Data Science & Artificial Intelligence

Calculus and Optimization

Single Variable Calculus

DPP 01 Discussion

(Part 02)

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Q21. If $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{x}}{2}$, then $\boxed{\int_0^\infty e^{-ax^2}}\,dx$ where $a > 0$ is

$\frac{\sqrt{\pi}}{2}$

$\sqrt{x} \longrightarrow \sqrt{\pi}$

$I = \int_0^\infty e^{-ax^2}\,dx$

Gamma Function

(a) $\frac{\sqrt{\pi}}{2}$

(b) $\frac{\sqrt{\pi}}{2a}$

(c) $2\frac{\sqrt{\pi}}{a}$

(d) $\boxed{\frac{1}{2}\sqrt{\pi/a}}$ ✓

$$I = \int_{x=0}^{\infty} e^{-ax^2}\,dx$$

put $ax^2 = t$ $\quad x = \sqrt{\frac{t}{a}} = \frac{\sqrt{t}}{\sqrt{a}}$

$$a \cdot 2x\,dx = dt$$

$$dx = \frac{dt}{2ax}$$

$$= \frac{dt}{2 \cdot a \cdot \frac{\sqrt{t}}{\sqrt{a}}}$$

$$= \frac{1}{2 \cdot \sqrt{t}\sqrt{a}}$$

$$\boxed{dx = \frac{1}{2\sqrt{a}} t^{-1/2}}$$

$$I = \int_0^{\infty} e^{-t} \cdot \frac{1}{2\sqrt{a}} t^{-1/2}\,dt$$

$$\Rightarrow \frac{1}{2\sqrt{a}} \underbrace{\int_0^{\infty} e^{-t}\, t^{-1/2}\,dt}_{\text{Gamma Function}}$$

$$\Rightarrow \frac{1}{2\sqrt{a}} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{\pi}}{2\sqrt{a}}$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

$$\Gamma(n) = \int_0^{\infty} e^{-t} t^{n-1}\,dt$$

$$n - 1 = -\frac{1}{2} \quad n = \frac{1}{2}$$

Q22. The expression $\frac{\int_0^n [x]dx}{\int_0^n \{x\}dx}$ is equal to

(where [*] and {*} denotes greatest integer function and fractional part function and n ∈ N).

(a) 1 / n - 1

(b) 1 / n

(c) n

(d) n - 1

$I_1 \Rightarrow \int_0^n [x]\,dx$

$I_2 = \int_0^n \{x\}\,dx$

$I_1 = \int_0^n [x]\,dx$ → Integer + Piecewise

$= \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \int_3^4 3\,dX + \cdots + \int_{(n-1)}^{n} (n-1)\,dx$

$= 0 + 1(2-1) + 2(3-2) + \cdots + (n-1)$ terms

$= 1 + 2 + \cdots + (n-1)$ terms

$\boxed{I_1 = \frac{n(n-1)}{2}}$

$$I_2 = \int_0^n \{x\}\,dx = \int_0^n (x - [x])\,dx$$

$x = [x] + \{x\}$

$\{x\} = x - [x]$

$$\Rightarrow \int_0^1 (x-0)\,dx + \int_1^2 (x-1)\,dx + \cdots + \int_{(n-1)}^n x-(n-1)\,dx$$

$$= \frac{n}{2} - \frac{n(n-1)}{2}$$

$$= \frac{n}{2} - \frac{n^2}{2} + \frac{n}{2}$$

$$I_2 = \left(n - \frac{n^2}{2}\right)$$

$=$

$$\frac{I_1}{I_2} = \left(\frac{\frac{n(n-1)}{2}}{\frac{n-n^2}{2}}\right) = \underline{(n-1)}$$

Ans

Q23. Let $A = \int_0^1 \frac{e^t dt}{1+t} dt$ then $\int_{a-1}^{a} \frac{e^{-t}}{t-a-1} dt$ has the value

Manipulate

(a) Ae^{-a}

(b) $-Ae^{-a}$

(c) $-ae^{-a}$

(d) Ae^{a}

$$I = \int_{a-1}^{a} \frac{e^{-t}}{(t-a-1)}$$

$$= -\int_{a-1}^{a} \frac{e^{-t}}{1+(a-t)} dt$$

$a-t = z$

$-dt = dz$

$\cancel{a}-(\cancel{a}-1) = z$

$1 = z$

$a-a = z$

$z = 0$

$$\Rightarrow -\int_1^0 \frac{e^{(z-a)} \cdot (-dz)}{1+z}$$

$$\Rightarrow \int_0^1 \frac{e^z \cdot e^{-a} (-dz)}{1+z}$$

$$= -e^{-a} \int_0^1 \frac{e^z}{1+z} = -Ae^{-a}$$

$f(2a-x)=f(x)$

Q24. $\int_0^{\pi} xf(\sin x)dx$ is equal to

$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$I = \int_0^{\pi} x f(\sin x)\, dx$

$I \Rightarrow \int_0^{\pi} (\pi - x) f(\sin[\pi - x])\, dx$

Add $2I = \int_0^{\pi} \pi f(\sin x)\, dx$

(a) $\pi\int_0^{\pi} f(\sin x)dx$

(b) $\frac{\pi}{2}\int_0^{\pi/2} f(\sin x)dx$

$\sin\left(\frac{\pi}{2}-x\right)$

$I = \frac{\pi}{2} \times 2 \int_0^{\pi/2} f(\sin x)\, dx$

(c) $\pi\int_0^{\pi/2} f(\cos x)dx$

(d) $\pi\int_0^{\pi} f(\cos x)dx$

$I = \pi \int_0^{\frac{\pi}{2}} f(\cos x)\, dx$

Q25. Let $f : R \to R$ be a differentiable function having $f(2) = 6, f'(2) = \left(\frac{1}{48}\right)$. Then $\boxed{\lim_{x\to 2} \int_6^{f(x)} \frac{4t^3}{x-2} dt}$ equals

Newton Leibnitz Rule

$$\underset{x\to 2}{\mathrm{Lt}} \left[\frac{4[f(x)]^3 \cdot f'(x)}{1}\right]$$

(a) 18

(b) 12

(c) 36

(d) 24

$$\underset{x\to 2}{\mathrm{lt}} \frac{1}{(x-2)} \boxed{\int_6^{f(x)} 4t^3 dt} = 4[f(2)]^3 \cdot f'(2) = \cancel{4} \times (6) \times \cancel{6} \times \cancel{6} \times \frac{1}{\cancel{48}}$$

$$= 18$$

Q26. The value of $\int_0^{\pi/2} \frac{(\sin x+\cos x)^2}{\sqrt{1+\sin 2x}}\,dx$ is

$$=\int_0^{\frac{\pi}{2}} \frac{(\sin x+\cos x)^2}{\sqrt{(\sin x+\cos x)^2}}$$

$$\frac{\sin^2 x+\cos^2 x+2\sin x\cos x}{(\sin x+\cos x)^2}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{(\sin x+\cos x)^2}{|\sin x+\cos x|} \Rightarrow 0 \text{ to } \frac{\pi}{2}$$

Positive

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\cancel{(\sin x+\cos x)^2}}{\cancel{\sin x+\cos x}}\,dx$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} (\sin x+\cos x)\,dx = 2$$

(a) 0

(b) 1

(c) 2

(d) 3

Q27. If $\underbrace{f(a+b-x) = f(x),}_{\text{Using this Property}}$ then $\int_a^b xf(x)dx$ is equal to

(a) $\frac{a+b}{2}\int_a^b f(b-x)dx$

(b) $\frac{a+b}{2}\int_a^b f(x)dx$

(c) $\frac{b-a}{2}\int_a^b f(x)dx$

(d) $\frac{a+b}{2}\int_a^b f(a+b+x)dx$

Q28. The value of $\lim_{x\to 0} \frac{\int_0^{x^2} \sec^2 t}{x \sin x} dt$ is

(a) 3

(b) 2

(c) 1

(d) -1

$$\lim_{x\to 0} \frac{\int_0^{x^2} \sec^2 t\, dt}{x \sin x} = \frac{\to 0}{\to 0} \text{ form}$$

derivative

Using L-Hospital Rule

$$= \lim_{x\to 0} \frac{\sec^2 x^2 \cdot 2x}{x\cos x + \sin x}$$

$$= \lim_{x\to 0} \frac{2\sec^2 x^2}{\frac{x}{x}\cos x + \frac{\sin x}{x}} = \lim_{x\to 0} \frac{2\sec^2 x^2}{\cos x + \frac{\sin x}{x}}$$

1

$$= \lim_{x\to 0} \frac{\cancel{2}\sec^2 x^2}{\cancel{2}} = 1$$

Q29. $\int_{\sin x}^{1} t^2 f(t)dt = 1 - \sin x \ \forall x \in (0, \pi/2)$, then $f\left(\frac{1}{\sqrt{3}}\right)$ is $\sin x = \frac{1}{\sqrt{3}}$

apply L-Hospital Rule:

(a) 3

(b) $\sqrt{3}$

(c) 1/3

(d) None of these

$= 1 f(1)\cdot 0 - \sin^2 x f(\sin x)\cos x = 0 - \cos x$

$\sin^2 x f(\sin x)\cos x = \cos x$

$f(\sin x) = \frac{1}{\sin^2 x}$

$\sin x = \frac{1}{\sqrt{3}}$ $f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2} = (\sqrt{3})^2 = 3$

Q30. $\lim_{x \to 0} \frac{\int_0^{x^2} \cos t^2 dt}{x \sin x}$ is equal to

Using Newton Leibnitz Rule:

$\frac{\to 0}{\to 0} = \lim_{x \to 0} \frac{\cos x^4 \cdot 2x}{x \cos x + \sin x} \Rightarrow 1$

L-Hospital Rule

(a) -1

(b) 1

(c) 2

(d) -2

Q31. If $\int_{ln\,2}^{x} \frac{dx}{\sqrt{e^x-1}} = \frac{\pi}{6}$, then x =

(a) 4

(b) ln 8

(c) ln 4

(d) None of these

$$\int_{\ln}^{x} \frac{dx}{\sqrt{e^x-1}} = \frac{\pi}{6}$$

Square Root

$$\boxed{e^x - 1 = t^2}$$

$$e^x = t^2+1$$

$$e^x dx = 2t\,dt$$

$$dx = \frac{2t\,dt}{e^x} = \frac{2t}{(t^2+1)}dt$$

$$= \int_1^{\sqrt{e^x-1}} \frac{2\cancel{t}\,dt}{(t^2+1)\sqrt{\cancel{t^2}}}$$

$$= 2\int_1^{\sqrt{e^x-1}} \frac{1}{(t^2+1)}dt = \frac{\pi}{6}$$

$$2\left[\tan^{-1} t\right]_1^{\sqrt{e^x-1}} = \frac{\pi}{6}$$

$$= 2\left[\tan^{-1}\sqrt{e^x-1} - \tan^{-1} 1\right] = \frac{\pi}{6}$$

Change The limit

$$e^{\ln 2} - 1 = t^2$$

$$2-1 = t^2$$

$$t = \pm 1$$

$$2\left[\tan^{-1}\sqrt{e^x-1}-\tan^{-1}1\right]=\frac{\pi}{6}$$

$$=2\left[\tan^{-1}\sqrt{e^x-1}-\frac{\pi}{4}\right]=\frac{\pi}{6}.$$

$$\tan^{-1}\sqrt{e^x-1}=\frac{\pi}{12}+\frac{\pi}{4}=\frac{\cancel{16}\pi}{\cancel{48}}=\frac{\pi}{3}.$$

$$\tan^{-1}\sqrt{e^x-1}=\frac{\pi}{6}$$

$$\sqrt{e^x-1}=\sqrt{3}$$

$$e^x-1=3 \qquad e^x=4$$

$$e^x=4 \qquad x=\ln 4$$

Q32. $\underset{x\to\infty}{Limit}\left(\frac{x^2-2x+1}{x^2-4x+2}\right)^x =$

(a) 1

(b) 2

(c) e^2

(d) e

$\underset{x\to\infty}{Lt}\left(\frac{x^2-2x+1}{x^2-4x+2}\right)^x \longrightarrow 1^{\to\infty}$ Ans

Using $(\to 1^{\to\infty}) \longrightarrow e^A$

Where $A = \underset{x\to\infty}{Lt}\left(f(x)-1\right]g(x)$

Q33. If α and β be the roots of $\boxed{ax^2 + bx + c = 0}$ then $\lim_{x\to\alpha}(1 + ax^2 + bx + c)^{\frac{1}{x-\alpha}}$ is

$\to \alpha, \beta$

$ax^2+bx+c=0$

$a(x-\alpha)(x-\beta)=0$

Two Roots

(a) $a(\alpha-\beta)$

(b) $\ln|a(\alpha-\beta)|$

(c) $e^{a(\alpha-\beta)}$ ✓

(d) $e^{a|\alpha-\beta|}$ → mod

$$\lim_{x\to\alpha}(1+ax^2+bx+c)^{\frac{1}{(x-\alpha)}} = \lim_{x\to\alpha}\left[1+a(x-\alpha)(x-\beta)\right]^{\frac{1}{(x-\alpha)}}$$

1^{∞} form

$= e^A$

Where $A = \lim_{x\to\alpha}[f(x)-1]\,g(x)$

$$= \lim_{x\to\alpha}\left[\cancel{1}+a\cancel{(x-\alpha)}(x-\beta)\cancel{-1}\right]\cdot\frac{1}{\cancel{(x-\alpha)}}$$

$= a(\alpha-\beta)$

Ans $= e^{a(\alpha-\beta)}$

Q34. $\lim\limits_{x\to\pi/2} \dfrac{2^{-\cos x}-1}{x(x-\pi/2)} =$

$$\lim_{x\to 0} \frac{a^x-1}{x} = \ln a$$

$$\lim_{x\to\frac{\pi}{2}} \frac{2^{-\cos x}-1}{x\left(x-\frac{\pi}{2}\right)} \times \frac{2^{\cos x}}{2^{\cos x}}$$

$$= \lim_{x\to\frac{\pi}{2}} \frac{1-2^{\cos x}}{x\left(x-\frac{\pi}{2}\right)2^{\cos x}}$$

(a) $\dfrac{2\ln 2}{\pi}$

(b) $\ln 2$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}} \frac{2^{\cos x}-1}{x\left(\frac{\pi}{2}-x\right)2^{\cos x}}$$

$$\frac{(2^x-1)}{x} = \ln 2$$

(c) $2/\pi$

(d) Does not exists

$$= \lim_{x\to\frac{\pi}{2}} \frac{\boxed{2^{\sin\left(\frac{\pi}{2}-x\right)}-1}}{x\boxed{\left(\frac{\pi}{2}-x\right)}2^{\cos x}}$$

$$= \frac{\ln 2 \times 2}{\pi} = \frac{2\ln 2}{\pi}$$

Q35. $\lim_{x \to \pi/2} \frac{\left(1-\tan\frac{x}{2}\right)(1-\sin x)}{\left(1+\tan\frac{x}{2}\right)(\pi-2x)^3}$ is $\longrightarrow$ Using L-Hospital Rule. Two times

$$\boxed{L = \frac{1}{32}}$$

(a) 1/16

(b) -1/16

(c) 1/32

(d) -1/32

Q36. $\lim_{x\to 0}(\cos m x)^{\frac{n}{x^2}}$

$\cos mx = \cos^2\frac{mx}{2} - \sin^2\frac{mx}{2}$

$= 1 - 2\sin^2\frac{mx}{2}$

$= 2\cos^2\frac{mx}{2} - 1$

(a) $e^{-m^2n/4}$

(b) $e^{-m^2n/2}$

(c) $e^{-mn^2/2}$

(d) $e^{-mn^2/4}$

$\lim_{x\to 0}(\cos mx)^{\frac{n}{x^2}}$ $\xrightarrow{\text{Limit}}$ $1^{\to\infty}$ form

Ans $= e^{A}$

Where $A = \lim_{x\to 0}[f(x)-1]g(x)$

$= \lim_{x\to 0}\left[\cos mx - 1\right]\frac{n}{x^2}$

$= \frac{-m^2n}{2}$

$\boxed{L = e^{-\frac{m^2n}{2}}}$

$\lim_{x\to 0}\frac{n(1-\cos mx)}{x^2}$

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \log a$$

Q37. $\underset{x\to 0}{Limit} \dfrac{(4^x-1)^3}{\sin\left(\frac{x}{p}\right) \ln\left(1+\frac{x^2}{3}\right)} =$

(a) 9 p (log 4)

(b) 3 p $(\log 4)^3$

(c) 12 p $(\log 4)^3$

(d) 27 p $(\log 4)^2$

$$\lim_{x\to 0} \frac{\frac{(4^x-1)^3}{x^3} \cdot x^3 \cdot 1}{\frac{\sin\left(\frac{x}{p}\right)}{\left(\frac{x}{p}\right)}\left(\frac{x}{p}\right)\frac{\ln\left(1+\frac{x^2}{3}\right)}{1+\frac{x^2}{3}}\left(1+\frac{x^2}{3}\right)}$$

$$= (\log 4)^3\, 3 \times p = 3p(\log 4)^3$$

Q39. $\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+3}\right)^x$ is equal to

(a) e^4

(b) e^2

(c) e^3

(d) e

$$\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+3}\right)^{x} \xrightarrow{} 1^{\to\infty}$$

$$\text{Ans} = e^{A}$$

Where $= A = \lim_{x\to\infty}[f(x)-1]g(x)$

$$\text{Ans} = e^{4}$$

Q40. Let α and β be the distinct roots of $ax^2 + bx + c = 0$,

Then $\lim\limits_{x\to\alpha} \frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$ is equal to

$\lim\limits_{x\to\alpha} \frac{1-\cos a(x-\alpha)(x-\beta)}{(x-\alpha)^2}$

$= \lim\limits_{x\to\alpha} \frac{\cancel{1}-\cancel{1}+2\sin^2\frac{(x-\alpha)(x-\beta)}{2}}{\frac{(x-\alpha)^2(x-\beta)^2}{4}}$

$\Rightarrow \frac{a^2}{2}(\alpha-\beta)^2$

(a) $\frac{1}{2}(\alpha-\beta)^2$

(b) $-\frac{a^2}{2}(\alpha-\beta)^2$

(c) 0

(d) $\frac{a^2}{2}(\alpha-\beta)^2$

Q41. $\lim_{x\to 0}\left(\cot\left(\frac{\pi}{4}+x\right)\right)^{\operatorname{cosec} x} =$

(a) e^{-1}

(b) e^{2}

(c) e^{-2} ✓

(d) e^{1}

$\underset{x\to 0}{\text{Lt}}\left[\cot\left(\frac{\pi}{4}+x\right)\right]^{\operatorname{cosec} x} = e^{-2}$

$= \underset{x\to 0}{\text{Lt}}\left[\frac{1-\tan x}{1+\tan x}\right]^{\frac{1}{\sin x}} \to e^{-2}$

$\boxed{\to 1^{\to\infty}}$ ✓

Q42. $\lim_{x\to\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^{x}$ is

(a) e

(b) e^2

(c) 1/e

(d) Does not exist

$\rightarrow 1^{\to\infty}$

$y \to \frac{1}{x}$ $\quad x\to\infty$, $y\to 0$

$= \lim_{y\to 0}\left[\sin y+\cos y\right]^{\frac{1}{y}}$ $\left(\to 1^{\to\infty}\right)$ form

Where $A = \lim_{y\to 0}\left[\sin y+\cos y-1\right]\frac{1}{y}$ $= e^{A}$

$= \lim_{y\to 0}\left[\frac{\sin y+\cos y-1}{y}\right]$ $\left(\frac{\to 0}{\to 0}\right)$ form

L-Hospital Rule

$\boxed{A = 1}$

Ans $= e^{A}$

$= e^{1} = e$

Q43. $\lim_{x \to \infty} \frac{\sin(6x^2)}{\ln\cos(2x^2 - x)} =$ L-Hospital Rule — Two times

misprint $x \to 0$

$$L = \lim_{x \to 0} \frac{\sin(6x^2)}{\ln\cos(2x^2 - x)} = -12$$

(a) 12

(b) -12

(c) 6

(d) -6

Q44. $\lim_{x \to 0} \frac{e^{-x^2/2} - \cos x}{x^3 \sin x} =$ Two Times Using L-Hospital Rule

$$\boxed{L = \frac{1}{12}}$$

(a) 1/4

(b) 1/6

(c) ✓ 1/12

(d) 1/8

✓ Leibnitz Rule:

$$f(x) = \int_{\phi(x)}^{\psi(x)} f(t)\,dt$$

remove The Integral

✓ $$\frac{d}{dx} f(x) = \frac{d}{dx}\int_{\phi(x)}^{\psi(x)} f(t)\,dt$$

$$\Rightarrow f[\psi(x)]\,\frac{d}{dx}\psi(x) - f[\phi(x)]\,\frac{d}{dx}[\phi(x)]$$

remove The Integral

$$f(x) = \int_{x^2}^{x^3} \sin t\,dt$$

$$= \sin x^3 \cdot 3x^2 - \sin x^2 \cdot 2x$$ ✓

Thank You!

GW Soldiers