HYDROSTATIC EQUILIBRIUM

Consider a mass element dm at a distance r from the center 'O'

$$dm = \rho(r)\, dAdr \quad \text{-----0}$$

dm is the mass element,

The force of gravity acting on the mass element downwards be

$$F_g = \frac{-GM(r)dm}{r^2} = -\frac{GM(r)\rho(r)drdA}{r^2} \quad \text{-----1}$$

G is the Universal gravitational constant.

Now, $\frac{Gm(r)}{r^2} = g(r) = \frac{g(R)}{R} r$ -----2

The surface gravity is defined as,

$$g(R) = \frac{GM(R)}{R^2} = g_s \quad \text{-----3}$$

M(R) is the total mass of the body. R is the mean radius of the large celestial body.

Force due to buoyancy on the mass element, acting outward

$$dp = \frac{F_g}{dA} = -\frac{GM(r)\rho(r)dr}{r^2} = -g(r)\rho(r)dr = \frac{g_s}{R} r\rho(r)dr \quad \text{-----4}$$

Which is the equation for hydrostatic equilibrium.It is also kown as the first stellar equation.

The equation of state for an ideal gas be,

$$p(r) = \frac{kT(r)}{Am_H}\rho(r) \quad \text{-----5}$$

P(r) is the pressure at r

ρ (r) is the density at r

T(r) is the temperature at r

k is the Boltzman constant

A is the atomic number of the elements involved in the process.

m_H is the rest mass of a proton

$$\rho(r) = \frac{Am_H}{kT(r)} p(r) \quad \text{-----6}$$

Again we have,

$$\frac{\rho_s}{p_s} = \frac{\rho(R)}{p(R)} = \frac{\rho(1)}{p(1)} = \frac{\rho_c}{p_c} = \frac{Am_H}{kT_s} \quad \text{-----7}$$

ρ_s is the mass density at surface

p_s is the surface pressure

$$p_c = \frac{2}{3}\pi G\bar{\rho}^2 R^2 \approx \frac{3GM^2(R)}{4\pi R^4} \text{ --8}$$

Equation 8 represents the expression for central pressure.
R is the radius of the body

$$\rho_c = \frac{Am_H}{kT_c}.\frac{3GM^2(R)}{4\pi R^4} \text{ ---9}$$

The temperature is very badly assumed to be in the form,

$$T(r) = \frac{a}{r} \text{ ---10}$$

The temperature profile is evaluated by the formula,

$$v_{orbit} = v_{rms}$$

$$R\sqrt{\frac{g_s}{r}} = \sqrt{\frac{3kT(r)}{Am_H}} \text{ ---10}$$

$$T(r) = \frac{a}{r}$$

Where a is a const., $a = \frac{Am_H g_s R^2}{3k}$

Puttung equation 6 in to equation 4

$$\frac{dp}{dr} = -\frac{g_s Am_H}{kR}\frac{p(r)}{T(r)}r \text{ ---11}$$

Using equation 10 and in the limit r >> 1, equation 14 reduces to

$$\frac{dp}{p} = -\frac{g_s Am_H}{akR}r^2 dr = -\frac{3}{R^3}r^2 dr \text{ --12}$$

Integrating on both sides,

$$\int_{p_c}^{p}\frac{dp}{p} = -\int_0^r \frac{3}{R^3}r^2 dr \text{ -------------------------------------13}$$

$$\ln\frac{p}{p_c} = -\alpha r^3 \text{ --14}$$

$$\alpha = \frac{1}{R^3} \text{ --15}$$

Therefore the pressure and density profile becomes,

$$p(r) = p_c \exp[-\alpha r^3] \text{ --16}$$

$$\rho(r) = \rho_c \exp[-\alpha r^3] \text{ --17}$$

The second stellar equation is,

$$\frac{d}{dr}M(r) = 4\pi . r^2 \rho(r) \text{ --18}$$

The mass included with in the radius r is,

$$M\ (r) = 4\pi\rho_c \int_0^r r^2 \exp[-\frac{r^3}{R^3}]dr \text{ ------------------------19}$$

After evaluating the integral on RHS the expression for M(r) becomes,

$$M(r) = \frac{4\pi\rho_c}{3\alpha}(1 - e^{-\alpha r^3}) = \frac{4\pi\rho_c}{\frac{3}{R^3}}(1 - e^{-\alpha r^3})$$

$$M(r) = (\frac{4}{3}\pi R^3)\rho_c(1 - e^{-\alpha r^3})$$

$$M(r) = V(R)\rho_c(1 - e^{-\alpha r^3})$$

The total mass contained with in V(R) of the celestial body is,

$$M(R) = (1 - \frac{1}{e})V(R)\rho_c = 0.63212V(R)\rho_c$$

The mean density of the celestial body becomes,

$$\overline{\rho}(R) = \frac{M(R)}{V(R)} = (1 - \frac{1}{e})\rho_c = 0.63212\ \rho_c$$

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