On sound generated aerodynamically
II. Turbulence as a source of sound

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The theory of sound generated aerodynamically is extended by taking into account the statistical properties of turbulent airflows, from which the sound radiated (without the help of solid boundaries) is called aerodynamic noise. The theory is developed with special reference to the noise of jets, for which a detailed comparison with experiment is made (§ 7 for subsonic jets, § 8 for supersonic ones).

The quadrupole distribution of part I (Lighthill 1952) is shown to behave (see § 3) as if it were concentrated into independent point quadrupoles, one in each 'average eddy volume'. The sound field of each of these is distorted, in favour of downstream emission, by the general downstream motion of the eddy, in accordance with the quadrupole convection theory of part I. This explains, for jet noise, the marked preference for downstream emission, and its increase with jet velocity. For jet velocities considerably greater than the atmospheric speed of sound, the 'Mach number of convection' M_c may exceed 1 in parts of the jet, and then the directional maximum for emission from these parts of the jet is at an angle of $\sec^{-1}(M_c)$ to the axis (§ 8).

Although turbulence without any mean flow has an acoustic power output, which was calculated to a rough approximation from the expressions of part I by Proudman (1952) (see also § 4 below), nevertheless, turbulence of given intensity can generate more sound in the presence of a large mean shear (§ 5). This sound has a directional maximum at 45° (or slightly less, due to the quadrupole convection effect) to the shear layer. These results follow from the fact that the most important term in the rate of change of momentum flux is the product of the pressure and the rate of strain (see figure 2). The higher frequency sound from the heavily sheared mixing region close to the orifice of a jet is found to be of this character. But the lower frequency sound from the fully turbulent core of the jet, farther downstream, can be estimated satisfactorily (§ 7) from Proudman's results, which are here reinterpreted (§ 5) in terms of sound generated from combined fluctuations of pressure and rate of shear in the turbulence. The acoustic efficiency of the jet is of the order of magnitude $10^{-4}M^5$, where M is the orifice Mach number.

However, the good agreement, as regards total acoustic power output, with the dimensional considerations of part I, is partly fortuitous. The quadrupole convection effect should produce an increase in the dependence of acoustic power on the jet velocity above the predicted U^8 law. The experiments show that (largely cancelling this) some other dependence on velocity is present, tending to reduce the intensity, at the stations where the convection effect would be absent, below the U^8 law. At these stations (at 90° to the jet) proportionality to about $U^{6\cdot5}$ is more common. A suggested explanation of this, compatible with the existing evidence, is that at higher Mach numbers there may be *less turbulence* (especially for larger values of nd/U, where n is frequency and d diameter), because in the mixing region, where the turbulence builds up, it is losing energy by sound radiation. This would explain also the slow rate of spread of *supersonic* mixing regions, and, indeed, is not incompatible with existing rough explanations of that phenomenon.

A consideration (§ 6) of whether the terms other than momentum flux in the quadrupole strength density might become important in heated jets indicates that they should hardly ever be dominant. Accordingly, the physical explanation (part I) of aerodynamic sound generation still stands. It is re-emphasized, however, that whenever there is a fluctuating *force* between the fluid and a solid boundary, a dipole radiation will result which may be more efficient than the quadrupole radiation, at least at low Mach numbers.

1. Introduction

Part I (Lighthill 1952) was concerned with building up a physical theory of aerodynamic sound production in general. In this paper the subject is specialized to the study of *turbulence* as a source of sound, and a more quantitative approach, involving comparison with the existing experimental data, is attempted.

The sound produced by turbulence (in the absence of any fluctuating forces between the fluid and solid boundaries)* may be called aerodynamic *noise*, for certainly, like the turbulence itself, it has always a broad frequency spectrum. The most familiar aerodynamic noise is from natural winds, but industrial workers are aware of a wide range of noise arising from turbulent boundary layers and wakes, and from turbulent flow in ducts and free jets. Most *hydrodynamic* noise is of a different character, being associated with cavitation, but there is a residual noise in hydraulic machinery, when cavitation has been eliminated, which may be ascribable to turbulence; and the noise of turbulence in the blood stream is of importance in clinical medicine.

However, fundamental experiments on aerodynamic noise have begun only recently, largely as a result of the public odium with which the aircraft industry is threatened on account of the noise of jet aeroplanes. In a review of the various components of aircraft noise Fleming (1946) spoke of the importance of the aerodynamic noise, which he expected to become predominant at the higher Mach numbers, but said that nothing was known concerning it. In the years that followed, the firms making jet aircraft confined their attention at first to noise data for whole engines, and a valuable correlation of these data has recently been given by Mawardi & Dyer (1953), but it is never clear in these experiments how much of the noise produced was aerodynamic in the sense here used (rather than, say, having its origin in the combustion chamber),† nor can the effects of the velocity and temperature of the jet be separated from one another. Recently, however, five studies of the noise of the cold jet, which is obtained by simply evacuating a reservoir of compressed air through a suitable orifice, have been published (Fitzpatrick & Lee 1952; Gerrard 1953; Lassiter & Hubbard 1952; Powell 1952*b*; Westley & Lilley 1952); in all these an effort was made to eliminate noise other than the strictly aerodynamic noise of the jet itself.

The results of these experiments are particularly full for the subsonic cold jet. Many points concerning it are well corroborated by being found by more than one investigator; these are as follows:

(i) The acoustic power output varies as a high power, near the eighth, of the jet velocity; as a consequence it becomes too small at low velocities for any measurements to have been found possible at Mach numbers below 0·3. A rough order of magnitude for the acoustic efficiency of the subsonic cold jet is $10^{-4}M^5$, where M is the orifice Mach number.

* It was perhaps not sufficiently stressed in the mathematical theory of part I, although it is evident from the physical discussion given there, that such forces can produce important dipole sound fields, not only when the solid boundaries move (e.g. sounding-boards, rotating blades), but also when they remain rigid; thus sound can be generated in this way by the fluctuating lift on a rigid cylinder in a uniform stream. Mathematically, this is because where boundaries are present the retarded potential solution must be supplemented by a surface integral, whose physical interpretation is easily verified to be the dipole radiation associated with the force between the solid boundary and the fluid.

† They find acoustic efficiencies increasing rapidly with jet power above a value of about 2000 kW, and remaining roughly constant below this value. It seems reasonable (in the light of the later work) to conclude from this that aerodynamic noise was dominant at the higher jet speeds, and that a source whose output bears a ratio to energy expended independent of jet Mach number (presumably, combustion noise) was dominant at the lower speeds.

(ii) The spectrum is a very broad one indeed, of the order of seven octaves. The peak frequency is always in the neighbourhood of $U/2d$, where U is the jet velocity and d the orifice diameter, but the peak is not sufficiently clearly defined for consistent results on its variation with U and d to have been obtained; probably it increases less rapidly than U.

(iii) Almost all the sound is radiated in directions making an *acute* angle with the jet; in fact, measurements behind the orifice were subject to some danger of error, because the relatively small intensity there was liable to be smothered by reflexions of the sound which had been radiated forward.

(iv) The directional maximum for the higher frequency sound is at an angle of 45°, or slightly less, to the jet axis. This higher frequency sound appeared to be emitted principally from near the orifice.

(v) The lower frequency sound has a directional maximum at a much smaller angle to the jet axis; this angle decreases as the frequency is reduced, or the Mach number increased, and is often below the limit (about 18°) for which measurements can be made without gusts blowing on to the microphone. The sound intensity at these small angles to the jet axis increases like a higher power of the jet velocity than the average sound intensity. The lower frequency sound appears to be emitted principally from the part of the jet of the order of five to twenty diameters downstream of the orifice.

In view of this substantial body of information on which the different experimenters are in agreement, it has been natural to develop the theory of aerodynamic noise with an eye constantly on this application, and to regard the comparison with those experimental results as an essential test of the theory. In § 7, general agreement with all the points noted above will be found, and detailed reference will also be made to particular results obtained in individual experiments.

The information on supersonic jets is much less complete, but Powell (1952 *b*) and Westley & Lilley (1952) have made valuable studies of different aspects of the subject, and a section on it is included below (§ 8). There is little precise evidence of the effect of heating a jet, but some comments on the theoretical aspects of this are made in §§ 6, 7 and 8.

2. Physical introduction to the theory of aerodynamic noise

It was established in part I that any turbulent flow can be regarded as equivalent, acoustically, to a distribution of quadrupoles, whose strength per unit volume at any instant is

$$T_{ij} = \rho v_i v_j + (p_{ij} - a_0^2 \rho \delta_{ij}), \tag{1}$$

placed in a uniform medium at rest. Here ρ is the density, v_i the fluid velocity, p_{ij} the compressive stress tensor, and a_0 the velocity of sound in the atmosphere. Only the first term in (1), representing momentum flux, is expected to be important in a 'cold' flow, i.e. one where no marked temperature differences exist.

The principal developments of the theory of part I, which are made in this paper, are as follows:

(i) It is necessary to make use of the statistical character of turbulence, and in particular of the fact that the values of the momentum flux at points with no eddy in common are uncorrelated, while its values at points with many eddies in common are well correlated. This fact leads to an expression for the sound field which can be interpreted by saying that separate 'average eddy volumes' can be thought of as giving out their quadrupole radiation independently.

(ii) A further consequence of the statistical character of turbulence is important when the turbulence is superimposed on a mean flow whose Mach number is not negligible. As explained under heading (i) of § 1, this includes all the experiments on jets (a Mach number of 0·3 not being negligible for these purposes). The fact that the rate of distortion of the local eddy pattern is least when viewed in a frame of reference moving with a certain 'local eddy convection velocity' is found to imply that the radiation field of those eddies will be modified, approximately, as was calculated in § 7 of part I for a quadrupole distribution analyzed with respect to a frame moving through the atmosphere (with that local convection velocity). One result of this conclusion (explained physically on p. 583 of part I, and more fully in § 3 below) is the observed fact that jets radiate far more sound forwards than backwards (heading (iii) of § 1).

(iii) Although turbulence without a mean flow has an acoustic power output, which Proudman (1952) has estimated for isotropic turbulence from the expressions of part I (with results which are summarized in § 4 below), nevertheless turbulence of given intensity can generate more sound in the presence of a large mean shear. Also the terms which then dominate the sound radiation field are considerably simpler than in the general case. These results might be expected from the fact that the fluctuations in momentum flux are increased when they consist principally of a shaking to and fro of the large *mean* momentum of the fluid by the smaller *fluctuations* in velocity; a process significant only when the mean momentum has a large gradient. Actually, the important term in the rate of change of momentum flux at a point can be shown to be the product of the pressure and the rate of strain. This is because high pressure at a point causes momentum to tend to flow in the direction of greatest expansion of a fluid element (see figure 2). Accordingly, the quadrupoles in a shear layer are mostly oriented along the principal axes of mean rate of strain, that is, at 45° to the motion. However, this lateral quadrupole field must be distorted (see (ii) above) as in figure 3 of part I when the eddy convection velocity is at non-negligible Mach number. In the case of a jet the heavy shear is near the orifice, and the turbulence there is probably responsible for the high-frequency noise, with its directional maximum at 45° or slightly less, noticed under heading (iv) of § 1.

At the beginning of part I the presence of solid sounding-boards to amplify the sound resulting from turbulent pressure fluctuations was excluded, but it is now seen that such amplification is provided also by a heavy mean shear, which may therefore be described loosely as an 'aerodynamic sounding-board'.

To some extent a similar amplification may arise from large inhomogeneities of temperature or fluid composition in the flow, resulting in a speed of sound notably different in the turbulent flow from that in the atmosphere. The second term in (1) is then important, because the pressure fluctuations in the turbulence are only partly balanced by a_0^2 times the density fluctuations. The unbalanced pressure fluctuations produce, according to the results of part I, §4 (and, indeed, as any externally applied pressures must do), a source field whose instantaneous strength per unit volume is a_0^{-2} times their second time derivative. However, the amplification described here is studied in detail in §6, where it is shown that even when the whole pressure fluctuation is unbalanced the effect may still be a small one because in turbulence the momentum flux fluctuates more vigorously than the pressure. Accordingly, it may be necessary to resist the temptation to use the result to explain the commonly observed excess noise of hot jets over cold jets at the same velocity, which may be due rather to causes like combustion noise (see §1). If the effect is ever important, the discussion of §6 indicates that it will be in the case of high-frequency sound from a turbulent mixing region, especially where heavy gases are used.

In interpreting the results for the cold jet, it should be noted that the lower frequency sound (which carries somewhat more than half the acoustic energy) comes principally (see §1, heading (v)) from that part of the jet where the turbulence is known to be most intense (see, for example, Corrsin & Uberoi 1949). Here the shear is not great enough for the terms described in (iii) above to predominate, and probably the sound field is a mixture of lateral quadrupole radiation, due to the interaction of the turbulence and the mean shear, and less strongly directional radiation due to the turbulence alone (which can be estimated roughly from Proudman's results for isotropic turbulence). The combination of these, when modified to allow for convection (see figures 2 and 3 of part I), would account for the directional maximum being at an angle considerably less than 45° and decreasing to very low values as the Mach number increases or the frequency decreases.

Paradoxically enough, the feature of the experimental results which is the hardest of all to explain, in the light of the theories briefly sketched above, is the accuracy with which the acoustic power output varies as U^8 (where U is the mean exit velocity), although this law was suggested in §6 of part I. For the conclusion of (ii) above (which seems to be inescapable, both on theoretical grounds and because it is impossible to think of any other reason for the greatly reduced sound behind the jet orifice) implies a further increase of power output with Mach number, over and above the U^8 law, as figure 1 of part I makes clear. Now the Mach number at which the principal sound-producing eddies are convected is probably not more than half the Mach number at the jet orifice; it is a familiar idea that eddies in the thin mixing region near the orifice travel at about half the speed of the jet (which one can infer by thinking of them as rigid rollers; observed values of the ratio (Brown 1937) are slightly less); again, the mean velocity on the axis in the region of intensest turbulence ten diameters from the orifice is about two-thirds of the exit velocity, and drops to one-third at twenty diameters. But even if the correction due to convection at only half the exit velocity U is put in, the expected slope of power output against U on a logarithmic scale, based on values between $M = 0{\cdot}5$ and $0{\cdot}9$, is increased from 8 to 9·5.

On the other hand, the measured slope does not exceed 8. Only Fitzpatrick & Lee (1952) have measured acoustic power output directly (by a reverberation chamber method), and they get a slope of almost exactly 8·0, especially at the higher Mach numbers. This deficiency in slope is borne out by measurements of intensity by other investigators, which give slopes comparably below the value expected by the theory as modified by convection (see § 7). The fact (§ 1, heading (v)) that the slope is greater at smaller angles makes it appear likely not that the convection effect (which would cause such a trend) is absent, but that the factors in the *turbulence* which cause sound production are reduced as the velocity increases.

This could be either a Reynolds number or a Mach number effect; the experimental evidence is inconclusive. From the theoretical point of view, however, one could hardly expect increased Reynolds number (which means less viscous damping) to *reduce* the acoustic output, and, indeed, there are arguments for expecting its effect to be small at the already relatively high Reynolds numbers (around 5×10^5) of the jets used. There are, however, two possible explanations of the discrepancy as an additional *Mach number* effect:

(*a*) The size of the effective quadrupoles in the jet (comparable by (i) above to an average eddy size in a certain sense, see § 3 below) may become comparable with the acoustic wave-length at high subsonic Mach numbers; this would lead to a reduction in the Stokes effect (part I, § 2) by which the efficiency of quadrupole radiation increases as the fourth power of the frequency (and hence as U^4). However, the expected values of the relevant eddy sizes at a given frequency do not quite confirm this view (see § 3).

(*b*) Possibly the intensity of the turbulence itself decreases at high subsonic Mach numbers. There is no definite evidence on this point, although it is well known that *supersonic* jets mix much more slowly with the outer air, and so presumably have less intense turbulence in the mixing region. (Typical observations are that in subsonic flow the total included angle of the turbulent mixing region is 15°, and that in supersonic flow it is 5°.) If the damping of turbulence in the mixing region increases *continuously* with Mach number, and already militates significantly against the rate of gain of energy from the mean shearing flow at Mach numbers slightly above 1, then presumably the resulting intensity of turbulence in the fully turbulent part of the jet would be reduced already at high subsonic Mach numbers. Reduced sound radiation from this part would then result.

Now when one inquires why there is greater damping in the turbulent mixing region of a supersonic jet, one is told that it is because disturbances in the jet boundary set up pressure distributions, characteristic of supersonic flow, which tend to reduce the disturbances. But since the linearized theory of supersonic flow is equivalent to the theory of sound, and the phenomenon is unsteady in any case, we see that the suggested cause of damping is in reality the sound field. This leads to the suggestion that the damping is really radiation damping resulting from the energy lost as sound. The present paper shows that turbulence in the layer of heavy shear would radiate an exceptional quantity of sound, increasing rapidly with Mach number, and it is possible that even at high subsonic Mach numbers it may thus lose sufficient energy (in the crucial region where the turbulence is being created most rapidly) to reduce the general level of the turbulence in the jet. (This would be in contradiction to the suggestion of part I, § 1, that the sound radiated would have no significant back-reaction on the flow.) Certainly the observed acoustic power output of the mixing region, at high subsonic Mach numbers, is great enough (see §7) to render plausible that it might reduce the rate of growth of turbulence in this crucial region to some measurable extent. It may be noted, too, that most of the experimental work indicates a fall in the dimensionless frequency parameter nd/U of the peak in the sound spectrum as the Mach number increases (in other words the peak frequency increases less rapidly than U); and this might be because the radiation damping mechanism is more effective at the higher frequencies, as the theory of §5 below would lead us to expect.*

To sum up, it seems likely that at higher Mach numbers the level of turbulence is reduced, especially for higher values of nd/U, where n is frequency and d diameter, because in the mixing region, where the turbulence builds up, it is losing energy by sound radiation. This is invoked to explain both the overall reduction in the intensity field from what would be expected on the convection theory, and also the change in the spectrum, based on the dimensionless frequency nd/U, as M increases.

The remainder of the paper will exhibit, in greater detail than has been possible above, first the extensions to the theory and then the comparison with experiment.

3. Effects of the statistical character of turbulence on the sound it produces

At a given instant the velocity (and hence also the momentum flux, which is the dominant term in the quadrupole strength density T_{ij}) has a highly random variation with position throughout a turbulent flow; and although its values at two points which are fairly close together show some statistical correlation, which, of course, tends to unity as the points tend to coincide, it is well established that there is a distance, say D (crudely, the 'diameter of the largest eddies'), such that the values of the velocity (and hence, doubtless, of T_{ij}) at points separated by distances greater than D are effectively uncorrelated. The first necessity is to make use of this fact in the statistical analysis of the radiation field of the quadrupoles.

The theory takes its simplest form in the case of small Mach number, which will be considered first. But since (§1) the experiments on jets were found possible only at Mach numbers exceeding 0·3, which are not small for the present purposes, it will be found that an extension of the theory to include the principal effects of Mach number is needed to obtain agreement.

A rough argument, which suggests the character of the necessary statistics, follows from the fact that, if the strengths of two point quadrupoles are statistically uncorrelated (in Rayleigh's terminology, 'unrelated in phase'), then the *intensity* field due to both together is the sum of the intensity fields of each separately.

* Of course any mechanism explaining a general reduction in frequency (relative to U/d) with increasing Mach number would explain also a falling away from the U^8 rule. Dr Powell has suggested to the author that such a mechanism might be connected with some statistical dependence of eddy creation at the orifice on the sound field of the jet. However, the author would expect rather a general increase of turbulence level, as the sound effects increased with increasing Mach number, to result from such a mechanism, whereas a decrease at the higher frequencies is what is required.

To apply this result one starts by replacing the correlation curve for T_{ij} by a square form; thus the values of T_{ij} within a certain volume V around a point are taken as perfectly correlated with the value there, and the values outside as perfectly uncorrelated. Here V is a sort of 'average eddy volume', whose value might vary through the field of flow. Now imagine the field divided up into a set of non-overlapping regions which fill it completely, each with volume equal to the local average eddy volume V. All the quadrupoles in one such region fluctuate in phase with the one at the centre; so they can be replaced by a single point quadrupole at the centre, of strength V times the local strength density. But, by hypothesis, all the resulting point quadrupoles are completely uncorrelated, so the intensity field is the sum of the intensity fields due to each separately. Further, each intensity field is proportional to the square of the strength of the quadrupole, and so to V^2. Hence the intensity field resulting from *unit* volume of turbulence is proportional to V, and, indeed, is V times that of a point quadrupole of strength equal to the local strength density T_{ij}.

This crude argument indicates at once that the acoustic power output from a given volume element of turbulence is proportional to some local average eddy volume. This is confirmed by equation (19) of part I for the intensity $I(\mathbf{x})$, namely,

$$I(\mathbf{x}) \sim \frac{1}{16\pi^2\rho_0 a_0^5}\iint \frac{(x_i-y_i)(x_j-y_j)(x_k-z_k)(x_l-z_l)}{|\mathbf{x}-\mathbf{y}|^3\,|\mathbf{x}-\mathbf{z}|^3} \times \overline{\frac{\partial^2}{\partial t^2}T_{ij}\left(\mathbf{y}, t-\frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right)\frac{\partial^2}{\partial t^2}T_{kl}\left(\mathbf{z}, t-\frac{|\mathbf{x}-\mathbf{z}|}{a_0}\right)}\,d\mathbf{y}\,d\mathbf{z}, \quad (2)$$

which shows that the intensity field of the volume element $d\mathbf{y}$ is proportional to the volume integral of the covariance of the value of $\partial^2 T_{ij}/\partial t^2$ at the element with its value at all points $\mathbf{z}$ of the turbulent field. Since the covariance (of which a correlation coefficient is a non-dimensional form) may be expected to fall from a maximum when $\mathbf{z} = \mathbf{y}$ to zero outside a sphere of radius D, the first factor in the integral has negligible variation with $\mathbf{z}$ from its value for $\mathbf{z} = \mathbf{y}$ when $x \gg D$. Further, the integral of the second factor with respect to $\mathbf{z}$ may evidently be regarded as a product of a sort of average eddy volume with the value of the integrand when $\mathbf{z} = \mathbf{y}$ (which is the mean square of $\partial^2 T_{ij}/\partial t^2$). This suggests a way of estimating the intensity field.

The last conclusion would evidently be false if there were a substantial region where the correlation of T_{ij} with its value at the element $d\mathbf{y}$ were negative, and especially if the integrated correlation over this part came near to balancing the integrated correlation over the part where it is positive. Then, too, the previous 'rough' argument would be invalidated, since the correlation curve could not reasonably be replaced by a square form. Actually the sound field would then have to be regarded as essentially an octupole field, as arguments on p. 576 of part I indicate. (In particular, if T_{ij} were of the form $\partial W_{ijk}/\partial x_k$, whose integrated covariance is zero because it can be written as the covariance of the value at a point with the surface integral of the normal component of W_{ijk} over a surface far from the point, then the equivalent octupole strength density would be W_{ijk}.) But the evidence of what follows (§5) will be that the difficulty does not appear to arise in practice.

More seriously, the argument as it stands neglects the differences between the times of emission from different points **y**, **z** of waves which reach simultaneously a given distant point x—between the 'retarded times', in fact, in equation (2). This, however, is permissible at small Mach numbers. For the covariance in (2) is negligible except for points **y**, **z** separated by distances less than D, and for such points the difference between the retarded times is certainly less than D/a_0. This time interval must be negligible compared with times significant in the turbulent fluctuations* if the velocity of sound a_0 is sufficiently great compared with the flow velocities. The covariance can then be treated as a 'simultaneous' covariance (mean product of values at two points at the *same* instant), as has been tacitly assumed above.

But if the Mach number is not small the analysis as given above would run into two difficulties. First, the difference in retarded times would not be negligible, implying (as §4 of part I makes clear) the presence of important octupole fields. Secondly, if eddies were being convected at an appreciable mean Mach number, then the intensity field at a distant point in the downstream direction would for statistical reasons be greater than on the foregoing theory; for the correlation between the values at the two points must then exceed the simultaneous correlation, because the retarded time of the downstream point is later, and at that time the eddy has moved a considerable distance towards the point from its position at the retarded time of the upstream point.

To obtain a better approximation to what happens under these conditions, the considerations of part I, §7, are used. For one can minimize the importance of the difference between the retarded times by analyzing the turbulence in a given neighbourhood in the frame of reference for which the time scale of the fluctuations of $\partial^2 T_{ij}/\partial t^2$ is greatest. This frame will move at what may be called the local 'eddy convection velocity', and its use will also remove the statistical difficulty noted above.

In a region where the mean velocity varies little from some average value, a frame moving with this average velocity would be used; note that in this frame the time scale of the turbulent fluctuations would be of the order of an eddy size divided by a typical *departure* of the velocity from its mean, and this would be large compared with D/a_0, provided only that the Mach number of the velocity fluctuations is small, which will almost always be the case in practice. On the other hand, even in a heavily sheared turbulent layer, as in the mixing region of a jet, it is generally believed that an 'eddy convection velocity' V_c exists, not in the sense that eddies are convected downstream with this velocity unchanged, but that they alter slowest when viewed by an observer moving with this velocity. A typical value for this velocity observed by Brown (1937) at low Reynolds numbers was $0{\cdot}4U$, where U is the mean velocity at the orifice. The phenomenon may not be as different as one might expect at high Reynolds numbers, even with turbulent oncoming flow, because jets have frequently been observed to form their large-scale eddies in the mixing region, due to dynamic instability, in a manner relatively uninfluenced by the small eddies already present. If the time scale of fluctuations in this new frame of reference

* Put another way, this means that the eddy size is small compared with the acoustic wave-length (see heading (a) of §2).

is large compared with D/a_0, as is probably the case, the difference between retarded times can be safely ignored, and the difficulty raised under heading (*a*) of §2 does not arise.

The effect of the use of the moving frame, as obtained in part I, §7, is to multiply the intensity field of an element dy by a factor

$$\left[1-\frac{\mathbf{M}_c\,.\,(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right]^{-6}, \tag{3}$$

where $\mathbf{M}_c$ (the 'Mach number of convection') signifies the eddy convection velocity divided by the *atmospheric* speed of sound a_0.

The question at once arises, however, whether one is justified in using the results of part I, §7, in view of the fact that the eddy convection $a_0\mathbf{M}_c$ is not in general uniform, but varies over the field of flow. One might reasonably claim to do so, provided that $\mathbf{M}_c$ varies only a little in a distance of order D, on the grounds that two points in the turbulent flow have been shown to radiate sound independently if the distance between them exceeds D. However, a further discussion is desirable, and it is shown in the appendix that, provided $\mathbf{M}_c$ is a solenoidal vector field, then by referring T_{ij} to the Lagrangian co-ordinates appropriate to the steady velocity field $a_0\mathbf{M}_c$ (this procedure is the equivalent, for a non-uniform field, of using moving axes), then the integral (2) transforms into an integral of the same form with an additional factor

$$\left[1-\frac{\mathbf{M}_c(\mathbf{y})\,.\,(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right]^{-3}\left[1-\frac{\mathbf{M}_c(\mathbf{z})\,.\,(\mathbf{x}-\mathbf{z})}{|\mathbf{x}-\mathbf{z}|}\right]^{-3}. \tag{4}$$

Provided that $\mathbf{M}_c$ varies only a little in a distance D the two factors in (4) can be equated, giving the result expressed in (3).*

If the angle between the direction of convection and the direction of emission is θ, the factor (3) may be written

$$(1-M_c\cos\theta)^{-6}. \tag{5}$$

This factor can signify a great increase in intensity for small θ, which was explained physically in part I as due to two causes:

(i) The positions of origin of those rays of sound from a single eddy which arrive at a distant point simultaneously fill a greater *volume* when the rays are emitted forward, since the foremost parts of the eddy have moved on before they need emit. This accounts for one factor $(1-M_c\cos\theta)^{-1}$ in the amplitude, and hence for two in the intensity.

(ii) The time interval between emission from the fore and aft parts of an eddy is greater for rays emitted forward and arriving simultaneously at a distant point; hence the 'Stokes effect' (cancelling out of the separate source fields because their

* At first the author felt inclined to use the result (4) directly, with $a_0\mathbf{M}_c$ taken as the local mean velocity (which for a subsonic jet would be very closely solenoidal). There is no theoretical objection to this, but it is questionable whether the extra complication in the mixing region (due to the rapid variation of $\mathbf{M}_c$) corresponds to any improvement of accuracy, owing to the commonly observed fact that there exists some sort of eddy convection velocity for the mixing region as a whole.

total strength is zero) is greatly mitigated in respect of those rays. This accounts for two factors $(1-M_c\cos\theta)^{-1}$ in the amplitude field, and hence for four in the intensity.

The arguments up to this point (including that by which the distribution of quadrupoles in an average eddy volume was combined into a single one) can be summarized graphically by regarding an eddy as four sources spread out over it to form a quadrupole, which is convected along at the local eddy-convection velocity.

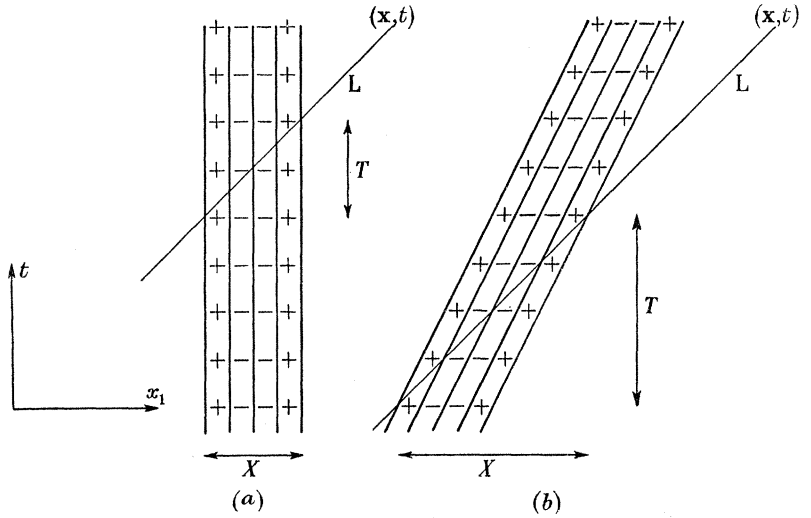

FIGURE 1. Effect of eddy convection in increasing the sound emitted forwards. L = locus of points emitting rays which reach **x** at time t. The quadrupole representing the eddy is stationary in (a), but in (b) it moves towards **x** at a Mach number $M_c = 0{\cdot}5$. This motion increases the effective volume (proportional to X) of the quadrupole, and so increases its strength (for given strength per unit volume T_{ij}). The motion also increases the time delay T between emissions from the four sources, which, by reducing cancellations, increases the intensity at **x** for given quadrupole strength.

The two points just made can then be represented in a simple diagram. Consider for simplicity the case of a longitudinal quadrupole with its axes in the direction of motion, and consider emission in that direction ($\theta = 0$). Figure 1 shows the motion of the quadrupole in the (x, t) plane, and shows also the locus of points which emit rays forward to reach a distant point simultaneously, in the two cases $M_c = 0$ and $M_c = 0{\cdot}5$. In the latter case the increased quadrupole strength due to increased volume and the reduced cancelling of the four sources due to increased time intervals between emission from each, are evident from the figure.

The reader should be warned against two possible misconceptions relating to this way of treating aerodynamic noise at non-negligible Mach number.

(a) It has been stressed that the general theory of part I, in representing the acoustic properties of turbulent flow by means of a distribution of quadrupoles, effectively 'reduces the flow to rest', and it may be wondered how the translation of quadrupoles by the mean motion of the fluid is compatible with this idea. The answer is clear when the situation is stated more exactly: the sound generated is that which the quadrupole distribution T_{ij} per unit volume would emit if placed in a uniform acoustic medium at rest—so that no effects of refraction by the varying velocity and temperature of the stream need be considered in calculating it*—but the variation of the quadrupole distribution with time may, as a result of convection, take the form of a combination of a translation of the whole with other changes which are more random but also more gradual, and when this is so the directional character of the radiated sound field will be affected. In the model, in fact, the quadrupoles can move, but not the fluid.

(*b*) It may be thought that the transformation will cause a great reduction in the value of the integral (which would be inadmissible, at least when M_c is small so that the factor $(1-M_c\cos\theta)^{-6}$ can make little corresponding increase) because the time variations described by $\partial^2 T_{ij}/\partial t^2$ will be much smaller in a frame of reference which moves along with the eddy convection velocity. In other words the false time variations, as measured by a stationary hot wire—which are really space variations—are then omitted. But these space variations would not in any case contribute significantly to the integral, for they constitute an octupole field whose radiation must be small compared with the quadrupole field due to the time variations, at least at low Mach number—in which limiting case the transformation therefore becomes an identity. Thus only the genuine frequencies of the turbulence, as measured in a frame moving with the eddy convection velocity, are relevant to the sound production.

The conclusions of this section can be summarized in a formula for the intensity field, per unit volume of turbulence situated at the origin. This is

$$i(\mathbf{x}) \sim \frac{x_i x_j x_k x_l}{16\pi^2 \rho_0 a_0^5 (x - \mathbf{M}_c . \mathbf{x})^6} \int \overline{\frac{\partial^2}{\partial t^2} T_{ij}(0,t) \frac{\partial^2}{\partial t^2} T_{kl}(\mathbf{z},t)}\, d\mathbf{z}, \tag{6}$$

where $i(\mathbf{x})$ has been written for the intensity at $\mathbf{x}$ per unit volume of turbulence at the origin. Equation (6) is obtained by putting $\mathbf{y} = 0$ in equation (2), dropping the integration with respect to $\mathbf{y}$, and neglecting the difference between $\mathbf{x}-\mathbf{y}$ and $\mathbf{x}-\mathbf{z}$ on the assumption that $\mathbf{x}$ is far from the origin compared with an average eddy size (as well as far compared with $(2\pi)^{-1}$ times an average acoustic wave-length as assumed in part I). The difference in times between the two values of $\partial^2 T_{ij}/\partial t^2$ is also neglected, for the reasons discussed above, and the additional factor $(1-\mathbf{M}_c.\mathbf{x}/x)^6$ is introduced into the denominator, changing x^6 into $(x-\mathbf{M}_c.\mathbf{x})^6$, on the understanding that T_{ij} is specified in a frame moving with the local eddy convection $a_0\mathbf{M}_c$ (so that it is what is called S_{ij} in the appendix).

4. Proudman's estimate of the intensity field of isotropic turbulence

Proudman (1952) has made an approximate calculation of the expression (6) in the case of isotropic turbulence without mean motion, taking the approximate form $\rho_0 v_i v_j$ for T_{ij} as in equation (7) of part I, and of course $M_c = 0$. Actually, he uses a slightly more accurate form than (6), because in writing down the covariance

* Actually, the extent to which this is true in practice was perhaps exaggerated in part I. If one neglects terms in T_{ij} resulting from the sound itself, and in particular the product of a mean-flow velocity at a point with the velocity associated with sound waves from more distant points in the flow, one may be neglecting phenomena which can be regarded essentially as refraction by the mean flow (cf. Lighthill 1953). But it appears that these refractions do not have as marked an effect on the directional distribution of (say) jet noise as does the quadrupole convection effect here discussed.

(which is the mean product of departures from the mean) he takes into account the fact that isotropic turbulence is necessarily decaying, so that time derivatives like $\partial^2 T_{ij}/\partial t^2$ may have a non-zero *mean*, due to the gradual reduction of the turbulent energy. However, this effect would be expected to be very small, and actually in his final estimate Proudman concludes that it is of the order of 1 % of the whole. So it is certainly permissible to neglect it, and to do the same by analogy in problems of noise due to shear flow turbulence, in which again the turbulence, viewed (as was found necessary in § 3) in a moving frame, may be decaying.

With the decay terms omitted, Proudman's transcription of equation (6) becomes

$$i(\mathbf{x}) \sim \frac{\rho_0}{16\pi^2 a_0^5 x^2} \int \overline{\frac{\partial^2}{\partial t^2}(v_x^2(0,t)) \frac{\partial^2}{\partial t^2}(v_x^2(\mathbf{z},t))}\, \mathrm{d}\mathbf{z}, \tag{7}$$

where $x_i v_i/x$ has been rewritten as v_x, the velocity component in the direction of emission. Clearly the integral in (7) is independent of the direction of $\mathbf{x}$ for isotropic turbulence, so that v_x can be replaced by an arbitrary component of $\mathbf{v}$, say v_1, and the directional distribution is that of a simple source field, whose power output (say p_i) per unit volume is

$$p_i = \frac{\rho_0}{4\pi a_0^5} \int \overline{\frac{\partial^2}{\partial t^2}(v_1^2(0,t)) \frac{\partial^2}{\partial t^2}(v_1^2(\mathbf{z},t))}\, \mathrm{d}\mathbf{z}. \tag{8}$$

The covariance in (8) is of the fourth degree in the velocities, but Proudman approximates to it, using an idea of Batchelor (1950), as that combination of covariances of the second degree in the velocities to which it would be equal if the values of v_1 and of its first two derivatives at 0 and $\mathbf{z}$ had a normal joint-probability distribution. There is good evidence that this should give a correct order of magnitude result. Further, with Heisenberg's form of the correlation function for v_1, which has reasonably good experimental support for large Reynolds numbers, and using the Batchelor type of approximation several more times in the analysis, the resulting integral is computed and expression (8) reduced to

$$p_i = \frac{38\rho_0 (\overline{v_1^2})^{\frac{5}{2}} \epsilon}{a_0^5}, \tag{9}$$

where ϵ is the mean rate of dissipation of energy per unit mass.

Proudman concludes also, from the details of his calculations, that at large Reynolds numbers the principal contribution to this power output comes from the larger, non-dissipating eddies; this is presumably because although the mean square of expressions like $\partial^2(v_1^2)/\partial t^2$ would be expected to be greatest for energy-dissipating eddies, the greatly reduced average volume of those eddies renders their total contribution negligible, for reasons made clear in § 3. The question why the coefficient 38 in (9) should be large compared with unity is discussed in § 5.

In the case of isotropic turbulence which is convected through the atmosphere at a mean velocity $a_0 \mathbf{M}_c$, one must expect the intensity field to be modified by a factor $(1 - M_c \cos\theta)^{-6}$ to become

$$i(\mathbf{x}) \sim \frac{p_i x^4}{4\pi(x - \mathbf{M}_c . \mathbf{x})^6}, \tag{10}$$

with power output (cf. (35) of part I)

$$p_i \frac{1 + 2M_c^2 + \frac{1}{5}M_c^4}{(1 - M_c^2)^5}. \tag{11}$$

5. The amplifying effect of shear

It is fairly evident that turbulence of given intensity can generate more sound in the presence of a large mean shear than it would in the absence of a mean flow. For there can be far wider variations of momentum flux if there is a large mean momentum to be shaken about by the turbulent fluctuations in velocity, and conversely a large mean velocity to transport turbulent fluctuations in momentum; in symbols, $\rho v_i v_j$ can fluctuate most widely if the fluctuations of ρv_i are amplified by a heavy mean value $\bar{v}_j$.

At first sight this argument might be thought to apply even if the mean velocity were merely uniform, but this is not so. For the time derivative of $(\rho v_i)\,\bar{v}_j$ is $\bar{v}_j$ times the rate of change of momentum per unit volume, so its integral over all space would be zero if $\bar{v}_j$ were uniform (since total momentum is conserved). The field would therefore be reduced to an octupole field, by § 4 of part I (or by § 3 above). Even if the mean velocity were nearly uniform, not everywhere (which indeed is impossible, since it is assumed in the basic theory to become negligible outside a limited region), but only over an average eddy volume, then the effect would still be absent, because there is a negligible interaction, as far as the generation of sound is concerned, between the values of the quadrupole strength at points a distance greater than D apart (§ 3).

However, where there is a large mean velocity *gradient* (causing considerable changes of velocity across a single eddy), large non-cancelling values of the quadrupole strength density may occur, as we shall see. These are presumably connected with the fact that momentum flux is always most important when it takes place along a large gradient of mean velocity.

To make the arguments precise, the momentum flux $\rho v_i v_j$ is now thrown into a form which brings out directly the importance of velocity gradient. More precisely, the *time derivative* of $\rho v_i v_j$ is used, because this is what is needed in equation (6) and because it was this whose integrated value was shown above to be zero in the case of uniform v_j.

Now evidently
$$\frac{\partial}{\partial t}(\rho v_i v_j) = v_j \frac{\partial(\rho v_i)}{\partial t} + v_i \frac{\partial(\rho v_j)}{\partial t} - v_i v_j \frac{\partial \rho}{\partial t}, \tag{12}$$
and the derivatives on the right-hand side can be expressed as space derivatives by the equations of motion (see part I, equations (2) for the equation of continuity and (3) for Reynolds's form of the momentum equation). In fact
$$\begin{aligned}\frac{\partial}{\partial t}(\rho v_i v_j) &= -v_j \frac{\partial}{\partial x_k}(\rho v_i v_k + p_{ik}) - v_i \frac{\partial}{\partial x_k}(\rho v_j v_k + p_{jk}) + v_i v_j \frac{\partial}{\partial x_k}(\rho v_k) \\ &= p_{ik}\frac{\partial v_j}{\partial x_k} + p_{jk}\frac{\partial v_i}{\partial x_k} - \frac{\partial}{\partial x_k}(\rho v_i v_j v_k + p_{ik} v_j + p_{jk} v_i).\end{aligned} \tag{13}$$

The last of the three terms in (13) is a space derivative and so represents an octupole field (again, by § 4 of part I or § 3 above), and must be expected to radiate relatively little sound, especially at the lower Mach numbers.* The quadrupole

* If the convection effect is non-negligible, the left-hand side of (13) should be replaced by the rate of change in a frame of reference moving with velocity $a_0\mathbf{M}_c$. In the original co-ordinates, this would be $(\partial/\partial t + a_0\mathbf{M}_c.\boldsymbol{\nabla})\,(\rho v_i v_j)$. Since $\mathbf{M}_c$ has been assumed solenoidal, the extra term is also a space derivative (the divergence of $\rho v_i v_j a_0 \mathbf{M}_c$), and in fact helps to reduce the strength of the octupole field, which may be an important consideration at the higher Mach numbers.

terms that remain emphasize the influence of velocity *gradient*, and will clearly fluctuate widely when there is a large mean velocity gradient.

Physically, equation (13) is an expression of the fact (foreshadowed in part I, §§2 and 4) that although there is conservation of mass and momentum—and in particular $\partial\rho/\partial t$ and $\partial(\rho v_i)/\partial t$ are exact space derivatives, so that their integral over a sphere of radius $> D$ is uncorrelated with their value at the centre—there is no similar conservation of momentum *flux*, whose time derivative per unit volume (13) indeed contains residual terms which are not space derivatives. Accordingly, genuine fluctuations in the total momentum flux in such a volume are possible (i.e. integrals of the kind occurring in (6) are not identically zero), leading to a true quadrupole radiation field.

Not only the last term in (13), but also the *viscous* contribution to the stresses p_{ik} and p_{jk} in the first two terms, will here be ignored. For the *mean* viscous stress is always small compared with the turbulent shearing stress $\rho\overline{v_1' v_2'}$, to which, however, the pressure fluctuations (the other term in the stress tensor) are known to be comparable. Again, the *fluctuations* in viscous stress are of order $\mu(\overline{v_1'^2})^{\frac{1}{2}}/\lambda$, where λ is Taylor's dissipation length, and this must be small compared with pressure fluctuations of order $\rho\overline{v_1'^2}$ if the Reynolds number $R_\lambda = \rho(\overline{v_1'^2})^{\frac{1}{2}}\lambda/\mu$ is large compared with 1, which is known to be true in fully developed turbulence.

Retaining therefore only the parts due to pressure fluctuations in the first two terms of (13), they become

$$p\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right) = pe_{ij}, \tag{14}$$

where e_{ij} is the rate-of-strain tensor. Thus as stated under heading (iii) of §2, *the important term in the rate of change of momentum flux at a point is the product of the pressure* and the rate of strain.*†

The reason for this becomes particularly clear if the rate-of-strain tensor is viewed in terms of its principal axes. (These are the axes along which a rod-shaped element of fluid is being simply extended or contracted parallel to itself.) Now, high pressure at a point will tend to increase the flow along an extending element of momentum parallel to itself, and to decrease the flow of such momentum along

* Throughout this paper and part I, the pressure p is understood to be measured from the atmospheric pressure as zero.

† One immediate consequence of this is that the 'source field' (see (13) of part I) resulting from the 'trace' ρv_i^2 of the momentum flux is normally negligible; for the trace of (14) is $2p\,\partial v_i/\partial x_i$, and the divergence of the velocity field is small compared with the individual velocity gradients in subsonic flows. Now (part I, p. 575, where, however, the word 'single' was a mistake) the part of T_{ij} other than the source term can be expressed as a combination of lateral quadrupoles. Thus Proudman's omnidirectional radiation from isotropic turbulence is not in any genuine sense a source field, but is rather a statistical assemblage of lateral quadrupole fields of all orientations, due to the combination of fluctuating pressures and rates of shear in the turbulent flow. (Note that, of course, no correlation between pressure and rate of shear at a point is necessary for the fluctuation of their product to be important.)

The full source field (including the part due to viscous terms in p_{ij}) arises essentially from local fluctuations in the rate of energy dissipation. Many different arguments indicate that these must be associated with sound generation, but it follows from the arguments presented above that this is unimportant compared with the sound associated with fluctuations in lateral momentum flux.

a contracting element. Hence there should be a term in the rate of change of momentum flux with the same principal axes as e_{ij}, and proportional both to it and to the pressure, as has been found above.

Before the result is applied to shear flows, it will first be used to throw some light on the significance of Proudman's results (§4) on isotropic turbulence without a mean flow. Now, replacement of $\partial T_{ij}/\partial t$ in Proudman's theory by pe_{ij} as in (14) indicates (after the usual simplifications resulting from isotropy) that the power output p_i per unit volume in isotropic turbulence is

$$p_i = \frac{1}{\pi\rho_0 a_0^5}\int \overline{\frac{\partial}{\partial t}\left(p\frac{\partial v_1}{\partial x_1}\right)_{\mathbf{x}=0}\frac{\partial}{\partial t}\left(p\frac{\partial v_1}{\partial x_1}\right)_{\mathbf{x}=\mathbf{z}}}\,\mathrm{d}\mathbf{z}. \tag{15}$$

Now, in the course of the calculations described by Proudman (1952) he estimated an integral somewhat similar to that in (15), and obtained the result*

$$\int \overline{\left(\frac{\partial p}{\partial t}\right)_{\mathbf{x}=0}\left(\frac{\partial p}{\partial t}\right)_{\mathbf{x}=\mathbf{z}}}\,\mathrm{d}\mathbf{z} \doteqdot 1\cdot 6\rho_0^2(\overline{v_1^2})^3 L, \tag{16}$$

where L is a scale of turbulence related to the energy-dissipation rate ϵ of equation (9) by the equation $(\overline{v_1^2})^{\frac{3}{2}}/\epsilon = L$; also, to within the accuracy of Proudman's results, L is identical with the so-called 'integral scale' of the turbulence, $\int_0^\infty f(r)\,\mathrm{d}r$ in the usual notation. Hence we can obtain the ratio of the two integrals in (15) and (16) (taking the value of the former from equation (9)) as

$$\frac{\int \overline{\frac{\partial}{\partial t}\left(p\frac{\partial v_1}{\partial x_1}\right)_{\mathbf{x}=0}\frac{\partial}{\partial t}\left(p\frac{\partial v_1}{\partial x_1}\right)_{\mathbf{x}=\mathbf{z}}}\,\mathrm{d}\mathbf{z}}{\int \overline{\left(\frac{\partial p}{\partial t}\right)_{\mathbf{x}=0}\left(\frac{\partial p}{\partial t}\right)_{\mathbf{x}=\mathbf{z}}}\,\mathrm{d}\mathbf{z}} \doteqdot \frac{38\pi}{1\cdot 6}\frac{\overline{v_1^2}}{L^2}. \tag{17}$$

One may reasonably infer from this that the order of magnitude of the fluctuations in velocity gradient that contribute most to the effectiveness of expression (14) in generating sound is the square root of expression (17),† that is, about $8(\overline{v_1^2})^{\frac{1}{2}}/L$. The interest of this result is its indication that, although velocity gradients typical of those in the 'energy-dissipating eddies' are *not* effective in generating sound (see §4), nevertheless, the effective velocity gradients are greater by a factor‡ of order 8 than those typical of the largest 'energy-bearing eddies'. It is an intermediate size of eddy that plays the most important role.

Now, in isotropic turbulence, velocity gradients in all directions contribute to expression (14), and the resulting noise field is omnidirectional. But in turbulence

* The second figure in the coefficient is not claimed to be significant.

† It might be thought that this argument is faulty because the covariance in the numerator would have a zero integral if the p's were removed, and hence (even though they are not) would be expected to be positive in some regions and negative in others. But approximate estimates indicate that the pressure covariance is already small at distances where the velocity-derivative covariance, integrated round a sphere, is negative, and so the 'negative' regions are relatively unimportant.

‡ Strictly, part of this large factor inferred from (17) may be due to the relevant *frequencies* of $\partial v_1/\partial x_1$ exceeding those of p.

with a large *mean* shear (say, a gradient in the x_2 direction of the $\bar{v}_1$ component of mean velocity) a single term $p\bar{e}_{12}$ will dominate expression (14). In words, the mean shear tends to orient the bulk of the quadrupoles along the principal axes of rate of strain, which for a shearing motion are at 45° to the direction of motion. Further,

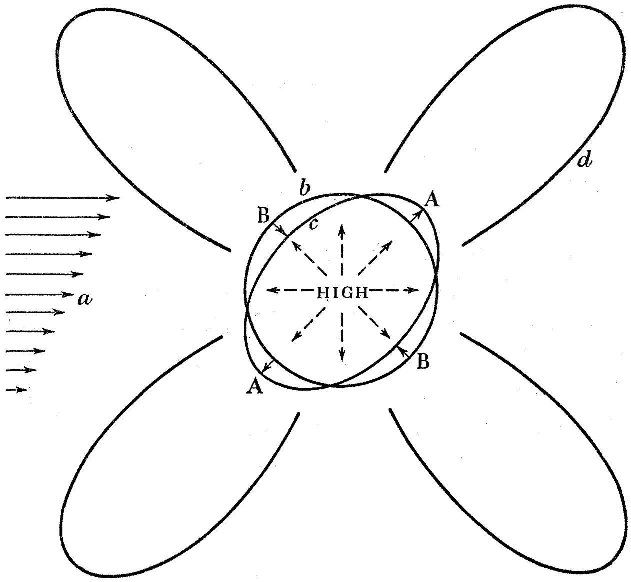

FIGURE 2. Basic lateral quadrupole resulting from combined excess pressure and shear at a point. Shear (a) in small time interval deforms spherical fluid element (b) into ellipsoid (c), with principal axes at 45° to the flow. Excess pressure (HIGH) causes momentum outwards from it to increase (broken-line arrows). At A momentum is being created in the direction of deformation, so there is *increasing* momentum flux outwards. At B momentum is being created in the direction opposite to that of deformation, so there is *decreasing* momentum flux outwards. Fluctuations in this pattern of changing momentum flux, due to fluctuations of either pressure or shear, produce sound radiation with the polar intensity diagram (d). (For the distortion of this directional pattern due to convection see part I, figure 3.) The work of § 5 shows that even sound from isotropic turbulence can be thought of as formed from elements like the one illustrated; but a large mean shear makes them conspicuous by giving them a predominant orientation.

one would expect the sound resulting from given pressure fluctuations to be amplified in this case, at least if $\bar{e}_{12}$ is large enough.* This is the result referred to in § 2 by describing shear as an 'aerodynamic sounding-board'.

The arguments which have shown that pressure fluctuations in shearing flow generate a lateral quadrupole field, with maximum rate of change of momentum

* To estimate how large, note that, according to the indications of the preceding discussion of isotropic turbulence, the *fluctuating* velocity gradients which are effective are of the order $8(\overline{v_1'^2})^{\frac{1}{2}}/L$, where L is the length scale of the energy-bearing eddies. However, there is also a factor $\sqrt{15}$ due to the fact that where only one (lateral) quadrupole orientation is present the sound energy radiated is $\frac{1}{15}$ as much as when all orientations are present in equal strength. (In symbols, $\int (x_1^2 x_2^2/x^6)\,\mathrm{d}S = \dfrac{1}{15}\int x^{-2}\mathrm{d}S$, where the integrals are taken over a large sphere.) Finally, then, amplification will result if the mean shear exceeds about $30(\overline{v_1'^2})^{\frac{1}{2}}/L$ (unless, perhaps, the inference from the discussion of isotropic turbulence is inapplicable to heavily sheared turbulence).

flux (and, in consequence, maximum radiated sound) at 45° to the direction of motion, are illustrated graphically in figure 2.

We now rewrite expression (6) for the intensity field, per unit volume of turbulence situated at the origin, on the assumption that the expression (14), which is used to replace $\partial T_{ij}/\partial t$, is dominated by the term

$$p\frac{\partial \bar{v}_1}{\partial x_2} = p\bar{e}_{12}. \tag{18}$$

It becomes

$$i(\mathbf{x}) \sim \frac{x_1 x_2 \bar{e}_{12}(0)}{8\pi^2 \rho_0 a_0^5 (x - M_c x_1)^6} \int x_k x_l \bar{e}_{kl}(\mathbf{z}) \overline{\frac{\partial p(0,t)}{\partial t}\frac{\partial p(\mathbf{z},t)}{\partial t}}\, d\mathbf{z} \tag{19}$$

(where the direction of eddy convection has been assumed parallel to the only significant mean velocity $\bar{v}_1$). It is clearly not too bad an approximation in (19) to neglect all components of $\bar{e}_{kl}(z)$ except $\bar{e}_{12}$ and $\bar{e}_{21}$, because the covariance in (19) is significant only when $\mathbf{z}$ is fairly close to the origin, so that in most cases the mean shears at $\mathbf{z}$ and the origin would be somewhat alike in orientation; and further, the approximation at most slightly distorts the *directional* dependence of the sound produced; it cannot alter the power output because the integral of

$$x_1 x_2 x_k x_l/(x - M_c x_1)^6$$

is zero unless k and l are 1 and 2 in some order.

Making this approximation, and writing $\tau(\mathbf{z})$ for $\bar{e}_{12}(\mathbf{z})$, the component of mean shear *in the direction of the shear at the origin*, (19) can be written as

$$i(\mathbf{x}) \sim \frac{x_1^2 x_2^2}{(x - M_c x_1)^6} \frac{\tau^2(0)\,\overline{(\partial p/\partial t)^2}\, V}{4\pi^2 \rho_0 a_0^5}, \tag{20}$$

where in accordance with the ideas of §3 an 'average eddy volume' has been introduced, whose precise definition is

$$V = \int \frac{\tau(\mathbf{z})}{\tau(0)} \frac{\overline{\partial p(0,t)/\partial t\, \partial p(\mathbf{z},t)/\partial t}}{\overline{(\partial p(0,t)/\partial t)^2}}\, d\mathbf{z}. \tag{21}$$

The power output p_s per unit volume, under these shear flow conditions, is obtained by integrating (20) over a large sphere centre $\mathbf{x} = 0$ (the integration was already carried out in part I, §7) as

$$p_s = \frac{1 + 5M_c^2}{(1 - M_c^2)^4} \frac{\tau^2(0)\,\overline{(\partial p/\partial t)^2}\, V}{15\pi \rho_0 a_0^5}. \tag{22}$$

The gain in power due to convection of the lateral quadrupole at the Mach number M_c was plotted in part I, figure 1, curve (ii); the directional distribution at different Mach numbers of convection M_c was shown in part I, figure 3.

The mode of expression of the results as in (20) and (22), which was convenient in the case of isotropic turbulence, has disadvantages in the case of shear-flow turbulence. To use it requires an approximate knowledge of $\overline{(\partial p/\partial t)^2}$ and V. The former might be estimated by measurements with a microphone at points successively closer to the mixing region, followed by an extrapolation to the mixing region itself (the microphone must not, of course, be placed in the flow).* It is difficult to do anything with V, however, except equate it to some other average eddy volume, measured more simply by integrating velocity correlations. The velocity correlation measurements could be carried out for convenience in a low Mach number flow (though such measurements have not yet been made, to the author's knowledge, in the mixing region of a round jet).

However, the procedure described suffers from a serious inaccuracy; it ignores the fact that the $\tau(\mathbf{z})/\tau(0)$ term in the integral (21) weights it in favour of places near the origin where the mean shear is greatest.† Also, it is unsuitable for giving the total power output of a shear layer because p_s will vary so much across the layer. A more satisfactory approximation for estimating the *total* power output of a thin shear layer, namely,

$$\frac{1+5M_c^2}{(1-M_c^2)^4}\frac{1}{15\pi\rho_0 a_0^5}\iint \tau(\mathbf{y})\,\tau(\mathbf{z})\,\overline{\frac{\partial p(\mathbf{y},t)}{\partial t}\frac{\partial p(\mathbf{z},t)}{\partial t}}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{z}, \tag{23}$$

is to assume that the covariance varies much more slowly, as $\mathbf{y}$ and $\mathbf{z}$ cross the layer, than $\tau(\mathbf{y})$ and $\tau(\mathbf{z})$ do. The integrations across the layer can then be carried out directly; if U is the change in velocity across the layer, (23) becomes

$$\frac{1+5M_c^2}{(1-M_c^2)^4}\frac{U^2}{15\pi\rho_0 a_0^5}\iint \overline{\frac{\partial p(\mathbf{y},t)}{\partial t}\frac{\partial p(\mathbf{z},t)}{\partial t}}\,\mathrm{d}S_{\mathbf{y}}\,\mathrm{d}S_{\mathbf{z}}, \tag{24}$$

where $\mathrm{d}S_{\mathbf{y}}$ is an element of area of the shear layer in $\mathbf{y}$ space, and similarly with $\mathrm{d}S_{\mathbf{z}}$. This gives a result

$$\Pi_s = \frac{1+5M_c^2}{(1-M_c^2)^4}\frac{U^2\overline{(\partial p/\partial t)^2}\,S}{15\pi\rho_0 a_0^5} \tag{25}$$

for the power output per unit *area* of turbulent shear layer, where

$$S = \int \frac{\overline{\partial p(\mathbf{y},t)/\partial t\,\partial p(\mathbf{z},t)/\partial t}}{\overline{(\partial p(y,t)/\partial t)^2}}\,\mathrm{d}S_{\mathbf{z}} \tag{26}$$

is an 'average eddy area' at the point $\mathbf{y}$. There would be somewhat fewer sources of error in an estimate of the total acoustic power output of a shear layer by means of (25) than in one based on (22). Data not available in the existing literature, in particular $\overline{(\partial p/\partial t)^2}$ and S (the latter to be replaced, presumably, by a simpler definition of average eddy area), are needed, but neither presents insuperable difficulties of measurement.

* Note that, since the time derivative is one in a frame of reference moving with the eddy convection velocity, one cannot obtain $\overline{(\partial p/\partial t)^2}$ by attaching a differentiating circuit to the stationary microphone. Rather one would have to measure $\overline{p^2}$, and then (cf. part I, §5) multiply by $(2\pi N)^2$, where N is a frequency typical of the radiation from the shear layer.

† Another procedure, if this term is ignored, would be to take $\overline{(\partial p/\partial t)^2}\,V$ as given by Proudman's estimate (16) of the resulting integral in the case of isotropic turbulence. This, however, is likely to be a serious underestimate for another reason. In a mixing region, pressure fluctuations are almost certainly not of order $\rho\overline{v_1'^2}$ (as they are in isotropic turbulence, and as (16) implies) but rather are more closely comparable with $\rho\bar{v}_1 v_1'$, just as for small disturbances to laminar flow.

It is easily shown that the result (25) applies not only to a plane shear layer, but also (for example) to the annular mixing region of a round jet, provided that in (26) area is taken to mean projected area on to the tangent plane at $\mathbf{y}$. (This is because $\tau(\mathbf{z})$ must be taken as the component of the shear in the direction of the shear at $\mathbf{y}$.) Further, as a result of the combination of the lateral quadrupoles in the (x_1, x_2) and (x_1, x_3) planes, the directional distribution is proportional to

$$\frac{x_1^2(x_2^2+x_3^2)}{(x-M_c x_1)^6} = \frac{\sin^2\theta\cos^2\theta}{x^2(1-M_c\cos\theta)^6}, \tag{27}$$

as in (36) of part I, where θ is the angle between the direction of emission and the jet axis.

Now the indication of the experimental work is that the sound from the annular mixing region is dominated by the lateral quadrupole terms discussed above at all subsonic Mach numbers (see §7 for the detailed evidence). This sound is at high frequency (note that frequencies in shear-flow turbulence are expected to be of the order of magnitude $(2\pi)^{-1}$ times the mean shear, because of the term $v_2\,\mathrm{d}\bar{v}_1/\mathrm{d}x_2$ in the acceleration $\mathrm{D}v_1/\mathrm{D}t$), and so is easily distinguished from the sound emanating from the 'core' of the jet. Without this experimental confirmation it would have been difficult to predict with confidence that the octupole term in (13) would not become important at the higher Mach numbers (but see the footnote following that equation for some considerations relevant to this issue).

One more point needs to be considered. Does the integrand in either the average eddy volume V (equation (21)) or the average eddy area S (equation (26)) remain, on the whole, positive, as was observed to be necessary in §3 if the methods of estimation of sound production are to be valid? Now, in these integrals, the covariance would be expected to remain positive in *incompressible* flow. For, first, the pressure covariance remains positive according to Batchelor's calculations (1950) for isotropic turbulence, and Proudman's (1952) indicated that this is probably also true of $\partial p/\partial t$. And, secondly, there is no known mechanism in incompressible flow tending to associate an increasing pressure at one point with a decreasing pressure at any neighbouring point. But for a *compressible* fluid the pressure trends are associated with density trends, and total conservation of mass implies that the integrated covariance of $\partial p/\partial t$ is zero. However, the distance between points such that increasing pressure at one is correlated with decreasing pressure at the other is of the order of half the acoustic wave-length. Hence, when this exceeds the size of the larger eddies, the central volume where the covariance has large positive values is surrounded by a widespread annular region where it is very small but negative, in such a way that the total volume integral is zero. But the integrand in (21) is weighted by the term $\tau(\mathbf{z})/\tau(0)$ and the integral (16) is only a surface integral. In both cases the values of the integrand which are relevant are confined to the thin shear layer. Hence the cancellation does not occur in either integral, because only a thin slice of the region of negative covariance, which is required to cancel the central core of positive covariance, is available. (Similar considerations may apply to equation (30) in §6 below.)

6. The amplifying effect of inhomogeneities of temperature or composition

This section treats of the sound associated with the terms in T_{ij} other than the momentum flux. These become significant only when the local mean velocity of sound varies widely in the fluid, as a result of inhomogeneities of temperature or of fluid composition. Thus, such inhomogeneities may be regarded as amplifying the sound due to turbulence, just as shear was shown to do in § 5. But evidently this amplification would be important only if the additional sound were at least comparable with the sound resulting from momentum flux; the theory given below indicates that in many important cases this is not so. In particular, it will be concluded that any large apparent extra noise from hot jets is probably due to causes (such as combustion noise, or simply higher speed) other than the direct amplification due to heat discussed in this section; the amplification, if observed at all, should affect principally the high-frequency components of the jet noise.

The main term in T_{ij} (see (1)) other than momentum flux is $(p - a_0^2\rho)\,\delta_{ij}$. (The unimportance of the viscous contribution to p_{ij} was fully discussed in part I.) The sound field resulting from this term has a simple physical interpretation. It is the source field due to that part of the turbulent pressure fluctuations which is *not* balanced by the multiple $a_0^2\rho$ of the local density fluctuations accompanying it in the *free* acoustic vibrations of the atmosphere. Evidently the part not so balanced can produce forced oscillations.

At a point where the mean velocity of sound is $\bar{a}$, the pressure fluctuations, being nearly adiabatic, would be approximately $(\bar{a})^2$ times the fluctuations in density. Hence they lead to a term in T_{ij} which is approximately $(1-(a_0/\bar{a})^2)\,p\delta_{ij}$, corresponding by part I, p. 575, to a source strength per unit volume

$$\left(1-\left(\frac{a_0}{\bar{a}}\right)^2\right)\frac{1}{a_0^2}\frac{\partial^2 p}{\partial t^2}. \tag{28}$$

It is interesting to note that the simple-minded idea that the sound emitted from turbulent flow would include 'the source field corresponding directly to the turbulent pressure fluctuations'—whose strength per unit volume, in the sense used in part I, p. 572 (*rate of change* of the rate at which mass is introduced), would be $\partial^2/\partial t^2$ of the corresponding density fluctuations, or approximately

$$\frac{1}{a_0^2}\frac{\partial^2 p}{\partial t^2} \tag{29}$$

—is correct only when $(\bar{a})^2 \gg a_0^2$, as, for example, in the limiting case of flows much hotter than the atmosphere into which they radiate sound. But even in this case it will be seen below that the noise due to momentum flux may be more important.

A larger effect is obtainable if values of $\bar{a}$ considerably *less* than a_0 are present; these are achieved most conveniently by the use of a gas of high molecular weight. Thus for SF_6 (Freon) $a_0/\bar{a}$ is about 2·3, so that (28) would be $-4{\cdot}3$ times (29), and the associated intensity fields would be in the ratio 18 : 1. The importance of the pressure terms relative to the momentum flux terms would be enhanced by this factor. Probably this is the main application in which the pressure terms would be really important. (Note that also the higher density of SF_6 increases the sound output, but to the same extent for both sets of terms.)

The power output of the source field (28), per unit volume of turbulence at the origin, is easily approximated by the methods of § 3 as

$$\frac{1}{4\pi\rho_0 a_0^5}\left[1-\left\{\frac{a_0}{\bar{a}(0)}\right\}^2\right]\int\left[1-\left\{\frac{a_0}{\bar{a}(\mathbf{z})}\right\}^2\right]\overline{\frac{\partial^2 p(0,t)}{\partial t^2}\frac{\partial^2 p(\mathbf{z},t)}{\partial t^2}}\,d\mathbf{z}, \tag{30}$$

where the factor due to quadrupole convection is omitted for simplicity. The basic argument, indicating that (30) is normally smaller than the power output due to momentum flux (at least in the cases where $\bar{a} > a_0/\sqrt{2}$, so that the square-bracketed factors have modulus less than 1), depends on the fact that the latter output takes a form in which, essentially, $\partial(pe_{ij})/\partial t$ replaces the $\partial^2 p/\partial t^2$ of (30). (See, for example, (15), in the case of isotropic turbulence.) Hence it will be larger if $(2\pi)^{-1} e_{ij}$ exceeds in magnitude the dominant frequencies for pressure fluctuations. If, in addition, the dominant frequencies for e_{ij} exceed those for the pressure, it will be larger still. Both suppositions are likely to be true, though more so in turbulence which is qualitatively similar to isotropic turbulence than in heavily sheared turbulence.

For isotropic turbulence one can be fairly precise. Replacing the square-bracketed factors in (30) by 1 (as in the case of very hot flows), the integral becomes expression (16), whose value is about $1{\cdot}6\rho_0^2(\overline{v_1^2})^3 L$, but with two time differentiations instead of one. Also one can estimate the integral without any time differentiations, from Batchelor's (1950) results, as $2\rho_0^2(\overline{v_1^2})^2 L^3$. These two results indicate that the dominant frequency for pressure fluctuations is of order $(\overline{v_1^2})^{\frac{1}{2}}/L$, and permits an estimate of the integral in (30) as $\rho_0^2(\overline{v_1^2})^4/L$ times a coefficient of order 1. Thus in this case (30) is less than the power output of isotropic turbulence (9) by a factor less than 10^{-2}. Even in the extreme case of Freon mentioned above (which introduces an extra factor of 18), the term (30) must still remain negligible.

These conclusions for isotropic turbulence can be assumed to hold qualitatively for the sound emitted from the 'core' of a jet, but not to that emanating from the heavily sheared mixing region. There the frequencies with which the pressure fluctuates may not fall so far below $(2\pi)^{-1}$ times the mean shear $\bar{e}_{12}$; further, the frequencies with which e_{ij} fluctuates are no longer relevant. In addition, the limitation to a lateral quadrupole field (as against a source field) gives the radiation due to $\bar{e}_{12}\partial p/\partial t$ a disadvantage by a factor $\frac{4}{15}$ (as a comparison of (22) and (30) shows). As an upper-bound estimate of the amount of amplification, due to high temperature, of the noise emanating from the mixing region, one may therefore take $10\log_{10}(1+\frac{15}{4}) = 6$ decibels, but one would regard a lower figure as more likely. The amplification will be greatest in the *highest* frequency bands of this noise (which is itself the high-frequency part of the *total* jet noise), because the relative importance of $\partial^2 p/\partial t^2$ to $\bar{e}_{12}\partial p/\partial t$ is greatest in those bands.

In the light of these estimates there is evidently some possibility of amplification due to the use of a Freon jet being really large (up to 19 db) at the higher frequencies, quite apart from the influence of the larger density. The evidence of Lassiter &

Hubbard's (1952) measurements with a Freon jet is somewhat negative on this point, but not conclusive since no frequency analysis was recorded (see § 8 for a further discussion).

Generally, the arguments of this section indicate that pressure fluctuations are likely to be of small importance in generating aerodynamic noise through their direct source field. (It is fairly evident that this conclusion would remain true even in the case of pressure fluctuations which were not, for some reason, approximately adiabatic, since it has been shown that the source field (29) is relatively ineffectual even when it is not mitigated by corresponding density fluctuations.) Accordingly one may reaffirm the general conclusion of part I, that where no fluctuating external forces or fluctuating sources of matter are present the principal source of noise generation is the fluctuation in the momentum flux across fixed surfaces.

7. Discussion of the experimental work on the cold subsonic jet

The experimental work (Fitzpatrick & Lee 1952; Gerrard 1953; Lassiter & Hubbard 1952; Westley & Lilley 1952) on the cold subsonic jet, referred to in general terms in §§ 1 and 2, will here be discussed in detail. The different aspects referred to in §§ 1 and 2 will be considered in the order adopted there.

Direct measurements of acoustic power output have been made only by Fitzpatrick & Lee (1952), who obtained them from measurements of intensity in a reverberation chamber which had been previously calibrated, in certain of the octave bands used, by means of a source of similar directional characteristics. The result (their figure 5) is close to an eighth-power law variation with the jet exit velocity U. The acoustic power coefficient

$$K = \frac{\text{acoustic power}}{\rho_0 U^8 a_0^{-5} d^2} \tag{31}$$

(which the authors adopt from (26) of part I) takes values between $0{\cdot}6 \times 10^{-4}$ and $1{\cdot}2 \times 10^{-4}$ as U varies by a factor of $2\frac{1}{2}$ and the jet diameter d by a factor of 2. There is very little evidence that the scatter is systematic, although the authors suggest tentatively a Reynolds number effect. For the larger Mach numbers, K was consistently close to $0{\cdot}9 \times 10^{-4}$.

Other authors measure only intensity, at various positions round the jet. Thus Lassiter & Hubbard (1952) give intensity as a function of the azimuth angle θ (measured from the orifice as origin) at a jet exit velocity $U = 1000\,\text{ft./s}$, for $d = \frac{3}{4}$ to 12 in. Performing a rough integration of their values, and making a guess at the values behind the orifice (which were not measured, but which other workers have found to be low), one obtains an average value of the acoustic power coefficient K, over the cases considered, as $0{\cdot}5 \times 10^{-4}$. For the smallest values of the orifice diameter d the value of K was sensibly lower, at $0{\cdot}3 \times 10^{-4}$.

But such a small variation of K over a large range of d indicates that acoustic power output is very closely proportional to d^2; in fact, Lassiter & Hubbard find intensities closely proportional to $(d/r)^2$ for large r/d, and other authors who have varied both d and r are in accord. Also, in two experiments, Lassiter & Hubbard varied the density of the jet from the atmospheric value ρ_0 to a value ρ_1 (that of He and SF_6). The intensities changed by a factor $(\rho_1/\rho_0)^2$. This is what might be expected from the theory (e.g. equation (2)), since T_{ij} contains a factor ρ_1. Thus, where the density of flow ρ_1 differs from that, ρ_0, of the atmosphere into which it radiates sound,* a modified acoustic power coefficient

$$K = \frac{\text{acoustic power}}{\rho_1^2 \rho_0^{-1} U^8 a_0^{-5} d^2} \tag{32}$$

is expected to remain approximately constant (at least if the amplification effects discussed in § 6 are unimportant).

Gerrard (1953) integrates his measurements of acoustic intensity round a large sphere, to obtain the acoustic power coefficient as a function of mean Mach number M at the jet orifice. He finds that $K = 0{\cdot}4 \times 10^{-4} M^{-0{\cdot}8}$ for a jet from a circular pipe 36 in. long, and $K = 0{\cdot}8 \times 10^{-4} M^{-0{\cdot}3}$ for a jet from a pipe 54 in. long (for both pipes the diameter $d = 1$ in.). Thus he finds that acoustic power increases as the (7·2)th and (7·7)th powers of the velocity in the two cases.

From these three bits of evidence (with varying degrees of turbulence in the oncoming stream, from that produced in the converging nozzle of Lassiter & Hubbard, through that of Gerrard's two straight pipes, up to that in Fitzpatrick & Lee's straight nozzle preceded by a sudden divergence) we have general agreement that in the range of Mach numbers covered ($0{\cdot}3 < M < 1$) the acoustic power coefficient K is between $0{\cdot}3 \times 10^{-4}$ and $1{\cdot}2 \times 10^{-4}$, with $0{\cdot}6 \times 10^{-4}$ as a typical value. The acoustic efficiency

$$\eta = \frac{\text{acoustic power}}{\frac{1}{2}\rho_0 U^3 \frac{1}{4}\pi d^2} = \frac{8}{\pi} K M^5 \tag{33}$$

is usually slightly greater than $10^{-4} M^5$, but this is certainly its order of magnitude, and the evidence is inadequate to justify stating more than this.

The spectrum of the sound is very broad; an estimate of this is given by the width of the frequency band in which the intensity per octave remains within 20 db of its peak value. From the curves of Fitzpatrick & Lee, performing a slight extrapolation, we may estimate this as seven octaves. Westley & Lilley (1952) obtain six octaves at small angles to the jet axis, and higher values at larger angles, for which, however, the extrapolation becomes impossible since measurements were not taken at high enough frequencies. Lassiter & Hubbard (1952) obtain between six and seven octaves.

The 'peak' of the spectrum is very flat. Gerrard (1953) obtains values of acoustic power within 2 db over a range of three octaves around the peak. Westley & Lilley (1952) obtain a similar result for their intensity measurements at 60 and 75° to the jet axis; they find more pronounced peaks at the higher frequencies in the range in question for larger angles, and at the lower frequencies for smaller angles. Fitzpatrick & Lee (1952) obtain sharper peaks, with a variation of 4 to 7 db over the top three-octave band, but one might in this matter prefer the verdict of the other authors, whose calibration procedure is to be preferred for such a purpose.

Evidently the peak frequency n_p is in all cases ill-defined by the measurements, but rough values can be obtained. It is natural to express the results in terms of a 'Strouhal number' $n_p d/U$, on the assumption that the frequency is related to

* A heated jet is an important case of this.

frequencies characteristic of the energy-bearing eddies in the turbulence, which would be closely proportional to U/d. Peak Strouhal numbers obtained by Fitzpatrick & Lee (1952) are between 0·3 and 0·6. Lassiter & Hubbard (1952) publish spectra with clear peaks, whose Strouhal number remains constant at 0·5 as U is varied; they remark, however, that these were found under conditions when the stream emerging from the orifice was more turbulent than in their main series of experiments, in which they state that less increase of peak frequency with velocity was observed. The peaks are especially ill-defined in the curves of Gerrard (1953) and Westley & Lilley (1952), which have a 'plateau' rather than a peak, but in each case the 'plateau' contains the frequency corresponding to Strouhal number 0·5.

As a summary of the situation we can say that the peak frequency is always *in the neighbourhood of* $U/2d$, but the evidence regarding its variation with U is inconclusive. The indications of the results of Gerrard and of Westley & Lilley are against proportionality to U. Fitzpatrick & Lee's spectra for $d = 1{\cdot}53$ in. are compatible with it to within the experimental scatter, but those for $d = 0{\cdot}765$ in. are interpretable in this way only if some systematic error be assumed; they are much more easily interpreted as indicating a peak frequency (4000 c/s) independent of jet velocity. Lassiter & Hubbard state that their spectra showing direct proportionality to velocity were not typical of experiments on jets issuing directly from a reservoir through a converging nozzle. Thus the balance of evidence is in favour of a slower increase of frequency with velocity than is given by direct proportionality, but the matter remains in doubt.* (The issue was discussed theoretically at the end of § 2.)

Passing to the directional distribution, the most striking fact, at once evident on walking round one of the jets, was that almost all the sound was radiated in directions making an acute angle with the jet. This is also well known for jet engines running on the ground (if the high-pitched 'compressor whine' is ignored).

Thus, for a jet at Mach number 1, Westley & Lilley (1952) obtain intensities about 15 db lower behind the jet (at angles $\theta > 90°$ to the jet axis) than at the directional maximum, and this is persistently true in each frequency band except the two highest used. Gerrard (1953) obtains similar results at high subsonic Mach numbers, with smaller differences at lower Mach numbers.

The detailed directional distribution in the forward region ($\theta \leqslant 90°$) is best understood for the sound of higher and lower frequencies separately. Lassiter & Hubbard (1952) show in their figure 10 that, when $M = 0{\cdot}8$, the band of Strouhal numbers $nd/U > 1{\cdot}5$ (where n = frequency) has a directional maximum at $\theta = 40°$, that the band from 0·4 to 0·8 has a maximum at $\theta = 30°$, and that the band from 0·2 downwards has intensities increasing monotonically as θ is reduced, at least as far as $\theta = 15°$. Westley & Lilley (1952), with $M = 1$, find a maximum at $\theta = 40°$ for Strouhal numbers $> 0{\cdot}7$, and a maximum at $\theta = 30°$ in the band 0·3 to 0·7, with monotonically increasing intensities in bands below this. Gerrard (1953), at the higher Mach numbers, finds a maximum at $\theta = 40°$ for a Strouhal number exceeding

* Even if the peak frequency is independent of velocity, as much of the evidence indicates, results must still, if they are to be useful, be referred to the only significant non-dimensional parameter nd/U. The result mentioned then signifies that the intensity increases more slowly with U (for the Mach number range of the experiments) at higher values of nd/U than at lower ones.

0·6, one at $\theta = 35°$ for a Strouhal number of 0·5, and a maximum at 20° or less from 0·3 downwards.

These observations indicate that a fairly definite division exists between 'high-frequency' sound (with frequencies sensibly above the peak frequency, say $0{\cdot}7U/d$ or more, and with a directional maximum slightly less than 45°, as expected theoretically from a heavily sheared flow), and 'low-frequency' sound (with frequencies sensibly below the peak frequency, say $0{\cdot}3U/d$ or less, and with a preference for forward emission). The phrases high and low frequency will be used in this sense henceforth.

In the light of the theory it would be natural to assume that the high-frequency sound, with its 'lateral quadrupole' type of directional distribution, emanates from the heavily sheared mixing region near the orifice; and that the low-frequency sound, with its distribution characteristic of omnidirectional radiation (as modified, of course, by the convection of the quadrupole sources), emanates from the region of more nearly isotropic turbulence farther downstream in the 'core' of the jet. Sound of intermediate frequency would be a combination of both types.

This interpretation is especially likely to be correct because it is known that the highest frequency turbulence occurs in the mixing region. A direct check is not, however, easy. Actual plots of contours of equal intensity, for a given frequency band, as made by Gerrard (1953) and by Westley & Lilley (1952), are not very helpful for this purpose, because the measurements near the jet contain sound other than that of the radiation field, and because the far field results are hard to interpret for this purpose owing to the large total extent of the source. Perhaps the best approach is on the lines hinted at by Lassiter & Hubbard (1952), who state without details that they tackled the problem by using 'a directional type of microphone and various recording equipment' (perhaps correlation measurements at two distant points in line with a point of the jet). They found (though again they give no details) that 'the higher frequency components emanate from the region immediately outside the jet pipe'.

Gerrard (1953) determines a 'source' of the higher frequency sound by studying the contours of equal intensity at large distances and drawing a *common perpendicular* to them(at about 40°, the directional maximum) to cut the axis at the source position. This varies from one to nine diameters downstream of the orifice as M increases from 0·3 to 1·0 at a frequency of 8500 c/s. (It is worth remarking that the largest value of M corresponds to a Strouhal number of 0·65 and so is barely in the high-frequency range.) The same approach applied to Westley & Lilley's measurements at $M = 1$ in the frequency range 6400 to 12 800 c/s gives a 'source' eight diameters downstream, which agrees with Gerrard. On the other hand, their intensity measurements at these frequencies in the near field are greatest at about five diameters from the orifice. Taken together the results quoted are sufficiently in accord with the general pattern suggested above.

Gerrard (1953) has also attempted to determine the 'eddy-convection Mach number' M_c (introduced in §3 above) for this high-frequency sound. To fit his observed radiation field he assumes a combination of a lateral quadrupole (as postulated in §5) with such a pair of longitudinal quadrupoles as will give an axisymmetrical intensity field whose directional maximum is at $\theta = 90°$. (Some term of this kind is necessary because without it there would be no sound at all at $\theta = 90°$.) He modifies the sum of the two intensity fields by the factor $(1-M_c \cos\theta)^{-6}$ and then gets best fit by taking $M_c = 0{\cdot}4M$, where M is the mean Mach number at the orifice. Such a result for the Mach number M_c of convection of the eddies is in good agreement with observation of the motion of eddies in a low-speed jet mixing region (Brown 1937).

The source of the low-frequency sound appears to be very extended; contour lines in the near field (Gerrard 1953; Westley & Lilley 1952) are indicative of a line source extending from say five to twenty diameters downstream of the orifice. Westley & Lilley conclude explicitly from their measurements that 'the jet contains a distribution of sound sources the extent of which increases in the downstream direction as the frequency is decreased'. Lassiter & Hubbard (1952) state that they located the source 'several' diameters downstream, in contrast to their findings on the high-frequency sound.

They also located the source of the *total* noise as being similarly placed, indicating that what we have called the low-frequency noise constituted the greater part of the total noise. The figures quoted above concerning the directional distribution in different frequency bands also indicated that rather (though not much) more than half the total energy is in the low-frequency noise.

It is appropriate therefore to seek to explain the observed order of magnitude $10^{-4} M^5$ for the acoustic efficiency in terms of the sound emitted from the core of the jet, where the turbulence is not strongly sheared, and so not qualitatively different from isotropic turbulence. Then it is reasonable to use Proudman's estimate $38(\overline{v_1'^2})/a_0^5$ (see (9)) for the ratio of the rate at which energy is radiated at a point to the rate at which it is dissipated. Now measurements by Corrsin & Uberoi (1949), of the mean velocity and turbulent energy distributions at a cross-section of a round jet twenty diameters downstream of the orifice, show that at that cross-section three-quarters of the initial energy flux has been dissipated. (Of the remaining quarter, about 20 % is in the form of turbulent energy.) Accordingly, one may insert for $\overline{v_1'^2}$ in Proudman's estimate a typical value in the range zero to twenty diameters downstream, to obtain the order of magnitude of the acoustic efficiency. Inserting a value $(\overline{v_1'^2})^{\frac{1}{2}} = 0{\cdot}08U$ (which Corrsin & Uberoi (1949) find is exceeded only for $6{\cdot}5 < x/d < 17$ on the axis, where it rises to a maximum of $0{\cdot}11U$, but with smaller values taken away from the axis, at least at $x/d = 20$, though possibly less so for smaller x/d), we obtain

$$38(0{\cdot}08M)^5 = 1{\cdot}2 \times 10^{-4}M^5 \tag{34}$$

for the acoustic efficiency, which is in agreement with the experiments.* Of course no more than order of magnitude agreement is significant here.

* Similarly, one would like to make use of (25) to explain that a fair proportion, perhaps one-third, or about $2\times 10^{-5}\rho_0 U^8 a_0^{-5} d^2$, of the total acoustic power output is radiated from the shear layer. But, as noted in §5, one is prevented at present by total ignorance of the value of $\overline{(\partial p/\partial t)^2}$ in the shear layer. Taking the average eddy area S as d^2, and the area of the shear layer as $\pi d \times 5d$, one would obtain the power output stated (neglecting the convection effect) if the root mean square of $\partial p/\partial t$ were $0{\cdot}008\rho_0 U^3/d$. Whether this is actually a correct order of magnitude result in the mixing region is, however, a matter of pure conjecture at present.

These considerations on acoustic efficiency do not take into account the effects predicted as due to quadrupole convection. As indicated in §2 these would be expected to produce a further addition to the power output (over and above the U^8 rule) at the higher subsonic Mach numbers, which is not observed in practice. Hence one must scrutinize the varying distribution of directional radiation to determine whether its trend is consistent with an explanation on which the translation effect is present at the higher Mach numbers but is balanced, as far as its effect on total power output is concerned, by some general reduction, either in turbulence level, or in those aspects of the turbulence which generate sound.

That this is so is at once indicated by results on the variation of intensity with velocity in the direction $\theta = 90°$, at which the convection effect must be absent since $(1-M_c\cos\theta)^{-6} = 1$. Measurements in this direction happen to be especially numerous, and yield intensities varying as powers of the velocity considerably lower than those typical of the total noise. Lassiter & Hubbard (1952) get an exponent 5·5 for their $\frac{3}{4}$ in. jet, 7·0 for their 3 in. jet, and 6·3 for their 12 in. jet (fair scatter would be expected in these as each value is based on only four points). Westley & Lilley get 6·3 (note that their $\mathfrak{P}$ varies like the (2·45)th power of the velocity at such a typical subsonic Mach number as 0·7). Gerrard gets from 6·3 to 6·8 under different conditions. These results indicate that the basic acoustic output is increasing less rapidly than U^8, and that it is only the quadrupole convection effect, causing more rapidly increasing intensities for smaller θ, which restores the measured power output to proportionality to U^8 or slightly less. Roughly, the intensity at $\theta = 90°$ (in the range of Mach numbers treated) is less by about a factor of $U^{1·5}$ than what is expected, which corresponds as pointed out in §2 to the factor by which U^8 is less than the expected total acoustic power output, including the convection terms.

The few measurements of the simultaneous dependence of intensity on velocity U and angle θ for lower values of θ support this conjecture. Gerrard (1953) finds (as do Westley & Lilley) that proportionality to $U^{8·2}$ is typical for $\theta \leqslant 30°$. It is significant, too, that while his measurements for $\theta = 30°$ show almost perfect proportionality to such a power of U, those for $\theta = 90°$ indicate that the intensity was increasing as a lower power of U for higher Mach numbers than for lower ones. Since the convection factor $(1-M_c\cos\theta)^{-6}$ which is present for $\theta < 90°$ increases as a higher power for the higher Mach numbers than for the lower ones, such a result is in accordance with the theory. Gerrard notes also that the behaviour which he observes for $\theta > 90°$ is qualitatively in accord with what would be expected from the presence of a factor $(1-M_c\cos\theta)^{-6}$ in the intensity.

To sum up, it seems likely that the interpretation of the directional distribution of intensity in terms of quadrupole convection is correct. Refraction by the mean flow in the jet may affect finer details, but it does not appear to be fundamental. However, the analysis leading to the proportionality of intensity to U^8 in the absence of the convection effect is at fault, at least at the higher subsonic Mach numbers. The possible sources of this error have been fully discussed in §2.

8. Some notes on the noise of supersonic jets

The information on the noise of supersonic jets is less systematic than in the subsonic case, and although some trends can be observed there is more divergence between results of different experiments.

Thus Powell (1952*b*) obtained a sound field dominated by a 'whistle or screech' with a more or less well-defined note. Westley & Lilley (1952) and Lassiter & Hubbard (1952) obtained no such effect. However, in the latter case the supersonic speed was obtained by the use of a helium jet, the jet speed being less than the speed of sound in helium, though considerably greater than that in air. Accordingly the shock wave structure, to which Powell (1952*b*) attributes the whistle, was absent. It is harder to see why Westley & Lilley (1952) obtained no whistle; the geometry in their experiments was similar to that in Powell's, except that a cylindrical pipe of length two or three diameters intervened in their case between the contraction cone and the orifice.

Lassiter & Hubbard measured intensity of noise from their helium jet principally at $\theta = 90°$, where no effect of quadrupole translation would be expected. Accordingly the sound intensity, after correction for the density of the jet (it was explained in § 7 that it would vary as the square of this density), would be expected to lie on a continuous curve through the subsonic points. This it does, and the curve takes the form of proportionality to $U^{5\cdot 2}$. But it is surprising that this exponent is so much below other measured values for $\theta = 90°$. The points on this curve obtained for Freon, air and helium could all be explained on the basis of a higher velocity exponent (say $U^{6\cdot 3}$) if dependence on a higher power of the density (say $\rho^{2\cdot 4}$) could be assumed, but this is very difficult to believe, especially for the low-density fluid. A tentative explanation could be that the velocity exponent really falls to 5·2 only at the *higher* speeds, owing to the radiation damping effect discussed in § 2, and that it appears to be so low in the case of the low-speed Freon jet only because this experiences the amplification discussed at the end of § 6.

Lassiter & Hubbard give also (for one jet speed, $U = 2620$ ft./s) a polar diagram for the intensity of noise from their helium jet. It is very strongly directional, with a maximum at $\theta = 42°$, where the intensity level is 15 db higher than at $\theta = 90°$.

Now the explanation of this maximum from the point of view of this paper bears no relation to the explanation of the directional maximum of the high-frequency sound from subsonic jets in terms of a lateral quadrupole field. The sound measured in the experiments was not of high frequency in this sense. The suggested explanation is rather that the phenomenon results from *supersonic convection of quadrupoles*. An eddy convected at Mach number M_c would produce, on the simple theory of § 3, infinite sound at an angle $\theta = \sec^{-1} M_c$. The infinity is not, of course, really present, because the theory contains approximations (especially the neglect of certain differences in times in the integral for the intensity) which if omitted would leave a finite value; but presumably this finite value would be large. Now $\sec^{-1} M_c$ would be 42° if M_c were 1·35, which is $0{\cdot}58M$, where $M = 2{\cdot}33$ is the ratio of the jet speed to the atmospheric speed of sound. Such a value is of the right order of magnitude.

Physically, the explanation given means that the circular wave fronts emitted by an eddy as it moves downstream have a conical envelope whose semi-angle is the well-known Mach angle, and whose direction of propagation is the complement, $\sec^{-1} M_c$, of this Mach angle. The intensity is greatest at such an envelope of wave fronts because a large number of sound pulses are heard simultaneously,* and because the times of emission are spread over a large interval, so that cancelling of different pulses is minimized.

Mr Lilley has shown the author unpublished Schlieren photographs of supersonic jets and their instantaneous sound-wave patterns, which show a strong maximum at an angle of about 30° to the jet, which he interprets in terms of the 'ballistic shock waves' emitted by the eddies while they are travelling supersonic. This explanation is very closely similar to that given above.†

Passing to the features of supersonic jets which are perhaps more closely associated with the steady shock-wave pattern in the jet (and so would be absent in Lassiter & Hubbard's helium jets, or in hot jets at speeds between the speed of sound in the atmosphere and that in the jet), one may notice first an additional sound maximum at $\theta = 100°$ found in unpublished work by Lilley & Peduzzi. The author would tentatively ascribe this to sound radiated as a result of turbulence passing through the stationary shock-wave pattern in the jet. In a recent paper he gave a theory of this sound radiation (Lighthill 1952), according to which the energy radiated would be a certain fraction, depending on the shock strength, of the kinetic energy of the turbulence. For a shock of pressure ratio 1·3 the fraction would be 6 %, and it would increase with shock strength, but unfortunately the theory was not expected to be accurate for pressure ratios much above 1·3. It is fairly evident, however, that the process might lead to acoustic efficiencies, for a supersonic jet, comparable with those already discussed. For the flux of turbulent energy through shocks in the jet might well amount to about 1 % of the total energy flux at the orifice, and if 6 % of this quantity was radiated, the efficiency would be 6×10^{-4}, which already exceeds the efficiency of a jet at $M = 1$. (Note, too, that the pressure ratio across any *normal* shock exceeds 1·3 for $M > 1{\cdot}12$.)

Directionally, the theory (Lighthill 1953) indicates that, in a frame of reference in which the air ahead of the shock is at rest, the sound radiated has a strong preference for emission in directions nearly parallel to the direction of shock-wave propagation (for reasons closely connected with the quadrupole convection effect discussed in this paper). Indeed, it forms part of a greater volume of sound, of which the rest is directed so close to the direction of motion of the shock that it catches up with it, and is largely absorbed by it. The source of this sound is at a distance behind the shock comparable with the length scale L of the turbulence. Evidently the 'freely scattered' sound which just misses the shock must, in a frame of reference in which the shock is stationary, be radiated in directions nearly parallel with the shock or making small angles with it in the upstream direction. The part associated

* Compare the well-known explanation of the so-called 'sonic bangs'.

† Again, hot jets from turbojet engines often show such a maximum when the jet speed sufficiently exceeds the atmospheric speed of sound, and the explanation probably holds good in this case too.

with the normal or nearly normal shocks (which are the strongest shocks in the jet) would then be expected to have a directional maximum close to the 100° observed by Lilley & Peduzzi, although evidently some of the sound in question would be refracted more into the upstream direction as it passed through the edge of the jet. The wave-length of the sound would be expected to be comparable with the scale of the turbulence, and this too agrees with the visual observations of Lilley & Peduzzi. However, this suggested explanation must remain very tentative.

To conclude this section some reference must be made to the analysis by Powell (1952*b*) of the 'whistle or screech' referred to above, although Powell's theory is already so complete that no further contribution is required. Briefly, the theory is that under favourable conditions the shock pattern in the jet may introduce a resonant frequency. For any specially strong eddy, created at the orifice, may (after amplification as it sweeps along the shear layer) send out a sound wave as it passes through the first shock, which, when it reaches the orifice, may cause the appearance of another strong eddy. This mechanism yields the observed frequency, and also suggests conditions for the appearance of the whistle which have some experimental support. The mechanism bears an obvious analogy to the mechanism of edge tones, which has recently received some attention (Curle 1953; Powell 1952*a*). The shock wave replaces the edge. As with edge tones the presence of the phenomenon greatly amplifies the sound produced, so that the first step towards reducing supersonic jet noise is to devise methods for avoiding the whistle, which Powell has successfully done.

References

Batchelor, G. K. 1950 *Proc. Camb. Phil. Soc.* **47**, 359.
Brown, G. B. 1937 *Proc. Phys. Soc.* **49**, 493.
Corrsin, S. & Uberoi, M. 1949 *Tech. Notes Nat. Adv. Comm. Aero., Wash.*, no. 1865.
Curle, N. 1953 *Proc. Roy. Soc.* A, **216**, 412.
Fitzpatrick, H. M. & Lee, R. 1952 *Rep. David W. Taylor Model Basin, Wash.*, no. 835.
Fleming, N. 1946 *J. R. Aero. Soc.* **50**, 639.
Gerrard, J. H. 1953 Ph.D. thesis, Manchester.
Lassiter, L. W. & Hubbard, H. H. 1952 *Tech. Notes Nat. Adv. Comm. Aero., Wash.*, no. 2757.
Lighthill, M. J. 1952 *Proc. Roy. Soc.* A, **211**, 564 (part I).
Lighthill, M. J. 1953 *Proc. Camb. Phil. Soc.* **49**, 531.
Mawardi, O. K. & Dyer, I. 1953 *J. Acoust. Soc. Amer.* **25**, 389.
Powell, A. 1952*a* *Rep. Aero. Res. Comm., Lond.*, no. 15333 (limited circulation), to appear in *Acoustica.*
Powell, A. 1952*b* *Rep. Aero. Res. Comm., Lond.*, no. 15623 (limited circulation), to appear in *Proc. Phys. Soc.*
Proudman, I. 1952 *Proc. Roy. Soc.* A, **214**, 119.
Westley, R. & Lilley, G. M. 1952 *Rep. Coll. Aero. Cranfield*, no. 53.

Appendix. Effect of non-uniformity of the eddy convection velocity on the distortion of the directional pattern due to convection

As stated in § 4, we refer T_{ij} to the Lagrangian variable associated with the steady velocity field $a_0 M_c(\mathbf{y})$. Thus we put

$$T_{ij}(\mathbf{y}, t) = S_{ij}(\mathbf{a}, t), \tag{A 1}$$

where the relation between $\mathbf{y}$ and $\mathbf{a}$ (for given t) is that the solution of

$$\frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = a_0 \mathbf{M}_c(\boldsymbol{\xi}), \quad \boldsymbol{\xi} = \mathbf{a} \text{ at } t = 0, \tag{A 2}$$

is $\boldsymbol{\xi} = \mathbf{y}$ at time t. If $\mathbf{M}_c$ is a solenoidal vector field then $\mathrm{d}\mathbf{y} = \mathrm{d}\mathbf{a}$, and also

$$\left(\frac{\partial}{\partial t}\right)_{\mathbf{a}} = \left(\frac{\partial}{\partial t}\right)_{\mathbf{y}} + a_0 \mathbf{M}_c \cdot \frac{\partial}{\partial \mathbf{y}}. \tag{A 3}$$

Now the integral in (2) is the mean square of the integral

$$\int \frac{(x_i - y_i)(x_j - y_j)}{|\mathbf{x}-\mathbf{y}|^3} \frac{\partial^2}{\partial t^2} T_{ij}\left(\mathbf{y}, t - \frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right) \mathrm{d}\mathbf{y}. \tag{A 4}$$

When T_{ij} is replaced by S_{ij}, the argument of S_{ij} will be the Lagrangian variable $\boldsymbol{\eta}$ associated with $\mathbf{y}$ at the time $t - |\mathbf{x}-\mathbf{y}|/a_0$. Now to obtain the Jacobian $\mathrm{d}\boldsymbol{\eta}/\mathrm{d}\mathbf{y}$, we can ignore the part *independent* of y in the time shift $t - |\mathbf{x}-\mathbf{y}|/a_0$ separating $\mathbf{y}$ from $\boldsymbol{\eta}$, because volumes are unchanged in the solenoidal velocity field $a_0 \mathbf{M}_c$, and concentrate only on the small variations in time shift in the small volume $\mathrm{d}\mathbf{y}$. But for these the mapping is simply the change to moving axes studied in part I, § 7, and so

$$\frac{\mathrm{d}\boldsymbol{\eta}}{\mathrm{d}\mathbf{y}} = 1 - \frac{\mathbf{M}_c \cdot (\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}. \tag{A 5}$$

Hence (A 4) becomes

$$\int \frac{(x_i - y_i)(x_j - y_j)}{|\mathbf{x}-\mathbf{y}|^3} \left(1 - \frac{\mathbf{M}_c \cdot (\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right)^{-1} \left(\frac{\partial}{\partial t} - a_0 \mathbf{M}_c \cdot \frac{\partial}{\partial \boldsymbol{\eta}}\right)^2 S_{ij}\left(\boldsymbol{\eta}, t - \frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right) \mathrm{d}\boldsymbol{\eta}. \tag{A 6}$$

In (A 6) the $\partial/\partial\boldsymbol{\eta}$ does not apply to the term $t - |\mathbf{x}-\mathbf{y}|/a_0$, although that varies with $\boldsymbol{\eta}$ for fixed t. Rewriting (A 6) so that it does apply to that term too, the

$$\partial/\partial t - a_0 \mathbf{M}_c \cdot \partial/\partial\boldsymbol{\eta}$$

becomes

$$\left(1 - \mathbf{M}_c \cdot \frac{\partial}{\partial \boldsymbol{\eta}} |\mathbf{x}-\mathbf{y}|\right) \frac{\partial}{\partial t} - a_0 \mathbf{M}_c \cdot \frac{\partial}{\partial \boldsymbol{\eta}}. \tag{A 7}$$

The second term in (A 7) can now be neglected, because $\partial \mathbf{M}_c/\partial\boldsymbol{\eta} = 0$ and using the divergence theorem. (Note that in this process no derivatives of terms on the first line of (A 6) will arise, because they vanish in the radiation field.)

But it may easily be calculated that

$$\mathbf{M}_c \cdot \frac{\partial}{\partial \boldsymbol{\eta}} |\mathbf{x}-\mathbf{y}| = - \frac{\mathbf{M}_c \cdot (\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}| - \mathbf{M}_c \cdot (\mathbf{x}-\mathbf{y})}. \tag{A 8}$$

Substituting, we obtain finally as the equivalent (in the radiation field) of (A 4),

$$\int \frac{(x_i - y_i)(x_j - y_j)}{|\mathbf{x}-\mathbf{y}|^3} \left(1 - \frac{\mathbf{M}_c \cdot (\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right)^{-3} \frac{\partial^2}{\partial t^2} S_{ij}\left(\boldsymbol{\eta}, t - \frac{|\mathbf{x}-\mathbf{y}|}{a_0}\right) \mathrm{d}\boldsymbol{\eta}. \tag{A 9}$$

On taking the mean square, by writing down the same integral with a different letter $\mathbf{z}$ instead of $\mathbf{y}$ (and consequentially a different letter $\boldsymbol{\zeta}$ instead of $\boldsymbol{\eta}$), and averaging their product, we obtain the result stated in § 3, equation (4).