SOUND GENERATION BY TURBULENCE AND SURFACES IN ARBITRARY MOTION

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CONTENTS

The Lighthill–Curle theory of aerodynamic sound is extended to include arbitrary convective motion. The Kirchhoff description of a homogeneous wave field in terms of surface boundary conditions is also generalized to surfaces in arbitrary motion. The extension is at variance with the two previously published accounts of this problem which are erroneous. When both the bounding surfaces and the turbulence are compact relative to the radiated length scales, the turbulence is acoustically equivalent to a volume distribution of moving quadrupoles and the surfaces to dipole and monopole distributions. At low convective speed, their field increases as powers of the Doppler factor $|1-M_r|^{-1}$. Convective acceleration generally gives rise to new source terms at this condition. At the Mach wave condition when the Doppler factor is singular, both the turbulence and surfaces are non-compact and are acoustically equivalent to monopole distributions. Convective acceleration then tends to limit the radiation. At this condition the surface sources are quite unrelated to the low-speed sources, being second order in the field variable contrasting with the linear low-speed terms. At high supersonic convective speeds, the field is dominated by an intensive beaming along the directions of Mach wave emission that lie normal to the surface. The magnitude of the field then varies inversely as the Gaussian surface curvature. If the surface has only single curvature the field is proportional to $r^{-\frac{1}{2}}$ and if it is locally plane at this condition, the field no longer decays with distance travelled. There are indications that the surface-induced intensity increases as the square of surface speed at high supersonic speeds.

1. Introduction

Lighthill (1952, 1954) has shown how the problem of aerodynamic sound can be posed as an acoustic analogy in which the turbulence provides a quadrupole distribution in an ideal atmosphere at rest. He described the general properties of the induced field and developed the dominant effect of steady low-speed solenoidal source convection. The field increases as $|1-M_r|^{-3}$, M_r being the Mach number at which the source approaches the field point.

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Ffowcs Williams (1963) extended this theory to account for high-speed steady solenoidal convection and showed that relatively intense but finite fields were radiated in the Mach wave direction where the Doppler factor, $|1-M_r|^{-1}$, is singular. Curle (1955) gave the general effect of static surfaces, showing them to be equivalent to surface dipole distributions. Lowson (1965) developed the theory of point multipoles in arbitrary convective motion inferring without proof that moving aerodynamic surfaces could be modelled as moving point dipoles. In this paper general expressions are developed for the equivalent sources of arbitrarily moving aerodynamic bounding surfaces with adjacent turbulent flow. It transpires that only at low subsonic speeds can surface effects be treated unambiguously by Lowson's model. This paper then goes on to discuss general features of the radiated field by source distributions in arbitrary motion and to discuss general features of the sound generated by turbulent flow around high-speed aerodynamic surfaces.

Curle's (1955) extension of Lighthill's theory to account for surface effects made use of the Kirchhoff boundary solution to the homogeneous wave equation. In principle a theory accounting for surfaces in arbitrary motion might start with extensions of the Kirchhoff problem to moving surfaces published by Morgans (1930) and Kromov (1963). However, these theories are mutually incompatible and both contain errors, Kromov's a fundamental error at an early stage in his analysis, and Morgans's a minor inconsistency at a later stage. A correct extension of the Kirchhoff problem is therefore not available as a starting-point, but is one of the results of this paper (equation (5·3)).

The following analysis makes extensive appeal to generalized function theory. Generalized forms of the field variables are established to hold over a continuous infinite space. In that part of the space occupied by fluid they are equal to the real field variables, but in regions within the surfaces they have a well-defined simple form. Discontinuities across the surfaces account for concentrated surface source distributions. In § 2, equations governing the generalized density fluctuation are arranged in an inhomogeneous wave equation of the Lighthill type, with quadrupole, dipole and monopole inhomogeneities, the latter two being concentrated on the bounding surfaces, if any.

Section 3 describes four alternative descriptions of the radiation field of arbitrarily moving multipole distributions. One of the forms (equation (3·21)) is a generalization to a distribution of the result given by Lowson (1965) for a point source. However, this particular form suffers from extreme interpretational difficulties in regions where $|1-M_r|^{-1} \to \infty$. It is a particular object of this paper to deal effectively with this Mach wave régime and for this the alternative forms, equations (3·22), (3·23) and (3·24), are far more appropriate; they contain no singularity at the Mach wave condition. These results are developed further in § 4 to represent surface multipole distributions in arbitrary motion, equation (4·6) being the general result. In this equation one singular point remains. The denominator of this equation vanishes if the surface approaches the field point along its normal at exactly sonic speeds. This condition is a special condition treated separately in § 7 where it is shown that an intense beaming of sound can then arise. Section 5 is devoted to a statement of the formal general solutions of the Kirchhoff and aerodynamic sound problems that emerge from an application of the foregoing analysis.

The formal solution is discussed in § 6 in so far as it affects the field generated by the distributed volume sources. The previous results of the Lighthill theory emerge as special cases as does the extension to cover high Mach number convection. Unsteadiness of the convective field is shown to induce additional source terms but also a moderating influence on the Mach wave field. Equation (6·12) gives a continuously valid general expression for the parametric dependence of the density field, and is identical to that obtained for special models of the flow by Ribner (1962) and Ffowcs Williams (1963). Equations (6·14) and (6·15) give the analogous expressions when the field point is close to the source distribution and when the source is undergoing convective acceleration.

Section 7 is devoted to a discussion of the general effects of moving surfaces. At low speed when the sources are all compact they are equivalent to quadrupoles distributed on the internal volume, together with a dipole of strength equal to the applied force plus the displaced inertia, a result shown explicitly in (7·4). At high speeds when the sources are no longer compact the situation is very different, the appropriate expression then being equation (7·16) though, for ease of demonstration, this is a particular form for steady solenoidal convective motion. The significant feature of this result is that the important source terms at high speed are completely divorced from those dominant at low speed. The surface effect is entirely controlled by the viscous term and by Lighthill's turbulence stress tensor. The one remaining singularity of that equation is then discussed to show how the surface curvature plays a crucial role in the high-speed problem and how an intense beam of sound can be radiated from high-speed surfaces.

Finally the paper is concluded with some qualitative implications of the theory to the question of sound generation by high-speed aerodynamic machines.

2. Derivation of the governing equations

The theory of aerodynamic sound is built upon the equations of mass and momentum conservation of a compressible fluid. These equations are valid in the region exterior to any closed internal surfaces that may be present, and can be combined to give an inhomogeneous wave equation governing the generation and propagation of sound waves in that region (Lighthill 1952; Curle 1955). Such a situation is essentially inhomogeneous in space, in that these equations are valid in the volume outside the surfaces, but are meaningless elsewhere. However, spatial homogeneity can be restored if this situation is abandoned in favour of the following one. An *unbounded* fluid is envisaged, but one which is partitioned into regions by *mathematical* surfaces that exactly correspond to the real surfaces. The motion of the new fluid on and outside the mathematical surfaces is defined to be completely identical with the real motion, whereas the interior flow can be specified arbitrarily. Thus the original situation is embedded in a more general one, and any problem in the real fluid is matched by one in the unbounded fluid, their respective solutions being identical in the exterior region. The interior motion is usually assumed to be very simple, and consequently does not match the exterior flow at the boundaries. Mass and momentum sources have to be introduced to maintain these discontinuities, and these ultimately act as sound generators. The equations governing the unbounded fluid are then conservation equations with sources, and are valid everywhere in space, thus restoring homogeneity to the problem. The mathematical description of the unbounded fluid is aided by the use of generalized functions (Jones 1966), which enable the discontinuities to be handled quite simply. The new mass and momentum equations for a fluid with discontinuities are derived by an extension of the usual techniques.

Consider a fixed volume of fluid V enclosed by a surface Σ. Suppose V is divided into regions 1 and 2 by a surface of discontinuity S encroaching on region 2 with velocity $\mathbf{v}$. S may consist of several closed surfaces. Let $\mathbf{l}$ be the outward normal from V, and let $\mathbf{n}$ be normal to S going from region 1 to region 2. The superscripts 1 and 2 refer to the two regions, and an overbar implies that the variable is to be regarded as a generalized function valid throughout V, e.g. $\bar{\rho}$ is equal to $\rho^{(1)}$ in $V^{(1)}$ and $\rho^{(2)}$ in $V^{(2)}$. If ρ represents the fluid density, then the rate of change of mass within V is,

$$\frac{\partial}{\partial t}\int_V \bar{\rho}\,\mathrm{d}V = \frac{\partial}{\partial t}\int_{V^{(1)}} \rho^{(1)}\,\mathrm{d}V + \frac{\partial}{\partial t}\int_{V^{(2)}} \rho^{(2)}\,\mathrm{d}V. \tag{2·1}$$

The two regions have a moving boundary S, so that for each region

$$\frac{\partial}{\partial t}\int_{V^{(1)}} \rho^{(1)}\,\mathrm{d}V = -\int_{\Sigma^{(1)}} (\rho u_i)^{(1)}\, l_i\,\mathrm{d}\Sigma - \int_S [\rho(u_i - v_i)]^{(1)}\, n_i\,\mathrm{d}S, \tag{2·2}$$

where u_i is the component of the fluid velocity in the direction x_i $(i = 1, 2, 3)$, and a repeated suffix implies a summation over these values. Hence the rate of change of the total mass within V is

$$\frac{\partial}{\partial t}\int_V \bar{\rho}\,\mathrm{d}V = -\int_\Sigma \overline{(\rho u_i)}\, l_i\,\mathrm{d}\Sigma + \int_S [\rho(u_i - v_i)]_{(1)}^{(2)}\, n_i\,\mathrm{d}S, \tag{2·3}$$

the symbol $[\ \ ]_{(1)}^{(2)}$ meaning the difference of the contents between regions 2 and 1. By applying the divergence theorem, equation (2·3) can be written

$$\int_V \left(\frac{\partial\bar{\rho}}{\partial t} + \frac{\partial}{\partial x_i}\overline{(\rho u_i)}\right)\mathrm{d}V = \int_S [\rho(u_i - v_i)]_{(1)}^{(2)}\, n_i\,\mathrm{d}S. \tag{2·4}$$

If an equation of the form $f = 0$ defines the surface S, and is such that $f < 0$ in region 1 and $f > 0$ in region 2, then a surface integral over S can be replaced by a volume integral over V with the integrand multiplied by the generalized function $\delta(f)\{(\partial f/\partial x_j)^2\}^{\frac{1}{2}}$ (see § 3). Here $\delta(f)$ is the one-dimensional delta function, which is zero everywhere except where $f = 0$. Now, $n_i\{(\partial f/\partial x_j)^2\}^{\frac{1}{2}}$ is equal to $\partial f/\partial x_i$, so that (2·4) leads to the generalized mass equation

$$\frac{\partial\bar{\rho}}{\partial t} + \frac{\partial}{\partial x_i}\overline{(\rho u_i)} = [\rho(u_i - v_i)]_{(1)}^{(2)}\,\delta(f)\,\frac{\partial f}{\partial x_i}. \tag{2·5}$$

This equation implies that, as far as mass conservation is concerned, to maintain the unbounded fluid in its defined state, a shell distribution of mass sources is required, whose strength is the difference between the mass flux requirements of each region. In the same manner, the generalized momentum equation can be deduced,

$$\frac{\partial}{\partial t}\overline{(\rho u_i)} + \frac{\partial}{\partial x_j}(\overline{\rho u_i u_j} + \overline{p_{ij}}) = [p_{ij} + \rho u_i(u_j - v_j)]_{(1)}^{(2)}\,\delta(f)\,\frac{\partial f}{\partial x_j}. \tag{2·6}$$

Here p_{ij} is the compressive stress tensor. Equations (2·5) and (2·6) are the general forms of the equations governing the unbounded fluid, and are valid throughout space. If there are no discontinuities, the mass and momentum sources vanish, leaving the usual conservation equations. It is emphasized that the only restriction placed upon the surface S is one of smoothness, it can move in an arbitrary fashion, and change its shape or orientation.

To investigate an arbitrary sound field in the presence of a moving surface S, the superscript (1) is taken to refer to the region of the unbounded fluid that corresponds to the volume inside S. In this region, the fluid is assumed to be at rest with density ρ_0 and pressure p_0. These values of density and pressure are those that would be found in the real fluid were it at rest. As the stress tensor p_{ij} has the same mean value $p_0 \delta_{ij}$ in both regions, this constant vanishes from equation (2·6), and the symbol p_{ij} can be reinterpreted as the difference of the stress tensor from its mean value. The interior condition is then $p_{ij} = 0$. It is also assumed that in all practical situations, the surface S is impermeable, so that in the exterior region $u_n = v_n$. As $\mathbf{n}$ is defined to go from region 1 to region 2, it represents the outward normal from S. After replacing the interior variables by their assigned values, and dropping the superscript (2), the mass and momentum equations become

$$\left.\begin{aligned} \frac{\partial \overline{\rho}}{\partial t}+\frac{\partial}{\partial x_i}(\overline{\rho u_i}) &= \rho_0 v_i \delta(f)\frac{\partial f}{\partial x_i}, \\ \frac{\partial \overline{\rho u_i}}{\partial t}+\frac{\partial}{\partial x_j}(\overline{\rho u_i u_j}+\overline{p_{ij}}) &= p_{ij}\delta(f)\frac{\partial f}{\partial x_j}. \end{aligned}\right\} \tag{2·7}$$

To obtain the wave equation governing the generation and propagation of sound, $\overline{\rho u_i}$ is eliminated from equations (2·7), to give

$$\left(\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x_i^2}\right)\overline{(\rho-\rho_0)} = \frac{\partial^2 \overline{T_{ij}}}{\partial x_i \partial x_j}-\frac{\partial}{\partial x_i}\left(p_{ij}\delta(f)\frac{\partial f}{\partial x_j}\right)+\frac{\partial}{\partial t}\left(\rho_0 v_i \delta(f)\frac{\partial f}{\partial x_i}\right). \tag{2·8}$$

The dependent variable has been changed to the generalized density perturbation $\overline{\rho-\rho_0}$, as this is a measure of the sound amplitude. The generalized function $\overline{T_{ij}}$ is equal to Lighthill's stress tensor $T_{ij} = \rho u_i u_j + p_{ij} - c^2(\rho-\rho_0)\delta_{ij}$ outside any surfaces, and is zero within them.

Equation (2·8) shows that in general sound can be regarded as generated by three source distributions. The first of these is a distribution of acoustic quadrupoles of strength density T_{ij} distributed throughout the region exterior to the surfaces (Lighthill 1952). This is supplemented by surface distributions of acoustic dipoles of strength density $p_{ij} n_j$ (Curle 1955), and if the surfaces are moving, by further surface distributions of sources essentially monopole in character representing a volume displacement effect. It is to be emphasized that although these may not be the physical origin of the sound, they do completely specify the field. This equation remains true if shock discontinuities are present; these can be treated in the same way, but as mass and momentum fluxes are continuous across a shock, there are no extra sources to be included in (2·8). However, $\overline{T_{ij}}$ now contains discontinuities other than at physical boundary surfaces.

Equation (2·8) can be obtained by a second method. A generalized function is set up that equals the required function in the relevant region, and is zero elsewhere. Such a function is formed with the aid of Heavyside's unit function $H(f)$ defined to be unity where $f > 0$ and zero where $f < 0$. Thus if f is a function that is positive in the region of interest and negative elsewhere, the required generalized form of ψ is $\psi H(f)$. We use this form to solve the homogeneous scalar wave equation,

$$\frac{\partial^2 \psi}{\partial t^2}-c^2\frac{\partial^2 \psi}{\partial x_i^2} = 0 \tag{2·9}$$

in the presence of a moving surface. This problem is an extension of that solved by Kirchhoff, and will be referred to by that name. It can easily be shown by direct differentiation that $\psi H(f)$ satisfies the equation,

$$\left(\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x_i^2}\right)\psi H(f) = -c^2\frac{\partial}{\partial x_i}\left(\psi\delta(f)\frac{\partial f}{\partial x_i}\right)-\frac{\partial}{\partial t}\left(\psi v_i\delta(f)\frac{\partial f}{\partial x_i}\right)-\left(c^2\frac{\partial\psi}{\partial x_i}+v_i\frac{\partial\psi}{\partial t}\right)\delta(f)\frac{\partial f}{\partial x_i}. \tag{2·10}$$

Thus the Kirchhoff problem is converted into that of solving the generalized wave equation with shell sources. Knowledge of ψ, $\partial\psi/\partial n$ and $\partial\psi/\partial t$ on the surface completely defines the source distribution, and these can be assumed to be given as the boundary conditions of the original problem. The density fluctuation $\rho-\rho_0$ of a compressible fluid satisfies Lighthill's inhomogeneous wave equation, and so this latter process would lead to an equation similar to (2·10) with an extra source $H(f)\,\partial^2T_{ij}/\partial x_i\,\partial x_j$ on the right-hand side. This equation is an alternative governing equation for the generation of aerodynamic sound. The formal equivalence of the two equations can be established if it is noted that $\overline{\rho-\rho_0}$ and $(\rho-\rho_0)\,H(f)$ represent the same generalized function, and the ordinary mass and momentum equations are employed together with the condition that the surface is impermeable.

Of the two methods of deriving equation (2·8) the first is the most efficient and physically illuminating. However, the latter method shows the relation between aerodynamic sound, and the scalar wave field of a moving surface. Clearly the only mathematical difference between these problems is the extra volume distribution of sources in aerodynamic sound generation. These sources are negligible if the fluid velocity is everywhere small, since T_{ij} is of second order in velocity. Consequently the two wave fields are essentially identical for a slowly moving surface. However, if the surface moves at high speed, its motion induces large fluid velocities, and these extra sources can no longer be neglected; the two wave fields are then different.

3. Field of a multipole distribution in arbitrary motion

The basic equation to be solved is a generalized wave equation of the form

$$\frac{\partial^2\phi}{\partial t^2}-c^2\frac{\partial^2\phi}{\partial x_i^2} = \frac{\partial^n Q_{ij\ldots}}{\partial x_i\,\partial x_j\ldots}, \tag{3·1}$$

ϕ and $Q_{ij\ldots}$ being generalized functions. This equation governs the field of a distribution of nth-order multipoles of strength density $Q_{ij\ldots}$. The solution is well known (Jones 1964, p. 38)

$$\phi(\mathbf{x},t) = \frac{1}{4\pi c^2}\int_{-\infty}^{+\infty}\frac{\partial^n Q_{ij\ldots}(\mathbf{y},\tau)}{\partial y_i\,\partial y_j\ldots}\,\frac{\delta(\tau-t+r/c)}{r}\,\mathrm{d}\mathbf{y}\,\mathrm{d}\tau, \tag{3·2}$$

where $r = |\mathbf{x}-\mathbf{y}|$ is the distance from the source point $\mathbf{y}$ to the field point $\mathbf{x}$. A basic property of such convolution integrals is that derivatives can be interchanged, so that (3·2) can immediately be rewritten as

$$4\pi c^2\phi(\mathbf{x},t) = \frac{\partial^n}{\partial x_i\,\partial x_j\ldots}\int_{-\infty}^{+\infty} Q_{ij\ldots}(\mathbf{y},\tau)\,\delta\left(\tau-t+\frac{r}{c}\right)\frac{\mathrm{d}\mathbf{y}\,\mathrm{d}\tau}{r}. \tag{3·3}$$

The usual way of presenting this result is to perform the integration over τ that yields the familiar retarded time result

$$4\pi c^2\phi(\mathbf{x},t) = \frac{\partial^n}{\partial x_i\,\partial x_j\ldots}\int Q_{ij\ldots}\left(\mathbf{y},t-\frac{r}{c}\right)\frac{\mathrm{d}\mathbf{y}}{r}, \tag{3·4}$$

the integral being taken over all space.

So far it has been assumed that the source strength $Q_{ij\ldots}$ is known as a function of the fixed coordinate system $\mathbf{y}$, corresponding to a view that the waves are generated by a system of stationary sources. However, it is often desirable to regard the sources as in motion, as this greatly simplifies the task of specifying the source strength. For example, the strength of the shell sources corresponding to a moving surface is clearly more easily specified in a coordinate system moving with the surface rather than in any fixed reference frame. To allow for this change of emphasis, Lagrangian coordinates $\boldsymbol{\eta}$ are introduced which move with the sources; the sources are then at rest in the $\boldsymbol{\eta}$ space. If the source convection velocity is written as $c\mathbf{M}$, the Lagrangian coordinates are related to the fixed system by the equation

$$\mathbf{y} = \boldsymbol{\eta}+\int^{\tau} c\mathbf{M}(\boldsymbol{\eta},\tau')\,\mathrm{d}\tau'. \tag{3·5}$$

By writing the source strength as $q_{ij\ldots}$ when referred to the $\boldsymbol{\eta}$ frame, the field integral (3·3) becomes

$$4\pi c^2\phi(\mathbf{x},t) = \frac{\partial^n}{\partial x_i\,\partial x_j\ldots}\int_{-\infty}^{+\infty} q_{ij\ldots}(\boldsymbol{\eta},\tau)\,\delta\left(\tau-t+\frac{r}{c}\right)J\frac{\mathrm{d}\boldsymbol{\eta}\,\mathrm{d}\tau}{r}; \tag{3·6}$$

where r is now a function of τ:

$$r = \left|\mathbf{x}-\boldsymbol{\eta}-\int^{\tau} c\mathbf{M}(\boldsymbol{\eta},\tau')\,\mathrm{d}\tau'\right|. \tag{3·7}$$

The Jacobian of the transformation J can be related to the convection velocity $c\mathbf{M}$ by the equation (Aris 1962, p. 84),

$$J = \exp\left\{\int^{\tau} \operatorname{div} c\mathbf{M}(\boldsymbol{\eta},\tau')\,\mathrm{d}\tau'\right\}, \tag{3·8}$$

the divergence being taken with respect to the $\mathbf{y}$ frame.

The time variable τ is now replaced by a new length variable λ, the two being related by

$$\left.\frac{\partial\lambda}{\partial\tau}\right|_{\boldsymbol{\eta}} = c\alpha. \tag{3·9}$$

The reasons for this are twofold. First, this unifies the dimensions of the field integral (3·6); and secondly, the arbitrary scaling factor α enables important space and time scales to be equated, considerably simplifying the interpretation of later results. This change yields the equation

$$4\pi c^2\phi(\mathbf{x},t) = \frac{\partial^n}{\partial x_i\,\partial x_j\ldots}\int_{-\infty}^{+\infty} q_{ij\ldots}(\boldsymbol{\eta},\lambda)\,\delta\left(\tau(\boldsymbol{\eta},\lambda)-t+\frac{r}{c}\right)J\frac{\mathrm{d}\boldsymbol{\eta}\,\mathrm{d}\lambda}{rc\alpha}. \tag{3·10}$$

Although equation (3·3) is one expression for the radiation field, alternative expressions can be given if some results of generalized function theory are employed. These concern m dimensional volume integrals containing delta functions, and are first (Jones 1966, p. 262)

$$\int_{-\infty}^{+\infty} F(\mathbf{z})\,\delta(g(\mathbf{z}))\,\mathrm{d}\mathbf{z} = \int_{\Sigma_k}\left\{\frac{F}{|\partial g/\partial z_k|}\right\}(\mathbf{z}^*)\,\mathrm{d}\mathbf{z}_k, \tag{3·11}$$

where $\mathbf{z}^*$ is a point on the hypersurface Σ defined by $g(\mathbf{z}^*) = 0$; Σ_k is the projection of Σ on to the coordinate hyperplane with normal in the k direction, an element of which is denoted by $\mathrm{d}\mathbf{z}_k$. Secondly (Jones 1966, p. 263),

$$\int_{-\infty}^{+\infty} F(\mathbf{z})\,\delta(g(\mathbf{z}))\,\mathrm{d}\mathbf{z} = \int_\Sigma \left\{\frac{F}{|\mathrm{grad}\,g|}\right\}(\mathbf{z}^*)\,\mathrm{d}\Sigma, \tag{3·12}$$

where $$|\mathrm{grad}\,g|^2 = (\partial g/\partial z_k)^2;$$

and thirdly $$\int_{-\infty}^{+\infty} F(\mathbf{z})\,\delta(g(\mathbf{z}))\,\mathrm{d}\mathbf{z} = \sum_{k=1}^{m} \int_{\Sigma_k} \left\{\frac{F\,|\partial g/\partial z_k|}{|\mathrm{grad}\,g|^2}\right\}(z^*)\,\mathrm{d}\mathbf{z}_k. \tag{3·13}$$

In (3·11) and (3·13) repeated suffices are not implicitly summed. Expression (3·12) implies that a volume integral containing a delta function is the same as a surface integral modified by the factor $|\mathrm{grad}\,g|^{-1}$, a result that would be obtained by working formally and utilizing the well-known property of the delta function

$$\int_{-\infty}^{+\infty} F(g)\,\delta(g)\,\mathrm{d}g = F(0). \tag{3·14}$$

The other two results are alternative expressions for calculating this surface integral, by either projecting Σ on to one coordinate hyperplane, or by resolving it on to all the hyperplanes. The projection of Σ fails if it lies perpendicular to the chosen hyperplane, and corresponds to $\partial g/\partial z_k$ vanishing in equation (3·11). The other two expressions never fail for a smooth surface, because $|\mathrm{grad}\,g|$ cannot then be zero. Although these three expressions yield identical values when integrated analytically, some may be more useful than others for estimating the magnitude of the radiated field (see § 5).

To utilize these expressions in the evaluation of the radiation integral (3·10), the relevant components of $|\mathrm{grad}\,g|$ must be calculated. In equation (3·10) g represents $\tau - t + r/c$ so that

$$\left.\frac{\partial g}{\partial \lambda}\right|_\eta = \left.\frac{\partial g}{\partial \tau}\right| \left.\frac{\partial \tau}{\partial \lambda}\right| = \frac{1}{c\alpha}\left(1 + \frac{1}{c}\left.\frac{\partial r}{\partial \tau}\right|\right). \tag{3·15}$$

However, from equation (3·7) it follows that $\partial r/\partial\tau|_\eta$ is $-cM_r$, M_r being the component of $\mathbf{M}$ in the direction of the radiation vector $\mathbf{r} = (\mathbf{x}-\mathbf{y})$. Thus

$$\frac{\partial g}{\partial \lambda} = \frac{1}{c\alpha}\{1 - M_r\}. \tag{3·16}$$

Similarly, $$\left.\frac{\partial g}{\partial \eta_i}\right|_\lambda = \frac{\partial g}{\partial \eta_i} + \left.\frac{\partial g}{\partial \tau}\right|_\lambda \left.\frac{\partial \tau}{\partial \eta_i}\right|_\lambda = \frac{1}{c}\frac{\partial r}{\partial \eta_i} + (1 - M_r)\left.\frac{\partial \tau}{\partial \eta_i}\right|_\lambda. \tag{3·17}$$

Again from equation (3·7) it follows that

$$\left.\frac{\partial r}{\partial \eta_i}\right|_\tau = -\frac{r_i}{r} - \frac{r_j}{r}\int^\tau c\,\frac{\partial M_j}{\partial \eta_i}\,\mathrm{d}\tau', \tag{3·18}$$

so that $$\left.\frac{\partial g}{\partial \eta_i}\right|_\lambda = -\frac{1}{c}\left\{\frac{r_i}{r} + a_i\right\},$$

where $$a_i = \frac{r_j}{r}\int^\tau c\,\frac{\partial M_j}{\partial \eta_i}\,\mathrm{d}\tau' - c(1 - M_r)\left.\frac{\partial \tau}{\partial \eta_i}\right|_\lambda. \tag{3·19}$$

Hence
$$\left.\begin{aligned}|\mathrm{grad}\, g| &= \frac{1}{c\alpha}\{(1-M_r)^2+\alpha^2(1+2a_r+a_i^2)\}^{\frac{1}{2}} \\ &= \frac{1}{c\alpha}\{(1-M_r)^2+\beta^2\}^{\frac{1}{2}}.\end{aligned}\right\} \tag{3·20}$$

Alternative expressions for the radiation integral can now be given. By choosing the k direction to coincide with the λ (time) axis, result (3·11) yields the expression

$$4\pi c^2\phi(\mathbf{x},t) = \frac{\partial^n}{\partial x_i\,\partial x_j\ldots}\int \frac{q_{ij\ldots}(\boldsymbol{\eta},\lambda^*)}{|1-M_r|}\,\frac{J}{r}\,\mathrm{d}\boldsymbol{\eta}, \tag{3·21}$$

λ^* being the value of λ corresponding to the retarded time $\tau^* = t-r/c$. The effect of source convection is clearly revealed in this result; convection effectively increases the source strength by a factor $|1-M_r|^{-1}$, the Jacobian J accounting for any divergence of the source during the motion. If other axes are chosen to coincide with the k direction, further expressions result which avoid the difficulties of the condition $|1-M_r| = 0$. For example, if the spatial component of the normal to Σ is chosen to coincide with the k direction, the radiation integral can be written as

$$4\pi c^2\phi(\mathbf{x},t) = \frac{\partial^n}{\partial x_i\,\partial x_j\ldots}\int \frac{q_{ij\ldots}(\boldsymbol{\eta}^*,\lambda)}{\{1+2a_r+a_i^2\}^{\frac{1}{2}}}\,\frac{J}{r\alpha}\,\mathrm{d}\eta_k\,\mathrm{d}\lambda, \tag{3·22}$$

a form appropriate to the estimation of Mach wave generation by supersonic flows (Ffowcs Williams 1963). If result (3·12) is employed, this leads to the formula

$$4\pi c^2\phi(\mathbf{x},t) = \frac{\partial^n}{\partial x_i\,\partial x_j\ldots}\int \frac{q_{ij\ldots}(\boldsymbol{\eta}^*,\lambda^*)}{\{(1-M_r)^2+\beta^2\}^{\frac{1}{2}}}\,\frac{J}{r}\,\mathrm{d}\Sigma, \tag{3·23}$$

whereas (3·13) gives the alternative result

$$\begin{aligned}4\pi c^2\phi(\mathbf{x},t) = {}& \frac{\partial^n}{\partial x_i\,\partial x_j\ldots}\int \frac{|1-M_r|}{\{(1-M_r)^2+\beta^2\}}\,q_{ij\ldots}(\boldsymbol{\eta},\lambda^*)\,\frac{J}{r}\,\mathrm{d}\boldsymbol{\eta} \\ &+\sum_{k=1}^{3}\frac{\partial^n}{\partial x_i\,\partial x_j\ldots}\int \frac{\alpha\,|r_k/r+a_k|}{\{(1-M_r)^2+\beta^2\}}\,q_{ij\ldots}(\eta_k^*,\lambda)\,\frac{J}{r}\,\mathrm{d}\eta_k\,\mathrm{d}\lambda.\end{aligned} \tag{3·24}$$

These three results are equivalent expressions for the solution of equation (3·1), and represent the field of a distribution of nth order multipoles. Although they are valid for any generalized distribution of sources, when the sources are concentrated on a shell, $Q_{ij\ldots}$ contains a further delta function which renders them of little practical use without further reduction.

4. Field of a surface multipole moving with arbitrary speed

An inspection of the equations (2·8) and (2·10) reveals that the source strength corresponding to a shell distribution of multipoles is better written as

$$Q_{ij\ldots}(\mathbf{y},\tau)\,|\mathrm{grad}_{\mathbf{y}} f|\,\delta(f(\mathbf{y},\tau)).$$

In these equations $\partial f/\partial y_i$ can always be replaced by $|\mathrm{grad}_{\mathbf{y}} f|\,n_i$. The subscript $\mathbf{y}$ is a reminder that the gradient is taken in the $\mathbf{y}$ frame, and does not include the τ component of f. The convection velocity $c\mathbf{M}$ of the surface sources is clearly the surface velocity $\mathbf{v}$, and it follows that f, regarded as a function of $\boldsymbol{\eta}$ is independent of τ. The early analysis of § 3 follows through unaltered, and the basic equation corresponding to (3·10) for a shell source is

$$4\pi c^2\phi(\mathbf{x},t) = \frac{\partial^n}{\partial x_i\,\partial x_j\ldots}\int q_{ij\ldots}(\boldsymbol{\eta},\lambda)\,|\mathrm{grad}_{\mathbf{y}} f|\,\delta(f)\,\delta(g)\,\frac{J}{rc\alpha}\,\mathrm{d}\boldsymbol{\eta}\,\mathrm{d}\lambda, \tag{4·1}$$

where g has again been written for $\tau - t + r/c$.

One way of reducing expression (4·1) is by repeated use of the general results (3·11) to (3·13). By employing the first formula with the k direction coinciding with the λ axis, the intermediate result is obtained,

$$4\pi c^2\phi(\mathbf{x},t) = \frac{\partial^n}{\partial x_i\,\partial x_j\ldots}\int \frac{q_{ij\ldots}(\boldsymbol{\eta},\lambda^*)}{r|1-M_r|}\,J\,|\mathrm{grad}_{\mathbf{y}} f|\,\delta(f)\,\mathrm{d}\boldsymbol{\eta}. \tag{4·2}$$

The remaining delta function is unaffected by this operation as f is independent of λ. This form of the integral allows a further use of the general results, and if (3·12) is now employed, a result analogous to (3·21) is obtained:

$$4\pi c^2\phi(\mathbf{x},t) = \frac{\partial^n}{\partial x_i\,\partial x_j\ldots}\int \frac{q_{ij\ldots}(\boldsymbol{\eta},\lambda^*)}{r|1-M_r|}\,J\,\frac{|\mathrm{grad}_{\mathbf{y}} f|}{|\mathrm{grad}_{\boldsymbol{\eta}} f|}\,\mathrm{d}S(\boldsymbol{\eta}). \tag{4·3}$$

The factor $J|\mathrm{grad}_{\mathbf{y}} f|\,|\mathrm{grad}_{\boldsymbol{\eta}} f|^{-1}$ appearing in this integral is no more than the ratio of the area elements of the surface S in the $\mathbf{y}$ and $\boldsymbol{\eta}$ spaces (just as J is the ratio of volume elements), and is denoted by A. This ratio can be related to the two dimensional, or surface, divergence of the convection velocity $c\mathbf{M}$, in the same way as J is related to its three-dimensional divergence. Differential expressions for this surface divergence are given in works on differential geometry or tensor calculus (Weatherburn 1927, 1950). If the surface is unextended in the motion, A is unity. Thus again convection of a surface source effectively increases its strength by a factor $|1-M_r|^{-1}$, A accounting for any expansion of the surface.

However, a more general expression corresponding to (3·23) can be obtained. Just as a volume integral containing one delta function $\delta(g)$ is equivalent to an integral over the subspace $g=0$, so an integral with two delta functions $\delta(g)$ and $\delta(f)$ is equivalent to an integral over the subspace $g=0, f=0$ (Gelfand & Shilov 1964, p. 239). If this subspace is denoted by σ, then it can be shown that the basic integral (4·1) can be written

$$4\pi c^2\phi(\mathbf{x},t) = \frac{\partial^n}{\partial x_i\,\partial x_j\ldots}\int_\sigma q_{ij\ldots}(\boldsymbol{\eta},\lambda)\,|\mathrm{grad}_{\mathbf{y}} f|\,\frac{J}{r\alpha c}\,\frac{\mathrm{d}\sigma}{D}, \tag{4·4}$$

where

$$D^2 = (\nabla g)^2\,(\nabla f)^2 - (\nabla f.\nabla g)^2. \tag{4·5}$$

The symbol ∇ is the usual gradient operator in the four-dimensional $\boldsymbol{\eta}$, λ space. If the four-dimensional normals to the hypersurfaces $g=0$ and $f=0$ are inclined at an angle Θ, then D can be rewritten as $|\mathrm{grad}_{\boldsymbol{\eta}} f|\,|\mathrm{grad}_{\boldsymbol{\eta}} g|\sin\Theta$. Alternatively, as f does not depend upon λ, D takes the form

$$|\mathrm{grad}_{\boldsymbol{\eta}} f|\,\{(\partial g/\partial\lambda)^2 + (\partial g/\partial\eta_i)^2\sin^2\theta\}^{\frac{1}{2}},$$

θ being the angle between the $\boldsymbol{\eta}$ components of ∇f and ∇g. Thus the alternative general expression for the field of a shell multipole is

$$4\pi c^2\phi(\mathbf{x},t) = \frac{\partial^n}{\partial x_i\,\partial x_j\ldots}\int \frac{q_{ij\ldots}(\boldsymbol{\eta},\lambda)}{\{(1-M_r)^2+\beta^2\sin^2\theta\}^{\frac{1}{2}}}\,\frac{A}{r}\,\mathrm{d}\sigma. \tag{4·6}$$

Other expressions can be obtained if this integral over the two dimensional subspace σ is projected or resolved into integrals over other subspaces, as for the hypersurface Σ. Equation (4·3) could be obtained in this way. However, other such expressions are extremely complicated, and are only useful in simplified circumstances, and are not given here.

The basic expression (4·6) for shell sources differs in one very important respect from the corresponding result (3·26) for volume sources. The integral becomes singular at the condition $\sin\Theta = 0$. This occurs whenever the surface S moves towards the field point at the wave speed c with its normal parallel to the radiation direction. This condition corresponds exactly to that of specifying the boundary conditions on a characteristic of the original governing differential equation, and a failure in the solution is to be expected. Although such singularities seriously complicate the analysis of shell sources, undue emphasis should not be placed upon this difficulty, as there are many practical situations in which this condition does not occur, and as will be shown in § 7, the singularities are integrable in the aerodynamic sound problem.

5. General solutions of the governing equations

The integral expressions for the radiated fields of arbitrary volume or surface multipoles developed in the previous sections can be used to write down the formal solutions of our two basic wave equations. Clearly these solutions can be given in many different forms, depending upon the exact formula chosen to evaluate each radiation integral. However, only those of general importance are recorded below. The first form of solution for equation (2·8) is obtained by utilizing expressions (3·21) and (4·3) to reduce the various radiation integrals:

$$4\pi c^2(\rho(\mathbf{x},t)-\rho_0) = \frac{\partial^2}{\partial x_i\,\partial x_j}\int\left[\frac{T_{ij}J}{r|1-M_r|}\right]\mathrm{d}\boldsymbol{\eta} - \frac{\partial}{\partial x_i}\int\left[\frac{p_{ij}n_jA}{r|1-M_r|}\right]\mathrm{d}S(\boldsymbol{\eta}) + \frac{\partial}{\partial t}\int\left[\frac{\rho_0 v_n}{r|1-M_r|}\right]\mathrm{d}S(\boldsymbol{\eta}), \quad (5·1)$$

where the square brackets imply that the contents are to be evaluated at the retarded time given implicitly by $\tau = t-(r/c)$. The volume integral is taken over the region exterior to the surfaces. Alternatively, if equations (3·23) and (4·6) are employed in the reduction of the radiation integrals, the following expression results:

$$4\pi c^2(\rho(\mathbf{x},t)-\rho_0) = \frac{\partial^2}{\partial x_i\,\partial x_j}\int\frac{\overline{T_{ij}}J}{\{(1-M_r)^2+\beta^2\}^{\frac{1}{2}}}\frac{\mathrm{d}\Sigma}{r} - \frac{\partial}{\partial x_i}\int\frac{p_{ij}n_jA}{\{(1-M_r)^2+\beta^2\sin^2\theta\}^{\frac{1}{2}}}\frac{\mathrm{d}\sigma}{r} + \frac{\partial}{\partial t}\int\frac{\rho_0 v_n A}{\{(1-M_r)^2+\beta^2\sin^2\theta\}^{\frac{1}{2}}}\frac{\mathrm{d}\sigma}{r}. \quad (5·2)$$

$\overline{T_{ij}}$ is zero at points corresponding to the inside of S.

These two expressions are important because they correspond to two different approaches to the generation of aerodynamic sound. In the first view, the spatial and temporal natures of the sources are treated as essentially different, regarding the sound as generated by a spatial distribution of time varying sources. Equation (5·1) is the solution incorporating this outlook. Any estimation of the radiated sound based on this equation clearly runs into serious difficulties whenever the factor $1-M_r$ vanishes, unless the equation is very carefully interpreted. Such an interpretation leads to the emphasis being transferred from the spatial to the temporal distribution of the sources (Ffowcs Williams 1963). These difficulties, however, stem from the separation of space and time, and disappear if such a distinction is abandoned in favour of a unified four-dimensional approach that regards the sources as distributed throughout the whole of space-time. Equation (5·2) reveals that the emphasis is now always to be placed upon the distribution of sources over a hypersurface of variable orientation. The previously troublesome vanishing of $1-M_r$ now simply implies that the hypersurface lies parallel to the time axis, immediately showing the importance of the temporal distribution of the sources, and allowing an unambiguous estimate to be made of the radiated sound field. Although the unified approach may be unfamiliar, this initial disadvantage is outweighed by its ability to shed light on situations of considerable complexity when posed within the more familiar framework. This is illustrated in the following section. However, a word of caution is necessary concerning the unified solution (5·2). The reason for developing this solution is to have a form that allows easy estimation of the sound field around the Mach wave condition $(1-M_r) = 0$. Before such an estimation can be carried out analytically, the field point derivatives must always be taken under the integral sign, where they become equivalent to differential operators acting upon the integrand. If there are any surfaces present, these operators act upon the discontinuities in $\overline{T_{ij}}$ to produce delta function singularities, and ultimately further integrals over σ. Thus if the sound field of a rapidly moving surface is investigated with the aid of (5·2), the whole equation must be considered and not just the last two integrals. This phenomenon does not occur with equation (5·1), since the extra σ integrals vanish when Σ coincides with the $\boldsymbol{\eta}$ space. On the other hand, equation (5·1) is useless in regions where $1-M_r$ vanishes. This importance of the first integral in (5·2) on mathematical grounds is in addition to its physical importance as a result of the high fluid velocities induced by the surface motion.

The essential character of the Kirchhoff problem is the derivation of an analytical expression for the wave field in terms of the given boundary conditions. It is automatically assumed that the resulting integrals can be evaluated exactly, and the more sophisticated expressions for the radiated field serve no useful purpose. Thus the basic solution of the scalar wave equation with boundary conditions prescribed on a moving surface is

$$4\pi c^2\psi(\mathbf{x},t) = -c^2\frac{\partial}{\partial x_i}\int\left[\frac{\psi n_i A}{r\,|1-v_r/c|}\right]\mathrm{d}S(\boldsymbol{\eta}) - \frac{\partial}{\partial t}\int\left[\frac{\psi v_n A}{r\,|1-v_r/c|}\right]\mathrm{d}S(\boldsymbol{\eta}) - \int\left[\left(c^2\frac{\partial\psi}{\partial n} + v_n\frac{\partial\psi}{\partial\tau}\right)\frac{A}{r\,|1-v_r/c|}\right]\mathrm{d}S(\boldsymbol{\eta}), \quad (5\cdot3)$$

where $\mathbf{v}$ is the surface velocity. Two previously published attempts at this problem are in error (Morgans 1930; Kromov 1963).

6. Sound field of an aerodynamic source in arbitrary motion

The unified four-dimensional approach can be used to advantage in investigating the far sound field generated by a single aerodynamic source in arbitrary motion. Not only do previously obtained results emerge quite naturally from this approach, but also new results concerning the effect of source acceleration are apparent. An aerodynamic source models the noise-generating properties of a turbulent eddy, which is equivalent to a quadrupole source of strength density T_{ij}. Consequently, the source is assumed to be coherent within spatial and temporal scales corresponding to the correlation length and lifetime of a turbulent eddy. It is further assumed that there are no surfaces present, and that the convection velocity is uniform throughout the source, so that $\mathbf{M}$ is independent of $\boldsymbol{\eta}$. It follows that J is unity and the vector $\mathbf{a}$ of equation (3·19) is zero. Equation (5·2) then shows that the density fluctuations in the sound field are given by the integral,

$$4\pi c^2(\rho-\rho_0) = \frac{\partial^2}{\partial x_i\,\partial x_j}\int \frac{T_{ij}}{\{(1-M_r)^2+\alpha^2\}^{\frac{1}{2}}}\frac{\mathrm{d}\Sigma}{r}. \tag{6·1}$$

To estimate the magnitude of these fluctuations, it is clear that not only is knowledge of the integrand required, but also a measure of the 'area' $\mathrm{d}\Sigma$. This is the area in which Σ intersects the source region, and varies with the orientation and shape of Σ. These in turn vary with the convection velocity. For example, if the source is at rest, $\mathrm{d}\Sigma$ is determined solely by the spatial length scale l, but in the Mach wave condition $(1-M_r) = 0$, Σ lies parallel to the time axis and $\mathrm{d}\Sigma$ now depends upon a typical temporal scale $\Delta\tau$. However, by choosing α to equate the temporal (λ) scale $\alpha c\Delta\tau$ and the spatial scale l, the area $\mathrm{d}\Sigma$ is always of magnitude l^3 regardless of the orientation or shape of Σ. A universal estimation of the radiation integral can then be accomplished for all convection velocities. Since these spatial and temporal scales are only defined approximately, α is also only an approximate order of magnitude, and must be regarded as constant throughout the integral. Before the detailed estimation can be completed, however, the derivatives with respect to $\mathbf{x}$ must be taken within the integral. The integrand's explicit dependence upon $\mathbf{x}$ through the factor r^{-1} is not important, as this only produces near field terms. Instead, the far field contribution arises from the dependence of the integral upon the variable position of the hypersurface Σ. To differentiate such an integral, it is best replaced by a volume integral containing a delta function, and the following lemma employed:

$$\int_{-\infty}^{+\infty} F(\mathbf{z})\,\delta'(g)\,\mathrm{d}\mathbf{z} = -\int_{-\infty}^{+\infty}\frac{\partial}{\partial z_i}\left(\frac{\partial g}{\partial z_i}\frac{F}{|\mathrm{grad}\,g|^2}\right)\delta(g)\,\mathrm{d}\mathbf{z}. \tag{6·2}$$

If $\mathbf{N}$ is written for the unit vector normal to the constant g hypersurface,

$$N_i = |\mathrm{grad}\,g|^{-1}\frac{\partial g}{\partial z_i}, \tag{6·3}$$

and the integral in (6·2) can be written

$$-\int_{-\infty}^{+\infty}\left\{\frac{N_i}{|\mathrm{grad}\,g|}\frac{\partial F}{\partial z_i}+F\frac{\partial}{\partial z_i}\left(\frac{N_i}{|\mathrm{grad}\,g|}\right)\right\}\delta(g)\,\mathrm{d}\mathbf{z}. \tag{6·4}$$

In the far field of a source whose convection velocity is independent of $\boldsymbol{\eta}$, the latter term in (6·4) reduces to its time component, so that after a double application of this result the radiation integral can be written

$$4\pi c^2(\rho-\rho_0) = \int\left[\left(\frac{\alpha}{\{(1-M_r)^2+\alpha^2\}^{\frac{1}{2}}}\frac{\partial}{\partial N}+\frac{\partial}{c\,\partial\tau}\left(\frac{1-M_r}{(1-M_r)^2+\alpha^2}\right)\right]^2 T_{rr}\frac{\mathrm{d}\Sigma}{r\{(1-M_r)^2+\alpha^2\}^{\frac{1}{2}}}. \tag{6·5}$$

Here T_{rr} is written for $T_{ij}\hat{r}_i\hat{r}_j$, where $\hat{\mathbf{r}}$ is the unit radiation vector.

Our previous remarks indicate that α is to be chosen to equate the relevant temporal (λ) and spatial scales of the intersection area $\mathrm{d}\Sigma$. The correct choice of temporal scale is crucial in estimating the Mach wave radiation since when $(1-M_r) = 0$, $\mathrm{d}\Sigma$ depends directly upon the time scale $\Delta\tau$. Away from this condition errors in the choice of $\Delta\tau$ are less critical. If Σ is plane, then it can intersect the source region over its whole lifetime, which thus defines the appropriate time scale. A typical lifetime of a turbulent eddy is l/bcM, where b is a small numerical constant, and so the value of α based on this time span is bM. The condition that Σ be plane is equivalent to the convection velocity being constant; if the source accelerates Σ is curved, and can no longer intersect the source region over the total lifetime but only for a much smaller time. Physically, this implies that the time spent by the eddy in the region from which Mach waves can be observed may be the total eddy lifetime, but alternatively may be a shorter time if the source rapidly accelerates through this region. This shorter time scale is determined as follows. The spatial realization of Σ is the contracting sphere $r = c(t-\tau)$, which passes through the source at a relative radial velocity of $c(1-M_r)$. In an interval of time $\Delta\tau$, this sphere moves a distance

$$\int_0^{\Delta\tau} c(1-M_r)\,\mathrm{d}\tau \tag{6·6}$$

through the source, and will have passed completely through the source when this distance equals the radial length scale l. This time scale is only important near the condition $(1-M_r) = 0$, so that $\Delta\tau$ must be determined by the equation

$$\left|c\frac{\partial(1-M_r)}{\partial\tau}\right|\frac{\Delta\tau^2}{2} = l. \tag{6·7}$$

Since $\Delta\tau$ is to be set equal to $l/c\alpha$ for space and time scales to be equal, α is given by

$$\alpha^2 = \frac{l^2}{c^2\Delta\tau^2} = \frac{l}{2c}\left|\frac{\partial M_r}{\partial\tau}\right|. \tag{6·8}$$

If M_r is written as $\mathbf{M}\,.\,\hat{\mathbf{r}}$, it follows from equation (3·7) that

$$\begin{aligned}\frac{\partial M_r}{\partial\tau} &= \frac{\partial\mathbf{M}}{\partial\tau}\,.\,\hat{\mathbf{r}}+\mathbf{M}\,.\,\frac{\partial\hat{\mathbf{r}}}{\partial\tau}\\ &= \frac{\partial\mathbf{M}}{\partial\tau}\,.\,\hat{\mathbf{r}}-\frac{c(M^2-1)}{r},\end{aligned} \tag{6·9}$$

and hence the value of α based on this timespan is

$$\alpha = \left\{\frac{l}{2c}\left|\frac{\partial\mathbf{M}}{\partial\tau}\,.\,\hat{\mathbf{r}}-\frac{c(M^2-1)}{r}\right|\right\}^{\frac{1}{2}}. \tag{6·10}$$

Thus this acceleration mechanism produces two time scales; the first is due to the basic acceleration of the source limiting the emission time, whereas the second arises from the changing geometry of the situation.

Some of the possible field régimes associated with an aerodynamic source can now be discussed. If the source is in steady rectilinear motion, in the far field the only relevant value of α is that based on eddy lifetime. For this model, the second part of the operator in (6·5) vanishes, and the density fluctuations are given by the simple integral

$$4\pi c^2(\rho-\rho_0) \simeq \int \frac{\alpha^2}{\{(1-M_r)^2+\alpha^2\}^{\frac{3}{2}}} \frac{\partial^2 T_{rr}}{\partial N^2} \frac{\mathrm{d}\Sigma}{r}. \tag{6·11}$$

The magnitude of this integral is easily established. The measure of $\mathrm{d}\Sigma$ and the normal derivative $\partial/\partial N$ are typically of magnitude l^3 and l^{-1} respectively. The turbulence stress tensor T_{ij} can be approximated by its first term $\rho u_i u_j$, which is of order $\rho c^2 M^2$, and the over-all magnitude of the density fluctuations is

$$4\pi c^2(\rho-\rho_0) \simeq c^2\rho \frac{l}{r} \frac{b^2 M^4}{\{(1-M_r)^2+b^2M^2\}^{\frac{3}{2}}}. \tag{6·12}$$

The mean square density fluctuation, which is proportional to the sound intensity, thus varies as

$$\overline{(\rho-\rho_0)^2} \sim \overline{\rho}^2 \frac{l^2}{r^2} \frac{b^4 M^8}{\{(1-M_r)^2+b^2M^2\}^3}. \tag{6·13}$$

This result embodies in a single formula the features of the sound field of a uniformly convected eddy that are well understood in connexion with jet noise theory. At low speeds, the intensity varies as the eighth power of the convection speed, coupled with a directional factor $|1-M_r|^{-6}$. At high speeds near the Mach wave condition $(1-M_r) = 0$, this power law is replaced by a second power variation at a much higher over-all level, since b is small. Away from the Mach wave condition, the intensity still varies with the square of the convection speed, but the over-all level is much reduced. These variations differ slightly from their counter-parts in the theory of jet noise because equation (6·13) refers to a single turbulent eddy. The results for a distribution of eddies can be obtained upon multiplication by the number of eddies heard concurrently, and this has the effect of reducing the power 3 in the denominator of (6·13) to a value of 5/2.

Nearer the source in the Mach wave régime, the relevant value of α is not that based upon eddy lifetime, but that defined by the changing geometry $\{l(M^2-1)/2r\}^{\frac{1}{2}}$. Again estimates of the Mach wave field can be based on equation (6·5) but now both factors in the operator are equally important, and vary dimensionally like l^{-1}. This leads to a mean square density fluctuation

$$\overline{(\rho-\rho_0)^2} \sim \overline{\rho}^2 \frac{l}{r} \frac{M^4}{M^2-1}. \tag{6·14}$$

This result differs by a factor M from that given by Ffowcs Williams (1965) because that estimate was based on a downstream rather than a radial length scale, but the two results are entirely equivalent. Thus the Mach wave field nearer the source falls off more slowly than it does in the very distant field. This variation of the (linear) density as $r^{-\frac{1}{2}}$ is also found in the conical wave field of a slender supersonic projectile, where the time scale is again set by the geometry. Again it should be remarked that (6·14) is the result for a single eddy, a distribution of eddies has a different dimensional variation.

It is clear that estimation of the Mach wave emission for an aerodynamic source in arbitrary motion is simple using the techniques outlined here. A choice is to be made between the possible emission time scales, and the relevant one used in the estimation of equation (6·5). If the eddy lifetime provides the appropriate time scale, the Mach wave field is given in equation (6·13). Alternatively, if acceleration effects are more important, the density fluctuations in the Mach wave field vary as

$$\overline{(\rho-\rho_0)^2} \sim \overline{\rho^2}\frac{l}{r^2}\frac{M^4}{\left|\dfrac{\partial \mathbf{M}}{\partial \tau}\cdot\dfrac{\hat{\mathbf{r}}}{c}-\dfrac{M^2-1}{r}\right|}. \tag{6·15}$$

It is significant that one general effect of source acceleration is to *reduce* the Mach wave strength by limiting the time spent by the source in the emission region. Together with this, there are Mach wave terms with a direct dependence on acceleration, as is clear from the time derivative term in equation (6·5). This substantiates the impression gained from Lowson's (1965) analysis that acceleration effects may well be important at high speeds, although his expression is unsuited to the field when $1-M_r$ tends to zero.

7. Sound field of a moving surface

The analysis of the previous section is essentially concerned with the sound field generated by an aerodynamic source such as a turbulent eddy. However, many sound fields are generated by surface motion at high speeds, and it is to such fields that this section is devoted. Most practical examples of high-speed sound-generating surfaces are rigid bodies, and for such bodies some preliminary simplifications of the general results can be made. A rigid body has a clearly defined convection velocity, and the simplest choice for the moving reference frame $\boldsymbol{\eta}$ is a Cartesian frame fixed in the body. This $\boldsymbol{\eta}$ frame may be translated and rotated from the fixed $\mathbf{y}$ frame, but it always remains Cartesian and always moves with a solenoidal convection velocity. Hence J and A are unity. Also,

$$\left(\frac{\partial g}{\partial \eta_i}\right)^2 = \left(\frac{\partial g}{\partial y_i}\right)^2 = \frac{1}{c^2}, \tag{7·1}$$

so that β reduces to the scale factor α. Further, the angle θ defined in §4 to be the angle between the spatial component of the normal to Σ and the normal to the surface, is the same whether measured in the $\boldsymbol{\eta}$ or $\mathbf{y}$ frames. In addition to these simplifications in the coordinate system, one other change is necessary. At present, the source term that represents the volume displaced by the surface is written in the monopole form. This form would be appropriate were the surface expanding, but is clearly inappropriate here. Instead, the surface is now equivalent to higher order volume sources, and this is brought out by a rearrangement of the original term. The convection velocity $\mathbf{v}$ is clearly well defined for points within as well as on the surface, and as this field is solenoidal, the following relation holds:

$$\frac{\partial}{\partial t}\left(v_i\frac{\partial H}{\partial x_i}\right) = -\frac{\partial}{\partial x_i}\{\dot{v}_i(1-H)\}+\frac{\partial^2}{\partial x_i\,\partial x_j}\{v_i v_j(1-H)\}. \tag{7·2}$$

Here $\dot{\mathbf{v}}$ is the Lagrangian time derivative of the velocity and so represents the acceleration of the source. The monopole surface source is thus equivalent to dipoles and quadrupoles distributed throughout the volume *within* the surface.

The two results (5·1) and (5·2) give alternative expressions for the sound field. Equation (5·2) is more general in that it involves no subtle limiting process in the region $(1-M_r) \sim 0$, and can be treated as an expression in which 'time retardation' effects are always negligible.

On the other hand, in general the subspace σ is somewhat abstract, so that it is not easy to analyse the whole of equation (5·2) in the manner developed in § 6. Further objections to such an analysis will appear presently. However, in a particular, and very broad class of problems, equation (5·2) is practically indistinguishable from (5·1). That is when the source distributions are compact. A distribution of dimension l is compact if the time taken by the sound in crossing the distribution $(1-M_r)^{-1}\, l/c$ is very much less than the source time scale $\Delta\tau$. This criterion is equivalent to the condition that the acoustic wavelengths be much greater than l, or alternatively that the value of α based on $\Delta\tau$ is a negligible fraction of $1-M_r$. Hence equation (5·2) is essentially the same as its formal limit when this ratio is zero; this limit is equation (5·1) which thus furnishes an adequate description of the sound field;

$$4\pi c^2(\rho-\rho_0) = \frac{\partial^2}{\partial x_i\,\partial x_j}\int\left[\frac{T_{ij}}{r\,|1-M_r|}\right]\mathrm{d}\eta - \frac{\partial}{\partial x_i}\int\left[\frac{p_{ij}\,n_j}{r\,|1-M_r|}\right]\mathrm{d}S(\eta)$$
$$-\frac{\partial}{\partial x_i}\int_{V_0}\left[\frac{\rho_0\,\dot{v}_i}{r\,|1-M_r|}\right]\mathrm{d}\eta + \frac{\partial^2}{\partial x_i\,\partial x_j}\int_{V_0}\left[\frac{\rho_0\,v_i\,v_j}{r\,|1-M_r|}\right]\mathrm{d}\eta. \quad (7·3)$$

The last two integrals are to be evaluated over the volume inside the surface, V_0. The spatial extent of the source distributions determine whether they can properly be replaced by a point source acting at a mean position; for those sources that may be so reduced, equation (7·3) becomes

$$4\pi c^2(\rho-\rho_0) = \frac{\partial^2}{\partial x_i\,\partial x_j}\left[\frac{Q_{ij}+\rho_0\,v_i\,v_j\,V_0}{r\,|1-M_r|}\right] - \frac{\partial}{\partial x_i}\left[\frac{P_i+\rho_0\,\dot{v}_i\,V_0}{r\,|1-M_r|}\right]. \quad (7·4)$$

Q_{ij} represents the integrated strength of the external quadrupoles, and P_i is the total force exerted by the surface on the fluid. V_0 is the volume enclosed by the surface. These results have been obtained and discussed by Lowson (1965), except that he takes no account of the volume displacement effect. However, as (7·4) shows, and as is already well known, this effect is easily accounted for by supplementing the main dipole and quadrupole by a further $\rho_0\,\dot{v}_i\,V_0$ and $\rho_0\,v_i\,v_j\,V_0$ respectively. If the quadrupole and dipole distributions have similar spatial and temporal scales, the quadrupole distribution can be neglected in regions where equation (7·4) is a good approximation to the field, since its effect is smaller than that of the dipole distribution by the factor $(1-M_r)^{-1}\, l/c\Delta\tau$. For such distributions the density is given by

$$4\pi c^2(\rho-\rho_0) = -\frac{\partial}{\partial x_i}\left[\frac{P_i+\rho_0\,\dot{v}_i\,V_0}{r\,|1-M_r|}\right]. \quad (7·5)$$

Of course, situations can be envisaged in which the scales appropriate to the two distributions differ; the quadrupoles may then be important sound sources and the whole of equation (7·4) must be retained.

The above results apply to compact source distributions. They apply to the sound fields generated at low rotational speeds by aircraft propellers, aero-engine fans and helicopter rotors. This is because the relevant time scales in these situations vary inversely with the convection speed, and the sources are compact at low speeds. The usual dimensional analysis of (7·5) shows that the sound intensity then varies as the sixth power of the convection velocity, and this agrees with practical observation. The above results may also apply to surfaces travelling at high supersonic speeds in a region where $M_r \gg 1$. However, as speeds increase typical time scales decrease, and it is not known whether practical source distributions remain compact at these high speeds. If they do, a dimensional analysis of (7·5) shows that the sound intensity now only increases as the square of the convection speed.

The above results fail if the source distributions are non-compact, i.e. the acoustic wavelengths are comparable with the source dimension. This type of distribution occurs in the investigation of the field of a supersonically moving surface in a region where $1-M_r$ is small. It may also occur for more moderate convection speeds if a mechanism is present which produces very short time scales. For non-compact sources, equation (7·3) is no longer a proper description of the sound field. It is still a valid formal expression, but retarded time differences can no longer be neglected. Instead, a better description of the field in these regions would be obtained by taking the limit of equation (5·2) as $1-M_r$ tended to zero. This equation is obviously more involved than (7·3), and a general analysis of it is very difficult. This is because the general equation (5·2) must first be cast into a form suitable for estimation requirements, and then the limit $1-M_r$ tending to zero taken, if all effects of acceleration are to be retained. Consequently, for simplicity it is to be assumed here that there are no significant effects of acceleration, so that $1-M_r$ can be set to zero at the outset. Also for the sake of simplicity, the time variable τ will be used here, rather than its scaled equivalent λ.

In the limit $(1-M_r) = 0$, the hypersurface Σ lies parallel to the time axis. The spatial realization of Σ is the contracting sphere $r = c(t-\tau)$, which is denoted by Ω. Thus $\mathrm{d}\Sigma$ becomes $c\,\mathrm{d}\Omega\,\mathrm{d}\tau$. This sphere cuts the real surface S in a curve Γ, and it follows from the definition of σ that $\mathrm{d}\sigma$ becomes $c\,\mathrm{d}\Gamma\,\mathrm{d}\tau$. Hence the limit of (5·2) as $1-M_r$ tends to zero is

$$4\pi c^2(\rho-\rho_0) = \frac{\partial^2}{\partial x_i\,\partial x_j}\int\{\overline{T_{ij}}+\rho_0 v_i v_j(1-H(f))\}\frac{c\,\mathrm{d}\Omega\,\mathrm{d}\tau}{r} - \frac{\partial}{\partial x_i}\int\frac{p_{ij}n_j}{\sin\theta}\frac{c\,\mathrm{d}\Gamma\,\mathrm{d}\tau}{r}. \tag{7·6}$$

The first integral in (7·6) is to be taken over the surface of the Ω sphere, whereas the second is to be integrated around the contour Γ in which Ω cuts S. Here θ is the angle between the outward normal to S and the radiation direction. This latter direction is the inward normal to Ω so that the angle θ used here is the supplement of that defined in § 4.

For estimation purposes, the field-point derivatives must again be taken under the integral sign. The only dependence of the quadrupole integral upon $\mathbf{x}$ in the far field is through the position of the sphere Ω which has $\mathbf{x}$ as its centre. The differentiation of such an integral is accomplished by replacing the surface integral by a volume integral containing a delta function, and utilizing the lemma (6·2). In the notation of this section, this leads to the result

$$\frac{\partial}{\partial x_i}\int F\,\mathrm{d}\Omega = \int\frac{\partial}{\partial r}(F\hat{r}_i)\,\mathrm{d}\Omega, \tag{7·7}$$

where, to avoid notational confusion, $\partial/\partial r$ is the derivative in the *radiation* direction, and not in the direction of increasing r. For the quadrupole integral, this radiation derivative acts upon the discontinuous functions $\overline{T_{ij}}$ and $\rho_0 v_i v_j(1-H)$, and so produces delta function singularities:

$$\left.\begin{aligned}\frac{\partial}{\partial r}(\overline{T_{ij}}) &= H(f)\frac{\partial T_{ij}}{\partial r}+T_{ij}\,\delta(f)\frac{\partial f}{\partial r},\\ \frac{\partial}{\partial r}\{\rho_0 v_i v_j(1-H(f))\} &= (1-H(f))\frac{\partial\rho_0 v_i v_j}{\partial r}-\rho_0 v_i v_j\,\delta(f)\frac{\partial f}{\partial r}.\end{aligned}\right\} \tag{7·8}$$

From the geometry of the situation, it follows that

$$\left.\begin{aligned} \partial f/\partial r &= |\mathrm{grad} f| \cos\theta \\ \text{and} \qquad \mathrm{d}\Omega &= \frac{\mathrm{d}f\,\mathrm{d}\Gamma}{|\mathrm{grad} f| \sin\theta}, \end{aligned}\right\} \tag{7·9}$$

and it is clear that after a single derivative has been taken under the integral sign, the density field can be expressed as

$$4\pi c^2(\rho-\rho_0) = \frac{\partial}{\partial x_i}\int\left\{H\frac{\partial T_{ir}}{\partial r}+(1-H)\frac{\partial(\rho_0 v_i v_r)}{\partial r}\right\}\frac{c\,\mathrm{d}\Omega\,\mathrm{d}\tau}{r} + \frac{\partial}{\partial x_i}\int\{(T_{ir}-\rho_0 v_i v_r)\cos\theta - p_{in}\}\frac{c\,\mathrm{d}\Gamma\,\mathrm{d}\tau}{r\sin\theta}. \tag{7·10}$$

Here p_{in} has been written for $p_{ij}n_j$. This process can be repeated to yield a further contour integral, and equation (7·10) becomes

$$4\pi c^2(\rho-\rho_0) = \int\left\{H\frac{\partial^2 T_{rr}}{\partial r^2}+(1-H)\frac{\partial^2(\rho_0 v_r v_r)}{\partial r^2}\right\}\frac{c\,\mathrm{d}\Omega\,\mathrm{d}\tau}{r} + \frac{\partial}{\partial x_i}\int\{(T_{ir}-\rho_0 v_i v_r)\cos\theta - p_{in}\}\frac{c\,\mathrm{d}\Gamma\,\mathrm{d}\tau}{r\sin\theta} + \int\cos\theta\frac{\partial}{\partial r}(T_{rr}-\rho_0 v_r v_r)\frac{c\,\mathrm{d}\Gamma\,\mathrm{d}\tau}{r\sin\theta}. \tag{7·11}$$

Equation (7·11) requires further reduction, as it still contains field-point derivatives acting upon the contour integral. Again the integral only depends upon $\mathbf{x}$ through the position of Γ; if the field point changes the sphere now cuts S in a different contour. As S is an arbitrary surface, it must be assumed that Γ also changes its shape and curvature, and the form of the lemma that enables these derivatives to be evaluated is slightly different from (7·7). If $\mathbf{m}$ is a unit vector on the surface S normal to Γ such that $\mathbf{m}.\mathbf{r} \geqslant 0$, $\mathbf{n}$, $\mathbf{m}$ and $\bar{\mathbf{\Gamma}}$ are mutually orthogonal, and,

$$\frac{\partial}{\partial x_i}\int F\,\mathrm{d}\Gamma = \int \mathrm{div}\,(\mathbf{m}F\hat{r}_i)\frac{\mathrm{d}\Gamma}{\sin\theta}. \tag{7·12}$$

The proof of this lemma is similar to that of (7·7), again accomplished by replacing the contour integral by a volume integral containing two delta functions. By utilizing the condition $v_r = c$ throughout the region of interest, (7·11) immediately becomes,

$$4\pi c^2(\rho-\rho_0) = \int H\frac{\partial^2 T_{rr}}{\partial r^2}\frac{c\,\mathrm{d}\Omega\,\mathrm{d}\tau}{r} + \int\left(\mathrm{div}\left\{\frac{\mathbf{m}}{\sin\theta}[(T_{rr}-c^2\rho_0)\cos\theta - p_{rn}]\right\} + \cos\theta\frac{\partial T_{rr}}{\partial r}\right)\frac{c\,\mathrm{d}\Gamma\,\mathrm{d}\tau}{r\sin\theta}. \tag{7·13}$$

The complicated form of the contour integral in (7·13) can be simplified if it is assumed that viscosity acts in all situations of interest. Then there is no slip between the fluid and the surface, and the radiation component of the fluid velocity u_r is equal to the equivalent component of the surface velocity v_r. From the definition of T_{ij} it follows that where $v_r = c$,

$$(T_{rr}-c^2\rho_0)\cos\theta - p_{rn} = p_{rr}\cos\theta - p_{rn}. \tag{7·14}$$

The right-hand side of (7·14) can be further simplified if the compressive stress tensor p_{ij} is written as the sum of a pressure term $p\delta_{ij}$ and a viscous stress tensor E_{ij}. We define a new unit vector $\mathbf{q}$ to be perpendicular to both the contour Γ and the radiation direction $\hat{\mathbf{r}}$ such that $\mathbf{q}.\mathbf{m} \geqslant 0$. Thus $\mathbf{r}$, $\mathbf{q}$ and $\mathbf{\Gamma}$ are mutually orthogonal. Then it can be shown that

$$(\sin\theta)^{-1}(p_{rr}\cos\theta - p_{rn}) = E_{ij}\hat{r}_i q_j = E_{rq}, \tag{7·15}$$

and (7·13) becomes

$$4\pi c^2(\rho-\rho_0) = \int H\frac{\partial^2 T_{rr}}{\partial r^2}\frac{c\,\mathrm{d}\Omega\,\mathrm{d}\tau}{r} + \int\left\{\operatorname{div} E_{rq}\,\mathbf{m} + \cos\theta\frac{\partial T_{rr}}{\partial r}\right\}\frac{c\,\mathrm{d}\Gamma\,\mathrm{d}\tau}{r\sin\theta}. \tag{7·16}$$

This result, as might be expected, is not simple. It shows that for non-compact sources the density field is related to various surface and contour integrals in the real space, and once again demonstrates the importance of the relevant time scale in determining the magnitude of the radiated field. An important conclusion can be drawn from (7·16). The earlier results derived for compact sources emphasize the importance of such parameters as the surface pressure and displaced volume in determining the magnitude of the radiated field. Also the presence of the Doppler factor $(1-M_r)^{-1}$ in these results suggests that as $1-M_r$ approaches zero these parameters become even more important. Equation (7·16) shows that this is not true. In fact these parameters that are so important at low speeds play *no* part in determining the magnitude of the Mach wave field of a high-speed surface. Instead, the details of the fluid flow around the surface, through their effect on T_{ij}, are critical in defining the Mach wave strength. This conclusion is clearly of great importance in the design of quiet high speed machinery. In contrast to these qualitative deductions, it becomes apparent from (7·16) that it is very difficult to infer any quantitative results using analytical methods though any particular model could be studied numerically. In fact, it is not possible in general to deduce a simple dimensional variation of the field strength with the convection Mach number M. The reason for this is that the condition $M_r = 1$ implies that the observer is at an angle $\cos^{-1}(1/M)$ to the direction of motion of the source. Thus the variation of the field strength with M is essentially the directional variation of the integrals in (7·16). Unless the latter can be exposed, a proper description of the variation of the sound field with convection speed is not possible.

It has been assumed above that the sphere Ω and the surface S intersect at a non-zero angle θ, and it remains to discuss the behaviour of (7·16) when this angle vanishes. In fact the contour integral in (7·16) still remains finite, because as $\sin\theta$ vanishes so does Γ. The reason for this is that provided Ω and S are not identically curved, they only have a common normal (which is the condition for $\sin\theta$ to vanish) if they touch at a single point P. Thus when $\sin\theta$ vanishes, the contour of intersection shrinks to P. The smooth surface S can always be approximated by an ellipsoid near P, and the curve Γ is then a small ellipse. For this contour the integral may be evaluated analytically, and the main contribution to the sound field from this integral is

$$\int\frac{\partial T_{rr}}{\partial r}\frac{2\pi c\,\mathrm{d}\tau}{r\{(\kappa_1+1/r)(\kappa_2+1/r)\}^{\frac{1}{2}}}. \tag{7·17}$$

κ_1 and κ_2 are the principal curvatures of S at P, and are measured positive if the centre of curvature is on the side of Γ opposite to x. The nature of this result is important for it shows that the curvature of the surface is critical in determining the character of the field along the normal at P. If the surface has double curvature, in the far field the factor r^{-1} is small compared with the two curvatures, and consequently the field strength falls off as r^{-1} with an over-all magnitude proportional to $(\kappa_1 \kappa_2)^{-\frac{1}{2}}$. Thus for a surface that is only slightly curved at P, this contribution to the field can be large, and may be the most important source of far field sound. This term is even more important if the surface has only a single curvature, for (7·17) shows that the field strength now only falls off as $r^{-\frac{1}{2}}$, and is ultimately more important than any source whose field varies as r^{-1}. Finally, if the surface at P is locally plane, the field along the normal at P does not vary at all with distance, and an intense beam is produced. Equation (7·17) is not valid if either term in the denominator is zero, indicating that S and Ω have identical curvatures so that $\mathbf{x}$ lies on the focal point of S, but it is likely that there is then a genuine singularity in the field. This type of dependence of the field upon curvature is also found in high-frequency diffraction theory. This is to be expected, since one effect of surface motion is that all radiated wavelengths become smaller by the factor $(1-M_r)$, and in our limit of $(1-M_r)$ tending to zero, the surface simply acts as an ultra high frequency radiator. None the less, it is interesting to see these results appearing without the use of harmonic analysis.

8. Conclusion

The paper gives in § 5 formal expressions for the density field radiated by turbulence in the presence of arbitrarily moving surfaces. The most general results are expressed as hypersurface integrals, which, though rather abstract, are generally computable if the source field and surface-boundary conditions are known in space and time. The latter sections are devoted to particular features of these general results that might have practical significance. The now well-known effects of uniformly convected turbulence emerge quite naturally without any assumption regarding detailed flow statistics. Convective acceleration induces additional sources, at low speed this being the principal effect of acceleration. At the Mach wave condition, however, convective acceleration also tends to limit the Mach wave strength by a restriction on the time for which the source can remain in the Mach wave phase.

The deductions regarding the sound of high-speed surfaces are quite new and could not easily be deduced by analogy with previously published accounts. The most significant effects are twofold. First, it is shown that features governing the surface generated sound at low speed, i.e. the applied force and displaced inertia, play no part at all in the high speed problem. Secondly, at very high speeds, surface curvature is crucial in determining the strength of an intense beam that radiates in the direction for which the surface normal coincides with the Mach wave direction. A further point that is implied by a dimensional analysis of equations (7·5) and (7·16) is that the intensity of the field at high supersonic speed increases only as the square of surface velocity. This implies that the acoustic output of a high-speed (supersonic) machine increases less rapidly than the mechanical power and that the acoustic efficiency falls off inversely as speed increases. However, this cannot be said with extreme conviction, there being some doubt that the component of the stress tensor T_{rr}, in (7·16), can reasonably be set proportional to $\rho_0 U^2$. These points emerge from the analysis of a surface in rectilinear motion, chosen here as the simplest example that illustrates the general effects of the results presented in § 5. We do not expect that these qualitative deductions will be seriously modified by a more complete evaluation of those results for the effects are clearly recognizable, but in a more abstract way, in that section. We would expect therefore that these points have some bearing on the noise radiated by the supersonically moving blades of an aero-engine fan, and we expect that blade curvature will then significantly influence what may well be the dominant part of the radiation field, the Mach wave radiated along the surface normal coinciding with the Mach angle.

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