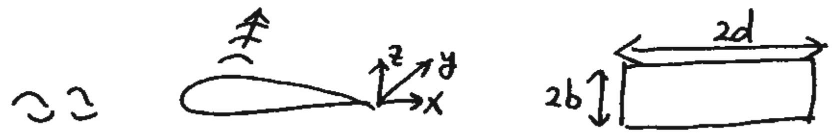
Turbulence - Airfoil Interaction Noise
(Amiet Theory)

An airfoil in a turbulent flow field experiences a fluctuating lift which should result in the generation of sound.



In principle, if the unsteady loading at each point on the airfoil were known as a function of time, the noise at any point in the field far could be calculated.

- An airfoil of chord $2b$ and span $2d$ is placed in a turbulent fluid with a mean flow U in the x direction. The y co-ordinates extends in the spanwise direction, and the origin of the co-ordinate system is at the center of the airfoil. The observer is in the far field.

The turbulence is assumed to be frozen so that in a co-ordinate system $x' = x - Ut$, which moves with the mean flow, the turbulent velocity in the $z=0$ plane can be written as $w(x', y)$.
In the airfoil fixed co-ordinate system, the turbulence velocity $\tilde{w}(x, y, t)$, written in terms of its wavenumber components, $\hat{w}_R(k_x, k_y)$ is

$$\tilde{w}(x, y, t) = \iint_{-\infty}^{\infty} \hat{w}_R(k_x, k_y)\, e^{i[k_x(x-Ut)+k_y y]}\, dk_x dk_y$$

where
$$\hat{w}_R(k_x, k_y) = \frac{1}{(2\pi)^2} \iint_{-R}^{R} w(x, y)\, e^{-i(k_x x + k_y y)}\, dx dy$$

where R is a large but finite number. R is not set equal to infinity at this point because of convergence difficulties if $w(x, y)$ does not go to zero at x and y go to infinity.

Analytical expressions for the distribution in the pressure jump, ΔP, across a flat plate airfoil of infinite span encountering a sinusoidal gust are available. For a gust of the form

$$w_g = w_0 e^{i[k_x(x-Ut)+k_y y]}$$

the distribution of the pressure jump can be written as

$$\Delta P(x.y.t) = 2\pi\rho_0 Ub\, w_0\, g(x, k_x, k_y)\, e^{i(k_y y - k_x Ut)}$$

where $g(x, k_x, k_y)$ is the transfer function between turbulent velocity and airfoil pressure jump.

The pressure jump at a given point on the airfoil due to all wavenumber components is then

$$\Delta P(x.y.t) = 2\pi\rho_0 Ub \iint_{-\infty}^{\infty} \hat{w}_R(k_x, k_y)\, g(x, k_x, k_y)\, e^{i(k_y y - k_x Ut)} dk_x dk_y$$

The Fourier transform with respect to time can be performed to give the frequency dependence. Since the turbulence was assumed to extend between $-R<x<R$, the time integration will be between $\pm T$ where $T = R/U$.

$$\Delta\hat{P}_T(x.y.\omega) = \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} \Delta P(x.y.t)\, e^{i\omega t}\, dt$$

Since $\int_{-T}^{T} e^{i\xi t} dt \to 2\pi\delta(\xi)$ as $T\to\infty$

$$\Delta\hat{P}_T(x.y.\omega) = 2\pi\rho_0 b \frac{U}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{w}_R(k_x.k_y)\, g(x.k_x.k_y)\, e^{ik_y y}\cdot e^{i(\omega - k_x U)t} dk_x dk_y dt$$

$$= 2\pi\rho_0 bU \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{w}_R(k_x, k_y)\, g(x, k_x, k_y)\, e^{ik_y y}\, \delta(\omega - k_x U)\, dk_x dk_y$$

Since $\int f(k)\,\delta(\omega - Uk)\,dk = \frac{1}{U} f(k=\frac{\omega}{U})$

$$\Delta\hat{P}_T(x.y.\omega) = 2\pi\rho_0 b \int_{-\infty}^{\infty} \hat{w}_R(K_x, k_y)\, g(x, K_x, k_y)\, e^{ik_y y}\, dk_y$$

where $K_x = \omega/U$

Since the turbulence is a random quantity, rather than work with deterministic quantities such as the time history of the pressure jump at a point, it is necessary to work with statistical quantities such as the cross-PSD, S_{QQ}, of the pressure jump at two points on the surface. The cross-PSD can be written as

$$S_{QQ}(x_1, x_2, y_1, y_2, \omega) = \lim_{T \to \infty} \left\{ \frac{\pi}{T} E\left[\Delta \hat{P}_T^*(x_1, y_1, \omega)\, \Delta \hat{P}_T(x_2, y_2, \omega) \right] \right\},$$

where $E[\cdots]$ denotes the expected value or ensemble average of a quantity. The only statistical or non-deterministic quantity on the right-hand side is $\hat{\omega}_R$. Because of the statistical orthogonality of the wavevectors, it can be shown that

$$E\left[\hat{\omega}_R(K_x, k_y)\, \hat{\omega}_R^*(K_x, k_y') \right] = \frac{R}{\pi} \delta(k_y - k_y')\, \phi_{ww}(K_x, k_y')$$

where

$$\phi_{ww}(K_x, k_y) = \int_{-\infty}^{\infty} \phi_{ww}(K_x, k_y, k_z)\, dk_z$$

and $\phi_{ww}(K_x, k_y, k_z)$ is the energy spectrum of the turbulence.

Then

$$S_{QQ}(x_1, x_2, y_1, y_2, \omega) = (2\pi \rho_0 b)^2 \cdot \frac{\pi}{T} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[\hat{\omega}_R(K_x, k_y)\, \hat{\omega}_R^*(K_x, k_y') \right] \times$$
$$g^*(x_1, K_x, k_y)\, g(x_2, K_x, k_y')\, e^{-i k_y y_1} \cdot e^{+i k_y y_2}\, dk$$

$$= (2\pi \rho_0 b)^2 \frac{R}{T} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(k_y - k_y')\, \phi_{ww}(K_x, k_y')\, g^*(x_1, K_x, k_y)\, g(x_2, K_x, k_y')$$
$$e^{i k_y (y_2 - y_1)}\, dk_y\, dk_y'$$

$$= (2\pi \rho_0 b)^2 U \int_{-\infty}^{\infty} g^*(x_1, K_x, k_y)\, g(x_2, K_x, k_y)\, \phi_{ww}(K_x, k_y)\, e^{i k_y \eta}\, dk_y$$

where $\eta = y_2 - y_1$ is the spanwise separation of the two points on the airfoil surface.

Since end effects of the airfoil are ignored, only the spanwise separation of the two points enters the equation, not the y co-ordinate of each point.

The theories of Kirchhoff and Curle now will be used to relate the cross-PSD of the surface pressure to the far-field sound.

The far-field sound produced by a point force of strength $F(x_0, y_0, \omega)e^{i\omega t}\vec{k}$ in a stream of Mach number M is

$$P_1(x,y,z,\omega;x_0,y_0) = \frac{i\omega z F(x_0,y_0,\omega)}{4\pi C_0 \sigma^2} e^{i\omega\left[t + \frac{M(x-x_0)-\sigma}{C_0\beta^2} + \frac{x x_0 + y y_0 \beta^2}{C_0\beta^2\sigma}\right]}$$

where $\sigma = \sqrt{x^2+\beta^2(y^2+z^2)}$ and $\beta = \sqrt{1-M^2}$.

For the present problem, the force $F(x_0, y_0)$ is the difference in pressure between the upper and lower surfaces of the airfoil (ΔP_T).

The far-field pressure can be found by integration of the pressure jump over x_0 and y_0: i.e., over the airfoil planform area.

$$S_{pp}(x.y.z,\omega) = \lim_{T\to\infty}\frac{\pi}{T}\int_{-d}^{d}\int_{-d}^{d}\int_{-b}^{b}\int_{-b}^{b} E[P_1^*(x.y.z,\omega;x_1,y_1)\, P_1(x,y,z,\omega;x_2,y_2)]\,dx_1\,dx_2\,dy_1\,dy_2$$

$$E[P_1^*\cdot P_1] = \left(\frac{\omega z}{4\pi C_0\sigma^2}\right)^2 \iiiint \lim_{T\to\infty}\left\{\frac{\pi}{T}E[\Delta P^*\cdot\Delta P]\right\} e^{-i\omega\left[\frac{M(x-x_1)}{C_0\beta^2} - \frac{M(x-x_2)}{C_0\beta^2} + \frac{x(x_2-x_1)+y(y_2-y_1)}{C_0\sigma}\right]} dx_1\,dx_2\,dy_1\,dy_2$$

$$= \left(\frac{\omega z}{4\pi C_0\sigma^2}\right)^2 \iiiint S_{QQ}(x_1,x_2,\eta,\omega)\, e^{\frac{i\omega}{C_0}\left[\beta^{-2}(x_1-x_2)(M - x/\sigma) + y\eta/\sigma\right]} dx_1\,dx_2\,dy_1\,dy_2$$

Define

$$\mathcal{L}(x, K_x, k_y) = \int_{-b}^{b} g(x_0, K_x, k_y)\, e^{-i\omega x_0 (M - x/\sigma)/C_0\beta^2}\, dx_0$$

Then,

$$S_{pp}(x,y,z,\omega) = \frac{1}{4}\left(\frac{\omega z \rho_0 b}{C_0 \sigma^2}\right)^2 \cdot U \int_{-\infty}^{\infty}\int_{-d}^{d}\int_{-d}^{d} |\mathcal{L}(x, K_x, K_y)|^2 \phi_{ww} \times e^{i\left[\frac{\omega}{C_0}\frac{(y_2-y_1)y}{\sigma} + k_y(y_2-y_1)\right]} dy_1 dy_2 dk_y$$

where $|\mathcal{L}(x, K_x, k_y)|^2 = \mathcal{L}^*(x_1, K_x, k_y)\, \mathcal{L}(x_2, K_x, k_y)$

$$\int_{-d}^{d}\int_{-d}^{d} e^{i\left[\frac{\omega}{C_0}\frac{(y_2-y_1)y}{\sigma} + k_y(y_2-y_1)\right]} dy_1 dy_2$$

$$= \int_{-d}^{d} e^{-i\left[\frac{\omega y}{C_0\sigma} + k_y\right] y_1} dy_1 \cdot \int_{-d}^{d} e^{i\left[\frac{\omega y}{C_0\sigma} + k_y\right] y_2} dy_2$$

Let $\frac{\omega y}{C_0\sigma} + k_y = X$

Then

$$= \int_{-d}^{d} e^{-iXy_1} dy_1 \cdot \int_{-d}^{d} e^{iXy_2} dy_2$$

$$= \frac{1}{-iX}\left[e^{-iXd} - e^{iXd}\right] \cdot \frac{1}{iX}\left[e^{iXd} - e^{-iXd}\right]$$

since $\sin Xd = \frac{e^{iXd} - e^{-iXd}}{2i}$

$$= \frac{1}{X^2} \cdot (-2i \sin Xd)(2i \sin Xd)$$

$$= \frac{4}{X^2} \sin^2 Xd$$

Therefore

$$S_{pp}(x,y,z,\omega) = \left(\frac{\omega z \rho_0 b}{C_0 \sigma^2}\right)^2 U d\pi \int_{-\infty}^{\infty} \left[\frac{\sin^2(d(k_y + \omega y/C_0\sigma))}{(k_y + \omega y/C_0\sigma)^2 \pi d}\right] |\mathcal{L}(x,K_x,k_y)|^2 \phi_{ww}(K_x,k_y)\,dk_y$$

The function $\mathcal{L}$ is related to the degree of non-compactness of the airfoil. If the frequency is small, the imaginary exponent is small, and $\mathcal{L}$ reduces to the sectional lift of the airfoil

To this point the analysis has been quite general. The principal assumption thus far, other than the assumption of linearized airfoil theory, has been to apply the infinite-span airfoil theory over the entire airfoil span, including the region near the tips. This should be a valid assumption for a large aspect ratio airfoil. However, for small turbulent length scales (high frequencies) it should not be necessary to make even the large aspect ratio assumption. As the wavelength of an incident gust decreases, the loading tends to concentrate near the leading edge, and the tip affects a smaller and smaller spanwise region of the airfoil.

We use

$$\lim_{d\to\infty}\left[\frac{\sin^2(\xi d)}{\xi^2 \pi d}\right] \rightarrow \delta(\xi)$$

and if we consider an observer $y=0$ plane.

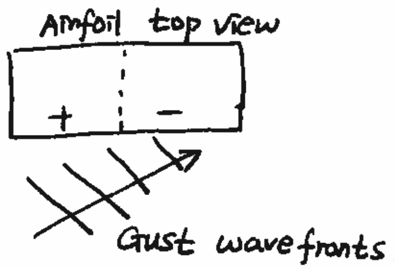
$$\boxed{S_{pp}(x,0,z,\omega) \rightarrow \left(\frac{\omega z \rho_0 b}{C_0 \sigma^2}\right)^2 \pi U d\, |\mathcal{L}(x,K_x,0)|^2 \phi_{ww}(K_x,0) \quad \text{as } d\to\infty}$$

since

$$\lim_{d\to\infty}\int_{-\infty}^{\infty} \frac{\sin^2(d(k_y+\frac{\omega y}{C_0\sigma}))}{(k_y+\omega y/C_0\sigma)^2\pi d} = \int_{-\infty}^{\infty}\delta\left(k_y+\frac{\omega y}{C_0\sigma}\right)dk_y = \int_{-\infty}^{\infty}\delta(K_y)\,dK_y$$

In the $y=0$ plane, only the $k_y=0$ gusts contribute to the sound.
(mid-span)

The skewed gust give cancelling effects and make no contribution to the sound.



− Let's define

$$R_{ww}(K_x, y) = \int_{-\infty}^{\infty} \phi_{ww}(K_x, k_y)\, e^{-ik_y y}\, dk_y$$

$$\phi_{ww}(K_x, k_y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{ww}(K_x, y)\, e^{ik_y y}\, dy$$

$$\phi_{ww}(K_x, 0) = \frac{1}{\pi} \int_{0}^{\infty} R_{ww}(K_x, y)\, dy$$

Correlation length :
(spanwise)

$$l_y(\omega) = \frac{1}{R_{ww}(K_x, 0)} \int_{0}^{\infty} R_{ww}(K_x, y)\, dy = \pi\, \phi_{ww}(K_x, 0) / R_{ww}(K_x, 0)$$

$R_{ww}(K_x, y)$ is U times the cross-PSD of the vertical velocity fluctuation at two points a distance y apart.

Let's define $S_{ww}(\omega) = R_{ww}(K_x, 0)/U$ is the PSD of the vertical velocity fluctuations.

$$\phi_{ww}(K_x, 0) = l_y(\omega)\, R_{ww}(K_x, 0)/\pi = l_y(\omega)\, S_w(\omega) \cdot \frac{U}{\pi}$$

$\hookrightarrow b_c \frac{\bar{U}_c}{\omega}$ $(b_c = 1.4,\ \bar{U}_c = 0.65\, U_\infty)$

Then

$$\boxed{S_{pp}(x, 0, z, \omega) \rightarrow \left(\frac{\omega z \rho_0 b M}{\sigma^2}\right)^2 d\, \left|\mathcal{L}(x, K_x, 0)\right|^2 l_y(\omega)\, S_{ww}(\omega)} \quad \text{as } \Delta \rightarrow \infty$$

where $\Delta \equiv M K_x d$. If Δ is large, both ϕ_{ww} and $\mathcal{L}$ become nearly independent of k_y, allowing them to be taken outside the integral.

The present theory becomes rigorous when

$$\Delta \equiv M K_x d \to \infty$$

For $k_y = 0$, the major portion of the turbulent energy is contained in a finite region $k_x < C/L$ where C is constant of the order 1 and L is the integral scale length of the turbulence.

The valid condition becomes $Md/L \to \infty$. Thus, for values of the Mach number which are not too small, the present theory should give a valid expression for the overall level when the integral scale, L, of the turbulence is significantly smaller than the airfoil spa

– Isotropic Turbulence.

The two most widely used models of isotropic turbulence are the Karman and the Liepmann models.

– Karman spectrum.

- Energy spectrum

$$E(k) = \frac{I k^4}{[1+(k/k_e)^2]^{17/6}}$$

Where $$I = \frac{55}{9\sqrt{\pi}} \frac{\Gamma(5/6)\,\overline{u^2}}{\Gamma(1/3)\,k_e^5}, \quad k_e = \frac{\sqrt{\pi}}{L}\frac{\Gamma(5/6)}{\Gamma(1/3)}$$

- Energy spectrum of the vertical velocity fluctuations

$$\phi_{ww}(k_x, k_y, k_z) = \frac{E(k)}{4\pi k^2}\left(1 - k_z^2/k^2\right)$$

By integrating over k_z, the function $\phi_{ww}(k_x, k_y)$

$$\phi_{ww}(k_x, k_y) = \frac{4}{9\pi}\frac{\overline{u^2}}{k_e^2}\frac{\hat{k}_x^2 + \hat{k}_y^2}{(1+\hat{k}_x^2+\hat{k}_y^2)^{7/3}}$$

Where $\hat{k} = k/k_e$

Taking the Fourier transform of $\phi_{ww}(k_x, k_y)$ with respect to k_y

$$R_{ww}(k_x, y) = \frac{2^{1/6}}{\Gamma(5/6)} (\tilde{y})^{5/6} \left[K_{5/6}(\tilde{y}) - \frac{3\tilde{y}}{3+8\hat{k}_x^2} K_{1/6}(\tilde{y}) \right] R_{ww}(k_x, 0),$$

where $\tilde{y} = y k_e \sqrt{1+\hat{k}_x^2}$

$$R_{ww}(k_x, 0) = \frac{L\bar{u}^2}{6\pi} \frac{3+8\hat{k}_x^2}{(1+\hat{k}_x^2)^{11/6}}$$

length scale

$$l_y(\omega) = \frac{8L}{3} \left[\frac{\Gamma(1/3)}{\Gamma(5/6)} \right]^2 \frac{\hat{k}_x^2}{(3+8\hat{k}_x^2)\sqrt{1+\hat{k}_x^2}}$$

Trailing Edge Noise

In eariler works the turbulence quadrupole sources have been assumed to be specified, and the noise such sources would produce in the presence of a scattering half-plane has been calculated. (Ref. Ffowcs Williams and Hall, 1970. JFM)

This procedure, however, gets involved ~~in~~ with difficulties inherent in the jet noise problem. Although the Lighthill-Curle formulation is mathematically exact, in a realistic approximate calculation the proper interpretation of the quadrupole source term can be difficult.

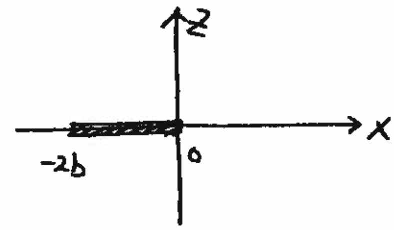
Chase is the first one who used the convecting surface pressure spectrum upstream of the trailing edge as input to trailing edge noise. (Ref. Chase, 1972, JASA). This approach is related to the quadrupole source approach: i.e., the surface pressure upstream of the trailing edge could, in principle, be calculated from a known quadrupole distribution plus its mirror image or the solution of Poisson equation for a half-plane surface. The surface pressure spectrum can be more easily obtained by measurement or empirical model and this approach is more useful than modelling the Lighthill stres tensor T_{ij}.

Howe developed a unified theory and showed that acoustic analogy and Chase model would ultimately provides the same result. He also discussed the kutta condition at the trailing edge.

Amiet developed a new formulation for trailing edge noise using the airfoil response function and considering mean flow effects. Amiet model is widely used to predict trailing edge noise. In his model, the basic assumption is that the turbulent velocity field is unaffected by the presence of the trailing edge: i.e., the turbulence is stationary in the statistical sense as it moves past the trailing edge. This assumption allows the edge noise to be calculated from the spectral characteristics of wall pressure which would exist in the absense of the trailing edge.

- The model to be used for calculating trailing-edge noise consists of a turbulent flow, with stationary statistical properties, convecting past a trailing edge. Upstream of the edge this produces a convecting pressure pattern on the airfoil surface. Near the trailing edge, in addition to the convecting pressure pattern, a radiating pressure field of comparable magnitude is produced. (scattered pressure).

The airfoil is reduced to a flat plate with zero thickness and angle of attack, and with chord length $c = 2b$.



Convected wave equation for the pressure on the surface

$$\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial z^2} - \frac{1}{C_0^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 p' = 0$$

Let $p'(x.z.t) = \bar{P}(x.z)\, e^{i\omega t}$

$$\beta^2 \frac{\partial^2 \bar{P}}{\partial x^2} + \frac{\partial^2 \bar{P}}{\partial z^2} - 2iKM\frac{\partial \bar{P}}{\partial x} + K^2 P = 0 \qquad \text{— Eq.(1)}$$

where $K = \frac{\omega}{C_0}$ and $\beta^2 = 1 - M^2$, $M = U/C_0$.

Use the change of variable

$$\bar{P}(x.z) = p(x.z)\, e^{i(KM/\beta^2)x} \qquad \text{— Eq.(2)}$$

$$\frac{\partial \bar{P}}{\partial x} = \frac{\partial p}{\partial x} e^{i(KM/\beta^2)x} + i(KM/\beta^2)\, p\, e^{i(KM/\beta^2)x}$$

$$\frac{\partial^2 \bar{P}}{\partial x^2} = \frac{\partial^2 p}{\partial x^2} e^{i(KM/\beta^2)x} + 2i(KM/\beta^2)\frac{\partial p}{\partial x} e^{i(KM/\beta^2)x} - \left(\frac{KM}{\beta^2}\right)^2 p\, e^{i(KM/\beta^2)x}$$

Eq.(1) becomes

Eq.(1) becomes

$$\beta^2 \left\{ \frac{\partial^2 p}{\partial x^2} + \cancel{2i \frac{KM}{\beta^2} \frac{\partial p}{\partial x}} - \left(\frac{KM}{\beta^2}\right)^2 p \right\} + \frac{\partial^2 p}{\partial z^2}$$

$$\cancel{-2iKM\frac{\partial p}{\partial x}} + 2\frac{(KM)^2}{\beta^2} p + K^2 p = 0 .$$

$$\beta^2 \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} + \left(K^2 + \frac{K^2M^2}{\beta^2}\right) p = 0 .$$

$$K^2 + \frac{K^2M^2}{\beta^2} = \frac{K^2(1-M^2) + K^2M^2}{1-M^2} = \frac{K^2}{1-M^2}$$

So that

$$\beta^2 \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} + \left(\frac{KM}{\beta}\right)^2 p = 0 \qquad - \text{Eq.(3)}$$

where $k = KM$ and $K = \omega/U$.

By further transforming the problem with

$$\bar{X} = \frac{x}{b}, \quad \bar{z} = \frac{\beta z}{b}, \quad \bar{K} = Kb, \quad \mu = \frac{\bar{K}M}{\beta^2} = \frac{\omega b M}{U\beta^2}$$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \bar{x}} \frac{\partial \bar{x}}{\partial x} = \frac{1}{b} \frac{\partial p}{\partial \bar{x}}, \quad \frac{\partial^2 p}{\partial x^2} = \frac{1}{b^2} \frac{\partial^2 p}{\partial \bar{x}^2},$$

$$\frac{\partial p}{\partial z} = \frac{\partial p}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial z} = \frac{\beta}{b} \frac{\partial p}{\partial \bar{z}}, \quad \frac{\partial^2 p}{\partial z^2} = \frac{\beta^2}{b^2} \frac{\partial^2 p}{\partial \bar{z}^2}$$

Then Eq.(3) becomes

$$\frac{\beta^2}{b^2} \frac{\partial^2 p}{\partial \bar{x}^2} + \frac{\beta^2}{b^2} \frac{\partial^2 p}{\partial \bar{z}^2} + \left(\frac{\bar{K} M}{b\beta}\right)^2 p = 0 .$$

$$\frac{\partial^2 p}{\partial \bar{x}^2} + \frac{\partial^2 p}{\partial \bar{z}^2} + \mu^2 p = 0$$

The airfoil extends over $-2 \le \bar{x} \le 0$ in non-dimensional variables.

Just upstream of the trailing edge, the incident pressure (normalized)

$$p'(x,0,t) = e^{i\omega t} e^{-i\frac{\omega}{U_c}x} = e^{i\omega t} \cdot e^{-i\underbrace{\frac{U}{U_c}}_{\alpha} \cdot \underbrace{\frac{\omega}{U}}_{K} x} = P_0 e^{i\omega t}.$$

$\bar{P}_0 = e^{-i\alpha K x}$. U_c is the convect speed, lower than U.

In order to derive the main scattering term from Schwarzschild's solution the airfoil is artificially extended to infinity upstream, covering $X<0$. $\bar{P}_0$ must be cancelled in the wake according to the Kutta condition. This is done by adding a disturbance pressure $\bar{P}_1$ such that $\bar{P} = \bar{P}_0 + \bar{P}_1$ is zero for $X \ge 0$. Since the airfoil is assumed perfectly rigid, the normal derivative of $\bar{P}_1$ must be zero for $X<0$.

$$\left(\frac{\partial \bar{P}}{\partial \bar{z}} = 0 \text{ and } \frac{\partial \bar{P}_0}{\partial \bar{z}} = 0, \quad \text{so that } \frac{\partial \bar{P}_s}{\partial \bar{z}} = 0 \right)$$

Let $\bar{P}_1 = P_1 e^{i(KM/\beta^2)x}$

Then,

$$\frac{\partial^2 P_1}{\partial \bar{X}^2} + \frac{\partial^2 P_1}{\partial \bar{z}^2} + \mu^2 P_1 = 0$$

$$\frac{\partial P_1}{\partial \bar{z}}(\bar{X},0) = 0, \quad \bar{x}<0.$$

$$P_1 = -e^{-i\alpha Kx} \cdot e^{-i(KM/\beta^2)x} \quad (\text{since } \bar{P}_1 = -\bar{P}_0)$$

$$Kx = \bar{K}\bar{x}, \quad k = KM.$$

$$P_1 = -e^{-i\bar{K}\bar{x}[\alpha + (M^2/\beta^2)]}, \quad X \ge 0$$

Schwarzschild's solution then yields P_1 and thus $\bar{P}_1$, for $X<0$ and $z=0$

$$P_1(\bar{X},0) = -\frac{1}{\pi}\int_0^{\infty} \sqrt{\frac{-\bar{X}}{\xi}} \frac{e^{-i\mu(\xi - \bar{X})}}{\xi - \bar{X}} e^{-i\bar{K}\xi[\alpha + (M^2/\beta^2)]} d\xi$$

$$= \frac{-e^{i\mu\bar{x}}}{\pi} \int_0^{\infty} \sqrt{\frac{-\bar{X}}{\xi}} \frac{e^{-i[\bar{K}\alpha + \underbrace{\bar{K}M^2/\beta^2 + \mu}_{\mu M}]\xi}}{\xi - \bar{X}} d\xi \quad \left(\mu = \frac{M\omega b}{U\beta^2} = \frac{KbM}{\beta^2} = \frac{\bar{K}M}{\beta^2}\right)$$

$$= -\frac{e^{i\mu\bar{x}}}{\pi} \int_0^{\infty} \sqrt{\frac{-\bar{X}}{\xi}} \frac{e^{-i[\alpha\bar{K} + (1+M)\bar{\mu}]\xi}}{\xi - \bar{X}} d\xi$$

We use

$$\int_0^{\infty} \sqrt{\frac{-X}{\xi}} \frac{e^{-iA\xi}}{\xi - X} d\xi = \pi e^{-iAX} \left[1 - \frac{e^{i\pi/4}}{\sqrt{\pi}} \int_0^{-AX} \frac{e^{-it}}{\sqrt{t}} dt \right]$$

Let $A = \alpha\bar{k} + (1+M)\bar{\mu}$

Then,

$$P_1 = -\frac{e^{i\mu\bar{x}}}{\pi} \cdot \pi e^{-i[\alpha\bar{k} + (1+M)\mu]\bar{x}} \left[1 - \frac{e^{i\pi/4}}{\sqrt{\pi}} \int_0^{-[\alpha\bar{k}+(1+M)\mu]x} \frac{e^{-it}}{\sqrt{t}} dt \right]$$

Let

$$E^*(x) = \int_0^x \frac{e^{-it}}{\sqrt{2\pi t}} dt = C_2(x) - iS_2(x)$$

where C_2 and S_2 are Fresnel integrals, and $\sqrt{2}\, e^{i\pi/4} = 1+i$.

$$P_1 = e^{-i[\alpha\bar{k} + M\mu]\bar{x}} \left[(1+i) E^*(-[\alpha\bar{k} + (1+M)\mu]\bar{x}) - 1 \right]$$

$$M\mu\bar{x} = \frac{\bar{k}M^2}{\beta^2}\bar{x} = \frac{kM^2}{\beta^2}x = \frac{kM}{\beta^2}x$$

Since $\bar{P}_1 = P_1 e^{i(kM/\beta^2)x}$

Therefore,

$$\bar{P}_1 = e^{-i\alpha\bar{k}\bar{x}} \left[(1+i) E^*(-[\alpha\bar{k} + (1+M)\mu]\bar{x}) - 1 \right]$$

Eq.(3)

or

$$g = \{(1+i)E^*[-\bar{x}((1+M)\mu + \bar{K}_x)] - 1\} e^{-i\bar{K}_x\bar{x}}$$

where $K_x = \frac{\omega}{U_c}$ and $\bar{K}_x = \frac{\omega}{U_c} \cdot b = \alpha\bar{K}$, $\alpha\bar{K}\bar{x} = \bar{K}_x\bar{x} = K_x x = \frac{\omega}{U_c}x$.

The total surface pressure jump is the sum of incident and induced pressure jumps. The incident pressure should grow gradually with the growth of the boundary layer instead of discontinously arising at the leading edge of the airfoil.

We modify the incident pressure with an exponential convergence factor, $\exp(\varepsilon\lambda x)$, so that the incident pressure at the trailing edge ($x=0$) is unchanged, but it is small at the leading edge ($x=-c$).

$$\bar{P}_0 = e^{-i\alpha K x} \cdot e^{\varepsilon\lambda x}.$$

Then, the total surface pressure jump is

$$\Delta P_S = P_I e^{-i\alpha\bar{k}\bar{x}} \left\{ e^{\varepsilon\lambda x} - 1 + (1+i)E^*[-\bar{x}((1+M)\mu + \bar{k}_x)] \right\}$$
$$= P_I \, g(\bar{x}, \omega, U_c)$$

Eq.(3) makes the assumption that the shed vorticity produced by the unsteady airfoil loading convects downstream with the flow speed U of the free stream. This assumption is standard in airfoil theory. In the present situation this assumption conflicts somewhat with the assumption that the turbulence on the airfoil and in the wake is convecting at a speed smalled than U.

Given the airfoil response function, g, standard spectral techniques can be applied to obtain the far-field noise if the spectral characteristics of the surface pressure field are known.

Note that $K_x U_c = \omega$. We assume a given frequency ω will be assumed to be associated with a single value of U_c and thus K_x. This is a reasonable assumption since U_c appears to be only a weak function of frequency.

For an airfoil in rectilinear motion the far-field sound can be calculated in the same manner as described for leading edge noise.

The normalization of g differs from by the factor $2\pi \rho_0 b U_c$. Also, the factor U should U_c for a surface pressure convecting at other than the freestream velocity.

Taking account of this factor

$$\left(\frac{\omega z \rho_0 b}{c_0 \sigma^2}\right)^2 U d\pi \rightarrow \left[\frac{\omega z}{2\pi c_0 \sigma^2}\right]^2 \frac{d\pi}{U_c}$$

$$S_{pp}(x,\omega) = \left(\frac{\omega z}{2\pi c_0 \sigma^2}\right)^2 \frac{\pi d}{U_c} \left|\mathcal{L}'(\vec{x}, k_x, \omega, U_c, k_y)\right|^2 \phi_{qq}(k_x, k_y)$$

$$|\mathcal{L}'| = \left|\int_{-2b}^{0} g(x_0, \omega, U_c)\, e^{-i\omega x_0 (M - x/\sigma)/c_0\beta^2}\, dx_0\right|$$

$$x_0 = b\xi \qquad dx_0 = b\, d\xi$$

$$= \left|\int_{-2}^{0} g(\xi, \omega, U_c)\, e^{-i\frac{\omega b \xi}{c_0\beta^2}(M - x/\sigma)}\, b\, d\xi\right|$$

$$= \left|\int_{-2}^{0} g(\xi, \omega, U_c)\, e^{-i\mu\xi(M - x/\sigma)}\, b\, d\xi\right| \qquad \left(\mu = \frac{\omega b M}{U\beta^2} = \frac{\omega b}{c_0\beta^2}\right)$$

So that

$$S_{pp}(x,\omega) = \left(\frac{\omega z b}{2\pi c_0 \sigma^2}\right)^2 \frac{\pi d}{U_c} \left|\mathcal{L}(x, \omega, U_c, k_y)\right|^2 \phi_{qq}(k_x, k_y)$$

$$|\mathcal{L}| = \left|\int_{-2}^{0} g(\xi, \omega, U_c)\, e^{-i\mu\xi(M - x/\sigma)}\, d\xi\right|$$

$$k_y = \omega y / c_0 \sigma$$

$$\sigma^2 = x^2 + \beta^2(y^2 + z^2)$$

ϕ_{qq} is the wavenumber spectrum of the airfoil surface produced by the turbulence. The function $\mathcal{L}$ can be thought of as an "effective lift".

$$|\mathcal{L}| = \frac{1}{\Theta} \left| e^{i2\theta} \left[1-(1+i) E^*[2((1+M)\mu + \bar{K}_x)] + \sqrt{\frac{(1+M+\bar{K}_x/\mu)}{(1+x/\sigma)}} E^*[2\mu(1+x/\sigma)] \right] \right|$$

where $\Theta = \bar{K}_x + \mu(M - x/\sigma)$

For $k_y = 0$; ($y=0$ plane)

$$\phi_{qq}(k_x, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{qq}(k_x, y, 0) e^{iky} \, dy.$$

$$l_y(\omega) = \frac{1}{R_{qq}(K_{xx}, 0)} \int_0^{\infty} R_{qq}(K_x, y=0) \, dy$$

$$\phi_{qq}(k_x, 0) = l_y(\omega) \cdot R_{qq}(k_x, 0) / \pi$$

$$= l_y(\omega) \, S_{qq}(\omega) \cdot \frac{U}{\pi}$$

where $S_{qq}(\omega) = R_{qq}(K_x, 0) / U$

$S_{qq}(\omega, y)$; spanwise cross-spectrum of surface pressure.

Then,

$$\boxed{S_p(x, 0, z, \omega) = \left(\frac{\omega b z}{2\pi C_0 \sigma^2}\right)^2 l_y(\omega) \, d \, |\mathcal{L}|^2 S_{qq}(\omega, 0)}$$

$$l_y(\omega) = \frac{1}{S_{qq}(\omega, 0)} \int_0^{\infty} S_{qq}(\omega, y) \, dy$$

$$\simeq 2.1 \, U_c / \omega$$

S_p must be multiplied by 8π to account for (i) a boundary layer on both upper and lower surface, (ii) to convert a single sided ($0<\omega<\infty$) spectrum and (iii) to convert to a 1 Hz bandwidth.