# Data Science & Artificial Intelligence

## Calculus and Optimization

**Single Variable Calculus**

**DPP 01 Discussion**

**(Part 01)**

[redacted]

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Q1. On the interval [0, 1] the function $x^{25}(1-x)^{75}$ takes its maximum value at

(a) 0

(b) 1/2

(c) 1

(d) 1/4 ✓

$f(x) = x^{25}(1-x)^{75}$ Polynomial [0,1] Closed Interval

Global Max or Global min

I × II

$f'(x) = 25x^{24}(1-x)^{75} + 75(1-x)^{74}\cdot(-1)\cdot x^{25}$

$f'(x) = 25x^{24}(1-x)^{74}[1-x+(-3x)]$

$f'(x) = 0$

$25x^{24}(1-x)^{74}(1-4x) = 0$

$1-4x = 0$

$x = \frac{1}{4}$

$x = \frac{1}{4}$ – max

$x > \frac{1}{4}$ – max

$x < \frac{1}{4}$ – min

$x = \frac{1}{4}$ Critical Pt

local max/min

Q2. The product of minimum $\boxed{\text{value of } x^x}$ and maximum value of $\left(\frac{1}{x}\right)^x$ is

(a) $e$

(b) $e^{-1}$

(c) 1

(d) $e^{-2}$

min value $f(x) = x^x \rightarrow$ (FUNCTION) function $f(x) = \left(\frac{1}{x}\right)^x$ $x^{-x}$

$f'(x) = 0$

$f'(x) =$ taking log both sides

$\log f(x) = \underset{\text{I}}{x} \times \underset{\text{II}}{\log x}$

$\Rightarrow \frac{1}{f(x)} \cdot f'(x) = x \cdot \frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}(x)$

$= f'(x) = f(x)\left[\cancel{x} \cdot \frac{1}{\cancel{x}} + \log x \cdot 1\right]$

$f'(x) = x^x [1 + \log x]$

$f'(x) = 0$

$x^x(1 + \log x) = 0$ $1 + \log x = 0$ $\log x = -1$ $\boxed{x = \frac{1}{e}}$ critical pt

$$f(x) = x^{-x}$$

$$\Rightarrow \log f(x) = -x \log x$$

$$\Rightarrow f'(x) = -x^{-x}[1 + \log x]$$

$$f(x) = x^{x}$$

$$x = \frac{1}{e}$$

$f''(x) > 0$ $x = \frac{1}{e}$ min

min value.

$$f(x) = x^{x}$$

min value. $f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{1/e} = (e^{-1})^{1/e} = e^{-\frac{1}{e}}$

$$f'(x) = 0$$

$$x^{-x}(1 + \log x) = 0$$

$1 + \log x = 0$ $\log x = -1$

$$\boxed{x = \frac{1}{e}}$$

$$f(x) = x^{-x}$$

critical Pt $x = \frac{1}{e}$

$f''(x) < 0$ max

max value. $f(x) = x^{-x}$

$$f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{-\frac{1}{e}}$$

$$\underline{\text{max}} = \left(e^{\frac{1}{e}}\right)$$

Min value $= e^{-\frac{1}{e}}$ max value $= e^{\frac{1}{e}}$

Product of $(\text{max})_{\text{value}} \times (\text{min value})$

$\Rightarrow e^{\frac{1}{e}} \cdot e^{-\frac{1}{e}}$

$\Rightarrow e^{0} = 1$

Q3. The minimum value of the function defined by f(x) = max (x, x + 1, 2 – x) is

$f(x) = \max\{x, x+1, 2-x\}$

Plot The curve

$y = x,\ x+1,\ 2-x$

$\max\{x, x+1, 2-x\}$

$= \frac{3}{2}$ Ans

(a) 0

$y = x+1$

$y = 2-x$

(b) 1/2

$x+1 = 2-x$

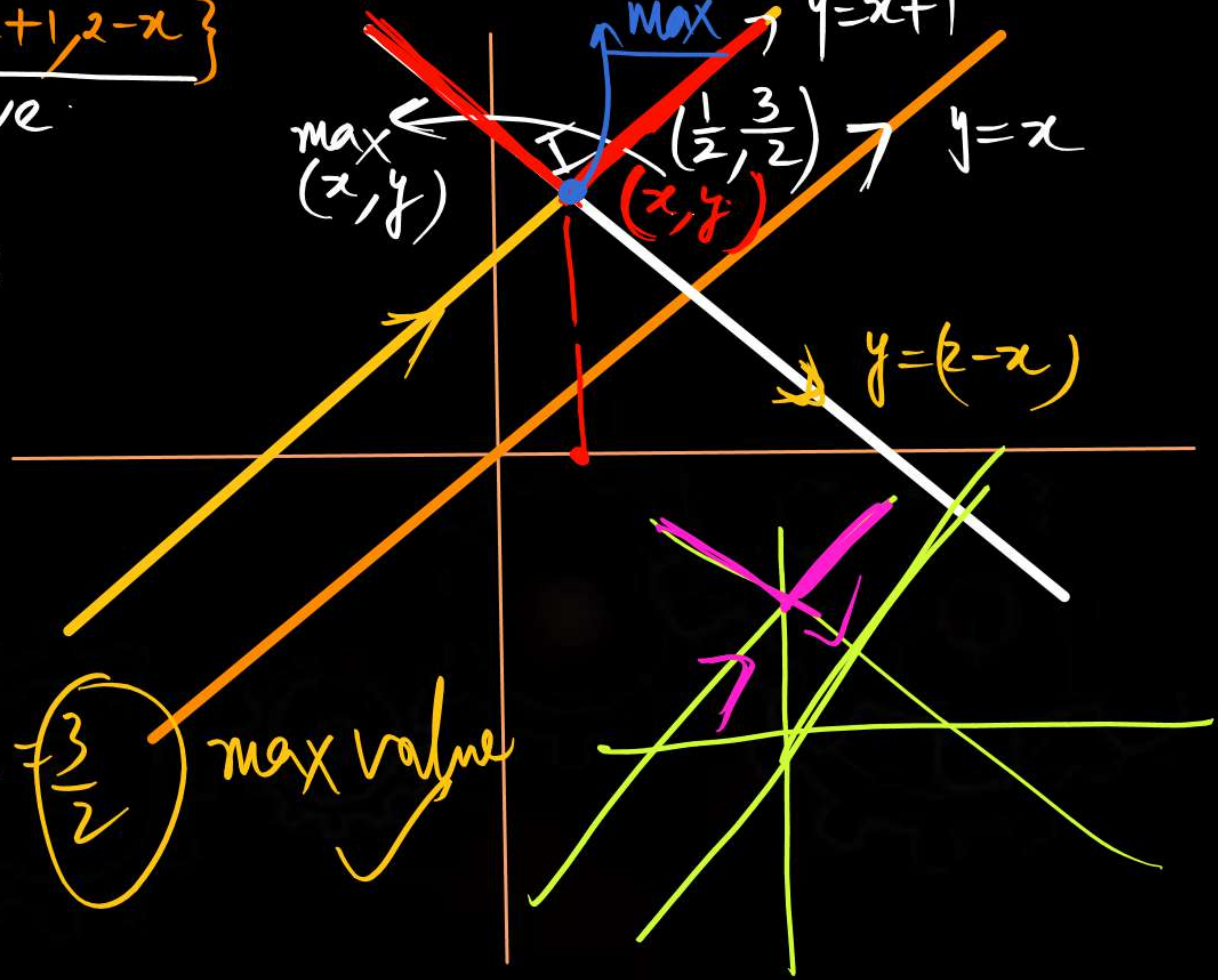
$2x = 1$

$\boxed{x = \frac{1}{2}}$

(c) 1

(d) 3/2

$y = (2-x) = 2-\frac{1}{2} = \frac{3}{2}$ max value



absolute max min

Q4. The greatest and the least values of the function,

$f(x) = 2 - \sqrt{1 + 2x + x^2}, x \in [-2, 1]$ are

$-1 \in [-2, 1]$

→ closed Interval ← Global max, global min

$x = -1$ ✓

$f(x) = 2 - \sqrt{1 + 2x + x^2}$

$f'(x) = 0 - \frac{1}{2\sqrt{1+2x+x^2}}(0+2+2x) = 0$

$\frac{-(2+2x)}{2\sqrt{1+2x+x^2}} = 0$

$2 + 2x = 0$

$x = -1$

$f(x) = 2 - \sqrt{1 + 2x + x^2}$

$f(-1) = 2 - \sqrt{1-2+1} = 0$

$f(-2) = 2 - \sqrt{1-4+4} = 1$

$f(1) = 2 - \sqrt{1+2+1} = 2$

max/min

Max value $f(1) = 2$

min $f(-1) = 0$

(a) 2, 1

(b) 2, –1

(c) 2, 0 ✓

(d) None of these

Q5. The difference between the greatest and least values of the function $f(x) = \sin 2x - x$ on $[-\pi/2, \pi/2]$ is

$-\frac{\pi}{6}, -\frac{\pi}{2}, \frac{\pi}{2}, +\frac{\pi}{6}$

(a) $\frac{\sqrt{3}+\sqrt{2}}{2}$

(b) $\frac{\sqrt{3}+\sqrt{2}}{2}+\frac{\pi}{6}$

(c) $\pi/2$

(d) $\pi$

$f(x) = \sin 2x - x$

$f'(x) = 0$

$2\cos 2x - 1 = 0$

$2\cos 2x = 1 \quad \cos 2x = \frac{1}{2}$

$\cos 2x = \cos\frac{\pi}{3} \quad x = \frac{\pi}{6}, -\frac{\pi}{6}$

$f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{6} \quad f\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$ min

$f\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2} + \frac{\pi}{6} \quad f\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$ Max

Difference between

$= \text{max} - \text{min}$

$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$

Q6. If p and q are positive real numbers such that $\boxed{p^2 + q^2 = 1}$, then the maximum value of $(p + q)$ is

(a) 2

(b) 1/2

(c) $\frac{1}{\sqrt{2}}$

(d) $\sqrt{2}$

max $f(p, q) = (p+q)$

$\boxed{f(p) = p + \sqrt{1-p^2}}$

$f'(p) = 0$

$1 + \frac{1}{2\sqrt{1-p^2}}(-2p) = 0$

$\sqrt{1-p^2} - p = 0$

$(\sqrt{1-p^2})^2 = (p)^2$

$1 - p^2 = p^2$

$q = \sqrt{1-p^2}$

A.M $\geq$ G.M

$\frac{p+q}{2} \geq \sqrt{p \cdot q}$

$1 = 2p^2 \Rightarrow p^2 = \frac{1}{2} \Rightarrow p = \frac{1}{\sqrt{2}}$

where $p = \frac{1}{\sqrt{2}}$ max $f''(p) < 0$

max value $= p + q$

$= p + \sqrt{1-p^2}$

$= \frac{1}{\sqrt{2}} + \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$= \sqrt{2}$

Q7. If x is real, the maximum value of $\dfrac{3x^2+9x+17}{3x^2+9x+7}$ is

$\dfrac{N^r}{D^r}$ = divide

$$d\left(\frac{u}{v}\right) = \frac{u\,dv - v\,du}{v^2}$$

$$f(x) = \frac{3x^2+9x+17}{3x^2+9x+7} = \boxed{1+\frac{10}{3x^2+9x+7}}$$

$$f'(x) = 0 + \frac{(6x+9)}{(3x^2+9x+7)^2} = 0$$

$x = -\frac{3}{2}$ critical Pt

$6x+9=0$

$x = -\frac{9}{6} = -\frac{3}{2}$

$f''(x) < 0$ at Pt $x = -\frac{3}{2}$

Max value $= 1 + \dfrac{10}{3\left(\frac{9}{4}\right) + 9\times\left(-\frac{3}{2}\right) + 7} = 1 + \dfrac{10}{\frac{27}{4} - \frac{27}{2} + 7}$

$= 41$

(a) 41

(b) 1

(c) 17/7

(d) 1/4

Q8. The maximum value $x^3 - 3x$ in the interval [0, 2] is

$f(x) = x^3 - 3x \quad [0,2]$

Critical Pts $= -1, 1 \quad \boxed{1 \in [0,2]}$ global max | global min

$f(0) = 0$

$f(1) = 1 - 3 = -2$

$f(2) = 8 - 6 = 2$

max value

$f(2) = 2$ max value

Point $x = 2$

(a) 1

(b) 2

(c) 0

(d) −2

Q9. Minimum value of $\dfrac{1}{3\sin\theta-4\cos\theta+7}$

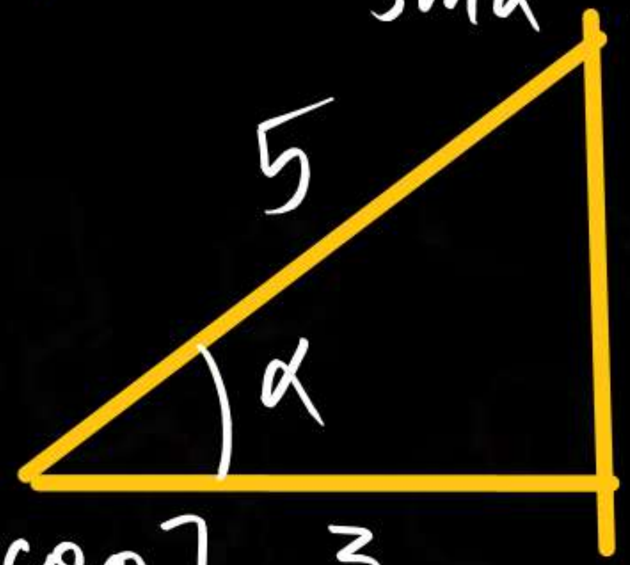
$f(\theta)=\dfrac{1}{\boxed{3\sin\theta-4\cos\theta+7}}$

✓ $\boxed{3\sin\theta-4\cos\theta}+7$

✓ $a\sin\theta-b\cos\theta$ OR ✓ $a\sin\theta+b\cos\theta$ ] make out $\dfrac{1}{\sqrt{a^2+b^2}}$ divide, $\sqrt{a^2+b^2}$ multiply

$\cos\alpha=\frac{3}{5}$

$\sin\alpha=\frac{4}{5}$




(a) 7/12

(b) 5/12

(c) 1/12 ✓

(d) 1/6

$5\left[\dfrac{3\sin\theta}{\sqrt{9+16}}-\dfrac{4\cos\theta}{\sqrt{9+16}}\right]$

$=5\left[\frac{3}{5}\sin\theta-\frac{4}{5}\cos\theta\right]=5\left[\cos\alpha\sin\theta-\sin\alpha\cos\theta\right]$

$=5\left[\sin\theta\cos\alpha-\cos\theta\sin\alpha\right]=5\sin(\alpha-\theta)$

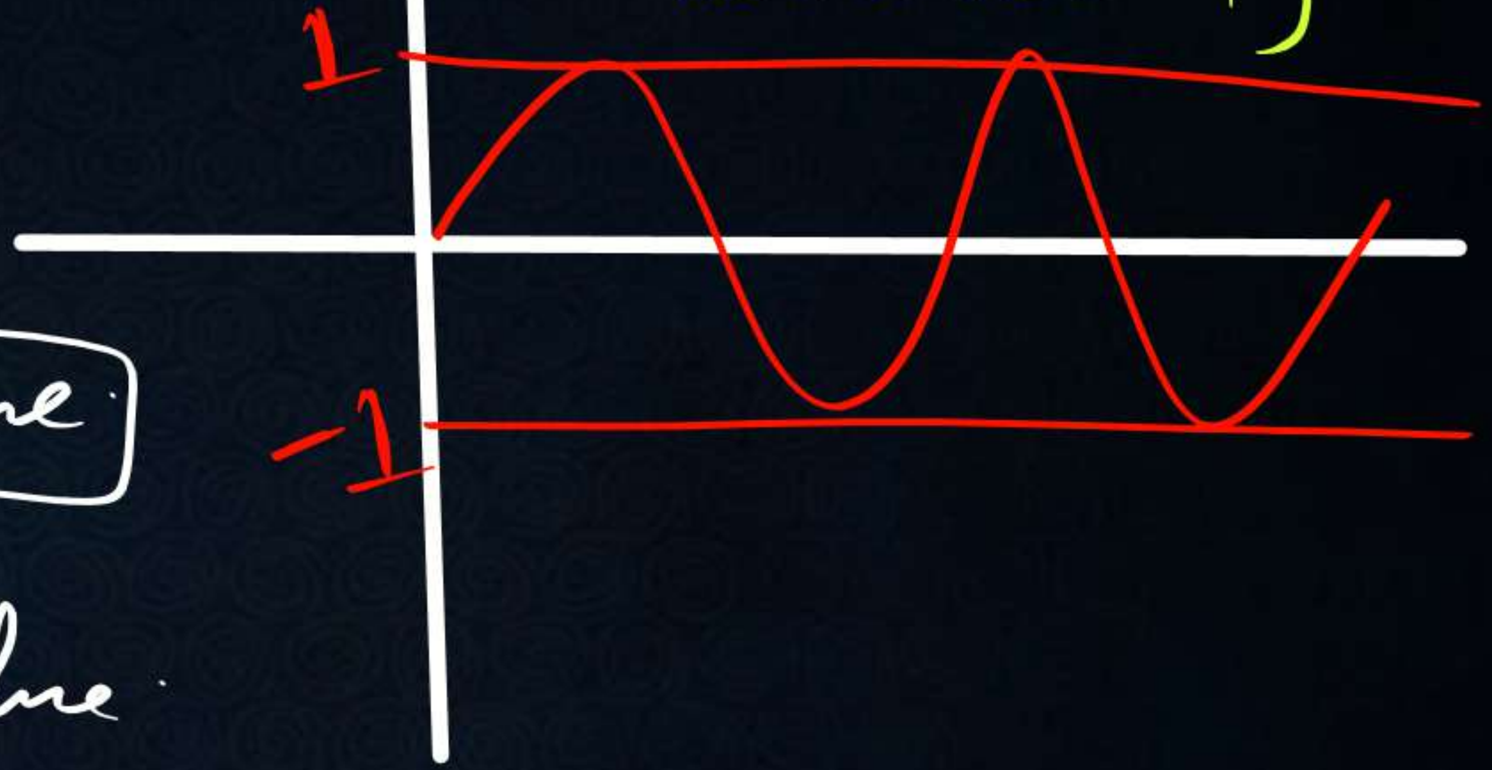
$$= \frac{1}{5\boxed{\sin(\alpha-\theta)}+7}$$

$$\boxed{-1 \le \sin(\alpha-\theta) \le 1}$$

min value $= \dfrac{1}{5\times 1+7} = \dfrac{1}{12}$ $\boxed{\text{min value}}$

max value $= \dfrac{1}{5\times -1+7} = \dfrac{1}{2}$ max value

max value $= 1$
min value $= -1$



Q10. The number of values of x where $f(x) = \cos x + \cos\sqrt{2}\, x$ attains its maximum value is

$x = 0$ ✓

HINT

$f(x) = \cos x + \cos(\sqrt{2})x$

max value

max value at $x = 0$ ✓ $\boxed{x = 0}$

$f(0) = \cos 0^{\circ} + \cos(\sqrt{2})0$

$= 1 + 1$

$= 2$

$x = 0$

single Pt.

(a) 1

(b) 0

(c) 2

(d) infinite

Q11. The greatest value of $f(x) = (x+1)^{1/3} - (x-1)^{1/3}$ in $\boxed{[0, 1]}$ is

global max / min

$$f'(x) = \frac{1}{3}\frac{1}{(x+1)^{2/3}} - \frac{1}{3}\cdot\frac{1}{(x-1)^{2/3}}$$

$$f'(x) = 0$$

$$\frac{1}{3}\left[\frac{(x-1)^{2/3} - (x+1)^{2/3}}{(x^2-1)^{2/3}}\right] = 0 \longrightarrow x = \pm 1$$

$f'(x)$ does Not exists at $x = \pm 1$

(a) 1

$\boxed{\text{(b) } 2}$

(c) 3

(d) $2^{1/3}$

$$(x-1)^{2/3} - (x+1)^{2/3} = 0 \qquad (x-1)^{2/3} = (x+1)^{2/3}$$

$$\boxed{x = 0}$$

$$f(0) = (0-1)^{2/3} - (0+1)^{2/3} = 2 \quad \boxed{\text{Max value}}$$

Q12. If $\underbrace{\int_1^x \frac{dt}{|t|\sqrt{t^2-1}}}_{}$ $= \frac{\pi}{6}$, then x can be equal to

$\left[\sec^{-1} t\right]_1^x = \frac{\pi}{6}$

$\int \frac{1}{|x|\sqrt{x^2-1}} = \sec^{-1} x$

$\boxed{\sec^{-1} 0}$

$\sec^{-1} x - 0 = \frac{\pi}{6}$

$\sec^{-1} x = \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

$\boxed{x = \frac{2}{\sqrt{3}}}$

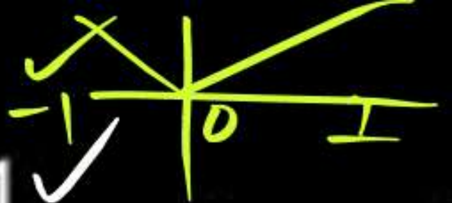
(a) $\frac{2}{\sqrt{3}}$ ✓ $\Rightarrow \sec^{-1} x - \sec^{-1} 1 = \frac{\pi}{6}$

(b) $\sqrt{3}$

(c) 2

(d) None of these

Q13. if $f(x) = \begin{cases} x; x < 1 \\ x - 1; x \geq 1 \end{cases}$, then $\int_0^2 x^2 f(x)dx$ is equal to




Piecewise

$$\begin{cases} x < 1 \longrightarrow x \\ x \geq 1 \longrightarrow (x-1) \end{cases}$$

(a) 1

(b) 4/3

(c) 5/3 ✓

(d) 5/2

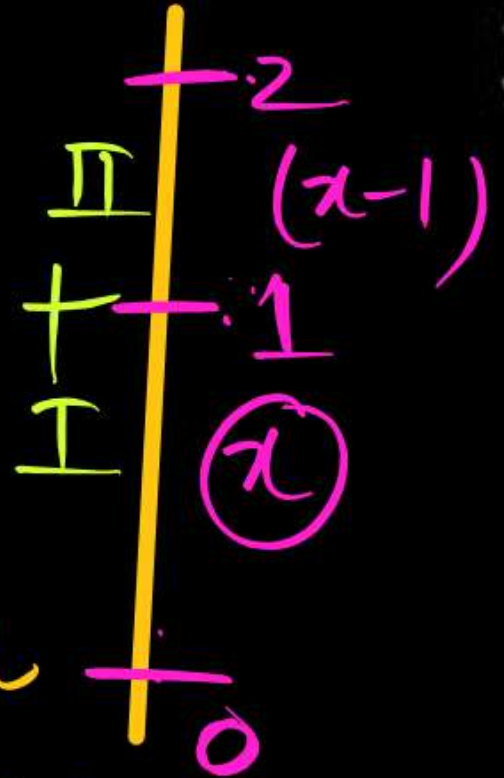
$$I_1 = \int_0^2 x^2 f(x)\, dx$$

$$\Rightarrow \int_0^1 x^2 (x)\, dx + \int_1^2 x^2 \cdot (x-1)\, dx$$

$$= \frac{1}{4} + \left(\frac{2^4}{4} - \frac{1^4}{4}\right) - \left[\frac{2^3}{3} - \frac{1^3}{3}\right]$$

$$= \frac{1}{4} + \left[4 - \frac{1}{4}\right] - \left[\frac{7}{3}\right] = \frac{1}{4} + \frac{15}{4} - \frac{7}{3}$$

$$= 4 - \frac{7}{3} = \frac{5}{3}$$



Q14. $\int_0^\pi |1 + 2\cos x|\, dx$ equal to :

Telegram

$I = \int_0^\pi |1 + 2\cos x|\, dx$

= Do yourself

$= \frac{\pi}{3} + 2\sqrt{3}$

(a) $2\pi / 3$

(b) $\pi$

(c) $2$

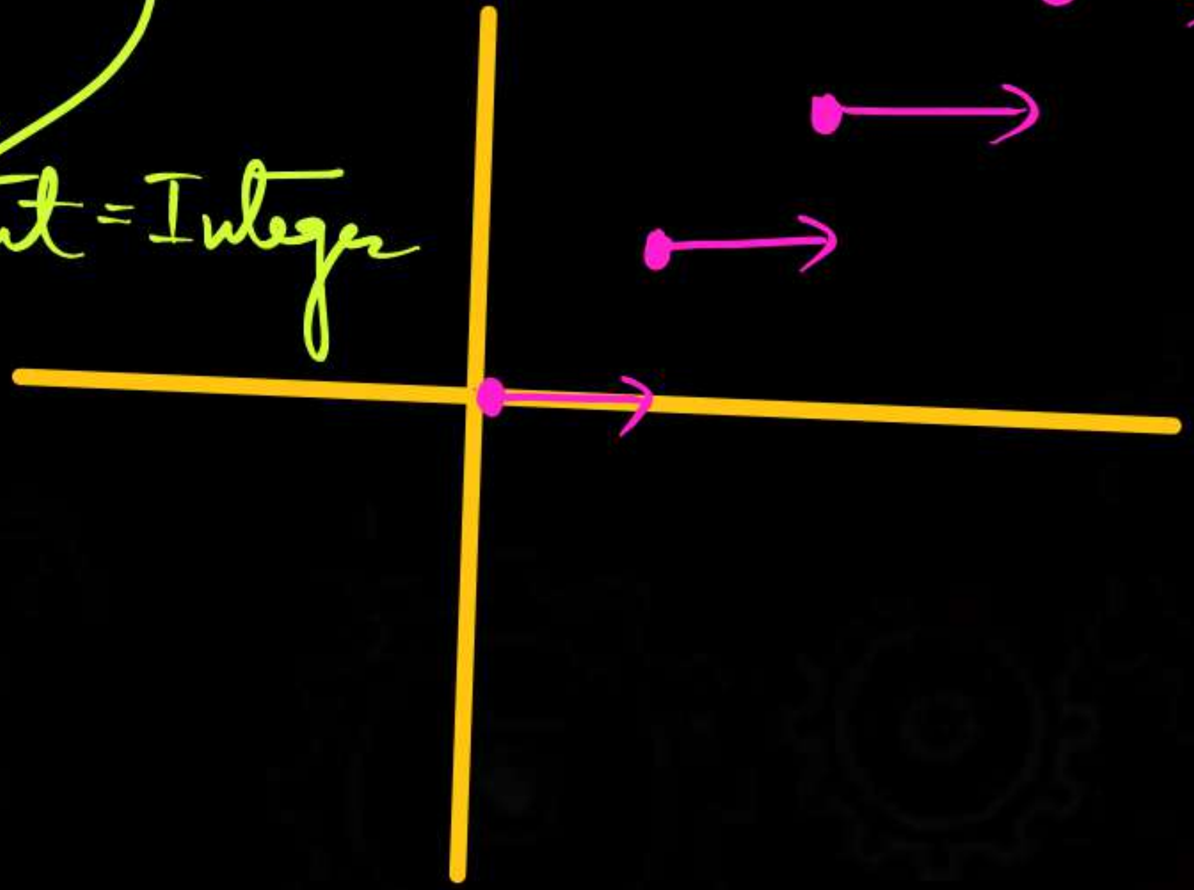
(d) $\frac{\pi}{3} + 2\sqrt{3}$

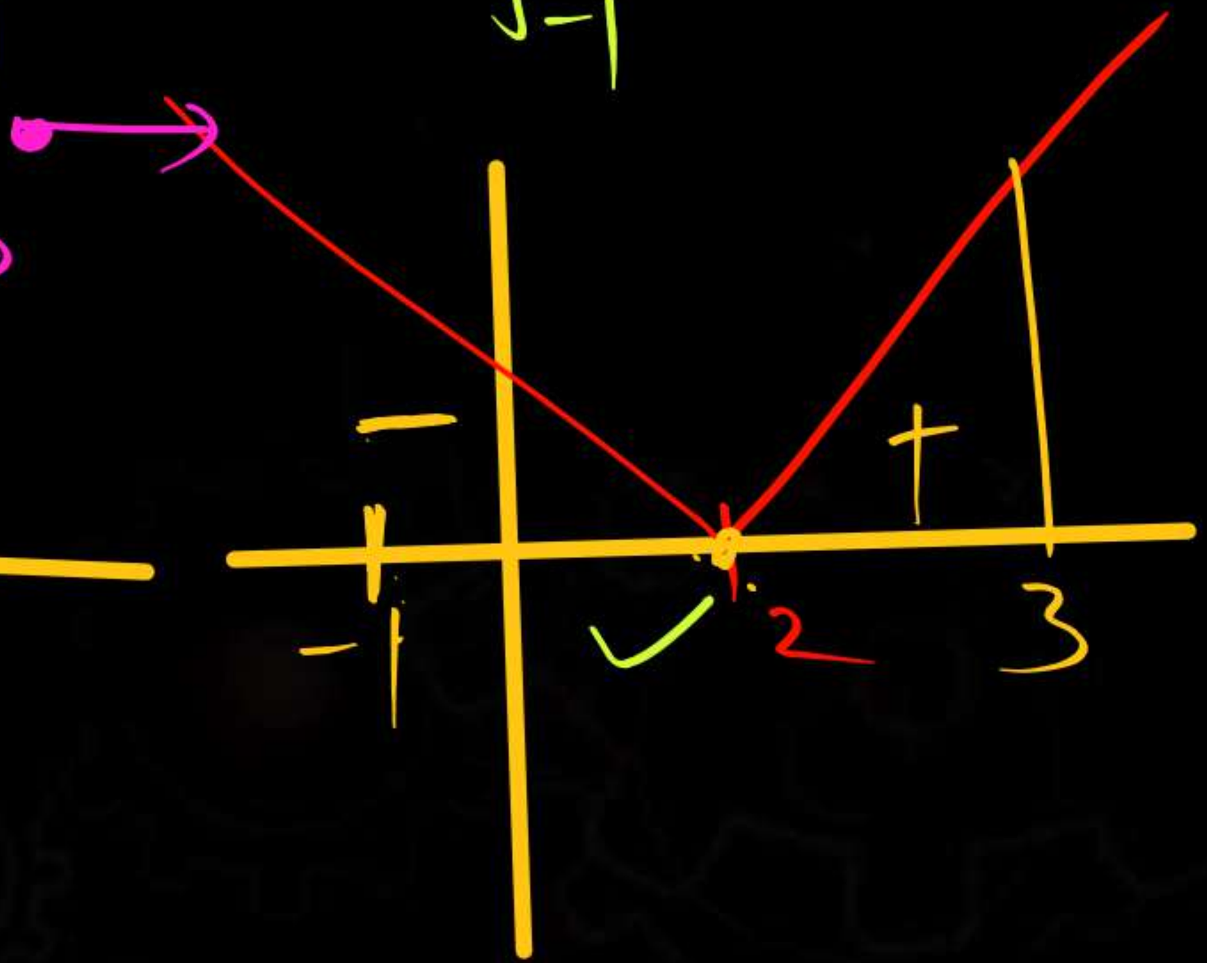
Q15. The value of $\int_{-1}^{3}(|x-2|+[x])dx$ is equal to (where [*] denotes greatest integer function)

$$\Rightarrow \int_{-1}^{3}|x-2|\,dx+\int_{-1}^{3}[x]\,dx$$

$y=[x]$

Input = Integer






(a) 7

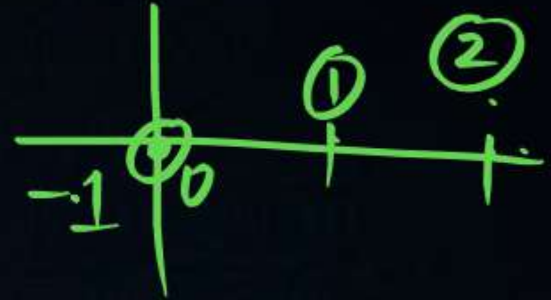
(b) 5

(c) 4

(d) 3

$$\int_{-1}^{3} |x-2|\,dx + \int_{-1}^{3} [x]\,dx$$

$[-1] = -2$




$$\Rightarrow \int_{-1}^{2} -(x-2)\,dx + \int_{2}^{3} +(x-2)\,dx + \int_{-1}^{0} -1\,dx + \int_{0}^{1} 0\,dx + \int_{1}^{2} 1\,dx$$

$$+ \int_{2}^{3} 2\,dx$$

$$= 7$$

Q17. $\int_{\log \pi - \log 2}^{\log \pi} \frac{e^x}{1-\cos\left(\frac{2}{3}e^x\right)} dx$ is equal to

(a) $\sqrt{3}$

(b) $-\sqrt{3}$

(c) $1/\sqrt{3}$

(d) $-\frac{1}{\sqrt{3}}$

$$\int_{\log \pi - \log 2}^{\log \pi} \frac{e^x \, dx}{1-\cos\left(\frac{2}{3}e^x\right)}$$

$$\frac{2}{3}e^x = t$$

$$e^x dx = \frac{3}{2} dt$$

$$\frac{2}{3} e^{\log\left(\frac{\pi}{2}\right)} = t$$

$$\frac{2}{3} \times \frac{\pi}{2} = t$$

$$t = \frac{\pi}{3}$$

$$t = \pi/3$$

$$\frac{2}{3} e^{\log \pi} = t$$

$$t = \frac{2\pi}{3}$$

$$= \frac{3}{2}\int_{\pi/3}^{2\pi/3} \frac{dt}{1-\cos t}$$

$$\Rightarrow \sqrt{3}$$

Q16. If $\int_{-1}^{3/2} |x \sin \pi x|\, dx = \frac{k}{\pi^2}$, then the value of k is

(a) $3\pi + 1$

(b) $2\pi + 1$

(c) $1$

(d) $4$

$f(x) = |x \sin \pi x|$

$f(-1) = |-1 \sin(-\pi)|$

$= 0$

$= \int_{-1}^{1} x \sin \pi x + \int_{1}^{3/2} -x \sin \pi x \, dx$

$= \frac{3\pi + 1}{\pi^2} = \frac{k}{\pi^2}$

$\boxed{k = 3\pi + 1}$

Q18. If $I_1 = \int_e^{e^2} \frac{dx}{\ln x}$ and $I_2 = \int_1^2 \frac{e^x}{x} dx$, then

Dummy variable

(a) $I_1 = I_2$

(b) $2 I_1 = I_2$

(c) $I_1 = 2 I_2$

(d) None of these

$\boxed{I_1 = I_2}$

$I_1 = \int_e^{e^2} \frac{dx}{\ln x}$

$\ln x = t$

$x = e^t$

$dx = e^t dt$

$e = e^t$

$t = 1$

$e^2 = e^t$

$t = 2$

$\Rightarrow \boxed{\int_1^2 \frac{e^t}{t} dt}$

$= I_2$

Q19. $\int_{2-\log 3}^{3+\log 3} \frac{\log(4+x)}{\log(4+x)+\log(9-x)} dx \Rightarrow \int_a^b \frac{f(x)}{f(a+b-x)+f(x)}$

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

$$= \frac{(b-a)}{2}$$

(a) Cannot be evaluated

(b) Is equal to 5/2

(c) is equal to 1+2 log 3

$\boxed{\text{(d) Is equal to ½ + log 3}}$

$$= \frac{3+\log 3-(2-\log 3)}{2}$$

$$= \frac{1+2\log 3}{2}$$

Q20. $\int_0^{\infty}[2e^{-x}]dx$ is equal to

(where [*] denotes the greatest integer function)

(a) 0

(b) ln 2

(c) $e^2$

(d) $2e^{-1}$

$\int_a^0 \longrightarrow -\int_0^a$

$2e^{-x} = t$

$2(-e^{-x})\,dx = dt$

$dx = -\dfrac{dt}{\boxed{2e^{-x}}} \to t$

$I = \int_0^{\infty}[2e^{-x}]\,dx \Rightarrow -\int_2^0 [t]\cdot\dfrac{dt}{2e^{-x}}$

Input = Integer

$\Rightarrow \int_0^2 \dfrac{[t]}{t}\,dt$

$2\times e^0 = t$

$2 = t$

$\Rightarrow \int_0^1 \dfrac{[t]}{t}\,dt + \int_1^2 \dfrac{[t]}{t}\,dt = \int_0^1 \dfrac{0}{t}\,dt + \int_1^2 \dfrac{1}{t}\,dt$

$2\times e^{-\infty} = t$

$t = 0$

$= [\ln t]_1^2 = \boxed{\ln 2 - \ln 1}$

$= \ln 2 - 0 = \ln 2$

# Thank You!

GW Soldiers