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ACOUSTIC RADIATION FROM AN AIRFOIL IN A TURBULENT STREAM

R. K. AMIET

United Aircraft Research Laboratories, East Hartford, Connecticut 06108, *U.S.A.*

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A theoretical expression for the far-field acoustic power spectral density produced by an airfoil in a subsonic turbulent stream is given in terms of quantities characteristic of the turbulence. For an observer directly above the airfoil the relevant quantities are the spanwise correlation length of the turbulence as a function of frequency and the PSD of the vertical velocity fluctuations, both quantities being measured in an airfoil fixed co-ordinate system. In the derivation it is assumed that the spanwise correlation length is much smaller than the airfoil span.

A more solid theoretical foundation is developed for a lift expression given by Liepmann for an airfoil in turbulence. Also, the analysis shows why it is not necessary, within certain limitations, to integrate over all gust wavenumbers in order to calculate the sound at a given observer location. Rather, for small turbulence length scales the observer hears only the sound produced by a particular wavenumber component of the turbulence, and thus the analysis can be simplified considerably.

A comparison of the theory with experimental acoustic and lift measurements available in the literature shows good agreement.

1. INTRODUCTION

An airfoil in a turbulent flow field experiences a fluctuating lift which according to the theories of Kirchhoff [1] and Curle [2] should result in the generation of sound. In principle, if the unsteady loading at each point on the airfoil were known as a function of time, the noise at any point in the field far could be calculated. This technique quite often presents measurement problems, however.

The principal difficulty can be understood most easily by considering the situation shown in Figure 1 with the observer in the far field directly above the airfoil in an open-jet tunnel and with a sinusoidal gust incident on the airfoil. With the observer directly overhead, retarded time differences can be neglected. (This point will be discussed further in a later section.) This

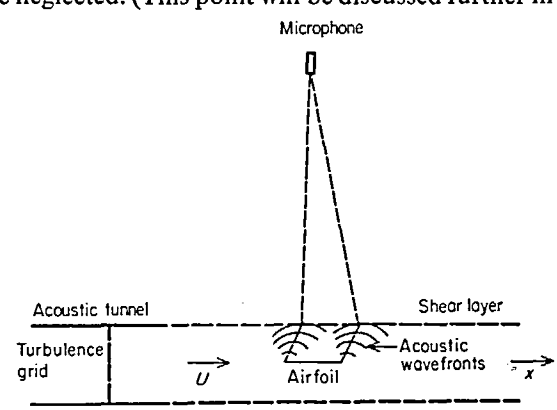

Figure 1. Airfoil in the free stream of an acoustic tunnel. Dashed lines from airfoil to microphone demonstrate near equality of all ray propagation distances.

then allows one to formulate the far field sound in terms of the total fluctuating lift of the airfoil. Figure 2(a) shows a gust with its wavefronts parallel to the airfoil leading edge, while Figure 2(b) shows a gust which is skewed relative to the leading edge. A typical turbulent eddy will be composed of both parallel gusts and skewed gusts. The parallel type gust should be an efficient sound producer, and should present no problem in calculation of the far-field sound from a few surface measurements if no skewed gusts were present. The skewed gust, however, does present a problem. Because it varies sinusoidally along the span, there is substantial cancellation of the lift produced by adjacent spanwise stations. Thus, one would need much more careful measurement of the loading as a function of position and cross-correlations of these measurements in order to separate out the local pressure fluctuations due to parallel gusts from those due to skewed gusts to enable one to predict the far-field noise overhead.

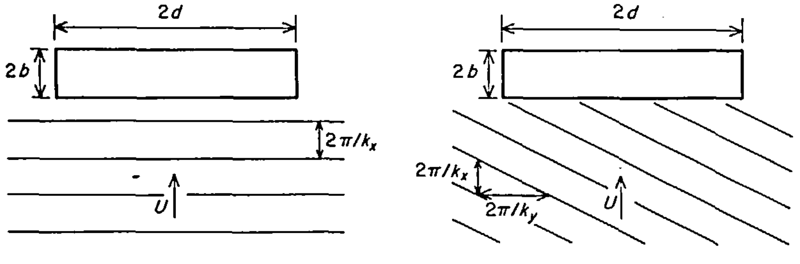

Figure 2. Parallel and skewed gusts incident on airfoil.

The phenomenon of source cancellation is a common occurrence in acoustics and often requires a reformulation of the problem. For example, Lighthill could have formulated his jet noise theory [3] in terms of monopole or dipole sources instead of quadrupoles. The reason for not doing so is that the resulting total monopole or dipole strength would be zero (just as the total lift is near zero for the skewed gust) and would thus require more careful measurements to differentiate the radiating fraction of the velocity fluctuations from those which merely produce near-field or pseudo-sound.

For our case, however, it will not be necessary to change from a dipole singularity to some other type. Since a turbulent eddy is composed of both parallel gusts and skewed gusts it would be expected, in view of the above discussion, that the sound radiation produced by the parallel gusts would be the dominant noise source. The present paper shows that for the purpose of calculating the sound level directly overhead it may be a good approximation to use two-dimensional airfoil theory for calculating the lift response. Although this procedure does not give a good representation of the sectional lift because of the presence of skewed gusts, it should give a good representation of the total lift which is more important for the present problem.

The problem will be formulated for the more general case of an arbitrary observer location with the free stream extending to infinity: i.e., no shear layer. The case of an observer directly overhead and with a shear layer between the airfoil and the observer will be derived as a special case of the more general solution.

2. ANALYSIS

An airfoil of chord $2b$ and span $2d$ is placed in a turbulent fluid with a mean flow U in the x direction. The y co-ordinate extends in the spanwise direction, and the origin of the co-ordinate system is at the center of the airfoil. The observer is in the far field. (A list of symbols is given in Appendix 3.)

The turbulence is assumed to be frozen so that in a co-ordinate system $x' = x - Ut$, which moves with the mean flow, the turbulent velocity in the $z = 0$ plane can be written as $w(x', y)$. In the airfoil fixed co-ordinate system the turbulent velocity, $\tilde{w}(x, y, t)$, written in terms of its wavenumber components, $\hat{\hat{w}}_R(k_x, k_y)$ is

$$\tilde{w}(x, y, t) = \iint\limits_{-\infty}^{\infty} \hat{\hat{w}}_R(k_x, k_y)\, e^{\mathrm{i}[k_x(x-Ut)+k_y y]}\, dk_x\, dk_y, \tag{1}$$

where the double hat on $\hat{\hat{w}}_R$ indicates the double spatial Fourier transform of w in the variables x and y. The Fourier components $\hat{\hat{w}}_R(k_x, k_y)$ can be determined from $w(x, y)$ by the inverse relation

$$\hat{\hat{w}}_R(k_x, k_y) = \frac{1}{(2\pi)^2} \iint\limits_{-R}^{R} w(x, y)\, e^{-\mathrm{i}(k_x x + k_y y)}\, dx\, dy, \tag{2}$$

where R is a large but finite number. R is not set equal to infinity at this point because of convergence difficulties if $w(x, y)$ does not go to zero as x and y go to infinity.

Analytical expressions for the distribution in the pressure jump, ΔP, across a flat plate airfoil of infinite span encountering a sinusoidal gust are available [4, 5]. For a gust of the form

$$w_g = w_0\, e^{\mathrm{i}[k_x(x-Ut)+k_y y]} \tag{3}$$

the distribution of the pressure jump can be written as

$$\Delta P(x, y, t) = 2\pi\rho_0\, U b w_0\, g(x, k_x, k_y)\, e^{\mathrm{i}(k_y y - k_x U t)}, \tag{4}$$

where $g(x, k_x, k_y)$ is the transfer function between turbulent velocity and airfoil pressure jump.

The pressure jump at a given point on the airfoil due to all wavenumber components is then

$$\Delta P(x, y, t) = 2\pi\rho_0\, U b \iint\limits_{-\infty}^{\infty} \hat{\hat{w}}_R(k_x, k_y)\, g(x, k_x, k_y)\, e^{\mathrm{i}(k_y y - k_x U t)}\, dk_x\, dk_y. \tag{5}$$

The Fourier transform with respect to time can be performed to give the frequency dependence. Since in equation (2) the turbulence was assumed to extend between $-R < x < R$, the time integration will be between $\pm T$ where $T = R/U$. With use of the fact that

$$\int_{-T}^{T} e^{\mathrm{i}\xi t}\, dt \to 2\pi\delta(\xi) \quad \text{as} \quad T \to \infty, \tag{6}$$

the result is

$$\Delta\hat{P}_T(x, y, \omega) = 2\pi\rho_0\, b \int_{-\infty}^{\infty} \hat{\hat{w}}_R(K_x, k_y)\, g(x, K_x, k_y)\, e^{\mathrm{i}k_y y}\, dk_y, \tag{7}$$

where $K_x = -\omega/U$. That is, a given frequency component of the pressure jump is produced by the $-\omega/U$ value of the chordwise turbulence wavenumber.

Since the turbulence is a random quantity, rather than work with deterministic quantities such as the time history of the pressure jump at a point, it is necessary to work with statistical quantities such as the cross-PSD, S_{QQ}, of the pressure jump at two points on the surface. The cross-PSD can be written as

$$S_{QQ}(x_1, x_2, y_1, y_2, \omega) = \lim_{T\to\infty} \left\{ \frac{\pi}{T} E[\Delta\hat{P}_T^*(x_1, y_1, \omega)\, \Delta\hat{P}_T(x_2, y_2, \omega)] \right\}, \tag{8}$$

where $E[\ldots\ldots]$ denotes the expected value or ensemble average of a quantity.

The only statistical or non-deterministic quantity on the right-hand side of equation (7) is $\hat{\hat{w}}_R$. Thus, if equation (7) is used to replace $\Delta\hat{P}_T$ in equation (8), all functions other than $\hat{\hat{w}}_R$

can be taken outside the expected value sign leaving $E[\hat{w}_R(K_x,k_y)\hat{w}_R^*(K_x,k_y')]$. However, because of the statistical orthogonality of the wavevectors, it can be shown that

$$E[\hat{w}_R(K_x,k_y)\,\hat{w}_R^*(K_x,k_y')] = \frac{R}{\pi}\delta(k_y-k_y')\,\Phi_{ww}(K_x,k_y'), \tag{9}$$

where

$$\Phi_{ww}(k_x,k_y) = \int_{-\infty}^{\infty}\Phi_{ww}(k_x,k_y,k_z)\,\mathrm{d}k_z \tag{10}$$

and $\Phi_{ww}(k_x,k_y,k_z)$ is the energy spectrum [6] of the turbulence. (See Appendix 1.) Combining equations (7) through (9) gives for the cross-PSD of the surface pressure jump

$$S_{QQ}(x_1,x_2,\eta,\omega) = (2\pi\rho_0 b)^2\,U\int_{-\infty}^{\infty} g^*(x_1,K_x,k_y)\,g(x_2,K_x,k_y)\,\Phi_{ww}(K_x,k_y)\,\mathrm{e}^{\mathrm{i}k_y\eta}\,\mathrm{d}k_y, \tag{11}$$

where $\eta = y_2 - y_1$ is the spanwise separation of the two points on the airfoil surface for which the cross-PSD is desired. Since end effects of the airfoil are ignored, only the spanwise separation of the two points enters the equation, not the y co-ordinate of each point.

The theories of Kirchhoff [1] and Curle [2] now will be used to relate the cross-PSD of the surface pressure to the far-field sound. These theories state that the acoustic response of the airfoil can be determined by distributing dipoles over the airfoil surface equal in strength to the force on the surface. The far-field sound produced by a point force of strength $F(x_0,y_0,\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathbf{k}$ in a stream of Mach number M is

$$P_1(x,y,z,\omega;x_0,y_0) = \frac{\mathrm{i}\omega z F(x_0,y_0,\omega)}{4\pi c_0\sigma^2}\,\mathrm{e}^{\mathrm{i}\omega\left[t+\frac{M(x-x_0)-\sigma}{c_0\beta^2}+\frac{xx_0+yy_0\beta^2}{c_0\beta^2\sigma}\right]} \tag{12}$$

where $\sigma = \sqrt{x^2+\beta^2(y^2+z^2)}$ and $\beta = \sqrt{1-M^2}$. For the present problem, the force $F(x_0,y_0)$ is the difference in pressure between the upper and lower surfaces of the airfoil. The far-field pressure can be found by integration of equation (8) over x_0 and y_0: i.e., over the airfoil planform area. If the result of this integration is multiplied by its complex conjugate and the expected value taken, the PSD of the far-field noise, S_{PP}, can be shown to be related to the cross-PSD of airfoil loading by

$$S_{PP}(x,y,z,\omega) = \left(\frac{\omega z}{4\pi c_0\sigma^2}\right)^2 \iiiint S_{QQ}(x_1,x_2,\eta,\omega)\,\mathrm{e}^{\frac{\mathrm{i}\omega}{c_0}[\beta^{-2}(x_1-x_2)(M-x/\sigma)+y\eta/\sigma]}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}y_1\,\mathrm{d}y_2. \tag{13}$$

If now equation (11) is substituted into equation (13), an expression for the far-field acoustic PSD in terms of the turbulence energy spectrum and airfoil response function results. Defining the following chordwise integral of the surface loading

$$\mathscr{L}(x,K_x,k_y) = \int_{-b}^{b} g(x_0,K_x,k_y)\,\mathrm{e}^{-\mathrm{i}\omega x_0(M-x/\sigma)/c_0\beta^2}\,\mathrm{d}x_0 \tag{14}$$

allows the far-field PSD to be written as

$$S_{PP}(x,y,z,\omega) = \left(\frac{\omega z\rho_0 b}{c_0\sigma^2}\right)^2 Ud\pi\int_{-\infty}^{\infty}\left[\frac{\sin^2(d(k_y+\omega y/c_0\sigma))}{(k_y+\omega y/c_0\sigma)^2\pi d}\right]|\mathscr{L}(x,K_x,k_y)|^2\,\Phi_{ww}(K_x,k_y)\,\mathrm{d}k_y. \tag{15}$$

The function $\mathscr{L}$ defined by equation (14) is related to the degree of non-compactness of the airfoil. If the frequency is small, the imaginary exponent is small, and $\mathscr{L}$ reduces to the sectional lift of the airfoil.

To this point the analysis has been quite general. Equation (15) is closely related to the total lift calculations given by equation (13) of reference [7] and equation (A9) of reference [8]. The principal assumption thus far, other than the assumption of linearized airfoil theory, has been to apply the infinite-span airfoil theory over the entire airfoil span, including the region near the tips. This should be a valid assumption for a large aspect ratio airfoil. However, for small turbulence length scales (high frequencies) it should not be necessary to make even the large aspect ratio assumption. As the wavelength of an incident gust decreases, the loading tends to concentrate near the leading edge, and the tip affects a smaller and smaller spanwise region of the airfoil.

Here, however, it is desired to simplify equation (15) further. The clue to the simplification is to note that as the semi-span, d, increases, the quantity in square brackets in equation (15) tends to a delta function: i.e.,

$$\lim_{d\to\infty}\left[\frac{\sin^2(\xi d)}{\xi^2 \pi d}\right] \to \delta(\xi). \tag{16}$$

For a physical explanation of this phenomenon, first consider the simpler case of an observer in the $y=0$ plane. Then

$$S_{PP}(x,0,z,\omega) \to \left(\frac{\omega z \rho_0 b}{c_0 \sigma^2}\right)^2 \pi U d |\mathscr{L}(x,K_x,0)|^2 \Phi_{ww}(K_x,0) \quad \text{as} \quad d\to\infty \tag{17}$$

In other words, only the $k_y=0$ gusts contribute to the sound. The skewed gusts give cancelling effects and make no contribution to the sound in the $y=0$ plane.

For the more general case of $y \neq 0$, as $d\to\infty$ the integral in equation (15) replaces the k_y argument of $\mathscr{L}$ and Φ_{ww} by $\omega y/c_0\sigma$. The physical explanation for this is shown in Figure 3 for the case of small axial Mach number which simplifies the interpretation. The velocity at which the gust sweeps the airfoil leading edge determines the propagation angle of the sound wave produced. As shown in the figure, the observer hears only that gust which produces an acoustic wavefront normal to the line joining airfoil and observer. (Recall that the observer is in the far-field, both geometric and acoustic, of the airfoil.)

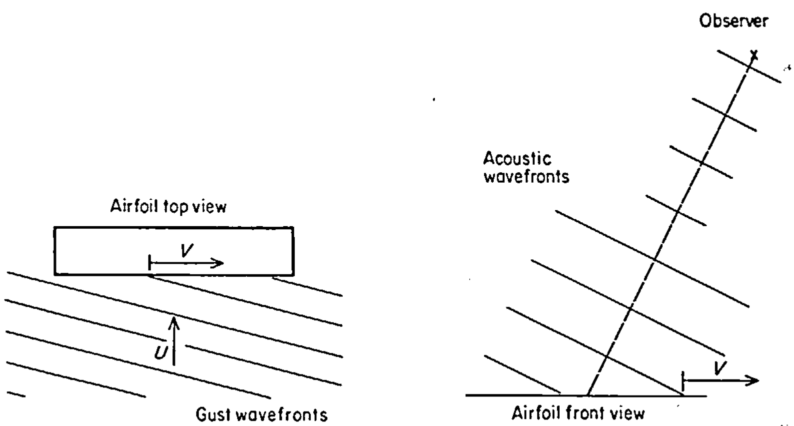

Figure 3. Effect of observer position in determining the separate wavenumber which produces the sound.

It is interesting to note that in this limit of d becoming large, a gust whose intersection point moves subsonically relative to the fluid produces negligible sound. As Graham [9] has shown when this intersection moves subsonically the airfoil response function is similar to that produced by a skewed gust in an incompressible fluid, while for a supersonically moving intersection point the airfoil response is similar to that produced by an unskewed gust in a compressible fluid. Thus, for the large span limit a two-dimensional compressible-flow airfoil theory is sufficient to determine the noise. Of course the surface pressure on the airfoil produced by its interaction with turbulence would depend on both the subsonic and supersonic gusts.

The previous analysis shows that the approximation is valid for large d, but it does not say large with respect to what. It is worthwhile to consider this point further. It is easiest to work with the case of an observer in the $y = 0$ plane, although it should be possible to derive a similar result for any value of y. It will be seen that the value of the integral

$$\int_0^C \sin^2(\xi d)\frac{\mathrm{d}\xi}{\xi^2}$$

for $C = 10/d$ is approximately 90% of the value for $C = \infty$. Thus, if $\mathscr{L}$ and Φ_{ww} do not vary significantly over the range $0 < k_y < 10/d$ and behave reasonably for $k_y > 10/d$, the result given by equation (17) should be a good approximation. An expression for Φ_{ww} to be used later shows that $\Phi_{ww}(K_x, k_y)$ becomes insensitive to k_y for k_y values significantly smaller than K_x. Thus, for $K_x d > 10$, Φ_{ww} can, without significant error, be taken outside the integral in equation (15). This gives the first indication that for fixed span the present result may be applicable at high frequencies.

For any fixed k_y, if K_x is made large enough the intersection point of the airfoil with the gust moves supersonically relative to the fluid. The problem then becomes similar to the two-dimensional compressible gust problem. Using the similarity rules presented by Graham [9], one can show that for $MK_x \gg k_y$ the airfoil response becomes essentially independent of k_y. Thus, if the parameter $\Lambda \equiv MK_x d$ is large, both Φ_{ww} and $\mathscr{L}$ become nearly independent of k_y allowing them to be taken outside the integral in equation (15), resulting in equation (17) for an observer in the $y = 0$ plane. For $y \neq 0$,

$$S_{PP}(x, y, z, \omega) \to \left(\frac{\omega z \rho_0 b}{c_0 \sigma^2}\right)^2 \pi U d \left|\mathscr{L}\left(x, K_x, \frac{\omega y}{c_0 \sigma}\right)\right|^2 \Phi_{ww}\left(K_x, \frac{\omega y}{c_0 \sigma}\right) \quad \text{as} \quad \Lambda \to \infty. \tag{18}$$

Equation (17) can be put in a somewhat more intuitive form by introducing the cross-correlation length, $l_y(\omega)$, as a function of frequency. If the Fourier transform of $\Phi_{ww}(K_x, k_y)$ with respect to k_y is performed, the resulting function $R_{ww}(K_x, y)$ is U times the cross-PSD of the vertical velocity fluctuation at two points a distance y apart. The y integral of this function would be expected to be proportional to a correlation length. In fact, a correlation length, $l_y(\omega)$, can be defined as

$$l_y(\omega) = \frac{1}{R_{ww}(K_x, 0)} \int_0^\infty R_{ww}(K_x, y)\,\mathrm{d}y = \pi \Phi_{ww}(K_x, 0)/R_{ww}(K_x, 0), \tag{19}$$

the second equality following because of the Fourier transform relation of Φ_{ww} and R_{ww}. Then, equation (17) can be rewritten as

$$S_{PP}(x, 0, z, \omega) \to \left(\frac{\omega z \rho_0 b M}{\sigma^2}\right)^2 d\,|\mathscr{L}(x, K_x, 0)|^2\, l_y(\omega)\, S_{ww}(\omega) \quad \text{as} \quad \Lambda \to \infty, \tag{20}$$

where $S_{ww}(\omega) = R_{ww}(K_x, 0)/U$ is the PSD of the vertical velocity fluctuations.

Equations (17) through (20) are the basic results for the sound produced by an airfoil in turbulence. With the assumption of isotropic turbulence, however, explicit expressions for the turbulence characteristics in terms of the rms intensity and integral scale length are available. These are given in Appendix 1. Appropriate expressions for the airfoil response function are given in Appendix 2.

As has been shown, the present theory becomes rigorous (within the assumptions of linearized theory) when

$$\Lambda \equiv MK_x d \to \infty. \tag{21}$$

The expression for $\Phi_{ww}(k_x, k_y)$ given by equation (31) in Appendix 1 shows that for $k_y = 0$, the major portion of the turbulent energy is contained in a finite region $k_x < C/L$ where C is a constant of the order of 1 and L is the integral scale length of the turbulence. This fact, together with equation (21), shows that equation (17) will be a valid expression for the overall level of the acoustic intensity if $Md/L \to \infty$. Thus, for values of the Mach number which are not too small, the present theory should give a valid expression for the overall level when the integral scale, L, of the turbulence is significantly smaller than the airfoil span.

3. AIRFOIL IN AN ACOUSTIC TUNNEL

Figure 1 shows the experimental set-up, the data from which will be compared to the theory derived herein. The airfoil is placed in a turbulent stream with the observer directly overhead in stationary air in the far field. The only significant difference between this situation and the situation to which the previous analysis applies is the presence of a shear layer through which the sound must pass before reaching the observer. The effect of this shear layer must be accounted for.

In reference [10] it is shown that for an observer directly above a sound source in an acoustic tunnel, the sound measured is the same as would be measured for the case of no tunnel flow if somehow one could create the same source intensity in the stationary fluid. For the present case, the sound source consists of a distribution of dipoles over the airfoil surface. If somehow the same dipole distribution on the airfoil could be induced with the tunnel turned off, the sound for an observer far overhead would be the same as if the tunnel were on. This is essentially due to the fact that an observer at a 90° angle to the retarded source position (as measured from the flow direction) sees no convective amplification of the sound.

The result of the above effect is to set $M=0$ in equations (14) and (17) giving for the predicted noise for an observer directly overhead in the far-field of an acoustic tunnel

$$S_{PP}(0,0,z,\omega)=\left(\frac{\omega\rho_0 b}{c_0 z}\right)^2 \pi U d\,|G(\hat{\omega})|^2\,\Phi_{ww}(K_x,0), \tag{22}$$

where $G(\hat{\omega})$ is the two-dimensional airfoil lift response function and $\hat{\omega}=\omega b/U$ is the reduced frequency. Note that $G(\hat{\omega})$ still contains an implicit Mach number dependence which is not set to zero since the strength of the dipole distribution over the airfoil depends on Mach number, and it is important to include this effect.

It will be useful also to have an expression for the overall airfoil lift. Equation (20) has contained in it an expression for the PSD of the overall lift. If the acoustic efficiency factor, $(\omega/4\pi c_0 z)^2$, is divided out, the PSD of the airfoil lift is found to be, by using equation (19),

$$S_{LL}(\omega)=(4\pi\rho_0 bU)^2\, l_y(\omega)\, d\,|G(\hat{\omega})|^2\, S_{ww}(\omega). \tag{23}$$

The square of an admittance, $|A(\hat{\omega})|^2$, can be defined as $S_{LL}(\omega)$ divided by $(4\pi\rho_0 bUd)^2 S_{ww}(\omega)$ (which is the PSD of the lift if the airfoil responded in a quasi-steady two-dimensional manner). $|A(\hat{\omega})|^2$ is found to be

$$|A(\hat{\omega})|^2=\frac{l_y(\omega)}{d}|G(\hat{\omega})|^2. \tag{24}$$

It should be pointed out that equation (23) is not new. An equation which is essentially the same has been used by previous authors (see, e.g., references [7] and [8]) but without discussion as to its limitations or as to why it should be a good approximation under these limitations. The present analysis shows that it should be a good approximation under the restriction of large A.

4. COMPARISON WITH EXPERIMENT

The first experimental data with which the present theory will be compared is that of reference [11]. An airfoil with a chord of 18 inches and a span of 21 inches was mounted between sideplates at zero angle of attack in the UARL Acoustic Tunnel. A turbulence-generating grid was placed upstream of the airfoil. Measurements of the turbulence in the test section of the tunnel showed that the turbulence properties were quite closely approximated by an isotropic homogeneous turbulence model [12]. The integral scale, L, of the turbulence, defined as

$$L = \int_0^\infty R_{uu}(x)\,dx, \tag{25}$$

was 1·25 inches. Streamwise turbulent intensity, $(\overline{u^2})^{1/2}/U$, was 4·4% for $U = 103$ ft/s, and was approximately equal to the transverse turbulent intensity. For the higher velocities the intensity of the turbulence followed approximately the expression

$$\sqrt{\overline{u^2}}/U = CU^{-0\cdot 2}. \tag{26}$$

Third octave sound measurements were made with a microphone 7 feet directly above the airfoil. The signal was higher than the background level by more than 3 dB up to about 2·5 kHz which was taken as the upper frequency limit for the data. The background level was subtracted from the signal level to give the data shown in Figure 4. The lower frequency limit was set by the anechoic cutoff of the test chamber which is about 200 Hz.

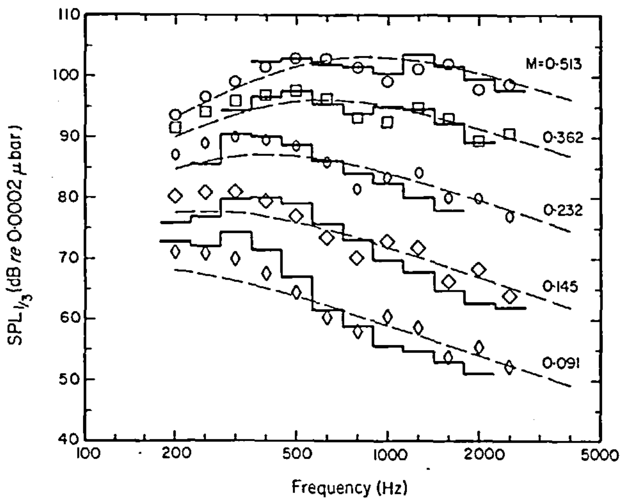

Figure 4. Experimental acoustic measurements, ——, *versus* results from theory when Karman turbulence model is used. ——, Theory with equation (44) used for airfoil response function; Symbols, theory with equations (42) and (43) used for airfoil response. (*Note added in proof.* An extra factor of β inadvertently was introduced in the calculations of the symbols in this Figure. Thus the circles should be raised by 1·3 dB, the squares by 0·6 dB and the other symbols by an insignificant amount.)

The frequency parameter $M\hat{\omega}/\beta^2$ discussed in Appendix 2 relative to the choice of airfoil theories thus varied from about 0·8 at low frequency to 11·0 at high frequency. For the present data this parameter thus lies above the value of $\pi/4$ given in Appendix 2 as the dividing line between the low-frequency and the high-frequency airfoil response functions, and the high-frequency solution was used for the present comparison.

If the high-frequency asymptote, equation (44), of the high-frequency solution is used along with the Karman model for the turbulence spectrum, the following simple expression for the PSD of the noise can be derived:

$$G_{PP}(0,0,z,\omega) = \frac{2d}{\pi c_0}\left(\frac{2L}{3\pi z}\right)^2 \frac{\overline{u^2}}{U^2}(\rho_0 U^2)^2 \left[\frac{\Gamma(1/3)}{\Gamma(5/6)}\right]^2 \frac{\hat{K}_x^2}{(1+\hat{K}_x^2)^{7/3}}, \tag{27}$$

where a factor of 2 has been introduced since G_{PP} represents the PSD for positive frequencies only: i.e., $G_{PP} = 2S_{PP}$. For comparison with data this first must be converted to third octave form by multiplication by a third octave bandwidth: i.e., $\Delta\omega = 2\pi\Delta f = 2\pi(0{\cdot}232)f$. The equation for the one-third octave level, $\mathrm{SPL}_{1/3}$, in dB relative to a pressure of 2×10^{-4} microbar is then

$$\mathrm{SPL}_{1/3} = 10\log_{10}\left[\frac{Ld}{z^2}M^5\frac{\overline{u^2}}{U^2}\frac{\hat{K}_x^3}{(1+\hat{K}_x^2)^{7/3}}\right] + 181{\cdot}3. \tag{28}$$

This equation is plotted as the dashed line in Figure 4, and shows quite good agreement with the data, especially in view of the fact that there are no adjustable parameters in the solution. The only input parameters, other than the Mach number and the airfoil geometry, are the integral scale length, L, and the intensity, $\overline{u^2}$, of the turbulence, both measured independently of the sound.

Equations (27) and (28) use the high frequency asymptote, equation (44), of the high frequency solution because of its simplicity. Presumably the full high frequency solution would give more accurate results. For this reason, a second calculation based on the second order high frequency solution (equations (41) and (42)) was made. The results for the sound based on this more exact airfoil theory are shown as the symbols in Figure 4. Agreement with the data is improved significantly at the lower frequencies.

It will be noted that for the lower velocities and higher frequencies the theoretical solution tends to overpredict the sound by a few decibels. One possible explanation for this effect is that the gust wavelength in this range is of the order of the leading edge radius of the airfoil. In particular, the gust wavelength is given by U/f which for a velocity of 100 ft/s and a frequency of 2400 Hz gives a gust wavelength of 1/2 inch as compared to the 1/2 inch leading edge radius of the airfoil. Given this fact, one might expect the airfoil response function used here to become inaccurate in this region. This could be checked by using an airfoil with a smaller leading edge radius.

The parameter $\Lambda \equiv MK_x d$ given as the parameter determining the applicability of the present theory varies from about 1·0 at 200 Hz to 10·0 at 2000 Hz. Although the present

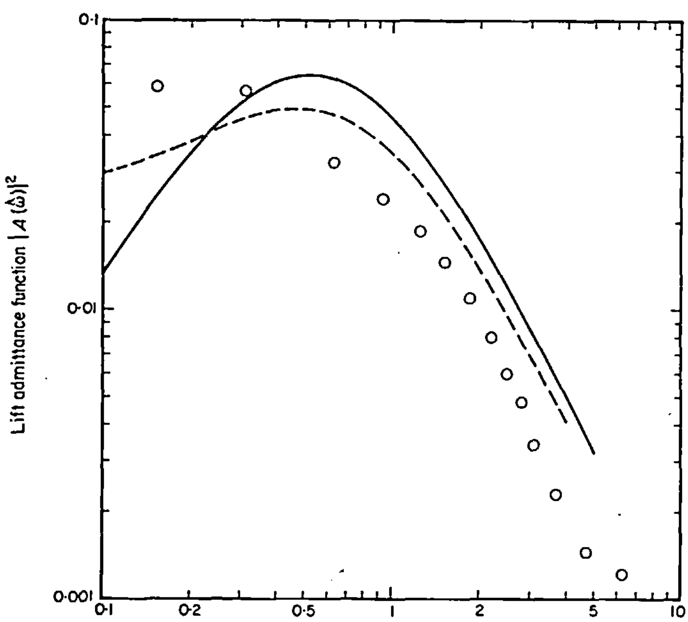

Figure 5. Experimental lift measurements, o, *versus* results from theory when Karman turbulence model is used. ——, Present theory with Sears function used for airfoil response; – – –, Graham's calculations in which three-dimensional incompressible airfoil theory is used.

theory was derived under the assumption of large Λ, the fact that the solution agrees quite well with the data even when $\Lambda = 1$ indicates that this condition may not be too restrictive.

Another indication of the effect of the value of Λ on the applicability of the present theory is obtained by considering the data of reference [8]. This describes an experiment in which the unsteady lift on an airfoil in a turbulent flow was measured, allowing comparison with equation (24) herein.

The airfoil had a chord 152 mm, a span of 406 mm, and a ratio of turbulent integral scale to chord of 0·41. The Reynolds number was $1{\cdot}6 \times 10^4$, giving a Mach number which is very small (approximately 0·004). Thus, the parameter Λ is much smaller than 1.

Figure 5 shows the comparison between theory and experiment. Since the Mach number was small, in the calculation presented here the Sears function was used for the airfoil response. Together with equation (23) for the Karman turbulence spectrum, equation (24) gives the solid line shown in Figure 5.

The dashed line represents calculations made by using the full three-dimensional airfoil theory together with the Karman spectrum. (This calculation was supplied by Graham in a private communication with the author. The calculations presented in reference [8] were based on a different turbulence spectrum model.) This calculation represents the lift that would be experienced by an airfoil section of aspect ratio 2·67 which is a segment of an airfoil of infinite aspect ratio. The experimental data shown by the symbols correspond to this situation in that dummy wings of the same cross-section and chord were fitted between the lift sensing element and the wind tunnel walls.

In view of the fact that Λ is small, the agreement between the present theory and the three-dimensional theory is reasonable, especially at higher frequency. The present theory shows somewhat more disagreement with the data. As was noted in reference [8], however, the turbulence was found to deviate somewhat from the isotropic assumption, and this fact explains most of the discrepancy between the experiment and the three-dimensional theory.

For this data, the parameter Λ is small principally because M is small. If Λ were made small by making $K_x d$ small, the agreement between experiment and theory might not be so good. This begins to become apparent at the lower frequencies in Figure 5 where $K_x d \sim 0{\cdot}3$.

5. CONCLUSIONS

The present theory for the sound generated by an airfoil in a turbulent stream agrees well with data when $\Lambda \equiv MK_x d > 1$. For very small values of Λ, comparison to measurements of unsteady lift shows agreement that is reasonable in view of the fact that the present theory was derived for large values of Λ. However, small values of $K_x d$ would be less likely to give good agreement. Also, the theory predicts the overall level when Md/L is large.

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APPENDIX 1: ISOTROPIC TURBULENCE

The two most widely used models of isotropic turbulence are the Karman and the Liepmann models. Without going into the details of derivation which can be found in texts such as that of Hinze [6], the necessary relations will be briefly summarized here.

A. KARMAN SPECTRUM

The equation for the energy spectrum, $E(k)$, as a function of the magnitude, k, of the wavevector is

$$E(k) = \frac{Ik^4}{[1+(k/k_e)^2]^{17/6}}, \tag{29}$$

where

$$I = \frac{55}{9\sqrt{\pi}} \frac{\Gamma(5/6)\,\overline{u^2}}{\Gamma(1/3)k_e^5}, \quad k_e = \frac{\sqrt{\pi}}{L}\frac{\Gamma(5/6)}{\Gamma(1/3)}.$$

As a function of the wavevector components, the energy spectrum, $\Phi_{ww}(k_x,k_y,k_z)$, of the vertical velocity fluctuations is related to $E(k)$ by

$$\Phi_{ww}(k_x,k_y,k_z) = \frac{E(k)}{4\pi k^2}(1 - k_z^2/k^2). \tag{30}$$

By integration over k_z, the function $\Phi_{ww}(k_x,k_y)$ introduced in equation (9) is found to be

$$\Phi_{ww}(k_x,k_y) = \frac{4}{9\pi}\frac{\overline{u^2}}{\hat{k}_e^2}\frac{\hat{k}_x^2+\hat{k}_y^2}{(1+\hat{k}_x^2+\hat{k}_y^2)^{7/3}}, \tag{31}$$

where $\hat{k} = k/k_e$. Taking the Fourier transform of $\Phi_{ww}(k_x,k_y)$ with respect to k_y gives

$$R_{ww}(k_x,y) = \frac{2^{1/6}}{\Gamma(5/6)}(\tilde{y})^{5/6}\left[\mathrm{K}_{5/6}(\tilde{y}) - \frac{3\tilde{y}}{3+8\hat{k}_x^2}\mathrm{K}_{1/6}(\tilde{y})\right]R_{ww}(k_x,0), \tag{32}$$

where $\tilde{y} = yk_e\sqrt{1+\hat{k}_x^2}$,

$$R_{ww}(k_x,0) = \frac{\overline{Lu^2}}{6\pi}\frac{3+8\hat{k}_x^2}{(1+\hat{k}_x^2)^{11/6}}. \tag{33}$$

The length scale, $l_y(\omega)$, can be found from equations (19), (31) and (33) to be

$$l_y(\omega)=\frac{8L}{3}\left[\frac{\Gamma(1/3)}{\Gamma(5/6)}\right]^2\frac{\hat{K}_x^2}{(3+8\hat{K}_x^2)\sqrt{1+\hat{K}_x^2}}. \tag{34}$$

B. LIEPMANN SPECTRUM

The corresponding quantities for the Liepmann spectrum are

$$E(k)=\frac{8}{\pi}\overline{u^2}L\frac{(Lk)^4}{(1+L^2k^2)^3}, \tag{35}$$

$$\Phi_{ww}(k_x,k_y)=\frac{3\overline{u^2}}{4\pi}L^2\frac{L^2(k_x^2+k_y^2)}{[1+L^2(k_x^2+k_y^2)]^{5/2}}, \tag{36}$$

$$R_{ww}(k_x,y)=\left[\hat{y}\mathrm{K}_1(\hat{y})-\frac{\hat{y}^2\mathrm{K}_0(\hat{y})}{1+3k_x^2L^2}\right]R_{ww}(k_x,0), \tag{37}$$

where $\hat{y}=y\sqrt{1+k_x^2L^2}/L$,

$$R_{ww}(k_x,0)=\frac{\overline{u^2}L}{2\pi}\frac{1+3k_x^2L^2}{(1+k_x^2L^2)^2}, \tag{38}$$

$$l_y(\omega)=\frac{3\pi L}{2\sqrt{1+K_x^2L^2}}\frac{K_x^2L^2}{1+3K_x^2L^2}. \tag{39}$$

APPENDIX 2: AIRFOIL RESPONSE FUNCTIONS

In order to calculate final expressions for the airfoil sound, expressions for the airfoil response function must be given. As discussed previously, because of the similarity rules presented by Graham [9], only the airfoil response to a two-dimensional gust in a compressible fluid is needed. At low Mach number, M, and reduced frequency, $\hat{\omega}$, the Sears function can be used. In many cases a solution will be needed for larger values of M and $\hat{\omega}$. Two solutions are presented below for different ranges of the parameter $M\hat{\omega}/\beta^2$. Both these solutions show good agreement with the numerical solutions of references [9] and [13]. Only the solutions for the lift are presented here. In order to calculate the function $\mathscr{L}$ of equation (14), one must know the pressure distribution on the airfoil. Further discussion of the solutions presented below, including the airfoil pressure distribution, is given in references [4], [5], [9] and [13]. The lift solutions below have the gust referenced to the leading edge of the airfoil rather than the midchord, which eliminates the familiar spiral nature of the Sears function. Also, the lift has been normalized by the factor $2\pi\rho_0 bUw_0$.

RANGE 1: $M\hat{\omega}/\beta^2<\pi/4$

Presented here is an approximate solution in which terms of order $(M\hat{\omega}/\beta^2)^2$ and higher are neglected. Reference [4] gives for the airfoil lift response to a sinusoidal gust

$$G(\hat{\omega})=\left\{\frac{1}{\beta}S(\hat{\omega}/\beta^2)\,[\mathrm{J}_0(M^2\hat{\omega}/\beta^2)-\mathrm{i}\mathrm{J}_1(M^2\hat{\omega}/\beta^2)]\right\}\mathrm{e}^{-\mathrm{i}\hat{\omega}}, \tag{40}$$

where $S(\hat{\omega})$ is the well known Sears function. The limitation on the range of $\hat{\omega}M/\beta^2$ has a physical interpretation for small M: i.e., as $M\to 0$ and $\beta\to 1$, the range can be interpreted as limiting the airfoil chord to 1/4 the acoustic wavelength.

RANGE 2: $M\hat{\omega}/\beta^2>\pi/4$

For this range a successive approximation solution procedure first described by Landahl

[14] can be used. The first solution involves treating the airfoil as a flat plate with a semi-infinite chord. Thus, there is no trailing edge and the Kutta condition cannot be applied. The second solution treats the airfoil as a semi-infinite flat plate with a trailing edge but no leading edge. The pressure jump at the trailing edge and in the wake is corrected to zero with a resulting incorrect pressure jump upstream of the airfoil which could be corrected by a third solution, etc. The first two terms of the solution derived by Adamczyk [13] are

$$G_1(\hat{\omega}) = \frac{1-\mathrm{i}}{\pi\hat{\omega}\sqrt{M}}\mathrm{E}^*\left(\sqrt{\frac{4\hat{\omega}M}{\pi(1+M)}}\right), \tag{41}$$

$$G_2(\hat{\omega}) \simeq \frac{\sqrt{1+M}}{\mathrm{i}M(\pi\hat{\omega})^{3/2}}\left\{\mathrm{E}^*\left(\frac{2}{\beta}\sqrt{2\hat{\omega}M/\pi}\right) - \frac{1-\mathrm{i}}{2} + \left[\frac{1-\mathrm{i}}{2} - \sqrt{\frac{2}{1+M}}\mathrm{E}^*\left(\sqrt{\frac{4\hat{\omega}M}{\pi(1-M)}}\right)\right]\mathrm{e}^{-\mathrm{i}(2\hat{\omega}M/1+M)}\right\}, \tag{42}$$

where

$$\mathrm{E}^*(x) = \mathscr{C}(x) - \mathrm{i}\mathscr{S}(x)$$

$$\mathscr{C}(x) = \int_0^x \cos\left(\frac{\pi}{2}\xi^2\right)\mathrm{d}\xi \quad \mathscr{S}(x) = \int_0^x \sin\left(\frac{\pi}{2}\xi^2\right)\mathrm{d}\xi. \tag{43}$$

The functions $\mathscr{C}$ and $\mathscr{S}$ are Fresnel integrals. For $\hat{\omega} \to \infty$

$$G_1(\hat{\omega}) \to \frac{-\mathrm{i}}{\pi\hat{\omega}\sqrt{M}}. \tag{44}$$

APPENDIX 3: LIST OF SYMBOLS

$A(\ )$	lift admittance function; see equation (24)
b	semi-chord
c_0	sound speed
C	constant
d	semi-span
$E(k)$	energy spectrum of turbulence
$E[\ldots]$	expected value
$\mathrm{E}^*(\ )$	combination of Fresnel integrals; see equation (43)
$g(x,k_x,k_y)$	airfoil pressure distribution for sinusoidal gust
$G(\hat{\omega})$	airfoil lift response for sinusoidal gust
$G_{PP}(\omega)$	PSD for positive frequency only; $= 2S_{PP}(\omega)$
J_0, J_1	Bessel functions of the first kind
$\mathbf{k}$	unit vector in the z direction
k_e	wavenumber defined by equation (29)
$\hat{k}$	wavenumber normalized by k_e
k_x, k_y	chordwise and spanwise wavenumbers of turbulence
K_x	particular value $k_x = -\omega/U$ for chordwise wavenumber
K_ν	Bessel function of imaginary argument
$l_y(\omega)$	frequency dependent correlation length of turbulence
$\mathscr{L}$	integral of airfoil pressure distribution; see equation (14)
L	integral scale length of turbulence; see equation (25)
M	free stream Mach number
P	pressure
ΔP	pressure jump across airfoil
$R_{uu}(x)$	axial cross-correlation of u
$R_{ww}(k_x,y)$	spanwise cross-correlation of w
R, T	integration limits eventually set to infinity

S_{PP}	PSD of far field sound
S_{QQ}	cross-PSD of airfoil loading
$S_{ww}(\omega)$	PSD of z velocity fluctuations; $= UR_{ww}(K_x, 0)$
$S(\omega)$	Sears function
u	axial velocity fluctuation
U	free stream velocity
V	velocity of gust-airfoil intersection
w	velocity fluctuation normal to airfoil
x, y, z	axial, spanwise, and normal Cartesian co-ordinates
β	$\sqrt{1-M^2}$
$\Gamma(\ )$	gamma function
$\delta(\ )$	delta function
η	spanwise separation
Λ	$MK_x d$
ρ_0	density
σ	$\sqrt{x^2+\beta^2(y^2+z^2)}$
ω	circular frequency
$\hat{\omega}$	reduced frequency $\omega b/U$
^	denotes either dimensionless variable or Fourier transform