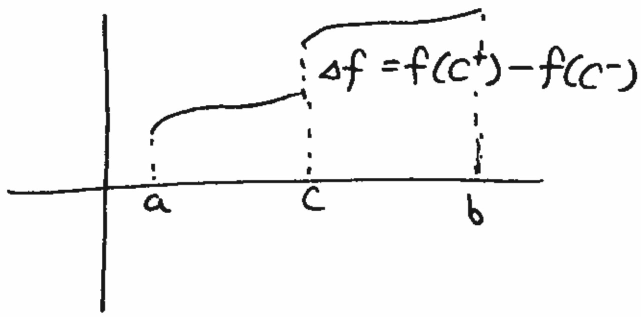
Ffowcs Williams – Hawkings Equation

– Generalized differentiation

Let $f(x)$ be an ordinary piecewise continuous function with a single ~~function~~ jump Δf at $x=c$.



Assume f' exists in $[a,c)$ and $(c,b]$ (f' does not exist at c)

Then, there is an ordinary function $g(x)$ continuous in $[a,b]$ and $g'(x)$ exists $(=f'(x))$ such that $f(x) = g(x) + \Delta f\, h(x-c)$

~~Therefore~~ where h is the ~~heavy~~ heaviside function.

We use a bar over the derivative symbol to denote generalized differentiation. Then, the generalized derivative of $f(x)$ is defined as

$$\boxed{\frac{\bar{d}f}{dx} = \bar{f}'(x) = g'(x) + \Delta f\,\delta(x-c) \quad \text{or} \quad \bar{f}'(x) = \underbrace{f'(x)}_{\hookrightarrow \text{ ordinary derivate of } f} + \Delta f\,\delta(x-c)} \quad \text{Eq. (1)}$$

where $g'(x)$ is the ordinary derivative of $f(x)$. $\delta(x)$ is ~~the~~ Dirac delta function.

One significant fact to remember about generalized differentiation is that for all x such that $a<c<x$, we have

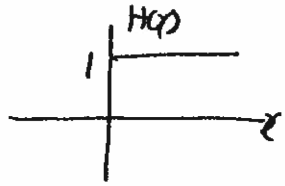
$$\int_a^x \frac{\bar{d}f}{dx}dx = \int_a^x \bar{f}'\,dx = \int_a^x [g'(x) + \Delta f\,\delta(x-c)]dx$$

$$= \Big[g(x) + \Delta f\, h(x-c)\Big]_a^x = f(x) - f(a) \neq \int_a^x f'(x)\,dx$$

This means that $\bar{f}'(x)$ retains the memory of the jump at $x=c$ but ~~$g'(x)$~~ $f'(x)$ does not.

To illustrate this result, we consider the Heaviside function

$$H(x) = \begin{cases} 0 & x<0 \\ 1 & x>1 \end{cases}$$



we have $H'(x)=0$ and $\Delta H = 1$ at $x=0$.

Therefore, $\bar{H}'(x) = H'(x) + \Delta H\, \delta(x-0) = \delta(x)$

It is seen that for $a<0$ and any x

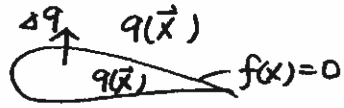
$$\int_a^x \bar{H}'(x)\,dx = \int_a^x \delta(x) = H(x) - H(a) = H(x)$$

which is not the same as

$$\int_a^x H'(x)\,dx = \int_a^x 0\,dx = 0.$$

Eq. (1) describes the extension of the differentiation process in generaliza function theory.

Now let $q(\vec{x})$ be a function of $\vec{x}(x_1, x_2, x_3)$ with a discontinuity across the surface $f(\vec{x})=0$



Let us define the jump Δq of $q(\vec{x})$ across $f=0$ as follows:

$$\Delta q = q(f=0+) - q(f=0-)$$

Note that Δq is a function of position on the surface of discontinuity $f=0$ and ∇f points into the region $f>0$

Then, the generalized partial derivative of $q(\vec{x})$ with respect to the variable x_i is

$$\frac{\bar{\partial} q}{\partial x_i} = \frac{\bar{\partial} q}{\partial f} \cdot \frac{\partial f}{\partial x_i} = \frac{\partial q}{\partial f} \frac{\partial f}{\partial x_i} + \Delta q \frac{\partial f}{\partial x_i} \delta(f) = \frac{\partial q}{\partial x_i} + \Delta q \frac{\partial f}{\partial x_i} \delta(f)$$

where $\delta(f)$ is the Dirac delta function. $\partial q/\partial x_i$ is the ordinary partial derivative of $q(\vec{x})$. The generalized gradient of $q(\vec{x})$ can be written as

$$\bar{\nabla} q = \nabla q + \Delta q \nabla f \, \delta(f)$$

- A useful result for the integration of $\delta(f)$ is the following:

$$\int Q(\vec{x}) \underset{\downarrow \atop \vec{n}}{|\nabla f|} \delta(f) \, d\vec{x} = \int_{f=0} Q(\vec{x}) \, dS \qquad \text{Eq.(2)}$$

where $Q(\vec{x})$ is an ordinary function. The integral on the right-hand side of Eq.(2) is a surface integral over the surface $f=0$ with $Q(\vec{x})$ evaluated on the surface.

Another important fact to remember is that the order of operations can be exchanged when working with generalized functions:

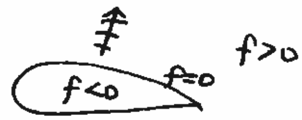
$$\frac{\partial^2 Q(\vec{x})}{\partial x_i \partial x_j} = \frac{\partial^2 Q(\vec{x})}{\partial x_j \partial x_i}$$

$$\frac{\partial}{\partial x_i} \int Q(\vec{x}, \vec{y})\, d\vec{y} = \int \frac{\partial}{\partial x_i} \left(Q(\vec{x}, y) \right) d\vec{y}$$

~~where~~ the limits of the integrals are not functions of the variable x_i.

- The Ffowcs Williams - Hawkings Equation

Let $f(\vec{x},t)=0$ be a moving surface enclosing the noise generating region of the flow, possibly including solid surfaces such as rotor blade. We will refer to $f(\vec{x},t)$ as a "data surface".



We are interested in computing the sound field in the exterior of the data surface. We assume that $f>0$ in the exterior of the data surface and $f<0$ in the interior. Furthermore, to simplify algebraic manipulations, we assume that $f(\vec{x},t)$ is defined such that $\nabla f=\hat{n}$, the outward unit normal vector. If the original definition of f does not satisfy this condition, then $f(\vec{x},t)$ can always be redefined as $f/|\nabla f|$ which does satisfy this condition on the surface. We do not assume that the data surface is impenetrable because this will give us the most general result for applications.

We assume that the fluid extends inside the data surface with the conditions of the undisturbed quiescent medium, which, with the exception of the sound speed c, we denote with the subscript 0. The reason for this extension (i.e., embedding the original problem into unbounded space) is that we want our wave equation to be valid in the entire three-dimensional space so that we can use the simple Green's function of the wave equation in unbounded space.

We have now introduced an artificial discontinuity in all fluid parameters across the data surface $f=0$.

- The discontinuity is

$$\tilde{\rho} = \begin{cases} \rho & f>0 \\ \rho_0 & f<0 \end{cases}$$

$$\Delta\tilde{\rho} = \rho - \rho_0$$

$$\widetilde{\rho u_i} = \begin{cases} \rho u_i & f>0 \\ 0 & f<0 \end{cases}$$

$$\Delta(\widetilde{\rho u_i}) = \rho u_i$$

$$\tilde{P}_{ij} = \begin{cases} P_{ij}\,\delta_{ij} & f>0 \\ P_0\,\delta_{ij} & f<0 \end{cases}$$

$$\Delta P_{ij} = (P_{ij} - P_0)\,\delta_{ij}$$ with neglecting the viscous term

- The mass continuity and conservation of momentum equations within the fluid (i.e., in the region $f>0$) are

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j + P_{ij}) = 0$$

where ρ is the fluid density, ρu_i is the component of fluid momentum and P_{ij} is the compressive stress tensor as defined by Lighthill. ($P_{ij} = P'\delta_{ij} + \tau_{ij}$).

In the case of the continuity equation with the discontinuity, we have

$$\frac{\bar{\partial}\tilde{\rho}}{\partial t}+\frac{\bar{\partial}}{\partial x_i}(\widetilde{\rho u_i})=\frac{\partial \rho}{\partial t}+(\rho-\rho_0)\frac{\partial f}{\partial t}\delta(f)+\frac{\partial}{\partial x_i}(\rho u_i)+(\rho u_i)\frac{\partial f}{\partial x_i}\delta(f)$$

$$\frac{\partial f}{\partial x_i}=\hat{n}\ .$$

With the definition of the total derivate, the absolute change of surface f with respect to time is

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\underbrace{\frac{\partial x_i}{\partial t}}_{V_i}\cdot\underbrace{\frac{\partial f}{\partial x_i}}_{n_i}=0$$

$\frac{df}{dt}$ is equal to zero because the surface is defined such that f always zero on the surface.

$$\therefore\ \frac{\partial f}{\partial t}=-V_n\ .$$

$$\frac{\bar{\partial}\tilde{\rho}}{\partial t}+\frac{\bar{\partial}}{\partial x_i}(\widetilde{\rho u_i})=[\rho_0 V_n+\rho(u_n-V_n)]\delta(f) \qquad -\text{Eq.}(3)$$

The generalized momentum equation is then

$$\frac{\bar{\partial}}{\partial t}(\widetilde{\rho u_i})+\frac{\bar{\partial}}{\partial x_j}(\widetilde{\rho u_i u_j}+\tilde{P}_{ij})=\frac{\partial}{\partial t}(\rho u_i)+\rho u_i\overset{\swarrow -V_n}{\frac{\partial f}{\partial t}}\delta(f)$$

$$+\frac{\partial}{\partial x_j}(\rho u_i u_j+P_{ij})+(\rho u_i u_j+\Delta P_{ij})\overset{\swarrow n_j}{\frac{\partial f}{\partial x_j}}\delta(f)$$

$$=\ [\rho u_i(u_n-V_n)+\Delta P_{ij}n_j]\,\delta(f) \quad -\text{Eq.}(4)$$

The next step in the derivative of the FW-H equation follows exactly Lighthill's derivation of the acoustic analogy except that all new derivatives taken are generalized derivatives.

$$\frac{\bar{\partial}}{\partial t}(3) - \frac{\bar{\partial}}{\partial x_i}(4)$$

then, substract $\bar{\nabla}^2 [c^2 \widetilde{(\rho - \rho_0)}]$ $\tilde{\rho}'$ from both sides of the equation

$$\boxed{\begin{aligned} &\frac{\bar{\partial}^2 \tilde{\rho}'}{\partial t} - c^2 \frac{\bar{\partial}^2 \tilde{\rho}'}{\partial x_i \partial x_i} \\ &= \frac{1}{c^2}\frac{\bar{\partial}^2 p'}{\partial t^2} - \frac{\bar{\partial}^2 p'}{\partial x_i \partial x_i} = \bar{\Box}^2 p'(\vec{x}, t) = \frac{\bar{\partial}}{\partial t}\{[\rho_0 V_n + \rho(u_n - V_n)]\delta(f)\} \\ &\quad - \frac{\bar{\partial}}{\partial x_i}\{[\Delta P_{ij}\hat{n}_j + \rho u_i(u_n - V_n)]\delta(f)\} + \frac{\bar{\partial}^2}{\partial x_i \partial x_j}[T_{ij} H(f)] \end{aligned}} \quad \text{Eq.(5)}$$

where $T_{ij} = \rho u_i u_j + P_{ij} - c^2 \rho'$ is the Lighthill stress tensor ($\rho' = \rho - \rho_0$ and $P_{ij} = p'\delta_{ij} - \tau_{ij}$). The Heaviside function $H(f)$ in the last term on the right-hand side is used here, following Ffowcs Williams and Hawkings, to remind that this volume source only exists exterior to the data surface. ~~By~~ $T_{ij} = 0$ inside the data surface. Note that, in general, the viscous term in ΔP_{ij} is a negligible sound source and one can often assume that $\Delta P_{ij} = (P - P_0)\delta_{ij}$ where δ_{ij} is the Kronecker delta. Ffowcs Williams and Hawkings chose to write p instead of $p - P_0$. They then regarded p as the gauge pressure in that source term. Practically, this term is local pressure term representing local lift and drag.

- Interpretation

The first two sources in Eq.(5) are monopole and dipole sources, respectively, based on their mathematical structure. Both of these sources are surface sources; i.e., they act only on the surface $f=0$ as indicated by the presence of the Dirac delta function $\delta(f)$. The third source term is a quadrupole source term that acts throughtout the volume that is exterior to the data surface - as indicated by the Heaviside function $H(f)$.

The three source terms on the right-hand side of Eq.(3) have become known as the thickness, loading, and quadrupole source terms, respectively, although this terminology is appropriate only if the data surface corresponds to a solid (impermeable) surface.

If the data surface $f=0$ coincides with a solid surface, then the normal velocity of the fluid is the same as the normal velocity of the surface ($u_n = v_n$). In this case, Eq.(3) can be written

$$\boxed{\bar{\Box}^2 p'(\vec{x},t) = \frac{\partial}{\partial t}\left\{[\rho_o v_n]\delta(f)\right\} - \frac{\partial}{\partial x_i}\left\{[\Delta P_{ij}\hat{n}_j]\,\delta(f)\right\} + \frac{\partial^2}{\partial x_i \partial x_j}\left[T_{ij}H(f)\right]} \quad \text{Eq.(6)}$$

The monopole or thickness source term in Eq.(6) models the noise generated by the displacement of fluid as the body passes (i.e., thickness noise). The dipole or loading source term in Eq.(6) models the noise that results from the unsteady force distribution on the body surface (i.e., loading noise including blade-vortex-interaction noise) ~~or broadban~~

The quadrupole source term in Eq(6) models the non-linearities due to ~~both~~ local sound speed variations, Reynolds shear stress &

If Eq.(6) is used with a permeable data surface that encloses the acoustic sources, then the surface source term in Eq.(5) also accounts for the sound generated by acoustic sources in the flow field outside of the blade surface but inside the data surface. In this case, the volume source in Eq.(5) may have a negligible contribution.

- It is interesting to note that the form of the various source term is not unique. One example of this is known as "Isom thickness noise"

$$\frac{\partial}{\partial t}[1-H(f)] = -\frac{\partial f}{\partial t}\delta(f) = V_n\delta(f)$$

$$\frac{\partial^2}{\partial t^2}[1-H(f)] = \frac{\partial}{\partial t}[V_n\delta(f)]$$

$$\nabla^2[1-H(f)] = -\frac{\partial}{\partial x_i}[\hat{n}_i\delta(f)]$$

Then,

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right]\{\rho_0 c^2[1-H(f)]\} = \frac{\partial}{\partial t}[\rho_0 V_n\delta(f)] + \frac{\partial}{\partial x_i}[\rho_0 c^2\hat{n}_i\delta(f)]$$

Because the trivial function $\rho_0 c^2[1-H(f)]$ is always zero outside the surface $f=0$, we see that

$$\underbrace{\frac{\partial}{\partial t}[\rho_0 V_n\delta(f)]}_{\text{thickness noise}} = \underbrace{-\frac{\partial}{\partial x_i}[\rho_0 c^2\hat{n}_i\delta(f)]}_{\text{loading noise}}$$

Remarks

- The FW-H equation is linear and many different closed form solutions can be derived and efficiently coded for numerical work.
- Most numerical schemes have dispersion and dissipation errors that depend on the spatial and temporal grid resolution. However, since the FW-H equation is solved using the Green's function, it is free of dispersion and dissipation errors.
- A much larger computational domain is needed for an acoustic problem than for an aerodynamic problem. This increased domain size increases computational resource requirements dramatically. Therefore, the acoustic analogy approach is an ideal partner to CFD; the near-field should be computed with unsteady CFD methods while the FW-H equation is appropriate for the prediction of the acoustic propagation.

Farassat's Formulations

Green's function of the wave equation in unbounded three dimensional space is given as

$$G(\vec{x}, t; \vec{y}: \tau) = \begin{cases} 0 & \text{for } \tau > t \\ \delta(\tau - t + r/c)/4\pi r & \text{for } \tau \leq t \end{cases}$$

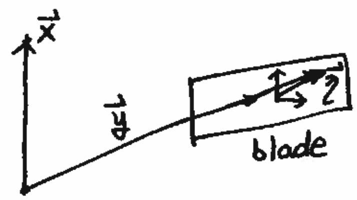
Let $g = \tau - t + r/c$ or $t - \tau - r/c$.

We are interested in the solution of the equation

$$\Box^2 p' = Q(\vec{x}, t)\, \delta(f)$$

$$4\pi p'(\vec{x}, t) = \int Q(\vec{y}, \tau)\, \delta(f) \frac{\delta(g)}{r} d\vec{y}\, d\tau$$

where $\int \cdots d\vec{y}\, d\tau = \int_{-\infty}^{t} \int_{R^3} \cdots d\vec{y}\, d\tau = \int_{-\infty}^{t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots dy_1 dy_2 dy_3\, d\tau$



$\vec{x}$-frame and $\vec{y}$-frame are fixed to the undisturbed medium.

Observer space-time variable $(\vec{x}, t)$ are kept fixed in all algebraic manipulations. Therefore, for all practical purposes, we are dealing with four variables $(\vec{y}, \tau)$.

We introduce $\vec{\eta}$-frame on which a point on surface (generally a blade) is described in a frame fixed relative to the surface.

$$\vec{y} = \vec{y}(\vec{\eta}, \tau) \qquad \text{— (1)}$$

$$\vec{\eta} = \vec{\eta}(\vec{y}, \tau) \qquad \text{— (2)}$$

Note that the equation of the moving surface $f(\vec{y},\tau)=0$ in the $\vec{\eta}$-frame is $f(\vec{y}(\vec{\eta},\tau),\tau)=\tilde{f}(\vec{\eta},\tau)=0$. In practice, $\tilde{f}=0$ is independent of time, i.e., the surface is described as $\tilde{f}(\vec{\eta})$.

In general, the analytic expressions for f and $\tilde{f}$ are different.

For example, for a moving (rigid) sphere, we have

$$f(\vec{y},\tau)=\sqrt{(y_1-vt)^2+y_2^2+y_3^2}-a=0 \quad \text{while}$$

$$\tilde{f}(\eta)=\sqrt{\eta_1^2+\eta_2^2+\eta_3^2}-a=0.$$ However, this distinction does not matter to us here because we are not able to integrate any of the acoustic integrals here in closed form for any nontrivial moving surface. We always evaluate our surface integrals numerically by finite difference scheme after we divide a data or a blade surface into panels.

For problems of interest to us in aeroacoustics, the transformation Eqs (1) and (2) are isometric, i.e., distance preserving, because they involve translations and rotations only. For this reason, the Jacobians of transformation are unity, that is, we have

$$\det\left(\frac{\partial y}{\partial \eta}\right)=1 \qquad \text{and} \qquad \det\left(\frac{\partial \eta}{\partial y}\right)=1$$

We first use the transformation $\vec{y}\to\vec{\eta}$

$$4\pi p'(\vec{x},t)=\int Q(\vec{y}(\vec{\eta},\tau),\tau)\,\delta(\tilde{f})\,\frac{\delta(g)}{r\det|\partial\eta/\partial y|}\,d\vec{\eta}\,d\tau$$

$$=\int_{-\infty}^{t}\int_{R^3}\tilde{Q}(\vec{\eta},\tau)\,\delta(\tilde{f})\,\frac{\delta(g)}{r}\,d\vec{\eta}\,d\tau$$

where $\tilde{Q}(\vec{\eta},\tau)=Q(\vec{y}(\vec{\eta},\tau),\tau)$

Next we use the transformation $\tau \to g$. The Jacobian of this transformation is $\partial g / \partial \tau$. Here the variable $\vec{\eta}$ is kept fixed in this partial differentiation.

$$g = \tau - t + \frac{|\vec{x} - \vec{y}(\vec{\eta}, \tau)|}{c}$$

$$\frac{\partial g}{\partial \tau} = 1 + \frac{1}{c} \frac{\partial r}{\partial y_i} \frac{\partial y_i}{\partial \tau} = 1 - \frac{\hat{r}_i V_i}{c} = 1 - M_r . \quad \left[\frac{\partial r}{\partial y_i} = -\hat{r}_i \right]$$

where $M_r = \hat{r}_i \cdot V_i / c$ is the Mach number of the point $\vec{\eta}$ in the radiation direction at the time τ. $V_i = \partial y_i(\vec{\eta}, \tau)/\partial \tau$ is the component of the velocity $\vec{V}$ of the point $\vec{\eta}$ with respect to the $\vec{y}$-frame fixed to the undisturbed medium.

Then,

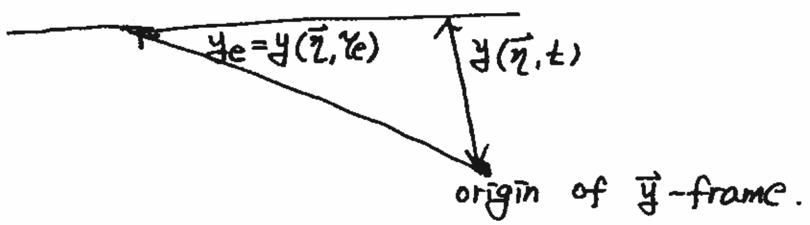
$$4\pi p'(\vec{x}, t) = \int \tilde{Q}(\vec{\eta}, \tau)\, \delta(\tilde{f})\, \frac{\delta(g)}{r|\partial g/\partial \tau|}\, d\vec{\eta}\, dg$$

$$= \int_{R^3} \int_{-\varepsilon}^{\varepsilon} \tilde{Q}(\vec{\eta}, \tau)\, \delta(\tilde{f})\, \frac{\delta(g)}{r|1 - M_r|}\, dg\, d\vec{\eta}$$

$$= \int_{R^3} \left[\frac{\tilde{Q}(\vec{\eta}, \tau)}{r|1 - M_r|}\, \delta(\tilde{f}) \right]_{g=0} d\vec{\eta}$$

Note that the limits of the inside integral (with respect to g) are from $-\varepsilon$ to ε ($\varepsilon > 0$) because $\delta(g)$ could only contribute to the integral in this region.

For subsonic moving surfaces $|1 - M_r| = 1 - M_r$, which is known as the Doppler factor.



$$g = \tau - t + \frac{|\vec{x} - \vec{y}(\vec{\eta}, \tau)|}{c} = 0 \qquad (\vec{x}, t) \text{ kept fixed}$$

$$|\vec{x} - \vec{y}(\vec{\eta}, \tau)| = c(t - \tau)$$

solutions : $\tau_e = \tau(\vec{x}, t; \eta)$ and $\vec{y}_e = \vec{y}(\vec{\eta}, \tau_e)$

$$\vec{r}_e = |\vec{x} - \vec{y}_e| = c(t - \tau_e)$$

τ_e : emission time, y_e : emission position

Then,

$$4\pi p'(\vec{x}, t) = \int_{R^3} \frac{\tilde{Q}(\vec{\eta}, \tau_e)}{r_e(1 - M_{re})} \delta[\tilde{f}(\vec{\eta})] d\vec{\eta}$$

where $M_{re} = \vec{M}(\vec{\eta}, \tau_e) \cdot \hat{r}_e$ and $\vec{M}(\vec{\eta}, \tau_e) = \vec{V}(\eta, \tau_e)/c$

For an arbitrary integrable function $q(\vec{y})$

$$\int_{R^3} q(\vec{y})\, \delta(f)\, d\vec{y} = \int_{f=0} q(\vec{y})\, ds$$

where $\vec{y}$ is a dummy variable

Then,

$$4\pi p'(\vec{x}, t) = \int_{\tilde{f}=0} \frac{\tilde{Q}(\vec{\eta}, \tau_e)}{r_e(1 - M_{re})} ds$$

It is customary to write the above result as :

$$4\pi p'(\vec{x}, t) = \int_{f=0} \left[\frac{Q(y, \tau)}{r(1 - M_r)} \right]_{ret} ds$$

↳ retarded time.

FW-H equation for an impermeable surface neglecting the quadrupole source term

$$\Box^2 p'(\vec{x},t) = \frac{\partial}{\partial t}\{[\rho_0 V_n]\delta(f)\} - \frac{\partial}{\partial x_i}\{\Delta P_{ij} n_j \, \delta(f)\}$$

- Thickness noise part of formulation 1 can be written as

$$4\pi p_T'(\vec{x},t) = \frac{\partial}{\partial t}\int_{f=0}\left[\frac{\rho_0 V_n}{r(1-M_r)}\right]_{ret} dS$$

We will say more about the mathematics behind keeping the time derivatives outside the integral sign later.

- For loading noise

$$4\pi P_L'(\vec{x},t) = -\frac{\partial}{\partial x_i}\int \Delta P_{ij} n_j \, \delta(f) \, \frac{\delta(g)}{r} \, d\vec{y} \, d\tau$$

$$= -\int \Delta P_{ij} n_j \, \delta(f) \frac{\partial}{\partial x_i}\left(\frac{\delta(g)}{r}\right) d\vec{y} \, d\tau \qquad g = \tau - t + r/c$$

$$\frac{\partial}{\partial x_i}\left(\frac{\delta(g)}{r}\right) = \frac{\partial}{\partial r}\left(\frac{\delta(g)}{r}\right)\frac{\partial r}{\partial x_i} = \left[\frac{1}{r}\frac{\partial \delta(g)}{\partial g}\frac{\partial g}{\partial r} - \frac{\delta(g)}{r^2}\right]\hat{r}_i = \left[\frac{\delta'}{cr} - \frac{\delta}{r^2}\right]\hat{r}_i$$

$$\frac{\partial}{\partial t}\left(\frac{\delta(g)}{r}\right) = \frac{1}{r}\frac{\partial}{\partial t}(\delta(g)) = \frac{1}{r}\frac{\partial \delta(g)}{\partial g}\cdot\frac{\partial g}{\partial t} = -\frac{\delta'}{r}$$

At this time $r = |\vec{x}(t) - \vec{y}(\tau)|$ is not a function of t (before integrating $\delta(g)$). It will be a function of t after integrating $\delta(g)$.

Therefore,

$$4\pi P_L'(\vec{x},t) = \frac{1}{c}\int \Delta P_{ij} n_j \, \delta(f) \frac{\partial}{\partial t}\left(\frac{\hat{r}_i \, \delta(g)}{r}\right) d\vec{y} \, d\tau + \int \Delta P_{ij} n_j \, \hat{r}_i \, \delta(f) \frac{\delta(g)}{r^2} \, d\vec{y} \, d\tau$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\int \Delta P_{ij} n_j \, \hat{r}_i \, \delta(f) \frac{\delta(g)}{r} \, d\vec{y} \, d\tau + \int \Delta P_{ij} n_j \, \hat{r}_i \, \delta(f) \frac{\delta(g)}{r^2} \, d\vec{y} \, d\tau$$

$$4\pi P_L'(\vec{x},t) = \frac{1}{c}\frac{\partial}{\partial t}\int_{f=0}\left[\frac{L_r}{r(1-M_r)}\right]_{ret} dS + \int_{f=0}\left[\frac{L_r}{r^2(1-M_r)}\right]_{ret} dS$$

where $L_r = \underbrace{\Delta P_{ij}}_{(P_{ij} - P_0)\delta_{ij}} n_j \cdot \hat{r}_i = \Delta P \underbrace{\cos\theta}_{\vec{n}_i \cdot \hat{r}_i}$

$\Delta P_{ij} n_j = \Delta P_{ii} n_i$

We have taken the observer time derivative out of the integral sign. We need to check the validity of the operation.

Leibniz's rule of differentiation under an integral sign

$$\frac{d}{d\theta}\left(\int_{a(\theta)}^{b(\theta)} f(x,\theta)\,dx\right) = \int_{a(\theta)}^{b(\theta)} \frac{\partial f(x,\theta)}{\partial \theta}dx + f(b(\theta),\theta)\cdot b'(\theta) - f(a(\theta),\theta)\cdot a'(\theta)$$

$$\frac{d}{dx}\left(\int_{y_0}^{y_1} f(x,y)\,dy\right) = \int_{y_0}^{y_1} \frac{\partial f(x,y)}{\partial x}\,dy.$$

Let $Q = \Delta P n_j \cdot \hat{r}_i$

$$I = \frac{\partial}{\partial t}\int Q(\vec{y},\tau)\,\delta(f)\,\frac{\delta(g)}{r}\,d\vec{y}\,d\tau = \frac{\partial}{\partial t}\int_{-\infty}^{t}\int_{R^3} Q(\vec{y},\tau)\,\delta(f)\,\frac{\delta(g)}{r}\,d\vec{y}\,d\tau$$

$$I = \int_{-\infty}^{t}\int_{R^3} \frac{Q(\vec{y},\tau)}{r}\,\delta(f)\,\frac{\partial\delta(g)}{\partial t}\,d\vec{y}\,d\tau + \int_{R^3}\frac{Q(\vec{y},t)}{r}\,\delta[f(\vec{y},t)]\,\delta\!\left(\frac{r}{c}\right)d\vec{y}$$

we used $g = \tau - t + r/c$. $\tau \to t$. $g \to r/c$,
since $\delta(r/c) = c\,\delta(r)$

$$I_1 = \int_{R^3}\frac{Q(\vec{y},t)}{r}\,\delta[f(\vec{y},t)]\,\delta(r)\,d\vec{y}$$

$$d\vec{y} = r^2\sin\theta\,dr\,d\varphi\,d\theta$$

$$I_1 = \int_0^{\pi}\int_0^{2\pi}\int_0^{\infty} Q(\vec{y},t)\,\delta[f(\vec{y},t)]\,r\sin\theta\,\delta(r)\,dr\,d\varphi\,d\theta = 0$$

As $r \to 0$, $I_1 \to 0$

Therefore,

$$\frac{\partial}{\partial t}\int Q(\vec{y},\tau)\,\delta(f)\,\frac{\delta(g)}{r}\,d\vec{y}\,d\tau = \int Q(\vec{y},\tau)\,\delta(f)\,\frac{\partial}{\partial t}\left(\frac{\delta(g)}{r}\right)d\vec{y}\,d\tau$$

Note that even though r is not a function of the observer time, we like to always associate r with $\delta(g)$ and write observer time derivate operating on $\delta(g)/r$ in the integrant.

- Derivation of Formulation 1A

Let us look at

$$E(\vec{x}, t; \vec{\eta}) = \left[\frac{Q(\vec{y}, \tau)}{r(1-M_r)}\right]_{ret} = \left[\frac{Q(\vec{y}, \tau)}{r(1-M_r)}\right]_{\tau=\tau_e}$$

we have shown that only the emission time τ_e is a function of the observer time.

Therefore

$$\left(\frac{\partial}{\partial t}\right)_x = \left[\frac{\partial \tau(\vec{x}, t; \vec{\eta})}{\partial t} \cdot \frac{\partial}{\partial \tau}\right]_{\tau=\tau_e} \qquad (3)$$

where $\tau_e(\vec{x}, t; \vec{\eta})$ is simply the solution of $\tau - t + |\vec{x} - \vec{y}(\vec{\eta}, \tau)|/c = 0$.

Now we find $\partial \tau / \partial t$

$$g = \tau - t + r(\vec{x}, \vec{y}(\vec{\eta}, \tau))/c = 0$$

where $\vec{\eta}$ is a given fixed point on the moving surface

$$\left(\frac{\partial \tau}{\partial t}\right)_{(\vec{x}, \vec{\eta})} - 1 + \frac{1}{c}\left(\frac{\partial r}{\partial t}\right)_{(\vec{x}, \vec{\eta})} = 0 \qquad (4)$$

$$\left(\frac{\partial r}{\partial t}\right)_{(\vec{x}, \vec{\eta})} = \left(\frac{\partial r}{\partial \tau}\right)_{(\vec{x}, \vec{\eta})} \left(\frac{\partial \tau}{\partial t}\right)$$

Now r is a function of t.

$$\left(\frac{\partial r}{\partial \tau}\right)_{(\vec{x}, \vec{\eta})} = \left(\frac{\partial r}{\partial y_i}\right)\left(\frac{\partial y_i}{\partial \tau}\right)_{(\vec{x}, \vec{\eta})} = -\hat{r}_i V_i = -V_r$$

where $\hat{r}_i$ is the component of unit radiation vector and V_r is the velocity of the point $\vec{\eta}$ in the radiation direction.

Eq. (4) becomes

$$\left(\frac{\partial \tau}{\partial t}\right)_{(\vec{x},\vec{\eta})} - 1 - M_r\left(\frac{\partial \tau}{\partial t}\right)_{(\vec{x},\vec{\eta})} = 0$$

So that

$$\left(\frac{\partial \tau}{\partial t}\right)_{(\vec{x},\vec{\eta})} = \frac{1}{1-M_r}$$

or Eq. (3) becomes

$$\boxed{\left(\frac{\partial}{\partial t}\right)_x = \left[\frac{1}{1-M_r}\frac{\partial}{\partial \tau}\right]_{\tau=\tau_e}}$$

Then,

$$\frac{\partial}{\partial t}\left[q(\vec{x},\vec{y},\tau)\right]_{ret} = \frac{\partial}{\partial t}\left[q(\vec{x},\vec{y}(\vec{\eta},\tau),\tau_e(\vec{x},t;\vec{\eta})\right] = \left[\frac{1}{1-M_r}\frac{\partial q(\vec{x},\vec{y},\tau)}{\partial \tau}\right]_{ret}$$

which must be interpreted as

$$\left[\frac{1}{1-M_r}\frac{\partial q(\vec{x},\vec{y},\tau)}{\partial \tau}\right]_{ret} = \left[\frac{1}{1-M_r(\vec{x};\vec{\eta},\tau)}\frac{\partial q(\vec{x},\vec{y}(\vec{\eta},\tau),\tau)}{\partial \tau}\right]_{\tau=\tau_e}.$$

- For thickness noise, formulation 1 is

$$4\pi p'_T(\vec{x},t) = \frac{\partial}{\partial t}\int_{f=0}\left[\frac{\rho_0 v_n}{r(1-M_r)}\right]_{ret} dS$$

$$= \int_{f=0}\left\{\frac{1}{1-M_r}\frac{\partial}{\partial \tau}\left(\frac{\rho_0 v_n}{r(1-M_r)}\right)\right\}_{ret} dS$$

$$\frac{\partial}{\partial \tau}\left[\frac{1}{r(1-M_r)}\right] = \frac{\partial}{\partial \tau}\left(\frac{1}{r}\right)\frac{1}{1-M_r} + \frac{1}{r}\frac{\partial}{\partial \tau}\left(\frac{1}{1-M_r}\right)$$

$$\frac{\partial}{\partial \tau}\left(\frac{1}{r}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial \tau} = -\frac{1}{r^2}(-v_r) = \frac{v_r}{r^2}$$

$$\frac{\partial}{\partial\tau}\left(\frac{1}{1-M_r}\right) = -\frac{1}{(1-M_r)^2}\frac{\partial}{\partial\tau}(-M_r)$$

$$\frac{\partial M_r}{\partial\tau} = \frac{\partial(\vec{M}\cdot\hat{r})}{\partial\tau} = \frac{\partial\vec{M}}{\partial\tau}\cdot\hat{r} + \vec{M}\cdot\frac{\partial\hat{r}}{\partial\tau}$$

$$\frac{\partial\hat{r}}{\partial\tau} = \frac{\partial}{\partial\tau}\left(\frac{\vec{r}}{r}\right) = \frac{1}{r}\frac{\partial\vec{r}}{\partial\tau} + \vec{r}\frac{\partial}{\partial\tau}\left(\frac{1}{r}\right) \qquad \frac{\partial\vec{r}}{\partial\tau} = \frac{\partial(x_i - y_i)}{\partial y_i}\frac{\partial y_i}{\partial\tau} = -\vec{V}.$$

$$= -\frac{\vec{V}}{r} + \vec{r}\cdot\frac{V_r}{r^2} = -\frac{\vec{V}}{r} + \frac{V_r\cdot\hat{r}}{r}$$

$$= -\frac{c}{r}(\vec{M} - M_r\hat{r})$$

$$\frac{\partial M_r}{\partial\tau} = \dot{\vec{M}}_r + \frac{c}{r}(-M^2 + M_r^2)$$

$$\frac{\partial}{\partial\tau}\left(\frac{1}{1-M_r}\right) = \frac{r\dot{\vec{M}}_r + c(-M^2+M_r^2)}{r(1-M_r)^2}$$

Then,

$$\frac{\partial}{\partial\tau}\left[\frac{1}{r(1-M_r)}\right] = \frac{cM_r}{r^2(1-M_r)} + \frac{r\dot{M}_r + c(-M^2+M_r^2)}{r^2(1-M_r)^2}$$

$$= \frac{cM_r(1-M_r) + r\dot{M}_r + c(-M^2+M_r^2)}{r^2(1-M_r)^2}$$

$$= \frac{r\dot{M}_r + c(M_r - M^2)}{r^2(1-M_r)^2} \qquad \frac{\partial}{\partial\tau}\left[\frac{1}{r^m(1-M_r)^n}\right] = ?$$

Therefore,

$$\boxed{4\pi p_T(\vec{x},t) = \int_{f=0}\left[\frac{\rho_0(\dot{V}_n + V_{\dot{n}})}{r(1-M_r)^2}\right]_{ret} ds + \int_{f=0}\left[\frac{\rho_0 V_n(r\dot{M}_r + c(M_r - M^2))}{r^2(1-M_r)^3}\right]_{ret} ds}$$

Formulation 1A – Thickness noise.