Unit III

Centroid & Centre of Gravity

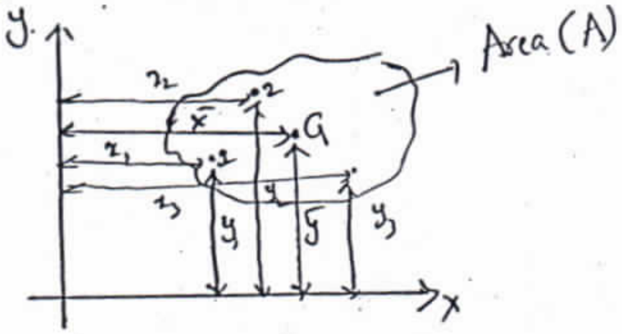
Centroid:- It is a single point about which entire Area is acting for a lamina (or) plane figure irrespective of the position of the plane figure.

* It is applicable to plane fig. having area but no volume

Ex: Rectangle, Square, circle, Semi-circle, Triangle ... etc.

Centre of gravity:- The point at which the whole weight of the body is supposed to act is called "centre of gravity". It is represented by C.G. (or) G.

Concept of Centroid:-



from the principle of moments

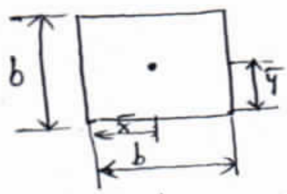
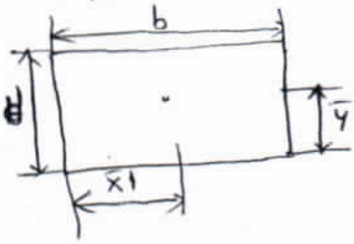
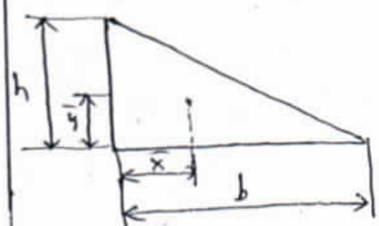
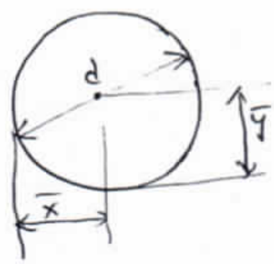
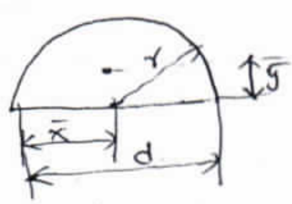
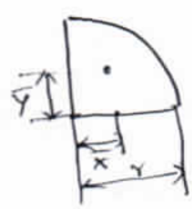
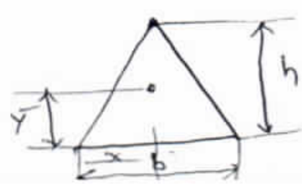
Taking positions of Centroid from y-axis, $\bar{x} = \dfrac{a_1x_1 + a_2x_2 + a_3x_3 + \cdots}{a_1 + a_2 + a_3 + \cdots}$

$$\boxed{\bar{X} = \frac{\Sigma ax}{\Sigma a}} = \frac{\int x\,dA}{\int dA \text{ (or) } A}$$

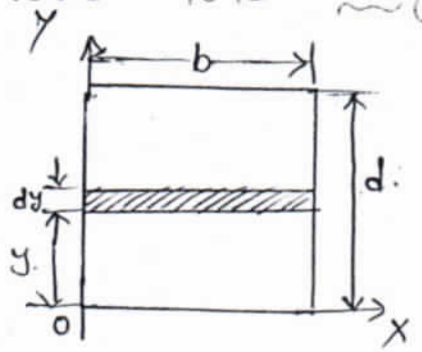
Similarly from x-axis $\bar{Y} = \dfrac{a_1y_1 + a_2y_2 + a_3y_3 + \cdots}{a_1 + a_2 + a_3 + \cdots}$

$$\boxed{\bar{Y} = \frac{\Sigma ay}{\Sigma a}} = \frac{\int y\,dA}{\int dA \text{ (or) } A}$$

Centroids of plane geometrical shapes:-

S.No	Shape	Shape	Area	$\bar{x}$	$\bar{y}$
1.	Square		b^2	$b/2$	$b/2$
2.	Rectangle		bd	$b/2$	$d/2$
3.	Right $\triangle$le Triangle		$\frac{bh}{2}$	$\frac{b}{3}$	$\frac{h}{3}$
4.	Circle		πr^2	$\frac{d}{2}$	$\frac{d}{2}$
5.	Semi circle		$\frac{\pi r^2}{2}$	$\frac{d}{2} = r.$	$\frac{4r}{3\pi}.$
6	Quadrant		$\frac{\pi r^2}{4}$	$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$
7.	Triangle		$\frac{1}{2} \times b \times h$	$\frac{b}{2}$	$\frac{h}{3}.$

Centroid of a Rectangular section by Integration:-



Consider a rectangular elementary strip of Thickness 'dy' at a distance y from the axis 'OX'.

Let $dA = b \times dy$

moment of Area 'dA' about axis 'OX'

$= dA \times y = (b \times dy)\, y = by.dy$

By Integrating whole area with limits 0 to d

$$= \int_0^d b.y.dy = b\left(\frac{y^2}{2}\right)_0^d = \frac{bd^2}{2}$$

Let 'A' total Area of rectangular.

$A = b \times d.$

Let $\bar{y}$ is distance of centroid of the rectangular section from ox.

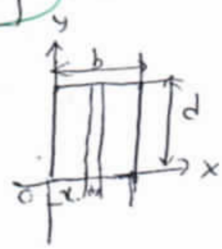
moment of total Area from ox

$$= A \times \bar{Y} = (b \times d)\,\bar{y}$$

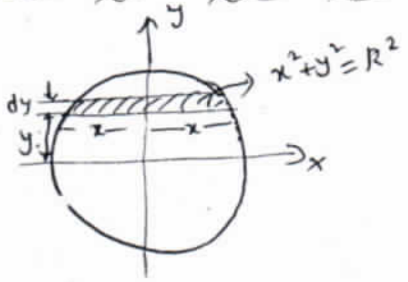
$$(b \times d)\,\bar{y} = \frac{bd^2}{2} \Rightarrow \boxed{\bar{y} = d/2}$$

$$\bar{y} = \frac{\int y\,dA}{dA} = \frac{bd^2/2}{bd} = d/2$$

Similary for $\bar{x}$ $\boxed{\bar{x} = \frac{b}{2}}$



Centroid of circular section:



Area of strip $dA = 2x.dy$.

moment of Area 'dA' about x-axis $= dA.y$

$= 2x.dy.y$

But $x^2+y^2 = R^2 \Rightarrow x = \sqrt{R^2-y^2}$

$= 2\left(\sqrt{R^2-y^2}\right).y\,dy$

Integrating with limits $-R$ to $+R$

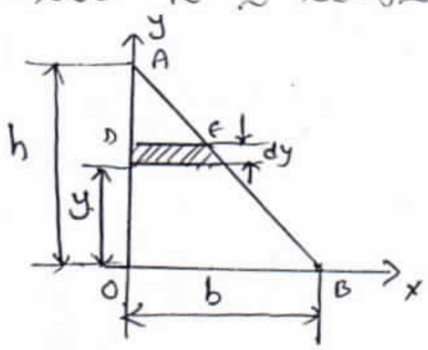
$$= \int_{-R}^{R} 2\sqrt{R^2-y^2}\,y\,dy = -\int_{-R}^{R} \left(R^2-y^2\right)^{3/2}$$

$$= \int_{-R}^{R} \sqrt{R^2-y^2}\,(-2y)\,dy = -\left[\frac{(R^2-y^2)^{3/2}}{3/2}\right]_{-R}^{R}$$

$$= -\frac{2}{3}\left[(R^2-R^2)^{3/2} - (R^2-(-R)^2)^{3/2}\right]$$

$$\bar{x} = -2/3\,[0-0] = 0 \quad , \; \bar{y} = 0 \; , \; \bar{x} = 0$$

→ Centroid of a triangular section -



Area of strip, dA = length $DE \times dy$.

from Δ^{le} ADE & Δ^{le} AOB are similar

$$\frac{DE}{OB} = \frac{AD}{AO}$$

$OB = b$, $AO = h$, $AD = (h-y)$

$$DE = \frac{b(h-y)}{h}$$

$$dA = \frac{b(h-y)}{h} \times dy$$

Moment of Area 'dA' about ox. $= dA \cdot y$.

$$= \frac{b(h-y)}{h} \times dy \cdot y = \frac{b}{h}(h-y)\,y\,dy$$

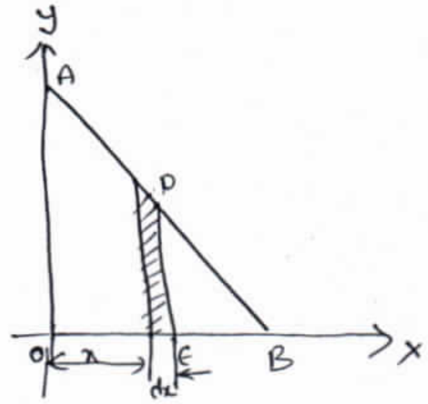
Integrate with limits 0 to h.

$$= \int_0^h \frac{b}{h}(h-y)\,y\,dy = \frac{b}{h}\int_0^h (h-y)y\,dy = \frac{b}{h}\int_0^h (hy - y^2)\,dy$$

$$= \frac{b}{h}\left[\frac{hy^2}{2} - \frac{y^3}{3}\right]_0^h = \frac{b}{h}\left[\frac{h^3}{2} - \frac{h^3}{3}\right] = \frac{bh^2}{6}$$

Moment of total Area of about Ox. $= A \times \bar{y}$

$$\left(\frac{b \times h}{2}\right)\bar{y} = \frac{bh^2}{6} \Rightarrow \bar{y} = \frac{h}{3}$$



Area of strip = length DE x dx.

~~from~~ Δ^{le} DBE, Δ^{le} AOB are similar

$$\frac{AO}{DE} = \cancel{AD}\ \frac{OB}{EB}$$

$$DE = \frac{h(b-x)}{b}$$

$$dA = \frac{h(b-x)}{b}\,dx.$$

Moment of Area about OY = $dA \cdot x$.

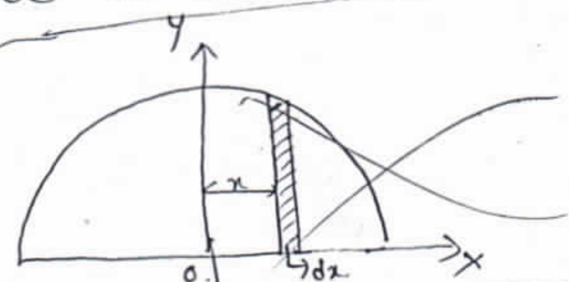
$$= \int_0^b \frac{h(b-x)}{b}\,dx \cdot x. \quad = \frac{h}{b}\int_0^b (bx - x^2)\,dx$$

$$= \frac{h}{b}\left(\frac{bx^2}{2} - \frac{x^3}{3}\right)_0^b \quad = \frac{hb^2}{6}$$

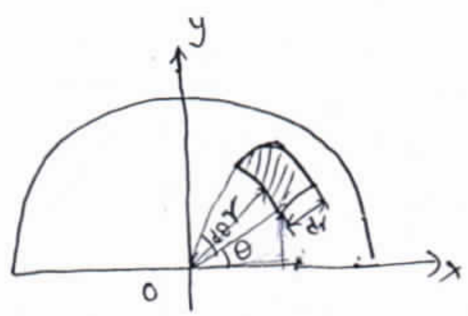
$$A \times \bar{x} = \frac{hb^2}{6} \rightarrow \bar{x} = b/3$$

$$(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{h}{3}\right)$$

→ Centroid of Semi Circular Section :-



Area of strip = $dA = y \cdot dx$.



Area of semi circle = $\frac{\pi R^2}{2}$

Area of elementary strip = $r \cdot d\theta \cdot dr$.

Moment about x-axis

$= dA \cdot r\sin\theta$.

Integrating ; $\int_0^{\pi}\int_0^{R} r\,d\theta.\,dr.\,r\sin\theta = \int_0^{\pi}\left(\frac{r^3}{3}\right)$

$\frac{\int y\,dA}{A}$

$= \int_0^{\pi}\sin\theta.\,d\theta \times \int_0^{R} r^2\,d\theta$

$\int y\,dA$ -

$= \left(-\cos\theta\right)_0^{\pi} \times \left(\frac{r^3}{3}\right)_0^{R}$

$= (-\cos\pi + \cos 0) \times \left(\frac{R^3}{3}\right)$

$= \frac{2R^3}{3}$.

Total Area 'A'.

Moment of Area = $A \times \bar{Y}$

$A \times \bar{Y} = \frac{2R^3}{3}$

$\frac{\pi R^2}{2} \times \bar{Y} = \frac{2R^3}{3} \rightarrow \bar{Y} = \frac{4R}{3\pi}$

$(\bar{X}, \bar{Y}) = \left(0, \frac{4R}{3\pi}\right)$

$\bar{Y} = \frac{\int y\,dA}{A}$

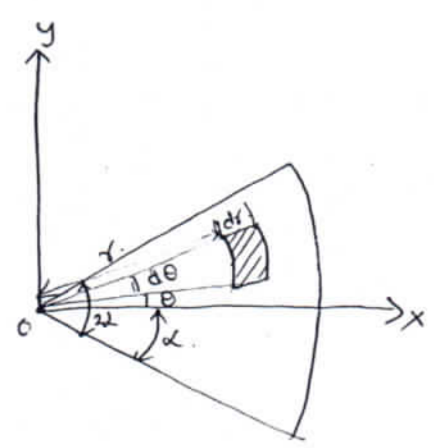
$= \frac{2R^3/3}{\pi R^2/2}$

$= \frac{2R^3}{3} \times \frac{2}{\pi R^2}$

$= \frac{4R}{3\pi}$

→ Similary for Quarter circle → $\bar{x} = \frac{4R}{3\pi}$, $\bar{Y} = \frac{4R}{3\pi}$

→ Centroid of Sector of a circle.



Due to symmetry, centroid lies on x-axis ($\bar{Y} = 0$)

Area of element = $r.d\theta.\,dr$.

moment about y-axis = $dA.\,r\cos\theta$

Total moment of Area about Y-axis =

$= \int_{-\alpha}^{\alpha}\int_0^{R} r^2\cos\theta\,dr.\,d\theta$

$= \left[\frac{r^3}{3}\right]_0^{R}\left[\sin\theta\right]_{-\alpha}^{\alpha}$

$= \frac{R^3}{3}\,2\sin\alpha$

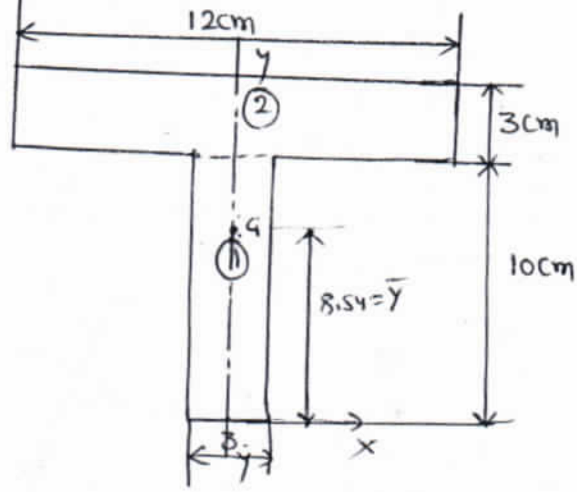
Total Area of Sector = $R^2\alpha$.

$$\bar{x} = \frac{\int x\,dA}{A}$$

$$\bar{x} = \frac{\frac{2R^3}{3}\sin\alpha}{R^2\alpha} = \frac{2R}{3\alpha}\sin\alpha.$$

$$(\bar{x}, \bar{y}) = \left(\frac{2R}{3\alpha}\sin\alpha, 0\right)$$

→ find the C.G. of the T-section as shown in fig.



The section is symmetric about y-axis, so $\bar{x} = 0$

$a_1 = 3 \times 10 = 30\,cm^2$

$a_2 = 12 \times 3 = 36\,cm^2$

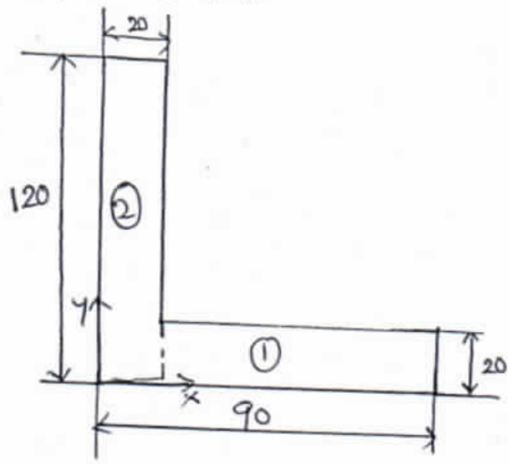
$y_1 = \frac{10}{2} = 5\,cm$

$y_2 = 10 + \frac{3}{2} = 11.5\,cm$

$$\bar{Y} = \frac{a_1y_1 + a_2y_2}{a_1 + a_2} = \frac{(30 \times 5) + (36 \times 11.5)}{30 + 36}$$

$$= 8.54\,cm.$$

→ find the C.G. of L-section



$a_1 = 70 \times 20 = 1400\,mm^2$

$a_2 = 120 \times 20 = 2400\,mm^2$

$x_1 = 20 + \frac{70}{2} = 55\,mm$

$x_2 = \frac{20}{2} = 10\,mm$

$y_1 = \frac{20}{2} = 10\,mm$

$y_2 = \frac{120}{2} = 60\,mm$

$$\bar{x} = \frac{a_1x_1 + a_2x_2}{a_1 + a_2}$$

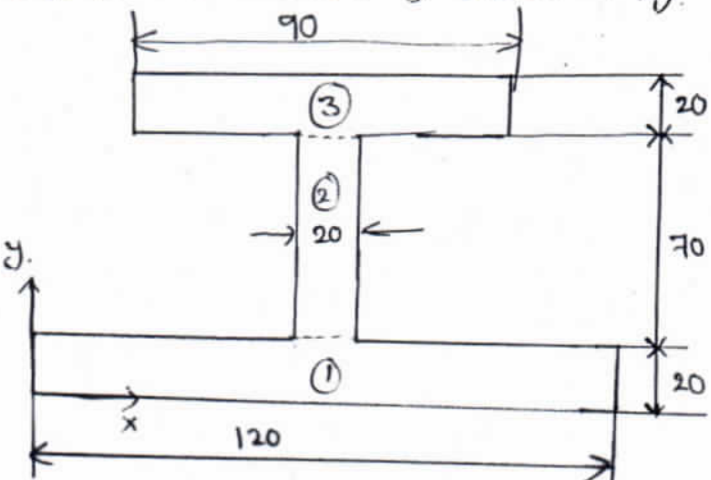
$$= \frac{(1400 \times 55) + (2400 \times 10)}{1400 + 2400} = 26.57\,mm$$

$$\bar{y} = \frac{a_1y_1 + a_2y_2}{a_1 + a_2}$$

$$= \frac{(1400 \times 10) + (2400 \times 60)}{1400 + 2400}$$

$$= 41.57\,mm$$

-3 find the C.G. of I-section of shown in fig.



C about

$a_1 = 120 \times 20 = 2400\,mm^2$, $a_2 = 20 \times 70 = 1400\,mm^2$ $a_3 = 90 \times 20 = 1800\,mm$

$x_1 = \frac{120}{2} = 60\,mm$ $x_2 = 50 + \frac{20}{2} = 60\,mm$ $x_3 = 15 + \frac{90}{2} = 60\,m$

$y_1 = \frac{20}{2} = 10\,mm$ $y_2 = 20 + \frac{70}{2} = 55\,mm$ $y_3 = 90 + \frac{20}{2} = 100\,m$

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3}{a_1 + a_2 + a_3}$$

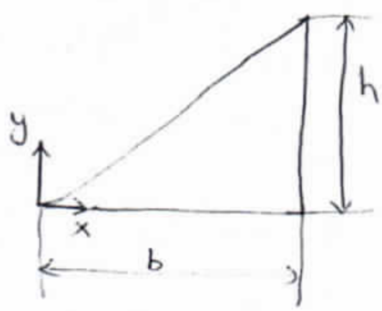
$$= \frac{(2400 \times 60) + (1400 \times 60) + (1800 \times 60)}{(2400 + 1600 + 1800)}$$

$$= \underline{\underline{60\,mm}}$$

$$\bar{Y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3}$$

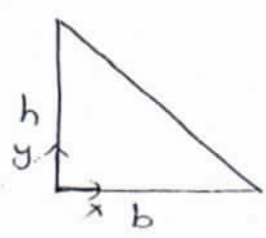
$$= \frac{(2400 \times 10) + (1400 \times 55) + (1800 \times 100)}{(2400 + 1400 + 1800)}$$

$\bar{Y} = 50.17\,mm$



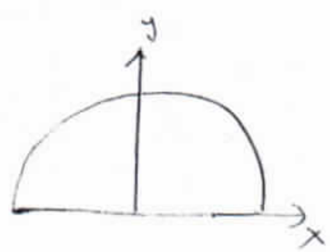
$\bar{x} = \frac{2}{3} b$

$\bar{Y} = \frac{h}{3}$



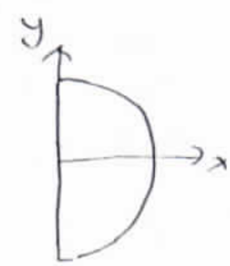
$\bar{x} = b/3$

$\bar{Y} = h/3$



$\bar{x} = 0$

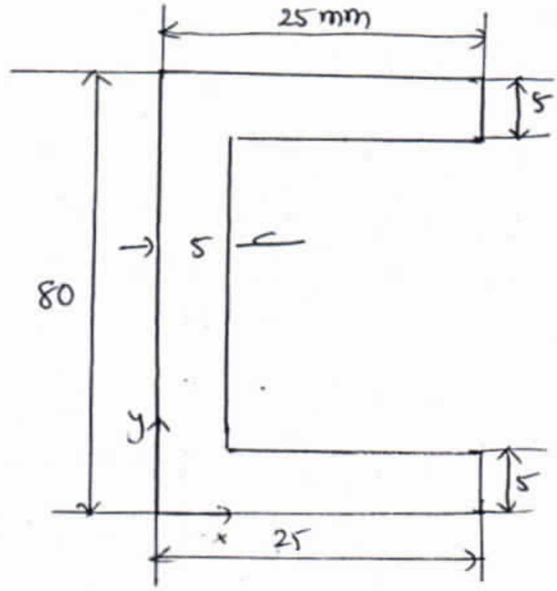
$\bar{Y} = \frac{4r}{3\pi}$



$\bar{Y} = 0$

$\bar{X} = \frac{4r}{3\pi}$

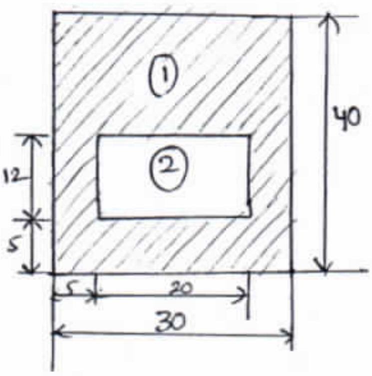
→ Determine the Centroid of channel section as shown in fig



$\bar{Y} = \frac{80}{2} = 40\text{mm}$

$\bar{X} = 6.67\text{ mm}$

→ find the centroid of shaded Area as shown in fig



$\bar{x} = \frac{30}{2} = 15\text{mm}$

$a_1 = 40 \times 30 = 1200\text{ mm}^2$

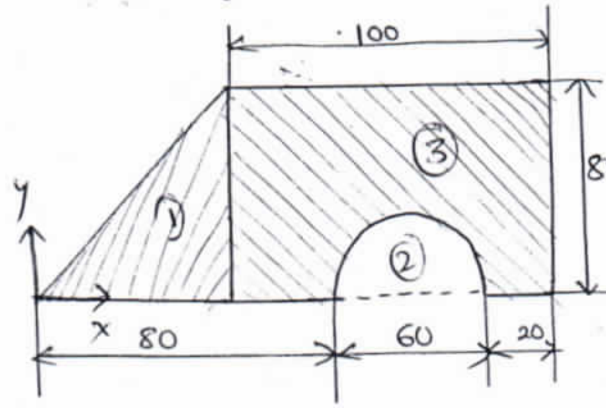
$y_1 = \frac{40}{2} = 20\text{mm}$

$a_2 = 20 \times 12 = 240\text{ mm}^2$

$y_2 = 5 + \frac{12}{2} = 11\text{mm}$

$\bar{Y} = \frac{a_1 y_1 - a_2 y_2}{a_1 + a_2} = 22.25\text{ mm}$

→ Determine the centroid of shaded portion of the shape as shown.



$a_1 = \frac{1}{2} \times b \times h = \frac{1}{2} \times 60 \times 80$

$= 2400\text{ mm}^2$

$a_2 = \frac{\pi R^2}{2} = \frac{\pi (30)^2}{2} = 1413.72\text{ mm}^2$

$a_3 = 100 \times 80 = 8000\text{ mm}^2$

$x_1 = \frac{2b}{3} = \frac{2 \times 60}{3} = 40\text{mm}$, $y_1 = \frac{80}{3} = 26.67\text{ mm}$

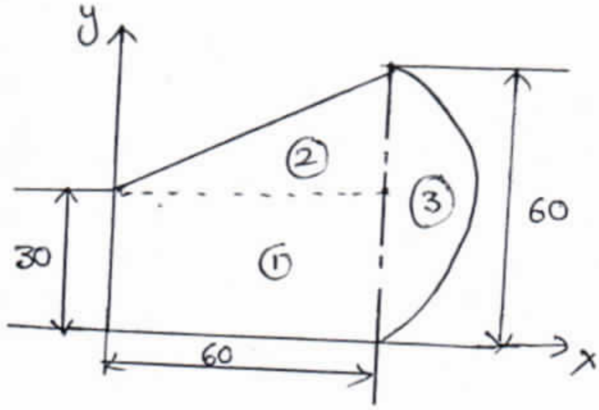
$x_2 = 80 + \frac{60}{2} = 110\text{ mm}$, $y_2 = \frac{4R}{3\pi} = \frac{4 \times 30}{3\pi} = 12.73\text{ mm}$

$x_3 = 60 + \frac{100}{2} = 110mm$, $y_3 = \frac{80}{2} = 40mm$.

$\bar{X} = \frac{a_1x_1 - a_2x_2 + a_3x_3}{a_1 - a_2 + a_3}$, $\bar{Y} = \frac{a_1y_1 - a_2y_2 + a_3y_3}{a_1 - a_2 + a_3}$

$\bar{X} = 91.3mm$ $\bar{Y} = 40.73mm$

→ Determine the Centroid of the composite Section as shown



$a_1 = 30 \times 60 = 1800 mm^2$

$a_2 = \frac{1}{2} \times 60 \times 30 = 900 mm$

$a_3 = \frac{\pi \times 30^2}{2} = 1413.67 mm^2$

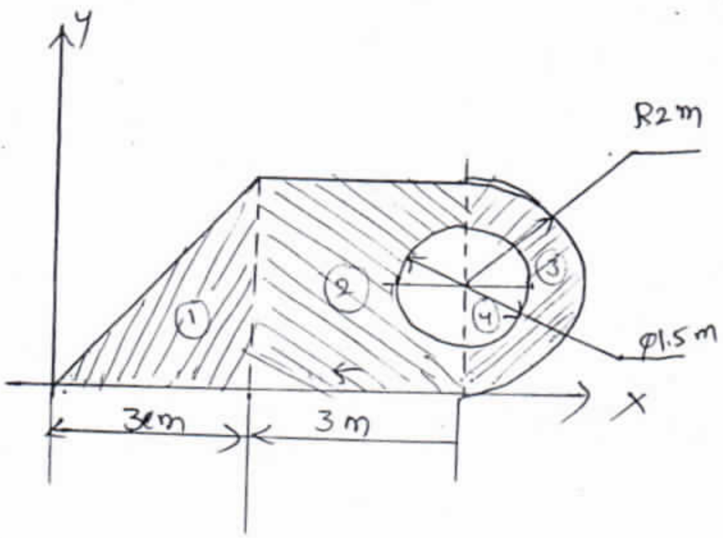
$x_1 = \frac{60}{2} = 30mm$, $y_1 = \frac{30}{2} = 15mm$

$x_2 = \frac{2 \times 60}{3} = 40mm$, $y_2 = 30 + \frac{30}{3} = 40mm$

$x_3 = 60 + \frac{4 \times 30}{3\pi} = 72.73mm$ $y_3 = \frac{60}{2} = 30mm$

$\underline{\bar{X} = 46.87mm}$, $\underline{\bar{Y} = 25.62mm}$

→ Determine the Centroid of Composite Section as shown in fig :



Sol

$a_1 = \frac{1}{2} \times 3 \times 4 = 6\,m^2$, $a_2 = 3 \times 4 = 12\,m^2$, $a_3 = \frac{\pi \times 2^2}{2} = 6.283\,m^2$

$a_4 = \pi (0.75)^2 = 1.767\,m^2$

$x_1 = 2b/3 = \frac{2 \times 3}{3} = 2\,m$ $\qquad x_2 = 3 + \frac{3}{2} = 4.5\,m$

$y_1 = \frac{4}{3} = 1.33\,m$ $\qquad y_2 = \frac{4}{2} = 2\,m$

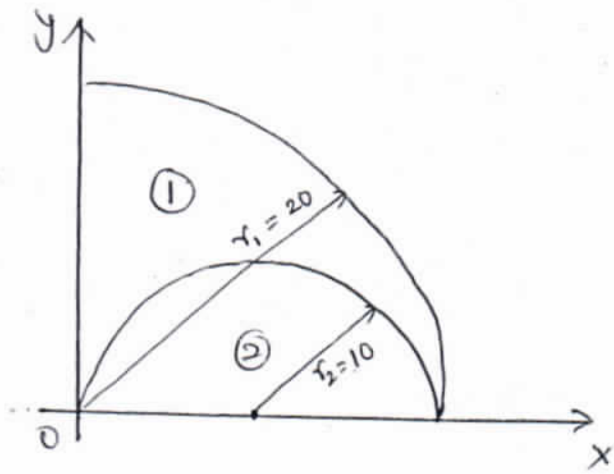
$x_3 = 6 + \frac{4r}{3\pi} = 6 + \frac{4 \times 2}{3\pi} = 6.848\,m$ $\qquad x_4 = 6\,m$

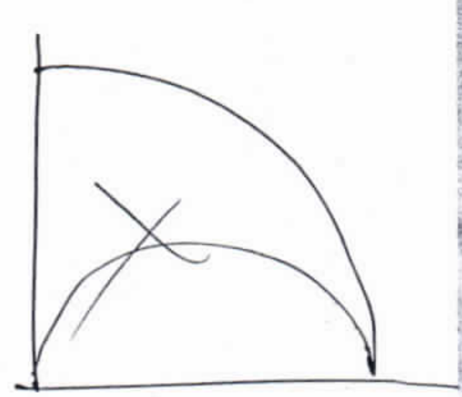
$y_3 = \frac{4}{2} = 2\,m.$ $\qquad y_4 = 1.25 + \frac{1.5}{2} = 2\,m$

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3 - a_4 x_4}{a_1 + a_2 + a_3 - a_4}, \quad \bar{Y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3 - a_4 y_4}{a_1 + a_2 + a_3 - a_4}$$

$= 4.37\,m$ $\qquad = 1.8\,m.$

→ find the co-ordinates of the centroid of the Area as shown in fig





Sol

$a_1 = \frac{\pi r^2}{4} = \frac{\pi \times 20^2}{4} = 314.16\,mm^2$, $a_2 = \frac{\pi r^2}{2} = \frac{\pi \times 10^2}{2} = 157.08\,mm^2$

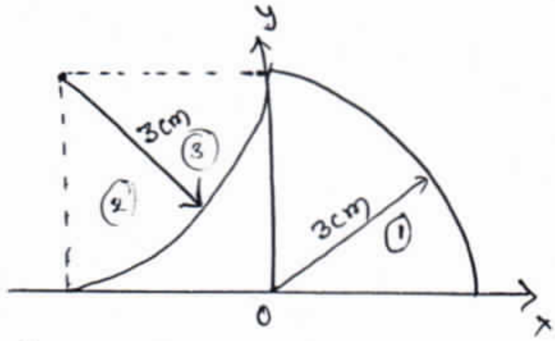
$x_1 = y_1 = \frac{4r}{3\pi} = 8.49\,mm$, $\qquad x_2 = 10\,mm$

$y_2 = \frac{4r}{3\pi} = \frac{4 \times 10}{3\pi} = 4.24\,mm$

$$\bar{x} = \frac{a_1 x_1 - a_2 x_2}{a_1 - a_2}, \quad \bar{Y} = \frac{a_1 y_1 - a_2 y_2}{a_1 - a_2}$$

$\bar{x} = 6.98\,mm$, $\bar{Y} = 12.74\,m$

→ find the centroid of shaded Area as shown in fig



③ & ① → Quarter circle
② → square

Sol

$A_1 = \frac{\pi r^2}{4} = \frac{\pi \times 3^2}{4} = 7.07\,cm^2$ $\qquad a_3 = \frac{\pi r^2}{4} = 7.07\,cm^2$

$a_2 = 3 \times 3 = 9\,cm^2$

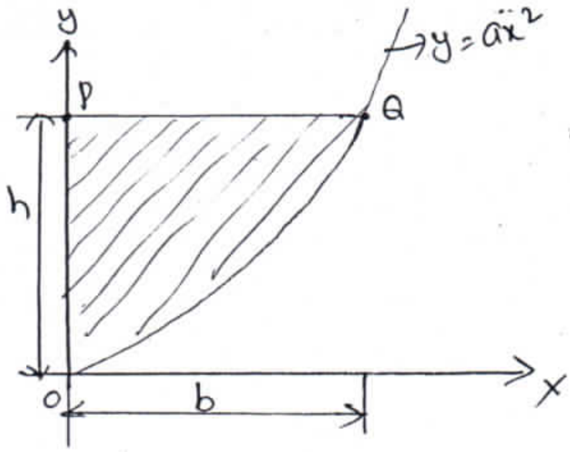
$x_1 = y_1 = \frac{4r}{3\pi} = 1.27\,cm$

$x_2 = -\frac{3}{2} = -1.5\,cm \qquad y_2 = 1.5\,cm$

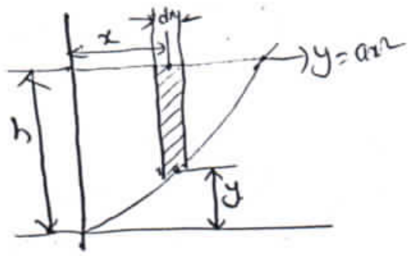
$x_3 = -\left(3 - \frac{4\times 3}{3\pi}\right) = -1.73\,cm, \quad y_3 = \left(3 - \frac{4\times 3}{3\pi}\right) = 1.73\,cm$

$\bar{x} = \frac{a_1x_1 + a_2x_2 - a_3x_3}{a_1 + a_2 - a_3}, \quad \bar{y} = a_1 y$

→ find the Centroid of shaded Area OPQ, shown in fig. The Curve OQ is parabolic.



i) To find $\bar{x}$:-



$$dA = (h-y)\,dx$$

$$\bar{x} = \frac{\int x\,dA}{\int dA} = \frac{\int_0^b x.(h-y)\,dx}{\int_0^b (h-y)\,dx}$$

$$\bar{x} = \frac{\int_0^b (xh - yx)\,dx}{\int_0^b (h-y)\,dx}$$

$y = ax^2$

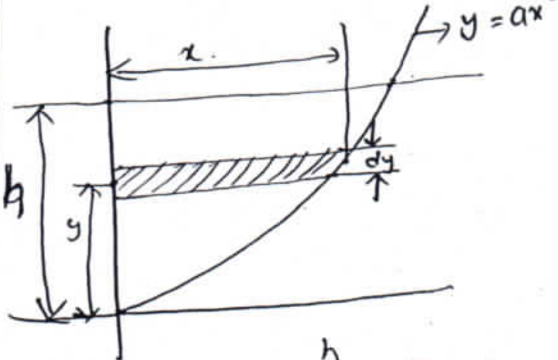
$$\bar{x} = \frac{\int_0^b (xh)\,dx - \int_0^b ax^3\,dx}{\int_0^b (h)\,dx - \int_0^b ax^2\,dx} = \frac{\left[\frac{hx^2}{2}\right]_0^b - \left[\frac{ax^4}{4}\right]_0^b}{\left[hx\right]_0^b - \left[\frac{ax^3}{3}\right]_0^b}$$

$$\bar{x} = \frac{hb^2/2 - ab^4/4}{hb - ab^3/3} = \frac{\frac{2hb^2 - ab^4}{4}}{\frac{3hb - ab^3}{3}}$$

$$= \frac{b^2(2h - ab^2)}{4} \times \frac{3}{b(3h - ab^2)}$$

$$\boxed{\bar{x} = \frac{3b\,(2h - ab^2)}{4\,(3h - ab^2)}}$$

ii) To find $\bar{y}$



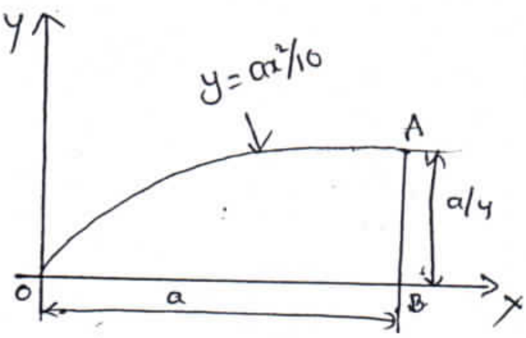
$$dA = (x.dy)$$

$$\bar{y} = \frac{\int y\,dA}{\int dA} = \frac{\int y.x.dy}{\int x\,dy}.$$

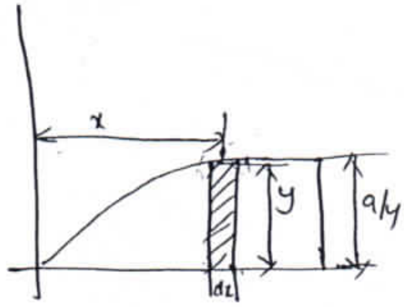
$$\bar{y} = \frac{\int_0^h y.\frac{\sqrt{y}}{\sqrt{a}}\,dy}{\int_0^h \frac{\sqrt{y}}{\sqrt{a}}.dy} = \frac{\frac{1}{\sqrt{a}}\int_0^h y^{3/2}\,dy}{\frac{1}{\sqrt{a}}\int_0^h y^{1/2}\,dy} = \frac{\left[2/5 . y^{5/2}\right]_0^h}{\left[2/3 . y^{3/2}\right]_0^h}$$

$$= \frac{\frac{2}{5}.h^{5/2}}{\frac{2}{3}h^{3/2}} = \frac{\cancel{2}}{5}\times\frac{3}{\cancel{2}}\times h \Rightarrow \underline{\underline{\bar{y} = \frac{3}{5}h}}$$

→ Locate the centroid of the area bounded by the curve as shown in fig



Sol) <u>To find $\bar{x}$:-</u>



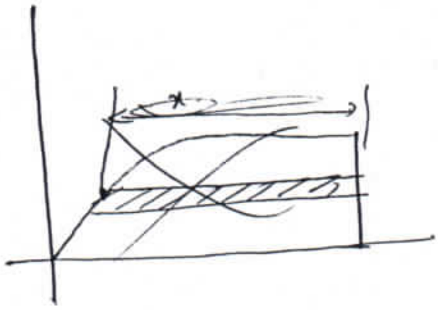
$$dA = y.dx.$$

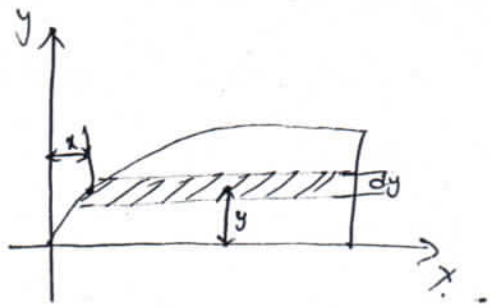
$$\bar{x} = \frac{\int x\,dA}{\int dA} = \frac{\int_0^a x.y\,dx}{\int_0^a y.dx}$$

$$= \frac{\int_0^a \frac{ax^3}{10}\,dx}{\int_0^a \frac{ax^2}{10}\,dx}$$

$$= \frac{\frac{a}{10}\left[\frac{x^4}{4}\right]_0^a}{\frac{a}{10}\left[\frac{x^3}{3}\right]_0^a} = a^4/4 \times 3/a^3 = 3a/4.$$

To find $\bar{y}$:-





$$dA = (a-x)\,dy \quad , \quad \bar{y} = \frac{\int y\,dA}{\int dA}$$

$$\bar{Y} = \frac{\int_0^{a/4} y(a-x)\,dy}{\int_0^{a/4} (a-x)\,dy} = \frac{\int_0^{a/4} y\left(a - \frac{\sqrt{10}\sqrt{y}}{\sqrt{a}}\right) dy}{\int_0^{a/4} \left(a - \frac{\sqrt{10}\sqrt{y}}{\sqrt{a}}\right) dy}$$

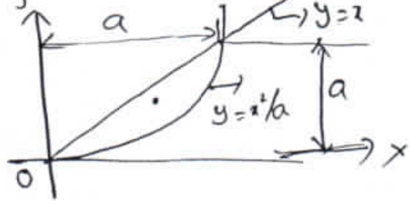
$$= \frac{\int_0^{a/4} ay\,dy - \int_0^{a/4} \frac{\sqrt{10}}{\sqrt{a}} y^{3/2}\,dy}{\int_0^{a/4} a\,dy - \int_0^{a/4} \frac{\sqrt{10}}{\sqrt{a}} y^{1/2}\,dy} = \frac{\left[a\frac{y^2}{2}\right]_0^{a/4} - \left[\frac{\sqrt{10}}{\sqrt{a}} y^{5/2} \times 2/5\right]_0^{a/4}}{\left[ay\right]_0^{a/4} - \left[\frac{\sqrt{10}}{\sqrt{a}} y^{3/2} \times 2/3\right]_0^{a/4}}$$

$$= \frac{\left[\frac{a}{2}\cdot\frac{a^2}{16}\right] - \left[\frac{2\sqrt{10}}{5\sqrt{a}} \times \frac{a^{5/2}}{32}\right]}{\left[a \times \frac{a}{4}\right] - \left[\frac{2\sqrt{10}}{3\sqrt{a}} \times \frac{a^{3/2}}{8}\right]} = \frac{\left[\frac{a^3}{32}\right] - \left[\frac{\sqrt{10}}{80} \times a^2\right]}{\left[a^2/4\right] - \left[\frac{\sqrt{10}}{12}\cdot a\right]}$$

$$= \frac{5a^3 - 2\sqrt{10}\,a^2/160}{\frac{3a^2 - \sqrt{10}\,a}{12}} = \frac{\frac{a^{\cancel{2}}}{\cancel{160}_{40}}\left[5a - 2\sqrt{10}\right]}{\frac{\cancel{a}}{\cancel{12}_{3}}\left[3a - \sqrt{10}\right]}$$

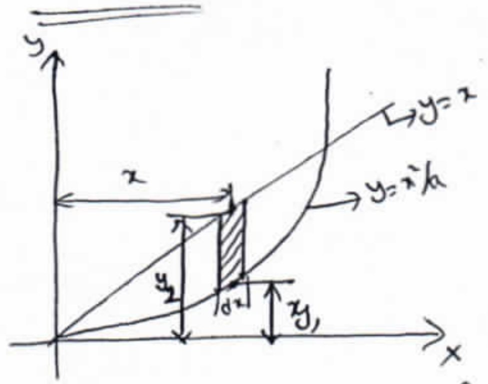
$$\bar{y} = \frac{3a}{40}\left(\frac{5a - 2\sqrt{10}}{3a - \sqrt{10}}\right)$$

→ Show that the coordinates of the centroid G of the area b/w the parabola $y = x^2/a$ & the straight line $y = x$ are $\bar{x} = a/2$, $\bar{y} = 2a/5$



Sol

Show $\bar{x} = a/2$



$$dA = (y_2 - y_1)\,dx.$$

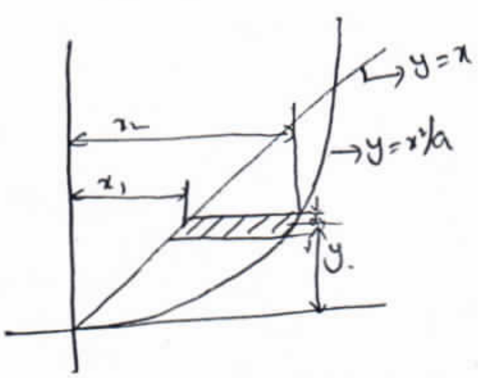
$$\bar{x} = \frac{\int x\,dA}{\int dA} = \frac{\int_0^a x\cdot(y_2-y_1)\,dx}{\int (y_2-y_1)\,dx}$$

$$\bar{x} = \frac{\int_0^a (x\cdot x)\,dx - \int_0^a \frac{x^2}{a}\cdot x\cdot dx}{\int_0^a \left(x - \frac{x^2}{a}\right) dx}$$

$$= \frac{\left[x^3/3\right]_0^a - \left[\frac{x^4}{4a}\right]_0^a}{\left[\frac{x^2}{2} - \frac{x^3}{3a}\right]_0^a} = \frac{a^3/3 - \frac{a^4}{4a}}{\frac{a^2}{2} - a^2/3} = \frac{\frac{4a^3-3a^3}{12}}{\frac{3a^2-2a^2}{6}}$$

$$= \frac{a^3}{12} \times \frac{6}{a^2} = a/2$$

show $\bar{y} = 2a/5$.



$$dA = (x_2 - x_1)\,dy.$$

$$\bar{y} = \frac{\int y\,dA}{\int dA} = \frac{\int_0^a y(x_2-x_1)\,dy}{\int_0^a (x_2-x_1)\,dy}$$

$$= \frac{\int_0^a y(\sqrt{ay}-y)\,dy}{\int_0^a (\sqrt{ay}-y)\,dy} = \frac{\int_0^a (\sqrt{a}\,y^{3/2}-y^2)\,dy}{\int_0^a (\sqrt{a}\,y^{1/2}-y)\,dy}$$

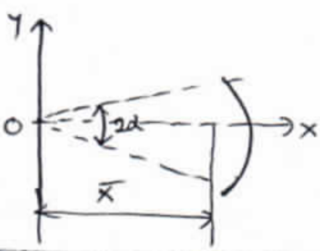
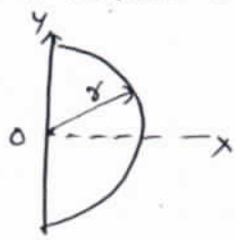
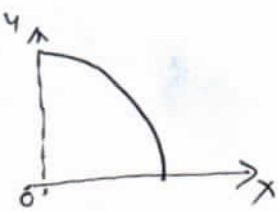
$$= \frac{\left[\sqrt{a}\,\frac{y^{5/2}}{5/2} - y^3/3\right]_0^a}{\left[\sqrt{a}\cdot\frac{y^{3/2}}{3/2} - y^2/2\right]_0^a} = \frac{\left[\frac{2}{5}\,a^{1/2}\times a^{5/2}\right] - \left[a^3/3\right]}{\left[2/3\cdot a^{1/2}\cdot a^{3/2}\right] - \left[a^2/2\right]}$$

$$\bar{y} = \frac{2/5\, a^3 - \frac{a^3}{3}}{\frac{2}{3}a^2 - \frac{a^2}{2}} = \frac{2a}{5}$$

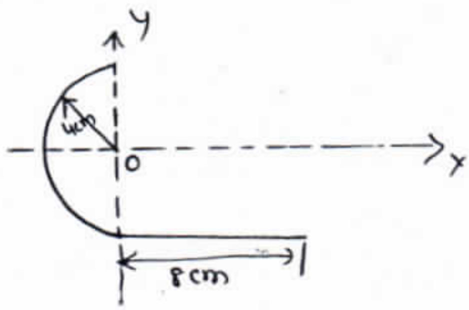
Centroids of Lines:

$$\bar{x} = \frac{l_1 x_1 + l_2 x_2 + \ldots}{l_1 + l_2}, \quad \bar{y} = \frac{l_1 y_1 + l_2 y_2 + \ldots}{l_1 + l_2}$$

Centroid for common lines (or) Arcs:-

Line	Length	$\bar{x}$	$\bar{y}$
a) Segment of Arc 	$2\alpha r$.	$\frac{r \cdot \sin\alpha}{\alpha}$	0
b) Semi-circular Arc. 	πr.	$\frac{2r}{\pi}$	0
c) Quarter-Circular Arc 	$\frac{\pi r}{2}$	$\frac{2r}{\pi}$	$\frac{2r}{\pi}$

→ A Slender homogeneous wire of uniform cross section is bent into the form shown in fig. Determine the position of centroid of the wire w.r.t. given axis.



So) $l_1 = 8cm$, $x_1 = 4cm$, $y_1 = -4cm$

$l_2 = \pi r = \pi(4) = 12.57\ cm$

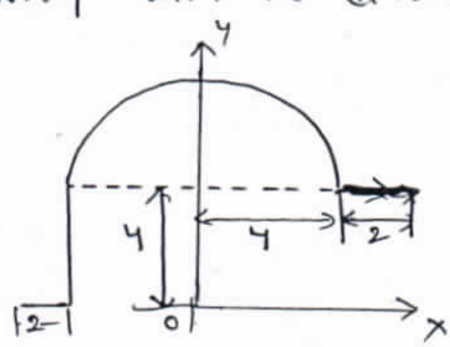
$$x_1 = -\frac{2r}{\pi} = -2.546\ cm\ ,\ y_1 = 0$$

$$\bar{x} = \frac{l_1 x_1 + l_2 x_2}{l_1 + l_2} = 0$$

$$\bar{y} = \frac{l_1 y_1 + l_2 y_2}{l_1 + l_2} = -1.556\ cm$$

(i.e) centroid lie on y-axis 1.556 cm below the x-axis.

→ Locate the centroid of the length of the mean center line of the stirrup with the dimensions shown in fig.



So) $l_1 = 2cm$, $x_1 = 4 + \frac{2}{2} = -5\ cm$, $y_1 = 0$

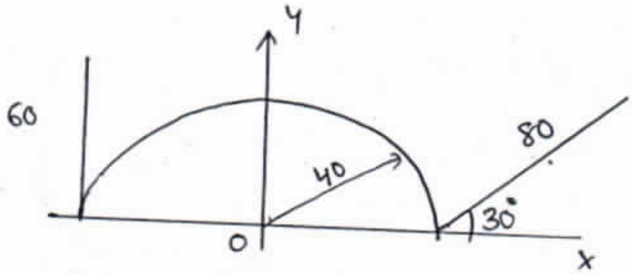
$l_2 = (4)\ cm$, $x_1 = -4cm$, $y_1 = 2cm$

$l_3 = 4\pi = 12.57\ cm$, $x_3 = 0$, $y_3 = 4 + \frac{2r}{\pi} = 6.54\ cm$

$l_4 = 2cm$, $x_4 = 5cm$, $y_4 = 4\ cm$

$$\bar{x} = -0.78\ cm\ ,\ \bar{y} = 4.78\ cm$$

→



Sol

$l_1 = 60\,mm, \quad x_1 = -40\,mm, \quad y_1 = 30\,mm$

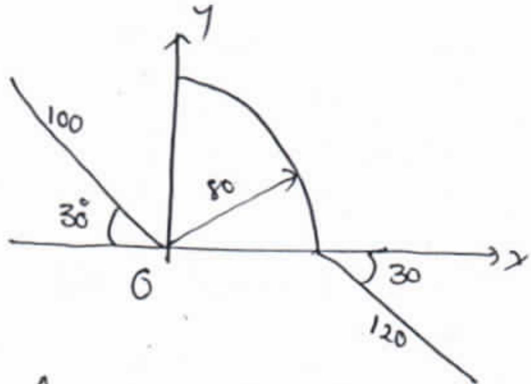
$l_2 = \pi r = 125.6\,mm, \quad x_2 = 0, \quad y_2 = \frac{2r}{\pi} = 25.46\,mm$

$l_3 = 80\,mm, \quad x_3 = (40 + 40\cos 30) = 74.64\,mm$

$y_3 = 40\sin 30 = 20\,mm$

$\bar{x} = 13.44\,mm, \quad \bar{y} = 24.84\,mm$

→



$l_1 = 100, \quad x_1 = -50\cos 30 = -43.3\,mm$

$y_1 = 50\sin 30 = 25\,mm$

$l_2 = \frac{\pi r}{2} = 125.66\,mm$

$x_2 = \frac{2r}{\pi} = 50.93\,mm$

$y_2 = 50.93\,mm$

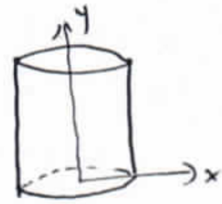
$l_3 = 120\,mm, \quad x_3 = 80 + 60\cos 30 = 131.96\,mm$

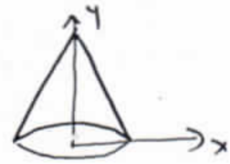
$y_3 = -60\sin 30 = -30\,mm$

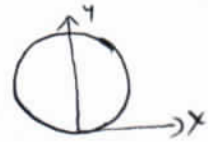
$\bar{x} = 51.8\,mm, \quad \bar{y} = 15.33\,mm.$

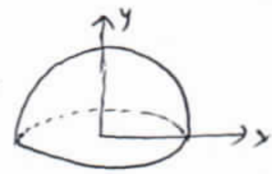
Centre of gravity of solids :-

$$\bar{x} = \frac{V_1 x_1 + V_2 x_2 + \dots}{V_1 + V_2 + \dots}, \quad \bar{y} = \frac{V_1 y_1 + V_2 y_2 + \dots}{V_1 + V_2}$$

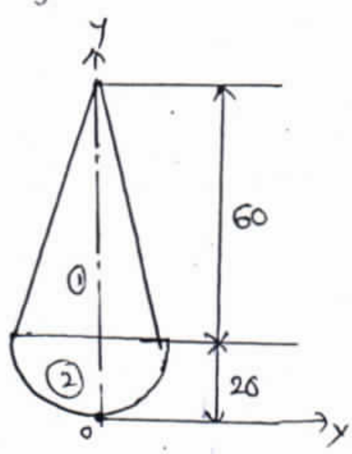
Cylinder  $V = \pi R^2 h$ $\quad \bar{y} = h/2$

Cone  $V = \frac{\pi R^2 h}{3}$ $\quad \bar{y} = \frac{h}{4}$

Sphere  $V = \frac{4}{3}\pi R^3$ $\quad \bar{y} = R$

Hemi-sphere  $V = \frac{2}{3}\pi R^3$ $\quad \bar{y} = \frac{3R}{8}$

→ A solid Hemi-Sphere of 20mm radius supports a solid cone of s base and height 60mm as shown in fig. Locate the C.G. of section



Sol

$$V_1 = \frac{\pi r^2 . h}{3} = \pi \frac{(20)^2 \times 60}{3} = 25132.74 \text{ mm}^3$$

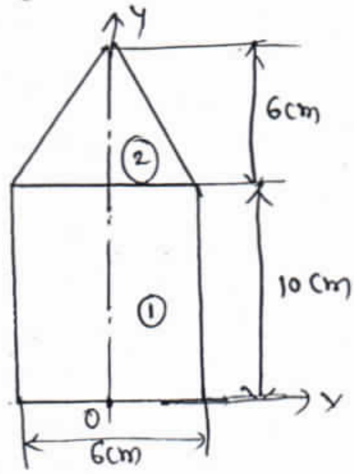
$$y_1 = \left(20 + \frac{60}{4}\right) = 35 \text{ mm}$$

$$V_2 = \frac{2}{3}\pi r^3 = \frac{2}{3}\pi (20)^3 = 16755.16 \text{ mm}^3$$

$$y_2 = \left(20 - \frac{3r}{8}\right) = \left(20 - \frac{3 \times 20}{8}\right) = 12.5 \text{ mm}$$

$$\bar{y} = \frac{V_1 y_1 + V_2 y_2}{V_1 + V_2} = 26 \text{mm}$$

→ A Solid cone having bye diameter 6 cm & height 6 cm is kept co-axially on a solid cylinder having 6 cm dia & 10 cm height find the C.G. of the combination



Sol

$V_1 = \pi r^2 h = \pi (3)^2 \times 10 = 282.74\ cm^3$

$y_1 = \frac{10}{2} = 5\ cm.$

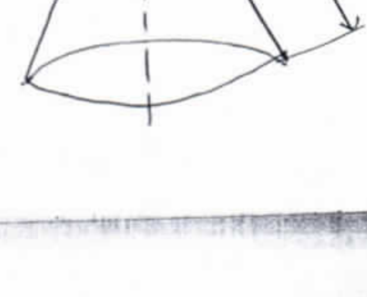
$V_2 = \frac{\pi r^2 h}{3} = \frac{\pi (3)^2 \times 6}{3} = 56.54\ cms$

$y_2 = 10 + \frac{6}{4} = 11.5\ cm$

$\bar{y} = \frac{V_1 y_1 + V_2 y_2}{V_1 + V_2} = \underline{\underline{6.08\ cm}}$

Pappus - Guldinus theorems:-

first theorem: The Area of surface of revolution is equal to the length of the generating curve times the distance travelled by the Centroid of the generating curve while the surface is generated.



$A = \underline{L(2\pi)x}$

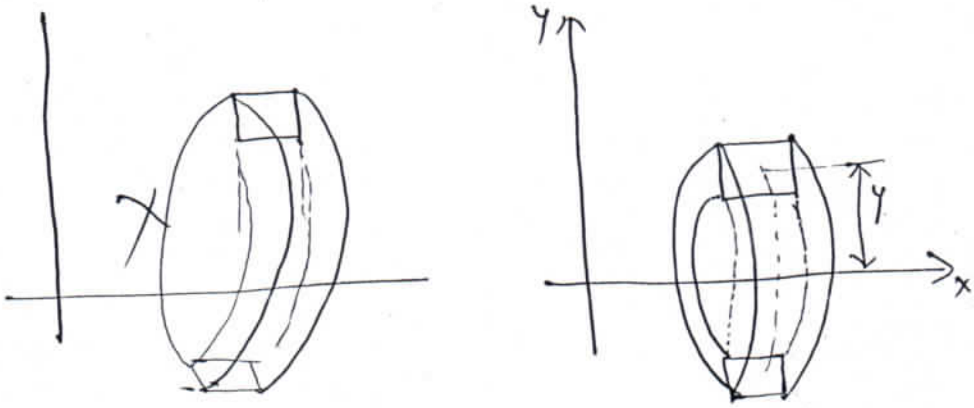
Proof: Consider an elemental length dl of the generator 'L' generates a surface area dA when revolves about Y-axis

$$dA = 2\pi x\, dl.$$

$$\int dA = 2\pi x \int_0^L dl$$

$$\underline{A = 2\pi x L}$$

Second theorem:- The volume of a body formed by rotating an area about an axis equal to area multiplied by the distance covered by the ~~distance~~ centroid during the rotation.



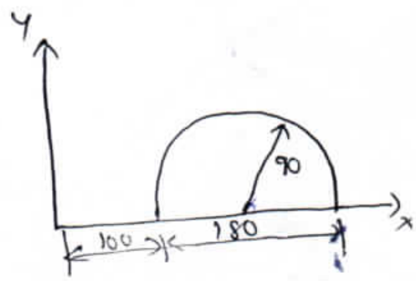
$$V = A \times (2\pi \times y)$$

$$dV = 2\pi y \times dA$$

$$\int dV = 2\pi y \int dA$$

$$\underline{V = 2\pi y A}$$

-) find the volume generated by revolving a semi-circular Area about x-axis & y-axis



Sol: Area of Semi-circle $= \frac{\pi r^2}{2} = \frac{\pi \times 90^2}{2} = 12723.45\ mm^2$

$$\bar{y} = 4r/3\pi = \frac{4 \times 90}{3\pi} = 38.2\ mm$$

$$V = (2\pi\bar{y})A = (2\pi \times 38.2) \times 12723.45$$

$$= 3053628\ mm^3$$

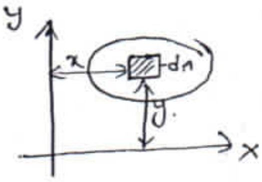
$\bar{x} = (100 + \frac{180}{2}) = 190\,mm$

$V = (2\pi\bar{x})A = (2\pi \times 190)(12723.45)$

$= 15189321\,mm^3$.

Moment of Inertia:

The M.I. of an area is the Sum of product of Area of section and square distance of the C.G. of the Area from that axis.



$I_{xx} = \Sigma dA . y^2$.

$I_{yy} = \Sigma dA . x^2$

Radius of gyration: The distance of a point where the whole Area of a section is assumed to be concentrated from a given axis is called the radius of gyration. It is denoted by k

$$k = \sqrt{\frac{I}{A}}.$$

Theorems of M.I:

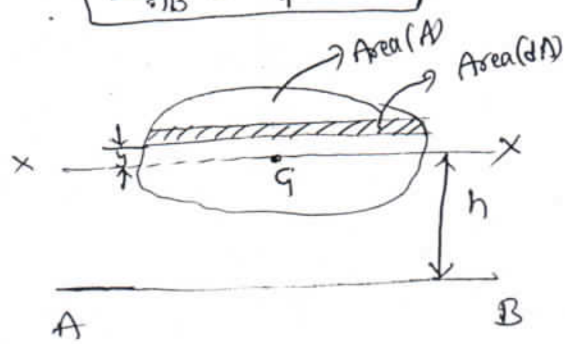
i) parallel-axis Theorem (Transfer formula)

ii) perpendicular axis " (Polar M.I).

Parallel axis Theorem: It states that the M.I. of a plane area about any axis is equal to the sum of the M.I about the centroidal axis & the product of the area x square of the distance b/w two axis & is given by

$$\boxed{I_{AB} = I_G + Ah^2}$$

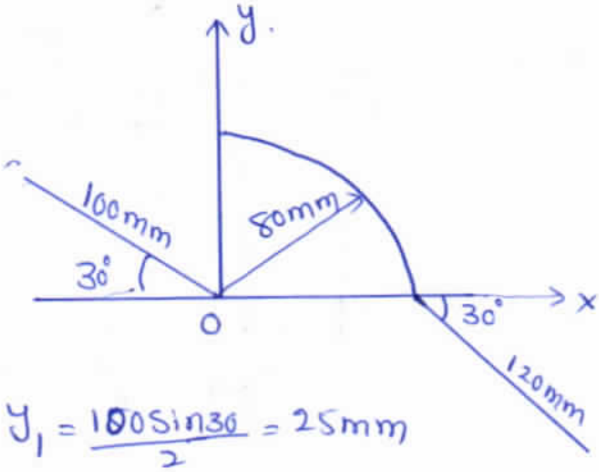
Proof:



Centroid & Centre of Gravity

Centroid of Lines:-

→ Locate the centroid of the wire bent as shown in fig.



Sol

$L_1 = 100mm$

$L_3 = \frac{\pi r}{2} = \frac{\pi \times 80}{2} = 125.56mm$

$L_2 = 80mm$

$L_4 = 120mm$

$x_1 = -\frac{100\cos 30}{2} = -43.3mm, \quad y_1 = \frac{100\sin 30}{2} = 25mm$

$x_2 = 0, \quad y_2 = \frac{80}{2} = 40mm$

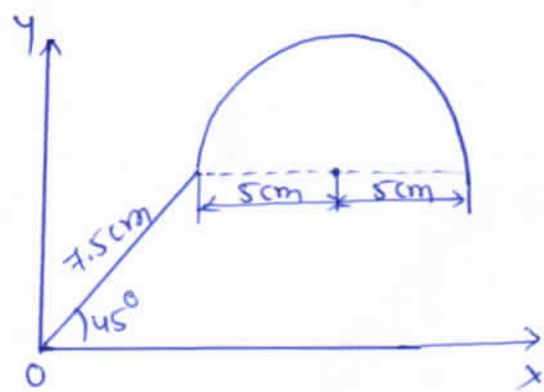
$x_3 = \frac{2r}{\pi} = \frac{2\times 80}{\pi} = 50.93mm, \quad y_3 = \frac{2r}{\pi} = 50.93mm$

$x_4 = 80 + \frac{120\cos 30}{2} = 131.96mm, \quad y_4 = -\frac{120\sin 30}{2} = -30mm.$

$$\bar{x} = \frac{L_1x_1 + L_2x_2 + L_3x_3 + L_4x_4}{L_1 + L_2 + L_3 + L_4}, \quad \bar{y} = \frac{L_1y_1 + L_2y_2 + L_3y_3 + L_4y_4}{L_1 + L_2 + L_3 + L_4}$$

$\bar{x} = \underline{42.06mm} \qquad \bar{y} = \underline{20mm}$

→ Locate the centroid of bent wire as shown in fig.



$L_1 = 7.5cm$

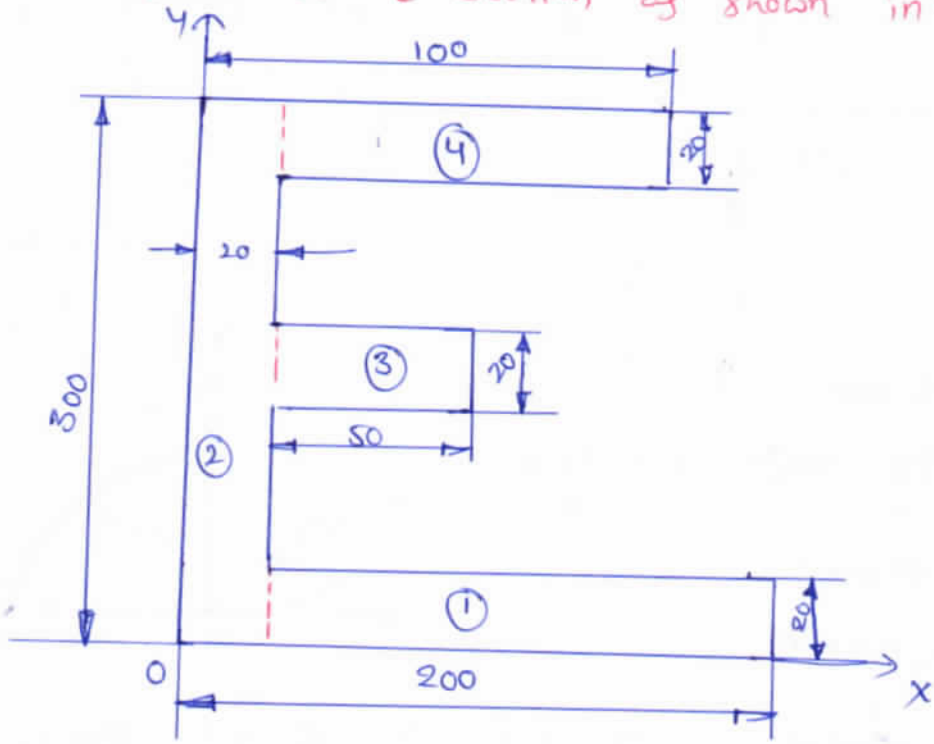
$L_2 = \pi r = \pi \times \cancel{5} = 15.7cm$

$x_1 = \frac{7.5\cos 45}{2} = 2.65cm, \quad y_1 = \frac{7.5\sin 45}{2} = 2.65cm$

$y_2 = 7.5\sin 45 + \frac{2\times 5}{\pi} = \cancel{5.83cm}\ 8.48cm, \quad x_2 = 7.5\cos 45 + 5 = 10.3cm$

$\bar{y} = 6.59cm, \quad \bar{x} = \underline{7.82cm}$

→ find the centroid of E-Section as shown in fig.



Sol

$a_1 = 180 \times 20 = 3600\ mm^2$, $a_2 = 300 \times 20 = 6000\ mm^2$

$a_3 = 50 \times 20 = 1000 mm^2$, $a_4 = 80 \times 20 = 1600\ mm^2$

$x_1 = 20 + \frac{180}{2} = 110mm$, $y_1 = \frac{20}{2} = 10mm$

$x_2 = \frac{20}{2} = 10mm$, $y_2 = \frac{300}{2} = 150mm$

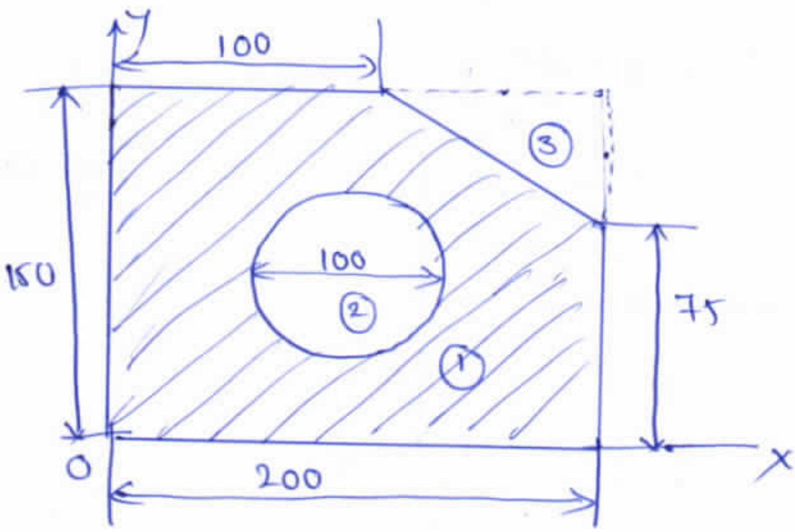
$x_3 = 20 + \frac{50}{2} = 45mm$, $y_3 = 140 + \frac{20}{2} = 150mm$

$x_4 = 20 + \frac{80}{2} = 60mm$, $y_4 = 280 + \frac{20}{2} = 290mm$

$$\bar{x} = \frac{a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4}{a_1 + a_2 + a_3 + a_4} = 48.9mm$$

$$\bar{y} = \frac{a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4}{a_1 + a_2 + a_3 + a_4} = 127.05mm$$

→ find the centroid of Shaded Area



$a_1 = 200 \times 150 = 30{,}000\ mm^2$, $a_2 = \pi r^2 = \pi (50)^2 = 7853.9mm$

$a_3 = \frac{1}{2} \times \cancel{2}75 \times 100 = 3750mm^2$

$x_1 = \frac{200}{2} = 100mm$, $y_1 = \frac{150}{2} = 75mm$

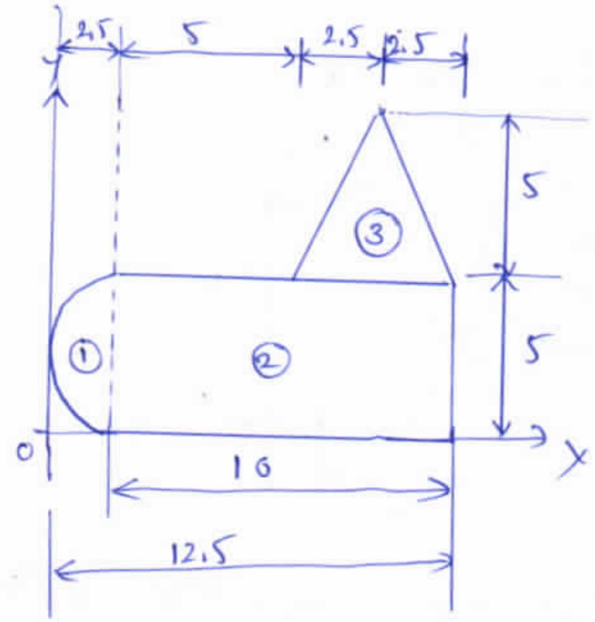
$x_2 = 50 + \frac{100}{2} = 100mm$, $y_2 = 25 + \frac{100}{2} = 75mm$

$x_3 = 200 \cancel{+} \frac{100}{3} = 166.67\ mm$, $y_3 = 150 - \frac{75}{3} = 125mm$

$\bar{x} = \frac{a_1 x_1 - a_2 x_2 - a_3 x_3}{a_1 - a_2 - a_3} = \cancel{106.00\ cm}\ 86.4\ cm$

$\bar{y} = \frac{a_1 y_1 - a_2 y_2 - a_3 y_3}{a_1 - a_2 - a_3} = 64.80\ cm$

→ find the Centroid of the Section as shown in fig.



In cm

$a_1 = \frac{\pi r^2}{2} = \frac{\pi \times 2.5^2}{2} = 9.81\ cm^2, \quad a_2 = 10 \times 5 = 50\ cm^2$

$a_3 = \frac{1}{2} \times 5 \times 5 = 12.5\ cm^2$

$x_1 = \left(2.5 - \frac{4 \times 2.5}{3\pi}\right) = 1.43\ cm, \quad y_1 = 5/2 = 2.5\ cm$

$x_2 = 2.5 + \frac{10}{2} = 7.5\ cm, \quad y_2 = 5/2 = 2.5\ cm$

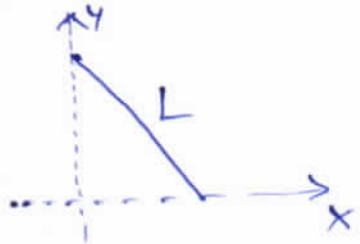
$x_3 = 7.5 + \frac{5}{2} = 10\ cm, \quad y_3 = 5 + \frac{5}{3} = 6.67\ cm$

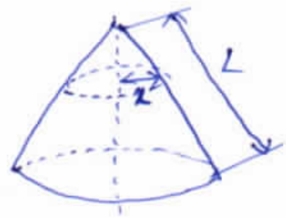
$$\bar{X} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3}{a_1 + a_2 + a_3} = 7.1\ cm$$

$$\bar{Y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3} = 3.22\ cm$$

Pappu's theorem - 1 :-

The Area generated by a curve/Line by rotating about an axis is Equal to the product of the length of the curve/line & the distance travelled by the centroid of the curve/line.

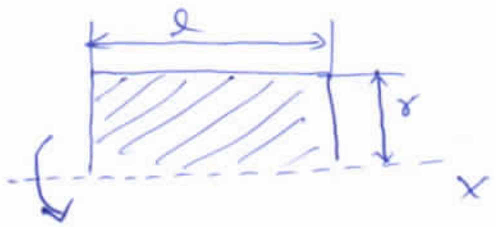


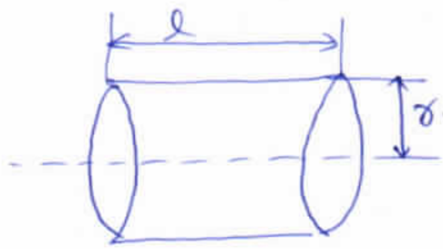


A = (Length) x (Distance moved by Centroid)

Pappu's theorem 2:-

The volume generated by a surface by rotating about an axis is Equal to the product of Area of the surface & distance travelled by the centroid of the surface.





V = (Area) x (distance moved by centroid)

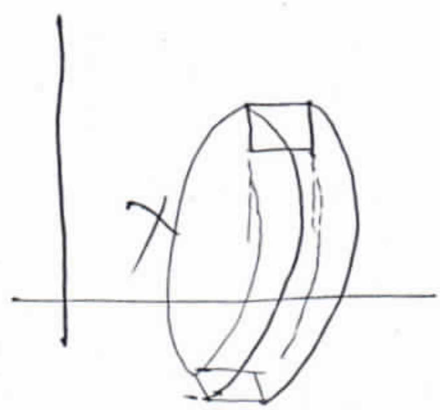
proof Consider an elemental length dl of the generator 'L' generating a Surface area dA when revolving about Y-axis

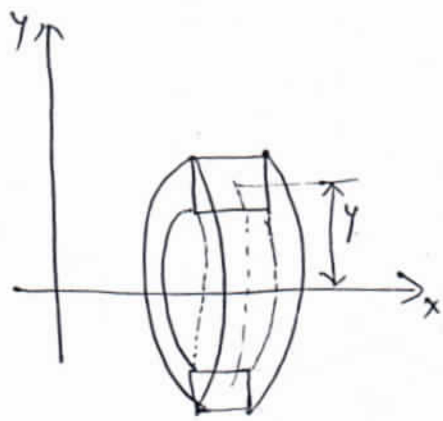
$$dA = 2\pi x \, dl.$$

$$\int dA = 2\pi x \int_0^L dl$$

$$\underline{A = 2\pi x L}$$

Second theorem:- The volume of a body formed by rotating an area about an axis Equal to area multiplied by the distance covered by the ~~distance~~ Centroid during the rotation.





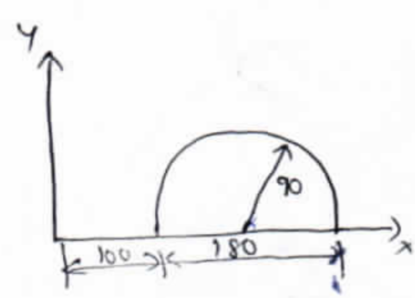
$$V = A \times (2\pi \times y)$$

$$dV = 2\pi y \times dA$$

$$\int dV = 2\pi y \int dA$$

$$\underline{V = 2\pi \bar{y} A}$$

-) find the volume generated by revolving a Semi-Circular Area about x-axis & Y-axis



Sol Area of Semi-Circle $= \frac{\pi r^2}{2} = \frac{\pi \times 90^2}{2} = 12723.45 \text{ mm}^2$

$$\bar{y} = 4r/3\pi = \frac{4 \times 90}{3\pi} = 38.2 \text{ mm}$$

$$V = (2\pi \bar{y}) A = (2\pi \times 38.2) \times 12723.45$$

$$= 3053628 \text{ mm}^3$$

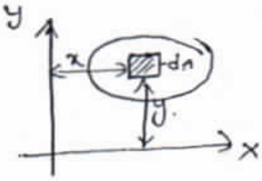
$$\bar{x} = \left(100 + \frac{180}{2}\right) = 190\,mm$$

$$V = (2\pi\bar{x})\,A = (2\pi \times 190)\,(12723.43)$$

$$= 15189321\ mm^3$$

Moment of Inertia:-

The M.I. of an area is the Sum of product of Area of section and Square distance of the C.G. of the Area from that axis.



$$I_{xx} = \Sigma dA.y^2$$

$$I_{yy} = \Sigma dA.x^2$$

Radius of gyration:- The distance of a point where the whole Area of a section is assumed to be concentrated from a given axis is called the radius of gyration. It is denoted by k

$$k = \sqrt{\frac{I}{A}}$$

Theorems of M.I:-

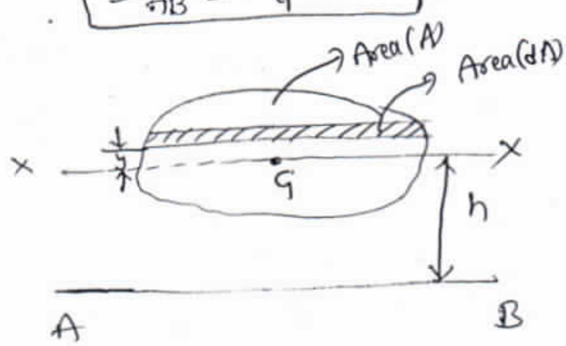
i) parallel-axis Theorem (Transfer formula)

ii) perpendicular axis " (Polar M.I).

Parallel axis Theorem:- It states that the M.I. of a plane area about any axis is equal to the sum of the M.I about the centroidal axis & the product of the area x square of the distance b/w two axis & is given by

$$\boxed{I_{AB} = I_G + Ah^2}$$

Proof:



M.I. of the total area 'A' about A-B axis

$$I_{AB} = \Sigma dA \cdot (y+h)^2$$
$$= \Sigma dA \; (y^2+h^2+2hy)$$
$$= \Sigma dA y^2 + \Sigma dA \cdot h^2 + 2h \; \Sigma dA \cdot y$$

$$\bar{y} = \frac{\Sigma dA \cdot y}{\Sigma dA}$$

But $\Sigma dA y^2 = I_G$ (or) I_{xx}

$\Sigma dA = A$.

$$I_{AB} = I_G + Ah^2 + 2h \; \Sigma dA \cdot y$$

$\Sigma dA \cdot y = 0$.

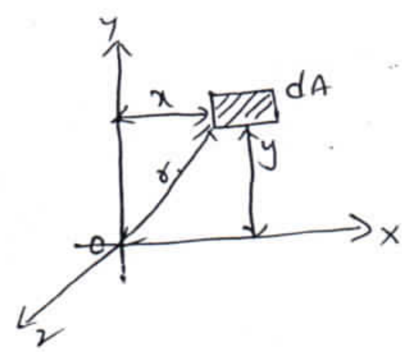
$$\boxed{I_{AB} = I_G + Ah^2}$$

Perpendicular axis Theorem:

It states that if I_{xx} & I_{yy} are the M.I. of a plane section about two mutually $\perp^{ar}$ axis x-x & y-y in the plane of section, then the M.I of the section I_{zz} about the axis z-z is given by.

$$I_{zz} = I_{xx} + I_{yy}.$$

Proof:



from fig. $r^2 = x^2+y^2$

M.I about x-axis $= I_{xx} = \Sigma dA \cdot y^2$

$I_{yy} = \Sigma dA \cdot x^2$

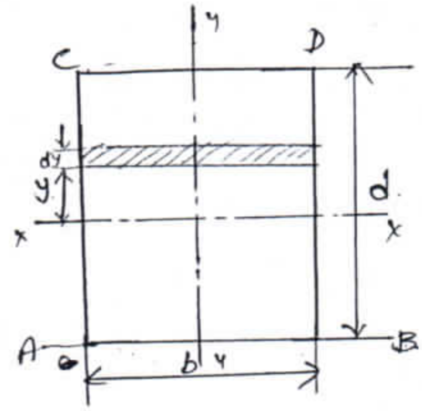
$I_{zz} = \Sigma dA \cdot r^2$

$$I_{zz} = \Sigma dA \, (x^2+y^2)$$
$$I_{zz} = \Sigma dA x^2 + \Sigma dA \cdot y^2$$
$$= \cancel{I_{yy}} + I_{xx}$$
$$\boxed{I_{zz} = I_{xx} + I_{yy}}$$

zz-axis is known as polar axis & the M.I about this polar is known as polar M.I.

M.I of some Standard Geometrical Shapes by Integration:-

1. M.I of Rectangle:-



Area of strip = $b.dy$.

M.I. of the elementary strip about XX

$$= dA \cdot y^2 = (b.dy) \cdot y^2$$

Total M.I about X-X

$$I_{xx} = 2\int_0^{d/2} b.y^2.dy$$

$$= 2b\left(\frac{y^3}{3}\right)_0^{d/2} = \frac{2b.d^3}{24} = \frac{bd^3}{12}$$

Similarly M.I about Y-Y $\quad I_{yy} = \frac{db^3}{12}$

M.I about base:-

$$I_{AB} = I_{xx} + Ah^2$$

$$= \frac{bd^3}{12} + (b\times d)\left(\frac{d}{2}\right)^2$$

$$= \frac{bd^3}{12} + \frac{bd^3}{4} = \frac{bd^3}{3}$$

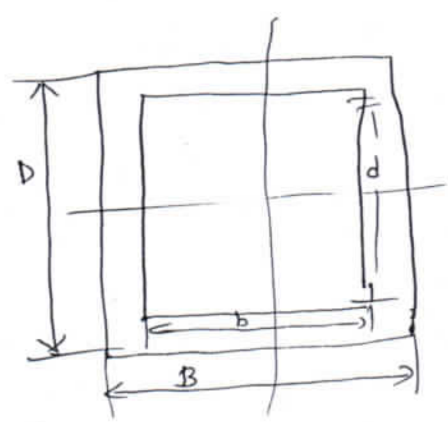
Similarly from 'AC' $\quad I_{AC} = \frac{db^3}{3}$

for square section $b = d = a$, a = side of the square

$$I_{xx} = I_{yy} = \frac{a^4}{12}$$

$$I_{AB} = I_{AC} = \frac{a^4}{3}$$

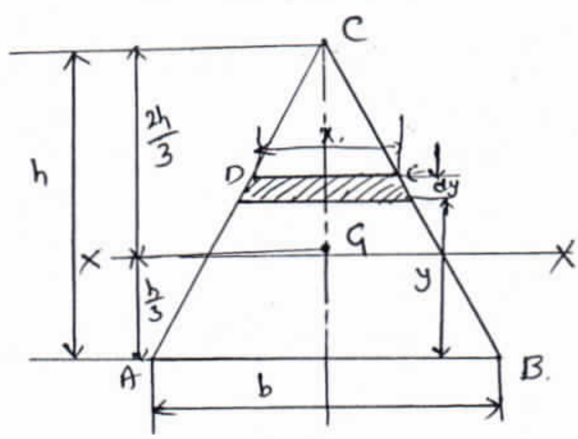
M.I of hollow rectangular section:



$$I_{xx} = \frac{BD^3 - bd^3}{12}$$

$$I_{yy} = \frac{DB^3 - db^3}{12}$$

→ M.I of triangular section:-



Area of strip $dA = x\,dy$

Similar Δ^{les} ABC & DEC

$$\frac{x}{b} = \frac{h-y}{h} \Rightarrow x = \frac{b(h-y)}{h}$$

$$dA = \frac{b(h-y)}{h} dy$$

$$I_{AB} = \int_0^h dA.y^2 = \int_0^h \frac{b(h-y)}{h} . y^2 . dy$$

$$= b\int_0^h y^2 dy - \frac{b}{h}\int_0^h y^3 dy$$

$$= b\left[\frac{y^3}{3}\right]_0^h - \frac{b}{h}\left[\frac{y^4}{4}\right]_0^h$$

$$= \frac{bh^3}{3} - \frac{bh^3}{4} = \frac{4bh^3 - 3bh^3}{12} = \frac{bh^3}{12} \Rightarrow I_{AB} = \frac{bh^3}{12}$$

$I_{xx} = ?$

Acc. to parallel axis theorem

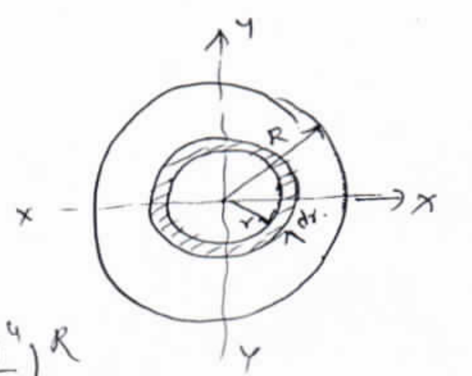
$$I_{AB} = I_{xx} + Ah^2$$

$$I_{xx} = I_{AB} - Ah^2$$

$$= \frac{bh^3}{12} - \frac{bh}{2}\left(\frac{h}{3}\right)^2$$

$$= \frac{bh^3}{12} - \frac{bh^3}{18} = \frac{bh^3}{36} \Rightarrow I_{xx} = \frac{bh^3}{36}$$

→ M.I of Circular section:



Area of circular ring $= 2\pi r . dr = dA$

$$I_{zz} = \int dA\, r^2$$

$$= \int_0^R 2\pi r . dr . r^2$$

$$= 2\pi \int_0^R r^3 dr = 2\pi \left(\frac{r^4}{4}\right)_0^R$$

$$I_{zz} = 2\pi \times \frac{R^4}{4/2} = \frac{\pi R^4}{2} \quad \text{where } R = D/2$$

$$I_{zz} = \frac{\pi (D/2)^4}{2} = \frac{\pi D^4}{32}.$$

from $\perp^{ar}$ axis Theorem.

$$I_{zz} = I_{xx} + I_{yy}$$

$$I_{xx} = I_{yy} \text{ (b/c of symmetry)}$$

$$I_{zz} = 2 I_{xx} \Rightarrow I_{xx} = \frac{\pi D^4}{32 \times 2} = \frac{\pi D^4}{64}$$

Semi-circle

$$I_{AB} = I_{yy} = \frac{\pi D^4}{128}$$

$$I_{xx} = 0.11 R^4$$

Quadrant

$$I_{AB} = I_{AC} = \frac{\pi R^4}{16}$$

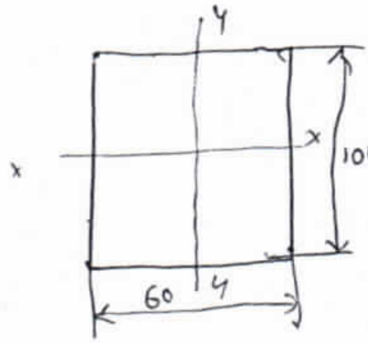
$$I_{xx} = I_{yy} = 0.055 R^4$$

for Hollow circular section -

$$I_{xx} = I_{yy} = \frac{\pi}{64}(D^4 - d^4)$$

$$I_{zz} = \frac{\pi}{32}(D^4 - d^4).$$

→) find the M.I of rectangle as shown in fig. about Centroidal axis (I_{xx} & I_{yy}) & its bye



Sol

$$I_{xx} = \frac{bd^3}{12} = \frac{60 \times 100^3}{12} = 5 \times 10^6 \text{ mm}^4$$

$$I_{yy} = \frac{db^3}{12} = \frac{100 \times 60^3}{12} = 18 \times 10^5 \text{ mm}^4$$

$$I_{AB} = \frac{bd^3}{3} = \frac{60 \times 100^3}{3} = 2 \times 10^7 \text{ mm}^4$$

→) find the m.I of rectangle 20mm wide & 30mm deep about a given axis, AB which is at a distance of 41 mm from its centroid

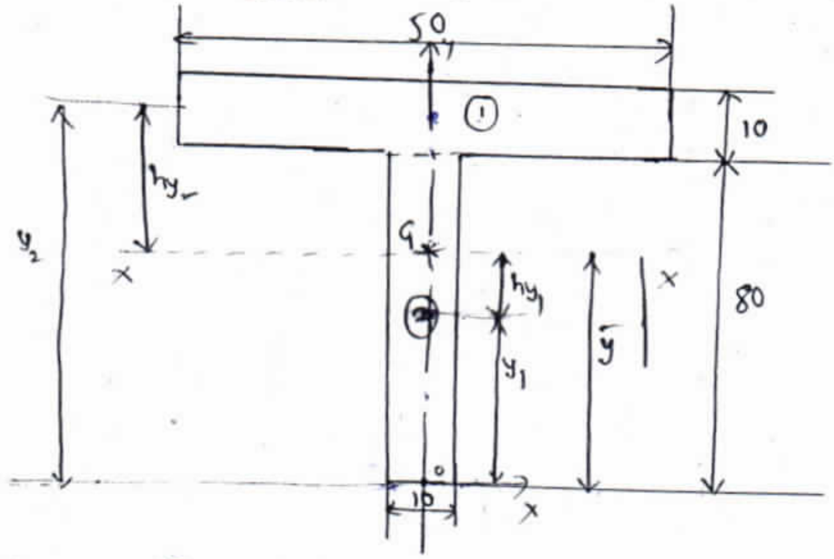
Sol

$$A = 20 \times 30 = 600 \text{ mm}^2$$

$$I_G = \frac{bd^3}{12} = \frac{20 \times 30^3}{12} = 45{,}000 \text{ mm}^4$$

$$I_{AB} = I_G + Ah^2 = 126 \times 10^4 \text{ mm}^4$$

→ A bar of T-section has a flange 50mm wide & 10mm thick. The web is 80mm deep & 10mm thick as shown in fig. find the M.I of the section about its centroidal axis (I_{xx}) & I_{yy}



$x_1 = 0$

$y_1 =$

Sol

$\bar{x} = 0, \quad \bar{y} = \dfrac{a_1 y_1 + a_2 y_2}{a_1 + a_2}$

$a_1 = 50 \times 10 = 500 mm^2, \quad a_2 = 10 \times 80 = 800 mm^2$

$y_1 = 80 + \dfrac{10}{2} = 85 mm, \quad y_2 = \dfrac{80}{2} = 40 mm$

$\bar{y} = 57.3 mm$

$I_{xx} = I_{xx_1} + I_{xx_2}$

$I_{xx_1} = I_{G_1} + A_1 h_{y_1}^2 \qquad I_{xx_2} = I_{G_2} + A_2 h_{y_2}^2$

$h_{y_1} = y_1 - \bar{y} = 85 - 57.3 \qquad h_{y_2} = \bar{y} - y_2 = 57.3 - 40$

$= 27.7 mm \qquad = 17.3 mm$

$I_{xx_1} = \dfrac{50 \times 10^3}{12} + (500)(27.7)^2 = 387811.67 mm^4$

$I_{xx_2} = \dfrac{10 \times 80^3}{12} + (800)(17.3)^2 = 666018.67 mm^4$

$I_{xx} = 1053910.3 mm^4$

$I_{yy} = I_{yy_1} + I_{yy_2}$ $\qquad h_{x_1} = 0$

$I_{yy_1} = \frac{d_1 b_1^3}{12} + A_1 h_{x_1}^2$ $\qquad h_{x_2} = 0$

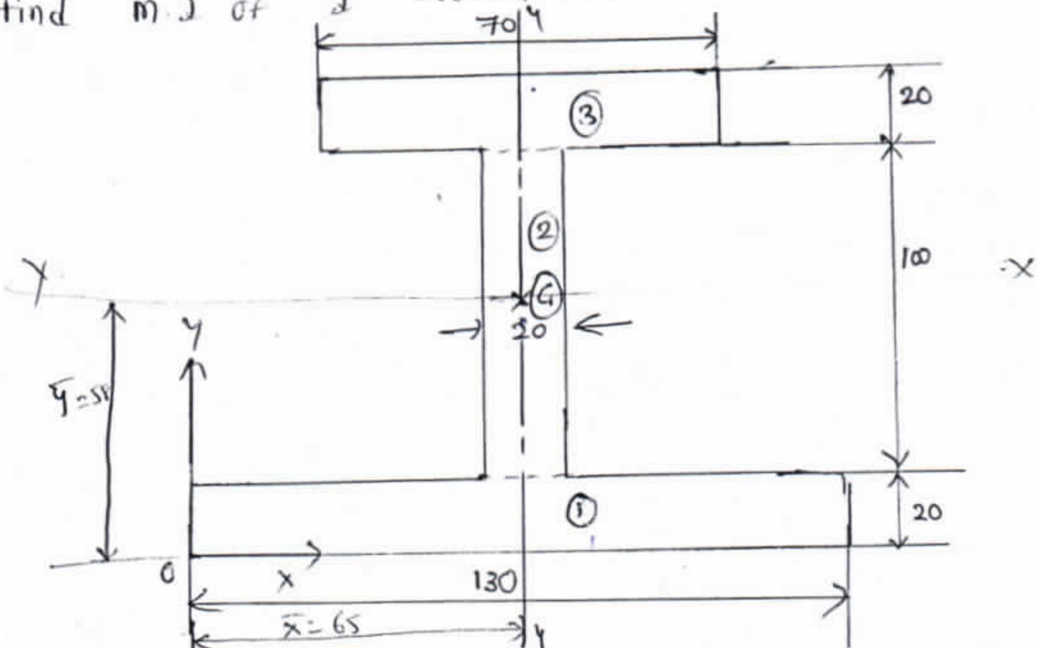
$= \frac{80 \times 10^3}{12} + (800 \times 0) = 6.6 \times 10^3 \text{ mm}^4$

$I_{yy_2} = \frac{d_2 b_2^3}{12} + A_2 h_{x_2}^2$

$= \frac{10 \times 50^3}{12} + 0 = 1.04 \times 10^5 \text{ mm}^4$

$I_{yy} = 1.1 \times 10^5 \text{ mm}^4$

2) find M.I of 'I' section about its centroidal axis as shown in fig



section is symmetric about y-axis.

$\bar{x} = \frac{130}{2} = 65 \text{ mm}$

$a_1 = 130 \times 20 = 2600 \text{ mm}^2, \quad a_2 = 20 \times 100 = 2000 \text{ mm}^2$

$a_3 = 70 \times 20 = 1400 \text{ mm}^2$

$y_1 = 20/2 = 10 \text{ mm}, \quad y_2 = 20 + \frac{100}{2} = 70 \text{ mm}, \quad y_3 = 120 + \frac{20}{2} = 130 \text{ mm}$

$\bar{y} = \frac{a_1 y_1 + a_2 y_2 + a_3 y_3}{a_1 + a_2 + a_3} = 58 \text{ mm}$

$h_{y_1} = 58 - 10 = 48 \text{ mm}, \quad h_{y_2} = 70 - 58 = 12, \quad h_{y_3} = 130 - 58 = 72 \text{ mm}$

$h_{x_1} = 0, \quad h_{x_2} = 0, \quad h_{x_3} = 0$

$$I_{xx} = I_{xx_1} + I_{xx_2} + I_{xx_3}$$

$$= \left(\frac{b_1 d_1^3}{12} + A_1 h_{y_1}^2\right) + \left(\frac{b_2 d_2^3}{12} + A_2 h_{y_2}^2\right) + \left(\frac{b_3 d_3^3}{12} + A_3 h_{y_3}^2\right)$$

$$= (6.07 \times 10^6) + (16.9 \times 10^6) + (7.3 \times 10^6)$$

$$\underline{\underline{I_{xx} = 30.27 \times 10^6 \ mm^4}}$$

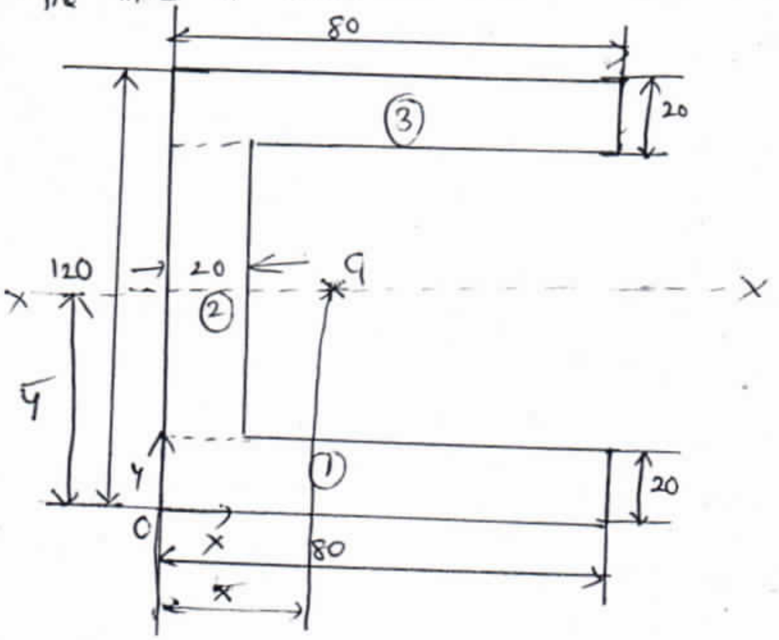
$h_{x_1} = h_{x_2} = h_{x_3} = 0$

$$I_{yy} = I_{yy_1} + I_{yy_2} + I_{yy_3}$$

$$= \left(\frac{d_1 b_1^3}{12} + A_1 h_{x_1}^2\right) + \left(\frac{d_2 b_2^3}{12} + A_2 h_{x_2}^2\right) + \left(\frac{d_3 b_3^3}{12} + A_3 h_{x_3}^2\right)$$

$$\underline{\underline{I_{yy} = 4.3 \times 10^6 \ mm^4}}$$

→ find the m.I of channel section as shown in fig.



Section is symmetric about x-axis

$$\bar{y} = \frac{120}{2} = 60mm$$

$a_1 = 80 \times 20 = 1600 mm^2$, $a_2 = 20 \times 80 = 1600 mm^2$, $a_3 = 1600 mm^2$

$x_1 = \frac{80}{2} = 40mm$, $x_2 = \frac{20}{2} = 10mm$, $x_3 = \frac{80}{2} = 40mm$

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2 + a_3 x_3}{a_1 + a_2 + a_3} = 30mm$$

$h_{x_1} = x_1 - \bar{x} = 40 - 30 = 10mm$ $\qquad$ $h_{y_1} = \bar{y} - y_1 = 60 - 10 = 50mm$

$h_{x_2} = \bar{x} - x_2 = 30 - 10 = 20mm$ $\qquad$ $h_{y_2} = \bar{y} - y_2 = 60 - 60 = 0$

$h_{x_3} = x_3 - \bar{x} = 40 - 30 = 10mm$ $\qquad$ $h_{y_3} = y_3 - \bar{y} = 110 - 60 = 50mm$

$$I_{xx} = I_{xx_1} + I_{xx_2} + I_{xx_3}$$

$$= \left(\frac{b_1 d_1^3}{12} + A_1 h_{y_1}^2\right) + \left(\frac{b_2 d_2^3}{12} + A_2 h_{y_2}^2\right) + \left(\frac{b_3 d_3^3}{12} + A_3 h_{y_3}^2\right)$$

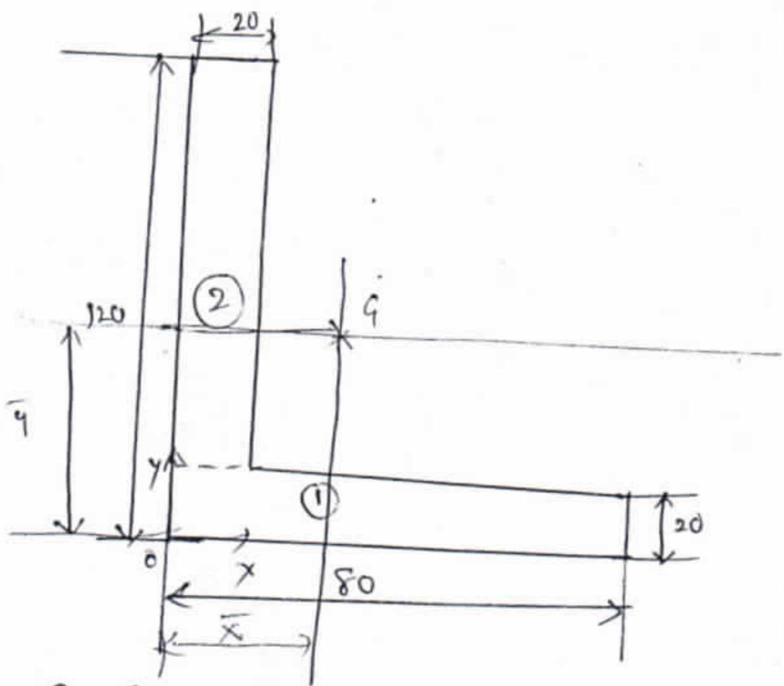
$$\underline{\underline{I_{xx} = 8.96 \times 10^6 \, mm^4}}$$

$$I_{yy} = I_{yy_1} + I_{yy_2} + I_{yy_3}$$

$$= \left(\frac{d_1 b_1^3}{12} + A_1 h_{x_1}^2\right) + \left(\frac{d_2 b_2^3}{12} + A_2 h_{x_2}^2\right) + \left(\frac{d_3 b_3^3}{12} + A_3 h_{x_3}^2\right)$$

$$\underline{\underline{I_{yy} = 2.7 \times 10^6 \, mm^4}}$$

→ find the M.I of 'L' section as shown in fig.



$a_1 = 80 \times 20 = 1600 mm^2$, $\qquad$ $a_2 = 100 \times 20 = 2000 mm^2$

$x_1 = \frac{80}{2} = 40mm$, $\qquad$ $y_1 = \frac{20}{2} = 10mm$

$x_2 = \frac{20}{2} = 10mm$, $\qquad$ $y_2 = 20 + \frac{100}{2} = 70mm$

$$\bar{x} = \frac{a_1x_1 + a_2x_2}{a_1 + a_2}$$

$$= 23.3\,mm$$

$$\bar{y} = \frac{a_1y_1 + a_2y_2}{a_1 + a_2}$$

$$\bar{y} = 43.3\,mm$$

$$h_{x_1} = x_1 - \bar{x} = 40 - 23.3 = 16.7\,mm$$

$$h_{x_2} = \bar{x} - x_2 = 23.3 - 10 = 13.3\,mm$$

$$h_{y_1} = \bar{y} - y_1 = 43.3 - 10 = 33.3\,mm$$

$$h_{y_2} = y_2 - \bar{y} = 70 - 43.3 = 26.6\,mm$$

$$I_{xx} = I_{xx_1} + I_{xx_2} + \cancel{I_{xx_3}}$$

$$= \left(\frac{b_1d_1^3}{12} + A_1h_{y_1}^2\right) + \left(\frac{b_2d_2^3}{12} + A_2h_{y_2}^2\right)$$

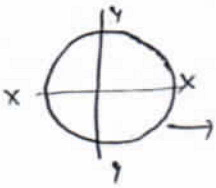
$$I_{xx} = \underline{4.9 \times 10^6\,mm^4}$$

$$I_{yy} = I_{yy_1} + I_{yy_2} + \cancel{I_{yy_3}}$$

$$= \left(\frac{d_1b_1^3}{12} + A_1h_{x_1}^2\right) + \left(\frac{d_2b_2^3}{12} + A_2h_{x_2}^2\right)$$

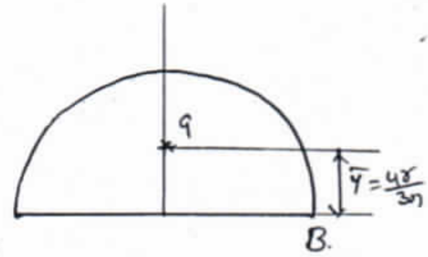
$$= \underline{\underline{1.72 \times 10^6\,mm^4}}$$

M.I of Semicircular section:-



$$I_{xx} = \frac{\pi}{64} d^4 \text{ (or) } \frac{\pi r^4}{4}$$

→ for circle



Similarly for Semi-circular Section

$$I_{AB} = \frac{1}{2}\left(\text{M.I of circle}\right)$$

$$= \frac{1}{2}\left(\frac{\pi r^4}{4}\right) = \frac{\pi r^4}{8}.$$

Acc. to paralel axis theorem

$$I_{AB} = I_G + Ah^2$$

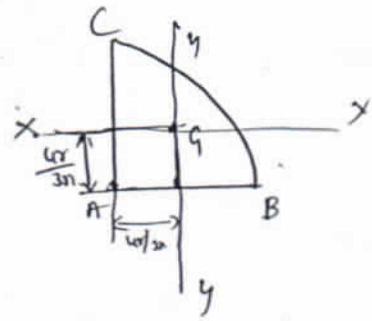
$$\frac{\pi r^4}{8} = I_G + \left(\frac{\pi r^2}{2}\right)\left(\frac{4r}{3\pi}\right)^2$$

$$I_G = \frac{\pi r^4}{8} - \frac{\cancel{16}^{8} r^4 \pi}{\cancel{18}_{9} \pi^2}$$

$$I_G = \frac{\pi r^4}{8} - \frac{8r^4}{9\pi} = r^4\left[\frac{\pi}{8} - \frac{8}{9\pi}\right]$$

$$I_{xx} \text{ (or) } \underline{\underline{I_G = 0.11 r^4}}$$

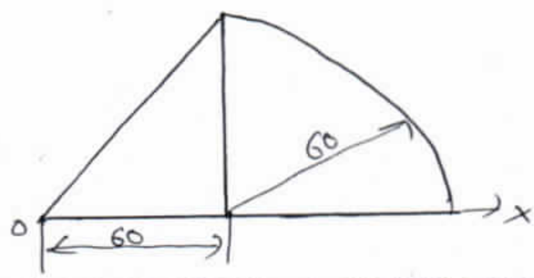
Quarter circle



$$I_{xx} = I_{yy} = 0.055 r^4$$

$$I_{AB} = I_{AC} = \frac{\pi r^4}{16}.$$

→ find the M.I of the area is given below about its centroidal axis parallel to x-axis



$a_1 = \frac{1}{2} \times 60 \times 60 = 1800 mm^2$

$a_2 = \frac{\pi r^2}{4} = \pi \frac{(60)^2}{4} = 2827.4 mm^2$

$y_1 = h/3 = 60/3 = 20mm$, $y_2 = \frac{4r}{3\pi} = \frac{4 \times 60}{3\pi} = 25.46mm$

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = 23.34mm$$

$h_{y_1} = \bar{y} - y_1 = 23.24 - 20 = 3.34mm$

$h_{y_2} = y_2 - \bar{y} = 25.46 - 2334 = 21.2mm$

$I_{xx} = I_{xx_1} + I_{xx_2}$

$= \left(\frac{bh^3}{36} + a_1 h_{y_1}^2\right) + \left(0.055 r^4 + a_2 h_{y_2}^2\right)$

$= 1.1 \times 10^6 \ mm^4$

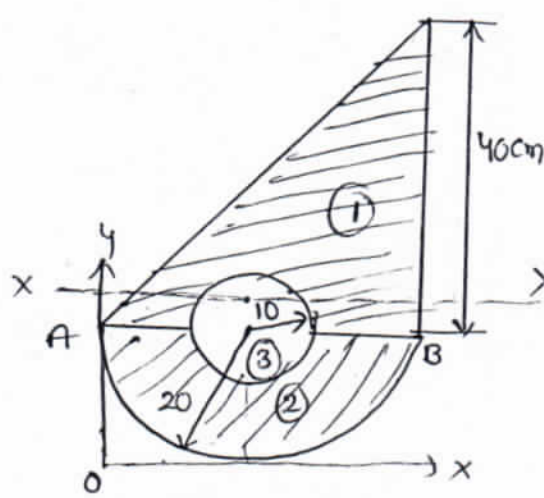
→ find the m.I of shaded area about horizontal centroidal axis

sol

$a_1 = \frac{bh}{2} = \frac{40 \times 40}{2} = 800 \ cm^2$

$a_2 = \frac{\pi r^2}{2} = \frac{\pi \times 20^2}{2} = 628.32 \ cm^2$

$a_3 = \pi r^2 = \pi (10)^2 = 314.16 \ cm^2$



$y_1 = 20 + \frac{40}{3} = 33.3 cm$

$y_2 = \left(20 - \frac{4 \times 20}{3\pi}\right) = 11.51 \ cm$

$y_3 = 10 + \frac{20}{2} = 20 cm$

$\bar{y} = 24.76 \ cm$

$h_{y_1} = y_1 - \bar{y} = 33.3 - 24.76 = 8.54$

$h_{y_2} = \bar{y} - y_2 = 13.25$

$h_{y_3} = \bar{y} - y_3 = 4.76$

$I_{xx} = I_{xx_1} + I_{xx_2} - I_{xx_3}$

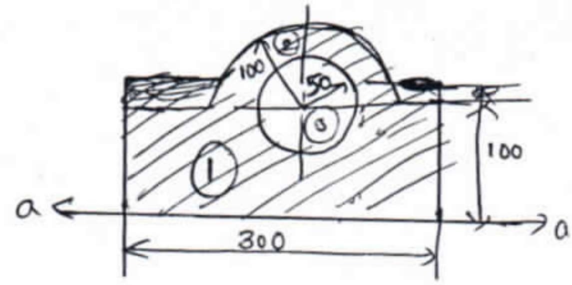
$= \left(\frac{b_1 h_1^3}{36} + a_1 h_{y_1}^2\right) + \left(0.11 r^4 + a_2 h_{y_2}^2\right) - \left(\frac{\pi r^4}{4} + a_3 h_{y_3}^2\right)$

$= (1.29 \times 10^5) + (1.27 \times 10^5) - (1.49 \times 10^4)$

$= 2.7 \times 10^5 \ cm^4$

→ Determine the second moment of area of the section as shown in fig about its base axis a–a.

$h = 100 + \frac{4r}{3\pi} = 142.4$



Sol

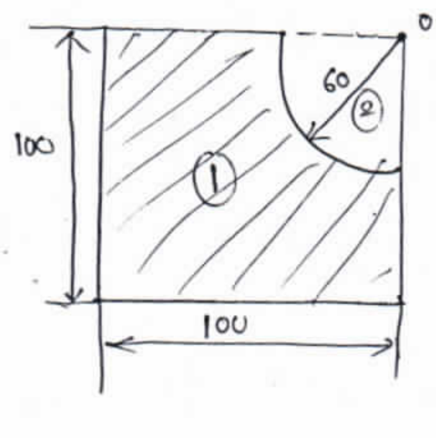
$$I_{aa_1} = \frac{bd^3}{3} = \frac{300 \times 100^3}{3} = 10^8 \text{ mm}^4$$

$$I_{aa_2} = I_G + Ah^2$$
$$= (0.11 \times 100^4) + \left(\frac{\pi(100)^2}{2} \times 142.4^2\right)$$
$$= 3.29 \times 10^8 \text{ mm}^4$$

$$I_{aa_3} = I_G + Ah^2$$
$$= \frac{\pi r^4}{4} + \left[\pi(50)^2 \times 100^2\right]$$
$$= 0.834 \times 10^8 \text{ mm}^4$$

$$I_{aa} = I_{aa_1} + I_{aa_2} - I_{aa_3}$$
$$= 3.46 \times 10^8 \text{ mm}^4$$

→ The cross section of a m/c part is as shown in fig. Determine its M.I & radius of gyration abt the horizontal centroidal axis



$$a_1 = 100 \times 100 = 10^4 \text{ mm}^2$$

$$a_2 = \frac{\pi r^2}{4} = \frac{\pi \times 60^2}{4} = 2.82 \times 10^3 \text{ mm}^2$$

$$y_1 = \frac{100}{2} = 50 \text{ mm}$$

$$y_2 = 40 + \left(60 - \frac{4r}{3\pi}\right) = 74.53 \text{ mm}$$

$$\bar{y} = \frac{a_1 y_1 - a_2 y_2}{a_1 - a_2} = 40.36 \text{ mm}$$

$h_{y_1} = 50 - 40.36 = 9.64$ mm, $h_{y_2} = 74.53 - 40.36 = 34.17$ mm.

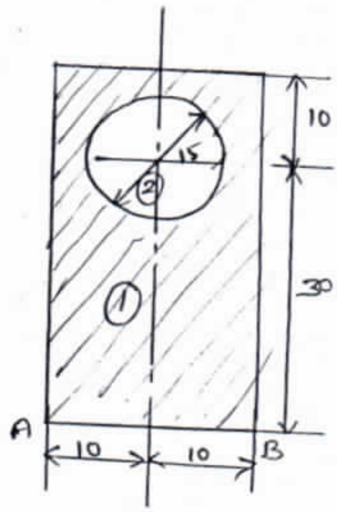
$$I_{xx} = I_{xx_1} - I_{xx_2}$$
$$= \left(\frac{100 \times 100^3}{12} + 10000(9.64)^2\right) - \left(0.055 r^4 + (2.82 \times 10^3 \times 34.17)\right)$$
$$= \cancel{4.005 \times 10^6} \ (9.26 \times 10^6) - (4.005 \times 10^6)$$
$$= 5.255 \times 10^6 \text{ mm}^4$$

Radius of gyration = $k = \sqrt{\frac{I_{xx}}{A}}$

$$A = a_1 - a_2 = 10^4 - (282 \times 10^3) = 7180$$

$$k = \sqrt{\frac{5.25 \times 10^6}{7180}} = 27.04\,mm$$

→ find the M.I of plate with a circular hole about the centroidal axis & about the base.



$a_1 = 20 \times 40 = 800\,cm^2$

$a_2 = \pi r^2 = \pi (7.5)^2 = 176.7\,cm^2$

$y_1 = 20\,cm, \; y_2 = 30\,cm$

$\bar{y} = \frac{a_1 y_1 - a_2 y_2}{a_1 - a_2} = 17.16\,cm$

$h_{y_1} = 20 - 17.6 = 2.84\,cm$

$h_{y_2} = 30 - 17.6 = 12.84\,cm$

about centroidal axis

$I_{xx_1} = \frac{b_1 d_1^3}{12} + A_1 h_{y_1}^2 = 1.13 \times 10^5\,cm^4$

$I_{xx_2} = \frac{\pi r^4}{4} + A_2 h_{y_2}^2 = 3.1 \times 10^4\,cm^4$

$I_{xx} = I_{xx_1} - I_{xx_2} = 8.15 \times 10^5\,cm^4$

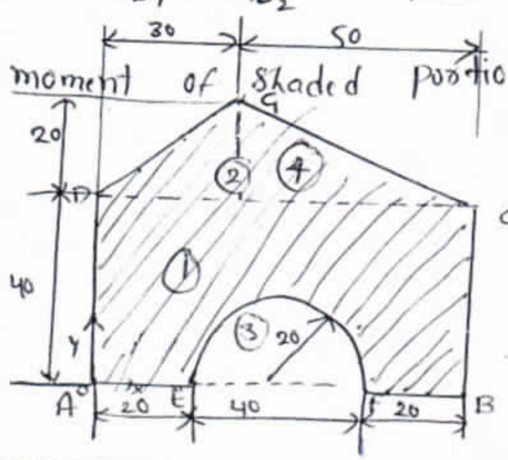
about its base

$$I_{AB_1} = \frac{b_1 d_1^3}{3} = \frac{20 \times 40^3}{3} = 4.26 \times 10^5\,cm^4$$

$$I_{AB_2} = \frac{\pi r^4}{4} + A_2 y_2^2 = \frac{\pi (7.5)^4}{4} + 176.7(30)^2 = 1.6 \times 10^5\,cm^4$$

$$I_{AB} = I_{AB_1} - I_{AB_2} = 2.6 \times 10^5\,cm^4$$

→ find the Second moment of shaded portion as shown in fig. about its centroidal axis.



a_1 = Rectangle = $(40\times80) = 3200\,mm^2$

a_2 = Triangle = $\frac{1}{2}\times b\times h = \frac{1}{2}\times 30\times 20 = 300\,mm^2$, $a_4 = \frac{bh}{2} = \frac{50\times20}{2} = 500\,mm^2$

$a_3 = \frac{\pi r^2}{2} = \frac{\pi\times20^2}{2} = 628.31\,mm^2$ (Semi circle)

$x_1 = \frac{80}{2} = 40mm$, $y_1 = \frac{40}{2} = 20mm$

$x_2 = \frac{2\times30}{3} = 20mm$, $y_2 = 40 + \frac{20}{3} = 46.6\,mm$

$x_3 = 20 + 20 = 40\,mm$, $y_3 = \frac{4r}{3\pi} = \frac{4\times20}{3\pi} = 8.48\,mm$

$x_4 = 30 + \frac{50}{3} = 46.6mm$, $y_4 = \frac{20}{3} + 40 = 46.6mm$

$$\bar{x} = \frac{a_1x_1 + a_2x_2 + a_4x_4 - a_3x_3}{a_1 + a_2 + a_4 - a_3} = 39.19mm$$

$$\bar{y} = \frac{a_1y_1 + a_2y_2 + a_4y_4 - a_3y_3}{a_1 + a_2 + a_4 - a_3} = 28.47\,mm$$

$h_{y_1} = 36.94 - 28.47 = 8.47$ $\qquad$ $h_{x_1} = 40 - 39.19 = 0.81$

$h_{y_2} = 46.6 - 28.47 = 18.13$ $\qquad$ $h_{x_2} = 39.19 - 20 = 19.19$

$h_{y_3} = 28.4 - 8.48 = 19.9$ $\qquad$ $h_{x_3} = 40 - 39.19 = 0.81$

$h_{y_4} = 46.6 - 28.47 = 18.13$ $\qquad$ $h_{x_4} = 46.6 - 39.19 = 7.41$

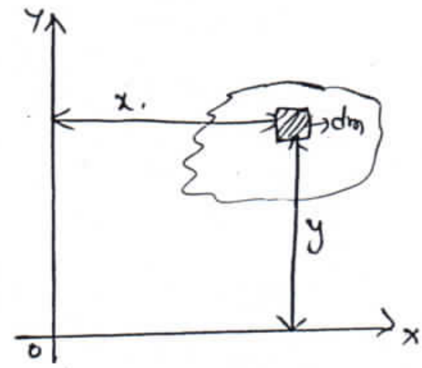
$$I_{xx} = I_{xx_1} + I_{xx_2} + I_{xx_3} + I_{xx_4}$$
$$= \left(\frac{b_1d_1^3}{12} + A_1h_{y_1}^2\right) + \left(\frac{bh^3}{36} + Ah_{y_2}^2\right) - \left(0.11\times r^4 + Ah_{y_3}^2\right) + \left(\frac{bh^3}{36} + Ah_{y_4}^2\right)$$
$$= (6.56\times10^5) + (1.05\times10^5) - (2.6\times10^5) + (1.75\times10^5)$$
$$= 6.76\times10^5\ mm^4$$

$$I_{yy} = I_{yy_1} + I_{yy_2} - I_{yy_3} + I_{yy_4}$$
$$= \left(\frac{d_1b_1^3}{12} + A_1h_{x_1}^2\right) + \left(\frac{hb^3}{36} + Ah_{x_2}^2\right) - \frac{1}{2}\left(\frac{\pi r^4}{4} + A_1h_{x_3}^2\right) + \left(\frac{hb^3}{36} + Ah_{x_4}^2\right)$$
$$= (1.7\times10^6) + (0.125\times10^6) - (0.63\times10^5) + (0.96\times10^5)$$
$$= 1.8\times10^6\ mm^4$$

Unit - IV

Mass Moment of Inertia

Mass Moment of Inertia of a body about an axis is defined as the sum of total of product of its elemental masses & square of their distance from the axis



Mass m.I about x-x $\Rightarrow (I_{xx})_m = \Sigma dm.y^2$

$= \int dm.y^2$

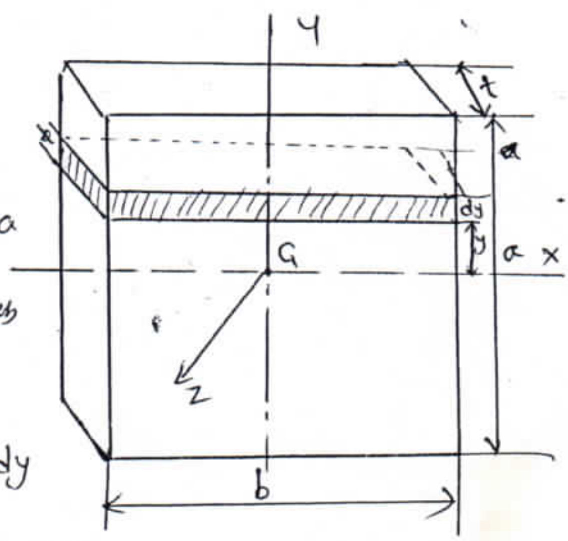
about y-y $\Rightarrow (I_{yy})_m = \Sigma dm.x^2$

$= \int dm.x^2$.

Units are $kg\text{-}m^2$.

→ Mass m.I of Rectangular plate bxa & thickness 't' about its centroidal axis

consider a rectangular plate bxa of dimensions, with thickness 't'



Mass of rectangular plate = $M = \rho \times b \times t \times a$

Consider an elemental strip of thickness 'dy' at a distance 'y' from the axis.

Mass of the elemental strip; $dm = \rho \times b \times t \times dy$

Mass m.I of elemental strip w.r.t. to x-axis

$$I_{xx} = \int y^2.dm$$

$$= \int_{-a/2}^{a/2} y^2.\rho bt\,dy = \rho bt\left[\frac{y^3}{3}\right]_{-a/2}^{a/2} = \rho bt\left[\frac{a^3}{24}+\frac{a^3}{24}\right]$$

$$= \frac{\rho bta^3}{12} = \frac{Ma^2}{12} \qquad (\because M = \rho bta)$$

Similarly $I_{yy} = mb^2/12$

Acc. to $\perp$ar axis theorem

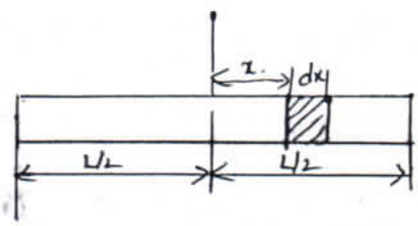
$$I_{zz} = I_{xx} + I_{yy}$$

$$= \frac{ma^2}{12} + \frac{mb^2}{12}$$

$$= \frac{m}{12}(a^2+b^2)$$

→ Mass m.I of a uniform rod of length 'L' about axis Normal to it at its a) centroid b) end.

a) centroid



Consider uniform rod of length 'L', cross section of rod is very small compare to Length.

$m = \rho \times A \times L$

Mass of the rod/~~unit length~~ $= L \times \rho = M$

Let us consider an elementary strip ~~that~~ length dx at a distance 'x' from the centre of the rod.

Mass of elementary strip/~~unit length~~ $dm = \rho \times dx$.

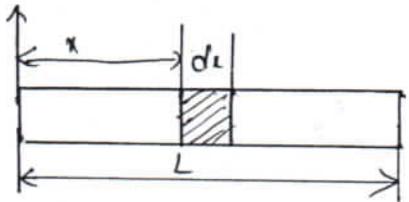
Mass m.I of rod about its centre

$$I = \int dm.x^2 = \int_{-L/2}^{L/2} (\rho \times dx)x^2$$

$$= \rho\left[\frac{x^3}{3}\right]_{-L/2}^{L/2} = \frac{\rho L^3}{12}$$

$$I = \frac{ML^2}{12} \quad (\because \rho \times L = m)$$

b) End:



Let us consider an elementary strip length dx at a distance 'x' from at its end of the rod.

Mass elemetary strip $dm = \rho \times dx$.

Mass m.I of rod about its end

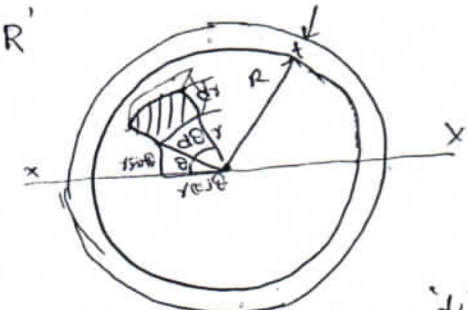
$$I = \int dm.\,x^2 = \int_0^L \rho \times dx \times x^2$$

$$= \rho \left[\frac{x^3}{3}\right]_0^L = \frac{\rho L^3}{3}.$$

$$I = \frac{mL^2}{3} \qquad (m = \rho . L)$$

→ Mass m.I of circular plate (or) disc about its centroidal axis

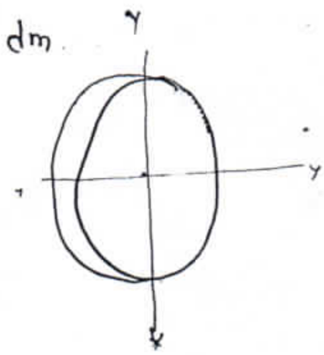
Consider a circular plate of Radius 'R' & thickness 't'.



Mass of circular plate $= M = \rho \times \pi R^2 \times t$.

let us consider a radial strip at a distance of 'r' from the centre of disc & radial strip thickness 'dr' an angle covered by radial strip is 'dθ'

Mass of radial strip $= \rho \times r d\theta \times dr \times t = dm$.



mass m.I of about x-axis;

$$I_{xx} = \int dm\,(r \sin\theta)^2$$

$$= \int \rho r t\, d\theta.dr.\,r^2 \sin^2\theta.$$

$$= \rho t \int_0^R r^3 dr \times \int_0^{2\pi} \sin^2\theta . d\theta \qquad (\because \sin^2\theta = \underline{1-c}$$

$$= \rho t \left[\frac{r^4}{4}\right]_0^R \times \left[\frac{1}{2}(\theta)_0^{2\pi} - \frac{1}{2}\left(\frac{\sin 2\theta}{2}\right)_0^{2\pi} \right]$$

$$= \rho t \frac{R^4}{4} \times \frac{2\pi}{2} = \frac{\rho t R^4 \pi}{4}$$

$$I_{xx} = MR^2/4 \qquad (\because M = \rho t \pi R^2)$$

Similarly $I_{yy} = \frac{mR^2}{4}$

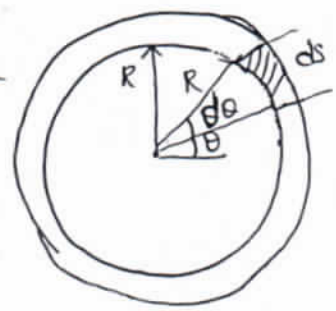
$$I_{zz} = I_{xx} + I_{yy}$$

$$= \frac{mR^2}{4} + \frac{mR^2}{4}$$

$$I_{zz} = \frac{mR^2}{2}$$

→ Mass M.I of uniform circular ring

Consider a uniform circular ring of radius 'R', cross section of ring is very small compared to Radius of ring.



Mass of the ring $= m = \rho \times V$

$$= \rho \times 2\pi R \times A$$

$$= \rho \times 2\pi R.$$

Consider a radial element of length 'ds' at an angle 'dθ'.

Mass of element strip $= dm = \rho \times d\theta \times R \times A$

$$= \rho \times d\theta \times R.$$

Mass M.I of about x-axis

$$I = \int dm.(R\sin\theta)^2 = \int_0^{2\pi} \rho \times d\theta \times R \times R^2 \sin^2\theta$$

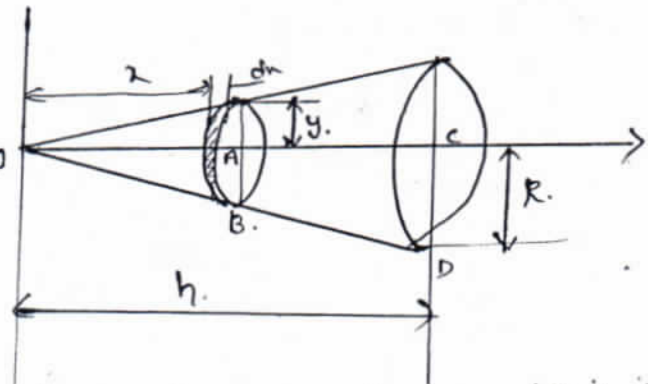
$$= \rho R^3 \left[\int_0^{2\pi} \frac{1-\cos 2\theta}{2}\, d\theta\right]$$

$$= \rho R^3 \left[\frac{1}{2}(2\pi) - [0\right] = \pi \rho R^3$$

$$I = \frac{mR^2}{2} \quad (\because m = \rho \times 2\pi R)$$

→ Mass M.I of Solid cone of height 'h' about its axis of rotation.

Consider a right circular cone of Height 'h' and base radius 'R'.



Mass of the cone $= M = \rho \times V$

$= \rho \times \frac{1}{3} \pi R^2 h$

Let us consider a thin circular plate at a distance 'x' from the apex as shown in fig. Thickness of plat is dx and radius of plate is 'y'

Mass of thin plate $dm = \rho \times \pi \times y^2 \times dx$

Mass M.I of thin plate about its axis of rotation of cone $=$

$\hookrightarrow \frac{1}{2} \times dm \times y^2$

$\left(I_{zz} = \frac{MR^2}{2} \right) \Rightarrow$

9. $= \int_0^h \frac{1}{2} \rho \pi y^2 dx\, y^2 = \frac{1}{2} \rho \pi y^4 dx$

$= \int_0^h \frac{1}{2} \rho \pi y^4 dx.$

from Similar $\Delta^{le's}$ OAB & OCD

$$\frac{x}{y} = \frac{h}{R} \Rightarrow y = \frac{Rx}{h}.$$

$$= \int_0^h \frac{1}{2} \rho \pi \left(\frac{Rx}{h} \right)^4 dx = \frac{1}{2} \rho \pi \frac{R^4}{h^4} \int_0^h x^4 dx$$

$$= \frac{1}{2} \frac{\rho \pi R^4}{h^4} \left[\frac{h^5}{5} \right] = \frac{\rho \pi R^4 h}{10}.$$

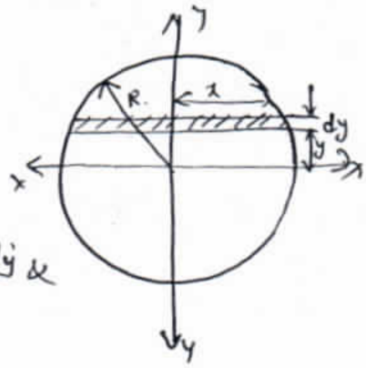
$$\left(\because M = \frac{\rho \pi R^2 h}{3} \right)$$

$$\boxed{I = \frac{3MR^2}{10}.}$$

→ Mass M.I. of Sphere

Consider a sphere of radius 'R'.

mass of sphere = $\rho \times \frac{4}{3}\pi R^3$



Let us consider a thin plate at a distance 'y' from x-axis. thickness of plate 'dy' & radius of thin plate 'x'.

mass of plate = $dm = \rho \times \pi \times x^2 \times dy$.

Mass M.I. of sphere about Y-axis $= \int_{-R}^{R} \frac{1}{2} \times dm \times x^2$

$$= \int_{-R}^{R} \frac{1}{2} \times \rho \times \pi \times x^2\, dy \times x^2 = \frac{1}{2} \times 2 \int_0^R \rho \pi x^4\, dy$$

from diagram $R^2 = x^2 + y^2 \Rightarrow x^2 = R^2 - y^2$.

$$I_{yy} = \int_0^R \rho \pi (R^2 - y^2)^2\, dy$$

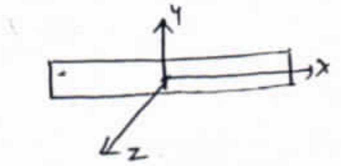
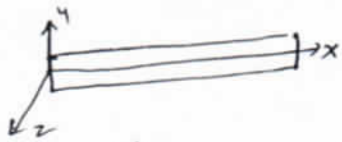
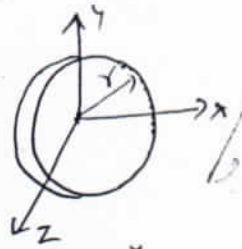
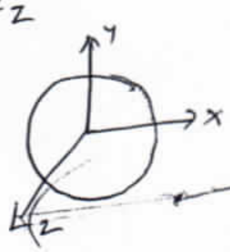
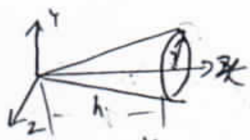
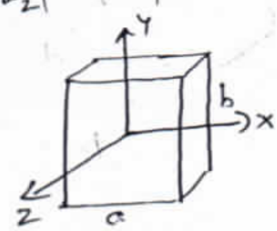
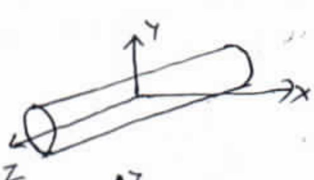
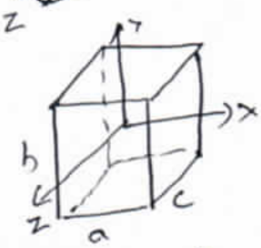
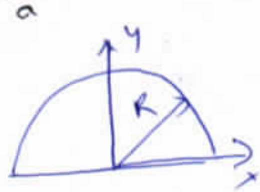
$$= \rho\pi \int_0^R (R^4 + y^4 - 2R^2y^2)\, dy$$

$$= \rho\pi \left[R^4 y + \frac{y^5}{5} - 2R^2\frac{y^3}{3} \right]_0^R$$

$$= \rho\pi \left[R^5 + \frac{R^5}{5} - \frac{2R^5}{3} \right] = \rho\pi \left[\frac{15R^5 + 3R^5 - 10R^5}{15} \right]$$

$$= \frac{\rho\pi\, 8R^5}{15} = \frac{4}{3}\pi R^3 \times \rho \times \frac{2R^2}{5}$$

$$\boxed{I = \frac{2MR^2}{5}}$$

Body (mass = m)	Mass M.I		
	I_x	I_y	I_z
Slender bar 	0	$\frac{mL^2}{12}$	$\frac{mL^2}{12}$
	0	$\frac{mL^2}{3}$	$\frac{mL^2}{3}$
circular disc 	$\frac{mr^2}{4}$	$\frac{mr^2}{4}$	$\frac{mr^2}{2}$
sphere 	$\frac{2}{5}mr^2$	$\frac{2}{5}mr^2$	$\frac{2}{5}mr^2$
cone 	$\frac{3mr^2}{10}$	$\frac{3m}{5}\left(\frac{r^2}{4}+h^2\right)$	$\frac{3m}{5}\left(\frac{r^2}{4}+h^2\right)$
Rectangular plate 	$\frac{mb^2}{12}$	$\frac{ma^2}{12}$	$\frac{m}{12}(a^2+b^2)$
Cylinder 	$\frac{m}{12}(3r^2+L^2)$	$\frac{m}{12}(3r^2+L^2)$	$\frac{m}{2}r^2$
prism (Rectangle) 	$\frac{m}{12}(b^2+c^2)$	$\frac{m}{12}(a^2+c^2)$	$\frac{m}{12}(a^2+b^2)$
Semi circular 	$0.07\,MR^2$	$\frac{MR^2}{4}$	

$I_{AB} = MR^2$

A cylinder of dia 500 mm & height 120 mm has mass density of 8000 kg/m³. find M.M.I of cylinder a) w.r.t. the axis of the cylinder b) about a line through centre, & $\perp^{ar}$ to longitudinal axis

Sol)

$$m = \rho \times V = 8000 \times \pi \times r^2 \times h = 8000 \times \pi \times 0.250^2 \times 0.120 = 1885 \text{ kg}$$

$$I_z = \frac{mr^2}{2} = \frac{1885(0.25)^2}{2} = 58.9 \text{ kg-m}^2$$

$$I_x = \frac{m}{12}(3r^2 + l^2) = \frac{1885}{12}\left(3(0.25)^2 + 0.12^2\right) = 255.65 \text{ kg-m}^2$$

→ Calculate the m.I & radius of gyration of grinding stone 90 cm in dia & 10 cm thick w.r.t. axis of rotation. Stone weight 0.0026 kg/cm³

Sol)

$$d = 90\text{cm},\ R = 45 \text{ cm},\ t = 10 \text{ cm}$$

$$m = \rho \times V = 0.0026 \times \pi \times R^2 \times t = 165.4 \text{ kg}$$

$$I_{zz} = \frac{mr^2}{2} = \frac{165.4 \times 0.45^2}{2} = 16.75 \text{ kg-m}^2$$

$$k = \sqrt{\frac{I}{m}} = \sqrt{\frac{16.75}{165.4}} = 0.318 \text{ m} = 31.82 \text{ cm}$$

→ Calculate the m I & radius of gyration w.r.t. axis of rotation of.

i) a circular lamina of dia 60cm & mass 0.001 kg/cm²

ii) A " cylinder of dia 50cm & height 15 cm & m = 0.002 kg/cm³

iii) A solid sphere of dia 40cm & mass 0.0015 kg/cm³

Sol

i) Circular lamina

$$d = 60\text{cm},\ r = 30\text{cm}$$

$$m = 0.001 \text{ kg/cm}^2$$

$$m = \rho \times V = 0.001 \times \pi r^2 = 0.001 \times \pi (30)^2 = 2.827 \text{ kg}$$

$$I = \frac{mr^2}{2} = \frac{2.827(30)^2}{2} = 1272.15 \text{ kg-cm}^2$$

$$k = \sqrt{\frac{I}{m}} = \sqrt{\frac{1272.15}{2.827}} = 21.2 \text{ cm}$$

ii) circular cylinder

$d = 80\,cm,\ r = 40\,cm,\ h = 15\,cm$

$m = 0.002 \times \pi \times R^2 \times h$

$= 0.002 \times \pi \times 40^2 \times 15 = 150.79\ kg$

$I = \frac{mR^2}{2} = \frac{150.79 \times 40^2}{2} = 120636.8\ kg\text{-}cm^2$

$k = \sqrt{\frac{I}{m}} = 28.28\,cm$

iii) for Solid Sphere

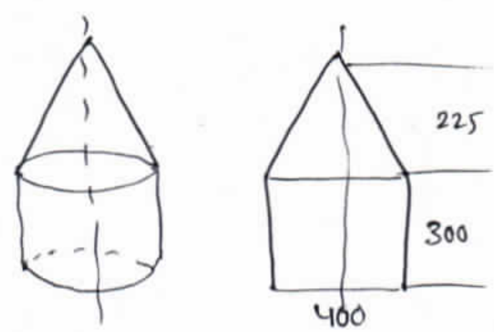
$d = 40\,cm,\ r = 20\,cm,$

$M = 0.0015 \times \frac{4}{3}\pi R^3 = 50.26\ kg$

$I = \frac{2}{5} mR^2 = \frac{2}{5} \times 50.26 \times 20^2 = 8042.4\ kg\text{-}cm^2$

$k = \sqrt{\frac{I}{m}} = \underline{12.64\,cm}$

→ A brass cone with base dia of 400mm & height 225mm is placed on a vertical aluminium cylinder of height 300mm & dia 400mm; $\rho_{brass} = 85\ kN/m^3$ & $\rho_{al} = 25.6\ kN/m^3$. Determine the M.M.I. of the body about vertical axis



Sol/ Consider brass cone.

$\rho = \frac{85 \times 1000}{9.81} = 8664.6\ kg/m^3$

$m = \rho \times V$

$= 8664.3 \times \frac{\pi r^2 h}{3} = 8664.3 \times \pi \frac{(0.2)^2}{3} \times 0.225$

$= 81.66\ kg$

$I_{YY_1} = \frac{3mr^2}{10} = \frac{3 \times 81.66 \times (0.2)^2}{10} = 0.98\ kg\text{-}m^2$

Consider Al. cylinder:

$$\rho = \frac{25.6 \times 1000}{9.81} = 2609.6 \ kg/m^3$$

$$m = \rho \times V = 2609.6 \times \pi r^2 \times h = 98.38 \ kg$$

$$I_{y_2} = \frac{mr^2}{2} = \frac{98.38 \times 0.2^2}{2} = 1.967 \ kg\text{-}m^2$$

$$I_y = I_{y_1} + I_{y_2}$$

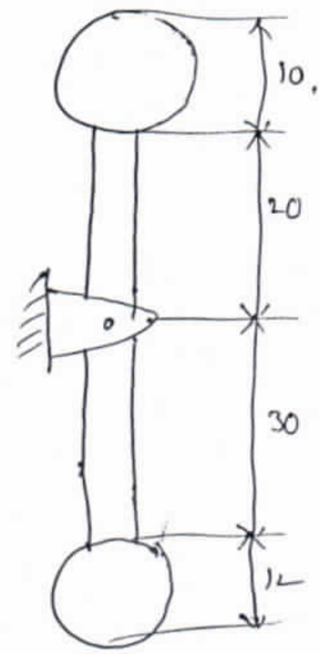
$$= 2.947 \ kg\text{-}m^2$$

→) A uniform rod of mass 10 kg is pinned at O & its ends are welded by sphere of mass 20 kg at upper end circular disc of mass 15 kg at lower end as shown in fig. find m.m.I about pinned axis

Sol

$$I_0 = I_{Rod} + I_{Sphere} + I_{Disc}$$

$$= \left[\frac{m_1 L_1^2}{12} + m_1 d_1^2\right] + \left[\frac{2 m_2 r_2^2}{5} + m_2 d_2^2\right] + \left[\frac{m_3 r_3^2}{2} + m_3 d_3^2\right]$$



$$= \left[\frac{10 \times 50^3}{12} + 10 \times 5^2\right] + \left[\frac{2 \times 20 \times 5^2}{5} + 20 \times 25^2\right] + \left[\frac{15 \times 6^2}{2} + 15 \times 36\right]$$