EAE 298 Aeroacoustics
Fall Quarter 2016
Homework #2

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Problem 1. Solution of Lilley's Equation and its Application

An axisymmetric jet of radius R_j has an exit mean velocity of W_j and an exit mean density of $\overline{\rho_j}$. The ambient mean velocity is zero and the ambient mean density is $\overline{\rho_0}$. Lilleys equation for a parallel axisymmetric flow can be written

$$\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)^3 p' - \left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)\left(\overline{a^2}\nabla^2 p'\right) - \frac{d\overline{a^2}}{dr}\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)\frac{\partial p'}{\partial r} + 2\overline{a^2}\frac{dW}{dr}\frac{\partial^2 p'}{\partial z \partial r} = S(\vec{x},t)$$

where

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}+\frac{\partial^2}{\partial z^2}$$

and $W(r)$ and $\overline{a^2}(r)$ are the radial distributions of the axial velocity and speed of sound squared.

Part (a)

[25 points] Seek solutions of Lilleys equation in the form

$$p'(r,\theta,z,t) \sim P(r)\exp[i(kz+n\theta-\omega t)]$$

Show that Lilleys equation reduces to

$$\frac{d^2P}{dr^2}+\left\{\frac{1}{r}-\frac{1}{\overline{\rho}}\frac{d\overline{\rho}}{dr}+\frac{2k}{(\omega-kW)}\frac{dW}{dr}\right\}\frac{dP}{dr}+\left\{\frac{(\omega-kW)^2}{\overline{a^2}}-k^2-\frac{n^2}{r^2}\right\}P = RHS$$

where $\overline{\rho}(r)$ is the radial distribution of the mean density. (Note: $\overline{a^2}=\gamma\overline{p}/\overline{\rho}$ and $\overline{p}$ is constant.)

Solution

Taking the solution for p' as

$$p'(r,\theta,z,t) = P(r)\exp[i(kz+n\theta-\omega t)]$$

we can solve for several terms. The first term is

$$\begin{aligned}
\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)^3 p' &= \left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)^2\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)p' \\
&= \left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)^2\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)P(r)\exp[i(kz+n\theta-\omega t)] \\
&= \left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)^2(-i\omega+Wik)\,P(r)\exp[i(kz+n\theta-\omega t)] \\
&= \left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)(-i\omega+Wik)^2\,P(r)\exp[i(kz+n\theta-\omega t)] \\
&= (-i\omega+Wik)^3\,P(r)\exp[i(kz+n\theta-\omega t)] \\
&= i\,(\omega-kW)^3\,P(r)\exp[i(kz+n\theta-\omega t)]
\end{aligned}$$

The second term is very lengthy. Starting with $\nabla^2 p'$, we have

$$\begin{aligned}
\nabla^2 p' &= \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]p' + \left[\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]p' + \left[\frac{\partial^2}{\partial z^2}\right]p' \\
&= p'_r + p'_\theta + p'_z
\end{aligned}$$

$$\begin{aligned}
p'_r &= \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]p' \\
&= \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]P(r)\exp[i(kz+n\theta-\omega t)] \\
&= \left[\frac{1}{r}\frac{\partial}{\partial r}(r)\right]\frac{dP}{dr}\exp[i(kz+n\theta-\omega t)] \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{dP}{dr}\exp[i(kz+n\theta-\omega t)]\right] \\
&= \frac{1}{r}\left[\frac{dP}{dr}\exp[i(kz+n\theta-\omega t)] + r\frac{d^2P}{dr^2}\exp[i(kz+n\theta-\omega t)]\right] \\
&= \frac{1}{r}\frac{dP}{dr}\exp[i(kz+n\theta-\omega t)] + \frac{d^2P}{dr^2}\exp[i(kz+n\theta-\omega t)] \\
&= \left[\frac{1}{r}\frac{dP}{dr} + \frac{d^2P}{dr^2}\right]\exp[i(kz+n\theta-\omega t)]
\end{aligned}$$

$$\begin{aligned}
p'_\theta &= \left[\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]p' \\
&= \left[\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]P(r)\exp[i(kz+n\theta-\omega t)] \\
&= \left[\frac{1}{r^2}\right]i^2n^2P(r)\exp[i(kz+n\theta-\omega t)] \\
&= -\frac{n^2}{r^2}P(r)\exp[i(kz+n\theta-\omega t)]
\end{aligned}$$

$$\begin{aligned}
p'_z &= \left[\frac{\partial^2}{\partial z^2}\right]p' \\
&= \left[\frac{\partial^2}{\partial z^2}\right]P(r)\exp[i(kz+n\theta-\omega t)] \\
&= i^2k^2P(r)\exp[i(kz+n\theta-\omega t)] \\
&= -k^2P(r)\exp[i(kz+n\theta-\omega t)]
\end{aligned}$$

$$\begin{aligned}
\nabla^2 p' &= \left[\frac{1}{r}\frac{dP}{dr} + \frac{d^2P}{dr^2}\right]\exp[i(kz+n\theta-\omega t)] + \left(-\frac{n^2}{r^2} - k^2\right)P(r)\exp[i(kz+n\theta-\omega t)] \\
&= \left\{\left(-\frac{n^2}{r^2} - k^2\right)P(r) + \left[\frac{1}{r}\frac{dP}{dr} + \frac{d^2P}{dr^2}\right]\right\}\exp[i(kz+n\theta-\omega t)]
\end{aligned}$$

Pluggin in the solution for $\nabla^2 p'$, we can further solve

$$
\begin{aligned}
-\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)\left(\overline{a^2}\nabla^2 p'\right) =& -\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)\times \\
& \overline{a^2}\left\{\left(-\frac{n^2}{r^2}-k^2\right)P(r)+\left[\frac{1}{r}\frac{dP}{dr}+\frac{d^2P}{dr^2}\right]\right\}\exp[i(kz+n\theta-\omega t)] \\
=& i\left(\omega-kW\right)\times \\
& \overline{a^2}\left\{\left(-\frac{n^2}{r^2}-k^2\right)P(r)+\left[\frac{1}{r}\frac{dP}{dr}+\frac{d^2P}{dr^2}\right]\right\}\exp[i(kz+n\theta-\omega t)]
\end{aligned}
$$

Finally, the third and fourth terms on the LHS can be solved

$$
\begin{aligned}
-\frac{d\overline{a^2}}{dr}\left[\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)\frac{\partial}{\partial r}\right]\rho' &= -\frac{d\overline{a^2}}{dr}\left[\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)\frac{\partial}{\partial r}\right]P(r)\exp[i(kz+n\theta-\omega t)] \\
&= -\frac{d\overline{a^2}}{dr}\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)\frac{dP}{dr}\exp[i(kz+n\theta-\omega t)] \\
&= -\frac{d\overline{a^2}}{dr}\left(i\left(kW-\omega\right)\frac{dP}{dr}\right)\exp[i(kz+n\theta-\omega t)] \\
&= \frac{d\overline{a^2}}{dr}\left\{i\left(\omega-kW\right)\frac{dP}{dr}\right\}\exp[i(kz+n\theta-\omega t)] \\
&= -\left(\frac{\gamma\overline{p}}{\overline{\rho}^2}\right)\frac{d\overline{\rho}}{dr}\left\{i\left(\omega-kW\right)\frac{dP}{dr}\right\}\exp[i(kz+n\theta-\omega t)] \\
&= -\left(\frac{\overline{a^2}}{\overline{\rho}}\right)\frac{d\overline{\rho}}{dr}\left\{i\left(\omega-kW\right)\frac{dP}{dr}\right\}\exp[i(kz+n\theta-\omega t)]
\end{aligned}
$$

$$
\begin{aligned}
+\left[2\overline{a^2}\frac{dW}{dr}\frac{\partial^2}{\partial z\partial r}\right]p' &= \left[2\overline{a^2}\frac{dW}{dr}\frac{\partial^2}{\partial z\partial r}\right]P(r)\exp[i(kz+n\theta-\omega t)] \\
&= \left[2\overline{a^2}\frac{dW}{dr}\frac{\partial}{\partial z}\right]\frac{\partial}{\partial r}P(r)\exp[i(kz+n\theta-\omega t)] \\
&= \left[2\overline{a^2}\frac{dW}{dr}\right]\frac{\partial}{\partial z}\frac{dP}{dr}\exp[i(kz+n\theta-\omega t)] \\
&= \left[2\overline{a^2}\frac{dW}{dr}\right]ik\frac{dP}{dr}\exp[i(kz+n\theta-\omega t)] \\
&= \left\{2ik\overline{a^2}\frac{dW}{dr}\frac{dP}{dr}\right\}\exp[i(kz+n\theta-\omega t)]
\end{aligned}
$$

Summing all these terms, we have

$$\begin{aligned}
&i\left(\omega - kW\right)^3 P(r)\exp[i(kz + n\theta - \omega t)] \\
&+ i\left(\omega - kW\right)\overline{a^2}\left\{\left(-\frac{n^2}{r^2} - k^2\right)P(r) + \left[\frac{1}{r}\frac{dP}{dr} + \frac{d^2P}{dr^2}\right]\right\}\exp[i(kz + n\theta - \omega t)] \\
&- \left(\frac{\overline{a^2}}{\overline{\rho}}\right)\frac{d\overline{\rho}}{dr}\left\{i\left(\omega - kW\right)\frac{dP}{dr}\right\}\exp[i(kz + n\theta - \omega t)] \\
&+ \left\{2ik\overline{a^2}\frac{dW}{dr}\frac{dP}{dr}\right\}\exp[i(kz + n\theta - \omega t)] = S(\vec{x}, t)
\end{aligned}$$

We can divide by $i\overline{a^2}\left(\omega - kW\right)\exp[i(kz + n\theta - \omega t)]$ to find

$$\begin{aligned}
&\frac{\left(\omega - kW\right)^2}{\overline{a^2}}P(r) + \left\{\left(-\frac{n^2}{r^2} - k^2\right)P(r) + \left[\frac{1}{r}\frac{dP}{dr} + \frac{d^2P}{dr^2}\right]\right\} \\
&\quad -\frac{1}{\overline{\rho}}\frac{d\overline{\rho}}{dr}\left\{\frac{dP}{dr}\right\}\left\{\frac{2k}{\left(\omega - kW\right)}\frac{dW}{dr}\frac{dP}{dr}\right\} = \frac{S(\vec{x}, t)}{i\overline{a^2}\left(\omega - kW\right)\exp[i(kz + n\theta - \omega t)]}
\end{aligned}$$

Rearranging for similar powers of P, we have

$$\begin{aligned}
&\frac{d^2P}{dr^2} + \left\{\frac{1}{r} - \frac{1}{\overline{\rho}}\frac{d\overline{\rho}}{dr} + \frac{2k}{\left(\omega - kW\right)}\frac{dW}{dr}\right\}\frac{dP}{dr} \\
&\quad + \left\{\frac{\left(\omega - kW\right)^2}{\overline{a^2}} - k^2 - \frac{n^2}{r^2}\right\}P(r) = \frac{S(\vec{x}, t)}{i\overline{a^2}\left(\omega - kW\right)\exp[i(kz + n\theta - \omega t)]}
\end{aligned}$$

We find

$$\frac{d^2P}{dr^2} + \left\{\frac{1}{r} - \frac{1}{\overline{\rho}}\frac{d\overline{\rho}}{dr} + \frac{2k}{\left(\omega - kW\right)}\frac{dW}{dr}\right\}\frac{dP}{dr} + \left\{\frac{\left(\omega - kW\right)^2}{\overline{a^2}} - k^2 - \frac{n^2}{r^2}\right\}P(r) = RHS$$

where

$$RHS = \frac{S(\vec{x}, t)}{i\overline{a^2}\left(\omega - kW\right)\exp[i(kz + n\theta - \omega t)]}$$

Part (b)

[25 points] Determine the general form of solution (solution of the homogeneous equation) for the pressure fluctuation outside the jet in the ambient medium where the sources vanish. Make sure the solution is chosen to ensure decaying solutions or outgoing waves.

For the region outside the jet, we have

$$W(r) = W_\infty = constant$$
$$\overline{\rho} = \overline{\rho_0} = constant$$

To solve a homogenous second order linear ordininary differential equation we find the characteristic polynomial. Plugging in the boundary conditions,

$$\begin{aligned}
\frac{d^2P}{dr^2} + \left\{\frac{1}{r} - \frac{1}{\overline{\rho}}\frac{d\overline{\rho}}{dr} + \frac{2k}{(\omega - kW)}\frac{dW}{dr}\right\}\frac{dP}{dr} + \left\{\frac{(\omega - kW)^2}{\overline{a^2}} - k^2 - \frac{n^2}{r^2}\right\}P(r) = 0 \\
\frac{d^2P}{dr^2} + \left\{\frac{1}{r}\right\}\frac{dP}{dr} + \left\{\frac{(\omega - kW_\infty)^2}{\overline{a^2}} - k^2 - \frac{n^2}{r^2}\right\}P(r) = 0 \\
\frac{d^2P}{dr^2} + \left\{\frac{1}{r}\right\}\frac{dP}{dr} + \left\{\lambda_\infty^2 - \frac{n^2}{r^2}\right\}P(r) = 0
\end{aligned}$$

where

$$\begin{aligned}
\lambda_\infty &= \sqrt{\frac{(\omega - kW_\infty)^2}{\overline{a^2}} - k^2} \\
&= \sqrt{\frac{\left(\omega - \frac{w}{\overline{a}}W_\infty\right)^2}{\overline{a^2}} - \frac{w^2}{\overline{\overline{a^2}}}} \\
&= \sqrt{\frac{\omega^2\left(1 - \frac{W_\infty}{\overline{a}}\right)^2}{\overline{a^2}} - \frac{\omega^2}{\overline{\overline{a^2}}}} \\
&= \sqrt{\frac{\omega^2}{\overline{\overline{a^2}}}\left[(1 - M_\infty)^2 - 1\right]} \\
&= \sqrt{\frac{\omega^2}{\overline{\overline{a^2}}}\left(1 - 2M_\infty + M_\infty^2 - 1\right)} \\
&= \sqrt{\frac{\omega^2}{\overline{\overline{a^2}}}\left(M_\infty^2 - 2M_\infty\right)} \\
&= \sqrt{\frac{\omega^2}{\overline{\overline{a^2}}}M_\infty(M_\infty - 2)}
\end{aligned}$$

is imaginary as the freestream velocity should be well below Mach 2. This ODE can be solved by the following equation

$$P = AH_n^{(1)}(\lambda_\infty r) + BH_n^{(2)}(\lambda_\infty r)$$

where $H_n^{(1)}$ and $H_n^{(2)}$ are the Hankel functions of the first and second kind, respectively, and A and B are unknown constants. The Hankel functions can be expressed as Bessel functions as

$$\begin{aligned}
H_m^{(1)}(x) = J_m(x) + iY_m(x) \\
H_m^{(2)}(x) = J_m(x) - iY_m(x)
\end{aligned}$$

Taking the limit, $\lim_{x\to\infty}$ for both equations

$$\lim_{x\to\infty} H_m^{(1)}(x) = (-i)^{m+1}\frac{e^{ix}}{x}$$
$$\lim_{x\to\infty} H_m^{(2)}(x) = (i)^{m+1}\frac{e^{-ix}}{x}$$

The input to the Hankel function $\lambda_\infty r$ is imaginary, causing $H_m^{(2)} \to \infty$ as $r \to \infty$, so B must be zero. The general form of the solution is therefore

$$P = AH_n^{(1)}(\lambda_\infty r)$$

Part (c)

[25 points] Determine the general form of solution (solution of the homogeneous equation) in the potential core region where the mean velocity and density are constant and equal to the jet exit values.

For the region inside the jet, we have

$$W(r) = W_j = constant$$
$$\overline{\rho} = \overline{\rho_j} = constant$$
$$0 < r < R_j$$

Plugging these in, we have

$$\frac{d^2P}{dr^2} + \left\{\frac{1}{r} - \frac{1}{\overline{\rho}}\frac{d\overline{\rho}}{dr} + \frac{2k}{(\omega - kW)}\frac{dW}{dr}\right\}\frac{dP}{dr} + \left\{\frac{(\omega - kW)^2}{\overline{a^2}} - k^2 - \frac{n^2}{r^2}\right\}P(r) = 0$$
$$\frac{d^2P}{dr^2} + \left\{\frac{1}{r}\right\}\frac{dP}{dr} + \left\{\frac{(\omega - kW_j)^2}{\overline{a^2}} - k^2 - \frac{n^2}{r^2}\right\}P(r) = 0$$
$$\frac{d^2P}{dr^2} + \left\{\frac{1}{r}\right\}\frac{dP}{dr} + \left\{\lambda_j^2 - \frac{n^2}{r^2}\right\}P(r) = 0$$

This ODE can be solved by the following equation

$$P = CJ_m(\lambda_j r) + DY_m(\lambda_j r)$$

where J_m and Y_m are the Bessel functions of the first and second kind, respectively, and C and D are unknown constants. The limit

$$\lim_{x\to 0} Y_m \to \infty$$

however, and so $D = 0$ and the general form for the solution is

$$P = CJ_m(\lambda_j r)$$

Part (d)

[25 points] Consider a case in which the real jet is replaced by a vortex sheet at $r = R_j$. If the solutions are to be matched at the vortex sheet, describe what matching conditions should be applied. Give both the physical description and the mathematical expressions.

The pressure on the vortex sheet must be equivalent from both the inside and outside of the jet (i.e., $P^+ = P^-$). Setting the solutions from part (b) and (c) equal to each other at $r = R_j$ gives

$$AH_n^{(1)}(\lambda_\infty R_j) = CJ_m(\lambda_j R_j)$$

Similarly, the velocity normal to the vortex sheet must be equivalent on both sides of the sheet (i.e., $v_r^+ = v_r^-$). Considering the momentum equation in the normal direction

$$\rho_0 \frac{D\vec{v}}{Dt} = -\nabla P$$
$$\rho_0 \frac{D\vec{v_r}}{Dt} = -\nabla P \cdot \hat{n} = -\frac{\partial P}{\partial r}$$

and

$$\vec{v} \cdot \hat{n} = \vec{v_r} = \hat{v}_r e^{-iwt+kz}$$

Expanding the total derivative

$$\rho_0 \left(\frac{\partial}{\partial t} + W \frac{\partial}{\partial z} \right) \vec{v_r} = -\frac{\partial P}{\partial r}$$

We can define the position of the vortex sheet, $\zeta(z,t)$, as

$$\zeta(z,t) = ae^{-iwt+kz}$$
$$\hat{v_r} = \frac{d\zeta}{dt}$$

We can find the normal velocity on the outside ($\hat{v_r}^+$) and inside ($\hat{v_r}^-$) the jet

$$\begin{aligned} \hat{v_r}^+ &= \left(\frac{\partial}{\partial t} + W \frac{\partial}{\partial z} \right) \zeta_\infty \\ &= -ia \left(w - W_\infty k \right) \zeta \\ \hat{v_r}^- &= \left(\frac{\partial}{\partial t} + W \frac{\partial}{\partial z} \right) \zeta_j \\ &= -ia \left(w - W_j k \right) \zeta \end{aligned}$$

Setting both of these equations equal to the constant a

$$\frac{\hat{v_r}^+}{-i \left(w - W_\infty k \right) \zeta} = \frac{\hat{v_r}^-}{-i \left(w - W_j k \right) \zeta}$$
$$\frac{\hat{v_r}^+}{(w - W_\infty k)} = \frac{\hat{v_r}^-}{(w - W_j k)} \tag{1}$$

Putting these solutions into the momentum equation yields

$$\begin{aligned}
\rho_0 \frac{D\vec{v_r}}{Dt} &= -\frac{\partial P}{\partial r} \\
\rho_0 \left(-ia\left(w - W_\infty k\right)\zeta\right)\vec{v_r}^+ &= -\frac{\partial P}{\partial r} \\
\rho_0 ia\left(w - W_\infty k\right)\zeta\vec{v_r}^+ &= \frac{\partial P}{\partial r}
\end{aligned}$$

Rearranging

$$\vec{v_r}^+ = \frac{1}{\rho_0 ia\left(w - W_\infty k\right)\zeta}\frac{\partial P^+}{\partial r}$$

We can similarly find

$$\vec{v_r}^- = \frac{1}{\rho_0 ia\left(w - W_j k\right)\zeta}\frac{\partial P^-}{\partial r}$$

Plugging these into Equation 1 results in

$$\frac{\frac{dP}{dr}^+}{\left(w - W_\infty k\right)^2} = \frac{\frac{dP}{dr}^-}{\left(w - W_j k\right)^2}$$

This gives us two equations (one from matching the pressure distributions, and one from matching the normal velocity equations) and two unknowns A and C. The general solutions found in parts (b) and (c) can be uniquely solved by setting these two equations equal to each other.