Function Definitions

Functions are one-to-one iff every element in X goes to a distinct element in Y

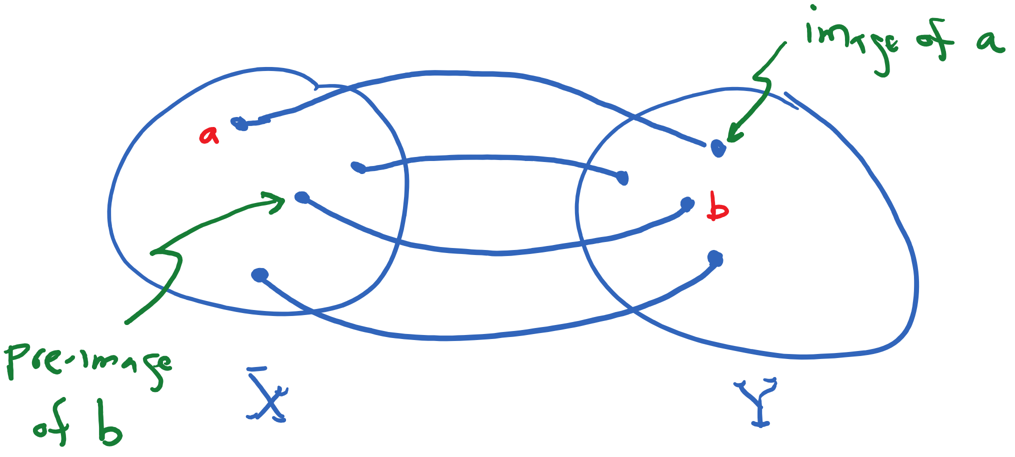
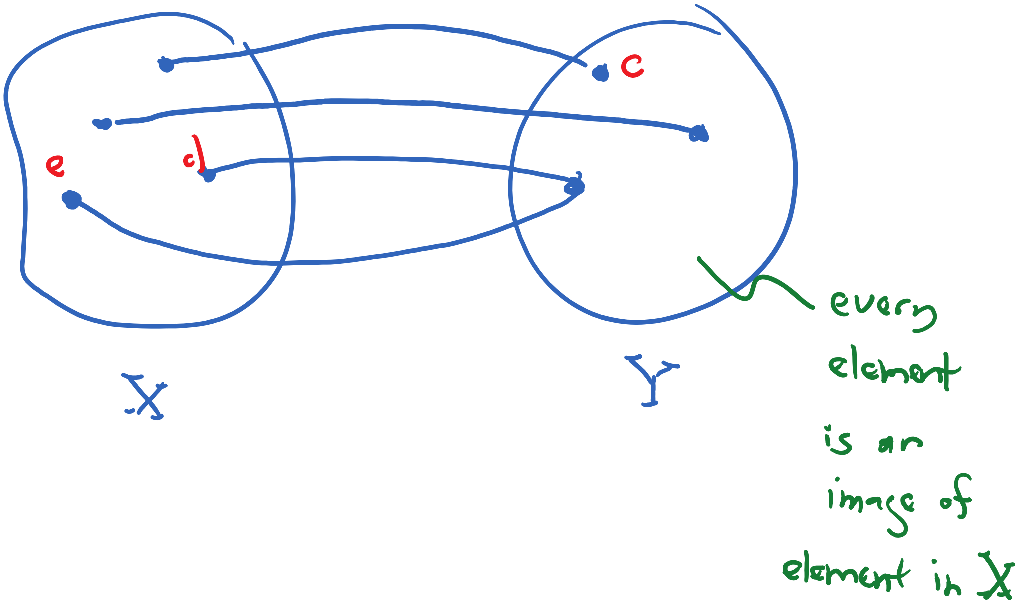


image of a

pre-image of b

Functions are onto iff every element of Y is an image of some element of X and thus

$\text{range}(f) = \text{codomain}(f)$

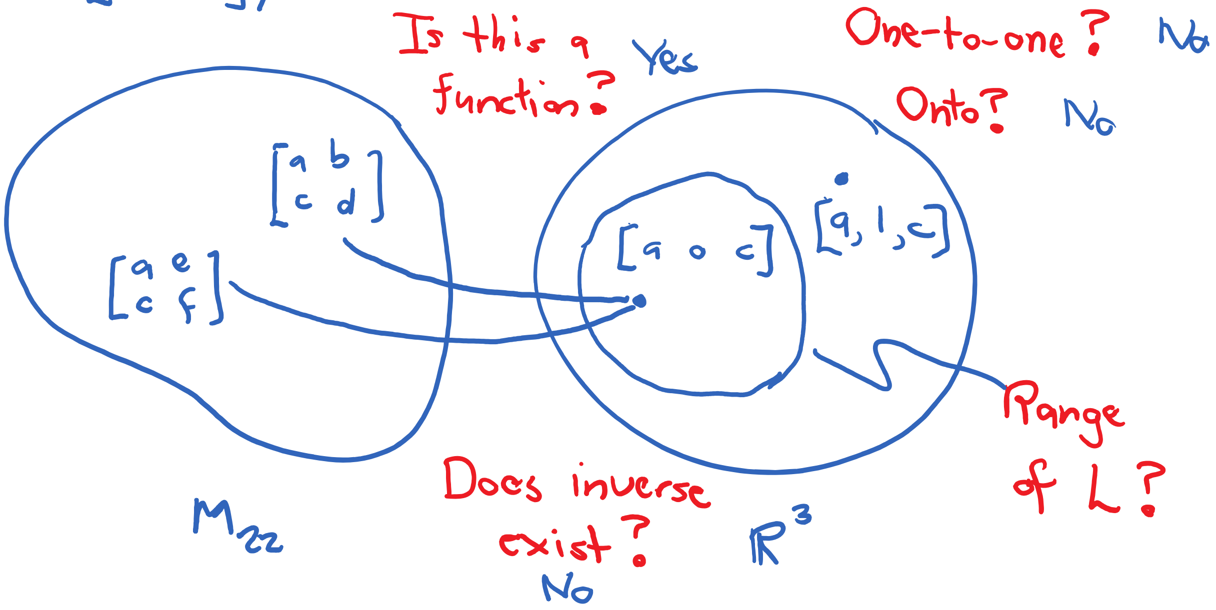


every element is an image of element in X

Linear Transformation Example

Example: Let $L: M_{22} \rightarrow \mathbb{R}^3$ (Linear Transformation)

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = [a, 0, c]$$ (from last lecture)



Is this a function? Yes

One-to-one? No

Onto? No

Range of L?

Does inverse exist? No

Let $\underline{M} = K_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + K_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + K_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + K_4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Use B_{11}, B_{12}, B_{21} + B_{22} as basis

$$\underline{M} = K_1 \underline{B}_{11} + K_2 \underline{B}_{12} + K_3 \underline{B}_{21} + K_4 \underline{B}_{22}$$

$$K_1 \underline{B}_{11} + K_2 \underline{B}_{12} + K_3 \underline{B}_{21} + K_4 \underline{B}_{22} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\therefore$ $K_1 + K_4 = a$, $K_2 = b$, $K_3 = c$, $K_4 = d$

+ $K_1 = a - d$

$$L(\underline{B}_{11}) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad L(\underline{B}_{12}) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$L(\underline{B}_{21}) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad L(\underline{B}_{22}) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\underline{A} = \begin{bmatrix} L(\underline{B}_{11})^T, L(\underline{B}_{12})^T, L(\underline{B}_{21})^T, L(\underline{B}_{22})^T \end{bmatrix}$$

$$\underline{A}\,\underline{k} = L(\underline{M})$$

$$(\underline{A}\,\underline{k})^T = \left(\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a-d \\ b \\ c \\ d \end{bmatrix} \right)^T$$

$$= \begin{bmatrix} a & 0 & c \end{bmatrix}$$

Linear Transformation & Orthogonality

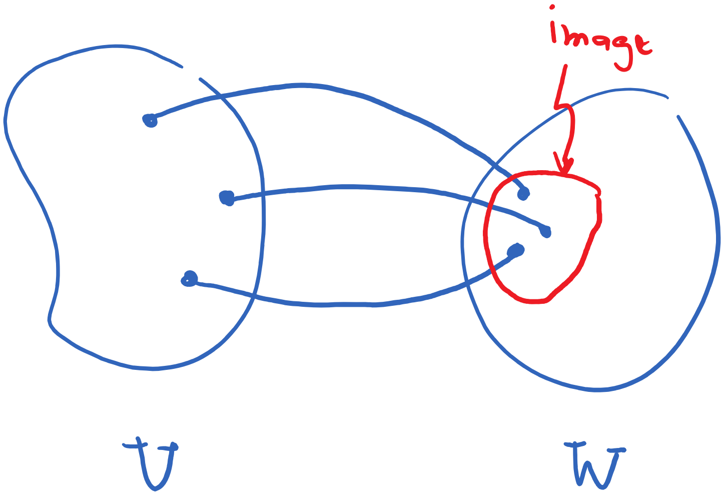
Let V & W be vector spaces with a linear transformation $L: V \to W$

Kernel of L: The $\ker(L)$ is the subspace of V, such that $\ker(L) = \{ \underline{v} \in V : L\underline{v} = \underline{0}_w \}$

The size (dimension) of the $\ker(L)$ is called the nullity of L: $\text{nullity}(L)$

Rank of L: The rank of a linear operator, $\text{rank}(L)$, is the dimension of its image.

Recall that the image of a vector space is the portion of W that it maps into



Rank-Nullity or Dimension Theorem

Rank-Nullity or Dimension Theorem

Let V + W be vector spaces with a linear transformation $L: V \to W$

Then

$$\text{rank}(L) + \text{nullity}(L) = \underbrace{|L|}_{\text{dimension (or size) of } V}$$

Apply to matrices: $\underline{A} \in M_{mn}$

$\underline{A}\,\underline{x} = \underline{b}$ $\qquad \underline{x} \in V \in \mathbb{R}^n$

$\qquad\qquad\qquad \underline{b} \in W \in \mathbb{R}^m$

$\underline{b}$ is the image of $\underline{x}$ under the linear transformation of $\underline{A}$

Linear combinations of $C(\underline{A})$ give all vectors in the image

$\Rightarrow$ $\text{rank}(\underline{A}) = |C(\underline{A})|$

↳ dimension (or size) of the column space of $\underline{A}$

Note: Each vector in $\mathcal{C}(\underline{A})$ must be an independent vector (i.e., $\mathcal{C}(\underline{A})$ contains the minimum # of vectors to span the columns of $\underline{A}$)

Example:

$$\underline{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Do not say

$$\mathcal{C}(\underline{A}) \neq \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

but rather

$$\mathcal{C}(\underline{A}) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\Rightarrow \text{rank}(\underline{A}) = 3$$

All linear combinations of $N(\underline{A})$ give vectors $\underline{b} = \underline{0} = \underline{0}_w$

$$\underline{A}\,\underline{x} = \underline{0}$$

$\Rightarrow$ nullity$(\underline{A}) = |N(\underline{A})|$

↳ dimension (or size)
of the null space of $\underline{A}$

Since for $\underline{A}\,\underline{x} = \underline{b}$, $\underline{x} \in \mathbb{R}^n$

Then rank$(\underline{A})$ + nullity$(\underline{A}) = n$

= # of columns

Theorem: Let $\underline{A} \in M_{mn}$, $C(\underline{A})$ be the column space, $N(\underline{A})$ be the nullspace, $C(\underline{A}^T)$ be the row space and $N(\underline{A}^T)$ be the left nullspace.

1) rank$(\underline{A}) = |C(\underline{A})| = |C(\underline{A}^T)|$

2) $|N(\underline{A})| = n -$ rank$(\underline{A})$

3) $|N(\underline{A}^T)| = m -$ rank$(\underline{A})$

Example: Let

$$\underline{A} = \begin{bmatrix} 8 & 2 & 1 & 23 \\ 4 & 2 & 9 & 19 \\ 10 & 1 & 6 & 35 \end{bmatrix}$$

$$\underset{3\times 4}{\underline{A}}\,\underset{4\times 1}{\underline{x}} = \underset{3\times 1}{\underline{b}}$$

$\underline{x} \in \mathbb{R}^4$

$\underline{b} \in \mathbb{R}^3$

rref (reduced row eshelon form)

$$\longleftrightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Independent columns

free variable

$$\text{rref}\,(\underline{A}^T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Independent columns

no free variables

$$\Rightarrow \text{rank}\,(\underline{A}) = |\mathcal{C}(\underline{A})| = 3 = |\mathcal{C}(\underline{A}^T)|$$

$$\Rightarrow |N(\underline{A})| = n - \text{rank}\,(\underline{A}) = 4 - 3 = 1$$

$$\Rightarrow |N(\underline{A}^T)| = m - \text{rank}\,(\underline{A}) = 3 - 3 = 0$$

Find all subspaces (Matlab check: rref)

$$C(\underline{A}) = \left\{ \begin{bmatrix} 8 \\ 4 \\ 10 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ 6 \end{bmatrix} \right\}$$

$$N(\underline{A}): \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$?

Let $x_4 = 1$ (free variable)

Then

$$\begin{aligned} x_1 + 3 &= 0 \\ x_2 - 1 &= 0 \\ x_3 + 1 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= -3 \\ x_2 &= 1 \\ x_3 &= -1 \end{aligned}$$

$$\therefore N(\underline{A}) = \left\{ \begin{bmatrix} -3 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Also, $N(\underline{A})$ contains the zero vector $\underline{0}_4$

Check

$$\begin{bmatrix} 8 & 2 & 1 & 23 \\ 4 & 2 & 9 & 19 \\ 10 & 1 & 6 & 35 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -24 + 2 - 1 + 23 \\ -12 + 2 - 9 + 19 \\ -30 + 1 - 6 + 35 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C(\underline{A}^T) = \left\{ \begin{bmatrix} 8 \\ 2 \\ 1 \\ 23 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 9 \\ 19 \end{bmatrix}, \begin{bmatrix} 10 \\ 1 \\ 6 \\ 35 \end{bmatrix} \right\}$$

$N(\underline{A}^T) = \{ \quad \}$ Contains $\underline{0}_3$ vector

Let $\underline{A} \in M_{mn}$

(1) The matrix $\underline{A}$ has full column rank if $\text{rank}(\underline{A}) = n$. If $\underline{A}$ has full column rank, then the following holds

a) All columns of $\underline{A}$ are independent

b) Only vector in $N(\underline{A})$ is $\underline{0}$

c) If $\underline{A}^{-1}$ exists, then the solution to $\underline{A}\,\underline{x} = \underline{b}$ is unique (i.e., only one $\underline{x}$ such that $\underline{A}\,\underline{x} = \underline{b}$)

(2) The matrix $\underline{A}$ has full row rank if

rank $(\underline{A}) = m$. Furthermore,

a) All rows of $\underline{A}$ are independent

b) $C(\underline{A})$ spans all of $\mathbb{R}^m$ $(\underline{b} \in \mathbb{R}^m)$

c) $\underline{A}\,\underline{x} = \underline{b}$ has at least one solution

for any $\underline{b}$

$C(\underline{A})$ spans all of $\mathbb{R}^m$

$\Rightarrow$ Any vector in $\mathbb{R}^m$ can be written as a linear combination of the columns of $\underline{A}$

$\Rightarrow$ Any $\underline{b}$ must be in $\mathbb{R}^m$ (for $\underline{A}\,\underline{x} = \underline{b}$)

$\Rightarrow$ $\underline{x}$ is that linear combination of the columns of $\underline{A}$ that gives $\underline{b}$

$$\underline{A}\,\underline{x} = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$= x_1 \underline{a}_1 + \underline{x}_2 \underline{a}_2 + \dots + x_n \underline{a}_n$

(3) Now let $\underline{A} \in M_{nn}$ (square matrix)

The matrix $\underline{A}$ has full rank if

$\text{rank}(\underline{A}) = n$ (i.e., both full row + column rank)

If $\underline{A}$ is full rank, then

a) $\underline{A}\,\underline{x} = \underline{b}$ has a solution for any $\underline{b}$

b) $C(\underline{A})$ span all of $\mathbb{R}^n$

c) $N(\underline{A})$ is only the $\underline{0}$

d) $\underline{A}\,\underline{x} = \underline{b}$ only has one solution for any $\underline{b}$

In other words, if $\underline{A}$ is full rank, then $\underline{A}^{-1}$ exists

Only one solution: $\underline{A}\,\underline{x} = \underline{b} \Rightarrow \underline{x} = \underline{A}^{-1}\underline{b}$

Now, all of the following are equivalent statements:

① $\underline{A}$ is invertible

② The columns of $\underline{A}$ are independent

③ The rows of $\underline{A}$ are independent

④ $\det(\underline{A}) \neq 0$

⑤ $\underline{A}\,\underline{x} = \underline{0}$ only has $\underline{x} = \underline{0}$ as a solution

⑥ $\underline{A}$ has n pivots for $\underline{A} \in M_{nn}$

⑦ $\underline{A}$ is full rank (i.e. $\text{rank}(\underline{A}) = n$)

⑧ $\text{rref}(\underline{A}) = \underline{I}$

⑨ $C(\underline{A})$ spans all of $\mathbb{R}^n$

⑩ $C(\underline{A}^T)$ spans all of $\mathbb{R}^n$

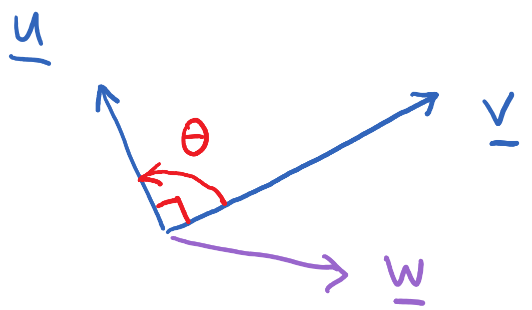
If any one of these is true, then
all are true for square matrix $\underline{A}$

If any one of these does not hold, then
none of these statements hold

Orthogonality

Recall: Two vectors are orthogonal (perpendicular) to each other iff

$$\underline{u} \cdot \underline{v} = \underline{u}^T \underline{v} = 0$$



Here, $\underline{u}$ and $\underline{v}$ are orthogonal

$$\underline{u} \cdot \underline{v} = \|\underline{u}\| \|\underline{v}\| \cos\theta = 0$$

while $\underline{u}$ and $\underline{w}$ are not orthogonal

Two subspaces are orthogonal to each other if any vector in one subspace is orthogonal to all vectors in the other subspace.

If $\underline{u}$ is in subspace S and $\underline{v}$ is in subspace T, then if for (all) any $\underline{u} \in S$ and $\underline{v} \in T$,

T, then if for any $\underline{u} \in S$ and $\underline{v} \in T$, we have $\underline{u} \cdot \underline{v} = 0$, then S and T are orthogonal.

For a matrix $\underline{A} \in M_{mn}$

(1) The row space $C(\underline{A}^T)$ is an orthogonal subspace in $\mathbb{R}^n$ of the nullspace $N(\underline{A})$

(a) To show this, consider

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

Let $\underline{x} \in N(\underline{A}) \Rightarrow \underline{A}\underline{x} = \underline{0}$

$$\underline{A}\underline{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} \underline{\text{row} 1} \cdot \underline{x} \\ \underline{\text{row} 2} \cdot \underline{x} \\ \vdots \\ \underline{\text{row} m} \cdot \underline{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The row space is the linear combination of the rows of $\underline{A}$

Since any $\underline{x}$ in $N(\underline{A})$ gives $\underline{A}\underline{x}=\underline{0}$ and since $\underline{A}\underline{x}$ is simply a dot product between rows of $\underline{A}$ and $\underline{x}$

$\Rightarrow C(\underline{A}^T)$ + $N(\underline{A})$ are othogonal to each other

(b) Another way to show this:

Let $\underline{y}$ be any vector compatible with $\underset{n\times m}{\underline{A}^T}$. Then

$\underset{n\times m}{\underline{A}^T}\,\underset{m\times 1}{\underline{y}}$ Is a linear combination of the rows of $\underline{A}$

Let $\underline{x}$ be in $N(\underline{A})$. Then

$$\underline{x}\cdot(\underline{A}^T\underline{y}) = \underline{x}^T\underline{A}^T\underline{y} = (\underline{A}\underline{x})^T\underline{y}$$

$$= \underline{0}^T \underline{y} = 0$$

(2) The column space $C(\underline{A})$ is an orthogonal subspace in $\mathbb{R}^m$ of the left nullspace $N(\underline{A}^T)$

Let $\underset{m\times n}{\underline{A}}\,\underset{n\times 1}{\underline{y}}$ represent any vector in $C(\underline{A})$

and let $\underline{x} \in N(\underline{A}^T)$: $\underset{1\times m}{\underline{x}^T}\,\underset{m\times n}{\underline{A}} = \underset{1\times n}{\underline{0}^T} = (\underset{n\times m}{\underline{A}^T}\,\underset{m\times 1}{\underline{x}})^T$

Then,

$$\underset{m\times 1}{\underline{x}} \cdot \underbrace{(\underset{m\times n}{\underline{A}}\,\underset{n\times 1}{\underline{y}})}_{m\times 1} = \underline{x}^T \underline{A}\, \underline{y} = (\underline{x}^T \underline{A})\, \underline{y} = \underline{0}^T \underline{y} = 0$$

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$$

Now, one step further

(1) $N(\underline{A})$ is the orthogonal complement of $C(\underline{A}^T)$ in $\mathbb{R}^n$

(2) $N(\underline{A}^T)$ is the orthogonal complement of $C(\underline{A})$ in $\mathbb{R}^m$

The orthogonal complement to a subspace

contains every possible vector that is perpendicular (orthogonal) to that subspace.

Example: continued from above

$$\underline{A} = \begin{bmatrix} 8 & 2 & 1 & 23 \\ 4 & 2 & 9 & 19 \\ 10 & 1 & 6 & 35 \end{bmatrix}$$

Recall

$$C(\underline{A}^T) = \left\{ \begin{bmatrix} 8 \\ 2 \\ 1 \\ 23 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 9 \\ 19 \end{bmatrix}, \begin{bmatrix} 10 \\ 1 \\ 6 \\ 35 \end{bmatrix} \right\}$$

↳ row space

$$N(\underline{A}) = \left\{ \begin{bmatrix} -3 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

↳ nullspace

$$\begin{bmatrix} -3 \\ 1 \\ -1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 8 \\ 2 \\ 1 \\ 23 \end{bmatrix} = -24 + 2 - 1 + 23 = 0 \checkmark$$

$$\begin{bmatrix} -3 \\ 1 \end{bmatrix}^T \begin{bmatrix} 4 \\ 2 \end{bmatrix} = -12 + 2 - 9 + 19 = 0 \checkmark$$

$$\begin{bmatrix}-3\\1\\-1\\1\end{bmatrix} \cdot \begin{bmatrix}4\\2\\9\\19\end{bmatrix} = -12+2-9+19=0 \checkmark$$

$$\begin{bmatrix}-3\\1\\-1\\1\end{bmatrix}^T \begin{bmatrix}10\\1\\6\\35\end{bmatrix} = -30+1-6+35=0 \checkmark$$

$\therefore$ $N(\underline{A})$ is the orthogonal complement to $C(\underline{A}^T)$

On the other hand,

$$\underset{\substack{\uparrow\\ \text{column space}}}{C}(\underline{A}) = \left\{ \begin{bmatrix}8\\4\\10\end{bmatrix}, \begin{bmatrix}2\\2\\1\end{bmatrix}, \begin{bmatrix}1\\9\\6\end{bmatrix} \right\}$$

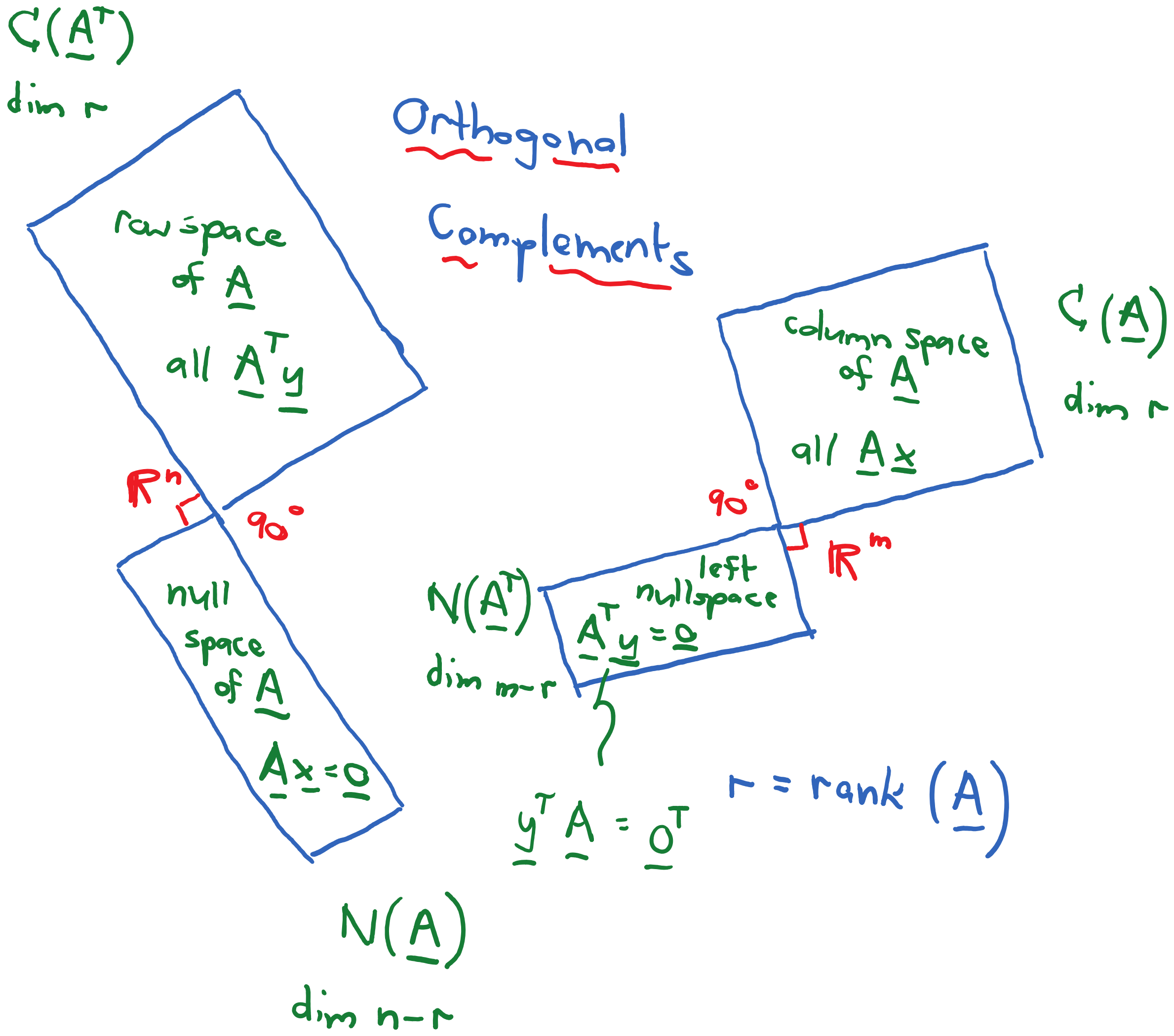
which spans all of $\mathbb{R}^3$. Consequently, there is no other vector orthogonal to $C(\underline{A})$ in $\mathbb{R}^3$ and the left nullspace $N(\underline{A}^T)$ consists of only the zero vector,

$\therefore$

that is,

$$N(\underline{A}^T) = \{\ \} \text{ or } N(\underline{A}^T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Pictorial View of the Four Subspaces of $\underline{A}$ (Strang, 2019)



r = rank $(\underline{A})$