EAE 298 Aeroacoustics, Fall Quarter 2016

Mid-Term Exam
(Open Note)

Name : ______________________

1. [15 points] A noise is generated by 80 pure tones, of different frequencies but identical power. Each pure tone has a sound pressure level of 60 dB. Determine the sound pressure level of the noise.

$$\overline{p'^2} = \frac{1}{2}(P_1^2 + P_2^2 + \cdots + P_{80}^2)$$

$$= 40 P_1^2$$

$$\overline{p'} = \sqrt{40}\, P_1$$

$$SPL = 20\log_{10}\left[\frac{\sqrt{40}\, P_1}{2\times 10^{-5}}\right] = 20\log_{10}\left[\frac{\sqrt{80}\, P_1}{\sqrt{2}\times 2\times 10^{-5}}\right]$$

$$= 20\log_{10}\left[\frac{P_1/\sqrt{2}}{2\times 10^{-5}}\right] + 20\log_{10}(\sqrt{80})$$

$$= 60 + 10\log_{10}(80)$$

$$= \underline{79.03\ dB}$$

Alternatively,

$$SPL = 10\log_{10}\left(\sum 10^{SPL_i/10}\right)$$

$$= 10\log_{10}(10^6 \times 80)$$

$$= \underline{79.03\ dB}$$

2. [35 points] Problem about sound intensity and power

2. (a) [20 points] Show that the time-averaged intensity radiated for the various harmonic sound fields whose pressure

$$p'(\vec{x},t) = p_0 l^{n+1} \frac{\partial^n}{\partial x_1^n}\left\{\frac{\cos\omega\left(t-\frac{r}{c}\right)}{r}\right\}$$

is

$$\frac{p_0^2 l^{2n+2}}{2\rho_0 c r^2}\left(\frac{\omega}{c}\right)^{2n}\cos^{2n}\theta$$

for different integer values of n. $p_0 l^{n+1}$represents the source strength, p_0 being a reference pressure and l an appropriate length scale. [Hint: Use a far-field approximation.]

In the far-field $u = P'/\rho_0 c$

$$I = \overline{P'u} = \frac{P_0^2 l^{2n+2}}{\rho_0 c}\,\overline{\frac{\partial^n}{\partial x_1^n}\left\{\frac{\cos\omega(t-r/c)}{r}\right\} \times \frac{\partial^n}{\partial x_1^n}\left\{\frac{\cos\omega(t-r/c)}{r}\right\}}$$

$$\frac{\partial}{\partial x_1} = \frac{\partial r}{\partial x_1}\cdot\frac{\partial}{\partial r} = \cos\theta\cdot\frac{\partial}{\partial r},\quad \frac{\partial^n}{\partial x_1^n} = \cos^n\theta\frac{\partial^n}{\partial r^n} \text{ for far-field}$$

For $n=1$

$$\frac{\partial}{\partial r}\left\{\frac{\cos\omega(t-r/c)}{r}\right\} = \cancel{-\frac{\cos\omega(t-r/c)}{r^2}} + \frac{1}{r}\left(-\frac{\omega}{c}\right)\left(-\sin\omega(t-r/c)\right)$$

$$\overline{\frac{\partial}{\partial r}\left\{\frac{\cos\omega(t-r/c)}{r}\right\}\frac{\partial}{\partial r}\left\{\frac{\cos\omega(t-r/c)}{r}\right\}} = \overline{\frac{1}{r^2}\left(\frac{\omega}{c}\right)^2\sin^2\omega(t-r/c)} = \frac{1}{2}\frac{1}{r^2}\left(\frac{\omega}{c}\right)^2$$

For $n=2$,

far-field

$$\frac{\partial^2}{\partial r^2}\left\{\frac{\cos\omega(t-r/c)}{r}\right\} = \frac{\partial}{\partial r}\left\{\frac{(\omega/c)\sin\omega(t-r/c)}{r}\right\} = \left\{\cancel{-\frac{(\omega/c)\sin\omega(t-r/c)}{r^2}} + \frac{(\omega/c)^2\cos\omega(t-r/c)}{r}\right.$$

$$\overline{\frac{\partial^2}{\partial r^2}\left\{\frac{\cos\omega(t-r/c)}{r}\right\}\frac{\partial^2}{\partial r^2}\left\{\frac{\cos\omega(t-r/c)}{r}\right\}} = \frac{1}{2}\frac{1}{r^2}\left(\frac{\omega}{c}\right)^4$$

For general n.

$$\overline{\frac{\partial^n}{\partial x_1^n}\left\{\quad\right\}\frac{\partial^n}{\partial x_1}\left\{\quad\right\}} = \cos^{2n}\theta\cdot\frac{1}{2}\frac{1}{r^2}\left(\frac{\omega}{c}\right)^{2n}$$

Therefore,

$$I = \frac{P_0^2 l^{2n+2}}{2\rho_0 c r^2}\left(\frac{\omega}{c}\right)^{2n}\cos^{2n}\theta$$

//

2. (b) [15 points] Show that the power radiated by the source from 2.(a) is

$$P = \frac{2\pi p_0^2 l^2}{\rho_0 c(2n+1)}\left(\frac{\omega l}{c}\right)^{2n}$$

[Hint: the surface integration for a sphere can be calculated by $2\pi \int_0^\pi A(r,\theta) r^2 \sin\theta \; d\theta$.]

$$P = 2\pi \frac{P_0^2 \, l^{2n+2}}{2\rho_0 C} \left(\frac{\omega}{C}\right)^{2n} \int_0^{\pi} \cos^{2n}\theta \sin\theta \, d\theta$$

$$A = \int_0^{\pi} \cos^{2n}\theta \sin\theta \, d\theta$$

$$= -\cos\theta \cdot \cos^{2n}\theta \Big|_0^{\pi} + 2n \int_0^{\pi} \cos\theta \cdot \cos^{2n-1}\theta \, (-\sin\theta) \, d\theta$$

$$= 2 - 2n \int_0^{\pi} \cos^{2n}\theta \sin\theta \, d\theta$$

$$\therefore \int_0^{\pi} \cos^{2n}\theta \sin\theta \, d\theta = \frac{2}{2n+1}$$

Therefore

$$P = \frac{2\pi P_0^2 \, l^{2n+2}}{\rho_0 C (2n+1)} \left(\frac{\omega}{c}\right)^{2n}$$

$$= \frac{2\pi P_0^2 l^2}{\rho_0 c (2n+1)} \left(\frac{\omega l}{c}\right)^{2n}$$

3. [30 points] Problem about a moving source

3. (a) [10 points] Show that the retarded time τ^* for a moving source with the constant velocity of U in the x_1 direction and the source location $x_2 = l$ and $x_3 = 0$ is

$$\tau^* = \left(t - \frac{|\vec{x}|}{c} + \frac{x_2 l}{|\vec{x}|c}\right)\frac{1}{1 - M\cos\theta} \quad for\ |\vec{x}| \gg l, |U\tau^*|$$

Show the detailed derivation and do not jump on the final equation without justification.

$$c(t-\tau^*) = \{(x_1 - U\tau^*)^2 + (x_2 - l)^2 + x_3^2\}^{1/2}$$

$$c(t-\tau^*) = |\vec{x}|\left\{1 - \frac{2x_1 \cdot U\tau^*}{|\vec{x}|^2} - \frac{2x_2 l}{|\vec{x}|^2} + \frac{U\tau^* + l^2}{|\vec{x}|^2}\right\}^{1/2}$$

where $|\vec{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$

For $|\vec{x}| \gg l, |U\tau^*|$

$$c(t-\tau^*) \simeq |\vec{x}|\left\{1 - \frac{x_1}{|\vec{x}|^2}U\tau^* - \frac{x_2 l}{|\vec{x}|^2}\right\}$$

$$c\tau^*\left\{1 - \frac{U}{c}\cos\theta\right\} = ct - |\vec{x}| + \frac{x_2 l}{|\vec{x}|}$$

where $\cos\theta = \frac{x_1}{|\vec{x}|}$

Therefore

$$\tau^* = \left(t - \frac{|\vec{x}|}{c} + \frac{x_2 l}{|\vec{x}|c}\right)\frac{1}{1 - M\cos\theta}$$

3. (b) [20 points] Show that the distant sound field produced by a source of strength

$$q(\vec{x},t) = \delta(x_1 - Ut)H(l^2 - x_2^2)\delta(x_3)Qe^{i\omega t}$$

is

$$p'(\vec{x},t) = \frac{cQ}{2\pi\omega x_2}\exp\left[\frac{i\omega\left(t - \frac{|\vec{x}|}{c}\right)}{1 - M\cos\theta}\right]\sin\left(\frac{\omega x_2 l}{|\vec{x}|c(1 - M\cos\theta)}\right)$$

when $|\vec{x}|$ is large in comparison with l and $|U\tau^*|$ and where $\cos\theta = x_1/|\vec{x}|$. Comment on the form when the source is compact, i.e. $\omega l \ll c(1 - M\cos\theta)$. [Hint: H is the Heaviside function where $H(x) = 1$ when $x \geq 0$, $H(x) = 0$ when $x < 0$ and $H(l^2 - x_2^2) = H\{(l - x_2)(l + x_2)\}$.]

$$p'(\vec{x},t) = \frac{1}{4\pi}\int_{-l}^{l} Q\,\frac{e^{i\omega\tau^*}}{r|1-M_r|}\,d\eta_2$$

where $\tau^* = \left(t - \frac{|\vec{x}|}{c} + \frac{x_i\eta_i}{|\vec{x}|c}\right)\frac{1}{1-M\cos\theta}$ for $|\vec{x}| \gg l,\ |U\tau^*|$

Then,

$$p'(\vec{x},t) = \frac{Q}{4\pi|\vec{x}|(1-M\cos\theta)}\underbrace{\int_{-l}^{l}\exp i\omega\left[\left(t-\frac{|\vec{x}|}{c}+\frac{x_2\eta_2}{|\vec{x}|c}\right)\frac{1}{1-M\cos\theta}\right]d\eta_2}_{A}$$

$$A = \frac{|\vec{x}|c(1-M\cos\theta)}{i\omega x_2}\cdot\exp\left[\frac{i\omega(t-\frac{|\vec{x}|}{c})}{1-M\cos\theta}\right]\left\{\exp\left(\frac{i\omega x_2 l}{|\vec{x}|c(1-M\cos\theta)}\right)-\exp\left(\frac{-i\omega x_2 l}{|\vec{x}|c(1-M\cos\theta)}\right)\right\}$$

since $e^{iax} - e^{-iax} = 2i\sin ax$.

$$A = \frac{2|\vec{x}|c(1-M\cos\theta)}{\omega x_2}\cdot\exp\left[\frac{i\omega(t-\frac{|\vec{x}|}{c})}{1-M\cos\theta}\right]\cdot\sin\left(\frac{\omega x_2 l}{|\vec{x}|c(1-M\cos\theta)}\right)$$

Therefore, $$p'(\vec{x},t) = \frac{cQ}{2\pi\omega x_2}\exp\left[\frac{i\omega(t-|\vec{x}|/c)}{1-M\cos\theta}\right]\sin\left(\frac{\omega x_2 l}{|\vec{x}|c(1-M\cos\theta)}\right)$$

For $\omega l \ll c(1-M\cos\theta)$ and $\sin\alpha \sim \alpha$ for $\alpha \ll 1$
or $l \ll \lambda$ (compact)

$$p'(\vec{x},t) \sim \frac{2lQ}{4\pi|\vec{x}|(1-M\cos\theta)}\exp\left[\frac{i\omega(t-|\vec{x}|/c)}{1-M\cos\theta}\right]$$

and the sound field is equivalent to that generated by a moving point source of strength $2lQ\,e^{i\omega t}$