(4) A Joint Probability Mass Function of x and y is given by $P(x,y) = k(2x+3y)$, $x = 0,1,2$ and $y = 1,2,3$

Find (i) k

(ii) Marginal Probability Function of x and y

(iii) Conditional Prob of x given $y=1$

(iv) Conditional Prob of y given $x=2$

(v) Probability distribution of $x+y$.

(i)

X \ Y	1	2	3	Total
0	3K	6K	9K	18K
1	5K	8K	11K	24K
2	7K	10K	13K	30K
Total	15K	24K	33K	72K

$72K = 1$

$K = 1/72$

(ii) Marginal Probability of x

x	P_i
0	18/72
1	24/72
2	30/72

Marginal Probability of y

y	P_j
1	15/72
2	24/72
3	33/72

(iii) Conditional Probability of x given $y=1$

$$P\left(\frac{X=x_i}{y=1}\right) = \frac{P(X=x_i \cap Y=1)}{P(Y=1)} = \frac{P(X=0 \cap Y=1)}{P(Y=1)} + \frac{P(X=1 \cap Y=1)}{P(Y=1)} + \frac{P(X=2 \cap Y=1)}{P(Y=1)}$$

$$= \frac{3K}{15K} + \frac{5K}{15K} + \frac{7K}{15K}$$

$$= \cancel{K+1+1} = \cancel{3}\ \frac{15}{15} = 1$$

(iv) Conditional Probability of y given $x=2$

$$P(Y=y_i / X=2) = \frac{P(X=2 \cap Y=1)}{P(X=2)} + \frac{P(X=2 \cap Y=2)}{P(X=2)} + \frac{P(X=2 \cap Y=3)}{P(X=2)}$$

$$= \frac{7K}{30K} + \frac{10K}{30K} + \frac{13K}{30K} = \frac{30K}{30K} = 1$$

(v) Probability distribution of $x+y$

Let $z = x+y$

$X = 0,1,2 \quad Y = 1,2,3$

$Z = 1,2,3,4,5$

~~Z = x+y~~	1	2	3	4	5

P(Z)

$Z = x+y$	$P(Z)$
$Z=1$	$P_{01} = 3/72$
$Z=2$	$P_{02}+P_{11} = 6K+5K = 11K = 11/72$
$Z=3$	$P_{03}+P_{12}+P_{21} = 9K+8K+7K = 25K = 25/72$
$Z=4$	$P_{13}+P_{22} = 11K+10K = 21K = 21/72$
$Z=5$	$P_{23} = 13K = 13/72$

(5) Verify that $f(x,y) = \begin{cases} 2/5\,(2x+3y) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

is a joint ~~prob~~ density function. If so find Probability $(X,Y) \in A$ where A is the region $\{(x,y) / 0<x<1/2,\ 1/4<y<1/2\}$

$$f(x,y) = \begin{cases} 2/5\,(2x+3y) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

To test whether it is a pdf we know that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$$

$$\Rightarrow \left[\int_{-\infty}^{0}\int_{-\infty}^{0} + \int_{0}^{1}\int_{0}^{1} + \int_{1}^{\infty}\int_{1}^{\infty}\right] f(x,y)\cdot dy\,dx$$

$$\Rightarrow \int_{-\infty}^{0}\int_{-\infty}^{0} f(x,y)\,dy\,dx + \int_0^1\int_0^1 f(x,y)\,dy\,dx + \int_1^{\infty}\int_1^{\infty} f(x,y)\,dy\,dx$$

$$\Rightarrow 0 + \int_{x=0}^{1}\int_{y=0}^{1} \frac{2}{5}(2x+3y)\,dy\,dx + 0$$

$$\Rightarrow \frac{2}{5}\int_{x=0}^{1}\left[2x(y)_0^1 + \left(\frac{3y^2}{2}\right)_0^1\right]dx$$

$$\Rightarrow \frac{2}{5}\int_{x=0}^{1}\left(2x(1-0) + \frac{3}{2}(1-0)\right)dx$$

$$\Rightarrow \frac{2}{5}\int_{x=0}^{1}\left(2x + \frac{3}{2}\right)dx$$

$$\Rightarrow \frac{2}{5}\left[2\left(\frac{x^2}{2}\right)_0^1 + \frac{3}{2}(x)_0^1\right]$$

$$\frac{2}{5}\left(\cancel{2}\left(\frac{1}{\cancel{2}}-0\right) + \frac{3}{2}(1-0)\right]$$

$$\frac{2}{5}\left(1+\frac{3}{2}\right) = \frac{2}{5}\left(\frac{5}{2}\right) = 1$$

$\therefore$ Given ~~pd~~ Joint probability function is a pdf.

$$P((x,y) \in A) = P(0 < x < 1/2, \; 1/4 < y < 1/2)$$

$$= \int_{x=0}^{1/2} \int_{y=1/4}^{1/2} \frac{2}{5}(2x+3y)\, dy \cdot dx.$$

$$= \frac{2}{5} \int_{x=0}^{1/2} \int_{y=1/4}^{1/2} (2x+3y)\, dy \cdot dx$$

$$= \frac{2}{5} \int_{x=0}^{1/2} \left(2x(y)_{1/4}^{1/2} + 3\left(\frac{y^2}{2}\right)_{1/4}^{1/2} \right) dx$$

$$= \frac{2}{5} \int_{x=0}^{1/2} \left(2x\left(1/2 - 1/4\right) + \frac{3}{2}\left(\frac{1}{4} - \frac{1}{16}\right) \right) dx$$

$$= \frac{2}{5} \int_{x=0}^{1/2} \left(2x\left(1/4\right) + \frac{3}{2}\left(\frac{3}{16}\right) \right) dx$$

$$= \frac{2}{5} \int_{x=0}^{1/2} \left(\frac{x}{2} + \frac{9}{32} \right) dx$$

$$= \frac{2}{5} \left[\frac{1}{2}\left[\frac{x^2}{2}\right]_0^{1/2} + \frac{9}{32}(x)_0^{1/2} \right]$$

$$= \frac{2}{5} \left[\frac{1}{4}\left[1/4 - 0\right] + \frac{9}{32}\left(1/2 - 0\right) \right]$$

$$= \frac{2}{5} \left[\frac{1}{16} + \frac{9}{64} \right]$$

$$= \frac{13}{160} = 0.08125$$

Marginal Pdf of Continuous R.V (X,Y):

If $F(x,y)$ is joint probability density function, then marginal probability function of 'x', $f_x(x)$ & is denoted by

it is given by $\int_y f(x,y)\, dy$.

Joint marginal probability density function of 'y' is $f_y(y)$ & is given by $\int_x f(x,y)\, dx$.

Joint Conditional Probability

$$f(x/y) = \frac{f(x,y)}{f_y(y)}$$

$$f(y/x) = \frac{f(x,y)}{f_x(x)}$$

Statistical Independence (strocastic)

Let X and Y be two random variables discrete or continuous with joint pdf $f(x,y)$ and marginal pdf $f_x(x)$ and $f_y(y)$ respectively. The R.V X and Y are said to be statistically independent if and only if

$$f_{xy}(x,y) = f_x(x)\, f_y(y)$$

Problem

(1) Given Joint Density Function $f(x,y) = \begin{cases} \frac{x(1+3y^2)}{4} & 0<x<2,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

(i) Find Marginal pdf of x and y
(ii) find $f(x/y)$
(iii) Verify whether x, y are statistically independent or not

Sol

(i) Marginal Pdf of x is $f_x(x) = \int_y f(x,y)\,dy$

$$= \int_0^1 \frac{x(1+3y^2)}{4}$$

$$= \frac{x}{4}\int_0^1 (1+3y^2)\,dy$$

$$= \frac{x}{4}\left[y + \cancel{3}\frac{y^3}{\cancel{3}}\right]_0^1$$

$$= \frac{x}{4}\left[1+1-0+0\right]$$

$$= \frac{x}{2}$$

Marginal pdf of Y is $f_y(y) = \int_x f(x,y)\,dx$

$$= \int_0^2 \frac{x(1+3y^2)}{4}\,dx$$

$$= \frac{1+3y^2}{4}\int_0^2 x\,dx$$

$$= \frac{(1+3y^2)}{4}\left[\frac{x^2}{2}\right]_0^2$$

$$= \frac{(1+3y^2)}{4}\left[\frac{4}{2} - 0\right]$$

$$= (1+3y^2)/2$$

(ii) $f(x/y) = \dfrac{f(x,y)}{f_y(y)}$

$$= \frac{\frac{x(1+3y^2)}{4}}{\left(\frac{1+3y^2}{2}\right)}$$

$$= \frac{x\cancel{(1+3y^2)}}{\cancel{4}_2} \times \frac{2}{\cancel{(1+3y^2)}}$$

$$= x/2$$

(iii) $f(x,y) = \dfrac{x(1+3y^2)}{4}$

$f_x(x) = x/2$

$f_y(y) = \dfrac{(1+3y^2)}{2}$

$f_{(x)}(x) f_y(y) = \dfrac{x(1+3y^2)}{4} = f(x,y)$

∴ x, y are statistically independent.

(2) Joint Pdf of X and Y is given by

$$f(x,y) = \begin{cases} 8xy & 0<x<y<1 \\ 0 & \text{otherwise} \end{cases}$$

find (i) Marginal Probability
(ii) Conditional distribution
(iii) Are X and Y are independent

Sol (i) Marginal Probability of X is

$$f_x(x) = \int_y f(x,y)\,dy$$

$$= \int_0^1 8xy\,dy$$

$$= \cancel{8}^{4}x\left[\frac{y^2}{\cancel{2}}\right]_0^1$$

$$= 4x(1-0)$$

$$= 4x$$

Marginal Probability of Y is

$$f_y(y) = \int_x f(x,y)\,dx$$

$$= \int_0^1 8xy\,dx$$

$$= \cancel{8}^{4}y\left[x^2/2\right]_0^1$$

$$= 4y(1-0]$$

$$= 4y$$

(ii) Condition Probability of x given y is

$$f(x/y) = \frac{f(x,y)}{f_y(y)}$$

$$= \frac{\cancel{8}^{2}x\cancel{y}}{\cancel{4y}}$$

$$= 2x$$

Conditional Probability of Y given x is

$$f(y/x) = \frac{f(x,y)}{f_x(x)}$$

$$= \frac{\cancel{8}^{2}\cancel{x}y}{\cancel{4x}}$$

$$= 2y.$$

(iii) $f(x,y) = 8xy$

$$= 4x \cdot 2y$$

$$= f_x(x)\,2y$$

$f(x,y) \neq f_x(x)\,f_y(y)$

$\therefore$ X, Y are not statistically independent.

(3) R.Vs X and Y have joint distribution $f(x,y)$

$$f(x,y) = \begin{cases} 6(1-x-y) & x>0\ \ y>0,\ x+y< \\ 0 & \text{otherwise} \end{cases}$$

Find (i) Marginal Probabilities

(ii) Examine x, y are independent

(i) Marginal Probability of X $f_x(x) = \int_y f(x,y)$

$$= \int_0^{1-x} 6(1-x-y)\,dy.$$

$$= 6\int_0^{1-x} (1-x-y)\,dy$$

$$= 6\left[y - xy - \frac{y^2}{2}\right]_0^{1-x}$$

$$= 6\left[(1-x) - x(1-x) - \frac{(1-x)^2}{2} - 0\right]$$

$= 6\left[1-x-x+x^2-\frac{1}{2}(x^2+1-2x)\right]$

$= 6\left[(x^2-2x+1)-\frac{1}{2}(x^2+1-2x)\right]$

$= 3(x^2-2x+1)$

$= 3(x-1)^2 \quad = 3(1-x)^2$

Marginal Probability of y $f_y(y) = \int_x f(x,y)\,dx$

$= \int_0^{1-y} 6(1-x-y)\,dx$

$= 6\int_0^{1-y} (1-x-y)\,dx$

$= 6\left[x-\frac{x^2}{2}-xy\right]_0^{1-y}$

$= 6\left[(1-y)-\frac{(1-y)^2}{2}-(1-y)y\right]$

$= 6(1-y)\left[1-\frac{(1-y)}{2}-y\right]$

$= \frac{6(1-y)^2}{2} = 3(1-y)^2$

(11)

$f_x(x) = 3(1-x)^2$

$f_y(y) = 3(1-y)^2$

$f_x(x)f_y(y) = 3(1-x)^2 3(1-y)^2$

$= 9(1-x)^2(1-y)^2$

$f(x,y) = 6[1-x-y]$

$f(x,y) \neq f_x(x)f_y(y)$

$\therefore$ x,y are not statisticaly independent.

(4) The RVs x,y have joint pdf

$f(x,y) = \frac{1}{2x^2y} \quad 1 \leq x < \infty, \ \frac{1}{x} < y < x$

derive marginal distributions of x and y. also find conditional probabilities y for x and x for y

Sol

Marginal pdf of x

$f_x(x) = \int_y f(x,y)\,dy$

$= \int_{1/x}^{x} \frac{1}{2x^2y}\,dy = \frac{1}{2x^2}\int_{1/x}^{x} \frac{1}{y}\,dy$

$= \frac{1}{2x^2}(\log y)_{1/x}^{x}$

$= \frac{1}{2x^2}\left[\log x - \log 1/x\right]$

$= \frac{1}{2}x^2 \log x/_{1/x} - \frac{1}{2x^2}\log x^2$

$$= \frac{1}{\cancel{2}x^2}\, x\log x$$

$$= \frac{\log x}{x^2}$$

Marginal probability of y

$$f_y(y) = \int_x f(x,y)\,dx$$

$$= \int_1^\infty \frac{1}{2x^2 y}\,dx$$

$$= \frac{1}{2y}\int_1^\infty \frac{1}{x^2}\,dx$$

$$= \frac{1}{2y}\left[\frac{x^{-1}}{-1}\right]_1^\infty$$

$$= \frac{-1}{2y}\left[\frac{1}{x}\right]_1^\infty$$

$$= \frac{-1}{2y}\,(0-1]$$

$$= \frac{1}{2y}$$

Conditional distribution of x for y

$$f(x/y) = \frac{f(x,y)}{f_y(y)} = \frac{\frac{1}{\cancel{2}x^2\cancel{y}}}{\frac{1}{\cancel{2}\cancel{y}}}$$

$$= \frac{1}{x^2}$$

$$f(y/x) = \frac{f(x,y)}{f_x(x)}$$

$$= \frac{\frac{1}{2x^{\cancel{2}}y}}{\frac{\log x}{\cancel{x^2}}}$$

$$= \frac{1}{2y\log x}$$

⑤ Find K so that $f(x,y) = Kxy$ $1 \le x \le y \le 2$ will be the pdf.

Sol $f(x,y) = kxy$ $1 \le x \le y \le 2$

W.K.T

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$$

$$\left[\left(\int_{-\infty}^{1}\int_{-\infty}^{1} + \int_1^2\int_1^2 + \int_2^\infty\int_2^\infty\right) f(x,y)\,dy\,dx\right] = 1$$

$$0 + \int_{x=1}^{2}\int_{y=1}^{2} (kxy)\,dy\,dx + 0 = 1$$

$$\int_{x=1}^{2} kx\left(\frac{y^2}{2}\right)_1^2 dx = \cancel{k}\,1$$

$$\int_1^2 kx\left[\frac{4}{2} - \frac{1}{2}\right] dx = 1$$

$$\int_1^2 \frac{3k}{2}\,x\,dx = 1 \Rightarrow \frac{3k}{2}\left[\frac{x^2}{2}\right]_1^2 = 1 = ?$$

$$\frac{3}{2}k\left[\frac{4}{2}-\frac{1}{2}\right]=1$$

$$\frac{3}{2}\times\frac{3k}{2}=1$$

$$k=\frac{4}{9}$$

⑥ If the joint cdf of x and y is given by

$$F(x,y)=\begin{cases}(1-e^{-x})(1-e^{-y}) & \text{for } x>0,\ y>0\\ 0 & \text{otherwise}\end{cases}$$

Find (i) JPdf

(ii) $P(1\le x<3,\ 1<y<2)$

Sol Joint P df of $f(x,y)=\frac{\partial}{\partial x}\frac{\partial}{\partial y}(1-e^{-x})(1-e^{-y})$

$$=\frac{\partial}{\partial x}(1-e^{-x})\frac{\partial}{\partial y}(1-e^{-y})$$

$$=e^{-x}e^{-y}$$

$$f(x,y)=\begin{cases}e^{-(x+y)} & x>0,\ y>0\\ 0 & \text{otherwise}\end{cases}$$

$$P(1<x<3,\ 1<y<2)=\int_1^3\int_1^2 f(x,y)\,dy\,dx$$

$$=\int_1^3\int_1^2 (1-e^{-x})(1-e^{-y})\,dy\,dx$$

$$=\int_1^3\int_1^2 (1-e^{-x})(y+e^{-y})_1^2\,dx$$

$$=\int_1^3\int_1^2 e^{-x}e^{-y}\,dy\,dx$$

$$=\int_1^3 e^{-x}\left[-e^{-y}\right]_1^2\,dx$$

$$=\int_1^3 e^{-x}\left[-e^{-2}+e^{-1}\right]dx$$

$$=\left[-e^{-x}\right]_1^3\left[e^{-1}-e^{-2}\right]$$

$$=\left[-e^{-3}+e^{-1}\right]\left[e^{-1}-e^{-2}\right]$$

$$=\left[e^{-1}-e^{-3}\right]\left[e^{-1}-e^{-2}\right]$$

$$=\left[\frac{1}{e}-\frac{1}{e^3}\right]\left[\frac{1}{e}-\frac{1}{e^2}\right]$$

$$=0.0739$$

⑦ If x and y are 2D.R.Vs. having the joint pdf $f(x,y)=\frac{1}{27}(2x+y)$ where x and y can assume the values of $x=0,1,2$, $y=0,1,2$. Find Marginal distributions & Conditional distribution of Y for x

Sol

$X \backslash Y$	0	1	2	total
0	0	$1/27$	$2/27$	$3/27$
1	$2/27$	$3/27$	$4/27$	$9/27$
2	$4/27$	$5/27$	$6/27$	$15/27$
Total	$6/27$	$9/27$	$12/27$	1

Marginal Prob function of x

X	P_i
0	$3/27$
1	$9/27$
2	$15/27$

Marginal Prob function of y

Y	P_j
0	$6/27$
1	$9/27$
2	$12/27$

Conditional distribution y for x

$$P(Y = y_i / X = x_i) = \frac{P(X = x_i \cap Y = y_i)}{P(X = x_i)}$$

$P(x=0)\ x=0$

Conditional distribution table

$X \backslash Y$	$Y=0$	$Y=1$	$Y=2$
$X=0$	0	$1/3$	$2/3$
$X=1$	$2/9$	$3/9$	$4/9$
$X=2$	$4/15$	$5/15$	$6/15$

UNIT-III Distributions

There are 2 types of theoretical Probability distributions

1. Discrete Probability Distribution
2. Continuous Probability Distribution

Discrete ~~Rando~~ Probability Distribution:

Binomial Distribution: (BD)

The conditions for the applicability of ~~for~~ the BD are

1. There are n independent trails
2. Each trail has only 2 possible outcomes
3. The probabilities of 2 outcomes remain constant and are denoted by p, q
4. Let the no. of trails be n, let trails be independent i.e. the sucess or failure doesnot effect the outcome of the other trails. Thus the prob. of success remains the same from trail to trail

We have $p+q=1$ i.e., total probability of success and failure is always equal to 1.

Let the probability of getting x successes and $(n-x)$ failures in one trail is $p^x q^{n-x}$

Similarly total prob. of getting x successes and $(n-x)$ failures in ${}^nC_x$ ways and probability of each is $p^x q^{n-x}$

Thus if x is the r.v representing the no. of successes and the prob. of getting x successes and $(n-x)$ failures is

$$P(X=x) = \left({}^nC_x\right) p^x q^{n-x}$$

Note Point:

(i) $(p+q)^n = \sum_{x=0}^{n} {}^nC_x p^x q^{n-x}$

(ii) $p+q=1$

(iii) $\sum_{x=1}^{n-1} {}^{n-1}C_{x-1} p^{x-1} q^{n-x} = (p+q)^{n-1}$

Properties:

i) Mean of a B.D is np

Proof: $E(x) = \text{Mean} = \mu = \sum_{x=0}^{n} x\, P(x)$

where $P(x) = \left({}^nC_x\right) p^x q^{n-x}$.

$$E(x) = \sum_{x=0}^{n} x \left({}^nC_x\right) p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \cancel{x} \frac{n(n-1)!}{\cancel{x}(x-1)!\,(n-x)!} p^{x-1} \times p q^{n-x}$$

$$= np \sum_{x=1}^{n-1} \frac{(n-1)!}{(x-1)!\,(n-x)!} p^{x-1} q^{n-x}$$

$$= np \sum_{x=1}^{n-1} \left({}^{n-1}C_{x-1}\right) p^{x-1} q^{n-x}$$

$$= np(p+q)^{n-1} \qquad (\because p+q=1$$

$$= np$$

2) Variance of BD is npq

$$V(x) = E(x^2) - E(x)^2$$

$$E(x^2) = \sum_{x=0}^{n} x^2 P(x)$$

$$P(x) = {}^nC_x \, p^x q^{n-x}$$

$$E(x^2) = \sum_{x=0}^{n} x^2 \left({}^nC_x \, p^x q^{n-x}\right)$$

$$= \sum_{x=0}^{n} x^2 \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} (x(x-1)+x) \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{(x)(x-1)\, n!}{x!(n-x)!} p^x q^{n-x} + \sum_{x=0}^{n} \frac{x\, n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{\cancel{x(x-1)}\, n!}{\cancel{x(x-1)}(x-2)!(n-x)!} p^x q^{n-x} + np$$

$$= \sum_{x=2}^{n-2} \frac{n!}{(x-2)!(n-x)!} p^x q^{n-x} + np$$

$$= \sum_{x=2}^{n-2} \frac{n!}{(x-2)!(n-x)!} p^2 p^{x-2} q^{n-x} + np$$

$$= \sum_{x=2}^{n-2} \frac{n(n-1)(n-2)!}{(x-2)!(n-x)!} p^2 p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n-2} \frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2} q^{n-x} + np$$

$$n(n-1)p^2 \sum_{x=2}^{n-2} {}^{n-2}C_{x-2} \, p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 (p+q)^{n-2} + np$$

$$= n(n-1)p^2 + np \qquad (\because p+q=1)$$

$$E(x^2) = np((n-1)p+1)$$

$$V(x) = E(x^2) - (E(x))^2$$

$$= \cancel{np}\, n(n-1)p^2 + np - n^2p^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p) \qquad (\because 1-p=q)$$

$$= npq$$

Mode of BD :

Mode of the BD is the value of x for which $P(x)$ is maximum

$$\text{Mode} = \begin{cases} \text{integral}[(n+1)p] & \text{if } (n+1)p \text{ is not an exact integer} \\ (n+1)p-1,\ (n+1)p & \text{if } (n+1)p \text{ is an exact integer} \end{cases}$$

Reccurance relation for Probabilities

$$P(x) = {}^nC_x\, p^x q^{n-x}$$

$$P(x+1) = {}^nC_{x+1}\, p^{x+1} q^{n-x-1}$$

$$\frac{P(x+1)}{P(x)} = \frac{{}^nC_{x+1}\, p^{x+1} q^{n-x-1}}{{}^nC_x\, p^{x} q^{n-x}}$$

$$= \frac{\frac{\cancel{n!}}{(x+1)!(n-x-1)!}\, \cancel{p^{x+1}}\, \cancel{q^{n-x-1}}}{\frac{\cancel{n!}}{\cancel{x!}(n-x)!}\, p^x q^{n-\cancel{x}}}$$

$$= \frac{\frac{1}{(x+1)\,\cancel{(n-x-1)}}\; p \cdot q^{-1}}{\frac{1}{(n-x)}}$$

$$\frac{P(x+1)}{P(x)} = \frac{p\,(n-x)}{q\,(x+1)}$$

$P(x+1) = \left(\frac{n-x}{x+1}\right)\frac{p}{q}P(x)$ is called reccurence relation ~~of~~ Probability

Problems

① 10 coins are tossed simultaneously. Find probability of getting atleast

(i) 7 heads

(ii) 6 heads

Sol $n = 10$ $\quad P(x) = {}^nC_x\, p^x q^{n-x}$

$p = 1/2,\ q = 1/2$

(i) Probability of atleast 7 heads $= P(x \geq 7)$

$\Rightarrow P(x=7) + P(x=8) + P(x=9) + P(x=10)$

$= {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3 + {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 + {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right) +$
${}^{10}C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^0$

$= \left(\frac{1}{2}\right)^{10} \left({}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}\right)$

$= \left(\frac{1}{2}\right)^{10} (176)$

$= \frac{11}{64}$

$= 0.1718$

(ii) Probability of atleast 6 heads $= P(x \geq 6)$

$= {}^{10}C_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^4 + {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3 + {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2 +$
${}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right) + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10} \left(\frac{1}{2}\right)^0$

$= \cancel{0.0352}\ 0.3769$

② If 10% of the ~~device~~ rivets produced by a machine are defective. Find Prob that out of 5 ~~device~~ rivets choosen at random (i) None will be defective
(ii) One will be defective
(iii) Atmost 2 will be defective
Atleast 2 will

$$P(X=x) = {}^{n}C_{x}\, p^{x} q^{n-x}$$

Sol $n = 6$

let x be the event of defective ~~de~~ rivets

p = probability of defective ~~device~~ rivets $= 10\% = 0.1$

$q = 1 - p = 0.9$

(i) None will be defective

$$P(x=0) = {}^{5}C_{0}\, p^{0} q^{5} = (0.9)^{5} = 0.5904$$

(ii) One will be defective

$$P(x=1) = {}^{5}C_{1}\, p^{1} q^{4} = 0.32805$$

(iii) Atmost 2 will be defective

$$P(x \le 2) = P(x=0) + P(x=1) + P(x=2)$$

$$= {}^{5}C_{0}\, p^{0} q^{5} + {}^{5}C_{1}\, p^{1} q^{4} + {}^{5}C_{2}\, p^{2} q^{3}$$

$$= 0.5904 + 0.32805 + \cancel{0.0007}\ 0.0729$$

$$= \cancel{0.9991}\ 0.9913$$

(iv) Atleast 2 will be defective

$$P(x \ge 2) = P(x=2) + P(x=3) + P(x=4) + P(x=5)$$

$$= 1 - P(x \le 2)$$

$$= 1 - (P(x=0) + P(x=1))$$

$$= 1 - (0.5904 + 0.3280)$$

$$= 0.0816$$

(3) The mean and variances of B.D are 4 and 4/3

Find prob of $P(x \ge 1)$

Sol Mean of B.D = 4

variance = 4/3

$np = 4$ $\quad npq = \frac{4}{3}$

$$\frac{\text{variance}}{\text{Mean}} = \frac{npq}{np} = q = \frac{\frac{4}{3}}{4} = \frac{1}{3}$$

$$p = 1 - q = 1 - \frac{1}{3} = \frac{2}{3}$$

$np = 4$

$$n = \frac{4}{p}$$

$$= \frac{4}{\frac{2}{3}}$$

$$= 4 \times \frac{3}{2} = 6$$

$$P(x \ge 1) = P(x=1) + P(x=2) + \dots + P(x=6)$$

$$= 1 - P(x < 1)$$

$$= 1 - (P(x=0))$$

$$= 1 - {}^{6}C_{0} \left(\frac{2}{3}\right)^{0} \left(\frac{1}{3}\right)^{6}$$

$$= 1 - \left(\frac{1}{3}\right)^{6}$$

$$0.9986$$

(4) A discrete R.V x has the mean 6 and variance 2 if it is assumed that the distribution is binomial Find Prob that $P(5 \le x \le 7)$

Sol

Mean of B.D $= np = 6$

variance $= npq = 2$

$$\frac{npq}{np} = \frac{2}{6} = \frac{1}{3}$$

$$q = \frac{1}{3}$$

$$p = 1 - q = 1 - \frac{1}{3} = 2/3$$

$$np = 6$$

$$n \frac{2}{3} = 6$$

$$n = 9$$

$$P(5 \le x \le 7) = P(x=5) + P(x=6) + P(x=7)$$

$$= {}^9C_5 (2/3)^5 \left(\frac{1}{3}\right)^4 + {}^9C_6 (2/3)^6 \left(\frac{1}{3}\right)^3 + {}^9C_7 (2/3)^7 \left(\frac{1}{3}\right)^2$$

$$= 0.2048 + 0.2731 + 0.2341$$

$$= 0.712$$

(5) In a B.D consisting of 5 independent trails Prob of 1 & 2 success are 0.4096 & 0.2048 respectively find parameters p, q of distribution

Sol

$$P(x=1) = 0.4096$$

$$P(x=2) = 0.2048$$

$$n = 5$$

$$P(x=1) = {}^5C_1 p^1 q^4$$

$$P(x=2) = {}^5C_2 p^2 q^3$$

$$\frac{P(x=2)}{P(x=1)} = \frac{{}^5C_2 p^2 q^3}{{}^5C_1 p^1 q^4}$$

$$\frac{0.2048}{0.4096} = \frac{2p}{q}$$

$$q = 4p$$

$$1 - p = 4p$$

$$5p = 1$$

$$p = 1/5$$

$$q = 4/5$$

(6) The overall percentage of failures in a certain Examination is 20%. If 6 candidates appear in examination what is prob that atleast 5 pass the examination.

Sol

let x is an event of students appearing in examination pass

$p =$ prob of pass% $= 80\% = 0.8$

$$q = 1 - p = 0.2$$

$n = 6$ $\quad P(X = x) = {}^nC_x p^x q^{n-x}$

P(at least 5 pass)

$P(x \geq 5) = P(x=5) + P(x=6)$

$= {}^6C_5 (0.8)^5 (0.2)^1 + {}^6C_6 (0.8)^6$

$= 0.392 + 0.2621$

$= 0.6553$

① The prob of man hitting a target is $1/3$. If he fires 5 times: what is probability of his hitting target atleast twice

Sol

let x be event of hitting target

$p = 1/3$

$q = 1 - 1/3 = 2/3$

$P(x \geq 2) = 1 - P(x<2)$

$= 1 - (P(x=0) + P(x=1))$ ~~+ P(x ≥ 2)~~

$= 1 - {}^5C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^5 + {}^5C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^4$

=

② In 800 families with 5 children each. How many families would you expect to have :-

(i) 3 boys.

(ii) No girl

(iii) Aleast 1 Boy

(iv) Atmost 2 girls

(v) Two Boys & 2 Girls

Sol

x is a event of giving a birth to boy

$n = 5$ $N = 800$ $P(x=x) = {}^nC_x p^x q^{n-x}$

$p = 1/2$ $q = 1/2$

(i) $P(3 \text{ boys}) = P(x=3) = {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2$

$= 0.3125$

No. of families of 3 boys $= 0.3125 \times 800$

$= 250$

(ii) $P(\text{No girl}) = P(x=5) = {}^5C_5 \left(\frac{1}{2}\right)^5$

$= 0.03125$

No. of families of No girl $= 0.03125 \times 800$

$= 25$

(iii) $P(\text{Aleast 1 Boy}) = P(x \geq 1) = 1 - P(x=0)$

$= 1 - {}^5C_0 \left(\frac{1}{2}\right)^5$

$= 1 - 0.03125$

$= 0.96875$

No. of families of atleast 1 Boy $=$ ~~0.03125~~ 0.96875×800

$= 775$

(iv) P(Atmost 2 girl) = $P(X$ … $P(2G,3B)$

$= P(X=5) + P(X=4) + P(X=3)$

$= {}^5C_5\left(\frac{1}{2}\right)^5 + {}^5C_4\left(\frac{1}{2}\right)^5 + {}^5C_3\left(\frac{1}{2}\right)^5$

$= 0.5$

No. of families Atmost 2 Girls = 400.

(v) Two Boys and Two Girls

$= \cancel{P(2B,3G) + P(3B,2G)}$

$= \cancel{P(X=2) + P(X=3)}$

$= \cancel{{}^5C_2\left(\frac{1}{2}\right)^5 + {}^5C_3\left(\frac{1}{2}\right)^5}$

$= \cancel{0.625}$

~~No. of families 2 boys and 2 girls = 500~~

$P(2B,3G)$

$= P(X=2)$

$= {}^5C_2\left(\frac{1}{2}\right)^5$

$= 0.3125$

No. of families 2 boys and 2 girls = 250

(A) Prob/ of ...

(a) Mean of a BD is 3 and variance is 9/4

Find (i) n.

(ii) $P(X \geq 7)$

(iii) $P(1 \leq X < 6)$

Sol

Mean $= np = 3$

Variance $= npq = \frac{9}{4}$

$\frac{npq}{np} = \frac{9/4}{3}$

$q = \frac{3}{4}$

$p = 1-q = p = \frac{1}{4}$

(i)

$np = 3$

$n = 3 \times 4$

$n = 12$

(ii) $P(X \geq 7) = P(X=7) + P(X=8) + P(X=9) + P(X=10) + P(X=11) + P(X=12)$

$= {}^{12}C_7\left(\frac{1}{4}\right)^7\left(\frac{3}{4}\right)^5 + {}^{12}C_8\left(\frac{1}{4}\right)^8\left(\frac{3}{4}\right)^4 + {}^{12}C_9\left(\frac{1}{4}\right)^9\left(\frac{3}{4}\right)^3 +$

${}^{12}C_{10}\left(\frac{1}{4}\right)^{10}\left(\frac{3}{4}\right)^2 + {}^{12}C_{11}\left(\frac{1}{4}\right)^{11}\left(\frac{3}{4}\right)^1 + {}^{12}C_{12}\left(\frac{1}{4}\right)^{12}$

$= 0.0114 + 0.0023 + 0.0003 + 0.00003$

$+ 0.000002 + 0.00000005$

$= 0.01403205$

(iii) $P(1 \le x < 6) = 1 - (P(x=0) + P(x=6) + \dots + P(x=12))$

$= \cancel{P(x=1) + P(x=2) + P(x=3)}$

$= 1 - \left[{}^{12}C_0 \left(\frac{3}{4}\right)^{12} + {}^{12}C_6 \left(\frac{1}{4}\right)^6 \left(\frac{3}{4}\right)^6 + 0.0140 \right]$

$= 1 - [0.0316 + \cancel{0.0407}\ 0.0401 + 0.0140]$

$= 1 - 0.0857$

$= 0.9143$

(10) Fit a BD for the following data.

$x = 0, 1, 2, 3, 4$

x	f	fx	$\frac{n-x}{x+1}$	$\left(\frac{n-x}{x+1}\right)\frac{p}{q}$	Expected frequency $F(x)$
0	28	0	4	1.9996	29.625
1	62	62	3/2	0.7498	59.2381
2	46	92	2/3	0.3332	44.4167
3	10	30	1/4	0.1249	14.7996
4	4	16	0	0	1.8484
	$\Sigma f = N = 150$	200			149.9278

Sol. Mean $= \mu = np$

Mean $= \mu = \frac{\Sigma fx}{\Sigma f}$

$np = \frac{\Sigma fx}{\Sigma f}$

$n = 4$

$4(p) = \frac{200}{150}$

$p = \frac{1.333\ldots}{4}$

$p = \cancel{4/3}\ 0.3333$

$q = 1 - p = 0.6667$

$\frac{p}{q} = \frac{0.3333}{0.6667} = 0.4999$

$P(x=0) = {}^{4}C_0 (0.3333)^0 (0.6667)^4$

$= 0.1975$

$F(0) = N \times P(0)$

$= 150 \times 0.1975$

$= 29.625$

Sum of the Given Frequencies is approximately equal to sum of the Expected frequencies

Hence Given data is fitted.

(11) Seven coins are tossed at a time 128 times the number of heads observed at each throw is given below ~~No. of~~ Fit a B.D By assuming the coins are

(i) Biased

(ii) Unbiased

No.of Heads	freq	$\frac{n-x}{x+1}$	$\left(\frac{n-x}{x+1}\right)\frac{p}{q}$	F(x)
0	7	7/1	7	1
1	6	6/2	3	7
2	19	5/3	5/3	21
3	35	4/4	1	35
4	30	3/5	3/5	35
5	23	2/6	1/3	21
6	7	1/7	1/7	7
7	1	0	0	1
	128			128

(ii) When the coins are unbiased

$p = 1/2 \quad q = 1/2$

$p/q = 1$

$n = 7$

$N = 128$

$P(X=x) = {}^nC_x \, p^x q^{n-x}$

$P(X=0) = {}^7C_0 \left(\frac{1}{2}\right)^7$

$= \cancel{0.0078} \; \frac{1}{128}$

$F(0) = N \times P(0) = 128 \times \cancel{0.0078} \; \frac{1}{128} = 1$

∴ Sum of Given Frequencies is exactly equal to sum of Expected frequency

Hence data is fitted

(i) UnBiased

When a coin is biased $\quad n = 7$

$\text{Mean} = \frac{\Sigma fx}{\Sigma f} = \frac{433}{128} = 3.3828 = 7p$

$p = 0.4832$

$q = 0.5168$

$\frac{p}{q} = \frac{0.4832}{0.5168} = 0.9349$

No.of Heads	freq	fx	$\frac{n-x}{x+1}$	$\left(\frac{n-x}{x+1}\right)\frac{p}{q}$	F(x)
0	7	0	7	6.5443	1.2544
1	6	6	6/2	2.8047 ~~8.7808~~	8.2091
2	19	38	5/3	1.5581 ~~26.3424~~	23.0240
3	35	105	4/4	0.9349 ~~43.904~~	35.8752
4	30	120	3/5	0.5609 ~~43.904~~	33.5397
5	23	115	2/6	0.3116	18.8137
6	7	42	1/7	0.1335	5.8629
7	1	7	0	0	0.7430
	128	433			127.3214

$P(0) = ({}^7C_0)(0.5816)^7$

$= 0.0098$

$F(0) = 1.2544$

∴ Sum of Given Frequencies is ~~exactly~~ approximately equal