1$$

$$L_f h(X) = \dot{y} = x_2$$

Then:

$$L_f^2 h(X) = \ddot{y} = F_1(x)$$

Final control law is as below

$$u = -\frac{1}{L_g L_f h(X)}\left(L_f^2 h(X) - V(x_1, x_2)\right) = -\frac{1}{G_1(X)}\left(F_1(X) - V(x_1, x_2)\right)$$

Now consider the zero dynamics of the system…

$X^* = [0 \quad 0 \quad x_3 \quad x_4]$

Then:

$$F_1(X^*) = \frac{-Gb}{ac - b^2} x_4$$

$$F_2(X^*) = \frac{Gc}{ac - b^2} x_4$$

$$G_1(X^*) = \frac{\eta_m \eta_g k_t k_g b}{R_m(ac - b^2)}$$

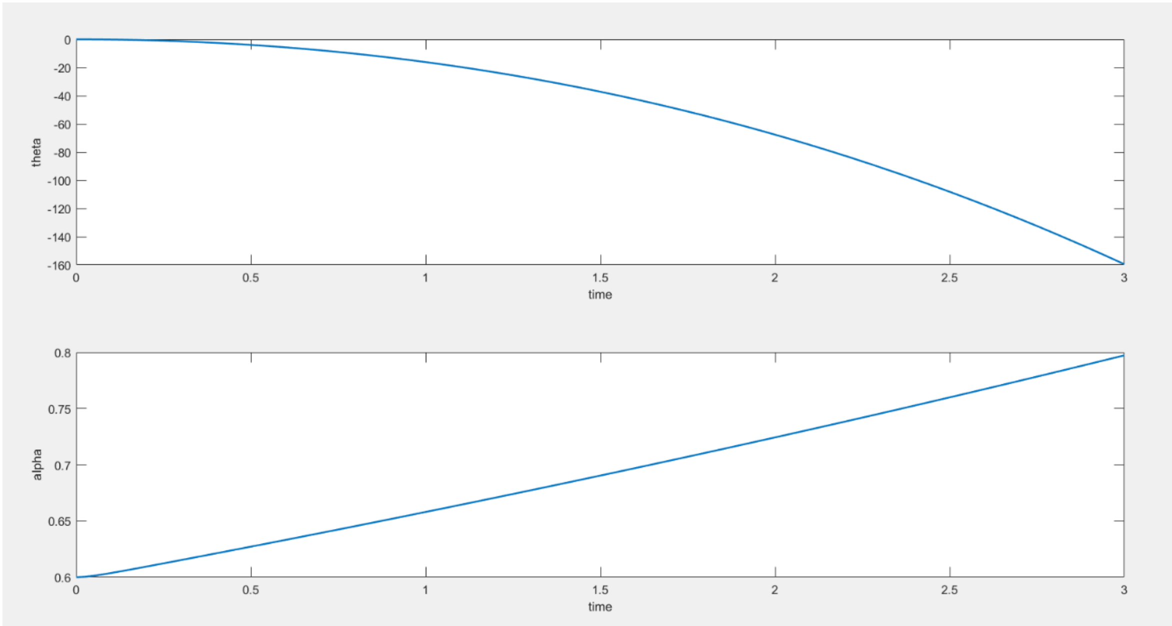
$$G_2(X^*) = \frac{\eta_m \eta_g k_t k_g c}{R_m(ac - b^2)}$$

Zero dynamics equation is…

$$\frac{d^2 x_3}{dt^2} = F_2(X^*) + G_2(X^*)u$$

The obtained zero dynamic of the system are unstable.

Result of State feedback linearization



We can observe the non-minimum phase so, we can’t implement this feedback linearization and go for sliding mode control

2. Sliding mode control

$$\dot{x}_1 + \lambda_1 x_1 = 0$$
$$\dot{x}_3 + \lambda_3 x_3 = 0$$

The sliding surfaces are chosen as below…

$$s_1 = x_2 + \lambda_1 x_1$$
$$s_2 = x_4 + \lambda_1 x_3$$

The Lyapunov function for the corresponding s1 and s2 sliding surfaces is given as

$$V = |s_1| + \lambda_2 |s_2|$$

$$\dot{V} = -\alpha \mathrm{sat}\left(\frac{V}{\Phi}\right)$$

Where…

$$\text{sat}\left(\frac{V}{\Phi}\right) = \begin{cases} \frac{V}{\Phi} & \text{if } \Phi < |V| \\ \text{sgn}\,(V) & \text{otherwise} \end{cases}$$

The control law is given as…

$$\begin{aligned} u = {} & [-\alpha \text{sat}\,(V/\Phi) - (\lambda_1 x_2 + F_1(X))\text{sgn}\,(s_1) \\ & -\lambda_2(\lambda_3 x_4 + F_2(X))\text{sgn}\,(s_2)] \\ & [G_1(X)\text{sgn}\,(s_1) + \lambda_2 G_2(X)\text{sgn}\,(s_2)]^{-1} \end{aligned}$$

The parameters apart from the one which present in system equations are given in below table.

Parameter	Value	Parameter	Value
α	0.3	λ_2	0.7
λ_1	0.2	λ_3	3
Φ	0.5		

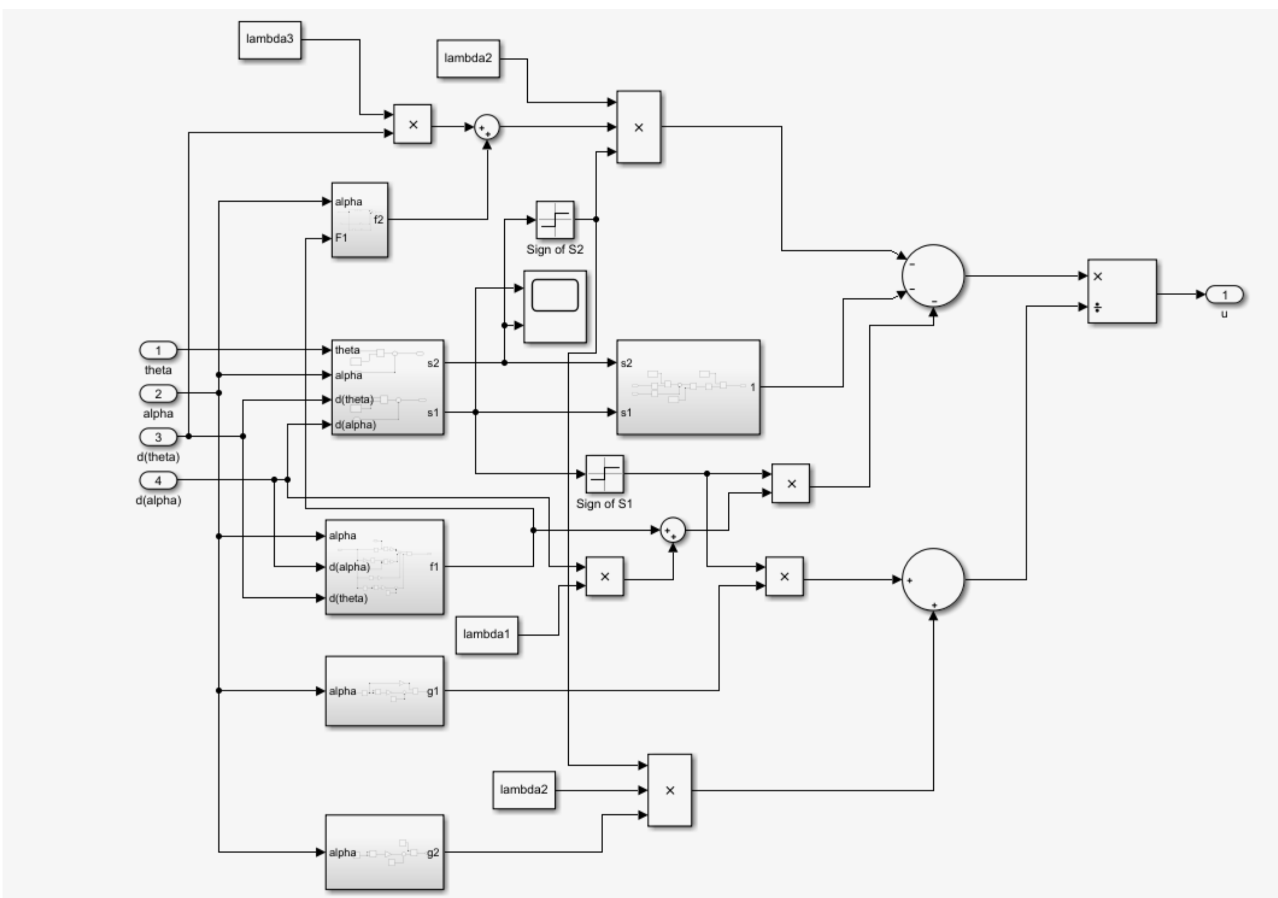
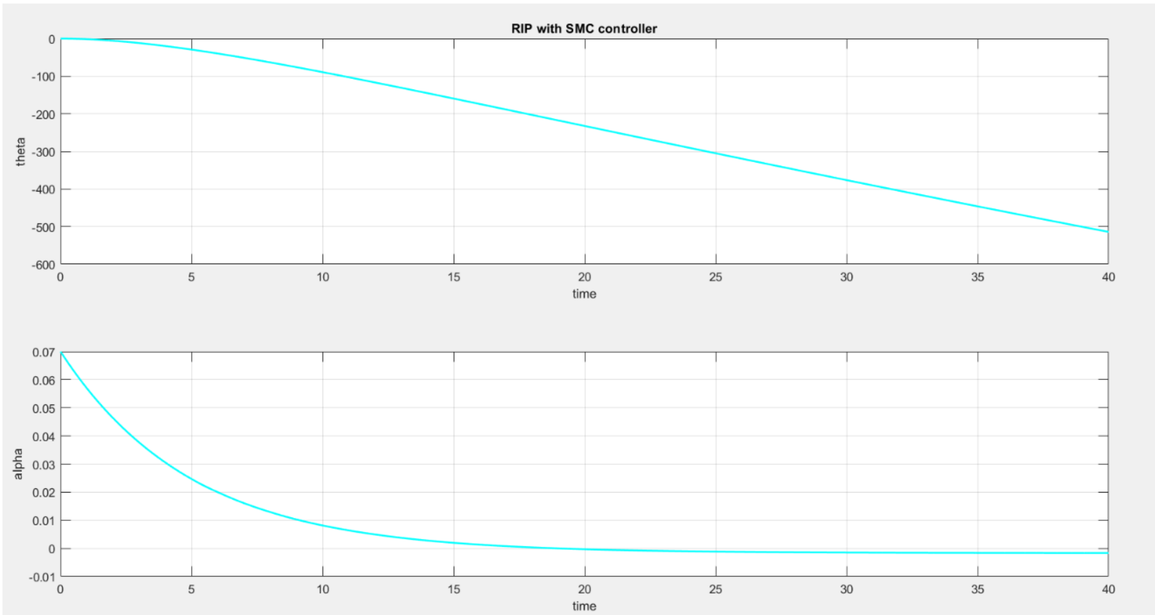


Figure 6 : Simulink model block diagram of sliding mode control

Output Graphs:



Comparison between graphs using different type of controllers for rotary inverted pendulum linear and nonlinear controllers.

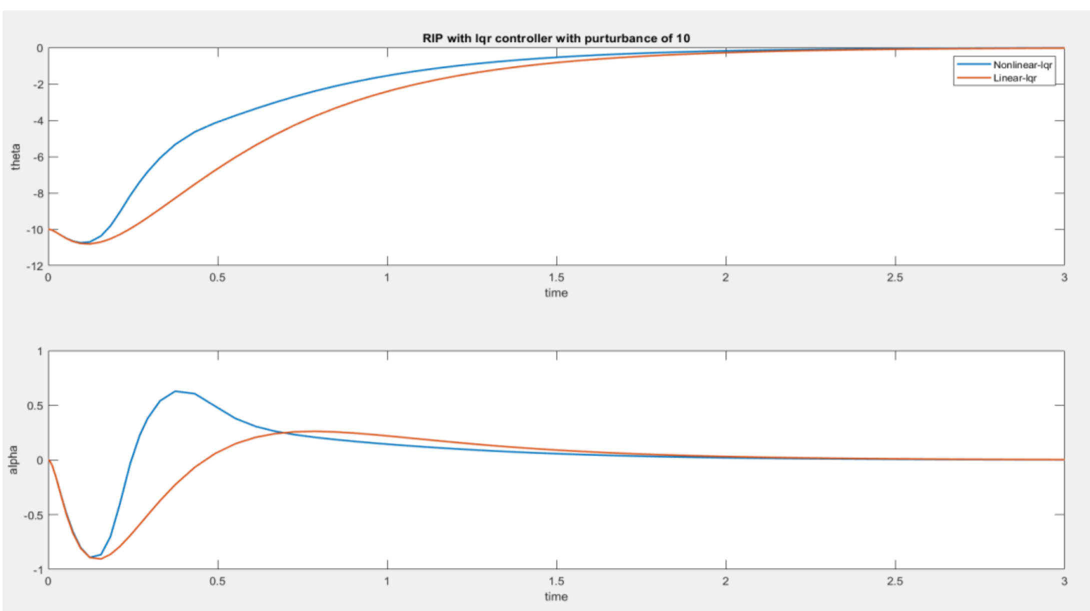


Figure 7: Closed loop RIP with LQR state feedback

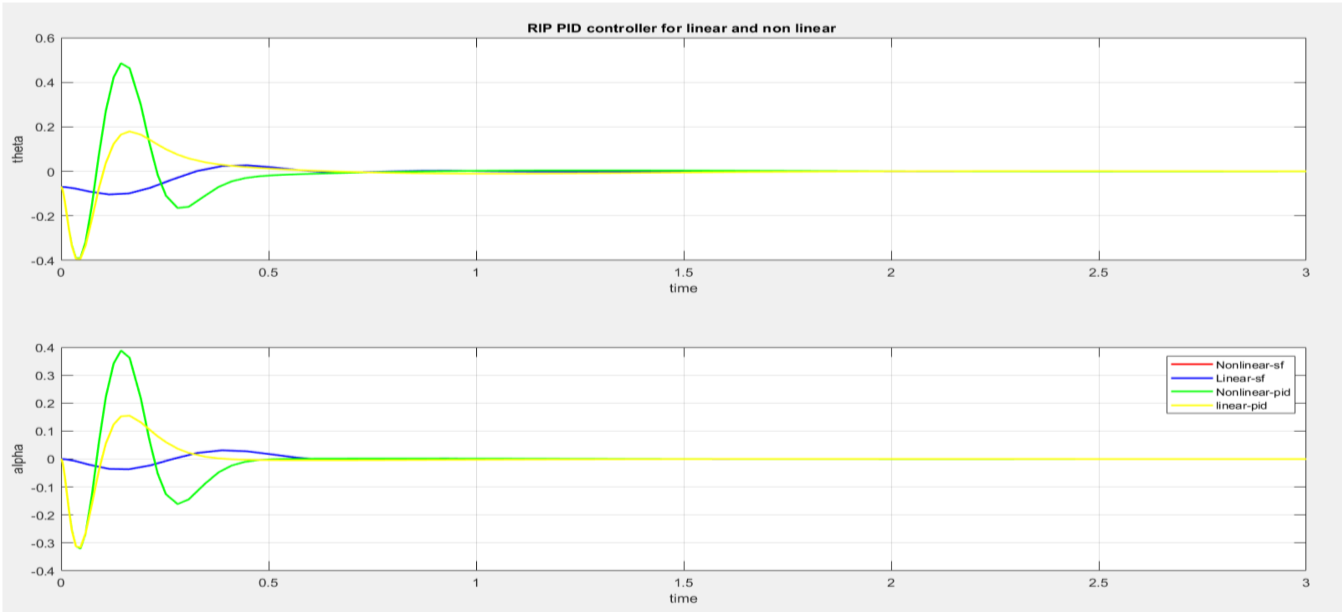


Figure 8: Comparison graph of RIP with FSF controller and 2DOF PID controller

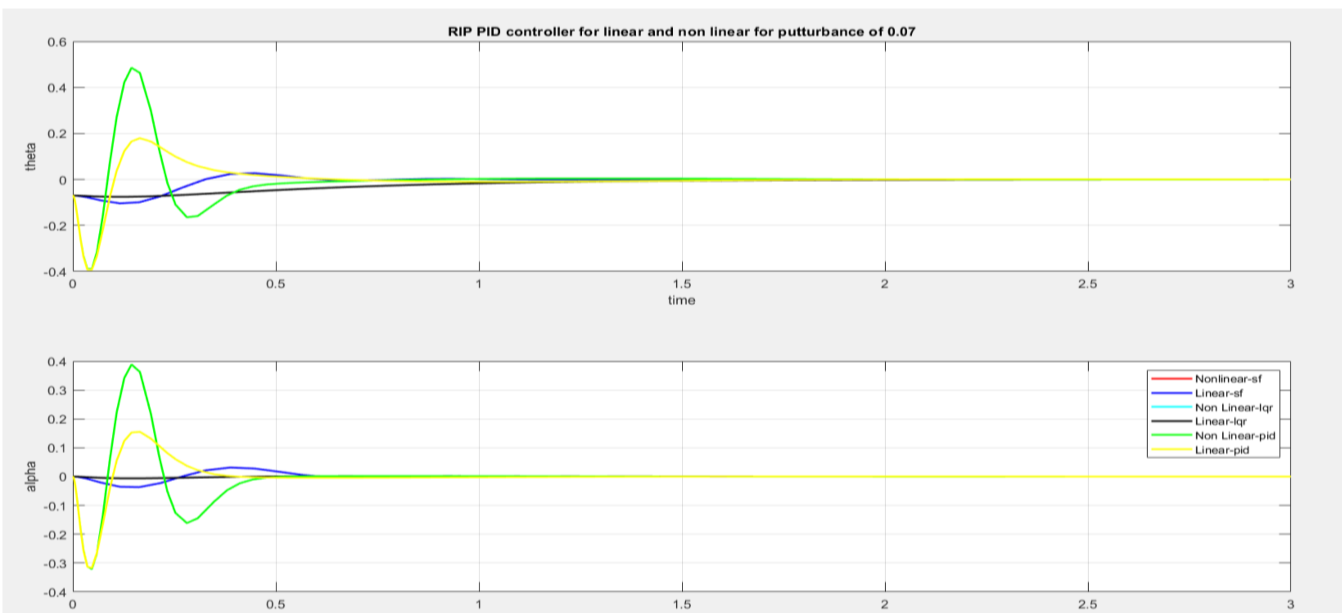
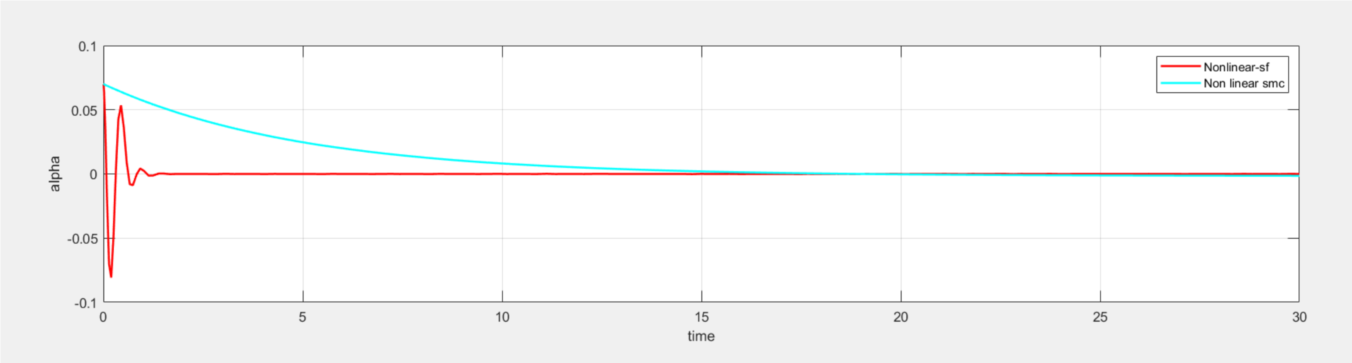


Figure 9: Comparison graph of RIP with FSF controller, LQR feedback and 2DOF PID controller.

Figure 10: Comparison between linear and non linear controllers for Rotary inverted pendulum

For angle alpha(angle made by inverted pendulum with vertical angle)



Controllers used **For nonlinear system (perturbance =0.07)**	**Settling time(sec)**	**Overshoot range**	**Range of allowed perturbance of** (θ), **(Robustness)**
2 DOF PID controller	0.535	More(0.35-0.4)	< 0.07
FSF controller	0.595	Less(0.04-0.05)	< 0.8
LQR controller	0.395	Very less(~0)	< 10

SMC controller	15 sec	(~0)	Any disturbances

Conclusions:

Based on our observation in linear controller ,LQR controller is working better and when it comes to nonlinear controller it is taking more time to stabilize but no overshoot and range of allowing disturbances is also more But SMC cant stabilize the theta state.

References :-

I.Navin John Mathew, K. Koteswara Rao, N. Sivakumaran,"Swing Up and Stabilization Control of a Rotary Inverted Pendulum,"IFAC Proceedings Volumes, Volume 46, Issue 32,2013

II.M. Akhtaruzzaman and A. A. Shafie, "Modeling and control of a rotary inverted pendulum using various methods, comparative assessment and result in analysis," 2010 IEEE International Conference on Mechatronics and Automation, 2010.

III.Mojtaba Ahmadieh Khanesar. Member, Mohammad Teshnehlab,
Mahdi Aliyari Shoorehdeli Sliding Mode Control of Rotary Inverted Pendulum , K.
N. Toosi, University of Technology Electrical Engineering faculty, IEEE,2007