Probability & Algebra.

U-I Basic Probability.

Random ~~Samples~~ Experiment : An experiment conducted repeatedly under the identical conditions is called Random Experiment

Ex: Tossing a coin 15 times

Rolling a dice 15 times

Trail : Each performance of the random experiment is called a trail.

Random Experiment definition in trail form: Trails under the same conditions is called Random Experiment.

Outcome: Result of a trail is called outcome.

Sample Space: The set of all possible outcomes of a random experiment is called sample space.

Ex: ① When we toss a coin the sample space consists of $S = \{H, T\}$

② When we toss 2 coins at a time, Sample space $S = \{HH, HT, TH, TT\}$

Event: Subset of a sample space is called event.

Ex: ① E_1 = event of considering ~~only~~ at least one Head when we toss 2 Coins at a time.

$E_1 = \{HH, HT, TH\}$

② E_2 = event of considering all tails when we toss 3 coins at a time

$E_2 = \{TTT\}$

Mutually Exclusive Events: Events are said to be mutually exclusive if the happening of any one event prevents the happening of the other events i.e., no two or more events occur simultaneously in the same trail.

Equally likely Events: If Preference of one event cannot expect the preference of other event then the two events are said to be equally likely. ie, when there is no reason to expect any one of them happens rather than any one of the others can happen.

Ex: When a card is drawn from a pack any card may be obtained. In this case all the 52 cards are equally likely to happen.

Exhaustive Events: All possible events in any trail are known as Exhaustive events.

Ex: In tossing a coin there are two exhaustive events namely Head and Tail.

In throwing a dice there are 6 exhaustive events namely 1, 2, 3, 4, 5, 6

Probability: In a random experiment if there are 'n' mutually exclusive and equally likely events. Let 'E' be an event of the experiment. If 'm' elementary events form event E (favourable to E) then the probability of E can be defined as

$$P(E) = \frac{m}{n} = \frac{\text{no. of elementary events in } E}{\text{Total number of elementary events in random exp}}$$

i.e., probability of E = $\dfrac{\text{No. of favourable chances of E}}{\text{Total no. of chances.}}$

Problems

① A class consist of 6 girls and 10 boys. If a committe of 3 choosen at random from class. Find probability that.

(i) 3 boys are selected

(ii) Exactly 2 girls are selected.

Sol: Total number of students = 16

No. of ways of choosing 3 from 16 = $^{16}C_3$ (16_3)

(i) Suppose 3 boys are selected.

This can be done in $^{10}C_3$ ways

∴ Probability of a committe selecting 3 boys

$$= \frac{^{10}C_3}{^{16}C_3} = 0.214$$

(ii) Suppose exactly 2 girls are selected then

Probability of a committe selecting exactly 2 girls in = $\dfrac{^{6}C_2 \cdot {}^{10}C_1}{^{16}C_3} = 0.267$

② A and B throw alternately with a pair of ordinary dice, if A wins if he throw 6 before B throws 7 and B winds if he throw 7 before A throws 6 If A begins the game find the probability of wining the game by A.

Sol When 2 dice are thrown total no. of elements in sample space $= \{ \ldots \} = 6^2 = 36$

The probability of A throwing 6 $= \frac{5}{36}$

[Sum 6 Chances are $= \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$

$\therefore$ probability of sum 6 $= \frac{5}{36}$]

The probability of A not throwing 6 $= 1-\frac{5}{36} = \frac{31}{36}$

The probability of B throwing 7 $= \frac{6}{36}$

[Sum 7 chances are $= \{(1,6)(2,5)(3,4)(4,3)(5,2)(6,1)\}$

$\therefore$ probability of sum 7 $= \frac{6}{36}$]

The probability of B not throwing 7 $= 1-\frac{6}{36} = \frac{30}{36}$

A	B
$\frac{5}{36}$	$\frac{31}{36} \times \frac{6}{36}$
$\frac{31}{36} \times \frac{30}{36} \times \frac{5}{36}$	$\frac{31}{36} \times \frac{30}{36} \times \frac{31}{36} \times \frac{6}{36}$
$\left(\frac{31}{36}\right)^2 \left(\frac{30}{36}\right)^2 \frac{5}{36}$	- - - - - - -

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Probability winning the game by A is

$$P(A) = \frac{5}{36} + \frac{31}{36} \times \frac{30}{36} \times \frac{5}{36} + \left(\frac{31}{36}\right)^2 \left(\frac{30}{36}\right)^2 \frac{5}{36} + \cdots$$

$$= \frac{5}{36}\left[1 + \frac{31}{36}\frac{\times 30}{36} + \left(\frac{31}{36}\right)^2 \left(\frac{30}{36}\right)^2 + \cdots\right]$$

which is great in geometric series with common ratio

$$r = \frac{31}{36} * \frac{30}{36} = 0.7175$$

$$P(A) = \frac{5}{36}\left[\frac{1}{1-0.7175}\right] = 0.4915$$

[Sum of infinite terms in G.P $S_\infty = \frac{a}{1-r}$]

③ A, B and C are in order toss a coin. The first one to toss head wins the game. What are their probabilities of winning the game assuming that game continues indefinitely.

Sol: When a coin is tossed no. of elements in sample space = 2

Sample Space = $\{H, T\}$ = 2

The probability of getting Head = $1/2$

{ The probability of getting Tail = $1/2$

A	B	C
$1/2$	$1/2 \times 1/2$	$1/2 \times 1/2 \times 1/2$
$1/2 \times 1/2 \times 1/2 \times 1/2$	$(1/2)^5$	$(1/2)^6$ ----
- -	- -	- - -

Probability of A winning the game is

$$P(A) = 1/2 * (1/2)^4 + (1/2)^7 + \dots$$

$$= 1/2\left(1 + (1/2)^3 + (1/2)^6 + \dots\right)$$

$$= \frac{1}{2}\left[\frac{1}{1-\left(\frac{1}{2}\right)^3}\right] \qquad \left[\because S_\infty = \frac{a}{1-r}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-\frac{1}{8}}\right]$$

$P(A) = 0.5714$

Probability of B winning of the game is

$$P(B) = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^8 + \cdots$$

$$= \left(\frac{1}{2}\right)^2\left[1 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 + \cdots\right]$$

$$= \left(\frac{1}{2}\right)^2\left[\frac{1}{1-\left(\frac{1}{2}\right)^3}\right]$$

$$= 0.2857$$

Probability of C winning the game is

$$P(C) = \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^9 + \cdots$$

$$= \left(\frac{1}{2}\right)^3\left[1 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 + \cdots\right]$$

$$= \frac{1}{8}\left[\frac{1}{1-\left(\frac{1}{2}\right)^3}\right]$$

$$= \frac{1}{8}\left[\frac{1}{1-\frac{1}{8}}\right] = 0.1428$$

④ Two ~~gourds~~ cards are selected at random numbers 1 to 10. Find the probability that sum is even

(i) Two cards are drawn together

(ii) Two cards are drawn one after other with replacement.

Sol

(i) Number of exhaustive cases = $^{10}C_2 = 45$

Sum is even if both are even or both are odd

There are 5 even and 5 odd cards

Both are even cases = $^5C_2$

Both are odd cases = $^5C_2$

∴ No. of favourable cases for ~~Sc~~ sum is even =

$$\frac{5c_2 + 5c_2}{\cancel{10c_2}}$$

Required probability for getting sum even =

$$\frac{5c_2 + 5c_2}{10c_2} = \frac{10 + 10}{45} = 0.4444$$

(ii) Two cards are drawn one after other with replacement.

Number of exhaustive cases = $^{10}c_1 \, ^{10}c_1$

~~Sum~~ No. of possible cases for both are even = $^5c_1 \, ^5c_1$

No. of possible cases for both are odd = $^5c_1 \, ^5c_1$

∴ No. of favourable cases for sum is even = $^5c_1 \, ^5c_1 + ^5c_1 \, ^5c_1$

Required probability $= \dfrac{{}^5C_1\,{}^5C_1 + {}^5C_1\,{}^5C_1}{{}^{10}C_1\,{}^{10}C_1}$

$= \dfrac{25+25}{100} = \dfrac{1}{2} = 0.5$

(5) Find the probability of getting sum of 10 if we throw 2 dice.

Sol Possibilities of getting sum $10 = \{(4,6),(5,5),(6,4)\}$
$= 3$

Total No. of chances when we ~~throw~~ 2 dice $= 6^2 = 36$

$\therefore$ Probability $= \dfrac{3}{36} = 0.0833$

(6) Out of 15 items 4 are not in good condition which 4 are selected at random. Find the probability that
(i) all are not good (ii) 2 are not good.

Sol Suppose 4 ~~obj~~-items are seleted at random

Total no. of chances $= {}^{15}C_4$

(i) all are not good

Favourable chances of all are not good $= {}^4C_4$

Required probability $= \dfrac{{}^4C_4}{{}^{15}C_4} = \dfrac{1}{\frac{32760}{24}} =$

7.3260×10^{-4}

$= 0.000732$

(ii) No. of possible cases for selecting 2 good and 2 not good $= {}^{11}C_2 \times {}^4C_2$

Required probability $= \dfrac{{}^{11}C_2\,{}^4C_2}{{}^{15}C_4} = \dfrac{55\times 6}{}$

$= \dfrac{330}{1365} = 0.2417$

(7) A bag contains 5 red balls 8 blue balls and 11 white balls. 3 balls are drawn together from the bag (i) Find the prob 1 is Red 1 is blue 1 is white
(ii) 2 W, 1 R
(iii) 3 white

Sol Total No. of chances to getting 3 balls $= {}^{24}C_3$

(i) getting 1R 1b 1w is $= {}^5C_1\,{}^8C_1\,{}^{11}C_1$

probability $= \dfrac{{}^5C_1\,{}^8C_1\,{}^{11}C_1}{{}^{24}C_3} = 0.2173$

(ii) getting 2W 1R is $= {}^{11}C_2\,{}^5C_1$

probability $= \dfrac{{}^{11}C_2\,{}^5C_1}{{}^{24}C_3} = 0.1358$

(iii) getting 3 white $= {}^{11}C_3$

probability $= \dfrac{{}^{11}C_3}{{}^{24}C_3} = 0.0815$

8) What is the probability that at least 2 out of n people have the same birthday assume 165 days in a year such that all days are equally likely.

Sol Since birthday of a person can fall on any of 365 days.

$\therefore$ No. of exhaustive cases $= \underbrace{{}^{365}C_1\, {}^{365}C_1 \ldots {}^{365}C_1}_{n\text{ times}}$

$= 365 \times 365 \times \ldots \ldots 365$ n times.

If ~~all~~ birthday of all n persons fall on different days of the year then the number of favourable cases will be $= {}^{365}C_1\, {}^{364}C_1\, {}^{363}C_1 \ldots \ldots {}^{(365-(n-1))}C_1$

$= 365 \times 364 \times 363 \times 364 \times \ldots \ldots \times (365-(n-1))$

Hence the required probability $= 1 -$ Probability that no two persons have same bday date

$$= 1 - \frac{{}^{365}C_1 \times {}^{364}C_1 \times \ldots (365-(n-1))_{C_1}}{{}^{365}C_1\, {}^{365}C_1 \ldots {}^{365}C_1}$$

$$= 1 - \left[\frac{365 \times 364 \times \ldots \ldots}{(365)^n}\right]$$

Axioms of Probability:

Let 'S' be a finite sample space a real value function 'p' from powerset of 'S' is called probability function on 'S' if the following three axioms are satisfied

(ii) Axiom of positivity $0 \le P(E) \le 1$

(i) Axiom of certainity $P(S) = 1$

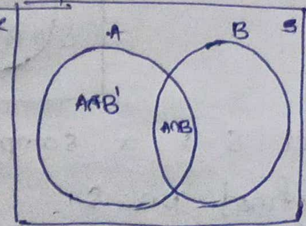
(iii) Axiom of Union: If A and B are two mutually exclusive events then

$$P(A \cup B) = P(A) + P(B)$$

Theorem 1:

$$\boxed{P(A \cap B') = P(A) - P(A \cap B)}$$

Here S is the sample space and A, B are 2 events in sample space S



A can be divide into 2 mutually exclusive events, $A \cap B'$ and $A \cap B$

$\therefore$ By third axiom of probability

$$P(A) = P(A \cap B') + P(A \cap B)$$

$$P(A \cap B') = P(A) - P(A \cap B)$$

Note:

~~Similarly~~ ~~we~~ can derive

A, B are 2 events then

~~probability~~ $P(A' \cap B) = P(B) - P(A \cap B)$

Theorem 2 :

Let S be the sample space which contains all the outcomes of the random experiment we know that S' (or) $\bar{S} = \phi$ which does not contain any element

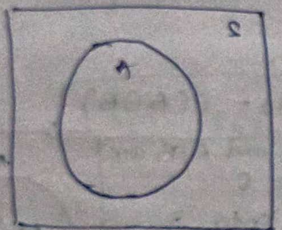
$\therefore P(S') = 1 - P(S)$

$\therefore S' = \phi$

$P(\phi) = 1 - 1$ [from axiom 1 : $P(S) = 1$]

$\boxed{\therefore P(\phi) = 0}$

Theorem 3 : $\boxed{P(A) = 1 - P(A')}$



Let S is a sample space and A is any event defined on S.

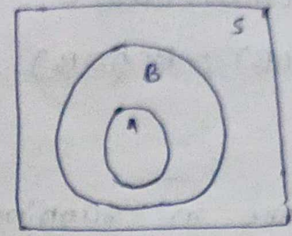
We know that $A \cup A' = S$ and $A \cap A' = \phi$

From 3^{rd} axiom $P(A \cup A') = P(S)$ ($\because$ A and A' are mutually exclusive)

$P(A) + P(A') = 1$

$\boxed{P(A) = 1 - P(A')}$

Theorem 4 : If $A \subset B$ then $P(A) \leq P(B)$.



Let S be a sample space ~~which contains~~ and A, B are two events defined on S

Since $A \subset B$

$A \cap B = A \Rightarrow P(A \cap B) = P(A)$

We know that

$P(A' \cap B) = P(B) - P(A \cap B)$

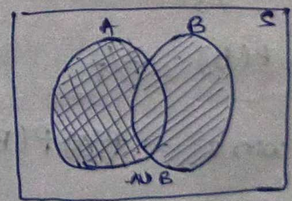
$P(A' \cap B) = P(B) - P(A)$

$P(A) = P(B) - P(A' \cap B)$

$\therefore \boxed{P(A) \leq P(B)}$ [since $0 \leq P(A' \cap B) \leq 1$]

Theorem 5: Addition Theorem for two events

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



Proof

Let S be a sample space and A, B are 2 events defined on S

$A \cup B$ can be divided ~~as~~ into 2 mutually exclusive events $A \cap B'$ and B

$\therefore (A \cap B') \cap B = \phi$

$(A \cap B') \cup B = (A \cup B)$

$P(A \cup B) = P(A \cap B') + P(B)$

$P(A \cup B) = P(A) - P(A \cap B) + P(B)$ [from theorem 1]

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Problems:

① 3 students A, B, C are in running race and each is twice likely to win as C. Find the probability that B or C wins.

A and B have same probabilities of winning and each is twice likely to win as C. Find the probability that B or C wins.

Sol

$A \cup B \cup C = S$

Total probability $P(A) + P(B) + P(C) = 1$

By given data

$P(A) = P(B) = 2P(C)$

$P(A) + P(B) + P(C) = 1$

$2P(C) + 2P(C) + P(C) = 1$

$5P(C) = 1$

$\therefore P(C) = 1/5$

$P(A) = P(B) = 2/5$

Probability that B or C wins = $\cancel{P(B)}\, P(B \cup C)$

$P(B \cup C) = P(B) + P(C) - P(B \cap C)$

$= 2/5 + 1/5 - 0$

$= 3/5 = 0.6$

② A card is drawn from a well shuffled pack of cards. What is the prob that it is ~~to~~ either a spade or an ace.

Let A be the event of getting a spade

B be the event of getting an ace

$P(A) = 13/52 \quad P(B) = 4/52$

then $A \cup B$ is the event of getting either a spade or an ace

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52}$

$= 16/52$

$= 0.3076$

Conditional Event:

If E_1 & E_2 are the events of sample space S & If E_2 occur after the occurance of E_1 then the event of occurance of E_2 after the event E_1 is called conditional event.

(E_2/E_1) – E_2 given E_1

Example:

Two coins are tossed the event of getting 2 tails given that there is atleast one tail is a conditional event.

Conditional Probability: If E_1 and E_2 are 2 events in a sample space 'S' and probability of event E_1, $P(E_1) \neq 0$ then $P(E_2)$ after the E_1 has occurred is called conditional probability of E_2 given E_1

$$P(E_2/E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)}$$

lly

$$P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

Multiplication theorem of Probabilities.

Statement: In a random experiment if E_1, E_2 are 2 events such that

$$\boxed{P(E_1 \cap E_2) = P(E_2/E_1)P(E_1)}$$

$$\boxed{P(E_1 \cap E_2) = P(E_1/E_2)\,P(E_2)}$$

Proof: Let S be the sample space associated with random experiment, Let E_1, E_2 be 2 events of S, such that $P(E_1) \neq 0$, $P(E_2) \neq 0$

By the ~~difection~~ defination of conditional probability

$$P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)}$$

$$P(E_1 \cap E_2) = P(E_1/E_2)\,P(E_2)$$

also

$$P(E_1 \cap E_2) = P(E_2/E_1)\,P(E_1) \quad \left(\because P(E_2/E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)}\right)$$

Problems

① Find $P(B/A)$ and $P(A/B')$ where A and B are 2 events with $P(A) = 1/3$, $P(B) = 1/4$, $P(A \cup B) = 1/2$

Sol By addition theorem

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$= 1/3 + 1/4 - 1/2$$

$$= \frac{7}{12} - \frac{1}{2}$$

$$= \frac{1}{12}$$

(i) $P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{1/12}{1/3} = 1/4$

(ii) $P(A/B') = \dfrac{P(A \cap B')}{P(B')}$

$$P(B') = 1 - P(B) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P(A \cap B') = P(A) - P(A \cap B) = \frac{1}{3} - \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$$

$$P(A/B') = \frac{P(A \cap B')}{P(B')} = \frac{1/4}{3/4} = \frac{1}{3}$$

Independent Events:

If the happening of an event E_2 is not affected by the happening or not happening of the event E_1 then event E_2 is said to be independent of E_1 and $P(E_2/E_1) = P(E_2)$

i.e., $P(E_2|E_1) = P(E_2)$

$$\frac{P(E_1 \cap E_2)}{P(E_1)} = P(E_2)$$

$$P(E_1 \cap E_2) = P(E_1)P(E_2)$$

Ex: The probabilities that the students A, B, C, D solve a problem 'R' $1/3, 2/5, 1/5, 1/4$ respectively. If all of them try to solve the problem, What is the probability that the problem is solved.

Sol: Given Probabilites of A, B, C, D are

$P(A) = 1/3 \quad P(B) = 2/5 \quad P(C) = 1/5 \quad P(D) = 1/4$

$\therefore P(A') = 1 - P(A) = 1 - 1/3 = 2/3$

$P(B') = 1 - P(B) = 1 - 2/5 = 3/5$

$P(C') = 1 - P(C) = 1 - 1/5 = 4/5$

$P(D') = 1 - P(D) = 1 - 1/4 = 3/4$

Probabilitys that the problem is not solved by A, B, C, D

$P(A \cup B \cup C \cup D)' = P(A') \cap P(B') \cap P(C') \cap P(D')$ [demorgan law]

$= P(A')\,P(B')\,P(C')\,P(D')$ [$\because$ independent events]

$= 2/3 \times 3/5 \times 4/5 \times 3/4$

$= \frac{6}{25}$

Probability that the problem solved

$P(A \cup B \cup C \cup D) = 1 - P(A \cup B \cup C \cup D)' = 1 - 6/25 = 19/25$

② Two aeroplane bomb a target in succession The Probability of each correctly scoring a hit is 0.3 and 0.2 respectively. The second will bomb only if the 1st misses the target. Find Prob that
(i) target is hit (ii) both failures to hit

Sol let A be the event of 1st plane hitting the target.
let B be the event of 2nd plane hitting the target.

The probability of 1st plane hitting the target

$P(A) = 0.3$

The probability of 2nd plane hitting the target

$P(B) = 0.2$

$P(A') = 1 - P(A) = 1 - 0.3 = 0.7$

$P(B') = 1 - P(B) = 1 - 0.2 = 0.8$

(i) Probability of target is hit = P(A hits or A fails and B hits)

$= P(A \cup (A' \cap B))$

$= P(A) + P(A' \cap B)$ [3^{rd} axiom]

$= P(A) + P(A')P(B)$ [independent events]

$= 0.3 + (0.7)(0.2)$

$= 0.3 + 0.14$

$= 0.44$

(ii) Probability of both fails = $P(A' \text{ and } B')$

$= P(A' \cap B')$

$= P(A')\,P(B')$

$= (0.7)(0.8)$

$= 0.56$

(3) A box 'A' contains 5 red and 3 white marbles. Box 'B' contains 2 red and 6 white marbles. If a marble is drawn from each both what is prob that they are

(i) both of same colour

Sol Let E_1 be the event that the marble is drawn from ~~the~~ box A and color is red.

E_2 be ~~the even~~

$P(E_1) = \frac{1}{2} \times \frac{5}{8}$

E_2 be the event that marble is drawn from box B and is red

$P(E_2) = \frac{1}{2} \times \frac{2}{8}$

The probability that both marbles are red is

$P(E_1 \cap E_2) = P(E_1) * P(E_2) = \frac{1}{2} \times \frac{5}{8} * \frac{1}{2} \times \frac{2}{8}$

$= \frac{10}{(16)^2} = 0.0390$

Let E_3 be the event that marble is drawn from box A and is white

$P(E_3) = \frac{1}{2} \times \frac{3}{8}$

Let E_4 be the event that marble is drawn from box B and is white

$P(E_4) = \frac{1}{2} \times \frac{6}{8}$

The probability that both marbles are white is

$P(E_3 \cap E_4) = P(E_3) \times P(E_4) = \left(\frac{1}{2} \times \frac{3}{8}\right)\left(\frac{1}{2} \times \frac{6}{8}\right)$

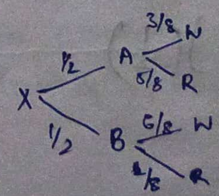
$= \frac{18}{16^2} = 0.0703$

Probability that the marbles are of same colour.

$\therefore$ Req Probability $= P(E_1 \cap E_2) + P(E_3 \cap E_4)$

$= 0.039 + 0.0703$

$= 0.1093$



(4) Probabilities of passing in subjects A, B, C and D are $\frac{3}{4}, \frac{2}{3}, \frac{4}{5}$ and $\frac{1}{2}$ respectively. To qualify in the exam a student should pass in A and in 2 subjects among '3'. What is probability of qualifying in that Examination.

Sol Probabilities of passing all the subjects.

$P(A) = \frac{3}{4}$ $\quad P(C) = \frac{4}{5}$ $\quad P(A') = 1/4$ $\quad P(B') = 1/3$

$P(B) = \frac{2}{3}$ $\quad P(D) = \frac{1}{2}$ $\quad P(C') = 1/5$ $\quad P(D') = \frac{1}{2}$

There are four possibilities

(i) to Pass A, B, C and fail in D

(ii) to Pass A, B, D and fail in C

(iii) to pass A, C, D and fail in B

(iv) to pass A, B, C, D

$\therefore$

Required probability $= P(A \cap B' \cap C \cap D) + P(A \cap B \cap C' \cap D) + P(A \cap B \cap C \cap D') + P(A \cap B \cap C \cap D)$

$= P(A)P(B')P(C)P(D) + P(A)P(B)P(C')P(D) + P(A)P(B)P(C)P(D') + P(A)P(B)P(C)P(D)$

$= \left(\frac{3}{4}\right)\left(\frac{1}{3}\right)\left(\frac{4}{5}\right)\left(\frac{1}{2}\right) + \left(\frac{3}{4}\right)\left(\frac{2}{3}\right)\left(\frac{1}{5}\right)\left(\frac{1}{2}\right) + \left(\frac{3}{4}\right)\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{1}{2}\right) + \left(\frac{3}{4}\right)\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{1}{2}\right)$

$= 0.1 + 0.05 + 0.1 + 0.2$

$= 0.55$

(5) Among the cookers in factory only 50 received bonus. Among those receiving the bonus 30% are skilled what is the prob of randomly selected workers whose skilled and receiving bonus

Sol Let A be an event of Receiving bonus

Let B be is an event of considering skilled cookers

$P(A) = 50\% = 0.5$

$P(B/A) = 30\% = 0.3$

Req Prob $P(A \cap B) = P(B/A)P(A) = 0.3 \times 0.5 = 0.15$

(6) If the probability that a communication system will have high fieldity is 0.81 and probability that it will have high fieldity and selectivity is 0.18. What is the prob. that a system will have high selectivity with high fieldity

Sol let A be the event of a communication system will have high fieldity

$P(A) = 0.81$

Let B be the event of a communication system will have high selectivity. $P(A \cap B) = 0.18$

$P(B/A) = P(A \cap B) / P(A) = 0.18/0.81 = 0.2222$

The proab of an system having high selectivity with high fieldity $= P(B/A)$

(7) 3 Machines A B C produces 40% 30% 30% of the total no. of items of a factory the percentage of defective items of these machines are 4% 2% 3% resp. If an item is selected at random what is the prob. that the item is defective.

Sol Given $P(A) = 40\% = 0.4$

$P(B) = 30\% = 0.3$

$P(C) = 30\% = 0.3$

Let X be an event that the selected item is defective

$P(X/A) = 4\% = 0.04$

$P(X/B) = 2\% = 0.02$

$P(X/C) = 3\% = 0.03$

$P(X) = P(A)P(X/A) + P(B)P(X/B) + P(C)P(X/C)$

$= (0.4)(0.04) + (0.3)(0.02) + (0.3)(0.03)$

$= 0.016 + 0.006 + 0.009$

$= 0.031$

Baye's theorem

** Statement: Suppose $E_1, E_2, \dots E_n$ are mutually exclusive events of a sample space 'S' such that $P(E_i) > 0$ $i = 1, 2, 3, \dots n$ and 'A' is any arbitary event of S such that probability of 'A' $P(A) > 0$

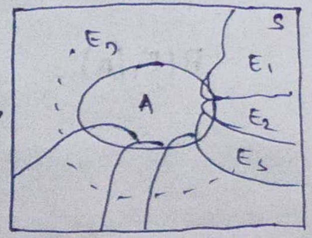
then $P(E_i/A) = \dfrac{P(A/E_i)(P(E_i))}{\sum_{i=1}^{n} P(A/E_i)P(E_i)}$

Proof:

Given A is subset of S

$\therefore A \subseteq S$

where 'A' is any arbitary event in 'S'



$A \subseteq \bigcup_{i=1}^{n} E_i$

where $E_1, E_2, E_3, \dots E_n$ are n mutually execlusive events in 'S'

$A \cap S = A$

$A \cap (\bigcup_{i=1}^{n} E_i) = A$

$A = A \cap (E_1 \cup E_2 \cup E_3 \dots \cup E_n)$

$A = (A \cap E_1) \cup (A \cap E_2) \dots \cup (A \cap E_n)$ ($\because$ distributive law)

Taking Probabilities on both sides

$P(A) = P((A \cap E_1) \cup (A \cap E_2) \dots \cup (A \cap E_n))$

$\therefore P(A) = P(A \cap E_1) + P(A \cap E_2) \dots$
$\dots + P(A \cap E_n)$

($\because E_1, E_2, \dots E_n$ are mutually exclusive events. $A \cap E_1, A \cap E_2, \dots A \cap E_n$ are also Mutually Exclusive)

($\because$ from 3rd axiom)

$$P(A) = \sum_{i=1}^{n} P(A \cap E_i)$$

$$P(A) = \sum_{i=1}^{n} P(E_i) P(A|E_i) \quad \text{(By multiplication theorem)} \quad — (1)$$

By conditional probability

$$P(E_i/A) = \frac{P(E_i \cap A)}{P(A)}$$

$$= \frac{P(E_i) P(A/E_i)}{P(A)} \quad \text{(By multiplication theorem)}$$

$$\boxed{P(E_i/A) = \frac{P(E_i) P(A/E_i)}{\sum_{i=1}^{n} P(E_i) P(A|E_i)}} \quad (\because \text{from } (1))$$

Problems

(1) Companies B_1, B_2 and B_3 produces 30%, 45%, 25% of the car respectively. It is known that 2%, 3%, 2% of these cars are defective.

(i) What is the prob that a car purchase is defective.

(ii) If a car purchased is found to be defective what is probablity that this car produced by company B_1

Sol: Let X be the event of defective items.

Probability of car produced by company B_1 is

$$P(B_1) = 30\% = \frac{30}{100} = 0.3$$

Probability of car produced by company B_2 is

$$P(B_2) = 45\% = \frac{45}{100} = 0.45$$

Probability of car produced by company B_3 is

$$P(B_3) = 25\% = \frac{25}{100} = 0.25$$

Probability of ~~ge~~ purchasing a defective car produced by company B_1 is $P(X/B_1) = 0.02$

Probability of purchasing a defective car produced by company B_2 is $P(X/B_2) = 0.03$

Probability of purchasing a defective car produced by company B_3 is $P(X/B_3) = 0.02$

(i) Probability of purchasing a defective car

$$P(X) = P(B_1) P(X/B_1) + P(B_2) P(X|B_2) + P(B_3) P(X/B_3)$$

$$= 0.3 \times 0.02 + 0.45 \times 0.03 + 0.25 \times 0.02$$

$$= \cancel{0.06 +} \; 0.0245$$

(ii) Probability of purchased car is from company B_1 which is found to be defective is

$$P(B_1/x) = \frac{P(B_1)P(x/B_1)}{P(B_1)P(x/B_1) + P(B_2)P(x/B_2) + P(B_3)P(x/B_3)}$$

$$= \frac{0.3 \times 0.02}{0.0245}$$

$$= \cancel{0.3061}\ 0.2448$$

(2) There are 3 boxes I, II, and III. Box I contains four red, 5 blue and 6 ~~six~~ white. Box II contains 3 red, 4 blue and 5 white balls. Box III contains 5 red, 10 blue and 5 white balls. One box is choosen and one ball is drawn from it. What is probability that the red ball is drawn from box 1 and also find prob of getting a red ball.

Sol Let x be an event ~~of~~ ~~get~~ that a red ball is choosen from 3 boxes.

let A, B, C be the events of ~~draw~~ a box is choosen.

Probability of choosing box I is $P(A) = 1/3$

Probability of choosing box II is $P(B) = 1/3$

Probability of choosing box III is $P(C) = 1/3$

Probability of ball is drawn is red from box I is

$$P(x/A) = 4/15$$

Probability of red ball from box II is $P(x/B) = \frac{3}{12} = \frac{1}{4}$

Probability of red ball from box III is $P(x/c) = \frac{5}{20} = \frac{1}{4}$

(i) Probability that red ball is drawn from box I is

$$P(A/x) = \frac{P(A)P(x/A)}{P(A)P(x/A) + P(B)P(x/B) + P(C)P(x/C)}$$

$$= \frac{\frac{1}{3} \times \frac{4}{15}}{\frac{1}{3} \times \frac{4}{15} + \frac{1}{3} \times \frac{1}{4} + \frac{1}{3} \times \frac{1}{4}}$$

$$= \frac{\frac{4}{45}}{\frac{4}{45} + \frac{1}{12} + \frac{1}{12}}$$

$$= 0.3478$$

(ii) Probability of drawing red ball is $P(A) = P(A)P(x/A) + P(B)P(x/B) + P(C)P(x/C)$

$$= \frac{4}{45} + \frac{1}{12} + \frac{1}{12}$$

$$= 0.2556$$

(3) In a class 2% of boys and 3% of girls are having Blue eyes. There are 30% girls in the class. If a student is selected and having blue eyes what is the probability that the student is a girl.

Probability of Boys $P(B) = 70\% = \frac{70}{100} = 0.7$

Probability of Girls $P(G) = 30\% = 0.3$

let x be an event of selected student having blue eyes.

Probability of selected blue eye student he is a boy

$$P(x \mid B) = 0.02$$

Probability of blue eye student is a girl

$$P(x \mid G) = 0.03$$

Probability of a girl having blue eyes

$$P(G/x) = \frac{P(G)\,P(x/G)}{P(G)P(x/G) + P(B)\,P(x/B)}$$

$$= \frac{0.3 \times 0.03}{0.3 \times 0.03 + 0.7 \times 0.02}$$

$$= 0.3913$$

④ Of the 3 men the chances that a politician, a business or an academician will be appointed as a vice chancellor (V.C) of a university are 0.5, 0.3, 0.2 respectively.

Probability that research is promoted by these persons if they are appointed as V.C are 0.3, 0.7, 0.8 respectively.

(i) Find prob that research is promoted.

(ii) If research is promoted what is probability that V.C is an academician.

Sol: let X be the event of promoting a research by V.C

let A, B, C be the events that a politician, a business man and an academician will be apointed as V.C.

Probability that a politician will be apointed as V.C. is $P(A) = 0.5$

ly $P(B) = 0.3$

$P(C) = 0.2$

$P(X/A) = 0.3 \quad P(X/B) = 0.7 \quad P(X/C) = 0.8$

(i) Probability that research is promoted is

$$P(X) = P(A)\,P(X/A) + P(B)\,P(X/B) + P(C)\,P(X/C)$$

$$= 0.5 \times 0.3 + 0.3 \times 0.7 + 0.2 \times 0.8$$

$$= 0.15 + 0.21 + 0.16$$

$$= 0.52$$

(ii) If research is promoted and VC is an academician

probability $\Rightarrow$ $$P(C/X) = \frac{P(C)\,P(X/C)}{P(A)P(X/A) + P(B)P(X/B) + P(C)P(X/C)}$$

$$= \frac{0.2 \times 0.8}{0.52}$$

$$= \frac{0.16}{0.52}$$

$$= 0.3076$$

(5) In a factory machine A produces 40% of the output and Machine B produces 60%. on the average 9 items in 1000 produced by A are defective and 1 item in 250 produced by B are defective. An item is drawn at random from a days output and is found to be defective. What is prob that it was produced by machine A or B

Let X be an event of defective items

Probability of A produces $P(A) = 40\% = 0.4$

Probability of B produces $P(B) = 60\% = 0.6$

Probability of defective from machine A =

$$P(X/A) = 0.009$$

Probability of defective from machine B = $P(X/B) = \frac{1}{250}$

$= 0.004$

Probability of getting defective from machine A is

$$P(A/X) = \frac{P(A)\,P(X/A)}{P(A)P(X/A) + P(B)P(X/B)}$$

$$= \frac{0.4 \times 0.009}{0.4 \times 0.009 + 0.6 \times 0.004} = 0.6$$

Probability of getting defective product from machine B is

$$P(B/X) = \frac{P(B)\,P(X/B)}{P(A)P(X/A) + P(B)P(X/B)} = \frac{0.6 \times 0.004}{0.4 \times 0.009 + 0.6 \times 0.004} = 0.4$$

Therefore probability of defective items produced by machine A or machine B $= P(A/X) + P(B/X)$

$= 0.6 + 0.4$

$= 1.0$

(6) Suppose 5 men out of 100 and 25 women 10,000 are color blind. A color blind person is choosen at random what is prob of the person being a male.

let X be a event of choosen person is color blind.

~~$P(M) = 5/100 = 0.05$~~

~~$P(W) = 25/10,000 = 0.0025$~~

A/c let us assume that male and female are equally populated.

$P(M) = 0.5$

$P(W) = 0.5$

Now

Probability of color blind person who is male =

$P(X/M) = 5/100 = 0.05$

$P(X/W) = 25/10000 = 0.0025$

Probability that male is color blind person.

$$P(M/X) = \frac{P(M)P(X/M)}{P(M)P(X/M) + P(W)P(X/W)} = \frac{0.5 \times 0.05}{0.5 \times 0.05 + 0.5 \times 0.0025}$$

=

7. A manufacturer firm produces ~~steel~~ steel pipes in 3 plants with daily production volumes of 500, 1000, 2000 units resp. According to past experience it is known that the defective o/p's produced by these plants are 0.005, 0.008 and 0.01 respectively. If a pipe is selected at random & if is found to be defective find out

(i) from which plant the pipe came

(ii) What is prob that it came from 1^{st} plant.

Let A, B, C be the 3 plo events that a pipe is manufactured in plant 1, 2, 3 resp.

let X be the event of defective pipe

$P(A) = 500/3500 = 0.142$

$P(B) = 1000/3500 = 0.285$

$P(C) = 2000/3500 = 0.571$

Prob of defective item from plant 1 $P(X/A) = 0.005$

$P(X/B) = 0.008$

$P(X/C) = 0.01$

(i) Prob of defective pipe $P(X) = P(A)P(X/A) + P(B)P(X/B) + P(C)P(X/C)$

$= 0.142 \times 0.005 + 0.285 \times 0.008 + 0.571 \times 0.01$

$= 0.008$

(ii) $P(A/X) = \dfrac{P(A)\,P(X/A)}{\sum_{i=A}^{C} P(A)P(X/A)} = \dfrac{0.142 \times 0.005}{0.142 \times 0.005 + 0.285 \times 0.0 + 0.571 \times 0.0}$

$=$

Extension of Addition Law of Probability.

[Addition theorem for n events]

Statement: For n events $A_1, A_2, \ldots A_n$ prob of

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} A_i - \sum\sum_{1 \le i \le j \le n} P(A_i \cap A_j) + \ldots + \cancel{P}(-1)^{n-1} P(A_1 \cap A_2 \ldots \cap A_n)$$

Proof:

We proove the theorem by using mathematical induction method.

let us assume that the theorem is true for $n = r$ events and we prove that it is true for $n = r+1$ events

By proving that it is true for $n = 2$ events

for two events A_1 and A_2 we have

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \quad -①$$

It is true for $n = 2$ events

Let us assume that the given theorem is true for $n = r$ events

$$P\left(\bigcup_{i=1}^{r} A_i\right) = \sum_{i=1}^{r} A_i - \sum\sum_{1 \le i \le j \le r} P(A_i \cap A_j) + \ldots + (-1)^{r-1} P(A_1 \cap A_2 \cap \ldots \cap A_r) \quad -②$$

Now

$$P\left(\bigcup_{i=1}^{r+1} A_i\right) = \cancel{\ldots} = P\left(\bigcup_{i=1}^{r} A_i \cup A_{r+1}\right)$$

$$= P\left(\bigcup_{i=1}^{r} A_i\right) + P(A_{r+1}) - P\left(\bigcup_{i=1}^{r} A_i \cap A_{r+1}\right)$$

$$= \left[\sum_{i=1}^{r} A_i - \sum\sum_{1\le i\le j\le r} (A_i \cap A_j) + \dots + (-1)^{r-1} P(A_1 \cap A_2 \cap \dots \cap A_r)\right] + P(A_{r+1})$$

$$- P\left(\bigcup_{i=1}^{r} (A_i \cap A_{r+1})\right)$$

$$= \sum_{i=1}^{r+1} P(A_i) - \sum\sum_{1\le i\le j\le r} P(A_i \cap A_j) - \dots (-1)^{r-1} P(A_1 \cap A_2 \cap \dots \cap A_r)$$

$$- \left[\sum_{i=1}^{r} P(A_i \cap A_{r+1}) - \sum\sum_{1\le i\le j\le r} P(A_i \cap A_{r+1} \cap A_j)\right.$$

$$\left. + \dots + (-1)^{r-1} \left(\bigcap_{i=1}^{r} (A_i \cap A_{r+1})\right)\right)$$

$$= \sum_{i=1}^{r+1} P(A_i) - \sum\sum_{1\le i\le j\le r} P(A_i \cap A_j) - \dots (-1)^{r} P(A_1 \cap A_2 \cap \dots \cap A_{r+1})$$

Unit - II

Random Variables.

Random Variable: The real variable X whose value is determined by the outcome of a random experiment is called a random variable.

This is also defined as a real value function on the sample space 's' of the random experiment such that for each point 'x' of the sample space the corresponding probability occurs.

Ex. When 2 coins are tossed at a time the sample space consist of S = {HH, HT, TH, TT}

let X be a random variable which takes the number of heads on S then X takes the values {0, 1, 2} ∴ X = {0, 1, 2}

X	0	1	2
$P(X=x_i)$ $P(x_i)$ P_i	1/4	2/4	1/4

Types of Random Variables

* Random variables are of 2 types
1. Discrete Random variables (R.V)
2. Continuous R.V

Discrete R.V:

A random variable X which can take only a finite number of discrete values in any domain is called as a Discrete R.V.

In other words if the random variable takes the values only on the set $X = \{0, 1, 2, \ldots, n\}$

Ex: Random variable denoting number of students in class

$X(s) = \{x / x \text{ is a positive integer}\}$

Continuous R.V:

Random variable X which can take values continuously i.e. which takes all possible values in a given interval is called continuous random variable.

Ex: The Height, Age, Weight of individuals are examples of Continuous R.V.

Probability Function of a discrete random variable:

(Probability Mass Function, Pmf):

If for a discrete random variable X, the real valued function $P(X = X_i) = P(x_i)$ is called probability function or probability mass function of a discrete random variable X.

probability function $P(x_i)$ of ~~a rand~~ possess the following properties (i) $P(x_i) \geq 0$

(ii) total probability $\sum_{i=1}^{n} P(x_i) = 1$

i.e., $P(x)$ can't be negative for any value of x.

The probability distribution of a random variable X given by the following table

X	$x_1 \; x_2 \; \ldots \ldots \; x_n$
$P(X = x_i)$	$P(X=x_1)\; P(X=x_2) \;\ldots\; P(X=x_n)$

$P(X < x_i) = P(X=x_1) + P(X=x_2) + \ldots \; P(X = x_{i-1})$

$P(X \leq x_i) = P(X = x_i) + P(x = x_2) + \ldots\ldots + P(x = x_i)$

Commulative distribution Function (Distribution Function) (cdf)

Suppose that x is a discrete random variable then the distribution function (or) Cummulative distribution function is denoted by $F(x)$ and it's defined as

$F(x) = P(x \leq x_i)$

Expectation, Mean, Variance of a probability distribution

Expectation of a probability Distribution:

Suppose a r.v assumes the value $x_1, x_2, \ldots x_n$ with respective probabilities $P_1, P_2, \ldots P_n$ then the mathematical expectation are mean expected values of X denoted by $E(x)$ and is defined as the sum of products of different values of x and the corresponding probabilities i.e., $\mu = E(x) = x_1P_1 + x_2P_2 + \ldots + x_nP_n$

$= \sum_{i=1}^{n} x_i P_i$

Theorem 1:

If X is a discrete r.v and K is a constant

then $E(X+K) = E(X) + K$

Proof: By the definition of expectation

$$E(x) = \sum_{i=1}^{n} x_i P_i$$

$$E(x+k) = \sum_{i=1}^{n} (x_i+k) P_i = \sum_{i=1}^{n} (x_i+k) P_i$$

$$= \sum_{i=1}^{n} x_i P_i + k\sum_{i=1}^{n} P_i$$

$$= E(X) + K(1)$$

$$E(X+K) = E(X) + K$$

Theorem 2:

If X is a discrete r.V and K is a constant then

$$E(KX) = KE(X)$$

Proof: By definition of expectation

$$E(X) = \sum_{i=1}^{n} x_i P_i$$

$$E(KX) = \sum_{i=1}^{n} K x_i P_i$$

$$= K\sum_{i=1}^{n} x_i P_i$$

$$E(KX) = K E(X)$$

Note: Mean of a discrete r.V is given by $\mu = \sum_{i=1}^{n} P_i x_i = E(x)$

Variance:

Variance characterises the variability in the distributions since 2 distributions with same mean can have different dispersions of data about their means.

Variance of the probability distribution of a r.V X is the mathematical expectation $((X - E(X))^2)$ then

$$V(X) = E[X - E(X)]^2$$

$$= E[X^2 + (E(X))^2 - 2X E(X)]$$

$$= E(X^2) + (E(X))^2 - 2(E(X))^2$$

$$= E(X^2) - (E(X))^2$$

Standard Deviation

It is the positive square root of variance denoted by σ

Problems

① A random variable X has the following probability function.

X	0	1	2	3	4	5	6	7	8
P(x)	$\frac{K}{45}$	$\frac{K}{15}$	$\frac{K}{9}$	$\frac{K}{5}$	$\frac{2K}{45}$	$\frac{6K}{45}$	$\frac{7K}{45}$	$\frac{8K}{45}$	$\frac{4K}{45}$

find (i) K (iii) $P(X \geq 5)$

(ii) $P(0 < X < 4)$ (iv) Mean.

Sol (i) We know that

$$\sum P(x) = 1$$

$$\frac{K}{45} + \frac{K}{15} + \frac{K}{9} + \frac{K}{5} + \frac{2K}{45} + \frac{6K}{45} + \frac{7K}{45} + \frac{8K}{45} + \frac{4K}{45} = 1$$

$$\frac{K+3K+5K+9K+2K+6K+7K+8K+4K}{45} \Rightarrow 1$$

$$\frac{45K}{45} = 1$$

$$K = 1$$

(ii) $P(0<X<4) = P(X=1) + P(X=2) + P(X=3)$

$$= \frac{K}{15} + \frac{K}{9} + \frac{K}{5}$$

$$\Rightarrow \frac{1}{15}+\frac{1}{9}+\frac{1}{5} = \frac{3+5+9}{45} = \frac{17}{45} = 0.3778$$

(iii) $P(X \geq 5) = P(X=5) + P(X=6) + P(X=7) + P(X=8)$

$$= \frac{6K}{45} + \frac{7K}{45} + \frac{8K}{45} + \frac{4K}{45}$$

$$= \frac{6}{45} + \frac{7}{45} + \frac{8}{45} + \frac{4}{45} = \frac{25}{45}$$

$$= 0.5556$$

(iv) Mean $= \sum_{i=1}^{n} P_i x_i = 0\left(\frac{K}{45}\right) + 1\left(\frac{K}{15}\right) + 2\left(\frac{K}{9}\right) + 3\left(\frac{K}{5}\right) +$

$$4\left(\frac{2K}{45}\right) + 5(6K/45) + 6\left(\frac{7K}{45}\right) + \left(\frac{8K}{45}\right)7 + 8\left(\frac{4K}{45}\right)$$

$$= 0 + \frac{1}{15} + \frac{2}{9} + \frac{3}{5} + \frac{8}{45} + \frac{30}{45} + \frac{42}{45} + \frac{56}{45} + \frac{32}{45}$$

$$= 4.622$$

(2) For the following probability distribution :

X	-3	-2	-1	0	1	2	3
P(x)	0.001	0.01	0.1	-	0.1	0.01	0.001

Find (i) the missing probability

(ii) mean

(iii) variance

Sol (i) Let missing probability be K

Since $\sum_{i=1}^{n} x_i = 1$

$$0.001 + 0.01 + 0.1 + K + 0.1 + 0.01 + 0.001 = 1$$

$$0.222 + K = 1$$

$$K = 0.778$$

(ii) Mean $= E(x) = \mu = \sum x P(x)$

$= -3(0.001) + (-2)(0.01) + (-1)(0.1) + 0(0.778) + 1(0.1) +$

$2(0.01) + 3(0.001)$

$E(x) = 0$

(iii) Variance $V(X) = E(x^2) - (E(x))^2$

$E(x^2) = 9(0.001) + 4(0.01) + 1(0.1) + 0(0.778) + 1(0.1) +$

$4(0.01) + 9(0.001)$

$= 0.298$

$V(X) = E(x^2) - (E(x))^2$

$= 0.298 - (0)^2 = 0.298$

③ If x denote the sum of 2 numbers that appear when a pair of fair dice is tossed.

Find (i) distribution function

(ii) Mean

(iii) Variance

Sol (i) If 2 dice are thrown the the sum x of the 2 numbers which turn up must be an integer b/w 2 and 12

For $x = 2$ there is only one favourable point (1,1)

and hence $P(X=2) = 1/36$

For $X=3$ there are 2 favourable points (1,2) (2,1)

and hence $P(X=3) = 2/36$

Similarly we calculate other probabilities

hence the distribution table is

Sum X	2	3	4	5	6	7	8	9	10	11	12
$P(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(ii) Mean $= E(x) = \mu = \sum x_i P_i$

$= 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{4}{36}\right) + 6\left(\frac{5}{36}\right) + 7\left(\frac{6}{36}\right) + 8\left(\frac{5}{36}\right)$
$+ 9\left(\frac{4}{36}\right) + 10\left(\frac{3}{36}\right) + 11\left(\frac{2}{36}\right) + 12\left(\frac{1}{36}\right)$

$= 7.\cancel{305}$

(iii) Variance $V(x) = E(x^2) - (E(x))^2$

$= 4\left(\frac{1}{36}\right) + 9\left(\frac{2}{36}\right) + 16\left(\frac{3}{36}\right) + 25\left(\frac{4}{36}\right) + 36\left(\frac{5}{36}\right) +$
$49 \times \frac{6}{36} + 64(5/36) + 81(4/36) + 100(3/36) + 121\left(\frac{2}{36}\right) +$
$144\left(\frac{1}{36}\right)$

$= 54.83$

$V(X) = E(x^2) - (E(x))^2$

$= 54.83 - (7)^2$

$= 54.83 - 49 = 5.83$

④ Find (i) K (ii) $P(X \le 4)$ (iii) $P(3 < X \le 6)$ (iv) find min of a so $P(X \le a) > 0.3$

x	0	1	2	3	4	5	6
$P(x)$	K	3K	5K	7K	9K	11K	13K

(i) $\sum p(x_i) = 1$

$K + 3K + 5K + 7K + 9K + 11K + 13K = 1$

$49k = 1$

$K = 0.0204$

(ii) $P(X \le 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= K + 3K + 5K + 7K + 9K$

$= 25K$

$= 25(0.0204)$

$= 0.51$

(iii) $P(3 < x \le 6) = P(x=4) + P(x=5) + P(x=6)$

$= 9K + 11K + 13K$

$= 33K = 33(0.0204$

$= 0.6732$

(iv) $P(x \le a) > 0.3$

let $a = 1$

$P(x \le 1) = P(x=0) + P(x=1)$

$= K + 3K = 4K = 4(0.0204)$

$= 0.0816 < 0.3$

let $a = 2$

$P(x \le 2) = P(x=0) + P(x=1) + P(x=2)$

$= K + 3K + 5K = 9K = 9(0.0204)$

$= 0.1836 < 0.3$

let $a = 3$

$P(x \le 3) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$= K + 3K + 5K + 7K = 16K = 16(0.0204)$

$= 0.326 > 0.3$

$\therefore a = 3$

(5)

x	0	1	2	3	4	5	6	7
P(x)	0	K	2K	2K	3K	3K	K^2	$7K^2+K$

find (i) k

(ii) mean

(i) $\Sigma P(x) = 1$

$0 + K + 2K + 2K + 3K + 3K + K^2 + 7K^2 + K = 1$

$8K^2 + 12K = 1$

[illegible]

$K = 0.0791$ or -1.579

K must be positive

$K = 0.0791$

(ii) Mean $= E(x) = \mu = \Sigma P_i x_i$

$= 0(0) + 1(K) + 2(2K) + 3(2K) + 4(3K) + 5(3K) + 6(K^2)$

$+ 7(7K^2 + K)$

$= 0 + K + 4K + 6K + 12K + 15K + 6K^2 + 49K^2 + 7K$

$= 55K^2 + 45K$

$= 55(0.0791)^2 + 45(0.0791)$

$= 3.9036$

(6) Five defective ~~bolte~~ bolts are accidently mixed with 20 good ones. Find probability distribution of the no. of defective bolts if 4 bolts are drawn at random from this lot. and Find mean

Sol: 4 bolts are randomly choosen
No. of non-defective bolts are 20
and defective bolts are 5

Probability of selecting 0 defectives

$$P(x=0) = \frac{{}^{20}C_4}{{}^{25}C_4} = 0.3830$$

Probability of selecting 1 defectives

$$P(x=1) = \frac{({}^{20}C_3)({}^{5}C_1)}{({}^{25}C_4)} = 0.4505$$

$^{20}C_3 \times 5$

Probability of selecting 2 defective

$$P(x=2) = \frac{({}^{20}C_2)({}^{5}C_2)}{{}^{25}C_4} = \cancel{0.3927}\ 0.1501$$

Probability of selecting 3 defective

$$P(X=3) = \frac{({}^{20}C_1)({}^{5}C_3)}{{}^{25}C_4} = \cancel{0.0412}\ 0.0158$$

Probability of selecting 4 defective

$$P(x=4) = \frac{{}^{5}C_4}{{}^{25}C_4} = 0.0003$$

x	0	1	2	3	4
p(x)	0.3830	0.4505	0.1501	0.0158	0.0003

$$\text{Mean} = E(x) = \mu = \sum P_i x_i$$

$$= 0(0.3830) + 1(0.4505) + 2(0.1501) + 3(0.0158) + 4(0.0003)$$

$$= 0.7993$$

(7) Find the mean and variance of the uniform probability distribution given by

$$F(x) = \frac{1}{n} \text{ for } x = 1, 2, 3, \ldots n$$

Sol

x	1	2	3	...	n
f(x)	1/n	1/n	1/n		1/n

$$\text{Mean} = E(x) = \mu = \sum P_i x_i$$

$$= 1\left(\frac{1}{n}\right) + 2\left(\frac{1}{n}\right) + \cdots n\left(\frac{1}{n}\right)$$

$$= \frac{(1+2+\cdots n)}{n} = \frac{\frac{\cancel{n}(n+1)}{2}}{\cancel{n}}$$

$$= \frac{n+1}{2}$$

$$\text{Variance} = E(x^2) - (E(x))^2$$

$$E(x^2) = 1^2\left(\frac{1}{n}\right) + 2^2\left(\frac{1}{n}\right) + \cdots n^2\left(\frac{1}{n}\right)$$

$$= \frac{1}{n}[1^2 + 2^2 + \cdots + n^2]$$

$$= \frac{1}{\cancel{n}}\left[\frac{\cancel{n}(n+1)(2n+1)}{6}\right] = \frac{(n+1)(2n+1)}{6}$$

$V(X) = E(X^2) - (E(X))^2$

$= \cancel{(n+1)n} \; \frac{(n+1)(2n+1)}{6} - \left(\frac{(n+1)}{2}\right)^2$

$= \frac{2(n+1)(2n+1) - 3(n+1)^2}{12}$

$= \frac{4n^2 + \cancel{12n} \overset{6}{} \overset{+2}{} - 3n^2 - 3 - 6n}{12}$

$= \frac{n^2 + \cancel{2n} - 3}{12} \quad \cancel{n^2 + 6n}$

$= \frac{n^2 - 1}{12}$

⑧ Find expected value and variance

x	8	12	16	20	24
$P(x)$	1/8	1/6	3/8	1/4	1/12

$E(x) = \Sigma x_i P_i$

$= 8\left(\frac{1}{8}\right) + 12\left(\frac{1}{6}\right) + 16\left(\frac{3}{8}\right) + 20\left(\frac{1}{4}\right) + 24\left(\frac{1}{12}\right)$

$= 1 + 2 + 6 + 5 + 2$

$= 16$

$E(x^2) = \Sigma x_i^2 P(x_i)$

$= 8^2\left(\frac{1}{8}\right) + (12)^2\left(\frac{1}{6}\right) + (16)^2\left(\frac{3}{8}\right) + (20)^2\left(\frac{1}{4}\right) + (24)^2\left(\frac{1}{12}\right)$

$= 8 + 24 + 96 + 100 + 48$

$= 276$

$V(x) = E(x^2) - (E(x))^2$

$= 276 - (16)^2$

$= 276 - 256$

$= 20$

Theorem: If x, y, are two independent random variables then

$E(xy) = E(x)\,E(y)$

Proof: let X assumes the values $x_1, x_2, \ldots x_n$

let y assumes the values $y_1, y_2, \ldots y_m$

the $E(x) = \sum_{i=1}^{n} x_i P_i$

$E(y) = \sum_{i=1}^{m} y_i P_i$

[the sum $x+y$ is also a random variable which takes $m+n$ values with $x_i + y_j$ $i = 1, 2, 3, \ldots n$ $j = 1, 2, 3 \ldots m$ with probabilities P_{ij}]

The product (xy) is also a independent variable which takes (nm) values with $x_i y_j$ $i = 1, 2, \ldots n$ $j = 1, 2, \ldots m$ with probabilities P_{ij}

$x P(x = x_i \cap y = y_i) \quad P(x = x_i) P(y = y_i)$

$[\because x \& y$ are independent r.v$]$

$\Rightarrow P_{ij} = P_i P_j$

$$E(xy) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j P_{ij}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} x_i y_j P_i P_j$$

$$= \sum_{i=1}^{n} x_i P_i \sum_{j=1}^{m} y_j P_j$$

$$= E(x) \, E(y)$$

$\therefore$ Hence proved $E(xy) = E(x) E(y)$

Theorem:

$$E(x+y) = E(x) + E(y)$$

Proof:

Let x assumes the values of $x_1, x_2, \ldots x_n$

y assumes the values of $y_1, y_2, y_3, \ldots y_m$

Then by definition

$$E(x) = \sum_{i=1}^{n} x_i P_i$$

$$E(y) = \sum_{j=1}^{m} y_j P_j$$

The sum $x+y$ is also a Random variable which takes $n+m)$ values with $(x_i + y_j)$ $i = 1, 2, \ldots n$ $j = 1, 2, 3, \ldots m$ with probabilities P_{ij}

Then by definition

$$E(x+y) = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i + y_j) P_i P_j$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} x_i P_{ij} + \sum_{i=1}^{n} \sum_{j=1}^{m} y_j P_{ij}$$

$$= \sum_{i=1}^{n} x_i \left(\sum_{j=1}^{m} P_{ij}\right) + \sum_{j=1}^{m} y_j \left(\sum_{i=1}^{n} P_{ij}\right)$$

$$= \sum_{i=1}^{n} x_i P_i + \sum_{j=1}^{m} y_j P_j$$

$$= E(x) + E(y)$$

$\therefore$ Hence proved $E(x+y) = E(x) + E(y)$

Note:

(i) $E(ax + by) = aE(x) + bE(y)$

(ii) $E(x - \bar{x}) = 0 \quad (\because E(x) - \bar{x} = \bar{x} - \bar{x} = 0)$

(iii) $E(ax + b) = aE(x) + b$

Results on Variances:

① Variance of a constant is 0

② Variance of $x+k$: $v(x+k)$ is $v(x)$

③ $v(ax+b) = a^2 v(x)$

Proof: let X is a r.v which takes the values $x_1, x_2, \dots x_n$ with probabilities $P_1, P_2, \dots P_n$

let $Y = ax+b$ —— ①

Taking Expectation on both sides

$E(Y) = E(ax+b)$

$E(Y) = aE(x)+b$ —— ②

① − ②

$Y - E(Y) = ax + \cancel{b} - aE(x) - \cancel{b}$

$Y - E(Y) = ax - aE(x)$

Squaring and taking expectation on both side

$E(Y-E(Y))^2 = E\left[a^2(x-E(x))^2\right]$

$v(y) = a^2 E(x-E(x))^2$ (from definition of variance)

$v(y) = a^2 v(x)$

$v(ax+b) = a^2 v(x)$

Note: If x and y are independent random variables then

$v(x \pm y) = v(x) \pm v(y)$

Ex:

① The probability distribution of random variable X is given by

X	−2	−1	0	1	2	3
P(x)	0.2	0.1	0.3	0.3	0.1	

find (i) $E(x)$
(ii) $v(x)$
(iii) $E(2x-3)$
(iv) $V(2x-3)$

Sol (i) $E(x) = \sum_{i=1}^{n} x_i P_i$

$= -2(0.2) + -1(0.1) + 0 + 1(0.3) + 2(0.1)$

$= 0$

(ii) $v(x) = E(x^2) - (E(x))^2$

$E(x^2) = 4(0.2) + 1(0.1) + 0 + 1(0.3) + 4(0.1)$

$= 1.6$

$v(x) = 1.6 - (0)^2 = 1.6$

(iii) $E(2x-3) = 2E(x) - 3$

$= 2(0) - 3 = -3$

(iv) $V(2x-3) = (2)^2 V(x) = 4(1.6) = 6.4$

② If 3 cars are selected from a lot of 6 cars containing 2 defectives find probability distribution of no. of defective cars. Hence find its mean and variances.

Sol Out of 6 cars 4 are non defective and 2 are defecting and we are selecting 3 cars that can be done in ${}^6C_3$ ways

Probability of zero defective cars is

$$P(X=0) = \frac{{}^4C_3}{{}^6C_3} = 0.200$$

Probability of 1 defective car is

$$P(X=1) = \frac{{}^4C_2 \, {}^2C_1}{{}^6C_3} = 0.6$$

Probability of 2 defective cars is

$$P(X=2) = \frac{{}^4C_1 \, {}^2C_2}{{}^6C_3} = 0.2$$

Distribution Table

X	0	1	2
P(x)	0.2	0.6	0.2

Mean $= E(x) = \sum x_i P_i = 0(0.2) + 1(0.6) + 2(0.2)$

$= 1.0$

variance $= V(x) = E(x^2) - (E(x))^2 = 0 + 1(0.6) + 4(0.2) - (1)^2$

$= 0.6 + 0.8 - 1$

$= 1.4 - 1$

$= 0.4$

③ A fair dice is tossed. Let the r.v x denotes the twice the number appearing on the dice. Write probability distribution of x. Hence find it S.D

Sol Let X is the r.v which takes twice the number appears on the dice.

∴ distribution table is

X	2	4	6	8	10	12
P(x)	1/6	1/6	1/6	1/6	1/6	1/6

Mean $E(X) = \sum x_i P_i$

$= 2(1/6) + 4(1/6) + 6(1/6) + 8(1/6) + 10(1/6) + 12(1/6)$

$= 7$

$V(X) = E(x^2) - (E(x))^2$

$= 4(\frac{1}{6}) + 16(1/6) + 36(1/6) + 64(1/6) + 100(\frac{1}{6}) + 144(1/6) - 49$

≈ 11.6667

$S.D = \sqrt{V(x)} = 3.4156$

Continuous random variable:

When a random variable takes every random value in the interval is called continuous random variable & is called continuous distribution of x. The distribution defined by the variates like temp. height and weights are continuous dist

• **Probability density Function**

$f(x)$ is called pdf or density funcⁿ of variable x if it has to satisfy the following 2 properties

(i) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(ii) $f(x) \geq 0$

where $-\infty \leq x \leq \infty$ & $f(x)$ is pdf of x

$P(a \leq x \leq b) = \int_a^b f(x)\,dx$

cummulative distribution function of a continuous random variable.

Cummulative distribution functions of continuous r.v

If x is continuous random variable with prob density funcⁿ $f(x)$ it's cdf is $F(x)$ and

$F(x) = \int_{-\infty}^{\infty} f(x)\,dx$

Properties of F(x)

(i) $0 \leq F(x) \leq 1$ where $-\infty < x < \infty$

(ii) $F(-\infty) = 0$

(iii) $F(\infty) = 1$

(iv) $P(a \leq x \leq b) = \int_a^b f(x)\,dx = F(b) - F(a)$

(v) Since $F(x) = \int_{-\infty}^{x} f(x) \Rightarrow F(x) = \int_{-\infty}^{x} f(x)\,dx$.

$f(x) = \frac{d}{dx} F(x) \Rightarrow f(x) = \frac{d}{dx} F(x)$

Mean, variance of a continuous r.v

If x is a c.r.v then

(1) $\text{Mean} = \mu = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx \qquad f(x) = \sum_{i=1}^{n} x_i P(x_i)$

(2) $\text{Variance} = E(x^2) - (E(x))^2$

$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$V(x) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} x f(x)\,dx\right]^2$

Problems:

(1) The prob density function of a C.R.V $f(x) = \begin{cases} 1/2(x+1) & -1<x< \\ 0 & \text{oth wi} \end{cases}$

find (i) Mean (ii) Variance

Sol: (i) $\text{Mean} = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

$= \int_{-\infty}^{-1} x f(x)\,dx + \int_{-1}^{1} x f(x)\,dx + \int_{1}^{\infty} x f(x)\,dx$

$= \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} x\left(\frac{1}{2}(x+1)\right) + \int_{1}^{\infty} 0\,dx$

$$= 0 + \frac{1}{2}\int_{-1}^{1}(x^2+x)\,dx + 0$$

$$= \frac{1}{2}\left[\left(\frac{x^3}{3}\right)_{-1}^{1} + \left(\frac{x^2}{2}\right)_{-1}^{1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{3} + \frac{1}{3} + \frac{1}{2} - \frac{1}{2}\right]$$

$$= \frac{1}{3}$$

(ii) Variance $V(x) = E(x^2) - (E(x))^2$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{-\infty}^{-1} x^2 f(x)\,dx + \int_{-1}^{1} x^2 f(x)\,dx + \int_{1}^{\infty} x^2 f(x)\,dx$$

$$= \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} x^2\left(\frac{1}{2}(x+1)\right)dx + \int_{1}^{\infty} 0\,dx$$

$$= \frac{1}{2}\int_{-1}^{1}(x^3+x^2)\,dx$$

$$= \frac{1}{2}\left[\left(\frac{x^4}{4}\right)_{-1}^{1} + \left(\frac{x^3}{3}\right)_{-1}^{1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} - \frac{1}{4} + \frac{1}{3} + \frac{1}{3}\right]$$

$$= 1/3$$

$$V(x) = E(x^2) - (E(x))^2$$

$$= \frac{1}{3} - \left(\frac{1}{3}\right)^2$$

$$= \frac{1}{3} - \frac{1}{9}$$

$$= \frac{2}{9}$$

** ② For continuous Probability fun $f(x) = \begin{cases} kx^2e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Find (i) K

(ii) Mean

(iii) Variance

Sol (i) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 1$$

$$0 + \int_{0}^{\infty}(kx^2e^{-x})\,dx = 1$$

$$k\left[\int_{0}^{\infty} x^2e^{-x}dx\right] = 1$$

$$k\left[x^2(-e^{-x}) - 2x(e^{-x}) + 2(-e^{-x})\right]_0^{\infty} = 1$$

$$k[0+2] = 1$$

$$2k = 1$$

$$k = 1/2$$

(ii) Mean $= \int_{-\infty}^{\infty} x f(x)\,dx$

$$= \int_{-\infty}^{0} x f(x)\,dx + \int_{0}^{\infty} x f(x)\,dx$$

$$= 0 + \int_{0}^{\infty} x f(x)\,dx$$

$$= \int_{0}^{\infty} x(kx^2e^{-x})\,dx$$

$$= \frac{1}{2}\int_{0}^{\infty} x^3e^{-x}\,dx$$

$$= \frac{1}{2}\left[x^3(-e^{-x}) - 3x^2(e^{-x}) + 6x(-e^{-x}) - 6(e^{-x})\right]_0^{\infty}$$

$= \frac{1}{2}[0 + \cancel{6}6]$

$= +\cancel{3}/2 \quad 3$

(iii) $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$= \int_{-\infty}^{0} x^2 f(x)\,dx + \int_{0}^{\infty} x^2 f(x)\,dx$

$= 0 + \int_{0}^{\infty} x^2 (Kx^2 e^{-x})\,dx$

$= \frac{1}{2}\int_{0}^{\infty} (x^4 e^{-x})\,dx$

$= \frac{1}{2}\left[-4x^3(e^{-x}) + 12x^2(-e^{-x}) \mp 24x(e^{-x}) + 24(-e^{-x})\right]_0^{\infty}$ ($x^4 - e^{-x}$)

$= \frac{1}{2}[0 + 24]$

$= 12$

$V(x) = E(x^2) - (E(x))^2$

$= 12 - 9$

$= 3$

(3) Pdf of a C.R.V $f(x) = \begin{cases} \frac{1}{16}(3+x)^2 & -3 \le x \le -1 \\ \frac{1}{16}(2-6x^2) & -1 \le x \le 1 \\ \frac{1}{16}(3-x)^2 & 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$

S.T the area under the curve above x axis is unity also find mean

Area under the curve $= \int_{-\infty}^{\infty} f(x)\,dx$

$= \int_{-\infty}^{-3} f(x)\,dx + \int_{-3}^{-1} f(x)\,dx + \int_{-1}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx$

$= 0 + \int_{-3}^{-1} \frac{1}{16}(3+x)^2\,dx + \int_{-1}^{1} \frac{1}{16}(2-6x^2)\,dx + \int_{1}^{3} \frac{1}{16}(3-x)^2\,dx + 0.$

$= \frac{1}{16}\left[\int_{-3}^{-1}(9+x^2+6x)\,dx + \int_{-1}^{1}(2-6x^2)\,dx + \int_{1}^{3}(9+x^2-6x)\,dx\right]$

$= \frac{1}{16}\left[\left[9x+\frac{x^3}{3}+\frac{6x^2}{2}\right]_{-3}^{-1} + \left[2x - \frac{6x^3}{2}\right]_{-1}^{1} + \left[9x+\frac{x^3}{3}-\frac{6x^2}{2}\right]_{1}^{3}\right]$

$= \frac{1}{16}\left[-9-\frac{1}{3}+\frac{6}{2}-\left(-27-\frac{27}{3}+\frac{54}{2}\right) + 2-\frac{6}{2}+2-\frac{6}{2}+27+\frac{27}{3}-\frac{54}{2} - 9-\frac{1}{3}+\frac{6}{2}\right]$

$= \frac{1}{16}\left[\cancel{-9}-\frac{1}{3}+\cancel{3}+\cancel{27}+\cancel{9}-\cancel{27}+2-\cancel{3}+2-\cancel{3}+\cancel{27}+\cancel{9}-\cancel{27}-\cancel{9} -\frac{1}{3}+\cancel{3}\right]$

$= \frac{1}{16}\left[4-\frac{2}{3}\right] = \frac{1}{16}\left[\frac{12-2}{3}\right] = \frac{\cancel{10}^{5}}{3} \times \frac{1}{\cancel{16}_{8}}$

$= 5/24 \neq 1$

$\therefore$ $f(x)$ is not a pdf

(4) The frequency function of a continuous random variable f(x) given by $f(x) = \begin{cases} kx(x-1) & 1 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$

Given that $P(1 \le x \le 3) = \frac{28}{3}$ Find value of k.

We know that

$$P(a \le x \le b) = \int_a^b f(x)\,dx$$

$$\therefore P(1 \le x \le 3) = \int_1^3 f(x)\,dx = \frac{28}{3}$$

$$\Rightarrow \int_1^3 k(x)(x-1)\,dx = \frac{28}{3}$$

$$\Rightarrow k\int_1^3 (x^2 - x)\,dx = \frac{28}{3}$$

$$k\left\{\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^3\right. = \frac{28}{3}$$

$$k\left[\frac{27}{3} - \frac{9}{2} - \frac{1}{3} + \frac{1}{2}\right] = \frac{28}{3}$$

$$k\left[\frac{26}{3} - \frac{8}{2}\right]$$

$$k = \frac{\cancel{28}^{\,2}}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{14}_{\,1}}$$

$$k = 2$$

(5) Show that the function $f(x) = \frac{1}{\pi(1+x^2)}$ $-\infty \le x \le \infty$ is a pdf

Sol
$$\int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)}\,dx$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx.$$

$$= \frac{1}{\pi}\left[\tan^{-1}(x)\right]_{-\infty}^{\infty}$$

$$= \frac{1}{\pi}\left[\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right]$$

$$= \frac{1}{\pi}\left[\pi/2 - \left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{\pi}{\pi} = 1$$

$\therefore$ f(x) is a pdf

(6) C.R.V x whose distibution function given by

$$F(x) = \begin{cases} 0 & \text{for } x \le 3 \\ 1 - \frac{9}{x^2} & \text{for } x > 3 \end{cases}$$

Find (i) Pdf

(ii) Mean

Sol (1) $f(x) = \frac{d}{dx}(F(x))$

$$f(x) = \begin{cases} 0 & x \le 3 \\ 0 - 9\left(\frac{-3}{x^4}\right) & x > 3 \end{cases} = \begin{cases} 0 & \text{for } x \le 3 \\ \frac{27}{x^4} & x > 3 \end{cases}$$

(ii) Mean $= E(x) = \int_3^\infty x\left(\frac{27}{x^4}\right)dx$

$= 27\int_3^\infty \frac{1}{x^3}dx$

$= 27\left[-\frac{1}{2x^2}\right]_3^\infty$

$= \frac{27}{2}\left[0+\frac{1}{9}\right]$

$= \frac{3}{2}$

(7) The cdf of a CRV X is $F(x) = \begin{cases} 1-e^{-2x} & x \geq 0 \\ 0 & x<0 \end{cases}$

find i) $f(x)$

ii) Mean

iii) Variance

Sol (i) $F(x) = \begin{cases} 1-e^{-2x} & x \geq 0 \\ 0 & x<0 \end{cases}$

$f(x) = \frac{d}{dx} F(x)$

$= \begin{cases} 0 - e^{-2x}(-2) & x \geq 0 \\ 0 & x<0 \end{cases}$

$= \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & x<0 \end{cases}$

(ii) Mean $= \int_{-\infty}^{0} x f(x)dx + \int_0^\infty x f(x)dx$

$= 0 + \int_0^\infty 2xe^{-2x}dx$

$= \cancel{2\left[\frac{e^{-2x}}{-2}\right]_0^\infty}$ $2\left[x\left(\frac{e^{-2x}}{-2}\right) - \frac{e^{-2x}}{-2(-2)}\right]_0^\infty$

$= 2\left[\cancel{\frac{0-1}{-2}}\right] = \left[0+\frac{1}{2}\right]$

$= \cancel{1}$ $= +1/2$

(iii) Variance $V(x) = E(x^2) - (E(x))^2$

$E(x^2) = \int_0^\infty x^2(f(x))dx$

$= \int_0^\infty 2x^2e^{-2x}dx.$

$= 2\left[x^2\left(\frac{-e^{-2x}}{-2}\right) - 2x\left(\frac{e^{-2x}}{-2(-2)}\right) + 2\left(\frac{-e^{-2x}}{-2(-2)(-2)}\right)\right]_0^\infty$

$= \left[0+1/2\right]$

$= 1/2$

$V(x) = E(x^2) - (E(x))^2$

$= 1/2 - (1/4)$

$= 1/4$

(8) If a CRV has following pdf find distribution

fu (i) $F(x)$ and (ii) $P(1 \leq x \leq 2)$

$$f(x) = \begin{cases} kx^2 & \text{in } 0 \leq x < 3 \\ 0 & \text{otherwise} \end{cases}$$

(i) $F(x) = \text{cdf} = \int_{-\infty}^{x} f(x)\,dx$

$$F(x) = \int_{-\infty}^{x} kx^2\,dx$$

to find k

$$\int_{-\infty}^{\infty} f(x)\,dx \Rightarrow$$

$$\left(\int_{-\infty}^{0} + \int_{0}^{3} + \int_{3}^{\infty}\right) f(x)\,dx = 1$$

$$0 + \int_{0}^{3} kx^2\,dx + 0 = 1$$

$$k\left(\frac{x^3}{3}\right)_0^3 = 1$$

$$\frac{k}{3}(27-0) = 1$$

$$k = 1/9$$

$$F(x) = \int_{-\infty}^{x} kx^2\,dx$$

$$= \int_{\infty}^{0} 0\,dx + \int_{0}^{x} kx^2\,dx$$

$$= 0 + k\left(\frac{x^3}{3}\right)_0^x$$

$$= \frac{1}{27}(x^3 - 0) = \frac{x^3}{27}$$

(ii) $P(1 < x \leq 2) = P(x = 2)$

$$= \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \left(\int_{-\infty}^{1} + \int_{1}^{2} + \int_{2}^{\infty}\right) f(x)\,dx$$

$$= 0 + \int_{1}^{2} kx^2\,dx + 0$$

$$= \frac{1}{9}\left(\frac{x^3}{3}\right)_1^2$$

$$= \frac{1}{27}(8-1) = 7/27$$

(9) $$f(x) = \begin{cases} \frac{1}{4} e^{-x/4} & \text{for } x > 0 \ (0, \infty) \\ 0 & \text{otherwise} \end{cases}$$

find mean and variance

Sol

Mean $= E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

$$= \left[\left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right) x f(x)\,dx\right]$$

$$= 0 + \int_{0}^{\infty} x \frac{1}{4}(e^{-x/4})\,dx$$

$$= \frac{1}{4}\int_{0}^{\infty} x \cdot e^{-x/4}\,dx$$

$$= \frac{1}{4}\left[x\left(\frac{e^{-x/4}}{-1/4}\right) - \frac{e^{-x/4}}{-1/4(-1/4)}\right]_0^{\infty}$$

$$= \frac{1}{4}\left[0 + \frac{1}{1/16}\right] = \frac{16}{4} = 4$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \left[\left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right) x^2 f(x)\,dx\right]$$

$$= \int_{-\infty}^{0} x^2 f(x)\,dx + \int_{0}^{\infty} x^2 f(x)\,dx$$

$$= 0 + \int_{0}^{\infty} x^2 \frac{1}{4} e^{-x/4}\,dx$$

$$= \frac{1}{4}\left[x^2 \frac{e^{-x/4}}{-1/4} - 2x \cdot \frac{e^{-x/4}}{-1/4 \times -1/4} + 2\frac{e^{-x/4}}{-\frac{1}{4}\times-\frac{1}{4}\times-\frac{1}{4}}\right]_0^{\infty}$$

$$= \frac{1}{4}\left[0 + \frac{2(1)}{\frac{1}{4^3}}\right]$$

$$= 2(16) = 32$$

$$V(x) = E(x^2) - (E(x))^2 = 32 - 4^2 = 16$$

(10) X is a CRV whose density fun is

$$f(x) = \begin{cases} x & 0<x<1 \\ 2-x & 1\le x<2 \\ 0 & \text{otherwise} \end{cases}$$

find (i) $E(25x^2 + 30x - 5)$

Sol $E(25x^2+30x-5)$

$\Rightarrow 25E(\cancel{25}x^2) + 30E(x) - 5$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \left[\int_{-\infty}^{0} + \int_{0}^{1} + \int_{1}^{2} + \int_{2}^{\infty}\right] x^2 f(x)\,dx$$

$$= \int_{-\infty}^{0} x^2 f(x)\,dx + \int_{0}^{1} x^2 f(x)\,dx + \int_{1}^{2} x^2 f(x)\,dx + \int_{2}^{\infty} x^2 f(x)\,dx$$

$$= 0 + \int_{0}^{1} x^2(x)\,dx + \int_{1}^{2} x^2(2-x)\,dx + 0$$

$$= \left[\frac{x^4}{4}\right]_0^1 + 2\left[\frac{x^3}{3}\right]_1^2 - \left[\frac{x^4}{4}\right]_1^2$$

$$= \frac{1}{4} + 2\left[\frac{8}{3} - \frac{1}{3}\right] - \left(\frac{16}{4} - \frac{1}{4}\right)$$

$$= \frac{1}{4} + \frac{14}{3} - \frac{15}{4}$$

$$= \frac{14}{3} - \frac{14}{4} = 7/6$$

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \left[\left(\int_{-\infty}^{0} + \int_{0}^{1} + \int_{1}^{2} + \int_{2}^{\infty}\right) x f(x)\,dx\right]$$

$$= 0 + \int_{0}^{1} x x\,dx + \int_{1}^{2} x(2-x)\,dx + 0$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \cancel{2}\left[\frac{x^2}{\cancel{2}}\right]_1^2 - \left[\frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{3} + [4-1] - \left(\frac{8}{3} - \frac{1}{3}\right)$$

$$= 3 - 6/3 = 1$$

$\Rightarrow 25\,E(x^2) + 30\,E(x) - 5$

$= 54.1667$

Cdf (Discrete R.V):

(11) Probability mass function of a Discrete R.V

$f(x) = \frac{1}{4}$, $x = 0, 1, 2, 3$

Find $F(x)$ and sketch its graph.

Sol: Given $f(x) = \frac{1}{4}$ $\quad x = 0, 1, 2, 3$

(i) $F(x) = 0 \quad x<0 \; (-\infty, 0)$

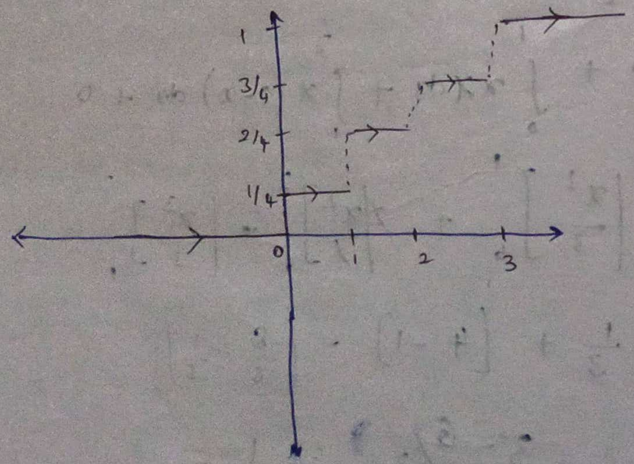
$F(x) = 1/4 \quad 0 \le x < 1 \quad [0,1)$

$F(x) = 1/4 + 1/4 = 2/4 \quad 1 \le x < 2 \quad [1,2)$

$F(x) = 1/4 + 1/4 + 1/4 = 3/4 \quad 2 \le x < 3 \quad [2,3)$

$F(x) = 1/4 + 1/4 + 1/4 + 1/4 = 1 \quad x \ge 3 \quad [3, \infty)$

(ii)



(12) A D.R.V x has the following distribution function.

$$F(x) = \begin{cases} 0 & \text{for } x<1 \\ 1/3 & \text{for } 1 \le x < 4 \\ 1/2 & \text{for } 4 \le x < 6 \\ 5/6 & \text{for } 6 \le x < 10 \\ 1 & \text{for } x \ge 10 \end{cases}$$

Find (i) $P(2 \le x \le 6)$ (ii) $P(x=5)$

(iii) $P(x \le 6)$ (iv) $P(x=6)$

(v) Draw the graph of $F(x)$

Sol (i) $P(2 < x \le 6) = F(6) - F(2)$

$= \frac{5}{6} - \frac{1}{3} = \frac{3}{6} = \frac{1}{2} = 0.5$

~~(ii) $P(x=5) = F(5) = 1/2 = 0.5$~~

~~(iii) $P(x<6) = F($~~

(ii) $P(x=5) = P(x \le 5) - P(x<5)$

$= F(5) - P(x<5)$

$= 1/2 - \frac{1}{2} = 0$

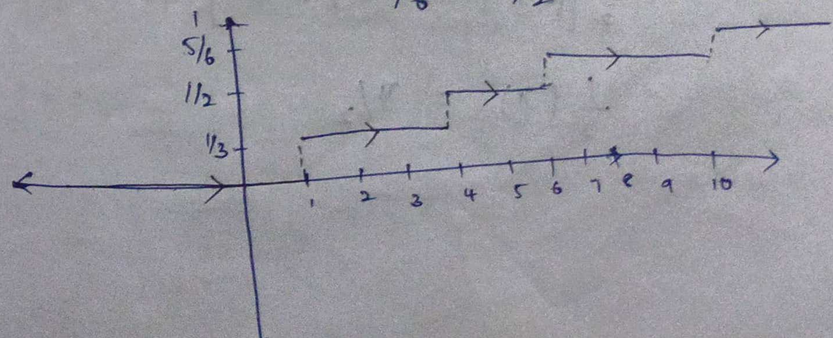
(iii) $P(x \le 6) = F(6) = 5/6$

(iv) $P(x=6) = P(x \le 6) - P(x<6)$

$= F(6) - P(x<5)$

$= 5/6 - 1/2 = 2/6 = 1/3 = 0.33$

(v)



(13) PDF of a R.V X is $f(x) = \frac{1}{2}\sin x$ for $0 \le x \le \pi$, 0 else where

$$f(x) = \begin{cases} \frac{1}{2}\sin x & 0 \le x \le \pi \\ 0 & \text{otherwise} \end{cases}$$

Find
(i) Mean
(ii) Mode

Sol (i) Mean $= E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

$$= \left[\left(\int_{-\infty}^{0} x + \int_{0}^{\pi} + \int_{\pi}^{\infty}\right) x f(x)\,dx\right]$$

$$= 0 + \int_0^{\pi} x\left(\frac{1}{2}\sin x\right) dx + 0$$

$$= \frac{1}{2}\int_0^{\pi} (x \sin x)\,dx$$

$$= \frac{1}{2}\left[x(-\cos x) - (-\sin x)\right]_0^{\pi}$$

$$= \frac{1}{2}\left[-x\cos x + \sin x\right]_0^{\pi}$$

$$= \frac{1}{2}\left[(-\pi(-1) + 0) \mp 0\right]$$

$$= \frac{1}{2}(\pi) = \pi/2$$

(ii) Mode (Mode is value of x for which $f(x)$ is maximum)

To find maximum value $f'(x) = \frac{1}{2}\cos x = 0$

$\cos x = 0$

$x = \pi/2, \frac{3\pi}{2}, \dots$

$$f''(x) = -\frac{\sin x}{2}$$

$$f''(\pi/2) = \frac{-\sin \pi/2}{2} = \frac{-1}{2} < 0$$

$\therefore$ $f(x)$ is maximum at $x = \pi/2$

$\therefore$ Mode of distribution is $\pi/2$

(14) Let x be a continuous random variable with pdf

$$f(x) = \begin{cases} 1/8 & 0 \le x \le 8 \\ 0 & \text{else where} \end{cases}$$

Find probability of (i) $P(2 \le x \le 5)$
(ii) $P(3 \le x \le 7)$
(iii) $P(x \le 6)$
(iv) $F(x)$

(i) $P(2 \le x \le 5) = \int_2^5 f(x)\,dx = \int_2^5 1/8\,dx = \frac{1}{8}[x]_2^5 = \frac{[5-2]}{8}$

$= 3/8$

(ii) $P(3 \le x \le 7) = \int_3^7 f(x)\,dx = \int_3^7 \frac{1}{8}\,dx = \frac{1}{8}(7-3) = \frac{1}{2}$

(iii) $P(x \le 6) = \int_{-\infty}^{6} f(x)\,dx = \left[\left(\int_{-\infty}^{0} + \int_0^6\right) f(x)\,dx\right] = 0 + \int_0^6 f(x)\,dx$

$= \frac{1}{8}[x]_0^6 = 6/8 = 3/4$

(iv) $F(x) = \int_{-\infty}^{x} f(x)\,dx$

$= \left[\left(\int_{-\infty}^{0} + \int_{0}^{8} + \int_{8}^{\infty}\right) f(x)\,dx\right]$

$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{8} f(x)\,dx + \int_{8}^{\infty} f(x)\,dx$

$= 0 + \int_{0}^{8} (1/8)\, f(x)\,dx$

$\frac{1}{8}\int_{-\infty}^{x} dx = \frac{1}{8}[x]_{-\infty}^{x}$

$= \frac{1}{8}[8 - 0]$

$F(x) = \begin{cases} x/8 & 0 \le x \le 8 \\ 0 & \text{otherwise} \end{cases}$

Multiple Random Variable:

Introduction

In multiple r.v techniques we use various r.v occuring at a time. But in many practical problems several r.v's interact with each other. So the idea of interacting more than one r.v is extended to study of Multiple R.Vs.

Definition:

Consider 2 r.v's X and Y or a 2 dimension r.v (X, Y) the outcome of a trial for the pair $X = x, Y = y$ ie (x, y) is called M.R.V

We call (X, Y) and it's distribution is Discrete if (X, Y) can assume only finite numbers of pairs.

Eg: Consider, the experiment of tossing 2 fair dice the sample space for this exp. has 36 equally likely points. Let X = Sum on 2 dice & Y = difference of 2 dice. Thus we can define bivarient r.v (X, Y) where X = sum on 2 dices, Y = difference of 2 dice

Joint Probability Function: JPF

Let (X, Y) be a discrete r.v on a sample space 'S' then probability function $f(x,y) = P(X = x, Y = y)$ is called J. P. mass F or Joint PMF

Joint Probability Distribution:

The joint probability distribution of 2 r.v (X, Y) is usually represented with the following table

X \ Y	y_1	y_2	y_3 --- y_j ---	y_m	total
x_1	P_{11}	P_{12}	--- P_{1j} ---	P_{1m}	P_1
x_2	P_{21}	P_{22}	--- P_{2j} ---	P_{2m}	P_2
⋮					⋮
x_i	P_{i1}	P_{i2}	--- P_{ij} ---	P_{im}	P_i
⋮					⋮
x_n	P_{n1}	P_{n2}	--- P_{nj} ---	P_{nm}	P_n
Total	[illegible]	[illegible]	P_j ---	P_m	1

$$\therefore \sum_{i=1}^{n} \sum_{j=1}^{m} P(x_i, y_j) = 1$$

Marginal Probability Function

Suppose joint distributions of 2 random variables x and y is given then the probability distribution of x is determine by

$$P(X = x_1) = P_X(x_i) = P(X=x_i \cap Y=y_j)$$

$$= P(X=x_i \cap Y=y_1) + P(X=x_i \cap Y=y_2) + \dots + P(X=x_i \cap Y=y_m)$$

$$= P_{i.}$$

$$= \sum_{j=1}^{m} P_{ij}$$

$$= \sum_{j=1}^{m} P(x_i, y_j)$$

$||^{ly}$

$$P(Y=y_j) = \cancel{P}\, P_Y(y_j)$$

$$= P(X=x_1 \cap Y=y_j) + P(X=x_2 \cap Y=y_j) + \dots + P(X=x_n \cap Y=y_j)$$

$$= P_{.j}$$

$$= \sum_{i=1}^{n} P_{ij}$$

$$= \sum_{i=1}^{n} P(x_i, y_j)$$

Here $P_{i.}$ and $P_{.j}$ are the marginal probability functions of x and y.

Conditional Probability Function:

X Given Y

$$P(X=x_i / Y=y_j) = \frac{P(X=x_i \cap Y=y_j)}{P(Y=y_j)} = \frac{P(x_i, y_j)}{P(y_j)}$$

Y Given X

$$P(Y=y_j / X=x_i) = \frac{P(X=x_i \cap Y=y_j)}{P(X=x_i)} = \frac{P(x_i, y_j)}{P(x_i)}$$

* Two r.v's X and Y are said to be independent if $P(X=x_i \cap Y=y_j) = P(X=x_i)P(Y=y_j)$

Joint Probability Distribution Function (CDF):

Let (X,Y) be a 2D r.v then their joint distribution is denoted by $F_{xy}(x,y)$. And it represents the probability that simultaneously the observation (X,Y) will have the property

(i) For D.R.V ~~Pt~~

$$F_{xy}(x,y) = P(X \le x, Y \le y) = P(-\infty \le X \le x, -\infty \le Y \le y)$$

(ii) For C.R.V

$$F_{xy}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y)\, dy\, dx$$

where $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$

Properties :

(i) $F(\infty,\infty) = 1$

(ii) $F(-\infty,-\infty) = 0$

$\Bigg\}\ -\infty \le x \le \infty\ ,\ -\infty \le y \le \infty$

(iii) $F_{xy}(-\infty, y) = F_{xy}(x, -\infty) = 0$

(iv) $0 \le F_{xy}(x,y) \le 1$

(v) $P(x_1 \le x \le x_2,\ y \le y_1) = F(x_2, y_1) - F(x_1, y_1)$

(vi) $F_x(x) = F_{xy}(x,\infty) = P(X \le x,\ Y \le \infty) = P(X \le x)$

(vii) $F_y(y) = F_{xy}(\infty, y) = P(X \le \infty, Y \le y)$

$= P(Y \le y)$

(viii) If density function is given $f(x,y)$ is continuous then $f(x,y) = \dfrac{\partial^2 F}{\partial x \partial y}$