EAE 298 Aeroacoustics
Fall Quarter 2016
Homework #2

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Problem 1. Solution of Lilley's Equation and its Application

An axisymmetric jet of radius R_j has an exit mean velocity of W_j and an exit mean density of $\overline{\rho_j}$. The ambient mean velocity is zero and the ambient mean density is $\overline{\rho_0}$. Lilleys equation for a parallel axisymmetric flow can be written

$$\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)^3 p' - \left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)\left(\overline{a^2}\nabla^2 p'\right) - \frac{d\overline{a^2}}{dr}\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)\frac{\partial p'}{\partial r} + 2\overline{a^2}\frac{dW}{dr}\frac{\partial^2 p'}{\partial z \partial r} = S(\vec{x},t)$$

where

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}$$

and $W(r)$ and $\overline{a^2}(r)$ are the radial distributions of the axial velocity and speed of sound squared.

Part (a)

[25 points] Seek solutions of Lilleys equation in the form

$$p'(r,\theta,z,t) \sim P(r)\exp[i(kz+n\theta-\omega t)]$$

Show that Lilleys equation reduces to

$$\frac{d^2P}{dr^2}+\left\{\frac{1}{r}-\frac{1}{\overline{\rho}}\frac{d\overline{\rho}}{dr}+\frac{2k}{(\omega-kW)}\frac{dW}{dr}\right\}\frac{dP}{dr}+\left\{\frac{(\omega-kW)^2}{\overline{a^2}}-k^2-\frac{n^2}{r^2}\right\}P = RHS$$

where $\overline{\rho}(r)$ is the radial distribution of the mean density. (Note: $\overline{a^2}=\gamma\overline{p}/\overline{\rho}$ and $\overline{p}$ is constant.)

Solution

Taking the solution for p' as

$$p'(r,\theta,z,t) = P(r)\exp[i(kz+n\theta-\omega t)]$$

we can solve for several terms. The first term is:

$$\begin{aligned}
\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)^3 p' &= \left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)^2\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)p' \\
&= \left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)^2\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)P(r)\exp[i(kz+n\theta-\omega t)] \\
&= \left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)^2(-i\omega+Wik)\,P(r)\exp[i(kz+n\theta-\omega t)] \\
&= \left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)(-i\omega+Wik)^2\,P(r)\exp[i(kz+n\theta-\omega t)] \\
&= (-i\omega+Wik)^3\,P(r)\exp[i(kz+n\theta-\omega t)] \\
&= i\,(\omega-kW)^3\,P(r)\exp[i(kz+n\theta-\omega t)]
\end{aligned}$$

The second term is very lengthy. Starting with $\nabla^2 p'$, we have:

$$\begin{aligned}\nabla^2 p' &= \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]p' + \left[\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]p' + \left[\frac{\partial^2}{\partial z^2}\right]p' \\ &= p'_r + p'_\theta + p'_z\end{aligned}$$

$$\begin{aligned}p'_r &= \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]p' \\ &= \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]P(r)\exp[i(kz+n\theta-\omega t)] \\ &= \left[\frac{1}{r}\frac{\partial}{\partial r}(r)\right]\frac{dP}{dr}\exp[i(kz+n\theta-\omega t)] \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{dP}{dr}\exp[i(kz+n\theta-\omega t)]\right] \\ &= \frac{1}{r}\left[\frac{dP}{dr}\exp[i(kz+n\theta-\omega t)] + r\frac{d^2P}{dr^2}\exp[i(kz+n\theta-\omega t)]\right] \\ &= \frac{1}{r}\frac{dP}{dr}\exp[i(kz+n\theta-\omega t)] + \frac{d^2P}{dr^2}\exp[i(kz+n\theta-\omega t)] \\ &= \left[\frac{1}{r}\frac{dP}{dr} + \frac{d^2P}{dr^2}\right]\exp[i(kz+n\theta-\omega t)]\end{aligned}$$

$$\begin{aligned}p'_\theta &= \left[\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]p' \\ &= \left[\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]P(r)\exp[i(kz+n\theta-\omega t)] \\ &= \left[\frac{1}{r^2}\right]i^2n^2P(r)\exp[i(kz+n\theta-\omega t)] \\ &= -\frac{n^2}{r^2}P(r)\exp[i(kz+n\theta-\omega t)]\end{aligned}$$

$$\begin{aligned}p'_z &= \left[\frac{\partial^2}{\partial z^2}\right]p' \\ &= \left[\frac{\partial^2}{\partial z^2}\right]P(r)\exp[i(kz+n\theta-\omega t)] \\ &= i^2k^2P(r)\exp[i(kz+n\theta-\omega t)] \\ &= -k^2P(r)\exp[i(kz+n\theta-\omega t)]\end{aligned}$$

$$\begin{aligned}\nabla^2 p' &= \left[\frac{1}{r}\frac{dP}{dr} + \frac{d^2P}{dr^2}\right]\exp[i(kz+n\theta-\omega t)] + \left(-\frac{n^2}{r^2} - k^2\right)P(r)\exp[i(kz+n\theta-\omega t)] \\ &= \left\{\left(-\frac{n^2}{r^2} - k^2\right)P(r) + \left[\frac{1}{r}\frac{dP}{dr} + \frac{d^2P}{dr^2}\right]\right\}\exp[i(kz+n\theta-\omega t)]\end{aligned}$$

Pluggin in the solution for $\nabla^2 p'$, we can further solve:

$$
\begin{aligned}
-\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)\left(\overline{a^2}\nabla^2 p'\right) =& -\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)\times \\
&\overline{a^2}\left\{\left(-\frac{n^2}{r^2}-k^2\right)P(r)+\left[\frac{1}{r}\frac{dP}{dr}+\frac{d^2P}{dr^2}\right]\right\}\exp[i(kz+n\theta-\omega t)] \\
=& i\left(\omega-kW\right)\times \\
&\overline{a^2}\left\{\left(-\frac{n^2}{r^2}-k^2\right)P(r)+\left[\frac{1}{r}\frac{dP}{dr}+\frac{d^2P}{dr^2}\right]\right\}\exp[i(kz+n\theta-\omega t)]
\end{aligned}
$$

Finally, the third and fourth terms on the LHS can be solved:

$$
\begin{aligned}
-\frac{d\overline{a^2}}{dr}\left[\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)\frac{\partial}{\partial r}\right]\rho' &= -\frac{d\overline{a^2}}{dr}\left[\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)\frac{\partial}{\partial r}\right]P(r)\exp[i(kz+n\theta-\omega t)] \\
&= -\frac{d\overline{a^2}}{dr}\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)\frac{dP}{dr}\exp[i(kz+n\theta-\omega t)] \\
&= -\frac{d\overline{a^2}}{dr}\left(i\left(kW-\omega\right)\frac{dP}{dr}\right)\exp[i(kz+n\theta-\omega t)] \\
&= \frac{d\overline{a^2}}{dr}\left\{i\left(\omega-kW\right)\frac{dP}{dr}\right\}\exp[i(kz+n\theta-\omega t)] \\
&= -\left(\frac{\gamma\overline{p}}{\overline{\rho}^2}\right)\frac{d\overline{\rho}}{dr}\left\{i\left(\omega-kW\right)\frac{dP}{dr}\right\}\exp[i(kz+n\theta-\omega t)] \\
&= -\left(\frac{\overline{a^2}}{\overline{\rho}}\right)\frac{d\overline{\rho}}{dr}\left\{i\left(\omega-kW\right)\frac{dP}{dr}\right\}\exp[i(kz+n\theta-\omega t)]
\end{aligned}
$$

$$
\begin{aligned}
+\left[2\overline{a^2}\frac{dW}{dr}\frac{\partial^2}{\partial z\partial r}\right]p' &= \left[2\overline{a^2}\frac{dW}{dr}\frac{\partial^2}{\partial z\partial r}\right]P(r)\exp[i(kz+n\theta-\omega t)] \\
&= \left[2\overline{a^2}\frac{dW}{dr}\frac{\partial}{\partial z}\right]\frac{\partial}{\partial r}P(r)\exp[i(kz+n\theta-\omega t)] \\
&= \left[2\overline{a^2}\frac{dW}{dr}\right]\frac{\partial}{\partial z}\frac{dP}{dr}\exp[i(kz+n\theta-\omega t)] \\
&= \left[2\overline{a^2}\frac{dW}{dr}\right]ik\frac{dP}{dr}\exp[i(kz+n\theta-\omega t)] \\
&= \left\{2ik\overline{a^2}\frac{dW}{dr}\frac{dP}{dr}\right\}\exp[i(kz+n\theta-\omega t)]
\end{aligned}
$$

Summing all these terms, we have

$$
\begin{aligned}
& i\left(\omega-kW\right)^{3} P(r) \exp[i(kz+n\theta-\omega t)] \\
+ i\left(\omega-kW\right)\overline{a^2}\left\{\left(-\frac{n^2}{r^2}-k^2\right)P(r)+\left[\frac{1}{r}\frac{dP}{dr}+\frac{d^2P}{dr^2}\right]\right\} & \exp[i(kz+n\theta-\omega t)] \\
-\left(\frac{\overline{a^2}}{\overline{\rho}}\right)\frac{d\overline{\rho}}{dr}\left\{i\left(\omega-kW\right)\frac{dP}{dr}\right\} & \exp[i(kz+n\theta-\omega t)] \\
+\left\{2ik\overline{a^2}\frac{dW}{dr}\frac{dP}{dr}\right\} & \exp[i(kz+n\theta-\omega t)] = S(\vec{x},t)
\end{aligned}
$$

We can divide by $i\overline{a^2}\left(\omega-kW\right)\exp[i(kz+n\theta-\omega t)]$ to find:

$$
\begin{aligned}
& \frac{\left(\omega-kW\right)^2}{\overline{a^2}}P(r)+\left\{\left(-\frac{n^2}{r^2}-k^2\right)P(r)+\left[\frac{1}{r}\frac{dP}{dr}+\frac{d^2P}{dr^2}\right]\right\} \\
& \qquad -\frac{1}{\overline{\rho}}\frac{d\overline{\rho}}{dr}\left\{\frac{dP}{dr}\right\}\left\{\frac{2k}{\left(\omega-kW\right)}\frac{dW}{dr}\frac{dP}{dr}\right\} = \frac{S(\vec{x},t)}{i\overline{a^2}\left(\omega-kW\right)\exp[i(kz+n\theta-\omega t)]}
\end{aligned}
$$

Rearranging for similar powers of P, we have:

$$
\begin{aligned}
& \frac{d^2P}{dr^2}+\left\{\frac{1}{r}-\frac{1}{\overline{\rho}}\frac{d\overline{\rho}}{dr}+\frac{2k}{\left(\omega-kW\right)}\frac{dW}{dr}\right\}\frac{dP}{dr} \\
& \qquad +\left\{\frac{\left(\omega-kW\right)^2}{\overline{a^2}}-k^2-\frac{n^2}{r^2}\right\}P(r) = \frac{S(\vec{x},t)}{i\overline{a^2}\left(\omega-kW\right)\exp[i(kz+n\theta-\omega t)]}
\end{aligned}
$$

We find:

$$
\frac{d^2P}{dr^2}+\left\{\frac{1}{r}-\frac{1}{\overline{\rho}}\frac{d\overline{\rho}}{dr}+\frac{2k}{\left(\omega-kW\right)}\frac{dW}{dr}\right\}\frac{dP}{dr}+\left\{\frac{\left(\omega-kW\right)^2}{\overline{a^2}}-k^2-\frac{n^2}{r^2}\right\}P(r) = RHS
$$

where

$$
RHS = \frac{S(\vec{x},t)}{i\overline{a^2}\left(\omega-kW\right)\exp[i(kz+n\theta-\omega t)]}
$$

Part (b)

[25 points] Determine the general form of solution (solution of the homogeneous equation) for the pressure fluctuation outside the jet in the ambient medium where the sources vanish. Make sure the solution is chosen to ensure decaying solutions or outgoing waves.

Part (c)

[25 points] Determine the general form of solution (solution of the homogeneous equation) in the potential core region where the mean velocity and density are constant and equal to the jet exit values.

Part (d)

[25 points] Consider a case in which the real jet is replaced by a vortex sheet at $r = R_j$. If the solutions are to be matched at the vortex sheet, describe what matching conditions should be applied. Give both the physical description and the mathematical expressions.