Discrete-Time Signals

Discrete-time signal $x(n)$ is a function of an independent variable that is an integer. It is important to note that a discrete-time signal is not defined at instants between two successive samples. i.e simply the signal $x(n)$ is not defined for non-integer values of n.

$x(n)$ is obtained from sampling an analog signal $x_a(t)$. Then $x(n) = x_a(nT)$ where T is the sampling period.

Alternative Representations

1. Functional Representations

$$x(n) = \begin{cases} 1 & \text{for } n = 1, 3 \\ 4 & \text{for } n = 2 \\ 0 & \text{elsewhere} \end{cases}$$

2. Tabular representation

n	-2	-1	0	1	2	3	4	5
$x(n)$	0	0	0	1	4	1	0	0

3. Sequence representation.

An infinite duration signal or sequence with time origin $(n=0)$ indicated by the symbol $\uparrow$ is represented as

$$x(n) = \{ \ldots 0, \underset{\uparrow}{0}, 1, 4, 1, 0, 0 \ldots \}$$

A sequence $x(n)$, which is zero for $n<0$, can be represented as

$$x(n) = \{ \underset{\uparrow}{0}, 1, 4, 1, 0\, 0 \}$$

A finite duration sequence can be represented as

$$x(n) = \{3, -1, \underset{\uparrow}{-2}, 5, 0, 4, 1\}$$

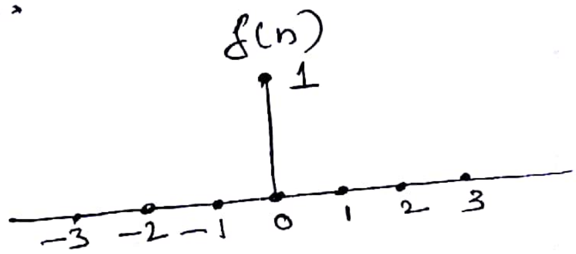
Some elementary Discrete-time Signals

These are some of the basic signals that appear often and play an important role.

1. Unit sample sequence $\delta(n)$

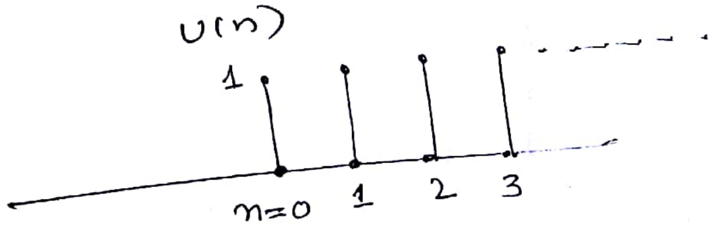
It is defined as
$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & n \neq 0 \end{cases}$$

Unit sample sequence is a signal that is zero everywhere, except at $n=0$ where its value is unity. This signal is sometimes referred to as a unit impulse.



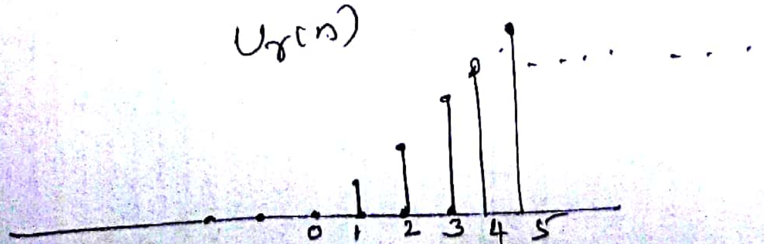
2. Unit step signal $u(n)$

It is defined as
$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$



3. Unit ramp signal $U_r(n)$

$$U_r(n) = n \quad \text{for } n \geq 0$$
$$= 0 \quad \text{for } n < 0$$



4. Exponential Signal

It is a sequence of the from $x(n) = a^n$ for all n.

If the parameter a is real, then $x(n)$ is a real signal.

when the parameter a is complex valued, it can be expressed as $a \equiv re^{j\theta}$

where r and θ are now the parameters. Hence we can expresse $x(n)$ as

$$x(n) = r^n e^{j\theta n}$$
$$= r^n (\cos n\theta + j \sin n\theta)$$

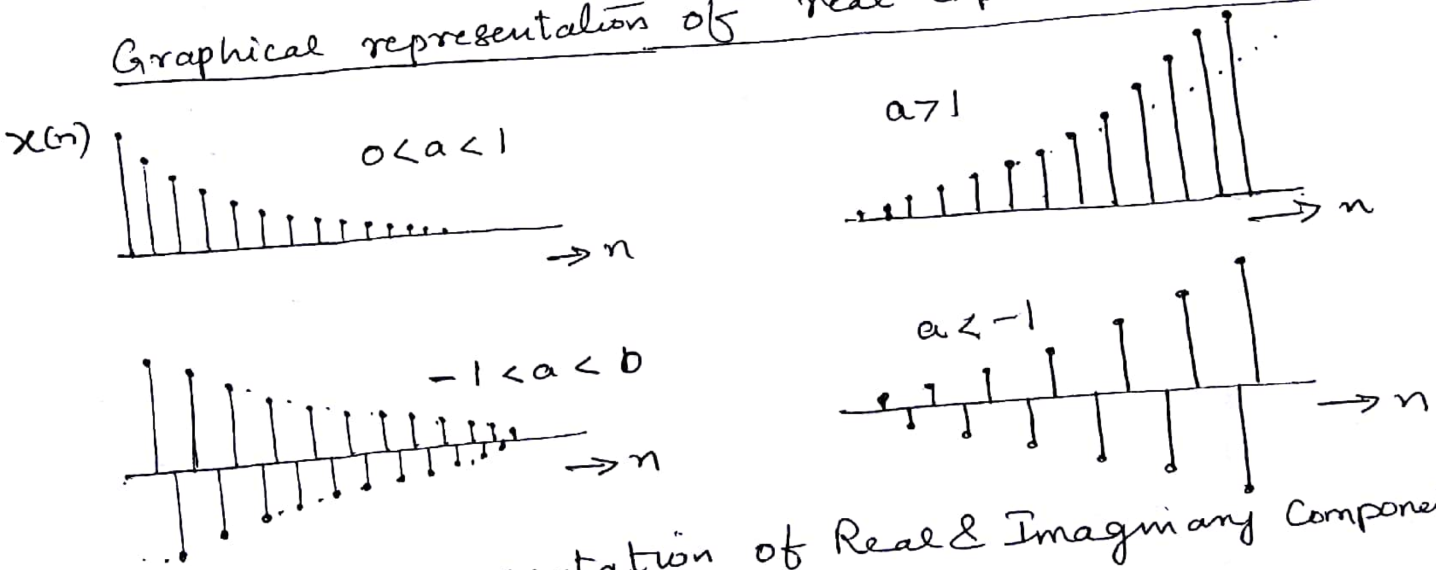
Since $x(n)$ is now complex valued. It can be represented graphically by plotting the real part as a function of n,

$$x_R(n) \equiv r^n \cos n\theta$$

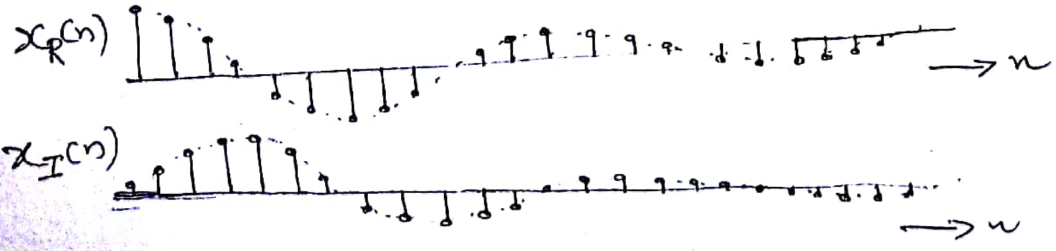
and seperately plotting the imaginary part as a function of n.

$$x_I(n) \equiv r^n \sin n\theta$$

Graphical representation of real exponential signals



Graphical representation of Real & Imaginary Components of a Complex-valued exponential Signal



Classification of Discrete-time Signals

Energy and power Signals

The energy E of a signal $x(n)$ is defined as

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

The energy of a signal can be finite or infinite. If E is finite (ie $0 < E < \infty$) Then $x(n)$ is Called as energy Signal.

Many of Signals that possess infinite energy have a finite average power. The average power of a Discrete-time signal $x(n)$ is defined as

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

The power of a signal Can be finite or infinite. If P is finite (ie $0 < P < \infty$) then $x(n)$ is Called as power signal.

* A signal Can be Energy Signal or power Signal alone.
* A Signal Can't be both at a time.
* A Signal Can be neither energy nor power Signal

Periodic Signals and aperiodic Signals

A Signal $x(n)$ is periodic with period N ($N>0$) if and only if $x(n+N) = x(n)$ for all n.

The Smallest value of N for which it holds is called the fundamental period. If there is no value of N that Satisfies the above eq. the Signal is Called nonperiodic or aperiodic.

The general expression of complex exponential signal is $x(n) = e^{j\omega_0 n}$

To find the period N for which this signal is periodic,

$$x(n) = x(n+N)$$

$$e^{j\omega_0 n} = e^{j\omega_0 (n+N)}$$

$$e^{j\omega_0 N} = 1$$

$$e^{j\omega_0 N} = e^{j2\pi m} \quad \text{(where m is integer)}$$

$$\omega_0 N = 2\pi m$$

$$\boxed{\frac{\omega_0}{2\pi} = \frac{m}{N}}$$ (where m is integer and N is also integer)

Hence the condition for signal to be periodic is

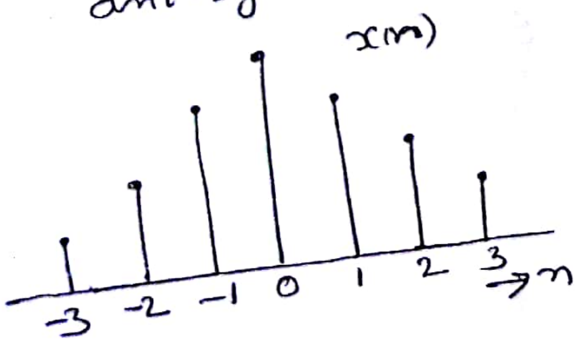
$\frac{\omega_0}{2\pi} = \frac{m}{N}$ where m, & N must be integers.

then period $\boxed{N = \frac{2\pi m}{\omega_0}}$

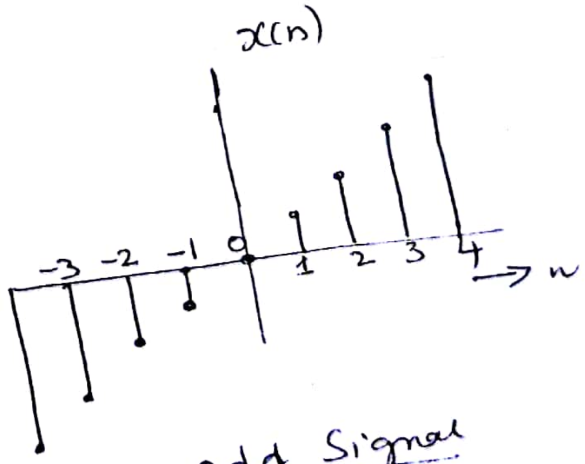
Symmetric (even) and antisymmetric (odd) signals.

A real-valued signal x(n) is called symmetric (even) if $x(-n) = x(n)$

On the other hand, a signal x(n) is called antisymmetric (odd) if $x(-n) = -x(n)$



even signal



odd signal

Any arbitrary signal can be expressed as the sum of two signal components, one of which is even and other odd.

$$x_e(n) = \frac{x(n) + x(-n)}{2}$$

$$x_o(n) = \frac{x(n) - x(-n)}{2}$$

$$\boxed{x(n) = x_e(n) + x_o(n)}$$

Simple Manipulations of Discrete-Time Signals

Transformation of the Independent Variable (Time)

(i) A signal $x(n)$ may be shifted in time by replacing the independent variable n by $n-k$, where k is an integer. If k is a positive integer, the time shift results in a delay of the signal by k units of time. If k is a negative integer, the time shift results in an advance of the signal by $|k|$ units in time.

(ii) Another useful modification of the time base is to replace the independent variable n by $-n$. The result of this operation is a folding or a reflection of the signal about the time origin $n=0$

$$x(n) = \{2, 4, \underset{\uparrow}{1}, -3, 0, 1, 0\}$$

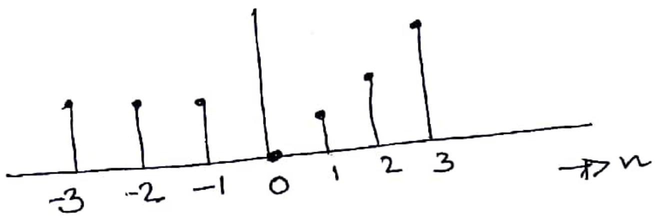
$$x(n-2) = \{\underset{\uparrow}{2}, 4, 1, -3, 0, 1, 0\}$$

$$x(n+2) = \{2, 4, 1, -3, \underset{\uparrow}{0}, 1, 0\}$$

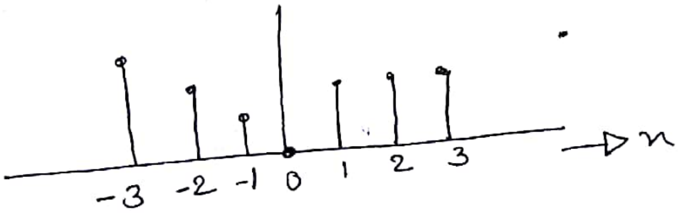
example for shifted versions

Graphical representations of signals $x(-n)$ and $x(-n+2)$ where $x[n]$ is the signal illustrated as shown.

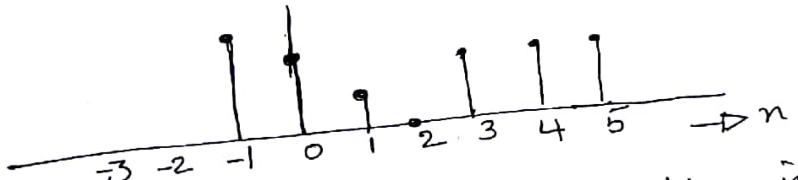
$x[n]$



$x(-n)$

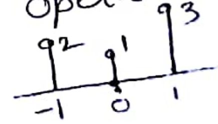


$x(-n+2)$



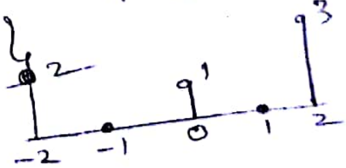
iii) A third modification of the independent variable involves replacing n by μn where μ is an integer. We refer to this time-base modification as time-scaling or down-sampling.

If the signal $x(n)$ was originally obtained by sampling an analog signal $x_a(t)$, then $x(n) = x_a(nT)$ where T is the sampling interval. Now $y(n) = x(2n) = x_a(2Tn)$. Hence the time scaling operation is equivalent to changing the sampling rate from $\frac{1}{T}$ to $\frac{1}{2T}$, that is, to decreasing the rate by a factor of 2. This is a down-sampling operation



for ex! $x(n) = \{ 2, \underset{\uparrow}{1}, 3 \}$ $0 \quad 3\}$

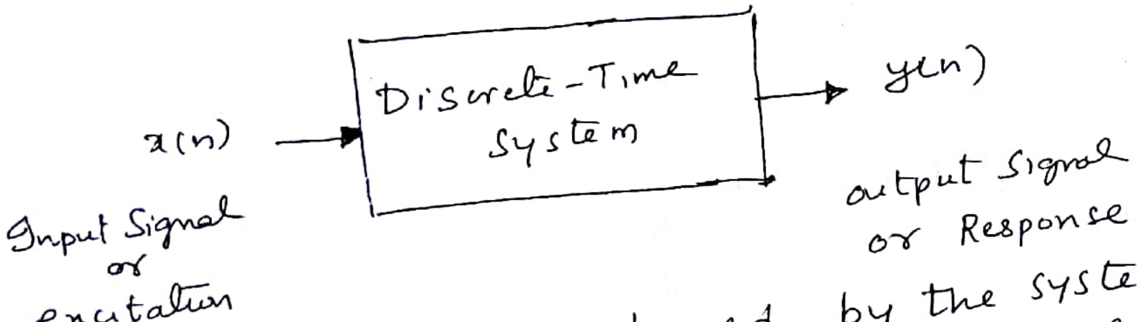
$x(2n) = \{ \quad 2 \quad 0 \quad \underset{\uparrow}{1}$



$x(\frac{n}{2}) = \{ \underset{\uparrow}{1} \}$

Discrete - Time Systems

In many applications of digital signal processing we wish you to design a device or an algorithm that performs some prescribed operation on a discrete-time signal. Such a device or algorithm is called discrete-time system.



Input signal $x(n)$ is transformed by the system into a signal $y(n)$ and express the general relationship between $x(n)$ and $y(n)$ as

$$y(n) \equiv T\{x(n)\}$$

where T denotes the transformation or processing performed by the system on $x(n)$ to produce $y(n)$.

Input - Output description of Systems

The input - output description of a DT system consists of a mathematical expression or a rule, which explicitly defines the relation between the input and output signals.

ex: (i) $y(n) = x(n)$ (Identity system)

(ii) $y(n) = x(n-1)$ (Unit delay system)

(iii) $y(n) = x(n+1)$ (Unit advance system)

(iv) $y(n) = \frac{1}{3}[x(n+1) + x(n) + x(n-1)]$ (moving average filter)

(v) $y(n) = \sum_{k=-\infty}^{n} x(k) = x(n) + x(n-1) + x(n-2) + \cdots$ (accumulator)

Classification of Discrete-Time Systems

1) Static Versus Dynamic Systems
2) Time-invariant Versus Time Variant Systems
3) Linear Versus Nonlinear Systems.
4) Causal Versus Noncausal Systems
5) Stable Versus Unstable Systems

Static Versus Dynamic Systems

A discrete-time System is called static or memoryless if its output at any instant n depends at most on the input sample at the same, but not on past or future samples of the input.

In any othercase, the system is said to be dynamic or to have memory.

Ex: $y(n) = ax(n)$
$y(n) = n x(n) + b x^3(n)$ } Static or Memoryless.

$y(n) = x(n) + 3x(n-1)$
$y(n) = \sum_{K=0}^{n} x(n-K)$ } Memory or Dynamic System.

Time-Invariant Versus Time-Variant Systems.

A System is called time-invariant if its Input-Output characteristic donot change with time

To elaborate, suppose that we have a system τ in a relaxed state which, when excited by an input signal $x(n)$, produces an output signal $y(n)$. Thus we write $y(n) = T\{x(n)\}$. Now suppose that the same input signal is delayed by K units of time to yield $x(n-K)$, and again applied to the same system. If the characteristics of the system do not change with time, the output of the relaxed system will be $y(n-k)$. ie the Output will be the same as the response to $x(n)$, except that it will be delayed by the same k Units in time that the input was delayed.

Definition

A relaxed system τ is time-invariant or Shift invariant if and only if $x(n) \xrightarrow{\tau} y(n)$

implies that $x(n-k) \xrightarrow{\tau} y(n-k)$

for every input signal $x(n)$ and every time shift k.

Test for determining Shift invariant or Time Invariant

1) Delay the input sequence by some amount k and Compute the output ie $\tau[x(n-k)]$. let us say $y_1(n)$

2) Next, determine the actual output at time(n) = n-k by replacing n by n-k in the equation;

ie $y(n-k)$

3) if $y_1(n) = y(n-k)$ then System is Shift invariant.

Linear Versus Nonlinear System

A linear system is one that satisfies the Superposition principle. It states that the response of the system to a weighted sum of signals to be equal to the corresponding weighted sum of the responses (outputs) of the system to each of the individual I/P signals.

Definition

$x_1(n) \xrightarrow{T} y_1(n)$

$x_2(n) \xrightarrow{T} y_2(n)$

$a_1x_1(n) + a_2x_2(n) \xrightarrow{T} a_1y_1(n) + a_2y_2(n)$

If a system does not satisfy the superposition principle as given by the definition above, it is called non-linear.

Causal Versus NonCausal Systems.

Defn A system is said to be causal if the output of the system at any time n depends only on present and past inputs [ie $x(n), x(n-1), x(n-2)\ldots$] but does not depend on future inputs [ie $x(n+1), x(n+2) \ldots$]:

$$y(n) = F[x(n), x(n-1), x(n-2) \ldots]$$

If a system does not satisfy this definition, it is called noncausal. Such a system has an output that depends not only on present and past inputs but also on future inputs.

Stable Versus Unstable Systems

Definition: An arbitrary relaxed system is said to be bounded input- ~~output~~ bounded output (BIBO) stable if and only if every bounded input produces a bounded output.

Mathematically Bounded Input- Bounded Output are represented as

$$|x(n)| \leq M_x < \infty \quad \text{for all } n.$$

$$|y(n)| \leq M_y < \infty$$

where M_x, M_y are some finite numbers.

Linear Shift (Time) Invariant System

LSI or LTI

A system which satisfies both the properties of Linearity and Shift Invariance is known as LTI

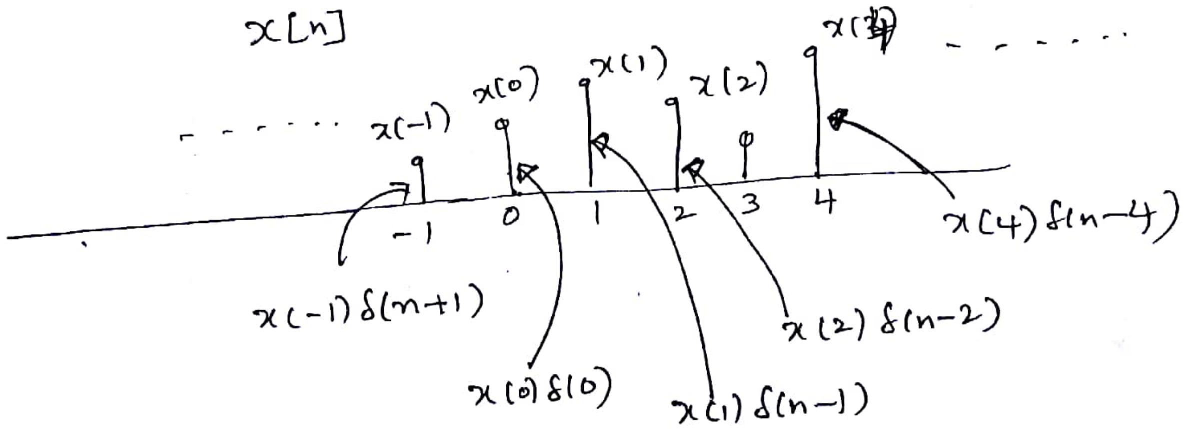
Linearity: $a_1 x_1(n) + a_2 x_2(n) \xrightarrow{T} a_1 y_1(n) + a_2 y_2(n)$

Shift Invariance $x(n-K) \xrightarrow{T} y(n-k)$

Resolution of a Discrete-Time Signal into Impulses

Suppose we have an arbitrary signal $x(n)$ that we wish to resolve into a sum of unit sample sequences. ~~To utilize the notation~~ as below.

for ex: consider a signal $x[n]$



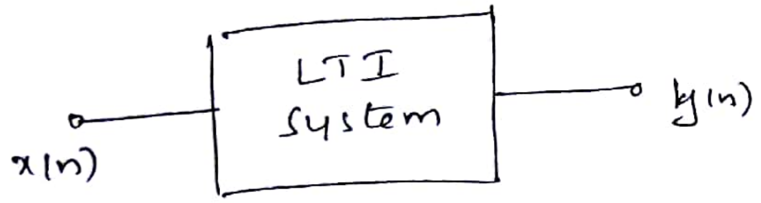
$$x(n) = \cdots + x(-1)\delta(n+1) + x(0)\delta(0) + x(1)\delta(n-1) + x(3)\delta(n-3) + x(4)\delta(n-4) + x(2)\delta(n-2) + \cdots$$

$$\boxed{x(n) = \sum_{K=-\infty}^{\infty} x(K)\,\delta(n-K)}$$

ie Any arbitrary signal $x[n]$ is completely resolved as sum of unit sample sequences.

Response of LTI System to arbitrary Inputs

Consider LTI system



Response of LTI system

Given that $h[n]$ is the response of the system to the input signal $\delta[n]$. Hence it is known as impulse response $h[n]$.

The arbitrary input signal $x[n]$ is resolved as sum of ~~input~~ ~~Sin~~ unit sample sequences as below. ie

$$x[n] = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$$

$$\delta(n) \longrightarrow h[n]$$

$$\delta(n-k) \longrightarrow h[n-k]$$

$$x(k)\,\delta(n-k) \longrightarrow x(k)\,h(n-k)$$

$$\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) \longrightarrow \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

$$x(n) \longrightarrow \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

Hence the Response ~~of~~ y(n) of the system to the input signal x(n) is

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)}$$

Response of LTI is Convolution

$$y(n) = \sum_{K=-\infty}^{\infty} x(K)\, h(n-K) \quad \text{--- (I)}$$

Given input signal $x(n)$ and impulse response of the system $h(n)$. Then how to find $y(n)$ using the equation (I)?

consider — $x(n)$ — $h(n)$

1) Change the Domains for conveinence — $x(K)$ — ~~h(K)~~ $h(K)$

2) Fold any one of the signal by keeping the other signal. ~~Consta~~ — $x(K)$ — $h(-K)$

3) Shift the folded signal by n — $x(K)$ — $h(n-K)$

for each shift of value n, find the product — $x(K)\,h(n-K)$

5) For all values of K find the product that gives output — $y(n) = \sum_{\text{all } K} x(K)\, h(n-K)$

ie In finding the response of the system, operations involved are folding, shifting and Multiplication. ~~Known~~ The combination of all three processes is defined as Convolution.

Hence output $y(n)$ is the convolution between the input $x(n)$ and impulse Response $h(n)$

$$y(n) = x(n) * h(n)$$

symbolic notation for convolution.

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Convolution and its properties.

(E) $y(n) = x(n) * h(n)$ for LTI system

① Commutative Law

$$x(n) * h(n) = h(n) * x(n)$$

proof $\quad x(n) * h(n) = \sum_{K=-\infty}^{\infty} x(K)\,h(n-K)$

Replace $n-K$ by n_1

$$= \sum_{n_1=\infty}^{-\infty} x(n-n_1)\,h(n_1)$$

$$= \sum_{n_1=-\infty}^{\infty} x(n-n_1)\,h(n_1)$$

Replace n_1 by K

$$= \sum_{K=-\infty}^{\infty} x(n-K)\,h(K)$$

$$= h(n) * x(n)$$

② Associative Law

$$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$$

③ Identity and Shifting properties

$$x(n) * \delta(n) = x(n)$$

$$x(n) * \delta(n-K) = x(n-K)$$

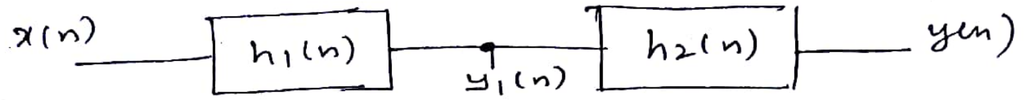
$$x(n-n_1) * \delta(n-n_2) = x(n-n_1-n_2)$$

④ Distributive Law

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$$

(6)

Systems are in cascade



Consider Two LTI systems which are connected in series.

$$y_1(n) = x(n) * h_1(n)$$

$$y(n) = y_1(n) * h_2(n)$$

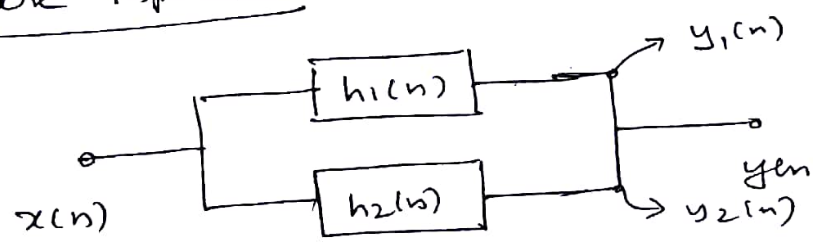
$$= x(n) * h_1(n) * h_2(n)$$

$$= x(n) * (h_1(n) * h_2(n))$$

$$= x(n) * h(n)$$

Hence overall impulse response $h(n) = h_1(n) * h_2(n)$

Systems are inparallel



Consider two LTI systems which are connected inparallel

$$y(n) = y_1(n) + y_2(n)$$

$$= x(n) * h_1(n) + x(n) * h_2(n)$$

$$= x(n) * [h_1(n) + h_2(n)]$$

$$= x(n) * h(n)$$

overall Impulse Response

$$\boxed{h(n) = h_1(n) + h_2(n)}$$

Causal LTI system

Consider an LTI system; $y(n) = x(n) * h(n)$

$$= h(n) * x(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$y(n) = \cdots\cdots + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \cdots\cdots \quad \text{—— (11)}$$

For a causal system, output depends on present and past values of the input signal but does not future values of the input signal.

From the equation (11), To not depend on future ~~...~~ i/p values of the input signal (ie $x(n+1)$, $x(n+2)$, $x(n+3)$ ......)

$$h(-1) = h(-2) = h(-3) = \cdots = 0$$

Hence $\boxed{h(n) = 0 \quad \text{for } n<0}$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

* Stable LTI system

Stable system property: BIBO

Assume that Bounded input ie $|x(n)| \leq M_x < \infty$

Then $|y(n)| = \left|\sum_{k=-\infty}^{\infty} h(k)\, x(n-k)\right|$

$$\leq \sum_{k=-\infty}^{\infty} |h(k)x(n-k)|$$

$$\leq \sum_{k=-\infty}^{\infty} |h(k)|\, |x(n-k)|$$

$$\leq \sum_{k=-\infty}^{\infty} |h(k)|\, M_x$$

$$|y(n)| \leq M_x \sum_{k=-\infty}^{\infty} |h(k)|$$

To have $|y(n)| \leq M_y < \infty$ Then it is necessary that $\boxed{\sum_{k=-\infty}^{\infty} |h(k)| < \infty}$

Systems with Finite duration and Infinite - duration Impulse Response

Upto this point we have characterized a linear Time-Invariant System in terms of its impulse response h(n). It is also convenient, however, to subdivide the class of LTI systems into two types.

i) FIR (finite duration impulse response)

ii) IIR (Infinite duration impulse Response)

Example : If the duration of the impulse response of the system is finite then it is called as FIR

ex! $h(n) = \{\underbrace{2, 0, 1, 0, 4, 3}_{\text{Duration is finite}}\}$

If the duration of the impulse response of the system is infinite then it is called as IIR

(i) $h(n) = a^n u(n)$

(ii). $h(n) = \{1, \underset{\uparrow}{2}, 2^2, 2^3 \ldots\}$

Recursive and Non Recursive Systems

Output of LTI is not only a function of present and past values of input, and also a function of past values of the output. Such Systems are said to be Recursive.

for ex! $y(n) = x(n) + y(n-1)$

If the output is not a function of the past values of the output, then it is known as Non-Recursive System.

for ex! $y(n) = x(n) + x(n-1)$

Linear Constant Coefficent Difference Equations

In this section, we focus our attention on a family of linear time-invariant systems described by an input-output relation called a difference equation with constant coefficients. To bringout the important ideas, Consider a recursive system with an input-output equation

$$y(n) = a\,y(n-1) + x(n)$$ where a is constant

Suppose that we apply an input signal $x(n)$ to the system for $n \geq 0$. we compute successive values of $y(n)$ for $n \geq 0$ beginning with $y(0)$, Thus

$$y(0) = a\,y(-1) + x(0)$$

$$y(1) = a\,y(0) + x(1) = a^2 y(-1) + a\,x(0) + x(1)$$

$$y(2) = a\,y(1) + x(2) = a^3 y(-1) + a^2 x(0) + a\,x(1) + x(2)$$

$\vdots$

$$y(n) = a\,y(n-1) + x(n)$$

$$= a^{n+1} y(-1) + a^n x(0) + a^{n-1} x(1) + \dots + x(n)$$

$$= \underbrace{a^{n+1} y(-1)}_{\text{Ist term}} + \underbrace{\sum_{k=0}^{n} a^k x(n-k)}_{\text{IInd term.}} \quad \text{--- I}$$

Ist term contains the term $y(-1)$, is a result of the initial condition $y(-1)$ of the system. If the input $x(n) = 0$ for all n, output of the system is $y_{zi}(n) = a^{n+1} y(-1)$. Hence this response is known as zero-input response or natural response and is denoted by $y_{zi}(n)$.

$$y_{zi}(n) = a^{n+1} y(-1) \quad n \geq 0$$

IInd term represents the output of the System with input applied for $n \geq 0$. If the System is initially relaxed (zero initial conditions), response of the system (ie eq 1) reduces to

$$y_{zs}(n) = \sum_{k=0}^{n} a^k x(n-k) \quad n \geq 0$$

Hence this response is called Zero State response or forced response.

$$\boxed{y(n) = y_{zs}(n) + y_{zi}(n)}$$

The general form for ~~LCCDE~~ system described by LCCDE is given by

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \quad a_0 = 1.$$

The integer N is called the order of the difference equation or the order of the system.

Solution of Linear Constant - Coefficient Difference equations

Given a LCCDE equation, our goal is to determine the explicit expression for the output $y(n)$.

Total Solution is the sum of two parts

$$y(n) = y_h(n) + y_p(n)$$

$y_h(n)$ is known as the homogeneous Solution or Complementary Solution

$y_p(n)$ is known as the particular Solution.

Homogeneous Solution of a difference equation

Homogeneous Solution is ~~the solution~~ obtained by making $x(n) = 0$.

General Expression of LCCDE is given by

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \qquad a_0 = 1$$

$$\sum_{k=0}^{N} a_k y(n-k) = 0$$

Solution of this equation is in the form of exponential

$$y_h(n) = \lambda^n$$

$$\sum_{k=0}^{N} a_k \lambda^{n-k} = 0$$

$$a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_N\lambda^{n-N} = 0$$

$$a_0\lambda^N + a_1\lambda^{N-1} + a_2\lambda^{N-2} + \dots + a_N = 0$$

$$\lambda^N + a_1\lambda^{N-1} + a_2\lambda^{N-2} + \dots + a_N = 0 \quad (\because a_0 = 1)$$

This polynomial is called characteristic equation of the System. If it has N roots, $\lambda_1, \lambda_2 \dots \lambda_N$.

1) $$y_h(n) = c_1\lambda_1^n + c_2\lambda_2^n + \dots + c_N\lambda_N^n$$

where $c_1, c_2, c_3 \dots c_N$ are weighting coefficients.

2) If λ_1 ~~has~~ is a root of multiplicity m.

$$y_h(n) = (c_1 + nc_2 + n^2c_3 + \dots + n^{m-1}c_m)\lambda_1^n + c_{m+1}\lambda_2^n + \dots + c_N\lambda_N^n$$

particular Solution of the difference equation

The particular Solution $y_p(n)$ is required to Satisfy the difference equation $\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$

for the Specific input Signal $x(n)$, $n \geq 0$.

General Form of the particular Solution for several Types of Input signals.

Input Signal $x(n)$	particular Solution $y_p(n)$
A	K
AM^n	KM^n
n^M	$K_0 n^M + K_1 n^{M-1} + \dots + K_M$
$A^n n^M$	$A^n (K_0 n^M + K_1 n^{M-1} + \dots + K_M$
$\left\{ \begin{matrix} A \cos w_0 n \\ A \sin w_0 n \end{matrix} \right\}$	$K_1 \cos w_0 n + K_2 \sin w_0 n$.

problems.

1) Check whether the Given Signal is Energy or power?

$x[n] = u[n]$

$$E = \sum_{n=-\infty}^{\infty} x[n]^2$$

$$= \sum_{0}^{\infty} u[n]^2$$

$$= \sum_{0}^{\infty} u[n]\, u[n]$$

$$= \sum_{0}^{\infty} (1)(1) = \sum_{0}^{\infty} 1 = 1+1+ \dots + \text{upto infinite terms}$$

$$= \infty \text{ (not an Energy signal)}$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} x[n]^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} u[n]^2.$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{0}^{N} (1)^2$$

$$= \lim_{N\to\infty} \frac{N+1}{2N+1} = \frac{1}{2}; \text{ Hence it is power signal}$$

Determine whether it is a ~~power signal~~ periodic signal)

$x(n) = \cos 1.1\pi n$

Condition for periodic signal $\frac{\omega_0}{2\pi} = \frac{m}{N}$ (m, N must be integers)

$$\omega_0 = 1.1\pi$$

$$\frac{\omega_0}{2\pi} = \frac{1.1\pi}{2\pi} = \frac{1.1}{2} = \frac{11}{20} = \frac{m}{N}$$

$m = 11$, $N = 20$ are integers.

Hence it is periodic signal. with period $N = 20$

(3) Check whether it is periodic signal?

$x[n] = \cos 7\pi n + \cos 3\pi n$

$\omega_{01} = 7\pi$ $\qquad$ $\omega_{02} = 3\pi$

$\frac{\omega_{01}}{2\pi} = \frac{7\pi}{2\pi} = \frac{m_1}{N_1}$ $\qquad$ $\frac{\omega_{02}}{2\pi} = \frac{3\pi}{2\pi} = \frac{3}{2} = \frac{m_2}{N_2}$

m_1, N_1 are integers $\qquad$ m_2, N_2 are integers.

Hence $\cos 7\pi n$, $\cos 3\pi n$ are periodic signals.

Hence $\cos 7\pi n + \cos 3\pi n$ is a periodic signal.

Total period $N = LCM(N_1, N_2)$

$= LCM(2, 2)$

$$\boxed{N = 2}$$

(4) Determine if the system is time-Invariant or TV.

$y(n) = x(n) - x(n-1)$

Condition for Time-Invariance is $\boxed{y(n,k) = y(n-k)}$

$y(n,k)$ is ~~the~~ output of the system with delayed input $x(n-k)$.

$y(n-k)$ is output of the system at time $n = n-k$.

$y(n,k) = x(n-k) - x(n-k-1)$

$y(n-k) = x(n-k) - x(n-k-1)$ (To find this replace n by n-k)

$y(n,k) = y(n-k)$ Hence Time-Invariant

(5) Check Time-Invariance

$$y(n) = n x(n)$$

$$y(n,k) = n x(n-k)$$

$$y(n-k) = n-k \; x(n-k)$$

$$y(n,k) \neq y(n-k)$$

Hence Time-Variant

(6) Check Time-Invariance

$$y(n) = x(-n)$$

$$y(n,k) = x(-n-k)$$

$$y(n-k) = x(-(n-k)) = x(-n+k)$$

$$y(n,k) \neq y(n-k)$$

Hence Time Variant

(7) Check Time-Invariance

$$y(n) = x(n) \cos w_0 n$$

$$y(n,k) = x(n-k) \cos w_0 n$$

$$y(n-k) = x(n-k) \cos w_0 (n-k)$$

$$y(n,k) \neq y(n-k)$$

Hence Time Variant

(8) Determine whether the following are Linear or Nonlinear

(a) $y(n) = n x(n)$

$x_1(n) \longrightarrow y_1(n) = n x_1(n)$

$x_2(n) \longrightarrow y_2(n) = n x_2(n)$

$x_1(n) + x_2(n) \longrightarrow y_3(n) = n(x_1(n) + x_2(n))$ Hence linear

$= y_1(n) + y_2(n)$

(b) $y(n) = x^2(n)$

$x_1(n) \longrightarrow y_1(n) = x_1^2(n)$

$x_2(n) \longrightarrow y_2(n) = x_2^2(n)$

$x_1(n) + x_2(n) \longrightarrow y_3(n) = [x_1(n) + x_2(n)]^2$

$= x_1^2(n) + x_2^2(n) + 2x_1(n) x_2(n)$

$\neq y_1(n) + y_2(n)$ Hence Non-linear

(c) $y(n) = Ax(n) + B$

$x_1(n) \longrightarrow y_1(n) = Ax_1(n) + B$

$x_2(n) \longrightarrow y_2(n) = Ax_2(n) + B$

$x_1(n) + x_2(n) \longrightarrow y_3(n) = A(x_1(n) + x_2(n)) + B$

$= y_1(n) + y_2(n) - B$

$\neq y_1(n) + y_2(n)$

Hence Non-linear.

(d) $y(n) = e^{x(n)}$

$x_1(n) \longrightarrow y_1(n) = e^{x_1(n)}$

$x_2(n) \longrightarrow y_2(n) = e^{x_2(n)}$

$x_1(n) + x_2(n) \longrightarrow y_3(n) = e^{x_1(n)+x_2(n)}$

$= e^{x_1(n)} e^{x_2(n)}$

$= y_1(n)\, y_2(n)$

$\neq y_1(n) + y_2(n)$

Hence Non-linear

(9) Determine the output $y(n)$ of a relaxed linear Time-Invariant System with impulse response $h(n) = a^n u(n)$ $|a|<1$ when the input is a unit step sequence. $x(n) = u(n)$

Sol. $y(n) = x(n) * h(n)$ [LTI system]

$$= \sum_{K=-\infty}^{\infty} x(K)\, h(n-K)$$

$$= \sum_{K=-\infty}^{\infty} u(K)\, a^{n-K}\, u(n-K)$$

$u(K) = 1$ for $K \geq 0$

$u(n-K) = 1$ for $n-K \geq 0 \Rightarrow K \leq n$

$\Rightarrow$ $u(K)\,u(n-K) = 1$ for $0 \leq K \leq n$

$$= \sum_{K=0}^{n} a^{n-K} = a^n + a^{n-1} + \cdots + a^0$$

$= 1 + a + a^2 + \cdots + a^n$

$=$

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$$y(n) = \frac{1-a^{n+1}}{1-a}.$$

(10) Determine the range of values of the parameter a for which the Linear Time-invariant System with impulse response is stable?

$$h(n) = a^n u(n)$$

Condition for stability is $\sum_{K=-\infty}^{\infty} |h(K)| < \infty$

$$= \sum_{n=-\infty}^{\infty} a^n u(n)$$

$$= \sum_{n=0}^{\infty} a^n) \qquad (\because u(n)=1)$$

$$= a^0 + a^1 + \cdots + a^\infty$$

$$= \frac{1}{1-a} \quad \text{for } |a|<1$$

System converges to finite for $|a|<1$

Hence Condition for a system to be stable is $\underline{|a|<1}$

(11) Determine the range of values of a and b for which the LTI with impulse response $h(n) = \begin{cases} a^n & n \geq 0 \\ b^n & n<0 \end{cases}$

Sol: $$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} \{a^n u(n) + b^n u(-n)\}$$

$$\sum_{n=-\infty}^{\infty} a^n u(n) = \sum_{0}^{\infty} a^n = a^0 + a^1 + \cdots + a^\infty$$

It converges if $\underline{|a|<1}$

$$= \frac{1}{1-a}$$

$$\sum_{n=-\infty}^{\infty} b^n u(-n) = \sum_{-\infty}^{-1} b^n = b^{-1} + b^{-2} + \cdots + b^{-\infty}$$

$$= \left(\frac{1}{b}\right) + \left(\frac{1}{b}\right)^2 + \cdots + \left(\frac{1}{b}\right)^\infty$$

it converges only if $\left|\frac{1}{b}\right| < 1$

$$= \frac{1}{1-\left(\frac{1}{b}\right)}$$

System converges (stable) if and only if $|a|<1$ & $\left|\frac{1}{b}\right| < 1$

Determine the solution of the system described by the LCCDE ?

$$y(n) + a_1 y(n-1) = x(n) \quad \text{with} \quad x(n) = u(n)$$

$$y(n) = y_h(n) + y_p(n)$$

Homogeneous Solution ; $y(n) + a_1 y(n-1) = 0$

$$y_h(n) = \lambda^n$$

$$\lambda^n + a_1 \lambda^{n-1} = 0$$

$$\lambda + a_1 = 0$$

$$\boxed{\lambda = -a_1}$$

$$y_h(n) = C_1 (-a_1)^n$$

Particular Solution.

$$y(n) + a_1 y(n-1) = u(n)$$; It is assume that particular solution is of the form. $y_p(n) = K u(n)$;

Suppose if it is the solution it must satisfy the difference equation

~~$y_p(n) =$~~

$$K u(n) + a_1 K u(n-1) = u(n)$$

for $n \geq 1$

$$K + a_1 K = 1$$

$$K = \frac{1}{1+a_1}$$

$$y_p(n) = \frac{1}{1+a_1} u(n)$$

Total Solution $y(n) = y_h(n) + y_p(n)$

$$= C_1 (-a_1)^n + \frac{1}{1+a_1} u(n)$$

Find C_1 $\quad y(0) = C_1 + \frac{1}{1+a_1}$

From Equation $y(0) + a_1 y(-1) = x(0) = u(0) = 1$

$y(0) = 1$ $\quad$ [$\because$ System is relaxed $y(-1) = 0$]

$$C_1 + \frac{1}{1+a_1} = 1$$

$$C_1 = 1 - \frac{1}{1+a_1} = \frac{a_1}{1+a_1}$$

$$y(n) = \frac{a_1}{1+a_1}(-a_1)^n + \frac{1}{1+a_1}u(n)$$

Determine the Homogeneous Solution of the System described by the homogeneous Second-order difference equation

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$$

$$y(-1) = 1, \quad y(-2) = 2$$

Homogeneous Solution is obtained ~~by~~ ~~x(n)~~ in the absence of input signal

$$y(n) - 3y(n-1) - 4y(n-2) = 0$$

$$y_h(n) = \lambda^n$$

$$\lambda^n - 3\lambda^{n-1} - 4\lambda^{n-2} = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda+1)(\lambda-4) = 0$$

$$\lambda = -1, 4.$$

$$y_h(n) = C_1(-1)^n + C_2 4^n$$

$$y(0) = C_1 + C_2$$

$$y(1) = -C_1 + 4C_2$$

From difference Equation

$$y(0) - 3y(-1) - 4y(-2) = 0$$

$$y(1) - 3y(0) - 4y(-1) = 0$$

$$y(0) = 3y(-1) + 4y(-2) = 3(1) + 4(2) = 11$$

$$y(1) = 3y(0) + 4y(-1) = 3(11) + 4(1) = 37$$

$$c_1 + c_2 = 11$$

$$-c_1 + 4c_2 = 37$$

By solving $5c_2 = 38$

$$c_2 = 38/5$$

$$c_1 = 11 - 38/5 = \frac{17}{5}$$

$$y_h(n) = \frac{17}{5}(-1)^n + \frac{38}{5}(4)^n \quad n \geq 0$$

Determine the response $y(n)$ $n \geq 0$ of the system described by the 2nd order difference equation

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$$

when the input sequence $x(n) = n4^n u(n)$

Homogeneous Solution

$$y(n) - 3y(n-1) - 4y(n-2) = 0$$

$$\lambda^n - 3\lambda^{n-1} - 4\lambda^{n-2} = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$\lambda = -1, 4$$

$$y_h(n) = C_1(-1)^n + C_2 4^n$$

particular Solution: It is of the form of the input

$$y_p(n) = Kn4^n u(n)$$

However, $y_p(n)$ is already contained in the homogeneous solution, so that this particular solution is redundant. Instead, we select the particular solution to be linearly independent of the terms contained in the homogeneous solution. In fact we treat this situation in the same manner as we have already treated multiple roots in the characteristic equation.

$$y_p(n) = Kn4^n u(n)$$

$$Kn4^n u(n) - K3(n-1)4^{n-1}u(n-1) - 4K(n-2)4^{n-2}u(n-2) =$$

$$4^n u(n) + 2\,4^{n-1}u(n-1)$$

$$Kn4^n - 3(n-1)K4^{n-1} - 4K(n-2)4^{n-2} = 4^n + 2.4^{n-1} \quad \text{for } n \geq 2$$

$$Kn - \frac{3(n-1)K}{4} - \frac{4(n-2)K}{16} = 1 + \frac{2}{4}$$

$$\frac{16nK - 12(n-1)K - 4(n-2)K}{16} = \frac{6}{4}$$

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$$\frac{12k + 8k}{16} = \frac{6}{4}$$

$$20k = 24$$

$$\boxed{k = \frac{6}{5}}$$

$$y(n) = y_h(n) + y_p(n)$$

$$= c_1(-1)^n + c_2 4^n + \frac{6}{5} n 4^n u(n)$$

$$y(0) = c_1 + c_2 + \frac{6}{5}(0) = c_1 + c_2 \quad \text{—— (1)}$$

$$y(1) = -c_1 + 4c_2 + \frac{6}{5} \cdot 1 \cdot 4^1 u(1)$$

$$= -c_1 + 4c_2 + \frac{24}{5} \quad \text{—— (2)}$$

From the difference equation

$$y(0) - 3y(-1) - 4y(-2) = x(0) + 2x(-1)$$

$$= 4^0 u(0) + 2 \cdot 4^{-1} u(-1)$$

Suppose System is relaxed

$$y(0) - 3(0) - 4(0) = 1$$

$$\Rightarrow y(0) = 1 \quad \text{—— (3)}$$

$$y(1) - 3y(0) - 4y(-1) = x(1) + 2x(0)$$

$$y(1) - 3y(0) = 4^1 u(1) + 2 \cdot 4^{1-1} u(1-1)$$

$$= 4 + 2 \cdot 4^0 (1) = 4 + 2 = 6$$

$$y(1) = 6 + 3(1) = 9 \quad \text{—— (4)}$$

From eq 1, 2, 3, & 4

$$c_1 + c_2 = 1$$

$$-c_1 + 4c_2 + \frac{24}{5} = 9$$

$$5c_2 = 10 - \frac{24}{5} = \frac{26}{5} \Leftrightarrow c_2 = \frac{26}{25}$$

$$c_1 = 1 - \frac{26}{25} = \frac{-1}{25}$$

$$\boxed{y(n) = -\frac{1}{25}(-1)^n + \frac{26}{25} 4^n + \frac{6}{5} n \cdot 4^n u(n)}$$

Determine the Impulse response $h(n)$ for the System described by the Second-order difference equation

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$$

Homogeneous Solution

$$y(n) - 3y(n-1) - 4y(n-2) = 0$$

$$y_h(n) = \lambda^n$$

$$\lambda^n - 3\lambda^{n-1} - 4\lambda^{n-2} = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$\lambda = -1, 4$$

$$y_h(n) = c_1(-1)^n + c_2 4^n$$

particular Solution

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$$

particular Solution takes the form $y_p(n) = K\delta(n)$

$$K\delta(n) - 3K\delta(n-1) - 4K\delta(n-2) = \delta(n) + 2\delta(n-1)$$

This equation does not hold for any value of n. Since each's term exist at different locations. This equation does not hold for any value of n.

Hence ~~K = 0~~ there is no particular Solution for the System with input signal as Impulse.

$$y_p(n) = 0.$$

Total Solution

$$y(n) = y_h(n) + y_p(n)$$

$$= c_1(-1)^n + c_2 4^n + 0$$

$$= c_1(-1)^n + c_2 4^n \quad n \geq 0$$

$$y(1) = -c_1 + 4c_2$$

$$y(0) = c_1 + c_2$$

$$y(0) - 3y(-1) - 4y(-2) = \delta(0) + 2\delta(-1)$$ (∵ Assume that System is relaxed)

$$y(0) = \delta(0) = 1$$

Illy $y(1) - 3y(0) - 4y(-1) = \delta(1) + 2\delta(0) = 2$

$$y(1) - 3(1) = 2 \Rightarrow y(1) = 5$$

$\therefore \quad c_1 + c_2 = 1$

$-c_1 + 4c_2 = 5$

$5c_2 = 6 \qquad\qquad c_1 = 1 - c_2$

$c_2 = \frac{6}{5} \qquad\qquad = 1 - \frac{6}{5} = \frac{-1}{5}$

$$\boxed{y(n) = \frac{-1}{5}(-1)^n + \frac{6}{5}(4)^n \quad n \geq 0}$$

Determine the Solution of System described by LCCDE

$y(n) - y(n-1) = x(n) + x(n-1)$

with input $x(n) = u(n)$

$y(n) = y_h(n) + y_p(n)$

Homogeneous Solution

$y_h(n) = \lambda^n$

$\lambda^n - \lambda^{n-1} = 0$

$\lambda - 1 = 0$

$\Rightarrow \boxed{\lambda = 1}$

$y_h(n) = c_1 (1)^n = c_1$

Particular Solution

Assume that $y_p(n) = K\, u(n)$

$K u(n) - K u(n-1) = u(n) + u(n-1)$

for $n \geq 1$

$K - K = 2$

Hence $y_p(n) = K\, u(n)$ is not a correct Solution

Assume $y_p(n) = K n u(n)$

$K n u(n) - K(n-1) u(n-1) = u(n) + u(n-1)$

for $n \geq 1$

$Kn - K(n-1) = 1 + 1$

$\boxed{K = 2}$

$\boxed{y_p(n) = 2 n u(n)}$

$y(n) = y_h(n) + y_p(n)$

Transform techniques are an important tool in the analysis of signals and linear time invariant systems. The z-transform plays the same role in the analysis of discrete time signals and LTI systems as the Laplace transform does in the analysis of continuous time signals and LTI systems.

In the z-domain the convolution of two time domain signals is equivalent to multiplications of their corresponding z-transforms this property greatly simplifies the analysis of the response of an LTI system to various signals. In addition, the z-transform provides us with a means of characterizing an LTI system, and its response to various signals, by its pole-zero locations.

*** DEFINATION OF THE Z-TRANSFORM :-**

The z-transform of a discrete time signal $x(n)$ is defined as the power series.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \quad \text{--- (1)}$$

where z is a complex variable. The relation is sometimes called the direct z-transform because it transforms the time domain signals $x(n)$ into its complex plane representation $X(z)$.

In the polar form z can be expressed as $z = r \cdot e^{j\omega}$ --- (2)

where r is the radius of the circle $X(r \cdot e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\omega n}$ --- (3)

→ For $r=1$, the expression (3) reduces to fourier transform of $x(n)$. That is, the z-transforms evaluated on the unit circle corresponds to the F.T.

→ The expression in eq (1) is refferred to as two-sided z-transforms.

→ If $x(n)$ is a causal sequence, i.e $x(n)=0$ for $n<0$ then the z-transforms

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}.$$

This above expression is called as a one-sided z-transform.

* **REGION OF CONVERGENCE (ROC):-**

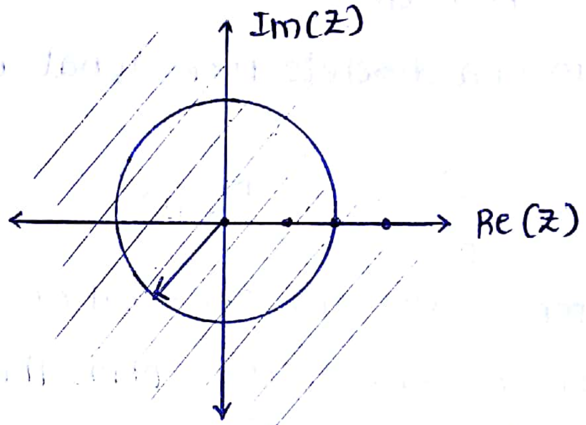
From definition, z-transform is an infinite power series, it exists only for these values of z for which this series converges. The region of convergence (ROC) of $X(z)$ is the set of all values of 'z' for which $X(z)$ attains a finite value.

* **SIGNIFICANCE:-**

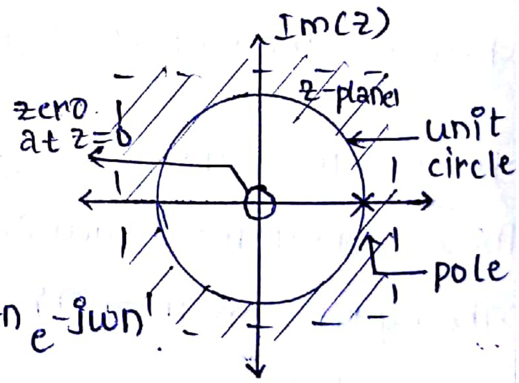
→ ROC gives an idea about values of z for which z-transform can be calculated.

→ ROC can be used to determine causality of the system.

→ ROC can be used to determine stability of the system.



The z-transforms method is used to analyse discrete systems for finding the transfer function, stability of digital network realisations of the system.



* **REGION OF CONVERGENCE :-**

$$X(z)\Big|_{z=re^{j\omega}} = X(r\cdot e^{j\omega})$$

$$= \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\omega n}$$

Which is the fourier transform of the modified sequence $[x(n)r^{-n}]$

If $r=1$, i.e $|z|=1$, $X(z)$ reduces to its fourier transform.

The series of the above equation converges if $x(n)r^{-n}$ is absolutely summable, i.e.

$$\sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \infty$$

→ If the output signal magnitude of the digital signal system, $x(n)$, is to be finite signal system, then the magnitude of its z-transform, $x(z)$ must be finite.

→ The set of z-values in the z-plane for which the magnitude of $x(z)$ is finite is called the ROC.

→ That is, convergence of $\sum_{n=0}^{\infty} x(n) r^{-n}$ generates the convergence of $x(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$. where $x(z)$ is a function of z^{-n}. Therefore, the condition for $x(z)$ to be finite is $|z| > 1$.

→ In other words, the ROC for $x(z)$ is the area outside the unit circle in the z-plane.

→ The ROC of a rational z-transform is bounded by the location of its poles. For example, the z-transform of the unit step response $u(n)$ is $x(z) = z/z-1$ which has a zero at $z=0$ and a pole at $z=1$ and the ROC of $|z|>1$ and extending all the way to ∞ as shown.

* IMPORTANT PROPERTIES OF THE ROC FOR THE Z-TRANSFORM:

1, $x(z)$ converges uniformly if and only if the ROC of the z-transform $x(z)$ of the sequence includes the unit circle.

The ROC of $x(z)$ consists of a ring in the z-plane centered about the origin. That is, the ROC of the z-transform of $x(n)$ has values of z for which $x(n) r^{-n}$ is absolutely summable.

$$\sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \alpha$$

2, The ROC does not contain any poles.

3, when $x(n)$ is of finite energy duration, then the ROC is the entire z-plane except possibly $z=0$ and/or $z=\infty$

4, If $x(n)$ is a right-sided sequence, the ROC is entire z plane except at $z=0$.

5, If $x(n)$ is a left sided sequence the ROC is entire z-plane.

at $z=\infty$.

6, Two-sided, entire z-plane except $z=0$ and $z=\alpha$

7, causal, $|z|>r_2$, it is infinite sequence.

8, anti causal, $|z|<r_1$, it is infinite sequence.

*Z-TRANSFORM AND ROC:

i, causal finite sequence (right handed side sequence)

$$x(n) = \{\underset{\uparrow}{1}, 3, -5\}$$

$$\rightarrow z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$= \sum_{n=0}^{2} x(n) z^{-n}.$$

$$\rightarrow x(z) = x(0) + x(1) z^{-1} + x(2) z^{-2}$$

$$= 1 + 3z^{-1} + (-5) z^{-2}$$

$$= 1 + \frac{3}{z} - \frac{5}{z^2}$$

ROC is entire z-plane except $z=0$.

ii, Non causal finite sequence - (left handed side sequence)

$$x(n) = \{3, -2, \underset{\uparrow}{1}\}$$

$$z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$= \sum_{n=-2}^{0} x(n) z^{-n}.$$

$$= x(0) + x(-1) z^{1} + x(-2) z^{2}$$

$$= 1 - 2z + 3z^2.$$

∴ ROC is entire z-plane except $z=\infty$.

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3, Two - sided finite sequence :

$$x(n) = \{1, 3, \underset{\uparrow}{5}, -1, 2\}$$
$$\quad -2\ -1\ 0\ \ 1\ \ 2$$

$$\rightarrow Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-2}^{2} x(n) z^{-n}$$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} + x(-1) z^{-1}$$

$$+ x(-2) z^{2}$$

$$= 5 - z^{-1} + 2z^{-2} + 3z + z^{2}$$

$$= z^{2} + 3z + 5 - z^{-1} + 2z^{-2}$$

→ ROC is entire z-plane except at $z=0$ and $z=\infty$.

*right handed side infinite sequence :

$$x(n) = a^n u(n)$$

$$Z[x(n)] = z/z-a, \quad |z|>|a|$$

→ROC is exterior to the circle with radius 'a'.

*left hand side infinite sequence :

$$x(n) = -\,n\,u(n-1)$$

$$\rightarrow Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [-b^n u(n-1)] z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} b^n u(n-1) z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} b^n z^{-n}$$

replace n with $-n$.

$$= -\sum_{n=1}^{\infty} b^{-n} z^{n}$$

$$= -\sum_{n=1}^{\infty} (b^{-1}z)^n.$$

$$= -\left[b^{-1}z + (b^{-1}z)^2 + (b^{-1}z)^3 + \cdots + \infty\right]$$

[W.K.T $A + A^2 + A^3 + \cdots = A(1 + A + A^2 + \cdots)$

$$= \frac{A}{1-A} \Big]$$

$$= -\frac{b^{-1}z}{1-b^{-1}z}.$$

$$= -\left[\frac{z/b}{1-z/b}\right], \left|\frac{z}{b}\right| < 1$$

$$= -\left[\frac{z}{b-z}\right], |z| < |b|$$

$$= \frac{z}{z-b}, |z| < |b|.$$

→ z-transforms of $a^n u(n)$ and $-a^n u(-n-1)$ are same, i.e $\frac{1}{1-az^{-1}}$, but their ROC are different.

Two sided infinite sequence :

$$x(n) = a^n u(n) + b^n u(-n-1).$$

$$x(z) = \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n}.$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n + \sum_{1}^{\infty} (b^{-1}z)^n.$$

$$= \frac{1}{1-az^{-1}} + \frac{1}{1-bz^{-1}}, \quad |z>a|$$

ROC : $|z| > |a|$ and $|z| < |b|$.

i.e, $|a| < |z| < |b|$.

*PROPERTIES OF ROC

i, $X(z)$ converges uniformly if and only if the ROC of the z-transform $X(z)$ of the sequence includes the unit circle.

The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin. That is, the ROC of the z-transform of $x(n)$ has value of z for which $x(n) r^{-n}$ is absolutely summable.

$$\sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \infty$$

2, The ROC does not contain any poles. $X(z) = \frac{(z-z_1)(z-z_2)\cdots(z-z_m)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$

→ poles are the locations where $X(z) = \infty$. Hence poles doesnot not lie in the ROC of $X(z)$.

→ zeros are the locations for which $X(z)=0$; which is finite. Hence zeros can be lie in the ROC of $X(z)$.

3, If $x(n)$ is right sided sequence, then the ROC is entire z-plane except at $z=0$, i.e causal sequence.

4, If $x(n)$ is non-causal finite sequence (left hand side sequence) then ROC is entire z-plane except $z=\infty$.

5, If $x(n)$ is a finite duration, two sided sequence the ROC is entire z-plane except at $z=0$ and $z=\infty$.

6, If $x(n)$ is an infinite duration, two sided sequence the ROC will consist of a ring in the z-plane, bounded on the interior and enter in by a pole, n of containing any poles.

7, The ROC of a LTI stable system contains the unit circle.

8, The ROC must be a connected region.

Find the z-transform of the following finite duration signals.

(a) $x(n) = \{3, 1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

Taking z-transform, we get, $X(z) = 3z^3 + z^2 + 2z + 5 + 7z^{-1} + z^{-3}$

ROC : entire z-plane except $z=0$ and $z=\infty$

(b) $x(n) = \delta(n)$

Sol $Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

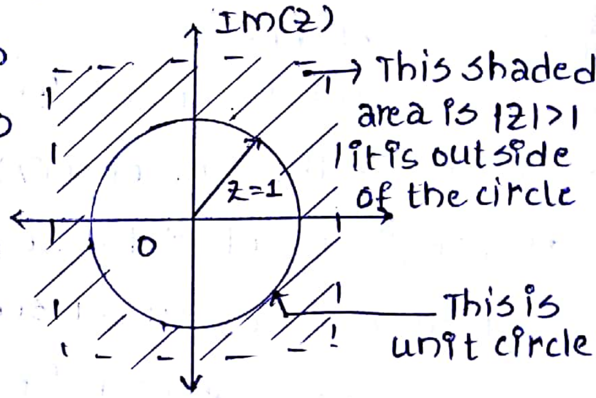
$$= \sum_{n=-\infty}^{\infty} \delta(n) z^{-n}$$

$\left[\begin{array}{l} \delta(n) = 1, n = 0 \\ = 0, n \neq 0 \end{array} \right]$ $\quad \sum_{n=-\infty}^{\infty} \delta(0) z^{0} = 1$

$\therefore$ ROC : entire z-plane.

② Determine the z-transform of unit step sequence, $u(n)$

Sol unit step sequence, $u(n) = 1$, for $n \geq 0$
$\qquad 0$, for $n < 0$



$$\therefore X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} 1 \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} (z^{-1})^{n} = 1 + z^{-1} + (z^{-1})^{2} + (z^{-1})^{3} + \cdots$$

$$\therefore X(z) = 1/1 - z^{-1}, \quad |z^{-1}| < 1 \text{ (or) } z > 1.$$

③ Determine the z-transform of the signal $x(n) = (1/2)^{n} u(n)$.

Sol The signal $x(n)$ consists of an infinite number of non zero values.

$$x(n) = \{1, (1/2), (1/2)^{2}, (1/2)^{3}, \cdots\}$$

The z-transform of $x(n)$ is the infinite power series

$$x(z) = 1 + \frac{1}{2} z^{-1} + \left(\frac{1}{2}\right)^{2} z^{-2} + \left(\frac{1}{2}\right)^{3} z^{-3} + \cdots$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} z^{-n}$$

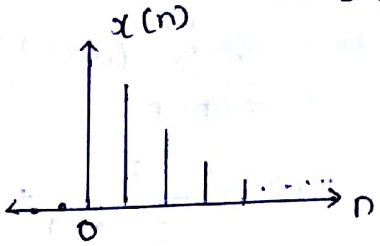
$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^{n}$$

This is an infinite geometric series we recall that.

$1 + A + A^2 + A^3 + \cdots = \frac{1}{1-A}$ if $|A| < 1$

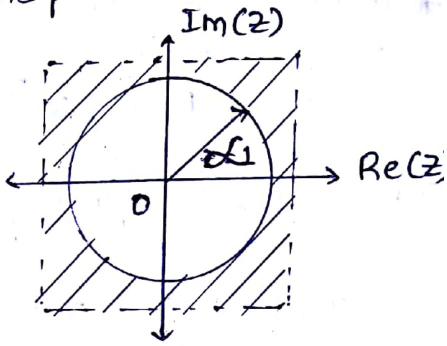
consequently, for $\left|\frac{1}{2} z^{-1}\right| < 1$ (or) equivalently for $|z| > \frac{1}{2}$, $X(z)$ converges to $X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}$, ROC : $|z| > \frac{1}{2}$.

④ Determine the z-transform of the signal $x(n) = \alpha^n u(n) = \alpha^n ; n \geq 0$
$= 0 ; n < 0$

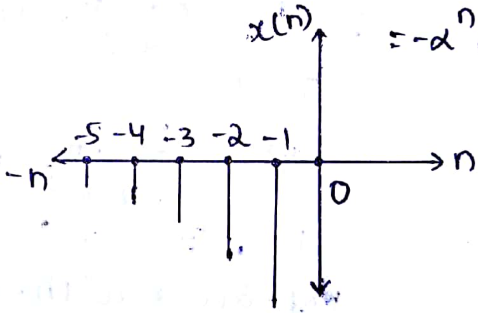


Sol
$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n}$$
$$= \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

If $\alpha z^{-1} < 1$ (or) equivalently $|z| > |\alpha|$, this power series converges to $\frac{1}{1 - \alpha z^{-1}}$. The ROC is the exterior of a circle having radius α.



⑤ Determine the z-transform of the signal $x(n) = -\alpha^n u(-n-1) = 0, n \geq 0$
$= -\alpha^n, n \leq -1$

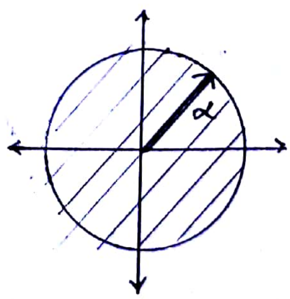


Sol from defination, we have.
$$X(z) = \sum_{n=-\infty}^{-1} (-\alpha)^n z^{-n}$$
$$= -\sum_{l=1}^{\infty} (\alpha^{-1} z)^l$$

where $l = -n$. using the formula $A + A^2 + A^3 + \cdots = A(1 + A + A^2 + \cdots)$
$$= \frac{A}{1-A}$$

when $|A| < 1$ gives, $X(z) = \frac{-\alpha^{-1} z}{1 - \alpha^{-1} z} = \frac{1}{1 - \alpha z^{-1}}$

provided that $|\alpha^{-1} z| < 1$ (or) equivalently $|z| < |\alpha|$



$x(n) = -\alpha^n u(-n-1) \xleftrightarrow{z} X(z) = \frac{1}{1 - \alpha z^{-1}}$

→ z transforms of both are same, i.e, $1/1 - \alpha z^{-1}$ but their ROC's are different.

(6) Determine the z transform of the signal $x(n) = \alpha^n u(n) + b^n u(-n-1)$

sol: Given $x(n) = \alpha^n u(n) + b^n u(-n-1)$

apply z-transforms

$$x(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n}.$$

$$= \sum_{n=0}^{\infty} (\alpha z^{-1})^n + \sum_{l=1}^{\infty} (b^{-1} z)^l$$

$$x(z) = \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-b z^{-1}}, \quad ROC : |z| > |a| \ \& \ |z| < |b|$$

i.e, $|a| < |z| < |b|$.

* if $x(n) = 2^n u(n) + 3^n u(-n-1)$.

ROC : $2 < |z| < 3$; poles at $z = 2 \ \& \ 3$, zeros is at $z = 0$.

(7) $x(n) = \left(\frac{1}{2}\right)^n u(-n)$

sol:

$$x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-x}.$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} u(-n)$$

$$= \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^n z^{-n}$$

replace n with -n

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{-n} z^{n}.$$

$$= \sum_{n=0}^{\infty} (2z)^n.$$

$$= 1 + 2z + (2z)^2 + \cdots + \alpha.$$

$$= \frac{1}{1-2z} \ ; \ 2z < 1$$

$$z < 1/2.$$

*PROPERTIES OF THE z-TRANSFORM :

The z-transform is a very powerful tool for the study of discrete time signals and systems.

The power of this transforms is a consequence of some very important properties that the transforms possesses.

Inthe treatment that follows, it should be remembered that when we combine several z-transforms, the ROC of the overall transform is the intersection of the ROC of the individual transforms.

1, **Linearity**:- If $x_1(n) \xleftrightarrow{z} X_1(z)$, $x_2(n) \xleftrightarrow{z} x_2(z)$.

then $x(n) = a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{z} X(z) = a_1 X_1(z) + a_2 X_2(z)$.

for any constansts a_1 and a_2.

The linearity property can easily be generalized for an arbitrary number of signals.

Basically, it implies that the z-t/f of a linear combination of signals is the same linear combination of their z-transforms. Thus the linearity property helps us to find the z-transforms of a signal by expressing the signal as a sum of elementary signals, for each of which, the z-t/f is already known.

1 Determine the z-transform of the signal. $x(n) = \delta(n+1) + 3\delta(n) + 6\delta(n-3) - \delta(n-4)$

Sol from the linearity property, we have $X(z) = z\{\delta(n+1)\} + 3z\{\delta(n)\}$
$+ 6z\{\delta(n-3)\} - z\{\delta(n-4)\}$.

using the z-transform pairs, we obtain.

$$X(z) = z + 3 + 6z^{-3} - z^{-4}.$$

therefore, $x(n) = \{1, \underset{\uparrow}{3}, 0, 0, 6, -1\}$

The ROC is the entire z-plane except $z=0$ and $z=\infty$.

→ The Result can be obtained by using the defination of the transforms

② Find the z-transform of $x(n) = \cos\omega_0 n$ for $n \geq 0$.

Sol $x(n) = \cos\omega_0 n = 1/2\left[e^{j\omega_0 n} + e^{-j\omega_0 n}\right]$

using the transform, for $n \geq 0$, $z[a^n] = \frac{1}{1-az^{-1}}$, $|z|>|a|$

therefore, for $n \geq 0$

$$z\left[(e^{j\omega_0})^n\right] = \frac{1}{1-e^{j\omega_0}z^{-1}}, \quad |z|>1$$

parallely for $n \geq 0$,

$$z\left[(e^{-j\omega_0})^n\right] = \frac{1}{1-e^{-j\omega_0}z^{-1}}, \quad |z|>1$$

therefore, $X(z) = z[\cos\omega_0 n] = z\left[\frac{1}{2}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right)\right]$

$$= \frac{\frac{1}{2}}{1-e^{j\omega_0}z^{-1}} + \frac{\frac{1}{2}}{1-e^{-j\omega_0}z^{-1}}$$

$$= \frac{1-\frac{1}{2}\left[e^{j\omega_0} + e^{-j\omega_0}\right]z^{-1}}{(1-e^{j\omega_0}z^{-1})(1-e^{-j\omega_0}z^{-1})}$$

$$= \frac{1-z^{-1}\cos\omega_0}{(1-e^{j\omega_0}z^{-1})(1-e^{-j\omega_0}z^{-1})}$$

$$= \frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0 + z^{-2}}.$$

$$= \frac{z(z-\cos\omega_0)}{z^2-2z\cos\omega_0+1}; \quad |z|>1$$

similarly, we can find $z[\sin\omega_0 n]$ using the property of linearity

i.e, $z[\sin\omega_0 n] = z\left[\frac{1}{2j}\left[e^{j\omega_0 n} - e^{-j\omega_0 n}\right]\right] = \frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$

$= z\sin\omega_0 / z^2 - 2z\cos\omega_0 + 1 \quad \therefore |z|>1.$

*TIME SHIFTING :-

If $x(n) \xleftrightarrow{z} X(z)$ then $x(n-k) \xleftrightarrow{z} z^{-k} X(z)$ the ROC of $z^{-k}X(z)$ is the same as that of $X(z)$ except for $z=0$ if $k>0$ and $z=\infty$ if $k<0$

→ The properties of linearity and time shifting are the key features that make the z-transform extremly useful for the analysis of discrete - time LTI systems.

Proof:-

$$z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$z[x(n-k)] = \sum_{n=-\infty}^{\infty} [x(n-k)] z^{-n}.$$

replace n-k with m ⇒

$n = m+k$ [k is a constant]

$$= \sum_{m=-\infty}^{\infty} x(m) z^{-(m+k)}$$

$$= \sum_{m=-\infty}^{\infty} x(m) z^{-m} \cdot z^{-k}.$$

$$= z^{-k} \sum_{m=-\infty}^{\infty} x(m) z^{-m}.$$

$$= z^{-k} X(z)$$

(1) By appling the time shifting property, determine the z-transform of the signal. $X(z) = z^{-1}/1-3z^{-1}$.

Sol $X(z) = \dfrac{z^{-1}}{1-3z^{-1}} = z^{-1} X_1(z)$, where $X_1(z) = \dfrac{1}{1-3z^{-1}}$

Here, from the time shifting property, we have $k=1$ &

$x_1(n) = (3)^n u(n)$

Hence $x(n) = (3)^{n-1} u(n-1)$.

② By applying the time shifting property, determine the z-transform of the signals $x_2(n)$ and $x_3(n)$ from the z-transform of $x_1(n)$.

Sol $x_1(n)=\{1,2,5,7,0,1\}$, $x_2(n)=\{1,2,\underset{\uparrow}{5},7,0,1\}$, $x_3(n)=\{0,0,1,2,5,7,0,1\}$.

it can easily be seen that $x_2(n)=x_1(n+2)$, $x_3(n)=x_1(n-2)$

$\therefore x_2(z)=z^2x_1(z)=z^2+2z+5+7z^{-1}+z^{-3}$

$\therefore x_3(z)=z^{-2}x_1(z)=z^{-2}+2z^{-3}+5z^{-4}+7z^{-5}+z^{-7}$.

*Time reversal:-

If $x(n)\xleftrightarrow{z}x(z)$ ROC: $r_1<|z|<r_2$ then $x(-n)\xleftrightarrow{z}x(z^{-1})$

ROC: $\frac{1}{r_2}<|z|<\frac{1}{r_1}$

Proof:-

$$z\{x(-n)\}=\sum_{n=-\infty}^{\infty}x(-n)z^{-n}. \Rightarrow \sum_{l=-\infty}^{\infty}x(l)\,(z^{-1})^{l}.=X(z^{-1})$$

when the change of variable $l=-n$ is made. The ROC of $X(z^{-1})$ is

$r_1<|z^{-1}|<r_2$ (or) equivalently $\frac{1}{r_1}<|z|<\frac{1}{r_2}$.

① Determine the z-transform of the signal $x(n)=u(-n)$.

Sol we know that $u(n)\xleftrightarrow{z}\frac{1}{1-z^{-1}}$ $\therefore$ ROC: $|z|>1$

by using time reversal, we easily obtain, $u(-n)\xleftrightarrow{z}\frac{1}{1-z}$, ROC: $|z|<1$

*Differentiation in the z-domain.:

If $x(n)\xleftrightarrow{z}x(z)$ then $nx(n)\xleftrightarrow{z}-z\frac{d}{dz}x(z)$

proof:- $z[x(n)]=x(z)=\sum_{n=-\infty}^{\infty}x(n)z^{-n}$.

differentiating on both sides w.r.t.z

$$\frac{d}{dz}(x(z))=\frac{d}{dz}\sum_{n=-\infty}^{\infty}x(n)z^{-n}$$

$$=\sum_{n=-\infty}^{\infty}x(n)\left[-n\cdot z^{-n-1}\right]$$

$$=-\sum_{n=-\infty}^{\infty}n\,x(n)z^{-n}\cdot\frac{1}{z}$$

$$-z \cdot \frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} [n \cdot x(n)] z^{-n}.$$

$$\therefore Z[n \cdot x(n)] = -z \cdot \frac{d}{dz} X(z).$$

① Determine the z-transform of the signal $x(n) = n a^n u(n)$.

Sol The signal $x(n)$ can be expressed as $n, x_1(n)$, where $x_1(n) = a^n u(n)$

$$x_1(n) = a^n u(n) \xleftrightarrow{z} X_1(z) = \frac{1}{1 - az^{-1}}$$

from differentiation property.

$$n a^n u(n) \xleftrightarrow{z} X(z) = -z \cdot \frac{d}{dz} X_1(z)$$

$$= -z \cdot \frac{d}{dz} \left[\frac{1}{1 - az^{-1}} \right]$$

$$= \frac{(1 - az^{-1})0 - 1(-a)(-1)(z^{-2})}{(1 - az^{-1})^2}$$

$$= \frac{(-z)^{-2} \cdot (-z)}{(1 - az^{-1})^2}$$

$$= \frac{z^{-1}}{(1 - az^{-1})^2} \qquad \therefore \text{ROC} : |z| > 1.$$

② Determine the signal $x(n)$ whose z-transform is given by

$$X(z) = \log(1 + az^{-1}), \ |z| > |a|$$

Sol by taking the first derivative of $X(z)$, we obtain

$$\frac{d}{dz} X(z) = \frac{-a \cdot z^{-2}}{1 + az^{-1}}$$

thus

$$-z \cdot \frac{d}{dz} X(z) = a \cdot z^{-1} \left[\frac{1}{1 - (-a) z^{-1}} \right]; \quad |z| > a$$

The inverse z-t/f of the term in $(-a)^n$. The multiplication by z^{-1} implies a time delay by one sample (time shifting), which results in $(-a)^{n-1} u(n-1)$.

Finally, from the differentiation property we have.

$$n x(n) = a(-a)^{n-1} u(n-1)$$

$$x(n) = (-1)^{n+1} \frac{a^n}{n} u(n-1).$$

* **convolution of two sequence** :

If $x_1(n) \xleftrightarrow{z} X_1(z)$

$x_2(n) \xleftrightarrow{z} X_2(z)$ then $x(n) = x_1(n) * x_2(n) \xleftrightarrow{z} X_1(z) \cdot X_2(z)$

proof:- The convolution of $x_1(n)$ and $x_2(n)$ is defined as.

$$x(n) = x_1(n) * x_2(n)$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k)$$

the transform of $x(n)$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k) x_2(n-k) \right] z^{-n}.$$

$$X(z) = \sum_{k=-\infty}^{\infty} x_1(k) \left[\sum_{n=-\infty}^{\infty} x_2(n-k) z^{-n} \right]$$

$$= X_2(z) \sum_{k=-\infty}^{\infty} x_1(k) z^{-k}.$$

$$= X_2(z) X_1(z)$$

① compute the convolution $x(n)$ of the signals $x_1(n) = 1, \; 0 \le x \le 5$

$= 0$, elsewhere.

sol $X_1(z) = 1 - 2z^{-1} + z^{-2}$, $X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$

carryout the multiplication of $X_1(z)$ and $X_2(z)$

thus $X(z) = X_1(z) \cdot X_2(z)$

$$= 1 - z^{-1} - z^{-6} + z^{-7}$$

hence $x(n) = \{1, -1, 0, 0, 0, 0, -1, 1\}$

*The system function of a linear time - invariant system :

The output of a relaxed Linear time - invariant system to an input sequence $x(n)$ can be obtained by computing the convolution of $x(n)$ with the unit sample response (impulse) the system.

The convolution property allows us to express this relationship in the z-domain as $Y(z) = H(z) \cdot X(z)$.

where $Y(z)$ is the z-t/f of the output sequence $y(n)$, $X(z)$ is the z-transform of the i/p sequence $x(n)$ and $H(z)$ is the z-transform of the unit sample response $h(n)$.

If we know $h(n)$ and $x(n)$, we can determine their corresponding z-transforms $H(z)$ and $X(z)$, multiply them to obtain $Y(z)$ and determine $y(n)$ by evaluating the inverse z-t/f of $Y(z)$.

-> Alternatively, if we know $x(n)$ and we observe the o/p $y(n)$ of the system, we can determine the unit sample response by first solving for $H(z)$ from the relation $H(z) = \frac{Y(z)}{X(z)}$ → ①

and then evaluating the inverse z-transform of $H(z)$

since $H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$.

-> $H(z)$ represents the z- domain characterization of a system, where the corresponding time - domain characterized in the system .

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The convolution property as one of the most powerful properties of the z-transform because it converts the convolution of two signals (time domain) to multiplication of their transforms.

→ computation of the convolution of two signals, using the z-transform requires the following steps:

1, compute the z-transforms of the signals to be converted.

$x_1(z) = z\{x_1(n)\}$, $x_2(z) = z\{x_2(n)\}$ (time domain)

2, multiply the two z-transforms $x(z) = x_1(z) + x_2(z)$ (z-domain).

3, Find the inverse z-transforms of $x(z)$, $x(n) = z^{-1}\{x(z)\}$ "

This procedure is computationally easier than the direct evalution of the convolution summation.

***Convolution property:**

If $x(n) \longleftrightarrow x(z)$ and $h(n) \overset{z}{\longleftrightarrow} H(z)$ then $x(n) * h(n) \longleftrightarrow x(z)\cdot H(z)$

$$z[x(n)] = x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$\therefore z[x(n) * h(n)] = \sum_{n=-\infty}^{\infty} [x(n) * h(n)] z^{-n}.$$

$$[x(n) * h(n)] = \sum_{k=-\infty}^{\infty} x(k) h(n-k) \quad z^{-k-n+k}.$$

$$z[x(n) * h(n)] = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k) h(n-k)\right] z^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k) h(n-k)\right] z^{-k}. z^{-(n-k)}.$$

$$= \sum_{k=-\infty}^{\infty} [x(k) z^{-k}]. \sum_{n=-\infty}^{\infty} h(n-k) z^{-(n-k)}$$

$$= x(z)\cdot H(z).$$

* **The system function of a linear time-invariant system:**

The output of a relaxed linear time-invariant system to an input sequence $x(n)$ can be obtained by computing the convolution of $x(n)$ with the unit sample response (input x) of the system.

The convolution property allows us to express this relationship in the z-domain is $Y(z) = H(z) \cdot X(z)$.

where $Y(z)$ is the z-t/f of the output sequence $y(n)$, $X(z)$ is the z-transform of the input sequence $x(n)$ and $H(z)$ is the z-transform of the unit sample response $h(n)$.

If we know $h(n)$ and $x(n)$, we can determine their corresponding z-transforms $H(z)$ and $X(z)$, multiply them to obtain $Y(z)$ and determine $y(n)$ by evaluating the inverse z-t/f of $Y(z)$

→ Alternatively, if we know $x(n)$ and we observe the o/p $y(n)$ of the system, we can determine the unit sample response by first solving for $H(z)$. from the relation $H(z) = Y(z)/X(z)$ —①

and then evaluating the inverse z-transform of $H(z)$

since $$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}.$$

→ $H(z)$ represents the z-domain characterization of a system, where as $h(n)$ is the corresponding time-domain characterization of the system.

→ In others words, $H(z)$ and $h(n)$ are equivalent descriptions of a system in the two domains. The transform $H(z)$ is called the system functions

→ The relation $H(z) = Y(z)/X(z)$ is particularly useful in the obtaining $H(z)$ when the system is described by a linear constant-coefficient difference equation of the form.

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \quad -②$$

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In this case the system function can be determined directly from ② by computing the z-transform of both sides of eq ②. Thus by applying the time shifting property, we obtain

$$Y(z) = -\sum_{k=1}^{N} a_k Y(z) z^{-k} + \sum_{k=0}^{M} b_k X(z) z^{-k}.$$

$$Y(z) = \left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] = X(z) \sum_{k=0}^{M} b_k z^{-k}.$$

$$\frac{Y(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k} \Big/ 1 + \sum_{k=1}^{N} a_k z^{-k}.$$

or equivalently,

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} \Big/ 1 + \sum_{k=1}^{N} a_k z^{-k}.$$

Therefore, a linear time invariant system described by a constant co-efficient difference equation has a rational system function.

This is the general form for the system function of a system described by a linear constanst co-efficient difference equation. From this general form we obtain two important special forms. First, if $a_k = 0$ for $1 \le k \le N$, eq (3) reduces to.

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}.$$

$$= \frac{1}{z^m} \sum_{k=0}^{M} b_k z^{m-k}.$$

In this case, $H(z)$ contains M zeros, whose values are determined by the system parameters $\{b_k\}$, and an m^{th} order pole at the origin $z=0$, since the system contains only trivial poles (at $z=0$) and m non-trivial zeros, it is called an all zero system. clearly, such a system has a finite-duration impulse response (FIR) and it is also called an FIR system (or) moving average system.

→ on the otherhand, if $b_K = 0$ for $1 \le K \le m$, the system function reduces to

$$H(z) = \frac{b_0}{1+\sum_{K=1}^{N} a_K z^{-N}}$$

$$= \frac{b_0 z^N}{\sum_{K=0}^{N} a_K z^{N-K}}, \quad a_0 \cong 1 \rightarrow (4)$$

In this case $H(z)$ consists of N poles, whose values are determined by the system parameters $\{a_K\}$ and an N^{th}-order zero at the origin $z=0$. we usually donot make reference to these trivial zeros. Consequently, the system function in eq (4) contains only non-trivial poles and the corresponding system is called an all-pole system.

→ Due to the presence of poles, the impulse response of such a system is infinite in duration, and hence it is an IIR system.

The general form of the system function by eq (3) contains both poles and zeros and hence the corresponding system is called a pole-zero system with N poles and m zeros. poles, zeros at $z=0$ and $z=\infty$ are implied but are not counted explicity. Due to the presence of poles, a pole-zero system is an IIR system.

(1) Determine the system function and the unit sample response of the system described by the difference equation.

$$y(n) = \frac{1}{2} y(n-1) + 2x(n).$$

(sol) By computing the z-transform of the difference equation, we obtain.

$$Y(z) = \frac{1}{2} z^{-1} Y(z) + 2X(z).$$

hence the system function is

$$\frac{Y(z)}{X(z)} = H(z) = \frac{2}{1-\frac{1}{2}z^{-1}}.$$

This system has a pole at $z = 1/2$ and a zero at the origin.

→ The unit sample response $h(n)$ is obtained by taking inverse z-t/f of the system function $H(z)$.

$$h(n) = 2\,(1/2)^n u(n).$$

→ Causality and stability :-

A causal linear time invariant system is one whose unit sample response $h(n)$ statisfies the condition.

$$h(n) = 0\,,\quad n<0.$$

→ A neccessary and sufficient condition for a linear time invariant system to be BIBO stable is

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

→ In turn, this conditions implies that $H(z)$ must contain the unit circle within its ROC.

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n}.$$

it follows that

$$|H(z)| \le \sum_{n=-\infty}^{\infty} |h(n) z^{-n}| = \sum_{n=-\infty}^{\infty} |h(n)|\,|z^{-n}|$$

when evaluated on the unit circle (i.e $|z|=1$)

$$|H(z)| \le \sum_{n=-\infty}^{\infty} |h(n)|$$

The partial fraction expansion is

$$\frac{H(z)}{z} = \frac{2}{z-3} + \frac{1}{z-0.5}$$

$$= \frac{2z}{z-3} + \frac{z}{z-0.5}$$

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* Draw the pole-zero plot for the system described by the differential equation.

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) - x(n-1).$$

Sol: apply z-transform

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = X(z) - z^{-1}X(z)$$

$$Y(z)\left[1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right] = X(z)\left[1 - z^{-1}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

$$= \frac{z^{-1}(z-1)}{z^{-2}\left[z^2 - \frac{3}{4}z + \frac{1}{8}\right]}$$

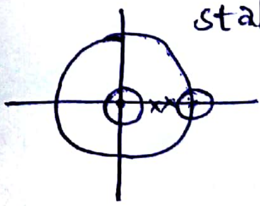
$$= \frac{8z(z-1)}{8z^2 - 6z + 1}$$

$$= \frac{8z(z-1)}{8z^2 - 8z + 2z + 1}$$

$$= \frac{8z(z-1)}{4z(2z-1) - 1(2z-1)}$$

$$= \frac{8z(z-1)}{(2z-1)(4z-1)}$$

stable.



the zeros are $z = 0,\ z = 1$

poles are $z = 1/4,\ z = 1/2$

② **Frequency response of first order system:**

The difference equation for a first-order system is given by

$$y(n) - ay(n-1) = x(n)$$

→ where $x(n)$ is the ~~output~~ in put and $y(n)$ is the output. Taking fourier transforms on both sides, for eq. (1.385) and assuming initial conditions to zero yield.

$$Y(e^{jw}) - ae^{-jw}Y(e^{jw}) = X(e^{jw})$$

$$Y(e^{jw})[1 - ae^{-jw}] = X(e^{jw})$$

→ the frequency response $\frac{Y(e^{jw})}{X(e^{jw})} = H(e^{jw}) = \frac{1}{1-ae^{-jw}}$

→ The impulse response $h(n) = F^{-1}[H(e^{jw})]$

$$= F^{-1}\left[\frac{1}{1-ae^{-jw}}\right]$$

$$= a^n u(n)$$

the frequency response of the first order system is given by

$$H(e^{jw}) = \frac{1}{1-ae^{jw}}$$

→ Now $H(e^{jw})$ is called the frequency response of the causal LTI system whose impulse response is $h(n)$.

$$H(e^{jw}) = \frac{1}{1 - a\cos w + aj\sin w}$$

the magnitude response

$$|H(e^{jw})| = 1\Big/\sqrt{(1-a\cos w)^2 + a^2\sin^2 w}$$

$$= 1\Big/\sqrt{1+a^2-2a\cos w}$$

The phase response $\angle H(e^{j\omega}) = -\tan^{-1} \frac{a \sin \omega}{1 - a\cos\omega}$

ω	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$
$\lvert H(e^{j\omega})\rvert$	5	1.402	0.78	0.6	0.6	0.55	0.6	0.78
$\angle H(e^{j\omega})$	0	-52.48°	-38.66°	-19.86°	-19.86°	0	19.86°	38.66°

5.7.4 Magnitude and Phase Spectrum

The magnitude response is the absolute value of a filter's complex frequency response. The phase response is the angle component of a filter's frequency response. For a linear time invariant system with a real-valued impulse response, the magnitude and phase functions possess symmetry properties which are detailed below. From the definition of z-transform, $H(e^{j\omega})$, a complex function of the real variable ω can be expressed as

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} h(n)\cos\omega n - j\sum_{n=-\infty}^{\infty} h(n)\sin\omega n$$

$$= H_R(e^{j\omega}) + jH_I(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{j\phi(\omega)}$$

$$= \sqrt{H_R^2(e^{j\omega}) + H_I^2(e^{j\omega})}\, e^{j\tan^{-1}[H_I(e^{j\omega})/H_R(e^{j\omega})]}$$

where $H_R(e^{j\omega})$ and $H_I(e^{j\omega})$ denote the real and imaginary components of $H(e^{j\omega})$.

Therefore, $\left|H(e^{j\omega})\right| = \sqrt{H_R^2(e^{j\omega}) + H_I^2(e^{j\omega})}$

$$\Phi(\omega) = \tan^{-1}\left[\frac{H_I(e^{j\omega})}{H_R(e^{j\omega})}\right]$$

Also, $\Phi(\omega)$ may be expressed as

$$\Phi(\omega) = \frac{1}{2j}\ln\left\{\frac{H(z)}{H(z^{-1})}\right\}\Big|_{z=e^{j\omega}}$$

The quantity $|H(e^{j\omega})|$ is called the magnitude function or ***magnitude spectrum*** and the quantity $\Phi(\omega)$ is called the phase function or ***phase spectrum***.

$$\left|H(e^{j\omega})\right|^2 = H(e^{j\omega})H^*(e^{j\omega})$$

$$= H(e^{j\omega})H(e^{-j\omega}) = H(z)H(z^{-1})\big|_{z=e^{j\omega}}$$

Here the function $H(z^{-1})$ has zeros and poles that are the reciprocal of the zeros and poles of $H(z)$. According to the correlation property for the z-transform, the function $H(z)H(z^{-1})$ is the z-transform of the autocorrelation of the unit sample response. Then it follows from the Wiener Khintchine theorem that $|H(z)|^2$ is the Fourier transform of the autocorrelation sequence of $h(n)$.

Example 5.42 *Obtain the frequency response of the first order system with difference equation $y(n)=x(n)+10y(n-1)$ with initial condition $y(-1)=0$ and sketch it. Comment about its stability.*

Solution The difference equation of the first order system is $y(n)=x(n)+10y(n-1)$

Taking z-transform, we get

$$Y(z) = X(z) + 10\,[z^{-1}Y(z) - y(-1)]$$

$$Y(z)[1-10z^{-1}] = X(z) \quad [\text{since } y(-1)=0]$$

Therefore, $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1}{1-10z^{-1}} = \dfrac{z}{z-10}$

Frequency Response

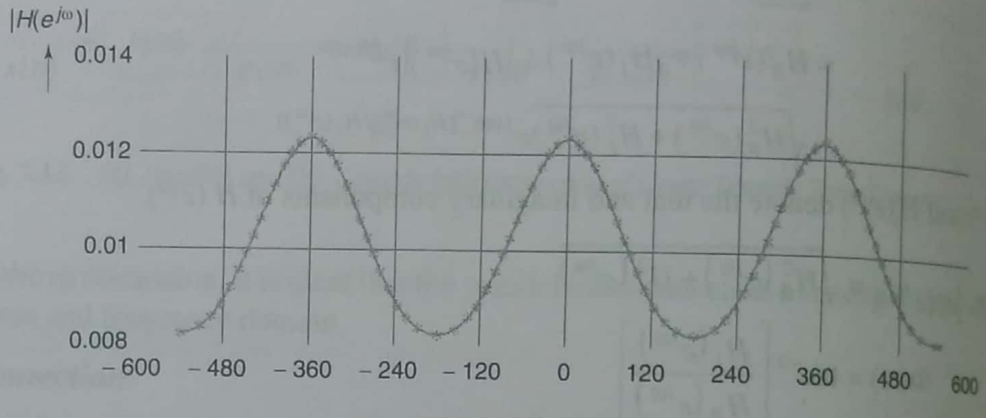
The frequency response is obtained by substituting $z = e^{j\omega}$.

Therefore, $H\left(e^{j\omega}\right) = H(\omega) = \dfrac{e^{j\omega}}{e^{j\omega} - 10}$

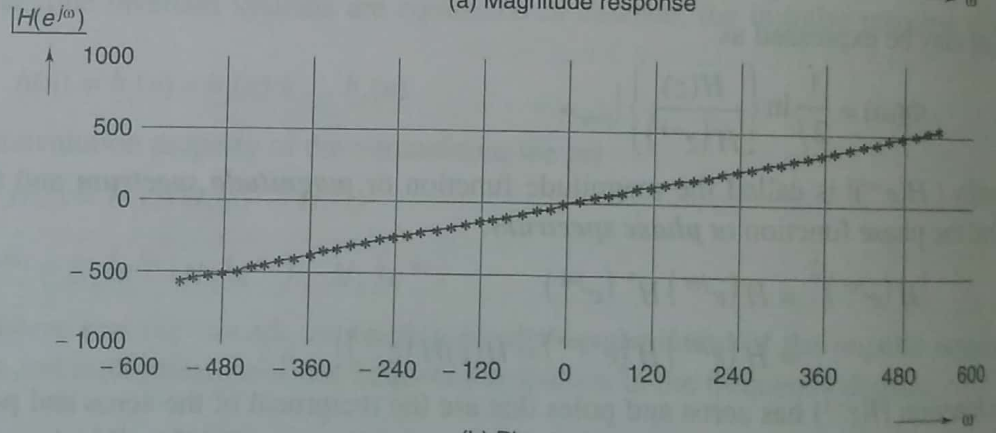
Magnitude Response

$$\left|H\left(e^{j\omega}\right)\right| = \frac{\left|e^{j\omega}\right|}{\left|e^{j\omega} - 10\right|} = \frac{1}{\sqrt{(\cos\omega - 10)^2 + \sin^2\omega}} = \frac{1}{101 - 20\cos\omega}$$

The magnitude response is shown in Fig. E5.42(a).



(a) Magnitude response



(b) Phase response

Fig. E5.42 *Magnitude Response and Phase Response*

Phase Response

$$H\left(e^{j\omega}\right) = \frac{e^{j\omega}}{e^{j\omega} - 10} = \frac{\cos\omega + j\sin\omega}{(\cos\omega - 10) + j\sin\omega}$$

$$\Phi(\omega) = \angle H\left(e^{j\omega}\right) = \omega - \tan^{-1}\left(\frac{\sin\omega}{\cos\omega - 10}\right)$$

The phase response is shown in Fig. E5.42(b).

Stability

The transfer function, $H(z) = \dfrac{z}{z - 10}$

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Therefore, $h(n) = (10)^n\, u(n)$

$$h\,(n) = 1,\ h\,(1) = 10,\ h\,(2) = 100,\ \ldots\ h(\infty) = \infty$$

$$\sum_{k=0}^{\infty} |n(k)| = \infty$$

Hence the system is constable.

5.7.5 Time Delay

The time delay of a filter is a measure of the average delay of the filter as a function of frequency. It is defined as the negative first derivative of a filter's phase response. If the complex frequency of a filter is $H(e^{j\omega})$, then the group delay is

$$\tau(\omega) = -\frac{d\Phi(\omega)}{d\omega} \tag{5.39}$$

where $\Phi(\omega)$ is the phase angle of $H\,(e^{j\omega})$.

Example 5.43 *Determine the impulse response h (n) for the system described by the second order difference equation.*

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$$

Solution For the impulse response, $x(n) = \delta(n)$ and hence $y(n) = h(n)$

Therefore, the given difference equation becomes

$$h(n) - 3h(n-1) - 4h(n-2) = \delta(n) + 2\delta(n-1)$$

Taking z-transform and rearranging, we get

$$H(z)[1 - 3z^{-1} - 4z^{-2}] = [1 + 2z^{-1}]$$

Therefore,
$$H(z) = \frac{1 + 2z^{-1}}{1 - 3z^{-1} - 4z^{-2}} = \frac{z(z+2)}{z^2 - 3z - 4} = \frac{z(z+2)}{(z+1)(z-4)}$$

$$F(z) = \frac{H(z)}{z} = \frac{(z+2)}{(z+1)(z-4)} = \frac{A_1}{(z+1)} + \frac{A_2}{(z-4)}$$

where
$$A_1 = (z+1)F(z)\big|_{z=-1} = \left.\frac{(z+2)}{(z-4)}\right|_{z=-1} = -\frac{1}{5}$$

$$A_2 = (z-4)F(z)\big|_{z=4} = \left.\frac{(z+2)}{(z+1)}\right|_{z=4} = \frac{6}{5}$$

Therefore,
$$\frac{H(z)}{z} = \frac{-\frac{1}{5}}{(z+1)} + \frac{\frac{6}{5}}{(z-4)}$$

Hence,
$$H(z) = \frac{-\frac{1}{5}z}{(z+1)} + \frac{\frac{6}{5}z}{(z-4)}$$

Taking inverse z-transform, we get the impulse response

$$h(n) = \left[-\frac{1}{5}(-1)^n + \frac{6}{5}(4)^n\right]u(n)$$

Example 5.44 *Find the magnitude and phase responses for the system characterised by difference equation*

$$y(n)=\frac{1}{6}x(n)+\frac{1}{3}x(n-1)+\frac{1}{6}x(n-2)$$

Solution Taking z-transform of the given difference equation, we get

$$Y(z)=\frac{1}{6}X(z)+\frac{1}{3}z^{-1}X(z)+\frac{1}{6}z^{-2}X(z)$$

Therefore, $$H(z)=\frac{Y(z)}{X(z)}=\frac{1}{6}+\frac{1}{3}z^{-1}+\frac{1}{6}z^{-2}$$

The frequency response is $H\left(e^{j\omega}\right)=\frac{1}{6}\left[1+2e^{-j\omega}+e^{-j2\omega}\right]$

$$=\frac{1}{6}e^{-j\omega}\left[e^{j\omega}+2+e^{-j\omega}\right]=\frac{1}{3}(1+\cos\omega)e^{-j\omega}$$

Hence, magnitude response is $|H(\omega)|=\frac{1}{3}(1+\cos\omega)$, and
Phase response is $\Phi(\omega)=-\omega$.

Example 5.45 *Find the impulse response, frequency response, magnitude response and phase response of the second order system*

$$y(n)-y(n-1)+\frac{3}{16}y(n-2)=x(n)-\frac{1}{2}x(n-1)$$

Solution For the impulse response, $x(n)=\delta(n)$ and hence $y(n)=h(n)$.
Therefore, the given function becomes

$$y(n)-y(n-1)+\frac{3}{16}y(n-2)=x(n)-\frac{1}{2}x(n-1)$$

Taking z-transform, we get

$$H(z)-z^{-1}H(z)+\frac{3}{16}z^{-2}H(z)=1-\frac{1}{2}z^{-1}$$

Therefore, $$H(z)=\frac{1-\frac{1}{2}z^{-1}}{1-z^{-1}+\frac{3}{16}z^{-2}}=\frac{z\left(z-\frac{1}{2}\right)}{z^2-z+\frac{3}{16}}=\frac{z\left(z-\frac{1}{2}\right)}{\left(z-\frac{1}{4}\right)\left(z-\frac{3}{4}\right)}$$

$$\frac{H(z)}{z}=\frac{A_1}{\left(z-\frac{1}{4}\right)}+\frac{A_2}{\left(z-\frac{3}{4}\right)}$$

Evaluating, we get $A_1=\frac{1}{2}$ and $A_2=\frac{1}{2}$

Hence $$\frac{H(z)}{z}=\frac{\frac{1}{2}}{\left(z-\frac{1}{4}\right)}+\frac{\frac{1}{2}}{\left(z-\frac{3}{4}\right)}$$

$$H(z)=\frac{\frac{1}{2}z}{\left(z-\frac{1}{4}\right)}+\frac{\frac{1}{2}z}{\left(z-\frac{3}{4}\right)}$$

Taking inverse z-transform, we have the impulse response

$$h(n)=\left[\frac{1}{2}\left(\frac{1}{4}\right)^n+\frac{1}{2}\left(\frac{3}{4}\right)^n\right]u(n)$$

Since the roots of the characteristic equation have magnitudes less than unity, we know that the impulse response is bounded.

Frequency Response

$$H(z)=\frac{\frac{1}{2}z}{\left(z-\frac{1}{4}\right)}+\frac{\frac{1}{2}z}{\left(z-\frac{3}{4}\right)}=\frac{1}{2}\left[\frac{1}{1-\frac{1}{4}z^{-1}}+\frac{1}{1-\frac{3}{4}z^{-1}}\right]$$

Therefore, $H(z)|_{z=e^{j\omega}}=H\left(e^{j\omega}\right)=\frac{1}{2}\left[\frac{1}{1-\frac{1}{4}e^{-j\omega}}+\frac{1}{1-\frac{3}{4}e^{-j\omega}}\right]$

Magnitude and Phase Responses

$$H(z)=\frac{z\left(z-\frac{1}{2}\right)}{\left(z-\frac{1}{4}\right)\left(z-\frac{3}{4}\right)}$$

$$H(e^{j\omega})=\frac{e^{j\omega}\left(e^{j\omega}-\frac{1}{2}\right)}{\left(e^{j\omega}-\frac{1}{4}\right)\left(e^{j\omega}-\frac{3}{4}\right)}$$

Therefore $\quad \left|H\left(e^{j\omega}\right)\right|=\frac{\left|\left(e^{j\omega}-\frac{1}{2}\right)\right|}{\left|\left(e^{j\omega}-\frac{1}{4}\right)\right|\left|\left(e^{j\omega}-\frac{3}{4}\right)\right|}$

$$\Phi(\omega)=\omega+\arg\left(e^{j\omega}-\frac{1}{2}\right)-\arg\left(e^{j\omega}-\frac{1}{4}\right)-\arg\left(e^{j\omega}-\frac{3}{4}\right)$$

Example 5.46 *The output $y(n)$ for an LTI system to the input $x(n)$ is*

$$y(n)=x(n)-2x(n-1)+x(n-2)$$

Compute and sketch the magnitude and phase of the frequency response of the system for $|\omega|\geq\pi$.

Solution $\quad y(n)=x(n)-2x(n-1)+x(n-2)$

Taking z-transform on both sides, we get

$$Y(z)=X(z)-2z^{-1}X(z)+z^{-2}X(z)$$

$$H(z)=\frac{Y(z)}{X(z)}=1-2z^{-1}+z^{-2}$$

Substituting $z=e^{j\omega}$, we get frequency response given by

ω	π	$7\pi/6$	$8\pi/6$	$9\pi/6$	$10\pi/6$	$11\pi/6$	$12\pi/6$	$13\pi/6$	$14\pi/6$	$15\pi/6$	$16\pi/6$	$17\pi/6$	$18\pi/6$
$\lvert H(e^{j\omega})\rvert$	4	3.73	3	2	1	0.268	0	0.268	1	2	3	3.73	4

$$H\left(e^{j\omega}\right)=1-2e^{-j\omega}+e^{-2j\omega}$$

$$=e^{-j\omega}\left[e^{j\omega}-2+e^{-j\omega}\right]=2e^{-j\omega}\left[\cos\omega-1\right]$$

Therefore, the magnitude response is

$$|H(e^{j\omega})|=|2[\cos\omega-1]|$$

The phase response is $\phi(\omega)=\angle H(e^{j\omega})=-\omega$

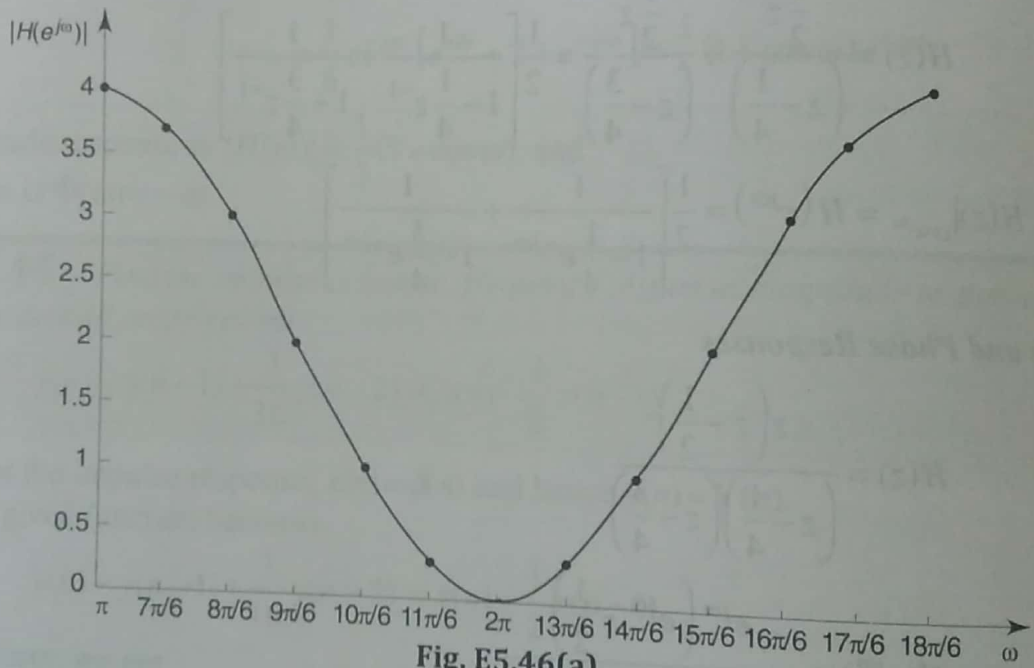

Fig. E5.46(a)

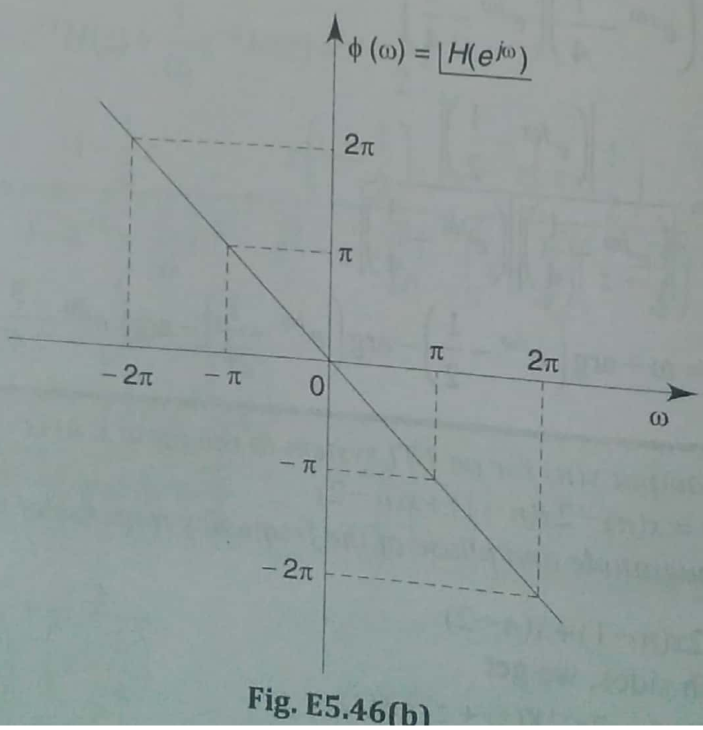

Fig. E5.46(b)

Example 5.47 *Determine the frequency response, magnitude response, phase response and time delay of the system given by*

$$y(n)+\frac{1}{2}y(n-1)=x(n)-x(n-1)$$

Solution

To find the frequency response H ($e^{j\omega}$)

Given, $$y(n)+\frac{1}{2}y(n-1)=x(n)-x(n-1)$$

Taking z-transform, we get

$$Y(z)+\frac{1}{2}z^{-1}Y(z)=X(z)-z^{-1}X(z)$$

$$Y(z)\left(1+\frac{1}{2}z^{-1}\right)=X(z)[1-z^{-1}]$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1-z^{-1}}{1+\frac{1}{2}z^{-1}}$$

Therefore, the frequency response is, $H\left(e^{j\omega}\right)=\dfrac{1-e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}$

To find the magnitude response |H ($e^{j\omega}$)|

Here,

$$H\left(e^{j\omega}\right)=\frac{1-e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}=\frac{1-\cos\omega+j\sin\omega}{1+\frac{1}{2}\cos\omega-\frac{1}{2}j\sin\omega}$$

$$\left|H\left(e^{j\omega}\right)\right|^2=\frac{(1-\cos\omega)^2+\sin^2\omega}{\left(1+\frac{1}{2}\cos\omega\right)^2+\left(\frac{1}{2}\sin\omega\right)^2}=\frac{2(1-\cos\omega)}{\frac{5}{4}+\cos\omega}=\frac{8(1-\cos\omega)}{5+4\cos\omega}$$

Therefore, $\left|H\left(e^{j\omega}\right)\right|^2=2\sqrt{2}\sqrt{\dfrac{1-\cos\omega}{5+4\cos\omega}}$

To find the phase response Φ(ω)

$$H\left(e^{j\omega}\right)=\frac{1-\cos\omega+j\sin\omega}{1+\frac{1}{2}\cos\omega-\frac{1}{2}j\cos\omega}$$

$$\Phi(\omega)=\tan^{-1}\left(\frac{\sin\omega}{1-\cos\omega}\right)-\tan^{-1}\left(\frac{-\frac{1}{2}\sin\omega}{1+\frac{1}{2}\cos\omega}\right)$$

$$=\tan^{-1}\left(\frac{\sin\omega}{1-\cos\omega}\right)+\tan^{-1}\left(\frac{\frac{1}{2}\sin\omega}{1+\frac{1}{2}\cos\omega}\right)=\Phi_1(\omega)+\Phi_2(\omega)$$

To find the time delay [τ(ω)]

$$\tau(\omega) = \tau_1(\omega) + \tau_2(\omega) = -\frac{d\Phi_1}{d\omega} - \frac{d\Phi_2}{d\omega}$$

$$\tau_1(\omega) = -\frac{d\phi_1}{d\omega} = -\frac{(1-\cos\omega)\cos\omega - \sin\omega(\sin\omega)}{(1-\cos\omega)^2 + \sin^2\omega}$$

$$= -\frac{\cos\omega - 1}{2 - 2\cos\omega} = \frac{1}{2}$$

$$\tau_2(\omega) = -\frac{d\phi_2}{d\omega} = -\frac{\left(1+\frac{1}{2}\cos\omega\right)\frac{1}{2}\cos\omega + \frac{1}{2}\sin\omega\left(\frac{1}{2}\sin\omega\right)}{\left(1+\frac{1}{2}\cos\omega\right)^2 + \frac{1}{4}\sin^2\omega}$$

$$= -\frac{\frac{1}{2}\cos\omega + \frac{1}{4}}{\cos\omega + \frac{5}{4}} = -\frac{1+2\cos\omega}{5+4\cos\omega}$$

Hence, $\tau(\omega) = \tau_1(\omega) + \tau_2(\omega)$

$$= \frac{1}{2} - \frac{1+2\cos\omega}{5+4\cos\omega} = \frac{3}{2(5+4\cos\omega)}$$

Note: If $\Phi(\omega) = \tan^{-1}\left[\frac{u(\omega)}{v(\omega)}\right]$, then

$$\tau(\omega) = -\frac{d\Phi(\omega)}{d\omega} = -\frac{(v\,du - u\,dv)}{v^2 + u^2}$$

Example 5.48 *The frequency response of a system is given by*

$$H(e^{j\omega}) = \frac{e^{j\omega} - a}{e^{j\omega} - b}$$

where a and b are real with $a \neq b$. Show that $|H(e^{j\omega})|^2$ is constant if $ab = 1$ and determine its value. Also, find the phase response and time delay.

Solution Given $H(e^{j\omega}) = \dfrac{e^{j\omega} - a}{e^{j\omega} - b} = \dfrac{(\cos\omega - a) + j\sin\omega}{(\cos\omega - b) + j\sin\omega}$

$$\left|H(e^{j\omega})\right|^2 = \frac{(\cos\omega - a^2) + \sin^2\omega}{(\cos\omega - b^2) + \sin^2\omega} = \frac{1 + a^2 - 2a\cos\omega}{1 + b^2 - 2b\cos\omega}$$

Substituting $b = \frac{1}{a}$ [since $ab = 1$], we have

$$\left|H(e^{j\omega})\right|^2 = \frac{1 + a^2 - 2a\cos\omega}{1 + \frac{1}{a^2} - \frac{2}{a}\cos\omega} = a^2 \cdot \frac{1 + a^2 - 2a\cos\omega}{1 + a^2 - 2a\cos\omega} = a^2$$

To find phase response $\Phi(\omega)$

$$H\left(e^{j\omega}\right)=\frac{e^{j\omega}-a}{ae^{j\omega}-\frac{1}{a}}=a.\frac{e^{j\omega}-a}{a\,e^{j\omega}-1}=-a\,e^{j\omega}\frac{1-ae^{-j\omega}}{1-ae^{j\omega}}$$

$$=-ae^{j\omega}\frac{1-a\cos\omega+ja\sin\omega}{1-a\cos\omega-ja\sin\omega}$$

$$\Phi(\omega)=\omega-\pi+\tan^{-1}\left[\frac{a\sin\omega}{1-a\cos\omega}\right]-\tan^{-1}\left[\frac{-a\sin\omega}{1-a\cos\omega}\right]$$

$$=\omega-\pi+2\tan^{-1}\left[\frac{a\sin\omega}{1-a\cos\omega}\right]$$

To find time delay $\tau(\omega)$

$$\text{i.e. } \tau(\omega)=-\frac{d\phi(\omega)}{d\omega}=-1-2\frac{(1-a\cos\omega)a\cos\omega-(a\sin\omega)a\sin\omega}{(1-a\cos\omega)^2+(a\sin\omega)^2}$$

$$=-1-\frac{a\cos\omega-a^2}{1-2\cos\omega+a^2}=\frac{-1+a\cos\omega}{1-2a\cos\omega+a^2}$$