MAE 298 – Homework #2
Lilley's Equation Solution and Application to Jet Noise

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Abstract about homework

Nomenclature

θ_{max}	Maxumum angle of pendulum oscillation
f	Frequency of pendulum oscillation
V_∞	Freestream velocity
V_T	Tangential velocity
V_{eff}	Effective velocity
α_G	Global (inertial) angle of attack
α_{eff}	Effective angle of attack
θ	Pitch angle
ϕ	Aerodynamic clock angle
$\dot{\theta}$	Pitch rate
α_T	Total angle of attack
$\dot{\alpha}_T$	Total angle of attack rate
C_X	Inertial frame X_i axis force coefficient
C_Z	Inertial frame Z_i axis force coefficient
C_A	Body frame X_b axis force coefficient
C_Y	Body frame Y_b axis force coefficient
C_N	Body frame Z_b axis force coefficient
$C_{N,m}$	Missile frame Z_b axis force coefficient
m_a	Apparent mass of air within parachute
ρ_∞	Freestream density
L_{ref}	Reference length (parachute diameter
R	Radius of parachute oscillation
$MPCV$	
$CPAS$	
$NFAC$	
EDU	
WTT	
FFT	
DCF	

I. Simple Pendulum

The oscillating Orion parachute is a complex system involving induced aerodyanmic forces, inertial responses, and, in some simulations, prescribed, motion-governing forces. Before delving into these intracacies, it is beneficial to first discuss the simplified problem of a gravetational pendulum (i.e. a mass on a string).

I.A. Free Body Diagram

In this example, a mass is held against gravity by a string attached to a roof. Some initial displacement from the neutral vertial position causes this system to exhibit simple pendulum motion due to the resulting force balance demonstrated in the free body diagram.

II. Process Raw OVERFLOW Data

Before the calculation of dynamic parameters, the raw run condition and force data output by the OVERFLOW CFD solver must be processed.

II.A. Read Case Data

Parameters pertaining to the run conditions are aquired by processing the output of the "'qinfo" script, which provides Mach number M, Reynold"s number Re, freestream specific heat ratio γ, freestream static temperature T_∞, freestream density ρ_∞, freestream dynamic pressure q_∞, OVERFLOW angle of attack α_{OVR}, and OVERFLOW sideslip angle β_{OVR}, among other parameters. Freestream parameters are converted into the grid reference unit of inches.

Parameters for simple pendulum oscillations including maximum amplitude θ_{max} and oscillation frequency f are read from the case"s "Scenario.xml" file. α_{OVR} is converted from the capsule reference frame to the parachute reference frame according to Eqn. 1.

$$\alpha_G = 180^o - \alpha_{OVR} \tag{1}$$

II.B. Condition Raw Data

Raw OVERFLOW data must also be conditioned before aerodynamic calculations are performed. All force data is read from "aero.chute" which is produced by the "overlst -p" function call. Raw data is first trimmed of any duplicates. Force and moment coefficients are then corrected for actual reference parameters as show in Eqn. 3 that may be different than those contained in "mixsur.i", which is used as input for surface integration during OVERFLOW solver iterations.

$$\begin{aligned} C_F =& C_{F,OVR}\frac{A_{ref,old}}{A_{ref,new}} \\ C_M =& C_{M,OVR}\frac{A_{ref,old}}{A_{ref,new}}\frac{l_{ref,old}}{l_{ref,new}} \end{aligned} \tag{2}$$

Cases that are run with a half-body geometry symmetric about the y-axis are then corrected for symmetry effects. Forces and moments that are expected to be non-zero must be doubled, to account for half of the geometry missing, and forces that are expected to be zero due to geometry symmetry must be forced to zero.

$$\begin{aligned} C_X =& 2 \cdot C_{X,OVR} \\ C_Y =& 0 \\ C_Z =& 2 \cdot C_{Z,OVR} \\ C_l =& 0 \\ C_m =& 2 \cdot C_{m,OVR} \\ C_n =& 0 \end{aligned} \tag{3}$$

Because these simulations are time accurate and have relative mesh motion ($DYNMCS = .T.$), it is neccessary to caclulate the simulation time from the iteration history. Non-dimensional simulation time can be calculated can be calculated by multiplying the iteration number $Iter$ subtracted by the iteration where grid motion began i_{start} by the non-dimensional time step $DTPHYS$, as shown in Eqn. 4. Dimensional time in seconds can then be calculated by multiplying by the reference length of one inch and dividing by the reference velocity, which is equal to the freestream velocity V_∞.

$$\begin{aligned} t_{non} &= (Iter - i_{start}) \cdot DTPHYS \\ t &= \left(\frac{L_{ref}}{V_{ref}}\right) t_{non} \end{aligned} \tag{4}$$

III. Process Dynamic Parameters

With OVERFLOW outputs corrected and brought into the time domain, actual aerodynamic properties can now be calculated.

III.A. Prescribed Motion Dynamics

First, attitude angles and rates are calculated according to the prescribed motion equation contained in "Scenario.xml". For these simulations, where oscillations are contained within the y-plane, the only attitude parameters to calculate are those pertaining to pitching motion θ and aerodynamic clock angle ϕ. θ and the associated rates can be calculated based on the simulation time for the simple, harmonic pendulum according to Eqn. 5 and for the Wind Tunnel Test (WTT) Track case according to Eqn. 6. The simple pendulum is constrained to simple, harmonic motion defined by a frequency and an amplitude, while the WTT Track case is designed to approximate the motion of the parachute wind tunnel test and is derived using a Fast Fourier Transform (FFT).

$$\begin{gathered}\omega = 2\pi f \\ \theta(t) = \theta_{max} \cdot \sin(\omega t) \\ \dot{\theta}(t) = \theta_{max} \cdot \omega \cdot \cos(\omega t) \\ \ddot{\theta}(t) = -\theta_{max} \cdot \omega^2 \cdot \sin(\omega t)\end{gathered} \tag{5}$$

where $\dot{\theta}$ is the pitch rate of the parachute, θ_{max} is the maximum angular amplitude of the oscillation, f is the frequency of the oscillation in Hertz, and t is the simulation time.

$$\begin{gathered}\omega_1 = 2\pi(0.186246),\ \omega_2 = 2\pi(0.200573) \\ A_1 = 0.011695,\ A_2 = 4.268271,\ A_3 = 1.001533,\ A_4 = 4.983758 \\ \theta(t) = -A_1 \sin(\omega_1 t) - A_2 \cos(\omega_1 t) + A_3 \sin(\omega_2 t) + A_4 \cos(\omega_2 t) \\ \dot{\theta}(t) = -A_1\omega_1 \cos(\omega_1 t) + A_2\omega_1 \sin(\omega_1 t) + A_3\omega_2 \cos(\omega_2 t) - A_4\omega_2 \sin(\omega_2 t) \\ \ddot{\theta}(t) = A_1\omega_1^2 \sin(\omega_1 t) + A_2\omega_1^2 \cos(\omega_1 t) - A_3\omega_2^2 \sin(\omega_2 t) - A_4\omega_2^2 \cos(\omega_2 t)\end{gathered} \tag{6}$$

where ω are the natural frequencies of the oscillation and A are the associated Fourier coefficients.

The rotational (tangential) velocity of the parachute V_T is calculated from the rate of pitching motion $\dot{\theta}$ and the fixed rotational radius $R = 2714.61in$, as depicted in Eqn. 7. V_T is in the normal body direction (Z_b-axis) and will always have the same sign as the pitching motion. Eqn. 7 produces the tangential velocity of the actual parachute, so this value will be negated to produce the relative wind velocity in the tangential direction, which is demonstrated in Fig. **??**. (Derivation of the tangential velocity can be found later on in Eqn. 25).

$$V_T = R\dot{\theta} \tag{7}$$

III.B. Aerodynamic Parameters

With all of the velocities and angular rates calculated, actual aerodynamic parameters such as effective velocity V_{eff} and effective angle of attack α_{eff} can be determined. Referencing Fig. **??**, it can be seen that the angle V_∞ makes with the body axis X_b of the parachute at any given time is equivalent to $\alpha_G - \theta$. Thus, the resulting relative air velocity components in the body frame axes are computed according to Eqn. 8.

$$\begin{gathered}V_{X_b,rel} = -V_\infty \cos(\alpha_G - \theta) \\ V_{Z_b,rel} = V_\infty \sin(\alpha_G - \theta) - V_T\end{gathered} \tag{8}$$

where velocity components are of the air surrounding the parachute and are in the opposite respective directions of the actual motion of the parachute.

From the component velocities, the effective relative velocity of the parachute V_{eff} can be calculated as the magnitude as in Eqn. 9.

$$V_{eff} = \sqrt{[V_{X_b,rel}]^2 + [V_{Z_b,rel}]^2} \tag{9}$$

The effective angle of attack is calculated from the ratio of the body velocity components.

$$\alpha_{eff} = \tan^{-1}\left[\frac{V_{Z_b,rel}}{-V_{X_b,rel}}\right] \tag{10}$$

where $V_{X_b,rel}$ is negated due to the direction of the X_b-axis to maintain consistency with the definition of angle of attack.

For a better comparison with the wind tunnel test, angle of attack is also calculated in the total frame, as seen in Eqn. 11.

$$\alpha_T = \cos^{-1}\left[\frac{V_{X_b,rel}}{V_{eff}}\right] \tag{11}$$

Because there is no side-slip in the CFD model, α_T is effectively the absolute value of α in these simulations.

Aerodynamic rates ($\dot{\alpha}$, $\dot{\alpha}_T$) are calculated from the α, α_T data series using numerical differencing, and is shown in Eqn. 12.

$$\dot{\alpha}_{T,i} = \frac{\alpha_{T,i+1} - \alpha_{T,i-1}}{2\Delta t} \tag{12}$$

Central differencing is used to calculate all middle derivatives, while forward and backward differencing are used to calculate the first and last derivatives in the time series, respectively.

To more closely compare to the wind tunnel test, $\dot{\alpha}_T$ is non-dimmensionalized according to Eqn. 13 by a reference length equal to the parachute skirt radius and the effective velocity V_{eff} of the parachute at the given instant in time.

$$\dot{\alpha}_{T,non} = \frac{|\dot{\alpha}_T| l_{ref}}{2V_{eff}} \tag{13}$$

Finally, the aerodynamic clock angle ϕ, which is defined in Eqn. 14, must be calculated to allow coordinate rotations into the missile frame.

$$\phi = \tan^{-1}\left(\frac{-V_{Y_b,rel}}{V_{Z_b,rel}}\right) \tag{14}$$

Though $V_{Y_b,rel}$ is always zero for the in-plane oscillations of the CFD simulations (as will C_Y), the changing sign of $V_{Z_b,rel}$ makes it such that ϕ will be either 0 or 2π, making the missile rotation matrix in Eqn. 18 in Section III.C either the identity or negative identity matrix.

III.C. Force and Moment Coefficient Rotations

Because the body and missile coordinate systems will be used to compare CFD results to the WTT, a series of coordinate frame rotations are required to transform OVERFLOW C_X, C_Z to C_A, $C_{N,m}$. However, all coeffiicents must first be corrected for inaccurate non-dimensionalization by the freestream dynamic pressure. Because OVERFLOW does not bookkeep the relative velocity of the grids during the simulation, all coefficients are non-dimensionalized simply by the freestream dynamic pressure q_∞ though they should actually be non-dimensionalized by the dynamic pressure of the flow in the frame of the actual parachute, defined in Eqn. 15.

$$q = {}^1\!/_2 \rho V_{eff}^2 \tag{15}$$

Assuming incompressible flow ($\rho = \rho_\infty$), this correction can be expressed simply as the ratio of the squares of the freestream and effective velocities, as derived in Eqn. 16.

$$\begin{gathered} C_F = \frac{F}{q} = \frac{F}{q_\infty} \cdot \frac{q_\infty}{q} = C_{F,OVR} \cdot \frac{q_\infty}{q} \\ \frac{q_\infty}{q} = \frac{\frac{1}{2}\rho_\infty V_\infty^2}{\frac{1}{2}\rho_\infty V_{eff}^2} = \left(\frac{V_\infty}{V_{eff}}\right)^2 \end{gathered} \tag{16}$$

After the non-dimensionalization correction, forces can then be rotated first from the inertial frame to the body frame, and then from the body frame into the missile frame

The rotation into the body frame shown in Fig. **??** and derived in Eqn. 17 is a simple, 2D rotation about the pitch axis for the CFD cases, as the yaw angle is always zero.

$$\begin{bmatrix} C_A \\ C_N \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} C_X \\ C_Z \end{bmatrix} \tag{17}$$

The missile frame rotation in Eqn 18, as previously explained, manifests as either the identity or negative identity matrix, effectively causing a discontinuity in the normal force coefficient C_N.

$$\begin{bmatrix} C_{Y,m} \\ C_{N,m} \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} C_Y \\ C_N \end{bmatrix} \tag{18}$$

IV. Apparent Mass Adjustment

To maintain consistency with the wind tunnel test, an apparent mass adjustment is applied the body forces to account for the displacement of the air trapped within the parachute as it pitches.

IV.A. Apparent Force

The apparent mass of the trapped air m_a is calculated as the mass of the air within a hemisphere of diameter L_{ref} multiplied by a scale factor, as seen in Eqn. 19.

apparent force diagram

$$m_a = \frac{1}{2}\left[\frac{4}{3}\pi\left(0.7\frac{L_{ref}}{2}\right)^3\right]\cdot \rho_\infty \tag{19}$$

The force the parachute exerts on this mass is then calculated according to Newton"s second law, using the axial and normal accelerations in Eqn. 20 derived from the precribed equations of motion.

$$\bar{a}_b = R\dot{\theta}^2\hat{X}_b + R\ddot{\theta}\hat{Z}_b \tag{20}$$

By Newton"s third law, this force must be subtracted from the aerodynamic forces acting on the parachute calculated by surface integration in OVERFLOW, resulting in Eqn. 21 for the final expression for the aerodynamic forces on the parachute.

$$\Sigma\overline{F} = \overline{F}_{aero} - m_a\bar{a}_b \tag{21}$$

IV.B. Accelerations

To calculate the forces required to accelerate the apparent mass, it is necessary to derive the equations of motion pertaining to the parachute"s oscillation. Representing the parachute as a single point rotating at a distance R behind its attachement point in the negative $\hat{X}_b$ direction, the expression for its position is given in Eqn. 22.

$$\bar{r}_b = -R\overline{\hat{X}_b} \tag{22}$$

The velocity of the parachute due to oscillations in θ can be determined by differentiating Eqn. 22 with respect to time. There is no change in the radius in these simulations, so the relative acceleration term in Eqn. 23 goes to zero, leaving the derivative of the $\hat{X}_b$ unit vector.

$$\bar{v}_b = \frac{{}^B d}{dt}\bar{r}_b = \cancelto{0}{\frac{{}^B dr}{dt}} - R\frac{d\hat{X}_b}{dt} \tag{23}$$

From Eqn. 17, $\hat{X}_b$ can be substituted into terms of inertial frame unit vectors and simplified as in Eqn. 24.

$$
\begin{aligned}
&= -R \cdot \frac{d}{dt}\left(\cos\theta \hat{X}_i - \sin\theta \hat{Z}_i\right) \\
&= -R\left(-\dot{\theta}\sin\theta \hat{X}_i - \dot{\theta}\cos\theta \hat{Z}_i\right) \\
&= R\dot{\theta}\underbrace{\left(\sin\theta \hat{X}_i + \cos\theta \hat{Z}_i\right)}_{\hat{Z}_b}
\end{aligned} \tag{24}
$$

$$\boxed{\overline{v}_b = R\dot{\theta}\hat{Z}_b}$$

Finally, the velocity can be differentiated to produce the acceleration components in the body axis system.

$$
\begin{aligned}
\overline{a}_b \quad &= \frac{{}^B d}{dt}\overline{v}_b = \frac{{}^B d}{dt}(R\dot{\theta}\hat{Z}_b) = R\left[\ddot{\theta}\hat{Z}_b + \dot{\theta}\frac{d\hat{Z}_b}{dt}\right] \\
&= R\left[\ddot{\theta}\hat{Z}_b + \dot{\theta}\frac{d}{dt}(\sin\theta \hat{X}_i + \cos\theta \hat{Z}_i)\right] = R\left[\ddot{\theta}\hat{Z}_b + \dot{\theta}(\dot{\theta}\cos\theta \hat{X}_i - \dot{\theta}\sin\theta \hat{Z}_i)\right] \\
&= R[\ddot{\theta}\hat{Z}_b + \dot{\theta}^2\underbrace{(\cos\theta \hat{X}_i - \sin\theta \hat{Z}_i)}_{\hat{X}_b}]
\end{aligned} \tag{25}
$$

$$\boxed{\overline{a}_b = R\dot{\theta}^2\hat{X}_b + R\ddot{\theta}\hat{Z}_b}$$