MAE 298 Aeroacoustics – Homework #2
Lilley's Equation Solution and Application to Jet Noise

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Abstract about homework

Nomenclature

R_j	Axisymmetric jet radius	E	Exponential term: $kz + n\theta - \omega t$
W_j	Jet exit mean velocity	P	
$\overline{\rho_j}$	Jet exit mean density	k	
W_0	Ambient mean velocity	ω	
$\overline{\rho_0}$	Ambient mean density	n	
$W(r)$	Radial distribution of axial velocity	r	
$\overline{a^2}(r)$	Radial distribution of speed of sound squared	θ	
p'	Perturbation pressure	z	
i	Imaginary number $\sqrt{-1}$		

I. Background

Lilley's equation for a parallel axisymmetric flow (Eqn 1):

$$\left(\frac{\partial}{\partial t} + W\frac{\partial}{\partial z}\right)^3 p' - \left(\frac{\partial}{\partial t} + W\frac{\partial}{\partial z}\right)\left(\overline{a^2}\nabla^2 p'\right) - \frac{d\overline{a^2}}{dr}\left(\frac{\partial}{\partial t} + W\frac{\partial}{\partial z}\right)\frac{dp'}{dr} + 2\overline{a^2}\frac{dW}{dr}\frac{\partial^2 p'}{\partial z \partial r} = S(\vec{x}, t) \tag{1}$$

$$\text{where } \nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

$W(r)$ is the radial distribution of axial velocity and $\overline{a^2}(r)$ is the radial distribution of speed of sound squared.

II. Problem 1 – Solution to Lilley's Equation

Seek solutions of Lilley's equation in the form:

$$p'(r, \theta, z, t) \sim P(r) \exp\left[i(kz + n\theta - \omega t)\right] \tag{2}$$

Assume:

1. $\overline{a^2} = \dfrac{\gamma \overline{p}}{\overline{\rho}}$
2. $\overline{p} = const$

where $\overline{\rho}(r)$ is the radial distribution of the mean density. Show that Lilley's equation reduces to (Eqn 3):

$$\frac{d^2P}{dr^2} + \left\{\frac{1}{r} - \frac{1}{\overline{\rho}}\frac{d\overline{\rho}}{dr} + \frac{2k}{(\omega - kW)}\frac{dW}{dr}\right\}\frac{dP}{dr} + \left\{\frac{(\omega - kW)^2}{\overline{a^2}} - k^2 - \frac{n^2}{r^2}\right\}P = RHS \tag{3}$$

To aid in this derivaiton, we find the results for the first-order partial derivatives of parameters relevant to the solution. First, we compute the derivative of the perturbation pressure p' with respect to (WRT) time t:

$$\begin{aligned}
\frac{\partial p'}{\partial t} &= \frac{\partial}{\partial t}\left\{P(r)\exp\left[i(kz + n\theta - \omega t)\right]\right\} \\
&= P\frac{\partial}{\partial t}[i(kz + n\theta - \omega t)]\exp[i(kz + n\theta - \omega t)] \\
&= iP\left[\cancelto{0}{\frac{\partial}{\partial t}(kz)} + \cancelto{0}{\frac{\partial}{\partial t}(n\theta)} - \frac{\partial}{\partial t}(\omega t)\right]\exp[i(kz + n\theta - \omega t)]
\end{aligned}$$

$$\boxed{\frac{\partial p'}{\partial t} = -iP\omega\exp[i(kz + n\theta - \omega t)] = -iP\omega e^{iE}} \tag{4}$$

where $E = kz + n\theta - \omega t$, $\dfrac{\partial E}{\partial t} = -\omega$, and $\dfrac{\partial E}{\partial z} = k$. Next, we compute the derivative of p' WRT the angular direction θ:

$$\begin{aligned}
\frac{\partial p'}{\partial \theta} &= \frac{\partial}{\partial \theta}\left\{P(r)\exp\left[i(kz + n\theta - \omega t)\right]\right\} \\
&= P\frac{\partial}{\partial \theta}[i(kz + n\theta - \omega t)]\exp[i(kz + n\theta - \omega t)] \\
&= iP\left[\cancelto{0}{\frac{\partial}{\partial \theta}(kz)} + \frac{\partial}{\partial \theta}(n\theta) - \cancelto{0}{\frac{\partial}{\partial \theta}(\omega t)}\right]\exp[i(kz + n\theta - \omega t)]
\end{aligned}$$

$$\boxed{\frac{\partial p'}{\partial \theta} = iPn\exp[i(kz + n\theta - \omega t)] = iPne^{iE}} \tag{5}$$

Next, we compute the derivative of p' WRT the axial flow direction z:

$$\begin{aligned}
\frac{\partial p'}{\partial z} &= \frac{\partial}{\partial z}\left\{P(r)\exp\left[i(kz + n\theta - \omega t)\right]\right\} \\
&= P\frac{\partial}{\partial z}[i(kz + n\theta - \omega t)]\exp[i(kz + n\theta - \omega t)] \\
&= iP\left[\frac{\partial}{\partial z}(kz) + \cancelto{0}{\frac{\partial}{\partial z}(n\theta)} - \cancelto{0}{\frac{\partial}{\partial z}(\omega t)}\right]\exp[i(kz + n\theta - \omega t)]
\end{aligned}$$

$$\boxed{\frac{\partial p'}{\partial z} = iPk\exp[i(kz + n\theta - \omega t)] = iPke^{iE}} \tag{6}$$

Next, we compute the derivative of p' WRT the radial direction r:

$$\frac{\partial p'}{\partial r} = \frac{\partial}{\partial r}\left\{P(r)\exp\left[i(kz+n\theta-\omega t)\right]\right\}$$

$$\boxed{\frac{\partial p'}{\partial r} = \frac{dP}{dr}\exp[i(kz+n\theta-\omega t)] = \frac{dP}{dr}e^{iE}} \tag{7}$$

II.A. Term 1

To simplify the derivation, we will apply the solution form individually to each term in Lilley's equation. For the first term, we must apply the multi-derivative operator a total of three times:

$$\begin{aligned}
\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)^3 p' &= \left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)^2\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)p' \\
&= \left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)^2\left(\frac{\partial p'}{\partial t}+W\frac{\partial p'}{\partial z}\right) \\
&= \left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)^2\left(-P\omega ie^{iE}+PkWie^{iE}\right) \\
&= \left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)(-\omega+kW)Pi(e^{iE}) \\
&= \left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)(-\omega+kW)Pi(-i\omega e^{iE}+ikWe^{iE}) \\
&= \left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)(-\omega+kW)^2Pi^2(e^{iE}) \\
&= (-\omega+kW)^3Pi^3(e^{iE}) = (-\omega+kW)^3P(-i)(e^{iE}) \\
&= (\omega-kW)^3iP(e^{iE})
\end{aligned}$$

The cubed imaginary number i^3 simplyfies to $-i$ and the -1 is distributed into the cubed factor. This results in the final expression for Term 1:

$$\boxed{\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)^3 p' = i\exp[i(kz+n\theta-\omega t)](\omega-kW)^3P} \tag{8}$$

II.B. Term 2

Application of the solution form to second term (Eqn 9) of Lilley's equation is slightly more involved.

$$\left(\frac{\partial}{\partial t} + W\frac{\partial}{\partial z}\right)(\overline{a^2}\nabla^2 p') \tag{9}$$

Term 2 requires computing the double divergence of perturbation pressure $\nabla^2 p'$:

$$\begin{aligned}
\nabla^2 p' &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p'}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p'}{\partial \theta^2} + \frac{\partial^2 p'}{\partial z^2} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{dP}{dr}e^{iE}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(Pnie^{iE}\right) + \frac{\partial}{\partial z}\left(Pkie^{iE}\right) \\
&= \frac{1}{r}e^{iE}\left(\frac{dP}{dr} + r\frac{d^2P}{dr^2}\right) + \frac{1}{r^2}Pn^2i^2e^{iE} + Pk^2i^2e^{iE} \\
&= e^{iE}\frac{d^2P}{dr^2} + e^{iE}\frac{1}{r}\frac{dP}{dr} + i^2e^{iE}\left(\frac{n^2}{r^2} + k^2\right)P
\end{aligned}$$

Thus, the double divergence of p' can be expressed in the following expression, which is seperated into like differential terms of P:

$$\boxed{\nabla^2 p' = \exp[i(kz + n\theta - \omega t)]\left[\frac{d^2P}{dr^2} + \frac{1}{r}\frac{dP}{dr} - \left(\frac{n^2}{r^2} + k^2\right)P\right]} \tag{10}$$

Substituting Eqn 10 into Term 2 (Eqn 9), we can perform the multi-derivative expression to derive the final term. All terms grouped with P as well as a are constant WRT t and z and can be carried outside of the derivative expression.

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + W\frac{\partial}{\partial z}\right)(\overline{a^2}\nabla^2 p') &= \left(\frac{\partial}{\partial t} + W\frac{\partial}{\partial z}\right)\overline{a^2}e^{iE}\left[\frac{d^2P}{dr^2} + \frac{1}{r}\frac{dP}{dr} - \left(\frac{n^2}{r^2} + k^2\right)P\right] \\
&= \overline{a^2}\left[\frac{d^2P}{dr^2} + \frac{1}{r}\frac{dP}{dr} - \left(\frac{n^2}{r^2} + k^2\right)P\right]\left(\frac{\partial}{\partial t} + W\frac{\partial}{\partial z}\right)e^{iE} \\
&= \overline{a^2}\left[\frac{d^2P}{dr^2} + \frac{1}{r}\frac{dP}{dr} - \left(\frac{n^2}{r^2} + k^2\right)P\right]\left(-\omega ie^{iE} + kWie^{iE}\right) \\
&= -\overline{a^2}ie^{iE}(\omega - kW)\left[\frac{d^2P}{dr^2} + \frac{1}{r}\frac{dP}{dr} - \left(\frac{n^2}{r^2} + k^2\right)P\right]
\end{aligned}$$

This results in the final expression for Term 2:

$$\boxed{\left(\frac{\partial}{\partial t} + W\frac{\partial}{\partial z}\right)(\overline{a^2}\nabla^2 p') = -\overline{a^2}i\exp[i(kz + n\theta - \omega t)](\omega - kW)\left[\frac{d^2P}{dr^2} + \frac{1}{r}\frac{dP}{dr} - \left(\frac{n^2}{r^2} + k^2\right)P\right]} \tag{11}$$

II.C. Term 3

Applying the solution form to third term of Lilley's equation:

$$\begin{aligned}
\frac{d\overline{a^2}}{dr}\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)\frac{dp'}{dr} &= \frac{d\overline{a^2}}{dr}\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)\frac{dP}{dr}e^{iE} \\
&= \frac{d\overline{a^2}}{dr}\frac{dP}{dr}\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)e^{iE} \\
&= \frac{d\overline{a^2}}{dr}\frac{dP}{dr}i(-\omega+kW)e^{iE}
\end{aligned}$$

Applying the isentropic relationship assumption for speed of sound and taking the derivative WRT r:

$$\begin{aligned}
\frac{d\overline{a^2}}{dr}\frac{dP}{dr}i(-\omega+kW)e^{iE} &= \frac{d}{dr}\left(\frac{\gamma\overline{p}}{\overline{\rho}}\right)\frac{dP}{dr}i(-\omega+kW)e^{iE} \\
&= -\left(\frac{\gamma\overline{p}}{\overline{\rho}^2}\right)\frac{d\overline{\rho}}{dr}\frac{dP}{dr}i(-\omega+kW)e^{iE} \\
&= \left(\frac{\overline{a^2}}{\overline{\rho}}\right)\frac{d\overline{\rho}}{dr}\frac{dP}{dr}i(\omega-kW)e^{iE}
\end{aligned}$$

This results in the final expression for Term 3:

$$\boxed{\frac{d\overline{a^2}}{dr}\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)\frac{dp'}{dr} = \overline{a^2}i\exp[i(kz+n\theta-\omega t)](\omega-kW)\frac{1}{\overline{\rho}}\frac{d\overline{\rho}}{dr}\frac{dP}{dr}} \tag{12}$$

II.D. Term 4

Apply solution form to fourth term of Lilley's equation:

$$\begin{aligned}
2\overline{a^2}\frac{dW}{dr}\frac{\partial^2 p'}{\partial z\partial r} &= 2\overline{a^2}\frac{dW}{dr}\frac{\partial}{\partial z}\frac{\partial p'}{\partial r} = 2\overline{a^2}\frac{dW}{dr}\frac{\partial}{\partial z}\left(\frac{dP}{dr}e^{iE}\right) \\
&= 2\overline{a^2}\frac{dW}{dr}\frac{dP}{dr}\frac{\partial}{\partial z}\left(e^{iE}\right) = 2\overline{a^2}\frac{dW}{dr}\frac{dP}{dr}ike^{iE}
\end{aligned}$$

This results in the final expression for Term 4:

$$\boxed{2\overline{a^2}\frac{dW}{dr}\frac{\partial^2 p'}{\partial z\partial r} = \overline{a^2}i\exp[i(kz+n\theta-\omega t)]2k\frac{dW}{dr}\frac{dP}{dr}} \tag{13}$$

II.E. Lilley's Equation Solution

To derive the final form of Lilley's equation, we combine Terms 1 through 4:

$$Eqn\ 8 - Eqn\ 11 - Eqn\ 12 + Eqn\ 13 = RHS$$

Which becomes the following in expanded form:

$$\left\{ie^{iE}(\omega - kW)^3 P\right\} - \left\{-\overline{a^2} ie^{iE}(\omega - kW)\left[\frac{d^2P}{dr^2} + \frac{1}{r}\frac{dP}{dr} - \left(\frac{n^2}{r^2} + k^2\right)P\right]\right\}$$
$$- \left\{\overline{a^2} ie^{iE}(\omega - kW)\frac{1}{\overline{\rho}}\frac{d\overline{\rho}}{dr}\frac{dP}{dr}\right\} + \left\{\overline{a^2} ie^{iE} 2k \frac{dW}{dr}\frac{dP}{dr}\right\} = RHS$$

III. Problem 2 – General Solution for Jet FlowFar-Field

Determine the general form of solution (solution of the homogeneous equation) for the pressure fluctuation outside the jet in the ambient medium where the sources vanish. Make sure the solution is chosen to ensure decaying solutions or outgoing waves.

IV. Problem 3 – General Solution for Jet Potential Core

Determine the general form of solution (solution of the homogeneous equation) in the potential core region where the mean velocity and density are constant and equal to the jet exit values.

V. Problem 4 – Matching of FlowFar-Field and Potential Core Solutions

Consider a case in which the real jet is replaced by a vortex sheet at = !. If the solutions are to be matched at the vortex sheet, describe what matching conditions should be applied. Give both the physical description and the mathematical expressions.