Lilley's equation for an axisymetric mean flow can be written

$$\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)^3 p' - \left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)(\bar{a}^2\nabla^2 p') - \frac{d\bar{a}^2}{dr}\left(\frac{\partial}{\partial t}+W\frac{\partial}{\partial z}\right)\frac{\partial p'}{\partial r}$$

$$+2\bar{a}^2\frac{dW}{dr}\frac{\partial^2 p'}{\partial r\partial z} = S(\vec{x},t)$$

where $$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}$$

1) Seek solutions of the form

$$p'(r.\theta.z.t) = P(r)\exp\left[i(kz+n\theta-\omega t)\right]$$

Then $\frac{\partial p'}{\partial z} \sim ikp'$, $\frac{\partial p'}{\partial\theta}\sim inp'$, and $\frac{\partial p'}{\partial t}\sim i\omega p'$

Then

$$(-i\omega+ikW)^3 P - \bar{a}^2(-i\omega+ikW)\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dP}{dr}\right)-\frac{n^2}{r^2}P-k^2P\right\}$$

$$-\frac{d\bar{a}^2}{dr}(-i\omega+ikW)\frac{dP}{dr}+2\bar{a}^2\frac{dW}{dr}(ik)\frac{dP}{dr} = RHS$$

Thus

$$(-i\omega+ikW)^2P-\bar{a}^2\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dP}{dr}\right)-\frac{n^2}{r^2}P-k^2P\right\}$$

$$-\frac{d\bar{a}^2}{dr}\frac{dP}{dr}+2\bar{a}^2\frac{ik}{(-i\omega+ikW)}\frac{dW}{dr}\frac{dP}{dr} = RHS$$

Now $$(-i\omega+ikW)^2 = -(-\omega+kW)^2 = -(\omega-kW)^2$$

and $$\frac{ik}{(-i\omega+ikW)} = -\frac{k}{(\omega-kW)}$$

So that

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dP}{dr}\right)+\frac{1}{\bar{a}^2}\frac{d\bar{a}^2}{dr}\frac{dP}{dr}+\frac{2k}{(\omega-kW)}\frac{dW}{dr}\frac{dP}{dr}+\frac{(\omega-kW)^2}{\bar{a}^2}P$$

$$-\frac{n^2}{r^2}P-k^2P = RHS$$

Now $\quad \bar{a}^2 = \dfrac{\gamma \bar{p}}{\bar{\rho}}, \quad \dfrac{d\bar{a}^2}{dr} = -\dfrac{\gamma \bar{p}}{\bar{\rho}^2} \dfrac{d\bar{\rho}}{dr} \quad$ for $\bar{p} =$ constant

and $\quad \dfrac{1}{\bar{a}^2} \dfrac{d\bar{a}^2}{dr} = -\dfrac{\gamma \bar{p}}{\bar{\rho}^2} \dfrac{d\bar{\rho}}{dr} \dfrac{\bar{\rho}}{\gamma \bar{p}} = -\dfrac{1}{\bar{\rho}} \dfrac{d\bar{\rho}}{dr}$

Also, $\quad \dfrac{1}{r} \dfrac{d}{dr}\left(r \dfrac{dp}{dr}\right) = \dfrac{d^2 p}{dr^2} + \dfrac{1}{r} \dfrac{dp}{dr}$

So that, finally,

$$\frac{d^2 p}{dr^2} + \left\{ \frac{1}{r} - \frac{1}{\bar{\rho}} \frac{d\bar{\rho}}{dr} + \frac{2k}{(\omega - kW)} \frac{dW}{dr} \right\} \frac{dp}{dr} + \left\{ \frac{(\omega - kW)^2}{\bar{a}^2} - k^2 - \frac{n^2}{r^2} \right\} p = RHS$$

2) Outside the jet $\frac{d\bar{\rho}}{dr} = \frac{d\bar{W}}{dr} = 0$ and $W = W_0$; $\bar{a}^2 = a_0^2$

Then Lilley's equation reduces to

$$\frac{d^2p}{dr^2} + \frac{1}{r}\frac{dp}{dr} + \left\{ \frac{(\omega - k W_0)^2}{a_0^2} - k^2 - \frac{n^2}{r^2} \right\} p = 0$$

Let $\lambda_0^2 = \frac{(\omega - k W_0)^2}{a_0^2} - k^2$

So that

$$\frac{d^2p}{dr^2} + \frac{1}{r}\frac{dp}{dr} + \left(\lambda_0^2 - \frac{n^2}{r^2}\right) p = 0$$

This is Bessel's equation. The solution can be written.

$$\underline{p(r) = A H_n^{(1)}(\lambda_0 r) + B H_n^{(2)}(\lambda_0 r)}$$

$H_n^{(1)} = J_n + i Y_n$
$H_n^{(2)} = J_n - i Y_n$.
H_n is useful for far-field wave propagation since it has asymptotic behavior.

Now, for large arguments

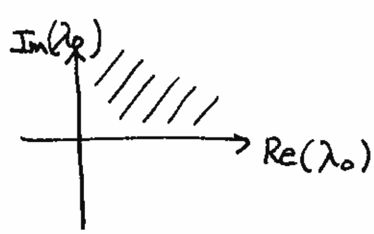
$$H_n^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} \exp\left[i \left(z - \frac{n\pi}{2} - \frac{\pi}{4} \right) \right]$$

$z = a + ib$
$\exp(iz) = \exp(ia) \cdot \exp(-b)$

Thus for decaying solution, $\mathrm{Im}(z) = b > 0$

and for outgoing waves $\mathrm{Re}(z) = a > 0$.

(remember $e^{-i\omega t}$ sign)



Thus $0 < \arg(\lambda_0) \leq \frac{\pi}{2}$

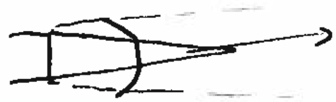
and $p(r) \sim H_n^{(1)}(\lambda_0 r)$

Since the solution $H_n^{(2)}(z)$ would give growing solution or incoming wave

$$\sim \sqrt{\frac{2}{\pi z}} \exp\left[-i \left(z - \frac{n\pi}{2} - \frac{\pi}{4} \right) \right]$$

3) In the potential core region, $W = W_j$, $\bar{a}^2 = a_j^2$ and

$$\frac{d\bar{p}}{dr} = \frac{dW}{dr} = 0$$

 potential core region

So the equation reduces to

$$\frac{d^2p}{dr^2} + \frac{1}{r}\frac{dp}{dr} + \left\{ \frac{(\omega - kW_j)^2}{a_j^2} - k^2 - \frac{n^2}{r^2} \right\} p = 0$$

Let $$\lambda_j^2 = \frac{(\omega - kW_j)^2}{a_j^2} - k^2$$

Then $$\frac{d^2p}{dr^2} + \frac{1}{r}\frac{dp}{dr} + \left(\lambda_j^2 - \frac{n^2}{r^2}\right) p = 0$$

Again, this is Bessel equation for which the solution can be written

J_n & Y_n expression is useful when including $r=0$

$$p(r) = A J_n(\lambda_j r) + B\, Y_n(\lambda_j r)$$

Now the Bessel function of the second kind $Y_n(\lambda_j r)$ has a logarithmic singurarity at $r=0$. So for bounded ~~Sol~~solution we obtain

$$p(r) \approx J_n(\lambda_j r)$$

~~and for completeness~~

$$\cancel{0 < \text{Arg}(\lambda_j) < \frac{\pi}{2}}$$

There is no decaying or growing solution since the field is bounded.

At the vortex sheet, we must have continuity of normal stress.

In an inviscid fluid, this means continuity of pressure.

Thus

$$\Delta\,[P] = 0$$

where $\Delta[\ ]$ denotes the change across the vortex sheet.

Also, at the vortex sheet, the fluid velocity in the y-direction must match the total rate of change of the interface position with time. Let the vortex sheet displacement be denoted by $\rho(z,t)$. Then the total rate of change in the y-direction is given by the sum of the local time change plus the contribution due to convection of fluid particle by the mean flow up or down the slope of the interface

Thus

$$V_+ = \left(\frac{\partial}{\partial t} + W_0 \frac{\partial}{\partial z}\right)\rho$$

$$V_- = \left(\frac{\partial}{\partial t} + W_j \frac{\partial}{\partial z}\right)\rho$$

If $\rho(z,t) = a \exp[i(kz - \omega t)]$

$$V^+ = -i(\omega - kW_0)\,a$$

$$V^- = -i(\omega - kW_j)\,a$$

$$\frac{iV^+}{(\omega - kW_0)} = \frac{iV^-}{(\omega - kW_j)} \qquad \text{i.e. } \Delta\left[\frac{V}{(\omega - kW)}\right] = 0.$$

From the radial momentum equation

$$\left(\frac{\partial}{\partial t} + W\frac{\partial}{\partial z}\right) v = -\frac{1}{\rho_0}\frac{\partial p'}{\partial r} \qquad v = \hat{v}\, e^{-i\omega t + kz}$$

i.e. $-i(\omega - kW)\hat{v} = -\frac{1}{\rho_0}\frac{dp}{dr}$ $\qquad \left[\frac{\hat{v}}{(\omega-kW)}\right]^{+} = \left[\frac{\hat{v}}{(\omega-kW)}\right]^{-}$

So the fluid's velocity in the y-direction is determined by the radial pressure gradient. Thus the jump condition for the pressure is

$$\Delta\left[\frac{1}{(\omega-kW)^2}\frac{dp}{dr}\right] = 0$$

Note that if $p^{+}(r) = A^{+} H_n^{(1)}(\lambda_0 r)$

and $p^{-}(r) = A^{-} J_n(\lambda_j r)$

Then $A^{+} H_n^{(1)}(\lambda_0 R_j) = A^{-} J_n(\lambda_j R_j)$

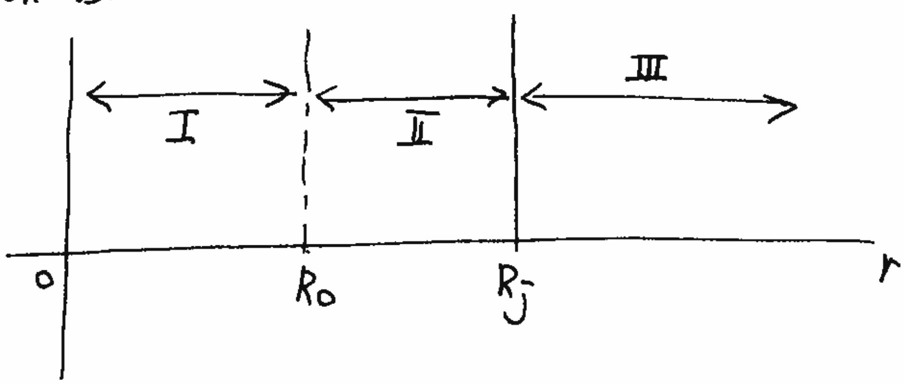
and $$\frac{A^{+}\lambda_0}{(\omega-kW_0)^2} H_n^{(1)\prime}(\lambda_0 R_j) = \frac{A^{-}\lambda_j}{(\omega-kW_j)^2} J_n'(\lambda_i R_j)$$

or
$$\begin{bmatrix} H_n^{(1)}(\lambda_0 R_j) & -J_n(\lambda_j R_j) \\ \frac{\lambda_0}{(\omega-kW_0)^2} H_n^{(1)\prime}(\lambda_0 R_j) & \frac{-\lambda_j}{(\omega-kW_j)^2} J_n'(\lambda_j R_j) \end{bmatrix} \begin{Bmatrix} A^{+} \\ A^{-} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For a non-trivial solution, the determinent of the matrix must be zero. That is

$$\frac{\lambda_0 J_n(\lambda_j R_j) H_n^{(1)\prime}(\lambda_0 R_j)}{(\omega - k W_0)^2} - \frac{\lambda_j J_n'(\lambda_j R_j) H_n^{(1)}(\lambda_0 R_j)}{(\omega - k W_j)^2} = 0$$

This is the "dispersion relationship." It determines what values of ω and k permit a non-trivial or eigensolutio
Note that, if there were a source at some location, say $r = R_0$, then the transmission through the vortex sheet would be determined by the Green function. This would be obtained by writing down the solution in the three regions as follows.



Region I : $\quad p(r) = A^- J_n(\lambda_j r)$

Region II : $\quad p(r) = A^+ J_n(\lambda_j r) + B^+ Y_n(\lambda_j r)$

Region III : $\quad p(r) = C^+ H_n^{(1)}(\lambda_0 r)$

Then all the coefficients are obtained by matching across the vortex sheet as above and the jump conditions at $r = R_0$