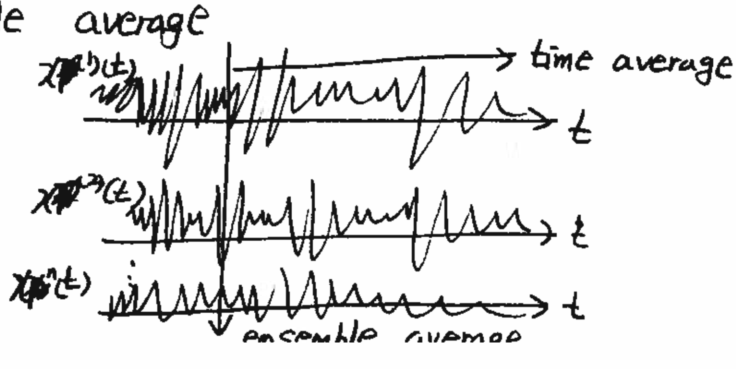
Turbulent Eddies. <u>Fine scale turbulent mixing noise</u>

Eddies in a non-uniformly moving stream

The fluid motion in a turbulent flow is generally chaotic, but nevertheless in some regions the flow is correlated. An eddy can be defined as a region in which flow conditions are relatively similar. Extremities of the eddy are points separated by more than the correlation scale. Within the eddy flow quantities are well correlated; they are uncorrelated over distances larger than the eddy dimension or, equivallently, the correlation scale. This usage of the term 'eddy' concerns a stochastic motion in which only statistical measures are significant. The eddy dimension indicates only the scale over which the stochastic variable is correlated.

- Stochastic ~~proce~~ analysis.

One cannot argue a single event at a certain time or position; one can ~~on~~ only discuss the averaged quantity of a single system over a certain time (or space) interval or the averaged quantity of many identical systems at a certain time instance (or spatial position). The former is called time (or space) average and the latter ensemble average



- Time average

$$\overline{x^{(i)}(t)} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x^{(i)}(t)\,dt \quad : \quad \text{mean}$$

$$\overline{x^{(i)}(t)^2} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} [x^{(i)}(t)]^2\,dt \quad : \quad \text{mean square}$$

$$\phi_x^{(i)}(\tau) = \overline{x^{(i)}(t)\,x^{(i)}(t+\tau)} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x^{(i)}(t)\,x^{(i)}(t+\tau)\,dt$$

: auto correlation

- Ensemble average

$$\langle x(t_1)\rangle = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} x^{(i)}(t_1) = \int_{-\infty}^{\infty} x_1 P_1(x_1, t_1)\,dx_1 \quad : \text{mean}$$

$$\langle x(t_1)^2\rangle = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} [x^{(i)}(t_1)]^2 = \int_{-\infty}^{\infty} x_1^2 P_1(x_1, t_1)\,dx_1 \quad : \text{mean square}$$

$$\langle x(t_1)x(t_2)\rangle = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} x^{(i)}(t_1)x^{(i)}(t_2)$$

$$= \int_{-\infty}^{\infty} x_1 x_2\, P_2(x_1, x_2; t_1, t_2)\,dx_1 dx_2 \quad : \text{covariance}$$

where $x_1 = x(t_1)$, $x_2 = x(t_2)$, $P_1(x_1, t_1)$ is the first-order probability density function (PDF), and $P_2(x_1, x_2; t_1, t_2)$ is the second-order joint probability density function.

- An ensemble average is a convenient theoretical concept since it is directly related to the probability density functions, which can be generally obtained by the theoretical analysis of a given physical system. On the other hand, a time-average is more directly related to real experiments.

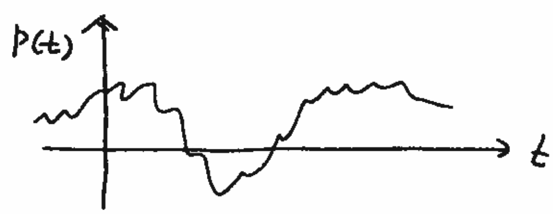
Theoretical predictions based on ensemble averaging are equivalent to experimental measurement results corresponding to time averaging when and only when, the system is a so-called "ergonic ensemble."

It is often said that ensemble averaging and time averaging are identical for a "statistically-stationary" system, but are different for a statistically-nonstationary system.

If a process is ergonic, any one sample function represents the entire process.

Then,

$$\overline{x(t)} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,dt = \langle x(t) \rangle$$



p(t), a random signal

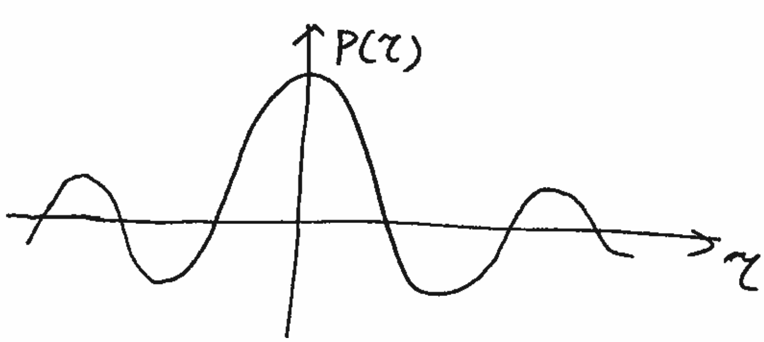
- Autocorrelation

$$P(\tau) = \overline{p(t)\,p(t+\tau)}$$

$$P(\tau) \underset{\tau\to\infty}{\sim} 0$$

$$P(0) = \overline{p^2}$$

$$P(\tau) = P(-\tau)$$



The Fourier transform of the autocorrelation function is called the "power spectral density"

$$\hat{P}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P(\tau)\, e^{i\omega\tau}\, d\tau$$

$$= \frac{1}{\pi}\int_{0}^{\infty} P(\tau)\cos\omega\tau\, d\tau \quad \text{since } P(\tau) \text{ is even.}$$

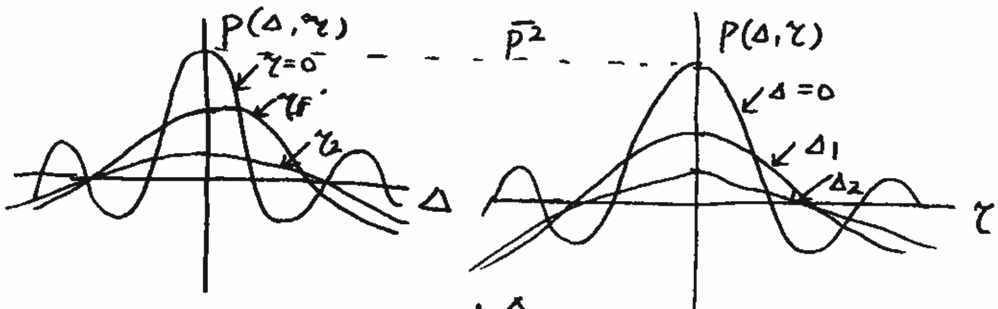
- Cross correlation

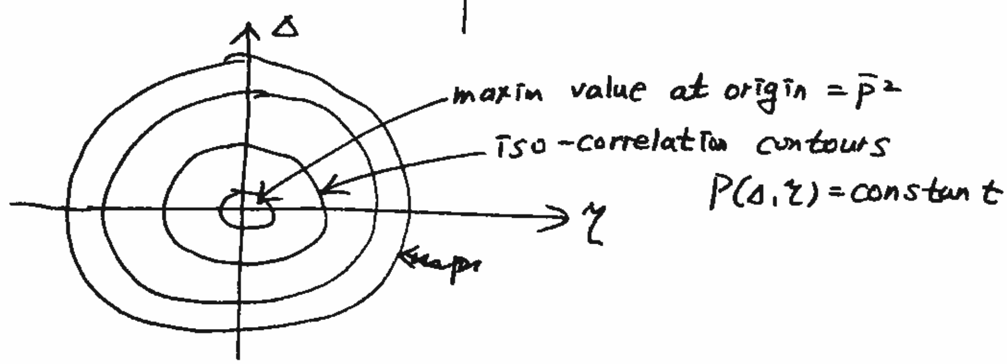
$$P(\Delta, \tau) = \overline{p(x,t)\, p(x+\Delta, t+\tau)}$$

This is a function only of the space separation Δ and time delay τ.

$$P(0,0) = \overline{p^2}$$

$$P(\Delta, \tau) = P(-\Delta, -\tau)$$





The Fourier transform of the cross correlation function is defined as the 'cross power spectral density'

$$\hat{P}(k,\omega) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(\Delta,\tau)\, e^{ik\Delta}\cdot e^{i\omega\tau}\, d\Delta\, d\tau$$

$$= \hat{P}(-k, -\omega)$$

Even though p is a chaotically varying function, there are nevertheless various ways in which scales that characterise the signal can be defined

Scales known as integral scales are particularly important.

The integral time scale L_τ is defined from the autocorrelation function

$$L_\tau = \int_{-\infty}^{\infty} \frac{P(\tau)}{P(0)} \, d\tau$$

and the integral length scale L_x is similarly defined from the space correlation $B(\Delta)$

$$L_x = \int_{-\infty}^{\infty} \frac{B(\Delta)}{B(0)} \, d\Delta$$

where $B(\Delta) = \overline{p(x,t)\, p(x+\Delta, t)} = P(\Delta, 0)$

Application of CFD to Jet Noise Predictions

There is a need to relate turbulent flow predictions using Computational Fluid Dynamics (CFD) to the radiated noise. The most commonly used approaches are based on solution to the Reynolds averaged Navier Stokes (RANS) equations. Often a two-equation turbulence model, such as the $k-\varepsilon$ model is used. In the following sections we will concentrate on noise radiation at $90°$ to the jet axis since, in this direction, the effects of source convection and mean flow refraction are minimized. First we consider methods that combine the Lighthill Acoustic Analogy with a CFD RANS solution.

- Methods based on the Lighthill acoustic analogy

We showed that

$$\rho'(\vec{x},t) = \frac{1}{4\pi c_0^2 x} \iiint_{V(\vec{y})} \frac{\partial^2}{\partial x_i \partial x_j} T_{ij}\left(\vec{y}, t - \frac{|\vec{x}-\vec{y}|}{c_0}\right) d\vec{y}$$

For a compact source $\underset{|\vec{x}-\vec{y}| \simeq |\vec{x}|}{m \ll 1,}$ let $\phi = t - x/c_0$ then

$$\frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left\{ \frac{\partial T_{ij}}{\partial t} \cdot \underset{(1)}{\frac{\partial t}{\partial \phi}} \cdot \underset{\left(-\frac{1}{c_0}\right)}{\frac{\partial \phi}{\partial x}} \cdot \underset{\left(\frac{x_j}{x}\right)}{\frac{\partial x}{\partial x_j}} \right\} = -\frac{1}{c_0}\frac{\partial}{\partial x_i}\left\{ \frac{x_j}{x}\frac{\partial T_{ij}}{\partial t} \right\}$$

$$= -\frac{1}{c_0}\underbrace{\frac{\partial}{\partial x_i}\left(\frac{x_j}{x}\right)}_{O(1/x)\ \text{near field}}\frac{\partial T_{ij}}{\partial t} - \frac{1}{c_0}\frac{x_j}{x}\underbrace{\frac{\partial}{\partial x_i}\left(\frac{\partial T_{ij}}{\partial t}\right)}$$

$$\hookrightarrow \frac{\partial}{\partial t}\left(\frac{\partial T_{ij}}{\partial x_i}\right) = \frac{\partial}{\partial t}\left\{-\frac{1}{c_0}\frac{x_i}{x}\frac{\partial T_{ij}}{\partial t}\right\}$$

O(1) far field

And
$$\frac{\partial}{\partial x_i}\left(\frac{x_j}{x}\right) = \frac{\partial x_j}{\partial x_i}\frac{1}{x} - \frac{x_j}{x^2}\frac{\partial x}{\partial x_i} = \frac{\delta_{ij}}{x} - \frac{x_j}{x^2}\frac{x_i}{x} = \frac{1}{x}\left\{\delta_{ij} - \frac{x_i x_j}{x^2}\right\}$$

$\rightarrow 0$ as $x \rightarrow \infty$

Thus, as $x \to \infty$

$$\rho'(\vec{x},t) = \frac{1}{4\pi c_0^4 x} \underbrace{\beta_i \beta_j}_{\downarrow \; x_i x_j / x^2} \iiint_{V(\vec{y})} \frac{\partial^2 T_{ij}}{\partial t^2}\left(\vec{y},\, t - \frac{|\vec{x}-\vec{y}|}{c_0}\right) d\vec{y}$$

$$T_{ij} \frac{x_i x_j}{x^2} = T_{rr}$$

where T_{rr} represents the component of the Lighthill stress tensor in the direction of the observer. Clearly, for noise radiation to 90°, this is related to the radial velocity fluctuations in a round jet. Thus we may write

$$\rho'(\vec{x},t) = \frac{1}{4\pi c_0^4} \frac{1}{x} \int \frac{\partial^2}{\partial t^2} T_{rr}\left(\vec{y},\, \frac{|\vec{x}-\vec{y}|}{c_0}\right) d\vec{y}$$

To obtain the far-field spectrum for the intensity it is necessary to form the auto-correlation function for the pressure

$$\bar{I} = \frac{\overline{p'(\vec{x},t)\, p'(\vec{x},t+\tau)}}{\rho_0 c_0} \qquad I_{av} = \frac{p_{av}^2}{\rho c}$$

where the overbar represents a time average. Now, since in the far-field $p' = c_0^2 \rho'$, we obtain

$$\bar{I}(\vec{x},\tau) = \frac{1}{16\pi^2 \rho_0 c_0^5 x^2} \int_{V(\vec{y}_1)} \int_{V(\vec{y}_2)} \overline{\frac{\partial^2 T_{rr}}{\partial t^2}(\vec{y}_1,t_1)\, \frac{\partial^2 T_{rr}}{\partial t^2}(\vec{y}_2,t_2)}\, d\vec{y}_1 d\vec{y}_2$$

where $t_1 = t - \dfrac{|\vec{x}-\vec{y}_1|}{c_0}$, $t_2 = t + \tau - \dfrac{|\vec{x}-\vec{y}_2|}{c_0}$

Now, if the turbulence statistics are stationary, it is readily shown that

$$\overline{\frac{\partial^2 T_{rr}}{\partial t^2}(\vec{y}_1, t_1)\,\frac{\partial^2 T_{rr}}{\partial t^2}(\bar{y}_2, t_2)} = \frac{\partial^4}{\partial \tau^4}\,\overline{T_{rr}(\bar{y}_1, t_1)\,T_{rr}(\bar{y}_2, t_2)}$$

Proof. Let $A = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \underbrace{\frac{\partial^2 T_{rr}}{\partial t^2}(\bar{y}_1, t_1)}_{T_{rr}'}\,\underbrace{\frac{\partial^2 T_{rr}}{\partial t^2}(\bar{y}_2, t_2)}_{T_{rr}''}\,dt$

$t_1 = t - \frac{|\bar{x}-\bar{y}_1|}{C_0}$, $\quad t_2 = t + \tau - |\bar{x}-\bar{y}_2|/C_0$

$$\frac{\partial}{\partial t}[f(t_2)] = \frac{\partial f}{\partial \tau}\cdot\underbrace{\frac{\partial \tau}{\partial t_2}}_{1}\cdot\underbrace{\frac{\partial t_2}{\partial t}}_{1}$$

$$\frac{\partial^2 T_{rr}''}{\partial t^2} = \frac{\partial^2 T_{rr}''}{\partial \tau^2}$$

$$A = \lim_{T\to\infty} \frac{1}{2T}\frac{\partial^2}{\partial \tau^2}\int_{-T}^{T} \frac{\partial^2 T_{rr}'}{\partial t^2}\,T_{rr}''\,dt$$

$$= \lim_{T\to\infty} \frac{1}{2T}\frac{\partial^2}{\partial \tau^2}\Bigg[\underbrace{\frac{\partial T_{rr}'}{\partial t}\,T_{rr}''\Bigg|_{-T}^{T}}_{0} - \int_{-T}^{T} \frac{\partial T_{rr}'}{\partial t}\cdot\underbrace{\frac{\partial T_{rr}''}{\partial t}}_{\frac{\partial T_{rr}''}{\partial \tau}}\,dt\Bigg]$$

0. boundary (stationary)

Then, $A = -\lim_{T\to\infty} \frac{1}{2T}\frac{\partial^3}{\partial \tau^3}\int_{-T}^{T} \frac{\partial T_{rr}'}{\partial t}\,T_{rr}''\,dt$

$$= -\lim_{T\to\infty} \frac{1}{2T}\frac{\partial^3}{\partial \tau^3}\Bigg[\underbrace{T_{rr}'\,T_{rr}''\Bigg|_{-T}^{T}}_{0} - \int_{-T}^{T} T_{rr}'\,\underbrace{\frac{\partial T_{rr}''}{\partial t}}_{\frac{\partial T_{rr}''}{\partial \tau}}\,dt\Bigg]$$

$$= \lim_{T\to\infty} \frac{1}{2T}\frac{\partial^4}{\partial \tau^4}\int_{-T}^{T} T_{rr}'\,T_{rr}''\,dt = \frac{\partial^4}{\partial \tau^4}\,\overline{T_{rr}'\,T_{rr}''}$$

we used $\int f(t_1)\frac{\partial g(t_2)}{\partial t}\,dt = \frac{\partial}{\partial \tau}\int f(t_1)\,g(t_2)\,dt \qquad t_1 = t,\; t_2 = t+\tau$

Now the cross-correlation of a stationary process is independent of absolute time translations, so that add $|\bar{x}-\bar{y}_1|/c_0$

$$\overline{T_{rr}(\bar{y},t_1)\, T_{rr}(\bar{y}_2,t_2)} = \overline{T_{rr}(\bar{y}_1,t)\, T_{rr}\left(\bar{y}_2,\, t+\tau+\frac{|\bar{x}-\bar{y}_1|-|\bar{x}-\bar{y}_2|}{c_0}\right)}$$

Now $\quad |\bar{x}-\bar{y}| = x - \frac{\bar{x}}{x}\cdot\bar{y} + O(x^{-1})$

so that $\quad \dfrac{|\bar{x}-\bar{y}_1|-|\bar{x}-\bar{y}_2|}{c_0} \simeq \dfrac{1}{c_0}\left[\left(\bar{x}-\frac{\bar{x}}{x}\cdot\bar{y}_1\right)-\left(\bar{x}-\frac{\bar{x}}{x}\cdot\bar{y}_2\right)\right]$

$$= \frac{\bar{x}}{x}\cdot\frac{(\bar{y}_2-\bar{y}_1)}{c_0}$$

Thus,

$$\bar{I}(\bar{x},\tau) = \frac{1}{16\pi^2\rho_0 c_0^5 x^2}\frac{\partial^4}{\partial\tau^4}\iint_{V(\bar{y}_1)V(\bar{y}_2)} \overline{T_{rr}(\vec{y}_1,t)\, T_{rr}(\vec{y}_2,\tau_0)}\, d\bar{y}_1\, d\bar{y}_2$$

where $\quad \tau_0 = t+\tau+\dfrac{\bar{x}}{x}\dfrac{(\bar{y}_2-\bar{y}_1)}{c_0}$

We can define the two point cross-correlation as

$$R(\vec{y}_1,\vec{\eta},\tau') = \overline{T_{rr}(\bar{y}_1,t)\, T_{rr}(\bar{y}_2,t+\tau')} \qquad \bar{y}_2 = \bar{y}_1+\bar{\eta}$$

where $\bar{\eta} = \bar{y}_2-\bar{y}_1$. Then

$$\bar{I}(\bar{x},\tau) = \frac{1}{16\pi^2\rho_0 c_0^5 x^2}\frac{\partial^4}{\partial\tau^4}\iint_{V(\bar{y}_1)V(\bar{y}_2)} R\left(\vec{y}_1,\vec{\eta},\,\tau+\frac{\vec{x}}{x}\cdot\frac{\vec{\eta}}{c_0}\right) d\vec{y}_1\, d\vec{\eta}$$

Now, the spectral density is given by

$$S(\vec{x},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{I}(\bar{x},\tau)\, e^{i\omega\tau}\, d\tau$$

So that

$$S(\vec{x},\omega) = \frac{1}{32\pi^3 \rho_0 C_0^5 x^2} \int_{-\infty}^{\infty} \left[\frac{-\partial^4}{\partial \tau^4} \iint_{V(\vec{y}_1)\, V(\vec{y}_2)} R\left(\vec{y}, \vec{\eta}, \tau + \frac{\vec{x}}{x} \cdot \frac{\vec{\eta}}{C_0}\right) d\vec{y}_1 d\vec{\eta} \right] e^{i\omega\tau} d\tau$$

$$= \frac{\omega^4}{32\pi^3 \rho_0 C_0^5 x^2} \int_{-\infty}^{\infty} \iint_{V(\vec{y}_1)\, V(\vec{y}_2)} R\left(\vec{y}_1, \vec{\eta}, \tau + \frac{\vec{x}}{x} \cdot \frac{\vec{\eta}}{C_0}\right) d\vec{y}_1 d\vec{\eta}\, e^{i\omega\tau} d\tau$$

we used $\int_{-\infty}^{\infty} \frac{\partial^4 f(\tau)}{\partial \tau^4} e^{i\omega\tau} d\tau$

$$= \underbrace{\frac{\partial^3 f}{\partial \tau^3} e^{i\omega\tau} \Big|_{-\infty}^{\infty}}_{\to 0} - i\omega \int_{-\infty}^{\infty} \frac{\partial^3 f}{\partial \tau^3} e^{i\omega\tau} d\tau$$

Now $\int_{-\infty}^{\infty} f(\tau + b) e^{i\omega\tau} d\tau \cancel{=} \cancel{\int_{-\infty}^{\infty} f(\tau) e^{i\omega(\tau - b)} d\tau}$

$(\tau^* = \tau + b,\ d\tau = d\tau^*,\ \tau = \tau^* - b)$

$$\int_{-\infty}^{\infty} f(\tau^*) e^{i\omega(\tau^* - b)} d\tau^* = \int_{-\infty}^{\infty} f(\tau) e^{i\omega(\tau - b)} d\tau.$$

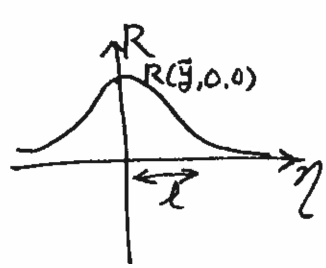
So that

$$S(\vec{x},\omega) = \frac{\omega^4}{32\pi^3 \rho_0 C_0^5 x^2} \int_{-\infty}^{\infty} \iint_{V(\vec{y}_1)\, V(\vec{y}_2)} R(\vec{y}_1, \vec{\eta}, \tau)\, e^{i\omega\left[\tau - \frac{\vec{x}}{x} \cdot \frac{\vec{\eta}}{C_0}\right]} d\vec{y}_1 d\vec{y}_2 d\tau$$

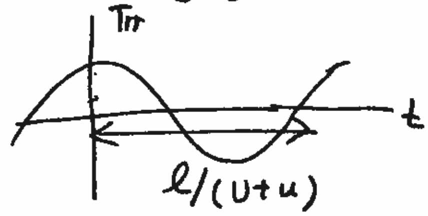
Now, for simplicity, since we are neglecting the effects of the mean flow directly ($\theta = 90°$), it will be assumed that the contributions to the two point cross-correlation are only velocity fluctuations and we neglect the mean components. $(\overline{V_0^2 + 2V_0 V' + V'V'}) = \overline{V'^2}$

Then, we introduce a length scale (l) such that

$$\frac{R(\vec{y}_1, \vec{\eta}, 0)}{R(\vec{y}, 0, 0)} \simeq 0 \quad \text{for } |\vec{\eta}| > l.$$



In writing down the far-field form of solution, we assumed that the source could be treated as compact. However, if we remain in a fixed reference frame, this is not the case, since in such a frame the eddies are changing with a frequency $O([U+u]/l)$.



compact. $\omega \sim \frac{u}{l} \sim O(1)$

non-compact $\omega \sim \frac{U+u}{l} \gg 1$

So we want to move with the eddies where the rates of changes are much slower: that is, the sources may be treated as compact.

Let

$$\vec{\xi} = \vec{\eta} - \vec{i}\, c_0 M_c \tau$$

where, $\vec{i}$ is a unit vector in the direction of the mean flow and M_c is the convection speed of the turbulent eddies. This gives

$$\tau - \frac{\vec{x}}{x}\cdot\frac{\vec{\eta}}{c_0} = \tau - \frac{\vec{x}}{x c_0}\left(\vec{\xi} + \vec{i}\, c_0 M_c \tau\right)$$

$$= \tau - \frac{\vec{x}\cdot\vec{\xi}}{x c_0} - \vec{i}\cdot\frac{\vec{x}\, c_0}{x c_0} M_c \tau$$

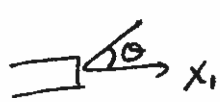
$$= \tau\left(1 - \frac{x_1}{x} M_c\right) - \frac{\vec{x}\cdot\vec{\xi}}{x c_0}$$

$$= \tau\left(1 - M_c \cos\theta\right) - \frac{\vec{x}\cdot\vec{\xi}}{x c_0}$$

$$S(\vec{x},\omega) = \frac{\omega^4}{32\pi^3 \rho_0 c_0^5 x^2} \int_{-\infty}^{\infty} \iint_{V(\vec{y}_1)\, V(\vec{\xi})} \exp\left\{ i\omega \left[(1 - M_c\cos\theta)\tau - \frac{\vec{x}}{x}\cdot\frac{\vec{\xi}}{c_0} \right] \right\} \times$$

$$R(\vec{y}_1, \vec{\xi}, \tau)\, d\vec{\xi}\, d\vec{y}_1\, d\tau$$

with $x_1/x = \cos\theta$



Now we introduce the wavenumber/frequency spectrum of the turbulence given by

$$H(\vec{y}_1, \vec{K}, \vec{\omega}) = \frac{1}{(2\pi)^4} \int_{V(\vec{\xi})} \int_{-\infty}^{\infty} e^{i(\omega\tau - \vec{K}\cdot\vec{\xi})} R(\vec{y}_1, \vec{\xi}, \tau)\, d\vec{\xi}\, d\tau$$

so that

$$R(\vec{y}_1, \vec{\xi}, \tau) = \int_{\vec{K}^*}\int_{\omega^*} H(\vec{y}_1, \vec{K}^*, \omega^*)\, e^{-i(\omega^*\tau - \vec{K}^*\cdot\vec{\xi})}\, d\vec{K}^* \cdot d\omega^*$$

This describes the spatial and temporal periodicity of the source.

Then,

$$S(\vec{X}, \omega) = \frac{\omega^4}{32\pi^3 \rho_0 C_0^5 \chi^2} \int_{-\infty}^{\infty} \int_{V(\vec{y}_1)} \int_{V(\vec{\xi})} \exp\left\{ i\omega \left[(1 - M_c\cos\theta)\tau - \frac{\vec{X}}{X} \cdot \frac{\vec{\xi}}{C_0} \right] \right\} \times$$

$$\int_{\vec{K}^*}\int_{\omega^*} H(\vec{y}_1, \vec{K}^*, \omega^*)\, e^{-i(\omega^*\tau - \vec{K}^*\cdot\vec{\xi})}\, d\vec{K}^*\, d\omega^*\, d\vec{\xi}\, d\vec{y}_1\, d\tau.$$

Now we use $\int_{-\infty}^{\infty} e^{\pm ixy}\, dy = 2\pi\delta(x)$

$$\int_{-\infty}^{\infty} \exp\{ i[\omega(1 - M_c\cos\theta) - \omega^*]\tau \} d\tau = 2\pi\delta[\omega(1 - M_c\cos\theta) - \omega^*]$$

$$\int_{V(\vec{\xi})} \exp\left\{ -i\left[\frac{\omega\vec{X}}{XC_0} - \vec{K}^* \right]\vec{\xi} \right\} d\xi = (2\pi)^3\, \delta\left[\frac{\omega\vec{X}}{XC_0} - \vec{K}^* \right]$$

So that

$$S(\vec{X}, \omega) = \frac{\pi\omega^4}{2\rho_0 C_0^5} \frac{1}{\chi^2} \int_{V(\vec{y}_1)} H\left[\vec{y}_1, \frac{\omega\vec{X}}{XC_0}, \omega(1 - M_c\cos\theta) \right] d\vec{y}_1$$

This show that the far field noise depends on the wavenumber frequency spectrum with a wavenumber that gives a sonic velocity to a far field observer and a Doppler shifted frequency.

Now it is necessary to introduce a model for the two point cross correlation. It is assumed that, in the moving frame of reference, the correlation takes on a Gaussian form

$$R(\vec{y}_1, \vec{\xi}, \tau) = \rho_s^2 u_s^4 \exp\left[-\frac{|\vec{\xi}|^2}{\ell^2} - \omega_s^2 \tau^2\right]. \qquad T_{rr} \sim \rho u^2.$$

where ℓ is the characteristic length scale and ω_s is a characteristic frequency. ρ_s and u_s are density and velocity scales that characterize the turbulence. Here, it assumed that the characteristic length scale is the same in all directions. This restriction can be easily relaxed. It is then possible to determine the wavenumber frequency spectrum

Recall that $\displaystyle\int_{-\infty}^{\infty} e^{-p^2 x^2 \pm q x}\, dx = \frac{\sqrt{\pi}}{p} \exp\left(\frac{q^2}{4p^2}\right)$

Then. $\displaystyle\int_{-\infty}^{\infty} e^{i\omega\tau} e^{-\omega_s^2 \tau^2}\, d\tau = \frac{\sqrt{\pi}}{\omega_s} \exp\left\{-\frac{\omega^2}{4\omega_s^2}\right\}$

and $\displaystyle\int_{-\infty}^{\infty} e^{-ik_1 \xi_1} e^{-\xi_1^2/\ell^2}\, d\xi_1 = \sqrt{\pi}\,\ell \exp\left\{-\frac{k_1 \ell^2}{4}\right\}$

$$\int_{V(\vec{\xi})} = (\sqrt{\pi}\,\ell)^3 \ldots$$

$$\sqrt{k_1^2 + k_2^2 + k_3^2}$$

Thus $\displaystyle H(\vec{y}_1, \vec{k}, \omega) = \frac{1}{(2\pi)^4} \rho_s^2 u_s^4 \frac{\pi^2 \ell^3}{\omega_s} \exp\left\{-\frac{(k\ell)^2}{4} - \frac{\omega^2}{4\omega_s^2}\right\}$

and $H\left\{\vec{y}_1, \frac{\omega \vec{x}}{x c_0}, \omega(1 - M_c \cos\theta)\right\}$

$$= \frac{1}{(2\pi)^4} \rho_s^2 u_s^4 \frac{\pi^2 \ell^3}{\omega_s} \exp\left\{-\frac{\omega^2 (1 - M_c \cos\theta)^2}{4\omega_s^2}\right\} \exp\left\{-\left(\frac{\omega \vec{x}}{x c_0} \cdot \ell\right)^2 / 4\right\}$$

$$k\ell = \frac{\omega \vec{x}}{x c_0} \cdot \ell \simeq \frac{\vec{x}}{x} \cdot \frac{\omega \cdot \ell}{c_0} \simeq \frac{u_s}{c_0} \sim m \ll 1$$

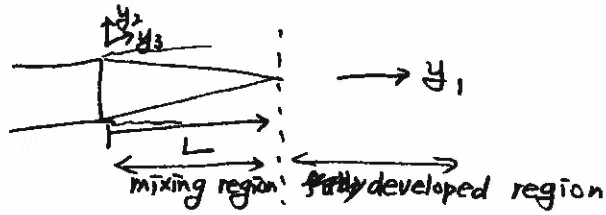
Thus $S(\vec{x},\omega) = \dfrac{\pi\omega^4}{2\rho_0 C_0^5 x^2}\dfrac{1}{16\pi^2}\displaystyle\int_{V(\vec{y}_1)} \dfrac{\rho_s^2 u_s^4 \ell^3}{\omega_s} \exp\left\{-\dfrac{\omega^2(1-M_c\cos\theta)^2}{4\omega_s^2}\right\} d\vec{y}_1$

$$= \frac{1}{32\pi\rho_0 C_0^5 x^2}\int_{V(\vec{y}_1)} \rho_s^2 u_s^4 \ell^3 \omega_s^3 \left(\frac{\omega}{\omega_s}\right)^4 \exp\left\{-\frac{\omega^2(1-M_c\cos\theta)^2}{4\omega_s^2}\right\} d\vec{y}_1$$

Since we are concentrating on the radiated noise at $90°$ to the jet axis where $\theta = \frac{\pi}{2}$, we obtain finally

$$S(\vec{x},\omega) = \frac{1}{32\pi\rho_0 C_0^5 x^2}\int_V \rho_s^2 u_s^4 \ell^3 \omega_s^3 \left(\frac{\omega}{\omega_s}\right)^4 \exp\left\{-\frac{\omega^2}{4\omega_s^2}\right\} d\vec{y}_1$$

- To make some estimates of the noise radiation from different regions of the jet, it will be assumed that the turbulence properties are unifor across each axial station. This assumption is removed when actual CFD data are used.



Then $$\iint dy_2\,dy_3 \sim \begin{cases} \pi D_j \ell & \text{for } 0 < y_1 < L \\ \pi \ell^2 & \text{for } L < y_1 < \infty \end{cases}$$

where, L is the length of the potential core.

Then.
$$S(\vec{x},\omega) = \frac{1}{32\pi\rho_0 C_0^5 x^2}\left\{\int_0^L \rho_s^2 u_s^4 \ell^3 \omega_s^3\, \pi D_j \ell \left(\frac{\omega}{\omega_s}\right)^4 \exp\left\{-\frac{\omega^2}{4\omega_s^2}\right\} dy_1 \right.$$
$$\left. + \int_L^\infty \rho_s^2 u_s^4 \ell^3 \omega_s^3\, \pi \ell^2 \left(\frac{\omega}{\omega_s}\right)^4 \exp\left\{-\frac{\omega^2}{4\omega_s^2}\right\} dy_1 \right\}$$

$$= \frac{1}{32\rho_0 C_0^5 x^2}\left\{\int_0^L \rho_s^2 u_s^4 \ell^4 \omega_s^3 D_j \left(\frac{\omega}{\omega_s}\right)^4 \exp\left\{-\frac{\omega^2}{4\omega_s^2}\right\} dy_1 \right.$$
$$\left. + \int_L^\infty \rho_s^2 u_s^4 \ell^5 \omega_s^3 \left(\frac{\omega}{\omega_s}\right)^4 \exp\left\{-\frac{\omega^2}{4\omega_s^2}\right\} dy_1 \right\}$$

Now, $\int_0^\infty x^{2n} e^{-px^2} dx = \frac{(2n-1)!!}{2(2p)^n}\sqrt{\frac{\pi}{p}}$

where $(2n-1)!! = 1\cdot 3 \cdots (2n-1)$

Thus $\int_0^\infty \left(\frac{\omega}{\omega_s}\right)^4 \exp\left\{-\frac{\omega^2}{4\omega_s^2}\right\} d\omega$ → integrating across all frequencies to get overall power.

$\left(n=2,\ x=\omega,\ p=\frac{1}{4\omega_s^2}\right)$

$$= \frac{1}{\omega_s^4}\cdot\frac{1\cdot 3}{2\cdot\left(\frac{1}{2\omega_s^2}\right)^2}\sqrt{\pi\cdot 4\omega_s^2}$$

$$= 12\sqrt{\pi}\,\omega_s$$

Then $S(\vec{x}) = \frac{3\sqrt{\pi}}{8\rho_0 c_0^5 x^2}\left\{\int_0^L \rho_s^2 u_s^4 \ell^4 \omega_s^4 D_j\, dy_1 + \int_L^\infty \rho_s^2 u_s^4 \ell^5 \omega_s^4\, dy_1\right\}$

Now in the annular mixing region of the jet, $u_s \sim U_j$.

So since $\omega_s \ell \sim u_s$, then $\omega_s \ell \sim U_j$. In the developed region of the jet $u_s \sim U_j \frac{L}{y_1}$ and $\ell \sim \ell(L)\frac{y_1}{L}$.

Thus

$$S(\vec{x}) \sim \frac{3\sqrt{\pi}}{8\rho_0 c_0^5 x^2}\left\{\int_0^L \rho_s^2 U_j^8 D_j\, dy_1 + \int_L^\infty \rho_s^2\left(\frac{U_j L}{y_1}\right)^8 \cdot \frac{\ell(L) y_1}{L}\, dy_1\right\}$$

$$= \frac{3\sqrt{\pi}}{8\rho_0 c_0^5 x^2}\rho_s^2 U_j^8\left\{L D_j + \left(\frac{L}{D_j}\right)^7 D_j\, \ell(L)\int_{L/D_j}^\infty \left(\frac{D_j}{y_1}\right)^7 d\left(\frac{y_1}{D_j}\right)\right\}$$

$\downarrow$ $\int_{L/D_j}^\infty x^{-7} dx \quad x = \frac{y_1}{D_j}$

$\downarrow$ $\frac{1}{6}\left(\frac{L}{D_j}\right)^{-6}$ or $\frac{1}{6}\left(\frac{D_j}{L}\right)^6$

$$= \cancel{\frac{3\sqrt{\pi}}{8\rho_0 c_0^5 x^2}}\ \frac{3\sqrt{\pi}\rho_0 c_0^3}{8x^2}\left(\frac{U_j}{c_0}\right)^8\left(\frac{\rho_s}{\rho_0}\right)^2 D_j^2\left\{\frac{L}{D_j}\left(1+\frac{\ell(L)}{6D_j}\right)\right\}$$

Note that, the second line of equation indicates that the "source strength" is constant in the annular mixing region and decay as y_1^{-7} in the developed region of the jet.

Finally $S(\vec{x}) \sim \frac{3\sqrt{\pi}\rho_0 C_0^3}{8x^2} M_j^8 D_j^2 \left(\frac{\rho_s}{\rho_0}\right)^2 \left\{\frac{L}{D_j}\left(1+\frac{\ell(L)}{6D_j}\right)\right\}$

Since $\ell(L) \sim D_j$, the sound generated in the region downstream of the end of the potential core is predicted to be only one sixth of that generated in the annular mixing region. Of course, this is a rather crude estimate as the transition from the annular mixing region to the fully developed jet has not been modeled in any detail.

- If a CFD solution is available it is possible to determine the contribution of each elemental volume in the numerical grid to the radiated noise spectrum. This contribution is given by

$$dS(\vec{x},\omega) \sim \frac{1}{32\pi\rho_0 C_0^5 x^2}\left\{\rho_s^2 u_s^4 \ell^3 \omega_s^3 \left(\frac{\omega}{\omega_s}\right)^4 \exp\left(-\frac{\omega^2}{4\omega_s^2}\right)\right\} dV$$

Now the length and time scale may be obtained from the $k-\varepsilon$ solution. However, we are assuming that $u_s/\ell \sim \omega_s$. So that with $u_s \sim k^{1/2}$
$(\omega_s \cdot \ell \sim u_s)$
we obtain

$$u_s^4 \ell^3 \omega_s^3 \sim u_s^7 \sim k^{7/2}$$

Thus,

$$dS(\vec{x},\omega) \sim \frac{1}{32\pi\rho_0 C_0^5 x^2}\left\{\rho_s^2 k^{7/2} \left(\frac{\omega}{\omega_s}\right)^4 \exp\left(-\frac{\omega^2}{4\omega_s^2}\right)\right\} dV$$

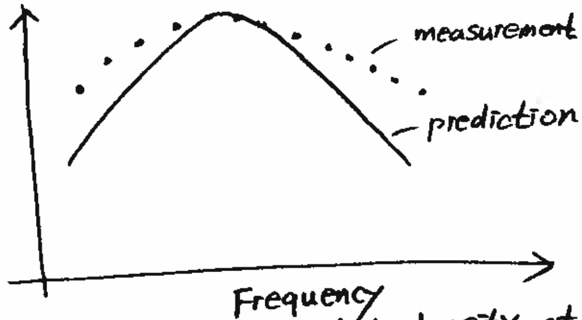
The factor

$$\left(\frac{\omega}{\omega_s}\right)^4 \exp\left(-\frac{\omega^2}{4\omega_s^2}\right)$$

describes how each volume element contributes to a range of frequencies about the local characteristic frequency ω_s.

ω_s is obtained from the local velocity and length scales with

$$\omega_s \sim k^{1/2}/\ell \quad \text{and} \quad \ell \sim k^{3/2}/\varepsilon$$



prediction of spectral density at 90° to the jet axis using acoustic analogy.

$x = 72D_j$, $M_j = 0.911$, $T_j/T_0 = 0.975$, $D_j = 0.0508\,m$