On sound generated aerodynamically

I. General theory

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A theory is initiated, based on the equations of motion of a gas, for the purpose of estimating the sound radiated from a fluid flow, with rigid boundaries, which as a result of instability contains regular fluctuations or turbulence. The sound field is that which would be produced by a static distribution of acoustic quadrupoles whose instantaneous strength per unit volume is $\rho v_i v_j + p_{ij} - a_0^2 \rho \delta_{ij}$, where ρ is the density, v_i the velocity vector, p_{ij} the compressive stress tensor, and a_0 the velocity of sound outside the flow. This quadrupole strength density may be approximated in many cases as $\rho_0 v_i v_j$. The radiation field is deduced by means of retarded potential solutions. In it, the intensity depends crucially on the frequency as well as on the strength of the quadrupoles, and as a result increases in proportion to a high power, near the eighth, of a typical velocity U in the flow. Physically, the mechanism of conversion of energy from kinetic to acoustic is based on fluctuations in the flow of momentum across fixed surfaces, and it is explained in §2 how this accounts both for the relative inefficiency of the process and for the increase of efficiency with U. It is shown in §7 how the efficiency is also increased, particularly for the sound emitted forwards, in the case of fluctuations convected at a not negligible Mach number.

1. Introduction

The subject of this paper is sound generated aerodynamically, that is, as a by-product of an airflow, as distinct from sound produced by the vibration of solids. The airflow may contain fluctuations as a result of instability, giving at low Reynolds numbers a regular eddy pattern which is responsible for the sound produced by musical wind instruments, and at high Reynolds numbers an irregular turbulent motion which is responsible for the roar of the wind and of jet aeroplanes; or they may be inherent in the mechanism for producing flow, as in the siren, or in machinery containing rotating blades. Since the pressure fluctuations within the airflow are in the main balanced by fluctuations of fluid acceleration, it is not clear even to within a very large factor what proportion of their energy is radiated as sound. It is true that the whole pressure fluctuation may play its part in generating sound if it is provided with a solid sounding-board;* but all such effects requiring the vibration of solid boundaries are here excluded, and not classed as aerodynamic.

Earlier studies of sound generated aerodynamically are almost all concerned with frequency. Experiments have been directed towards showing that the frequencies in the flow are identical with those of the sound produced, and towards relating them with other constants of the flow; theory has been concerned with explaining the production of such frequencies in the flow by instability. The theory of the instability was further illuminated by the experiments showing how oscillations

* For example, a loose panel in the wall of a wind-tunnel may produce very intense noise in sympathy with pressure fluctuations in the boundary layer, greater by a large factor than the noise emitted when the wall is rigid.

introduced acoustically at the orifice of a jet or flame may be rapidly amplified under certain conditions; in this case the fluid flow shows a very sensitive response to sound waves from an external source.

But the evolution of a general procedure for estimating the *intensity* of the sound produced in terms of the details of the fluid flow, which this paper attempts, is one fundamental question, perhaps the only one, which escaped the attention of the great physicists who in the last century created the science of sound. It is true that, with many instruments which generate sound aerodynamically, the acoustic power output is known as a function of the various conditions of operation; but, nevertheless, a study of its relation to the actual flow has not been attempted, largely because in most cases the latter has not been seriously investigated. The problem is a fundamental one because it is concerned with uncovering the mechanism of conversion of energy between two of its forms, namely, the kinetic energy of fluctuating shearing motions and the acoustic energy of fluctuating longitudinal motions.

This paper is concerned with the general problem: given a fluctuating fluid flow, to estimate the sound radiated from it. Part II will take up the question of turbulent flows proper, with special reference to the sound field of a turbulent jet, for which a comparison with experiment is possible.

The problem's utility may be questioned on the grounds that we never know a fluctuating fluid flow very accurately, and that therefore the sound produced in a given process could only be estimated very roughly. Indeed, one could hardly expect, even with the great advances in knowledge of turbulent flow which have lately been made, that such a theory could be used with confidence to predict acoustic intensities within a factor of much less than 10. But, on the other hand, one could certainly make no confident estimate even to within a factor of 1000 on existing knowledge, and further, the range of intensities in which one is interested is at least 10^{14}. Also, nothing is known at present concerning how different sorts of changes in a flow pattern may be expected to alter the sound produced, and this is a serious impediment to those experimenting in novel fields of aerodynamic sound production. Clearly such knowledge can arise only from a process in two parts, the first considering what sort of fluctuating flow will be generated, and the second what sound the flow will produce.

The proposed method of attack, in which first the details of the flow are to be estimated, from aerodynamic principles not concerned with the acoustic propagation of fluctuations in the flow, and secondly the sound field is to be deduced, precludes the discussion of phenomena where there is a significant back-reaction of the sound produced on the flow field itself. But such back-reaction is only to be expected when (as in wind instruments) there is a resonator close to the flow field. All the evidence of experiment, and of the theory to be developed below, is that the sound produced is so weak relative to the motions producing it that no significant back-reaction can be expected unless there is such a resonator present to amplify the sound.

Actually it seems likely from the theory in its present form that quantitative estimates of the sound field will be obtainable (as will be seen in part II) only for the sound radiated into free space; and thus they will neglect not only neighbouring resonators but also all effects of reflexion, diffraction, absorption or scattering by solid boundaries. But the general result of these effects could often be sketched in subsequently, in the light of existing knowledge. Again, the estimates refer only to the energy which actually *escapes* from the flow as sound, and to its directional distribution; thus the departures from inverse square-law radiation which are to be expected within a very few wave-lengths of the flow, due to a standing wave pattern, will *not* appear in the estimates, although they are implicit in the general theory. As a final restriction the theory is effectively confined in its application to completely subsonic flows, and could hardly be used to analyze the change in character of the sound produced which is often observed on transition to supersonic flow, at least if, as Cave-Browne-Cave suggests, it is due to high-frequency emission of shock waves.

Because the material of parts I and II is likely to be of interest to workers in acoustics, turbulence theory, gas dynamics and aeronautical engineering, and because it draws ideas from a number of other parts of the physical sciences, the author has thought it desirable to develop the theory less briefly than is customary, so as not to assume an intimate knowledge on the part of the reader of the less salient points of any of these subjects. Since also the method of part I is intended to supply a fundamental basis for further work on the subject (of which part II will be only a first *ad hoc* attempt), as well as to answer the question concerning the mechanism of energy conversion, it has been thought desirable to devote a further introductory section towards justifying the choice of method and giving a preliminary explanation of it.

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2. Description and justification of the approach used

Considering a fluctuating fluid flow occupying a limited part of a very large volume of fluid of which the remainder is at rest. Then the equations governing the fluctuations of *density* in the real fluid will be compared with those which would be appropriate to a *uniform acoustic medium at rest*, which coincided with the real fluid outside the region of flow. The difference between the two sets of equations will be considered as if it were the effect of a fluctuating external force field, known if the flow is known, acting on the said uniform acoustic medium at rest, and hence radiating sound in it according to the ordinary laws of acoustics.

This scheme has two advantages. First, since we are not concerned (see § 1) with the back-reaction of the sound on the flow, it is appropriate to consider the sound as produced by the fluctuating flow after the manner of a forced oscillation. Secondly, it is best to take the free system, on which the forcing is considered to occur, as a uniform acoustic medium at rest, because otherwise, after the sound produced has been estimated, it would be necessary to consider the modifications due to its convection with the turbulent flow and propagation at a variable speed within it, which would be difficult to handle. But by the method just described all these effects are replaced by equivalent forcing terms and incorporated in the hypothetical external force field.

The comparison is made most easily if we write the equation expressing conservation of momentum in the form first used extensively by Reynolds. This is equivalent simply to considering the momentum contained within a fixed region of space as changing at a rate equal to the combined effect of (i) the stresses acting at the boundary and (ii) the flow across the boundary of momentum-bearing fluid. It is easy to see that the latter part is equivalent to an additional stress system. This may be represented symbolically by $\rho v_i v_j$, and referred to either as the 'momentum flux tensor' (i.e. the rate at which momentum in the x_i direction crosses unit surface area in the x_j direction; it is worth noting that similarly the real stresses* are simply a mean momentum flux tensor for the peculiar motions of the molecules) or as the 'fluctuating Reynolds stresses' (to distinguish them from their mean value, minus the product of means $\bar{\rho}\bar{v}_i\bar{v}_j$, which is called 'Reynolds stress' in the theory of turbulence).

Thus the Reynolds momentum equation already expresses that the momentum in any fixed region of space changes at a rate exactly the same as if the gas were *at rest* under the combined action of the real stresses,* say p_{ij}, and the fluctuating Reynolds stresses $\rho v_i v_j$. On the other hand, a *uniform acoustic medium* at rest would experience stresses only in the form of a simple hydrostatic pressure field, whose variations would be proportional to the variations in density, the constant of proportionality being the square a_0^2 of the speed of sound. Hence the density fluctuations in the real flow must be exactly those which would occur in a uniform acoustic medium subject to an external stress system given by the *difference*

$$T_{ij} = \rho v_i v_j + p_{ij} - a_0^2 \rho \delta_{ij} \tag{1}$$

between the effective stresses in the real flow and the stresses in the uniform acoustic medium at rest.

It is important to notice (as the equations will show in § 3) that this analogy approach to the problem of aerodynamic sound production is an exactly valid one. It merely assumes that the mass in a given region changes at a rate equal to the total inward normal momentum, and that the momentum changes at a rate equal to the total inward normal component of the complete stress system of Reynolds. It makes no simplifying assumption concerning the real stresses and their relation to rates of strain. The external stress system T_{ij} incorporates not only the generation of sound, but its convection with the flow (in part of the term $\rho v_i v_j$), its propagation with variable speed and gradual dissipation by conduction (each in part of the difference between pressure variations and a_0^2 times the density variations), and its gradual dissipation by viscosity (in the viscous contribution to the stress system p_{ij}).

In practice the dissipation of acoustic energy into heat, by viscosity and heat conduction, is a slow process; in the atmosphere only half the energy is lost in the first mile of propagation even at the frequency (4 kc/s) of the top note of the piano-

* These are made up of hydrostatic pressure $p\delta_{ij}$ and viscous stresses.

forte, while for other frequencies the required distance varies as their inverse square. The contribution of the viscous stresses to T_{ij} is therefore probably unimportant, at least for phenomena on a terrestrial scale of distance. For flows in which the temperature departs little from uniformity the differences between the exact pressure field $p\delta_{ij}$ and the approximate one $a_0^2\rho\delta_{ij}$ are similarly unimportant, and then the principal generators of sound are the fluctuating Reynolds stresses, corresponding to variable rates of momentum flux across surfaces fixed in the fluctuating fluid flow.

Now at this stage the question concerning the mechanism of conversion of energy from the kinetic energy of fluctuating shearing motions into the acoustic energy of fluctuating longitudinal motions is already effectively answered. The answer is most illuminating if one lists three different ways in which one can cause kinetic energy to be converted into acoustic energy, as follows:

(i) By forcing the mass in a fixed region of space to fluctuate, as with a loudspeaker diaphragm embedded in a very large baffle.

(ii) By forcing the momentum in a fixed region of space to fluctuate, or, which is the same thing, forcing the rates of mass flux across fixed surfaces to vary; both these occur when a solid object vibrates after being struck.

(iii) By forcing the rates of momentum flux across fixed surfaces to vary, as when sound is generated aerodynamically with no motion of solid boundaries.

This is a linear sequence of methods of energy conversion, concerning which two facts are important. First, each is less efficient than the preceding one. Secondly, this statement becomes increasingly true as the frequency is lowered, or more precisely as the wave-length of the sound produced is increased.

These two facts, as far as the relation between methods (i) and (ii) is concerned, are now a commonplace of acoustics, having been first fully realized and understood by Stokes, whose explanation of the reduced acoustic power of a bell when hydrogen is added to the air in which it is sounded (so that the wave-length is augmented) is quoted at length in Rayleigh's *Theory of sound*. In brief the physical explanation is that any forcing motion on a scale comparable with the wave-length is balanced partly by a local reciprocating motion, or standing wave, and partly by compressions and rarefactions of the air whose effect is propagated outwards. The larger the wave-length in comparison with the scale of the forcing motion, the more completely can the motion be fully reciprocated by the local standing wave. Mathematically, the difference is between the fields of an acoustic source and an acoustic dipole, which represent fluctuating rates of production at a point of mass and momentum respectively, so that motions of types (i) and (ii) can be represented as due to distributions of sources and dipoles respectively.

For method (iii), the Stokes effect is even more marked. Indeed, Stokes himself showed that if the surface of a sphere vibrates in such a way that its volume and the position of its centroid remain fixed, and the associated wave-length is twice the circumference, the acoustic power is about $\frac{1}{1000}$ of what it would be if motion near the sphere were forced to be purely radial (so that the lateral reciprocating motion could not be set up and the forcing were like that due to a number of independent point sources). The corresponding factor for a *rigid* vibration is $\frac{1}{13}$. A similar effect occurs in the more complicated conditions of aerodynamic sound production; it is the radiation due to the minute fraction of the fluctuation in momentum flux which is *not* balanced by a local reciprocating motion that we must seek to determine. Mathematically, we have to find the radiation field of a distribution of acoustic quadrupoles.

Now the approach described above, in which the sound is shown to be that which would be produced by the externally applied stresses (1), gives at once the local quadrupole strength per unit volume. For since a dipole is equivalent to a concentrated fluctuating force, it follows that a stress field, which produces equal and opposite forces on both sides of a small element of fluid, is equivalent to a distribution of quadrupoles whose instantaneous strength per unit volume is proportional to the local stress. The radiation field of this quadrupole distribution can at once be written down. These deductions will be given at greater length in § 4.

Thus in this approach the influence of the minute fraction of the fluctuations which is not balanced is at once isolated. The author believes that any approach which differed from his up to this stage, by approximating too early, might well throw away the one small part of the sound field which it is desirable to keep, or swamp it with much larger terms, as, for example, if eddies were supposed to emit sound like vibrating rigid spheres.

One important conclusion from the above qualitative discussion concerns those flows, referred to in § 1, the fluctuations in which are inherent in the mechanisms for producing them, these mechanisms being fluctuating sources of matter for the siren, or of momentum for machinery containing rotating blades. The conclusion in question is that the *strictly* aerodynamic contribution to the sound produced, namely, the quadrupole field arising from the oscillations in momentum flux across fixed surfaces in the flow, will be less important than the direct sound due to the source or dipole distributions corresponding to the puffs of air or the motion of the blades respectively. This conclusion is borne out by the success of theories of propeller noise (at least as regards the more intense, low frequency, part) which simply replace the propeller by a rotating line of dipoles. The conclusion could perhaps bear further investigation, but nevertheless, on the basis of it, the remainder of part I and, later, part II will treat only of flows with fixed boundary conditions, i.e. flows in which the fluctuations are solely the result of instability.

In part I the formula for sound radiation is not applied to a particular case, but only used to make a general dimensional analysis of the intensity field in terms of typical velocities and lengths in the flow. The result of this analysis is strongly affected by the considerations just given concerning quadrupole fields and the Stokes effect. The amplitude of the quadrupole strength per unit volume T_{ij} is evidently proportional to the square of a typical velocity U in the flow; but the amplitude of the radiation field due to a quadrupole is proportional to its strength multiplied by the square of its frequency (the Stokes effect). Since in many cases a typical frequency will be roughly proportional to the velocity U, it follows that the amplitude of the sound generated by a given fluctuating flow will increase roughly like the *fourth* power of a typical velocity U in the flow. Hence the intensity will increase roughly like U^8. This is the cardinal result of § 6, which also considers the effect of Reynolds number and Mach number upon it, and discusses the acoustic efficiency.

Lastly, in § 7, the formulae for the radiation field are rewritten for the case in which it is convenient to describe the fluctuating flow in terms of a frame of reference which is not stationary with respect to the undisturbed atmosphere; this work is somewhat analogous to the Lienard-Wiechert theory of the field of a moving electron. It is found that for fluctuations in momentum flux across a surface moving through the undisturbed atmosphere the Stokes effect may be greatly mitigated in respect of radiation emitted forwards. This theory will be made use of in part II to estimate the influence of Mach number in promoting departures from the power laws indicated by dimensional analysis.

3. The equivalent external stress field

The propagation of sound in a uniform medium, without sources of matter or external forces, is governed by the equations

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_i}(\rho v_i)=0, \quad \frac{\partial}{\partial t}(\rho v_i)+a_0^2\frac{\partial \rho}{\partial x_i}=0, \quad \frac{\partial^2 \rho}{\partial t^2}-a_0^2\nabla^2\rho=0, \tag{2}$$

of which the first is the exact equation of continuity, the second is an *approximate* equation of momentum, and the third follows by eliminating the momentum density ρv_i from the other two. Here ρ is the density, v_i the velocity in the x_i direction, a_0 the speed of sound in the uniform medium, and any suffix repeated in a single term is to be summed from 1 to 3.

On the other hand, the *exact* equation of momentum in an arbitrary continuous medium under no external forces is

$$\frac{\partial}{\partial t}(\rho v_i)+\frac{\partial}{\partial x_j}(\rho v_i v_j+p_{ij})=0, \tag{3}$$

in Reynolds's form. Here p_{ij} is the compressive stress tensor, representing the force in the x_i direction acting on a portion of fluid, per unit surface area with inward normal in the x_j direction. Equation (3) is most simply derived from the physical argument given in § 2. Alternatively, it can be obtained from the momentum equation in the more familiar Eulerian form by adding a multiple of the equation of continuity.

Hence the equations of an arbitrary fluid motion can be rewritten, as suggested in § 2, as the equations of the propagation of sound in a uniform medium at rest due to externally applied fluctuating stresses, namely, as

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x_i}(\rho v_i)=0, \quad \frac{\partial}{\partial t}(\rho v_i)+a_0^2\frac{\partial \rho}{\partial x_i}=-\frac{\partial T_{ij}}{\partial x_j}, \quad \frac{\partial^2 \rho}{\partial t^2}-a_0^2\nabla^2\rho=\frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}, \tag{4}$$

where the instantaneous applied stress at any point is

$$T_{ij}=\rho v_i v_j+p_{ij}-a_0^2\rho\delta_{ij}. \tag{5}$$

Equations (4), of which the middle one is simply (3) rewritten, are chosen as the basic equations of the theory of aerodynamic sound production for reasons fully discussed in § 2.

For a Stokesian gas the stress tensor p_{ij} is given in terms of the velocity field by the equations

$$p_{ij} = p\delta_{ij} + \mu\left\{-\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} + \frac{2}{3}\left(\frac{\partial v_k}{\partial x_k}\right)\delta_{ij}\right\}, \tag{6}$$

where μ is the coefficient of viscosity, and the pressure p is related to the other thermodynamic variables as for the gas at rest. Atmospheric air may be taken as a Stokesian gas for practical purposes.

It is emphasized again that all effects such as the convection of sound by the turbulent flow, or the variations in the speed of sound within it, are taken into account, by incorporation as equivalent applied stresses, in equations (4); this fact is evident, since the equations are exactly true for any arbitrary fluid motion. However, for an airflow embedded in a uniform atmosphere at rest (a case important in practice) the stress system (5) can be neglected outside the flow itself. For there the velocity v_i consists only of the small motions characteristic of sound, and it appears quadratically in (5). Also the viscous stresses in p_{ij}, and the conduction of heat (which causes departures of $p - p_0$ from $a_0^2(\rho - \rho_0)$, where the suffix zero signifies atmospheric values) are both very small effects. (Actually the solution of the equations of sound, taking into account these effects of viscosity and heat conduction, was effected by Kirchhoff. His analysis, given in Rayleigh's *Theory of sound*, shows that the stresses equivalent to the effects of viscosity and heat conduction simply cause a damping of the sound due to the conversion of acoustic energy into heat by these processes, which as indicated in § 2 is negligible except for very large-scale phenomena.) Thus outside the airflow the density satisfies the ordinary equations of sound (2), and the fluctuations in density, caused by the effective applied stresses within the airflow, are propagated acoustically.

If it is assumed that the viscous stresses in T_{ij} (see (6)) can also be ignored in the flow (this point will be returned to), then it should be noted that at low Mach number, provided that any difference in temperature between the flow and the outside air is due simply to kinetic heating or cooling (that is, heating by fluid friction or cooling by rapid acceleration), T_{ij} is approximately $\rho_0 v_i v_j$, with a proportional error of the order of the square of the Mach number M. This results from the fact that relative changes in density under these conditions are known to be of order M^2, while the ratio of fluctuations in pressure to fluctuations in density departs from a_0^2 by a proportional error of order M^2. The resulting approximate form

$$T_{ij} \simeq \rho_0 v_i v_j \tag{7}$$

of the equivalent applied stress field might have been found from an approach which made approximations in the equations of motion right from the start, but there would then have been no guarantee that, in obtaining this main contribution to the quadrupole field, source and dipole fields of small strength might not have been neglected, whose contribution to the sound radiated might (§ 2) be relatively large.

4. Source, dipole and quadrupole radiation fields

The theory of the quadrupole radiation field due to the equivalent applied stresses will be given at some length. This is because quadrupole radiation has hitherto played a relatively small part in physics, and therefore the subject is not covered adequately in books, and is probably unfamiliar to most readers. The confusion concerning the subject is demonstrated particularly by the fact that the term 'quadrupole radiation' is used in at least two senses besides its proper one: namely, as the part of the intensity field which falls off as the inverse fourth (or, sometimes, sixth) power of the distance from the source, and as the part of the amplitude field expressible in terms of spherical harmonics of the second order. The proper meaning is of course a certain limiting case of the field of four sources; at large distances this obeys the inverse *square* law of radiation; and its amplitude field includes spherical harmonics of zero order as well as of the second order.

But to analyze the subject clearly and relate it to the generation of sound by externally applied stresses it is convenient to recall parts of the theory of the generation of sound by simpler mechanisms. First, if fluctuating sources of additional matter are continuously distributed throughout part of the medium, so that a mass $Q(\mathbf{x},t)$ per unit volume per unit time is introduced at $\mathbf{x}$ at time t, then equations (2) are modified by an additional term Q on the right of the first equation, and hence an additional term $\partial Q/\partial t$ on the right of the third equation. If the sources of matter are concentrated into a point, where the total rate of introduction of mass is $q(t)$, and if the medium is unbounded, then the density field is given by the equation

$$\rho-\rho_0=\frac{1}{4\pi a_0^2}\frac{q'(t-r/a_0)}{r}, \tag{8}$$

where r is the distance from the source. When the sources of matter are not so concentrated, the density field is given by a volume integral of terms such as (8), namely,

$$\rho-\rho_0=\frac{1}{4\pi a_0^2}\int\frac{\partial}{\partial t}Q\left(\mathbf{y},t-\frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right)\frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}, \tag{9}$$

where the integral is taken over all space, and is of the kind referred to in electromagnetic theory as a retarded potential. But if solid boundaries are present, reflected and diffracted waves must be added to (8) and (9).

Since in (8) it is only the rate of change with time of the rate of introduction of mass $q(t)$ which affects the sound produced, it is this derivative $q'(t)$ which will here be called the instantaneous *strength* of the concentrated source. Similarly, $\partial Q/\partial t$ is called the source strength per unit volume; note that this source strength density is precisely the term appearing on the right of the *third* of equations (2) when continuously distributed fluctuating sources of matter are present. The nomenclature just introduced is not by any means standard, but (at least in this paper) is very convenient, especially since it means that the instantaneous strength of a dipole is equal to the equivalent applied force, as will shortly be seen.

If, in fact, sources of matter are absent, and the sound is generated instead by a fluctuating external force field F_i per unit volume in part of the medium, then equations (2) are modified by an additional term F_i on the right of the second equation, and so by an additional term $-\partial F_i/\partial x_i$ on the right of the third equation. Thus such a fluctuating force field is equivalent (in the density fluctuations it produces) to a source distribution whose strength per unit volume is equal to the *flux of force inwards*, $-\partial F_i/\partial x_i$.

But it would be most misleading to base any estimate of the acoustic power output of the fluctuating force field on the order of magnitude of this equivalent source strength per unit volume. This is because, at any one instant, the total source strength is zero, as being the integrated flux of a quantity which vanishes outside a limited region. Hence the sound given by (9) at large distances (where $|\mathbf{x}-\mathbf{y}|^{-1}$ can effectively be replaced by x^{-1}) is simply due to the fact that the values of Q therein do not quite cancel out because they are not *simultaneous* values throughout the fluctuating force field. Evidently if F_i varies with time only slowly, that is, if the frequencies are low, this may mean relatively little energy radiation (compare § 2).

Actually of course the sound field generated is a dipole field, and the arguments of the last paragraph merely show that it is essential to take this into account before any approximate estimation of acoustic power output be made. For example, the term $-\partial F_1/\partial x_1$ in the source distribution is equivalent (in the limit as $\epsilon \to 0$) to a distribution $\epsilon^{-1}F_1(x_1, x_2, x_3)$ and a distribution $-\epsilon^{-1}F_1(x_1+\epsilon, x_2, x_3)$; so that any one *value*, say $\epsilon^{-1}F_1(x_1, x_2, x_3)$, occurs with positive sign at (x_1, x_2, x_3) and negative sign at $(x_1-\epsilon, x_2, x_3)$. These two together constitute in the limit a dipole of strength F_1 with axis in the positive x_1 direction.

It follows that the force field is equivalent to a field of dipoles with axis in the x_1 direction and strength F_1 per unit volume, together with two similar fields with axes in the x_2 and x_3 directions. But the whole may of course be regarded, from a more fundamental point of view, as a single dipole field whose strength per unit volume is a *vector* F_i (whose direction as well as magnitude may fluctuate). One may see at once the significance of this vector strength density by choosing, at any instant, the x_1-axis in the direction of F_i, in which case the argument of the last paragraph shows that the dipole is of strength equal to the magnitude of F_i and has axis in the direction of F_i. A force field F_i per unit volume emits sound like a volume distribution of dipoles, whose strength vector per unit volume is F_i.

If the force is concentrated at a point, with value $f_i(t)$, and if the medium is unbounded, the density field is given by

$$\rho-\rho_0 = -\frac{1}{4\pi a_0^2}\frac{\partial}{\partial x_i}\left(\frac{f_i(t-r/a_0)}{r}\right), \tag{10}$$

as follows immediately from (8) when values of q' equal to $\pm\epsilon^{-1}f_1$ are placed at $(0, 0, 0)$ and $(-\epsilon, 0, 0)$ (as in the argument just given), and similarly for f_2 and f_3, and ϵ is allowed to tend to zero. It follows from (10) that for the general volume distribution of dipoles

$$\rho-\rho_0 = -\frac{1}{4\pi a_0^2}\frac{\partial}{\partial x_i}\int F_i\left(\mathbf{y}, t-\frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right)\frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}. \tag{11}$$

(As a mathematical point it may be noticed that the identity of (11) with (9), if in the latter $\partial Q/\partial t$ at $\mathbf{y}$ be replaced by the flux $-\partial F_i/\partial y_i$, can also be deduced from the divergence theorem as applied to the vector integrand of (11).)

Now in carrying out the differentation with respect to x_i in (11), one sees that the part due to differentiating the $|\mathbf{x}-\mathbf{y}|$ in the denominator falls off like the inverse square of the distance from the field of fluctuating forces. But the part due to differentiating F_i itself falls off only like the inverse first power of this distance. Hence at large distances from the field of flow the density fluctuations are dominated by this latter part, namely,

$$\rho-\rho_0 \sim \frac{1}{4\pi a_0^2}\int \frac{x_i-y_i}{|\mathbf{x}-\mathbf{y}|^2}\frac{1}{a_0}\frac{\partial}{\partial t}F_i\left(\mathbf{y}, t-\frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right)d\mathbf{y}. \tag{12}$$

Since fluctuations of $\partial F_i/\partial t$ differ from fluctuations of F_i by a factor of the order of magnitude 2π times a typical frequency, it is easily seen that the term neglected in (12) is truly negligible if x is at a distance from the field of flow large compared with $(2\pi)^{-1}$ times a typical wave-length. It is then in the radiation field of each dipole separately. One may note that if, in addition, this distance were large compared with the dimenions of the field of flow itself, then by choosing an origin within the field of flow one could approximate to the fraction $(x_i-y_i)/|\mathbf{x}-\mathbf{y}|^2$ in (12) by $x_i/|\mathbf{x}|^2$ and take it outside the integral. This approximation would be perfectly sufficient for obtaining the radiation field of the flow as a whole, and hence the acoustic power output, since only terms of order x^{-2} in the amplitude are neglected.

Equation (12) also shows explicitly how the sound radiated to large distances depends on the fact that rays of sound reaching a distant point simultaneously were *not emitted simultaneously*. For it depends only on the rate of change of dipole strength with time.

After these preliminaries the properties of the generation of sound by applied fluctuating stresses T_{ij} are understood more easily. As equations (4) show, the stresses produce a force per unit volume equal to their flux inwards $-\partial T_{ij}/\partial x_j$. Hence they generate sound like a dipole field of strength $-\partial T_{ij}/\partial x_j$ per unit volume.

But again it is essential that the sound radiated be not estimated from the order of magnitude of the dipole strength per unit volume, since at any instant the total dipole strength is zero. Actually the sound produced is quadrupole field. For example, the term $-\partial T_{i1}/\partial x_1$ is equivalent (in the limit as $\epsilon \to 0$) to a dipole field $\epsilon^{-1}T_{i1}(x_1, x_2, x_3)$ and a second dipole field $\epsilon^{-1}T_{i1}(x_1+\epsilon, x_2, x_3)$, so that any one *value*, say $\epsilon^{-1}T_{i1}(x_1, x_2, x_3)$, occurs with positive sign at (x_1, x_2, x_3) and negative sign at $(x_1-\epsilon, x_2, x_3)$. The two together, in the limit, may be said to constitute a quadrupole whose strength is the magnitude of the vector T_{i1}, and whose axes are in the direction of T_{i1} and in the x_1 direction. (Generally, if a quadrupole is formed by equal and opposite dipoles with axes in one direction, whose relative position is in another direction, these two directions are called the axes of the quadrupole. When the axes coincide the quadrupole may be called longitudinal, and when they are perpendicular it may be called lateral.)

It has been shown that the stress field emits sound like three quadrupole fields, namely, the one just described and two others similarly associated with T_{i2} and T_{i3}. But the whole may be regarded, from a more fundamental point of view, as a single quadrupole field *whose strength per unit volume is the stress tensor T_{ij}*. The division into three quadrupoles with specified axes which has just been mentioned is merely one of many possible analyses of the quadrupole into simpler elements. First, each of the nine elements of the tensor T_{ij} is a quadrupole whose strength is the scalar quantity T_{ij} and whose axes are in the x_i and x_j directions. This gives an analysis into three longitudinal quadrupoles, e.g. T_{11} with both axes in the x_1 direction, and three lateral quadrupoles, e.g. $2T_{23}$ with axes in the x_2 and x_3 directions. Secondly, if the principal axes of stress are used locally and instantaneously, then the three lateral quadrupoles disappear and only three mutually orthogonal longitudinal quadrupoles remain (whose orientations as well as strengths will in general be fluctuating).

Thirdly (and perhaps most significantly for the present problem), an analysis of the stress T_{ij} into a pressure and a single pure shearing stress may be made, which leads to an analysis of the quadrupole into three *equal* mutually orthogonal longitudinal quadrupoles, each of strength $T = \frac{1}{3}T_{ii}$, and one lateral quadrupole. The three longitudinal quadrupoles add up to form a simple source of strength $a_0^2 \partial^2 T/\partial t^2$, at least as far as the effect they produce *outside* the stress field is concerned. For if a *source* distribution of strength T per unit volume produces a field $\rho = R$, then a longitudinal quadrupole distribution of strength T per unit volume and axes in the x_1 direction produces a field $\rho = \partial^2 R/\partial x_1^2$. Hence the three longitudinal distributions together produce a field $\rho = \nabla^2 R$. But outside the stress field

$$\nabla^2 R = a_0^{-2} \partial^2 R/\partial t^2,$$

which is the density field due to a source distribution $a_0^{-2} \partial^2 T/\partial t^2$ per unit volume.

Thus the sound field, due to the applied fluctuating stresses T_{ij}, may be split up into a source field of strength

$$\frac{1}{3}\left(\frac{1}{a_0^2}\frac{\partial^2 T_{ii}}{\partial t^2}\right), \tag{13}$$

due to the equivalent applied fluctuating pressures, which for the stress field (5) are

$$\tfrac{1}{3}T_{ii} = \tfrac{1}{3}\rho v_i^2 + p - a_0^2 \rho, \tag{14}$$

and a field of lateral quadrupoles due to the applied fluctuating shearing stresses. The latter are due to lateral momentum flux (fluctuating Reynolds shearing stresses) and to viscous stresses.

Note that the fact that quadrupoles can combine to form a source does not contradict the important principle that quadrupole radiation is less effective, especially for large wave-lengths, than source radiation. For as (13) shows the equivalent source strength is inversely proportional to the square of the wave-length.

From the point of view of spherical harmonic analysis the result is due to the fact that the field of a single longitudinal quadrupole like T_{11} can be analyzed into a term proportional to a spherical harmonic of the second order, and a term proportional to one of zero order, corresponding to the field of a source of strength $\frac{1}{3}a_0^{-2} \partial^2 T_{11}/\partial t^2$. On the other hand, a lateral quadrupole field is simply proportional to a spherical harmonic of the second order.

Now when the medium is unbounded, an expression analogous to (10) can at once be written down for the field of a concentrated quadrupole. The expression analogous to (11) for the field of a continuous distribution with tensor strength density T_{ij} can then be deduced in the form

$$\rho - \rho_0 = \frac{1}{4\pi a_0^2}\frac{\partial^2}{\partial x_i \partial x_j}\int T_{ij}\left(\mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right)\frac{d\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}. \tag{15}$$

Note that mathematically (15) could have been deduced from the retarded potential solution (namely (9) with $\partial^2 T_{ij}(\mathbf{y}, t)/\partial y_i \partial y_j$ for $\partial Q(\mathbf{y}, t)/\partial t$) to the third of equations (4), by the relatively short process of applying the divergence theorem twice. The author prefers the arguments of the present section as being more illuminating as well as more general (showing the field to be a quadrupole field even in the presence of solid boundaries).

At points far enough from the flow to be in the radiation field of each quadrupole, that is, at a distance large compared with $(2\pi)^{-1}$ times a typical wave-length, the differentiation in (15) may be applied (compare (12)) to T_{ij} only, giving

$$\rho - \rho_0 \sim \frac{1}{4\pi a_0^2} \int \frac{(x_i - y_i)(x_j - y_j)}{|\mathbf{x} - \mathbf{y}|^3} \frac{1}{a_0^2} \frac{\partial^2}{\partial t^2} T_{ij}\left(\mathbf{y}, t - \frac{|\mathbf{x} - \mathbf{y}|}{a_0}\right) d\mathbf{y}. \tag{16}$$

This formula for the sound radiation field must give an exact result for the total energy radiated and its directional distribution, since only terms falling away more rapidly than the inverse first power of the distance are excluded. The formula, depending principally on the second time derivative of the equivalent applied stress T_{ij}, is the basic result of this paper.

It is worth emphasizing again that because in this problem the total effective dipole strength (i.e. the integral of $-\partial T_{ij}/\partial x_j$ over all space) is zero, the radiation field in the form (12) is only non-zero because the values of $\partial F_i/\partial t$ in the integrand are not *simultaneous* values. In fact, rays of sound from different dipoles reaching a distant point simultaneously were not in general emitted simultaneously. For this reason, as (16) shows, the true form of the radiation field involves a further differentiation with respect to time.

Provided that the quadrupole distribution T_{ij} is not itself approximately a space derivative and therefore approximately replaceable by an octupole field, it will be possible to estimate the radiation field (16) if the order of magnitude of the fluctuations of $\partial^2 T_{ij}/\partial t^2$ in the flow are known. (Certainly there is no general theoretical reason why at any instant the total quadrupole strength should be practically zero, and in all the problems so far considered by the author and co-workers (as will be seen in part II) there is substantial evidence that it is not so. But the possibility should always be borne in mind.)

Actually the total quadrupole strength arising from the contribution of the *viscous* stresses to T_{ij} (see (5) and (6)) is certainly very small, for clearly their integral over the whole flow is of the order μ times a typical velocity *outside* the flow. This gives support to the suggestion made in § 3 that the viscous stresses are just as unimportant inside the flow as they are known to be outside it—independently of the fact that in most flows the macroscopic momentum flux will greatly exceed the viscous stress.

At distances large compared with the dimensions of the flow one may approximate $x_i - y_i$ by x_i in (16), provided that the origin is taken within the flow, without neglecting any terms of order x^{-1} (where x is the magnitude of $\mathbf{x}$). This gives the simpler form

$$\rho - \rho_0 \sim \frac{1}{4\pi a_0^2} \frac{x_i x_j}{x^3} \int \frac{1}{a_0^2} \frac{\partial^2}{\partial t^2} T_{ij}\left(\mathbf{y}, t - \frac{|\mathbf{x} - \mathbf{y}|}{a_0}\right) d\mathbf{y} \tag{17}$$

for the radiation field of the complete flow. But for extensive flows the replacement of $x_i - y_i$ by x_i may be inadequate at distances from the flow at which one can conveniently make measurements.

5. Intensity and frequency analysis of the sound field

The quantities which can be estimated by the human ear, or measured by other phase-insensitive instruments, are the intensity at any point, and its frequency spectrum. The intensity of sound at a point where the density is ρ is a_0^3/ρ_0 times the mean-square fluctuation of ρ (i.e. with bars signifying average values at a point, times $\overline{(\rho-\bar{\rho})^2}$, which is also called the variance of ρ and written $\sigma^2\{\rho\}$, where $\sigma\{\rho\}$ is the standard deviation). In symbols, then, the intensity is

$$I(\mathbf{x}) = \frac{a_0^3}{\rho_0}\sigma^2\{\rho(\mathbf{x}, t)\}. \tag{18}$$

In the radiation field the intensity signifies the rate at which energy is crossing unit surface area at the point. To obtain the total acoustic power output of the field of flow one must integrate the intensity over a sphere of radius large compared with the dimensions of the flow, that is, one so large that the formula (17) may be used for the variations in density on it.

The natural units for intensity are watts/sq.cm, but a rather more convenient scale for most purposes is the decibel scale. The intensity level on the decibel scale is now usually defined as $10\log_{10}[I/(10^{-16}\,\text{watts/sq.cm})]$.

Now in fields with a discrete frequency spectrum the variance $\sigma^2\{\rho\}$ would be simply half the sum of the squares of the amplitudes in the various frequencies. For such problems the theoretical worker would be well advised to concentrate on determining these amplitudes, and to leave the determination of intensity and its distribution over the various frequencies until the final question arises of displaying his results and comparing them with experiment.

But at higher Reynolds numbers, when the flow field is fully turbulent, so that presumably the sound field partakes of the same chaotic quality, there is no meaning to be attached to the concept of 'amplitude' for any frequency, or even for any band of frequencies however small. This is because the phase is completely random and discontinuous, so that the fluctuating physical quantities themselves (as distinct from the intensity) possess no spectral density, because the limiting process which is used to define this concept does not converge. Hence intensity and its spectrum are all that can be given a meaning in such cases, not simply all that we need to know.

Now since the value of $\rho - \rho_0$ in the radiation field, by (16), is a time-derivative, its mean with respect to time at any point is 0 and the mean density $\bar{\rho}$ is simply ρ_0. Hence the variance $\sigma^2\{\rho\}$ is simply the mean value at a point of the square of (16). To write down the square of (16) one may simply write it down twice, but it is necessary to use different symbols in the two cases for the variables of summation and integration i, j and $\mathbf{y}$ if confusion is to be avoided. The two integrals can then be combined into a double integral. Then the intensity field derived from (16) and (18) is

$$I(\mathbf{x}) \sim \frac{1}{16\pi^2 \rho_0 a_0^5} \iint \frac{(x_i - y_i)(x_j - y_j)(x_k - z_k)(x_l - z_l)}{|\mathbf{x}-\mathbf{y}|^3 \, |\mathbf{x}-\mathbf{z}|^3} \times \overline{\frac{\partial^2}{\partial t^2} T_{ij}\left(\mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right) \frac{\partial^2}{\partial t^2} T_{kl}\left(\mathbf{z}, t - \frac{|\mathbf{x}-\mathbf{z}|}{a_0}\right)} \, \mathrm{d}\mathbf{y} \, \mathrm{d}\mathbf{z}. \tag{19}$$

The mean product, or 'covariance',* in (19) is the type of quantity which can be measured by hot-wire techniques in a turbulent flow, though it is a lot more complicated than any such quantity which has yet been studied. The physical interpretation of (19), and the methods for approximating to it by simpler expressions, especially by making use of the fact that the covariance is negligibly small except when the points $\mathbf{y}$ and $\mathbf{z}$ in the turbulent flow are rather close together, are all postponed to part II.

However, a general expression for the total acoustic power output of the field will here be obtained. For this one need only use the simplified form (17) for the ultimate radiation field, and integrate its mean square (which is (19), with the fraction inside the integral replaced by $x_i x_j x_k x_l / x^6$) over the surface of a large sphere Σ. Now the fraction $x_i x_j x_k x_l / x^6$ may easily be integrated over Σ; the answer is evidently zero unless i, j, k, l are equal in pairs; it is $4\pi/15$ if the pairs are unequal, and $4\pi/5$ if the pairs are equal. In fact

$$\int_\Sigma \frac{x_i x_j x_k x_l}{x^6} \, \mathrm{d}S = \frac{4\pi}{15} (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{20}$$

Hence the total acoustic power output P is given as

$$\begin{aligned} P &= \frac{1}{60\pi\rho_0 a_0^5} \iint \Bigg[\overline{\frac{\partial^2}{\partial t^2} T_{ii}\left(\mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right) \frac{\partial^2}{\partial t^2} T_{kk}\left(\mathbf{z}, t - \frac{|\mathbf{x}-\mathbf{z}|}{a_0}\right)} \\ &\qquad + 2\, \overline{\frac{\partial^2}{\partial t^2} T_{ij}\left(\mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right) \frac{\partial^2}{\partial t^2} T_{ij}\left(\mathbf{z}, t - \frac{|\mathbf{x}-\mathbf{z}|}{a_0}\right)} \Bigg] \mathrm{d}\mathbf{y} \, \mathrm{d}\mathbf{z} \\ &= \frac{1}{60\pi\rho_0 a_0^5} \Bigg[\sigma^2 \left\{ \int \frac{\partial^2}{\partial t^2} T_{ii}\left(\mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right) \mathrm{d}\mathbf{y} \right\} \\ &\qquad + 2 \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma^2 \left\{ \int \frac{\partial^2}{\partial t^2} T_{ij}\left(\mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right) \mathrm{d}\mathbf{y} \right\} \Bigg]. \end{aligned} \tag{21}$$

It should be noticed that in the second term in (21) each of the nine quadrupole fields T_{ij} ($i = 1$ to 3, $j = 1$ to 3) makes its contribution to the power output independently. The first term is connected with the equivalent source strength (13) when the field is split up into a source field and a field of lateral quadrupoles; it involves cross-terms between the different longitudinal quadrupole fields T_{11}, T_{22}, T_{33}. It may also be noticed that the time derivatives may be taken outside the

* Of which a 'correlation coefficient' is a non-dimensional form.

integrals in (21). This gives a rough method of simplification for the purposes of estimation, by making use of the general result that

$$\sigma^2\{f'(t)\} = 4\pi^2 N^2 \sigma^2\{f(t)\}, \tag{22}$$

where N is a frequency somewhere within the band of frequencies appearing in the oscillations of $f(t)$.

Now, most of the work of part I and of part II will be concerned with the intensity field and total power output, but it is also desirable to be able to predict their frequency spectra. To modify the theory for this purpose the following principle may be used: If, from the quadrupole strength density T_{ij}, all fluctuations with frequency outside a certain band were removed,* then the intensity field of the modified quadrupole distribution would be nothing else but the integral of the intensity spectrum over the said band in the original field. This tells us a definite procedure to be adopted if the latter quantity is to be estimated.

Many will find the above principle intuitively obvious, but for the benefit of others it may be noted that a proof in terms of the retarded potentials solutions is possible, using the statistical theorems that the part of a fluctuating function $f(t)$ in the frequency band from 0 to $a/2\pi$ is $\int_{-\infty}^{\infty} f(t+s)\sin(as)\frac{\mathrm{d}s}{\pi s}$, and that the portion of its mean square relating to frequencies in the same band is $\int_{-\infty}^{\infty} \overline{f(t)f(t+s)}\sin(as)\frac{\mathrm{d}s}{\pi s}$.

One use of the principle might be that in experimental comparison of the properties of T_{ij} (or its approximation (7)) in turbulent flow with those of the sound field generated, it would be permissible to use instruments which excluded fluctuations of T_{ij} outside a certain band of frequencies, provided that they were compared with the sound intensities in this band.

Notice that the frequency spectrum of T_{ij} in turbulent motion, which the above analysis shows to be dominant in determining that of the sound field, is different from that of the velocity field since it depends on it quadratically. Thus it will include summation and difference tones of the velocity frequencies, and generally may be expected to be a considerably flatter spectrum.

If one is interested in a relatively small band of frequencies, the use of the principle (22) for estimating expressions like (21) becomes much more accurate.

6. Dimensional analysis of aerodynamic sound production

Certainly the simplest, and perhaps at first the most practically useful deduction which can be made from the theory of §§ 3 to 5 is an analysis of the dependence of the sound field, for geometrically similar mechanisms of flow production, on a typical velocity U in the flow and a typical linear dimension l, and also on constants of the gas such as ρ_0, a_0 and the kinematic viscosity ν_0. Only in the light of some such analysis can experiments be co-ordinated.

* And it is known that *this* kind of spectral analysis *is* (theoretically) possible for a turbulent flow, even though the fluctuating flow quantities do not, perhaps, have a spectral *density* in the strict sense.

Now by (5) the amplitude of the fluctuations in T_{ij} will, at corresponding points in similar flows, be in the main proportional to $\rho_0 U^2$, although there will be some additional dependence on Reynolds number $R = Ul/\nu_0$, Mach number U/a_0, and on the ratio of a typical temperature in the flow to its atmospheric value; actually changes with Reynolds number are usually very gradual, and changes with Mach number are small unless it approaches 1. But to infer from these facts anything concerning the dependence of the density variations (16) on U, l, ρ_0, a_0 and ν_0 it is necessary to know how typical frequencies of the flow depend on these quantities, so as to be able to relate the fluctuations in $\partial^2 T_{ij}/\partial t^2$ to those in T_{ij}.

Since in § 2 it was decided that the discussion could be confined to flows whose fluctuations are generated by instability rather than by any direct external cause, the dominant frequency or band of frequencies must of course vary in an ascertainable manner with the other constants of the flow. Now in all cases, if n is a typical frequency, the non-dimensional product nl/U (sometimes known as the Strouhal number) has been found to vary far less with changing conditions than n itself. For example, at low Reynolds numbers, when a regular eddy pattern appears, the product nl/U rises only slowly with Reynolds number; in particular for the eddies shed by a wire of diameter l in a stream of speed U, nl/U is about $0{\cdot}2 - 4R^{-1}$ for $40 < R < 40000$. At the upper end of this range of R the frequency spectrum spreads out more and more, while the most prominent frequencies have a slightly higher value of nl/U. The appearance of frequencies very much less than $0{\cdot}2U/l$ is impeded by the scale of the system, but the appearance of higher frequencies is limited only by viscous damping, and so the range of values of nl/U continues to grow at its upper end as R increases. However, the turbulent energy continues to be borne principally by frequencies with nl/U less than 1, although the fluctuations of velocity *gradient* are greatest for rather higher frequencies. For example, the mean square vorticity is carried principally by motions with nl/U proportional to $R^{\frac{1}{2}}$, but these motions have relatively little energy and a significant rate of energy loss by viscous dissipation.

These considerations indicate that to obtain a preliminary rough idea of how the sound produced varies with the constants of the flow, one may take frequencies as proportional to U/l on the whole, and so take the fluctuations in $\partial^2 T_{ij}/\partial t^2$ as roughly proportional to $(U/l)^2 \rho_0 U^2$. One may then conclude that at a distance x from the centre of the flow, in a given direction, the density variations (16) are roughly proportional to the product

$$\frac{1}{a_0^2}\frac{1}{x}\frac{1}{a_0^2}\left(\frac{U}{l}\right)^2 \rho_0 U^2 l^3 = \rho_0 \left(\frac{U}{a_0}\right)^4 \frac{l}{x}. \tag{23}$$

The most striking fact about this formula is the dependence of the density changes in the sound radiation field on the *fourth* power of the Mach number $M = U/a_0$. By contrast, density changes in the flow itself (where l/x is of order 1) are known to be of order $\rho_0 M^2$. The additional factor M^2 at distances large compared with $(2\pi)^{-1}$ wave-lengths, showing that sound radiation is a 'Mach number effect', is due entirely to the quadrupole nature of the field (see § 2).

From (18) it follows that the intensity is roughly proportional to a_0^3/ρ_0 times the square of (23), i.e. to

$$\rho_0 U^8 a_0^{-5} \left(\frac{l}{x}\right)^2, \tag{24}$$

and hence that the total acoustic power output is roughly proportional to

$$\rho_0 U^8 a_0^{-5} l^2. \tag{25}$$

The prediction that sound intensities increase like some high power, near the eighth, of a typical velocity U in the flow, is borne out by experiment (as will be seen in part II).

On the other hand, in a careful experimental study, one would expect to be able to detect departures from laws such as (25), because of the approximate character of the arguments used to support them. Hence, to get the maximum benefit from such experiments, by laying bare the precise nature of such departures, the data should be expressed in terms of an 'acoustic power coefficient'

$$K = \frac{\text{acoustic power}}{\rho_0 U^8 a_0^{-5} l^2}. \tag{26}$$

The dependence of this quantity K on Reynolds number and Mach number could be studied by varying U and l independently, while retaining geometrically similar flow-producing mechanisms. The dependence of K on the ratio of a typical temperature in the flow to the atmospheric value should also be investigated by preheating or pre-cooling the flow. Finally, analyses of K may be made both with respect to direction of propagation and with respect to frequency. (Some of these procedures will be used in part II in analyzing certain data obtained by Gerrard.) Here again a frequency analysis of K would have most value if made in terms of the non-dimensional frequency parameter nl/U, especially if such analyses were performed at different Reynolds numbers, when changes of the shape of the spectrum might give important information as to which aspects of the turbulent flow contribute most to the sound produced.

At this stage it is impossible to make predictions concerning the variations discussed above. It might be thought that K will increase with Reynolds number because as explained above the frequencies bearing the major fluctuations of derivatives like $\partial^2 T_{ij}/\partial t^2$ tend to grow gradually (relative to U/l) with Reynolds number. But this is (at least partly) counteracted by the fact that the eddy-sizes corresponding to these frequencies, and hence the range of values of $|\mathbf{y}-\mathbf{z}|$ for which the covariance in (19) is not negligible, are correspondingly smaller.

To conclude this section it may be noted that in a steadily maintained flow the energy per unit volume will be roughly proportional to $\rho_0 U^2$, and the total rate of supply of energy to $(\rho_0 U^2)(Ul^2)$. Hence the ratio of the acoustic power output to the supply of power, which ratio can be described as the *efficiency* of aerodynamic sound production, will satisfy (to the same sort of accuracy as (25), that is only very roughly)

$$\eta \propto M^5. \tag{27}$$

Of course acoustic efficiencies are always very low indeed, and doubtless that of aerodynamic sound production, even at Mach numbers near the top of the range in which (27) is expected to have some validity (i.e. with M approaching 1) is no exception. (The experiments of Gerrard indicate an order of magnitude 10^{-4} for the coefficient η/M^5.) But (27) makes it clear that turbulence at *low* Mach numbers is a quite exceptionally inefficient producer of sound.

7. Modifications required when flow is analyzed with respect to a moving frame

One can imagine various applications of the theory of §§3 to 5 in which the necessary time-derivatives, and space-integrals, could be estimated more easily, and accurately, if they referred to co-ordinate axes in uniform motion relative to the undisturbed atmosphere. As an example one may quote flow fields carried along with a moving aircraft; and a more complicated application, involving the use of different frames of reference for different parts of a turbulent flow, will be given in part II. In any case the necessary modifications to the theory are easily made, and are therefore worth giving here.

The symbol T_{ij} will continue to have its old significance (5), with the velocities v_i measured relative to the undisturbed atmosphere. However, it will here be considered as given in terms of a co-ordinate system (of constant orientation) whose origin moves with uniform velocity $a_0\mathbf{M}$, where $M < 1$. Thus T_{ij} refers to momentum flux across surfaces moving uniformly through the fluid. It will be particularly interesting to compare the acoustic effect of fluctuations in *this* with that of identical fluctuations of momentum flux across surfaces fixed in the fluid.

Now, if the moving axes are chosen to coincide with the fixed axes at time t, then the retarded value of T_{ij} appearing in the fundamental expression (15) must be rewritten as

$$T_{ij}\left(\boldsymbol{\eta}, t-\frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right), \quad \text{where} \quad \boldsymbol{\eta} = \mathbf{y}+\mathbf{M}\,|\mathbf{x}-\mathbf{y}|. \tag{28}$$

This is because the axes have moved on a distance $\mathbf{M}\,|\mathbf{x}-\mathbf{y}|$ during the time taken for a ray of sound to go from $\mathbf{y}$ to $\mathbf{x}$.

When the integral (15) is transformed into the $\boldsymbol{\eta}$ space the element of volume is altered by a factor equal to the Jacobian of the transformation (28). One easily calculates that

$$\mathrm{d}\boldsymbol{\eta} = \mathrm{d}\mathbf{y}\left(1-\frac{\mathbf{M}.(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right), \tag{29}$$

and so by (15)

$$\rho-\rho_0 = \frac{1}{4\pi a_0^2}\frac{\partial^2}{\partial x_i \partial x_j}\int T_{ij}\left(\boldsymbol{\eta}, t-\frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right)\frac{\mathrm{d}\boldsymbol{\eta}}{|\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})}. \tag{30}$$

Now at points far enough from the flow to be in the radiation field of each quadrupole, the differentiation in (30) may be applied to T_{ij} only. This requires a knowledge of the derivative of $|\mathbf{x}-\mathbf{y}|$ with respect to x_i, keeping $\boldsymbol{\eta}$ constant, where x and y are related as in (28). This is easily calculated as

$$\frac{\partial}{\partial x_i}|\mathbf{x}-\mathbf{y}| = \frac{x_i-y_i}{|\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})}. \tag{31}$$

Hence (30) becomes

$$\rho-\rho_0 \sim \frac{1}{4\pi a_0^2}\int \frac{(x_i-y_i)(x_j-y_j)}{\{|\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})\}^3}\frac{1}{a_0^2}\frac{\partial^2}{\partial t^2}T_{ij}\left(\boldsymbol{\eta}, t-\frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right)\mathrm{d}\boldsymbol{\eta}. \tag{32}$$

Comparing (32) with the sound field (16) due to fluctuations in momentum flux across *fixed* surfaces, we see that given fluctuations, when they take place across moving surfaces, will send an increased quantity of sound in directions making an acute angle with the direction of motion, and a decreased quantity in those making an obtuse angle with it. If the said angle is θ then the factor multiplying the amplitude, due to motion of the axes, is $(1-M\cos\theta)^{-3}$. (Note that the added sound emitted forwards is more than the reduction in sound emitted backwards.)

There are two physical factors contributing to these differences, corresponding to the two mathematical factors emerging from the above (the Jacobian of the transformation and the effect of double differentiation). These factors are the position and time of origin (respectively) of those rays of sound from a single eddy* which arrive at a distant point simultaneously. First, the positions of origin of such rays fill a greater *volume* when the rays are emitted forward, since the foremost parts of the eddy have moved on before they need emit. Secondly, the cancelling out of rays from different parts of a quadrupole is less effective for rays emitted forward, owing to the *time* interval between emission from fore and aft parts being increased. Speaking more crudely, waves emitted forward by an object in motion pile up (the Doppler effect) and this makes cancelling of successive waves less effective.

The *intensity* field derived from (32) takes the form (19), with T_{ij} a function of $\boldsymbol{\eta}$ rather than $\mathbf{y}$, and similarly T_{kl} of $\boldsymbol{\zeta}$, and the integrations taken over $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ space. Also the denominator $|\mathbf{x}-\mathbf{y}|^3$ is replaced by $\{|\mathbf{x}-\mathbf{y}|-\mathbf{M}.(\mathbf{x}-\mathbf{y})\}^3$, and similarly with $|\mathbf{x}-\mathbf{z}|^3$. Hence to obtain the power output, as in § 5, one must evaluate not the integral (20) but rather the integral

$$\int_\Sigma \frac{x_i x_j x_k x_l}{(x-\mathbf{M}.\mathbf{x})^6}\,\mathrm{d}S. \tag{33}$$

It is convenient for this purpose to choose the x_1-axis in the direction of the vector $\mathbf{M}$. Then it is clear that the integral (33) vanishes, by symmetry, unless i, j, k, l are equal in pairs. The evaluation when they *are* is somewhat tedious, but straightforward if spherical polars are used. One finds that the corresponding values for the case $M = 0$ are multiplied by

$$(a)\ \frac{1}{(1-M^2)^3},\quad (b)\ \frac{1+5M^2}{(1-M^2)^4},\quad (c)\ \frac{1+10M^2+5M^4}{(1-M^2)^5}, \tag{34}$$

according as (a) neither, (b) one or (c) both of the pairs of suffixes is 1.

This means that the power output is given by an expression precisely of the form (21), but with each term multiplied by one of the factors (34), according as it results

* Here the reader may understand 'eddy' as meaning simply a small (imaginary) volume carried along with velocity $a_0 M$.

from quadrupoles with (*a*) neither, (*b*) one or (*c*) both of their axes in the direction of motion. The cross-terms between two longitudinal quadrupoles are multiplied by factor (*b*) when one is in the direction of motion and by factor (*c*) when neither is.

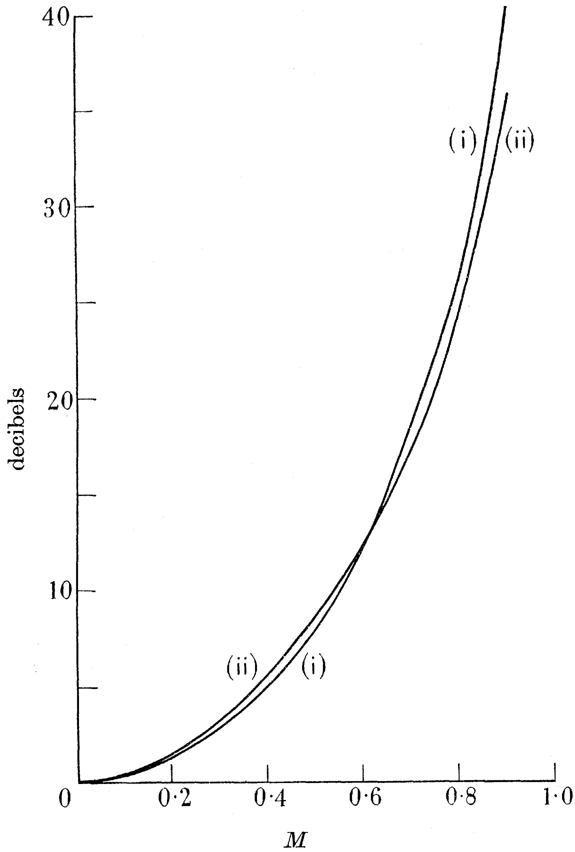

FIGURE 1. Change, as a result of translation at a Mach number M, in the total acoustic power output of (i) three equal mutually orthogonal longitudinal quadrupoles and (ii) a lateral quadrupole with one axis in the direction of translation.

To illustrate the above theory, the increase in power output, expressed in decibels, is shown as a function of Mach number in figure 1 for (i) three equal longitudinal quadrupoles (which for $M = 0$ would produce a simple source field, see § 4), (ii) a single lateral quadrupole of which one of the axes is in the direction of motion. The effect of translation of the fluctuating field in case (ii) is multiplication by the factor (*b*), so that it is 10 times the logarithm to base 10 of factor (*b*) which is plotted. In case (i), by combining the effects of the factors (34) according to the laws given above, one finds that the intensity is multiplied by

$$\frac{1+2M^2+\frac{1}{5}M^4}{(1-M^2)^5}. \tag{35}$$

The increase in intensity level in cases (i) and (ii) is further analyzed in figures 2 and 3 respectively, by a plot of variation in intensity level with direction θ (measured from the direction of motion*) and with Mach number M. The quantities plotted are ten times the logarithm to base 10 of

$$\text{(i)}\ \frac{1}{(1-M\cos\theta)^6},\quad \text{(ii)}\ \frac{4\sin^2\theta\cos^2\theta}{(1-M\cos\theta)^6} \tag{36}$$

in figures 2 and 3 respectively (so that in each case the maximum of the curve for $M = 0$ is arbitrarily chosen as the intensity level 0). These figures illustrate the

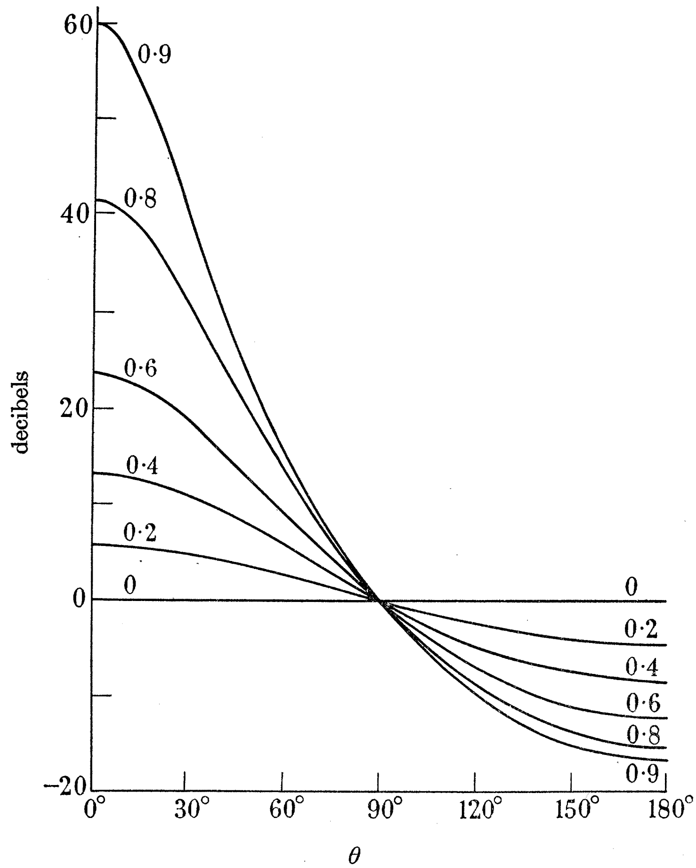

FIGURE 2. Change in directional intensity distribution due to three equal mutually orthogonal longitudinal quadrupoles as a result of translation at a Mach number, the value of which is indicated against each curve, the direction θ being measured from the direction of translation.

change with Mach number of the directional intensity field of a moving fluctuating eddy, at points far from it compared both with $(2\pi)^{-1}$ times a typical wave-length and also with the largest distance between points $\mathbf{y}$ and $\mathbf{z}$ in the flow such that a

* Note that the direction θ is measured relative to the position of the eddy when it emits, not its position when the sound arrives at the point $\mathbf{x}$.

covariance such as that in (19) is significant. For when this is so the difference between $x_i - y_i$ and $x_i - z_i$ in (19) can be neglected, and both replaced by

$$|\mathbf{x}-\mathbf{y}|\cos\theta \quad \text{or} \quad |\mathbf{x}-\mathbf{y}|\sin\theta$$

according as i is 1 or not.

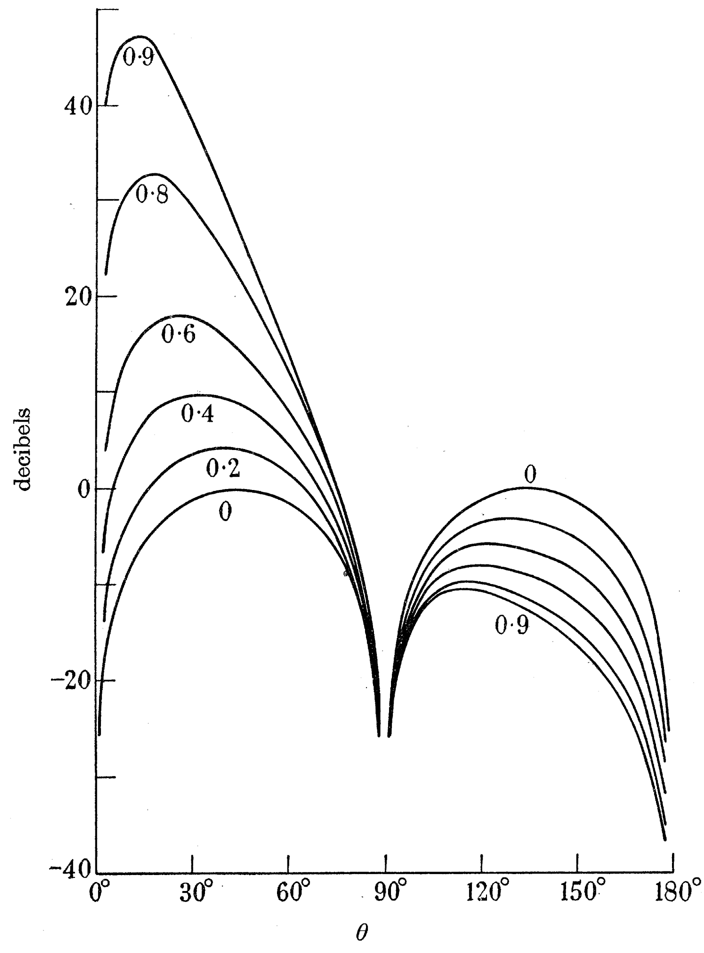

FIGURE 3. Change in the directional intensity distribution due to a lateral quadrupole as a result of translation at a Mach number, indicated against each curve, in the direction of one of its axes, the direction θ being measured from the direction of translation.

Figure 2 shows the preference for forward emission increasing with M. Figure 3 shows this effect superimposed on the basic directional pattern of a lateral quadrupole. Notice that the parts of the curves above the level 0 are more effective in increasing the power outputs (figure 1) than are the parts below in decreasing them, owing to the logarithmic scale used in plotting the curves.

Finally, the frequency analysis of the sound produced by fluctuations of T_{ij} at points in motion relative to the atmosphere will be governed by the principle stated at the end of § 5 only if it is modified to include the Doppler effect. Thus the band of frequencies retained in T_{ij} $(\boldsymbol{\eta}, t)$ will be responsible for the sound radiation to a distant point x with frequencies in a band obtained by multiplying those in the original band by

$$\left(1-\frac{\mathbf{M}.(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right)^{-1}. \tag{37}$$

The frequency is of course increased for sound emitted forwards, and decreased for sound emitted backwards.

The drawing of practical conclusions from the results of this section is postponed to part II.