Mathematical Foundations

Instructions

Please share your answers wherever applicable in-line with the word document. Submit code separately wherever applicable. Mathematical calculations which are manually performed should be uploaded with a picture along with the explanation in a word document.
Please ensure you update all the details:

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Topic: Mathematical Foundations

Note: Submit pictures of mathematical calculations

Problem Statements

Q1) Find the maximum and minimum values of the function: $x^3 - 3x^2 - 9x + 12$?

To find the maximum and minimum values of the function
$f(x)=x^3-3x^2-9x+12$, we can follow these steps:

Step 1: Find the first derivative $f'(x)$

The first derivative helps us find the critical points, where the function could have maxima or minima.

$f'=d/dx(x^3-3x^2-9x+12)=3x^2-6x-9$

Step 2: Set the first derivative equal to zero to find critical points

$3x^2-6x-9=0$

Divide by 3 to simplify:

$x^2-2x-3=0$

Now, factor the quadratic equation:

$(x-3)(x+1)=0$

So, the critical points are: $x=3$ and $x=-1$

Step 3: Find the second derivative $f''(x)$

The second derivative will help determine the nature of the critical points (maxima or minima).

$f''(x)= d/dx(3x^2-6x-9)=6x-6$

Step 4: Determine the nature of the critical points

At $x=3$:

f″(3)=6(3)−6=18−6=12>0

Since

$f''(3)>0$,x=3 is a local minimum.

At x=-1

f″(-1)=6(-1)−6=-6−6=12<0

Since

f″(−1)<0, x=−1 is a local maximum.

Step 5: Calculate the function values at the critical points

Now, substitute the critical points back into the original function f(x):

At x=3:

$f(3) = 3^3-3(3^2)-9(3)+12=27-27-27+12 = -15$

At x=-1:

$f(-1) = (-1)^3-3(-1)^2-9(-1)+12 =-1-3+9+12=17$

Conclusion

The local maximum value is f(−1)=17.

The local minimum value is f(3)=−15.

Q2) Calculate the slope and the equation of a line which passes through the points (-1, -1), (3, 8)

To find the slope and equation of the line passing through the points (−1,−1) and (3,8), follow these steps:

Step 1: Find the slope m

The formula for the slope of a line passing through two points (x1,y1) and (x2,y2) is:

$$m = \frac{x2 - x1}{y2 - y1}$$

Substitute the given points (−1,−1) and (3,8):

$$m=\frac{8-1(-1)}{3-1(-1)} = \frac{8+1}{3+1} = \frac{9}{4}$$

$$\text{slope is :- } m = \frac{9}{4}$$

Step 2: Use the point-slope form to find the equation of the line

The point-slope form of the equation of a line is:

$$y-y1=m(x-x1)$$

Substitute m= $\frac{9}{4}$ m and one of the points, say (−1,−1):

$$y-(-1)= \frac{9}{4}(x-(-1))$$

Simplify:

$$y+1=\frac{9}{4}(x+1)$$

Step 3: Simplify the equation

Expand the equation:

$$y+1=\frac{9}{4}x+\frac{9}{4}$$

Now, subtract 1 from both sides:

$$y=\frac{9}{4}x+\frac{9}{4}-1$$

Convert 1 to a fraction with denominator 4:

$$y=\frac{9}{4}x+\frac{9}{4}-\frac{4}{4}$$

Simplify:

$$y=\frac{9}{4}x+\frac{5}{4}$$

Final Equation

The equation of the line is:

$$y=\frac{9}{4}x+\frac{5}{4}$$

Q3) Solve for w'(z) when

$$w(z)=\frac{4z-5}{2-z}$$

To solve for w′(z), where $w(z) = \frac{4z-5}{2-z}$,we'll use the quotient rule for differentiation.

Quotient Rule:

If you have a function $w(z)=\frac{f(z)}{g(z)}$, then the derivative is given by:

$$w'(z) = \frac{f'(z)g(z)-f(z)g\prime(z)}{[g(z)]2}$$

Step 1: Identify f(z) and g(z)

For the given function:

F(z) = 4z-5, g(z) = 2 – z

Step 2: Compute f′(z) and g′(z)

- f′(z)=4 (since the derivative of 4z−5 is 4)
- g′(z)=−1 (since the derivative of 2−z is -1)

Step 3: Apply the quotient rule

Substitute the values into the quotient rule formula:

$$w'(z) = \frac{(4)(2-z)-(4z-5)(-1)}{(2-z)^2}$$

Step 4: Simplify the expression

Expand and simplify:

$$w'(z) = \frac{8-4z+4z-5}{(2-z)^2}$$

$$w'(z) = \frac{3}{(2-z)^2}$$

Final Answer

So, the derivative w′(z)w'(z)w′(z) is:

$$w'(z) = \frac{3}{(2-z)^2}$$

Q4) Consider Y = $2x^3+6x^2+3x$. Identify the critical values and verify if it is the maxima or minima.

To find the critical values and determine whether they are maxima or minima for the function:

$Y=2x^3+6x^2+3x$

Step 1: Find the first derivative $Y'(x)$

To locate critical values, we first compute the first derivative of Y:

$$Y'(x)=\frac{dy}{dx}(2x^3+6x^2+3x) = 6x^2+12x+3$$

Step 2: Set the first derivative equal to zero

To find the critical points, set the derivative equal to zero and solve for x:

$$6x^2+12x+3=0$$

Step 3: Solve the quadratic equation

We'll solve the quadratic equation using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For the equation $6x^2+12x+3=0$, a=6, b=12, and c=3. Substitute these values into the quadratic formula:

$$x = \frac{-12 \pm \sqrt{12^2 - 4(6)(3)}}{2(6)}$$

$$x = \frac{-12 \pm \sqrt{144 - 72}}{12}$$

$$x = \frac{-12 \pm \sqrt{72}}{12}$$

$$x = \frac{-12 \pm 6\sqrt{2}}{12}$$

$$x = \frac{-12}{12} \pm \frac{6\sqrt{2}}{12}$$

$$x = -1 \pm \frac{\sqrt{2}}{2}$$

So, the critical values are:

$$x1 = -1 + \frac{\sqrt{2}}{2} \quad , \quad x2 = -1 - \frac{\sqrt{2}}{2}$$

Step 4: Verify maxima or minima using the second derivative

To verify if these critical points correspond to maxima or minima, we compute the second derivative of Y;

$$y''(x) = \frac{d}{dx}(6x^2 + 12x + 3) = 12x + 12$$

Now evaluate Y″(x)Y"(x)Y″(x) at the critical points:

For $x1 = -1 + \frac{\sqrt{2}}{2}$

$$y''(x1) = 12\left(-1 + \frac{\sqrt{2}}{2}\right) + 12$$

This expression needs to be evaluated numerically to check the sign.

For $x2 = -1 - \frac{\sqrt{2}}{2}$

$$y''(x2) = 12\left(-1 - \frac{\sqrt{2}}{2}\right) + 12$$

- *Similarly, evaluate numerically to check if it's positive or negative.*

Once you evaluate the second derivative at these points, the sign will determine whether the critical values correspond to maxima (if negative) or minima (if positive).

Q5) Determine the critical points and obtain relative minima or maxima of a function defined by

$$y = 2x_1^2 + 2x_1x_2 + 2x_2^2 + 6x_1$$

The given function is: $y = 2x_1^2 + 2x_1x_2 + 2x_2^2 + 6x_1$

Step 1: Find the partial derivatives

To find the critical points, we need to calculate the partial derivatives of y with respect to x1 and x2 and set them equal to zero.

Partial derivative with respect to x1:

$$\frac{dy}{dx_1} = 4x_1 + 2x_2 + 6$$

Partial derivative with respect to x2:

$$\frac{dy}{dx_1} = 2x_1 + 4x_2$$

Step 2: Set the partial derivatives equal to zero

We now set both partial derivatives equal to zero to find the critical points.

1. $4x_1 + 2x_2 + 6 = 0$
2. $2x_1 + 4x_2$

Step 3: Solve the system of equations

From equation (2):

$2x_1 + 4x_2 = 0$ => $x_1 = -2x_2$

Substitute $x_1 = -2x_2$ into equation (1):

$$4(-2x_2) + 2x_2 + 6 = 0$$

$$-8x_2 + 2x_2 + 6 = 0$$

$$-6x_2 + 6 = 0$$

$$x_2 = 1$$

Now , substitute $x_2 = 1$ into $x_1 = -2x_2$

x1=−2(1)=−2

Thus, the critical point is (x1,x2)=(−2,1).

Step 4: Find the second partial derivatives to determine maxima or minima

To determine whether this critical point is a relative maximum, minimum, or a saddle point, we compute the second partial derivatives.

Second partial derivatives:

$$\frac{d^2y}{dx_1^2} = 4, \quad \frac{d^2y}{dx_2^2} = 4, \quad \frac{d^2y}{dx_1\,dx_2} = 2$$

Step 5: Use the second derivative test

The second derivative test involves computing the discriminant D, which is given by:

$$D = \frac{d^2y}{dx_1^2} . \frac{d^2y}{dx_2^2} - \left(\frac{d^2y}{dx_1\,dx_2}\right)^2$$

Substitute the values of the second partial derivatives:

$$D = (4)(4) - (2)^2 = 16 - 4 = 12$$

Since D>0 and $\frac{d^2y}{dx_1^2} > 0$,the function has a **relative minimum** at the critical point (−2,1)(-2, 1)(−2,1).

Conclusion

The function has a relative minimum at (x1,x2)=(−2,1)