Turbomachinery Noise

– Circular duct

Consider a circular duct of radius, a. We take a cylindrical coordinate system $\{x, r, \theta\}$ where the x axis is along the duct axis. With no flow velocity, the acoustic pressure is governed by the wave equation:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 0, \qquad \nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial x^2}$$

The pressure must satisfy an initial condition at $x = x_0$ and a wall boundary condition at $r = a$. We use the method of separati of variable and assume

$$p(x,r,\theta,t) = X(x)\, R(r)\, \Theta(\theta)\, T(t)$$

$$X''R\Theta T + R''X\Theta T + \frac{R'}{r}X\Theta T + \frac{1}{r^2}XRT\Theta'' - \frac{1}{c^2}T''XR\Theta = 0$$

$$\frac{X''}{X} + \frac{R'' + R'/r}{R} + \frac{\Theta''}{r^2\Theta} - \frac{1}{c^2}\frac{T''}{T} = 0 \quad \text{— Eq. (1)}$$

If we take

$$\Theta''/\Theta = -m^2$$

$$X''/X = -k^2$$

$$T''/T = -\omega^2$$

where m is an integer. This implies a solution of the form

$$P_{mk\omega} = R_m(r)\, e^{i(kx + m\theta - \omega t)}$$

Eq.(1) can be written as

$$-K^2 + \frac{R'' + R'/r}{R} - \frac{m^2}{r^2} + \frac{\omega^2}{c^2} = 0$$

Then $$r^2 R'' + rR' + R\left\{ r^2 \underbrace{\left(\frac{\omega^2}{c^2} - k^2\right)}_{\mu^2 \text{ (eigenvalue)}} - m^2 \right\} = 0$$

$$r^2 R_m'' + rR_m' + (\mu^2 r^2 - m^2) R_m = 0 \qquad -(2)$$

For a rigid duct, this equation must satisfy an impermeability conditi

$$\left(\frac{dR_m}{dr}\right)_{r=a} = 0 \qquad -(3)$$

$$\rho_0 \frac{\partial u}{\partial t} = -\nabla p$$
$$\rho_0 \frac{\partial(\vec{u}\cdot\vec{n})}{\partial t} = -\nabla p \cdot \vec{n}$$
$$\left.\frac{\partial p}{\partial r}\right|_{r=a} = 0$$

Introducing the non-dimensional variable $\tilde{r} = \mu r$

$$d\tilde{r} = \mu dr, \quad r = \tilde{r}/\mu, \quad \frac{dR}{dr} = \frac{dR}{d\tilde{r}}\frac{d\tilde{r}}{dr} = \frac{dR}{d\tilde{r}}\mu$$

Eq. (2) becomes

$$\tilde{r}^2 \frac{d^2 R_m}{d\tilde{r}^2} + \tilde{r}\frac{dR_m}{d\tilde{r}} + (\tilde{r}^2 - m^2) R_m = 0$$

$R_m = J_m(\tilde{r})$, bessel function. We dropped $Y_m(\tilde{r})$ for singurarity at $r=0$.

The wall condition Eq.(3) implies $J_m'(\mu a) = 0$

The boundary-value problem is a Strum-Liouville problem whose solutions form a complete set. The derivatives of the Bessel function has an infinite number of zeros which we denote as $\{\alpha_{mn}\}$

$$J_m'(\alpha_{mn}) = 0 \qquad , m = 0, 1, \cdots$$

Hence, the eigenvalues are

$$\mu_{mn} = \frac{\alpha_{mn}}{a}$$

This ~~implie~~ defines the axial wave number as

$$K_{mn} = \sqrt{\left(\frac{\omega}{c}\right)^2 - \mu_{mn}^2} \qquad \text{from } \frac{\omega^2}{c^2} - K^2 = \mu^2 .$$

The eigen function

$$P_{mn} = J_m\left(\frac{\alpha_{mn} r}{a}\right) e^{i(K_{mn}x + m\theta - \omega t)}$$

is called the $\{mn\}$ mode. For every frequency ω, the solution is then

$$P_\omega = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} C_{mn} P_{mn}$$

The expression for the coefficients C_{mn} is determined using the initial condition

$$P_\omega(0, r, \theta, t) = f_\omega(r, \theta) e^{-i\omega t}$$

↳ known function

At $x=0$,

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} C_{mn} J_m\left(\frac{\alpha_{mn} r}{a}\right) e^{i(m\theta - \omega t)} = f_\omega(r, \theta) e^{-i\omega t}$$

We use the orthogonality property of the Bessel function

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \sum_{m'=-\infty}^{\infty} \sum_{n'=0}^{\infty} \int_0^a r J_m\left(\frac{\alpha_{mn} r}{a}\right) J_{m'}\left(\frac{\alpha_{m'n'} r}{a}\right) dr$$

$$= \begin{cases} 0 & m \neq m', \ n \neq n' \\ \dfrac{a^2(\alpha_{mn}^2 - m^2)}{2\alpha_{mn}^2} J_m^2(\alpha_{mn}) & m = m', \ n = n' \end{cases}$$

$$C_m \frac{a^2(\alpha_{mn}^2 - m^2)}{2\alpha_{mn}^2} J_m^2(\alpha_{mn}) \int_0^{2\pi} d\theta = \int_0^{2\pi}\int_0^a f_\omega(r,\theta) J_m\left(\frac{\alpha_{mn} r}{a}\right) e^{-im\theta} r\,dr\,d\theta$$

$$C_{mn} = \frac{1}{\pi a^2} \frac{\alpha_{mn}^2}{(\alpha_{mn}^2 - m^2) J_m^2(\alpha_{mn})} \int_0^{2\pi}\int_0^a f_\omega(r,\theta) J_m\left(\frac{\alpha_{mn} r}{a}\right) e^{-im\theta} r\,dr\,d\theta$$

Note the condition for propagation of an acoustic mode is that the wavenumber, k_{mn}, must be real. Otherwise the wave will decay exponentially and is known as an evanescent wave.

$$e^{ikx} = e^{-k_i x} \to 0 \quad \text{as} \quad x \to \infty .$$

Therefore, an $\{mn\}$ mode propagates if

$$\boxed{\frac{\omega a}{c} > \alpha_{mn}}$$

At low frequencies, only the fundamental mode (plane wave)

$$P_{00} = e^{i[(\omega/c)x - \omega t]}$$

propagates. $m=0$, $n=0$, $\alpha_{mn}=0$ $\quad J_{00}(0)=1. \quad J_{00}'(0)=0.$

As ω increases, an additional mode propagates. The frequency at which a mode $\{mn\}$ begins to propagate is known as the "cutoff" frequency of the mode. As the frequency increases (decreases) and is equal to the cutoff frequency of a mode $\{mn\}$, the mode $\{mn\}$ is said to cut-on (cut-off). → more details later.

As an example, consider a duct of radius $a = 0.5\,m$, $c = 340\,m/s$, and the sound frequency is 3000 rpm $(\omega = 3600 \times \frac{2\pi}{60})$, ~~of~~

Then, $a\omega/c = \cancel{3}\,0.462$.

From the tables of zeros of Bessel functions, the lowest zero is $\alpha_{11} = 1.8412$, $J_1(\alpha_{11}) = \cancel{1.8412}\,3.8317$

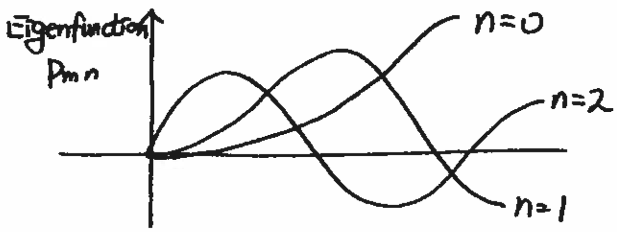
$$\frac{a\omega}{c} < \alpha_{11}.$$

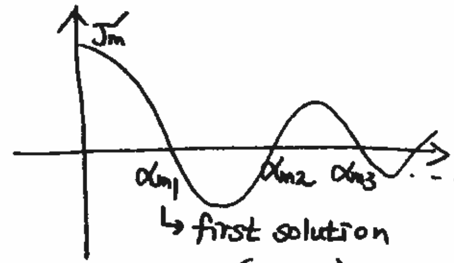
Hence, only the fundamental mode (plane wave) will propagate.

$\{11\}$ mode : cut-off

In practice and Tyler & Sofrin paper,

We use $n=0$ for the first solution of the Bessel function (not funandmental mode). $n=0$ implies that there is no zero crossing in eigenfunction or $J_m\left(\frac{\alpha_{mn} r}{a}\right)$





$$P_{mn} = \begin{cases} J_m(\mu_{mn} r) & : \text{circular duct} \\ J_m(\mu_{mn} r) + \dfrac{J_m'(\mu_{mn}\underset{\sim}{\lambda})}{Y_m'(\mu_{mn}\lambda)} Y_m(\mu_{mn} r) & : \text{annular duct} \end{cases}$$

↳ more detail later

↳ hub/a
hub-to-tip ratio

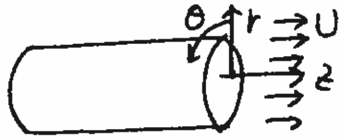
$$P_{mn}' = 0 : \begin{cases} J_m'(\mu_{mn} a) = 0 \\ J_m'(\mu_{mn} a) Y_m(\mu_{mn} a) - J_m'(\mu_{mn} a) Y_m'(\mu_{mn} a) = 0 \end{cases}$$

Fundamental mode

$m=0$, $n=0$, $\mu_{mn}=0$, $J_{00}(0)=1$. $J_{00}'(0)=0$

so that $P_{mn}=1$, $P = e^{i(kx-\omega t)}$

– Assume uniform flow in the duct: $U = \text{const}$



pressure perturbation satisfies the convected wave equation

$$\frac{1}{c_0^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial z}\right)^2 p = \nabla^2 p$$

where, in polar cylindrical coordinates

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial}{\partial r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

For a rigid-walled duct, the wall boundary condition is

$$\frac{\partial P}{\partial r} = 0 \qquad @ \; r = a$$

Also, p finite for all r, in particular $P|_{r=0}$ is finite.

Seek separable solutions of the form

$$p'(r,\theta,z,t) = P(r)\exp\left[i\omega t - i k_z z \pm i m \theta\right]$$

Then,

$$-\frac{1}{c_0^2}(i\omega - U i k_z)^2 P = \frac{d^2 P}{dr^2} + \frac{1}{r}\frac{dP}{dr} - \frac{m^2}{r^2}P - k_z^2 P$$

Let $\quad k = \frac{\omega}{c_0} \quad$ and $\quad M = \frac{U}{c_0}$

Then

$$\frac{d^2 P}{dr^2} + \frac{1}{r}\frac{dP}{dr} + \left(\kappa^2 - \frac{m^2}{r^2}\right)P = 0$$

where $\quad \kappa^2 = k^2 - 2kk_z M - k_z^2(1-M^2) \quad \leftarrow$ Mach # effect

— (4)

For $M=0$, $\kappa^2 = k^2 - k_z^2$

Let $s = Kr$ then

$$s^2 \frac{d^2p}{ds^2} + s\frac{dp}{ds} + (s^2 - m^2)p = 0$$

The general solution to Bessel's equation may be written

$$P(r) = A_m J_m(Kr) + B_m Y_m(Kr)$$

Now $Y_m(z)$ has a logarithmic singularity at $z=0$, so that $B_m = 0$.

To satisfy the condition at $r=a$, $\frac{dP}{dr} = 0$ at $r=a$. $\frac{\partial P}{\partial r} = k\frac{\partial P}{\partial s} = 0$

$$J_m'(Ka) = 0$$

Thus $K_{mn}a$ is the $n+1$-th zero of $J_m'(z)$

Remember $n=0$ is the first solution of $J_m'(z)$.

and it implies zero crossing of $J_m(z)$

~~and~~ Thus,

$$P_{mn}(r) = A_{mn} J_m(K_{mn} r)$$

Also, From Eq. (4)

$$\frac{K_z}{K} = \frac{-M \pm \sqrt{1-(1-M^2)(K_{mn}/K^2)}}{(1-M^2)}$$

and $$p'(r,\theta,z,t) = \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} A_{mn} J_m(K_{mn}r)\exp[i\omega t - K_z z + im\theta]$$

The equivalent result for a rectangular duct is

$$P'(x.y.z.t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right) \times$$

$$\left[A_{mn} \exp(-ik_z z) + B_{mn} \exp(ik_z z) \right] \exp(i\omega t)$$

where

$$\frac{K_z}{K} = \frac{-M \pm \sqrt{1-(1-M^2)(K_{mn}/K^2)}}{(1-M^2)}$$

and

$$K_{mn} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

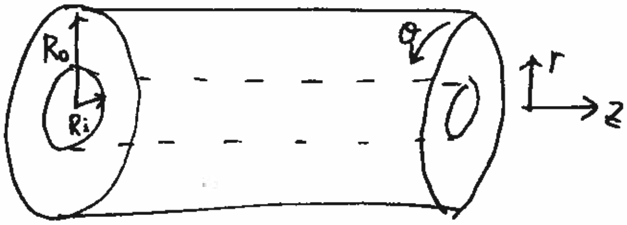
Consider the case with $M=0$ (no flow), then

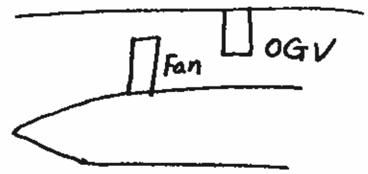
$$\frac{K_z}{K} = \pm\sqrt{1-(K_{mn}/K)^2}$$

If $K_{mn} < K$, K_z is ~~purely~~ purly real → propagat~~ing~~on can occur in the upstream or downstream

If $K_{mn} > K$, K_z is ~~purely~~ purly imaginary → physically acceptable solution indicates exponential decay. Modes are said to be "cut-off".

- Annular hard-walled duct modes in uniform mean flow : practical





Aircraft engine

The derivation is similar with circular duct

$$p(z,t,\theta,r) = \sum_m \sum_n \underset{\text{mode amplitude}}{A_{mn}} \underset{\text{eigenfunction}}{\psi_{mn}(r)} \exp(-i\omega t + i k_z z + i m\theta)$$

$$\frac{k_z}{k} = \frac{-M \pm \sqrt{1-(1-M^2)(\mu_{mn}/k)^2}}{(1-M^2)} \quad \text{where } k=\frac{\omega}{c},\ \mu_{mn}: \text{eigenvalue}$$

$$\psi_{mn}(r) = a J_m(\mu_{mn} r) + b Y_m(\mu_{mn} r)$$

$b \neq 0$ since $r \not\to 0$. (for circular duct $b=0$ because of a finite value of $\psi_{mn}(r)$ at $r=0$)

B.C. $\quad \dfrac{d\psi_{mn}}{dr} = 0$, at $R=R_i$ and $R=R_o$

$$a J_m'(\mu_{mn} R_o) + b Y_m'(\mu_{mn} R_o) = 0$$

$$a J_m'(\mu_{mn} R_i) + b Y_m'(\mu_{mn} R_i) = 0$$

$$\begin{bmatrix} J_m'(\mu_{mn} R_o) + Y_m'(\mu_{mn} R_o) \\ J_m'(\mu_{mn} R_i) + Y_m'(\mu_{mn} R_i) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For non-trivial solution, the determinant of the left matrix should be zero. Otherwise $[a, b] = [0, 0]$

Then, $J_m'(\mu_{mn} R_o) Y_m'(\mu_{mn} R_i) - J_m'(\mu_{mn} R_i) Y_m'(\mu_{mn} R_o) = 0$

You can find out μ_{mn}, eigenvalue.

Note
$J_m' = \frac{1}{2}[J_{m-1} - J_{m+1}]$
$Y_m' = \frac{1}{2}[Y_{m-1} - Y_{m+1}]$

Also,
$$b = -\frac{J_m'(\mu_{mn} R_i)}{Y_m'(\mu_{mn} R_i)} a$$

So that
$$\boxed{\psi_{mn} = J_m(\mu_{mn} r) - \frac{J_m'(\mu_{mn} R_i)}{Y_m'(\mu_{mn} R_i)} Y_m(\mu_{mn} r)}$$

Note that the constant a is not included as ~~~~ it can be included in A_{mn}.

Let $P(z,r,\theta) = \sum_m \sum_n A_{mn} \psi_{mn} e^{i(k_z z + m\theta)}$

At $z=0$ or inlet
$$P(0,r,\theta) = \sum_m \sum_n A_{mn} \psi_{mn} e^{im\theta}$$

Take the average in θ direction after multiplying $e^{-im\theta}$
$$\frac{1}{2\pi}\int_0^{2\pi} p e^{-im\theta} d\theta = \frac{1}{2\pi}\int_0^{2\pi} \sum_m \sum_n A_{mn} \psi_{mn} d\theta$$
↳ Fourier transform in θ
$$\tilde{p}(0,r) = \sum_m \sum_n A_{mn} \psi_{mn}$$
Integrate over r (inner product) after multiplying ψ_{mn}
$$\int_{R_i}^{R_o} \tilde{p}(0,r) \psi_{mn} r dr = \int_{R_i}^{R_o} \sum_m \sum_n A_{mn} |\psi_{mn}|^2 r dr$$

$$I_{mn} = \int_{R_i}^{R_o} |\psi_{mn}(\mu_{mn} r)|^2 r dr = \begin{cases} \frac{1}{2}(R_o^2 - R_i^2) & \text{if } m=n=0 \\ \frac{1}{2}\left[R_o^2 - \frac{m^2}{\mu_{mn}^2}\right]\psi_m^2(\mu_{mn} R_o) - \frac{1}{2}\left[R_i^2 - \frac{m^2}{\mu_{mn}^2}\right]\psi_m^2(& \text{otherwise} \end{cases}$$

Then, $A_{mn} = \frac{1}{\Gamma_{mn}} \int_{R_i}^{R_o} \tilde{P}(0,r)\, \psi_{mn}\, r\, dr$

- Often, we are more interested in sound power level (PWL) rather than pressure itself.

Intensity is given as

$$\underbrace{\langle I_z \rangle}_{\text{area-averaged}} = \frac{\overbrace{2\pi}^{\text{integration around theta}}}{\underbrace{2}_{\text{time-averaging}}\ \underbrace{A_z}_{\text{duct area plane}}} \mathrm{Re}\left[\int_{R_i}^{R_o} \left\{ (1+M^2) P V_z^* + M \left(\frac{|P|^2}{\rho_0 c_0} + \rho_0 c_0 |V_z|^2 \right) \right\} r\, dr \right]$$

For $M=0$, $[\ ] \to P V_z^*$

$\langle I \rangle = \frac{1}{2} \mathrm{Re}\{ P V_z^* \}$

Let's consider linearized momentum equation

$$\rho_0 \frac{D V_z}{D t} = - \frac{\partial P}{\partial z} \qquad V_z = \tilde{V}_z e^{-i\omega t + i k_z z}, \quad P_z = \tilde{P}_z e^{-i\omega t + i k_z z}$$

$$\rho_0 c_0 \left[-\frac{i\omega}{c_0} + M\, i k_z \right] V_z = -i k_z P \quad , \quad \text{Let } k_z = k_{z,mn}$$

$$V_z = \frac{1}{\rho_0 c_0} \frac{k_{z,mn}}{(\omega/c_0 - k_{z,mn} M)} P$$

Therefore, Modal Power becomes

$$\boxed{W_{mn} = \langle I_z \rangle A_z = \frac{\pi}{\rho_0 c_0} \Gamma_{mn} A_{mn} A_{mn}^* \left[(1+M^2) \mathrm{Re}\left(\frac{k_{z,mn}}{\omega/c_0 - k_{z,mn} M} \right) + M \left[1 + \left| \frac{k_{z,mn}}{\omega/c_0 - k_{z,mn} M} \right|^2 \right] \right]}$$

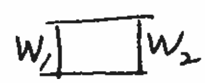
$$\Gamma_{mn} = \int_{R_i}^{R_o} \psi_{mn} \psi_{mn}^* r\, dr$$

Total Power

$$W = \sum_m \sum_n W_{mn} \quad \text{(only for cut-on modes)}$$

- Usually m is related to BPF. So for each m, we have multiple ns (radial modes).

The power is only defined at one z plane of interest.



We showed that the cut-off frequency for $M=0$

(in circular duct)

$$\frac{\omega}{c_0} = \frac{\alpha_{mn}}{a} = \mu_{mn} \qquad \text{where } J_m'(\mu_{mn} a) = 0$$

(for annular duct)

$$J_m'(\mu_{mn} R_o) Y_m'(\mu_{mn} R_i) - J_m'(\mu_{mn} R_i) Y_m'(\mu_{mn} R_o) = 0$$

Now consider the solutions for $M \neq 0$

$$\frac{K_z}{K} = \frac{-M \pm \sqrt{1-(1-M^2)(\mu_{mn}/K)^2}}{(1-M^2)}$$

for both circular and annular duct

For propagation eithe upstream or downstream

$$K^2 > (1-M^2)\mu_{mn}^2 \rightarrow K_z \text{ real "cut-on"}$$

Note that

$$\frac{K_z^+}{K} = \frac{-M + \sqrt{1-(1-M^2)(\mu_{mn}/K)^2}}{(1-M^2)}$$

$\rightarrow$ $K_z^+ < 0$ possible for certain frequencies and M

$\rightarrow$ All waves $+z$ decaying

Only propagating $-z$ directi

and

$$\frac{K_z^-}{K} = \frac{-M - \sqrt{1-(1-M^2)(\mu_{mn}/K)^2}}{(1-M^2)}$$

We should select $Im(K_z) > 0$ to ensure decaying wave

$\rightarrow$ Can't be solution unless pure real values. Otherwise it is exponentially growing soluti

So that

$$|K_z^+| < |K_z^-|$$

The condition for downstream propagation is

$$K^2 > (1-M^2)\mu_{mn}^2 \quad \text{and} \quad -KM + \sqrt{K^2-(1-M^2)\mu_{mn}^2} > 0 \text{ or } K_z^+ >$$

$$K^2 - (1-M^2)\mu_{mn}^2 > K^2 M^2$$

$$K^2 - \mu_{mn}^2 - M^2(K^2 - \mu_{mn}^2) > 0$$

$$(K^2 - \mu_{mn}^2)(1-M^2) > 0$$

For subsonic $M<1$, $K^2 > \mu_{mn}^2$

Now for large values of m, the lowest zero of $J_m'(z)$ is given by

$$\mu_{mo}\, a \simeq m+1$$

In fact

$$\mu_{mo} a = j_{mo} \sim m\,(1 + 0.8086\, m^{-2/3} + \cdots\ )$$

m	$\mu_{mo} a$	$m+1$
6	7.501	7
8	9.647	9

Now the spining mode pattern sweeps the duct walls with a tangential velocity

$$V_p = \frac{\omega}{m} a$$

ωa : radial velocity

m : mode ~~#~~ number

Small m, V_p large

large m, V_p small

Then, the cut-off threshhold is given by

$$\frac{\omega_{mo}}{C_o} = \mu_{mo} \simeq \frac{m+1}{a}$$

For a mode to be cut-on

$$\omega > \frac{m+1}{a} C_o$$

or

$$\frac{V_p m}{a} > \frac{m+1}{a} C_o$$

or

$$\frac{V_p}{C_o} = M_p > \frac{m+1}{m} \simeq 1 \quad \text{for } m \gg 1$$

For a mode to be cut-on, it must sweep the duct wall with supersonic phase velocity.

Example. $m=8$, $a=1m$ $C_0=340$ m/s.

$$\frac{\omega_{cut\text{-}off}}{C_0} = \mu_{m0}$$

$$\omega_{cut\text{-}off} = \mu_{m0}\cdot C_0 = (\mu_{m0}\cdot a)\left(\frac{1}{a}\right)C_0 = 9.647\cdot 340 = 3280 \text{ rad/sec}$$

- Decay rates of cut-off modes

Below cut-off $K_z^2<0$, and K_z is imaginary.

$$p(r,\theta,z,t) = A_{mn} J_m(\mu_{mn} r)\exp[i(m\theta-\omega t)]\exp[-|K_z|z]$$

exponential decay as the wave propagates down the duct

with $|K_z| = \sqrt{\mu_{mn}^2 - k^2}$ (no flow)

If $\mu_{mn}a \simeq m+1 \simeq m$ and also $(\omega/m)(a/C_0) = M_p$

$$|K_z| \simeq \sqrt{\frac{m^2}{a^2} - \frac{\omega^2}{C_0^2}}$$

$$= \frac{m}{a}\sqrt{1-\left(\frac{\omega a}{mC_0}\right)^2}$$

$$= \frac{m}{a}\sqrt{1-M_p^2}$$

The ratio of the pressure in the duct separated in the axial direction by one radius is

$$\frac{p(z+a)}{p(z)} = \exp[-|K_z|a] \simeq \exp\left[-\frac{m}{a}\sqrt{1-M_p^2}\,a\right]$$

So that

$$\Delta dB = 20\log_{10}\left[\frac{p(z+a)}{p(z)}\right]$$

decrease in sound pressure level

$$= 20\log_{10}\left[\exp\left(-m\sqrt{1-M_p^2}\right)\right]$$

Now $\log_{10}e^x = x\log_{10}e = 0.4342x$

Finally $\Delta dB = 20\left[-0.4342\, m\sqrt{1-M_p^2}\right] = -8.69\, m\sqrt{1-M_p^2}$ dB/duct radiu

Remember if $M_p>1$, mode is cut-on (does not decay)

Let $m=25$, $M_p=0.5$, Then $\Delta dB = 188.14$ dB!

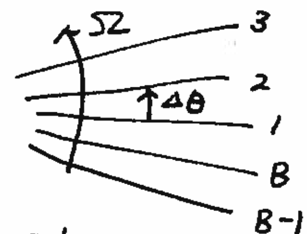
Mode Generation Mechanisms

- Rotating fan

Of particular interest, especially in aircraft noise reduction, is the following model of a propeller or fan with B identical blades, equally spaced $\Delta\theta = 2\pi/B$ radians apart, rotating with angualar speed Ω. If at some point $t=0$ at a fixed position x, the field due to one blade is given by the shape function $q(\theta, r)$, then from periodicity total field is described by

$$p(r,\theta,0) = \underbrace{q(\theta, r)}_{\text{1st blade}} + \underbrace{q(\theta-\Delta\theta, r)}_{\text{2nd blade}} + \cdots + \underbrace{q(\theta-(B-1)\Delta\theta, r)}_{B\text{-th blade}}$$

$$= \sum_{K=0}^{B-1} q\left(\theta - \frac{2\pi}{B}K, r\right)$$



This function, periodic in θ with period $2\pi/B$, may be expanded in a Fourier series:

$$p(r,\theta,0) = \sum_{n=-\infty}^{\infty} q_n(r)\, e^{-inB\theta}$$

(Note that Fourier series has a form of $e^{-i\frac{2\pi n}{T}\theta}$)

Since the field is associated to the rotor, it is a function of $\theta - \Omega t$. So at a time t

[positive direction of Ω]

$$p(r,\theta,t) = \sum_{K=0}^{B-1} q\left(\theta - \Omega t - \frac{2\pi}{B}K, r\right) = \sum_{n=-\infty}^{\infty} q_n(r)\, e^{inB\Omega t - inB\theta}$$

(with $q_{-n} = q_n^*$ because p is real).

Evidently, the field is built up from harmonics of the blade passing frequency $B\Omega$. Note that each frequency $\omega = nB\Omega$ is now linked to a circumferential periodicity $m = nB$, and we jump with steps B through the modal m-spectrum. Since the plane wave ($m=0$) is generated with frequency $\omega = 0$, it is acoustically not interesting, and we may ignore this component.

$n=1$: fundamental harmonic (some people call this 1st harmonic mistakenly)

$n=2$: 1st harmonic

An interesting consequence for a rotor in a duct is the observation that it is not obvious if there is (propagating) sound generated at all: The frequency must be higher than the cut-off frequency.

The cut-on threshhold is

$$f_m = \frac{m\Omega}{2\pi} > \frac{j_{mo} c_o}{2\pi a} \qquad (\omega = m\Omega, \ j_{mo} = \alpha_{mo} \text{ when } J'_m(\alpha_{mo}) = \text{ or 1st solution}$$

for no flow or subsonic since $\frac{\omega_{mo}}{c_o} = k_{mo} > \mu_{mo}\cancel{(1-M^2)} = \frac{\alpha_{mo}}{a}\cancel{(1-M^2)}$

which is for the tip Mach number M_{tip}, the condition

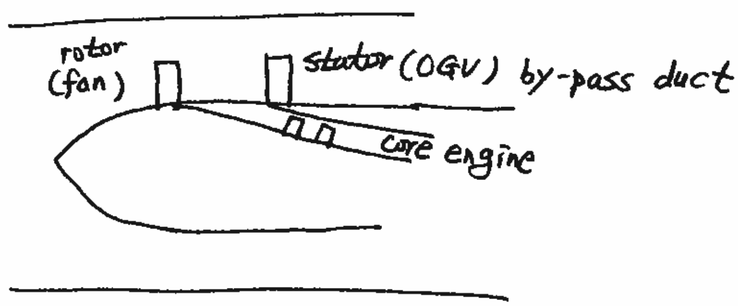
$$M_{tip} = \frac{a\Omega}{c_o} > \frac{\alpha_{mo}}{m} \simeq \frac{m+1}{m} \qquad \text{for large } m.$$

Since the first zero of J_m' is alwalys (slightly) larger than m ($\sim m+1$ for large m), it implies that the tip must rotate supersonically ($M_{tip} > 1$) for the fan to produce sound.

Of couse, in practice a ducted fan with ~~sub~~ subsonically rotating blades will not be entirely silent. For example, ingested turbulence and the turbulent wake of the blades are not periodic and will therefore not follow this cut-off reduction mechanism.

On the other hand, if the perturbations resulting from blade thickness and lift forces alone are dominating as in aircraft engines, the present result is significant, and indeed the inlet fan noise level of many aircraft turbo fan engines is greatly enhanced at takeoff by the inlet fan rotating supersonically (together with other effects leading to the so-called buzzsaw noise.)

– Tyler and Sofrin rule for rotor-stator interaction



The most important noise source of an aircraft turbo fan engine at inlet side is the noise due to interaction between inlet rotor and neighboring stator. Behind the inlet rotor, or fan, a stator is positioned to compensate for the rotation in the flow due to the rotor. The viscous and inviscid wakes from the rotor blades hit the stator vanes which results into the generation of sound. A very simple but at the same time very important and widely used device to reduce this sound is the "Tyler and Sofrin" selection rule." It is based on elegant manipulation of Fourier series, and amounts to nothing more than a clever choice of the rotor blade and stator vane numbers, to link the first (few) harmonics of the sound to duct modes that are cut-off and therefore do not propagate

Consider the same rotor as before, with B identical blades, equally spaced $\Delta\theta = 2\pi/B$ radiants apart, rotating with angular speed Ω, and a stator V identical vanes, equally spaced $\Delta\theta = 2\pi/V$ radians apart. First, we observe that the field generated by rotor-stator interaction must have the time dependence of the rotor, and is therefore built up from harmonics of the blade passing frequency $B\Omega$.

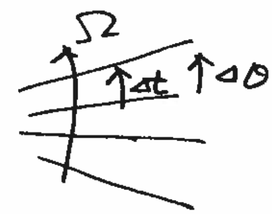
Furthermore, it is periodic in θ, so it may be written as

$$p(r,\theta,t) = \sum_{n=-\infty}^{\infty} Q_n(r,\theta)\, e^{inB\Omega t} = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} Q_{mn}(r)\, e^{inB\Omega t - im\theta}$$

$\neq nB$

However, we can do better than that, because most of m-components are just zero. The field is periodic in θ with the stator periodicity $2\pi/V$ in such a way that when we travel with the rotor over an angle $\Delta\theta = 2\pi/V$ in a time step $\Delta t = \Delta\theta/\Omega$, the field must be the same:

$$p(r,\theta,t) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} Q_{nm}(r)\, e^{inB\Omega(t-\Delta t) - im(\theta-\Delta t)}$$



This ~~yele~~ yields for any m, the restriction:

$$-inB\Omega\Delta t + im\Delta\theta = 2\pi i K$$

$$-nB + m\underbrace{\frac{\Delta\theta}{\Omega\Delta t}}_{=1} = \underbrace{\frac{2\pi}{\Omega\Delta t}}_{=2\pi/\Delta\theta = V} K \quad (\Delta\theta = 2\pi/V)$$

$$-nB + m = VK$$

or $\boxed{m = KV + nB}$ others use $mB + KV$ or $mB - KV$

where K is any integer, and n the harmonic of interest.

By selecting B and V such that the lowest $|m|$ possible is high enough for the harmonic of interest to be cut-off, this component is effectively absent for a long enough inlet duct. In practice, only the first harmonic $(n=1)$ is reduced in this way. A recent development is that the second harmonic, which is usually cut-on, is reduced by selecting the mode number m to be opposite sign of n, which means: counter rotating with respect to the rotor. In this case, the rotor itself acts as a shield obstructing the spiralling modes to leave the duct.

In detail, for a cut-off n-th harmonic, we need

$$\frac{nB\Omega}{2\pi} < \frac{\alpha_{mo} c_0}{2\pi a} \quad \text{or} \quad nBM_{tip} < \alpha_{mo} \quad \text{(cut-off)}$$

Since typically M_{tip} is slightly smaller than 1 and α_{mo} is slightly larger than $|m|$, we get the evanescent wave condition

$$nB \leq |m| = |KV + nB|$$

The only values of KV for which this inequality is not satisfied automatically is in the interval $-2nB < KV < 0$.

If we make the step size V big enough so that we avoid this interval for $K=-1$, we avoid it for any K.

we need for $K=-1$

So, we have finally the condition: $V \geq 2nB$ [$KV \leq -2nB$, $KV \geq$

Consider, as a realistic example, the following configuration of a rotor with $B=22$ blades and a stator with $V=55$ vanes.

The generated m-modes are for the first two harmonics:

for $n=1$: $m = \cdots, -33, 22, 77, \cdots$ $(nB=22)$

for $n=2$: $m = \cdots, \underline{-11}, 44, 99, \cdots$ $(nB=44)$

which indeed corresponds to only cut-off modes of the first harmonic ($m=22$ and larger) and a counter rotating cut-on at second harmonic ($m=-11$).

- Speed ratio :

$$\left|\frac{nB}{nB+KV}\right| \quad \begin{array}{ll} >1 & : \text{cut-on, high efficiency} \\ <1 & : \text{cut-off, low efficiency} \end{array}$$

Ex 1. $B=18$, $V=1$ for distortion noise or pylon-rotor interaction noise

$$m = 18n + K$$

$$SR = \left|\frac{18n}{18n+K}\right|$$

$n=1$: for $K=-1,-2,\cdots$, $SR = \frac{18}{17}, \frac{18}{16}, \cdots$

higher gust harmonics show higher efficiency

Ex 2. $B=18$ $V=44$ (rotor-stator interaction noise)

$n=1$: cut-off

$n=2$: $SR = \left|\frac{18\times 2}{18\times 2 + 44K}\right|$

For $K=-1$. $SR = \left|\frac{36}{8}\right| = 4.5$ (high efficiency & cut-on)