Chapter 3 Homework

Directions: Work on the problems in this order: yellow, green, blue, gray. Do this for Sections 3.1-3.3 first, then Sections 3.4-3.7, then Sections 3.8-3.11.

You will generally be given about two days' notice that a certain color grouping will be due in Top Hat. The expectation is that you are working on a few of these problems every day, so two days should be plenty of time to wrap up your work and submit your answers.

Some of the red problems (if any) may be discussed in class as time permits.

The uncolored problems can be done for additional practice.

Section 3.1

Problem 1. A coin is tossed 5 times. Let X count the number of heads.

(a) Determine the support of X.

(b) Determine $X(\text{HHTHT})$.

Problem 2. A coin is tossed 3 times and then a die is rolled. Let X be the number of heads times the number of spots on the die.

(a) Determine the support of X.

(b) Determine $X(\text{HHT6})$, $X(\text{TTT4})$, and $X(\text{HTT3})$.

Problem 3. A coin is tossed 3 times and then a die is rolled. Let X be the number of heads plus the number of spots on the die.

(a) Determine the support of X.

(b) Determine $X(\text{HHT6})$, $X(\text{TTT4})$, and $X(\text{HTT3})$.

Problem 4. A red die is rolled, followed by a green die. Let $X = 2(\text{number of spots on red}) + (\text{number of spots on green})$.

(a) Determine $X(3,4)$.

(b) Determine $X(4,3)$.

Problem 5. Five cards are selected from a deck of cards. Let X denote the number of aces selected. Determine the support of X.

Section 3.2

Problem 6. What does pmf stand for?

Problem 7. What does CDF stand for?

Problem 8. What is the mathematical term for the story of probability?

Problem 9. Determine the pmf $f(x)$ for the random variable in Problem 1.

Problem 10. Determine the pmf $f(x)$ for the random variable in Problem 2.

Problem 11. Determine the pmf $f(x)$ for the random variable in Problem 3.

Problem 12. Determine the pmf $f(x)$ for the random variable in Problem 4.

Problem 13. Determine the pmf $f(x)$ for the random variable in Problem 5.

Problem 14. Two dice are to be rolled. Let X be the largest value you see. Determine $f(x)$.

Problem 15. Suppose that a random variable X has pmf $f(x) = cx$ for $x = 1, 2, 3, 4$. Determine the value of c.

Problem 16. Suppose that a random variable X has pmf $f(x) = c(0.3)^x$ for $x = 1, 2, 3, \ldots$. Determine the value of c.

Problem 17. Determine the value of c that makes the function given in the table below a valid pmf.

x	1	2	3	4	5	6	7	8
$f(x)$	0.16	0.05	0.06	c	0.08	0.02	0.22	0.14

Problem 18. Use the pmf of the random variable X provided in the table below to determine $F(x)$.

x	1	2	3	4	5	6	7	8
$f(x)$	0.3	0.05	0.06	0.13	0.08	0.02	0.28	0.08

Problem 19. Use the pmf of the random variable X provided in the table below to determine the CDF of X.

x	1	2	3	4	5	6	7	8
$f(x)$	0.2	0.05	0.16	0.23	0.03	0.02	0.23	0.08

Problem 20. Determine the value(s) of c that makes the function given in the table below a valid CDF.

x	1	2	3	4	5	6	7	8
$F(x)$	0.2	0.25	0.45	0.55	c	0.87	0.95	1.00

Problem 21. Determine the value(s) of c that makes the function given in the table below a valid CDF.

x	1	2	3	4	5	6	7	8
$F(x)$	0.1	0.25	0.46	c	0.71	0.72	0.86	1.00

Problem 22. The CDF of a random variable X is given by the function $F(x) = \dfrac{cx^2}{3x^2 + x}$ with support $x = 1, 2, 3, \ldots$. [Hint: what must be true about $F(x)$ as x gets large?]

(a) Determine $P(X = 1)$.

(b) Determine $P(X = 2)$.

Problem 23. The CDF of a random variable X is given by the function $F(x) = \dfrac{x^2}{25}$ with support $x = 1, 2, 3, \ldots, c$. Determine the value of c.

Problem 24. The CDF of a random variable X is given in the table below. Determine the probabilities that follow.

x	1	2	3	4	5	6	7	8
$F(x)$	0.08	0.15	0.26	0.43	0.78	0.82	0.88	1.00

(a) $P(X < 5)$

(b) $P(X \leq 3)$

(c) $P(6 \leq X)$

(d) $P(2 \leq X)$

(e) $P(3 \leq X < 7)$

(f) $P(3 < X \leq 7)$

(g) $P(X \leq 4.5)$

Problem 25. Two coins are tossed and then a die is rolled. The random variable X will be the number of heads tossed times the value on the die. Determine the pmf for X.

Section 3.3

Problem 26. The pmf for a random variable X is given in the table below. Determine the mean of X.

x	1	2	3	4	5
$f(x)$	0.3	0.18	0.06	0.14	0.32

Problem 27. Determine $\mathrm{E}[X]$ for the random variable in Problem 24.

Problem 28. The pmf for a random variable X is given in the table below. Determine the mean of X.

x	1	2	3	4	5
$f(x)$	0.2	0.25	0.15	0.22	0.18

Problem 29. A game starts with a player rolling a die. If the result is 1, 2, or 3, the player moves ahead 1 space on the game board. If the result is a 4 or 5, the player moves ahead 2 spaces. If the result is a 6, the player moves ahead 3 spaces. Let X denote the number of spaces that a player moves on a given turn.

(a) Write out the pmf for X in tabular form.

(b) Determine $\mathrm{E}[X]$.

Problem 30. In the game of roulette, there are 18 red spaces, 18 black spaces, and 2 green spaces. Each of the spaces also has a unique number on it. A ball is placed on a wheel, the wheel is spun, and the ball lands in one of the spaces.

(a) One of the bets that can be made is to bet on black or red. If you bet \$1 on red and the ball lands on one of the red spaces, you win \$1 (profit). If the ball does not land on red, you lose the \$1 you bet. Determine the expected value of this bet.

(b) Another bet you can make is to bet on a specific number. If the ball lands on that number, you win \$35 (profit). If not, you lose the \$1 you bet. Determine the expected value of this bet.

Problem 31. A single die is rolled. If the result is a 1, 2, or 3, the player loses \$5. If the result is a 4 or 5, the player wins \$3. If the result is a 6, the player wins \$8. Determine the expected value of the game.

Problem 32. In a game called "craps", two dice are rolled and the sum of the two dice is noted. If the sum is 2, the player wins \$2. If the sum is 12, the player wins \$3. If the sum is 3, 4, 9, 10, or 11, the player wins \$1. For all other sums, the player loses \$1. Determine the expected value of this game.

Problem 33. Determine the probability of a negative batch test result for each combination of batch size n and positivity rate p in the table below:

	$p = 1\%$	$p = 2\%$	$p = 3\%$	$p = 4\%$	$p = 5\%$
$n = 5$	(a)	(b)	(c)	(d)	(e)
$n = 8$	(f)	(g)	(h)	(i)	(j)
$n = 10$	(k)	(l)	(m)	(n)	(o)
$n = 12$	(p)	(q)	(r)	(s)	(t)
$n = 15$					

Problem 34. Determine the expected number of tests for each of the batch size/positivity rate combinations in Problem 33: (a), (b), (c), (d), (e), (f), (g), (h), (i), (j), (k), (l), (m), (n), (o), (p), (q), (r), (s), (t).

Problem 35. Determine the variance for the random variable in:

(a) Problem 26

(b) Problem 28

Problem 36. Use the full-page CDF from the Chapter 3 notes (page 11) to determine the probabilities listed below:

(a) $P(X < 5)$

(b) $P(X \leq 5)$

(c) $P(16 \leq X)$

(d) $P(16 < X)$

(e) $P(3 \leq X < 7)$

(f) $P(3 < X \leq 7)$

(g) $P(X \leq 4.5)$

(h) $P(12.5 < X)$

Problem 37. Use the full-page CDF from the Chapter 3 notes (page 11) to determine the appropriate value of k in the probabilities listed below:

(a) $P(k \leq X) \approx 0.91$

(b) $P(k < X) \approx 0.91$

(c) $P(X < k) \approx 0.05$

(d) $P(X \leq k) \approx 0.05$

(e) $P(X < k) \approx 0.10$

Section 3.4

Problem 38. The pmf for a random variable X is given in the table below.

x	1	2	3	4	5
$f(x)$	0.30	0.18	0.05	0.15	0.32

(a) Determine $\mathrm{E}[X]$.

(b) Determine $\mathrm{E}[X^2]$.

(c) Determine $\mathrm{E}[Y]$ if $Y = 3X + 2$.

(d) Determine $\mathrm{E}[Y]$ if $Y = \ln X$.

(e) Determine $\mathrm{E}[Y]$ if $Y = \frac{1}{X}$.

(f) Determine $\mathrm{E}[Y]$ if $Y = |X|$.

Problem 39. The pmf for a random variable X is given in the table below.

x	1	2	3	4	5
$f(x)$	0.20	0.25	0.15	0.22	0.18

(a) Determine $\mathrm{E}[X]$.

(b) Determine $\mathrm{E}[X^2]$.

(c) Determine $\mathrm{E}[Y]$ if $Y = 3X + 2$.

(d) Determine $\mathrm{E}[Y]$ if $Y = 3|X| + 4$.

(e) Determine $\mathrm{E}[Y]$ if $Y = \frac{1}{X}$.

(f) Determine $\mathrm{Var}[X]$.

Problem 40. The pmf for a random variable X is given in the table below.

x	-2	-1	1	2	3	4	5
$f(x)$	0.10	0.15	0.20	0.05	0.10	0.20	0.20

(a) Determine $\mathrm{E}[X]$.

(b) Determine $\mathrm{E}[X^2]$.

(c) Determine $\mathrm{E}[Y]$ if $Y = 3X + 2$.

(d) Determine $\mathrm{E}[Y]$ if $Y = \ln|X|$.

(e) Determine $\mathrm{E}[Y]$ if $Y = 3X^2$.

(f) Determine $\mathrm{E}[Y]$ if $Y = |X|$.

Problem 41. The pmf for a random variable X is given in the table below.

x	-3	-2	-1	0	1	2	3	4
$f(x)$	0.20	0.05	0.10	0.20	0.10	0.05	0.20	0.10

(a) Determine $\mathrm{E}[X]$.

(b) Determine $\mathrm{E}\left[X^2\right]$.

(c) Determine $\mathrm{E}[Y]$ if $Y = 3X + 2$.

(d) Determine $\mathrm{E}[Y]$ if $Y = \ln|X + 4|$.

(e) Determine $\mathrm{E}[Y]$ if $Y = 6X^2 + 4$.

(f) Determine $\mathrm{Var}[X]$.

Problem 42. Consider the function

$$g(x) = \begin{cases} 0 & \text{if } x = 1, 2 \\ 50 & \text{if } x = 3 \\ 100 & \text{if } x = 4 \\ 150 & \text{if } x = 5. \end{cases}$$

(a) Determine $\mathrm{E}[g(X)]$ using the random variable X from Problem 38.

(b) Determine $\mathrm{E}[g(X)]$ using the random variable X from Problem 39.

Section 3.5

Problem 43. Determine the variance of the random variables in Problems 39 and 41.

Problem 44. Determine the variance of the random variables in Problems 38 and 40.

Problem 45. Determine the 3^{rd} and 4^{th} moments for the random variable in Problem 39.

Problem 46. Determine the 3^{rd} and 4^{th} moments for the random variable in Problem 38.

Problem 47. Determine the 3^{rd} and 4^{th} moments for the random variable in Problem 41.

Problem 48. Determine the 3^{rd} and 4^{th} moments for the random variable in Problem 40.

Section 3.6

Problem 49. Determine the moment generating function for the random variable in Problem 38.

Problem 50. Determine the moment generating function for the random variable in Problem 40.

Section 3.7

For Problems 51 and 52: the distribution of two random variables is given below. A new random variable is defined as a mixture of these two random variables.

x_1	$f_{X_1}(x_1)$
1	0.30
2	0.30
3	0.20
4	0.10
5	0.05
6	0.05

x_2	$f_{X_2}(x_2)$
5	0.10
7	0.15
9	0.25
11	0.50

Problem 51. The pmf of a mixture distribution for a random variable X is given by $f_X(x) = 0.3f_{X_1}(x) + 0.7f_{X_2}(x)$ for all $x \in \mathcal{S}_X$.

(a) Determine the support of X.

(b) Determine $f_X(4)$.

(c) Determine $f_X(5)$.

(d) Determine $f_X(7)$.

(e) Determine $\mathrm{E}[X]$.

(f) Determine $\mathrm{Var}[X]$.

Problem 52. The pmf of a mixture distribution for a random variable X is given by $f_X(x) = 0.7f_{X_1}(x) + 0.3f_{X_2}(x)$ for all $x \in \mathcal{S}_X$.

(a) Determine the support of X.

(b) Determine $f_X(4)$.

(c) Determine $f_X(5)$.

(d) Determine $f_X(7)$.

(e) Determine $\mathrm{E}[X]$.

(f) Determine $\mathrm{Var}[X]$.

For Problems 53 and 54: the distribution of two random variables is given below. A new random variable is defined as a mixture of these two random variables.

x_1	$f_{X_1}(x_1)$
1	0.05
2	0.05
3	0.10
4	0.20
5	0.30
6	0.30

x_2	$f_{X_2}(x_2)$
5	0.50
7	0.25
9	0.15
11	0.10

Problem 53. The pmf of a mixture distribution for a random variable X is given by $f_X(x) = 0.3f_{X_1}(x) + 0.7f_{X_2}(x)$ for all $x \in \mathcal{S}_X$.

(a) Determine the support of X.

(b) Determine $f_X(4)$.

(c) Determine $f_X(5)$.

(d) Determine $f_X(7)$.

(e) Determine $\mathrm{E}[X]$.

(f) Determine $\mathrm{Var}[X]$.

Problem 54. The pmf of a mixture distribution for a random variable X is given by $f_X(x) = 0.7f_{X_1}(x) + 0.3f_{X_2}(x)$ for all $x \in \mathcal{S}_X$.

(a) Determine the support of X.

(b) Determine $f_X(4)$.

(c) Determine $f_X(5)$.

(d) Determine $f_X(7)$.

(e) Determine $\mathrm{E}[X]$.

(f) Determine $\mathrm{Var}[X]$.

For Problems 55 and 56: the distribution of two random variables is given below. A new random variable is defined as a mixture of these two random variables.

x_1	$f_{X_1}(x_1)$
1	0.40
2	0.30
3	0.20
4	0.10

x_2	$f_{X_2}(x_2)$
3	0.10
4	0.15
5	0.25
6	0.50

Problem 55. The pmf of a mixture distribution for a random variable X is given by $f_X(x) = 0.4 f_{X_1}(x) + 0.6 f_{X_2}(x)$ for all $x \in \mathcal{S}_X$.

(a) Determine the support of X.

(b) Determine $f_X(2)$.

(c) Determine $f_X(4)$.

(d) Determine $f_X(5)$.

(e) Determine $\mathrm{E}[X]$.

(f) Determine $\mathrm{Var}[X]$.

Problem 56. The pmf of a mixture distribution for a random variable X is given by $f_X(x) = 0.6 f_{X_1}(x) + 0.4 f_{X_2}(x)$ for all $x \in \mathcal{S}_X$.

(a) Determine the support of X.

(b) Determine $f_X(2)$.

(c) Determine $f_X(4)$.

(d) Determine $f_X(5)$.

(e) Determine $\mathrm{E}[X]$.

(f) Determine $\mathrm{Var}[X]$.

For Problems 57 and 58: the distribution of two random variables is given below. A new random variable is defined as a mixture of these two random variables.

x_1	$f_{X_1}(x_1)$
1	0.40
2	0.30
3	0.20
4	0.10

x_2	$f_{X_2}(x_2)$
13	0.10
14	0.15
15	0.25
16	0.50

Problem 57. The pmf of a mixture distribution for a random variable X is given by $f_X(x) = 0.4 f_{X_1}(x) + 0.6 f_{X_2}(x)$ for all $x \in \mathcal{S}_X$.

(a) Determine $\mathrm{E}[X]$.

(b) Determine $\mathrm{Var}[X]$.

Problem 58. The pmf of a mixture distribution for a random variable X is given by $f_X(x) = 0.6 f_{X_1}(x) + 0.4 f_{X_2}(x)$ for all $x \in \mathcal{S}_X$.

(a) Determine $\mathrm{E}[X]$.

(b) Determine $\mathrm{Var}[X]$.

Problem 59. A die is rolled. If you roll a 1, 2, or 3, you will toss 10 coins. If you roll a 4, 5, or 6, you will toss 20 coins. Let X denote the number of heads obtained.

(a) Determine the support of X.

(b) Determine the mean of X.

(c) Determine the variance of X.

Problem 60. A die is rolled. If you roll a 1 or 2, you will toss 10 coins. If you roll a 3, 4, 5, or 6, you will toss 20 coins. Let X denote the number of heads obtained.

(a) Determine the support of X.

(b) Determine the mean of X.

(c) Determine the variance of X.

Sections 3.8-3.10

Problem 61. A die is to be rolled 8 times. Determine the probability that a 5 is rolled exactly 3 times.

Problem 62. A die is to be rolled 8 times. Let X count the number of times we roll a 5 or 6. Determine $P(X = 3)$.

Problem 63. The defect rate in a certain manufacturing process is known to be 1%. We now take a sample of 100 items. Determine the probability that there will be 3 or more defects among the 100 sampled items.

Problem 64. Based on past data, we feel that the probability of a full recovery from a certain type of injury is 0.80. Suppose that we have 18 patients in our rehab center with this type of injury. Let X count the number of full recoveries. Use the binomial CDF to determine the following probabilities:

(a) $P(X = 9)$

(b) $P(13 \leq X)$

(c) $P(8 \leq X < 13)$

(d) $P(X < 5)$

Problem 65. The probability that a person will be helped by a certain medicine is 0.90. A doctor will be seeing 15 patients today. Use the binomial CDF to determine the following:

(a) $P(\text{the medicine will help all 15 patients})$

(b) $P(\text{the medicine will help exactly 12 patients})$

(c) $P(\text{the medicine will help more than 9 patients})$

(d) $P(\text{the medicine helps all of the first 10 patients})$

(e) $P(\text{the medicine helped the first patients} \,|\, \text{the medicine helped exactly 14 patients})$

Problem 66. A fair coin is to be tossed 20 times. Let X count the number of heads tossed. Using the binomial CDF, determine the value of k so that $P(X \leq k) \approx 0.95$ (as close as you can get).

Problem 67. A fair coin is to be tossed 18 times. Let X count the number of heads tossed. Using the binomial CDF, determine the value of k so that $P(X \leq k) \approx 0.95$ (as close as you can get).

Problem 68. A fair coin is to be tossed 16 times. Let X count the number of heads tossed. Using the binomial CDF, determine the value of k so that $P(X \leq k) \approx 0.95$ (as close as you can get).

Problem 69. An urn contains 5 white chips, 4 red chips, and 3 blue chips. We will now select 8 chips **with replacement**. Let X count the number of red chips selected. Determine the following probabilities:

(a) $P(X = 4)$

(b) $P(X = 6)$

(c) $P(1 \leq X)$

Problem 70. An urn contains 5 white chips, 4 red chips, and 3 blue chips. We will now select chips, **with replacement**, until we get a blue chip. Let X count the number of chips needed to end this experiment. Determine the following probabilities:

(a) $P(X = 4)$

(b) $P(X = 6)$

(c) $P(2 \leq X)$

Problem 71. An urn contains 5 white chips, 4 red chips, and 3 blue chips. We will now select chips, **with replacement**, until we get a blue chip. Determine the following probabilities:

(a) P(the first blue chip is selected on the third trial)

(b) P(the first blue chip is selected on the fifth trial)

Problem 72. An urn contains 5 white chips, 4 red chips, and 3 blue chips. We will now select chips, **with replacement**, until we get a blue chip for the third time. Let X count the number of chips needed to end this experiment. Determine the following probabilities:

(a) $P(X = 4)$

(b) $P(X = 6)$

(c) $P(1 \leq X)$

(d) $P(5 \leq X)$

Problem 73. An urn contains 5 white chips, 4 red chips, and 3 blue chips. We will now select chips **with replacement**. Determine the following probabilities:

(a) P(the second blue chip is selected on the fifth trial)

(b) P(the fourth blue chip is selected on the tenth trial)

Problem 74. An urn contains 5 white chips, 4 red chips, and 3 blue chips. We will now select 8 chips **without replacement**. Let X count the number of red chips selected. Determine the following probabilities:

(a) $P(X = 4)$

(b) $P(X = 2)$

(c) $P(1 \leq X)$

Problem 75. An urn contains 5 white chips, 4 red chips, and 3 blue chips. We will now select 8 chips **without replacement**. Determine the probability that we get exactly 3 red chips.

Problem 76. An urn contains 5 white chips, 4 red chips, and 3 blue chips. We will now select chips, **without replacement**, until we get a blue chip. Let X count the number of chips needed to end this experiment. Determine the following probabilities:

(a) $P(X = 4)$

(b) $P(X = 6)$

(c) $P(2 \leq X)$

Problem 77. An urn contains 5 white chips, 4 red chips, and 3 blue chips. We will now select chips, **without replacement**, until we get a blue chip. Determine the following probabilities:

(a) P(the first blue chip is selected on the third trial)

(b) P(the first blue chip is selected on the fifth trial)

Problem 78. An urn contains 5 white chips, 4 red chips, and 3 blue chips. We will now select chips, **without replacement**, until we get a blue chip for the third time. Let X count the number of chips needed to end this experiment. Determine the following probabilities:

(a) $P(X = 4)$

(b) $P(X = 6)$

(c) $P(1 \leq X)$

Problem 79. An urn contains 5 white chips, 4 red chips, and 3 blue chips. We will now select chips **without replacement**. Determine the following probabilities:

(a) P(the second blue chip is selected on the fifth trial)

(b) P(the fourth blue chip is selected on the tenth trial)

Problem 80. A coin is to be tossed until we get a heads. Determine the probability that this happens on:

(a) the third toss

(b) the fifth toss

(c) the tenth toss

Problem 81. A coin is to be tossed 25 times. Determine the probability that we get:

(a) exactly 10 heads

(b) exactly 15 heads

(c) exactly 20 heads

Problem 82. A coin is to be tossed 30 times. Determine the probability that we get:

(a) exactly 10 heads

(b) exactly 15 heads

(c) exactly 20 heads

Problem 83. A coin is to be tossed 30 times. Determine the probability that:

(a) the fourth head occurs on the tenth toss

(b) the eighth head occurs on the twelfth toss

Problem 84. A coin is to be tossed 40 times. Determine the probability that:

(a) the fourth head occurs on the tenth toss

(b) the eighth head occurs on the twelfth toss

Problem 85. Seven cards are selected from a standard deck of cards. Determine the probability that exactly 5 are hearts.

Problem 86. Seven cards are selected from a standard deck of cards. Determine the probability that exactly 6 are hearts.

Problem 87. Seven cards are selected from a standard deck of cards. Determine the probability that exactly 7 are hearts.

Problem 88. Seven cards are selected from a standard deck of cards. Determine the probability of getting a flush.

Problem 89. Seven cards are selected from a standard deck of cards. Determine the probability that three of them are aces.

Problem 90. Seven cards are selected from a standard deck of cards. Determine the probability that two of them are aces.

Problem 91. Seven cards are selected from a standard deck of cards. Determine the probability that the second ace occurs on the sixth card selected.

Section 3.11

Problem 92. Suppose that the random variable X follows a Poisson distribution with parameter $\lambda = 2.5$. Determine $P(X = 4)$.

Solution: $P(X = 4) = \frac{e^{-2.5}(2.5)^4}{4!} \approx 0.1336$ □

Problem 93. Suppose that the random variable X follows a Poisson distribution with parameter $\lambda = 7$. Determine the following:

(a) $P(X = 6)$

(b) $P(5 \leq X \leq 12)$

(c) the value of k such that $P(k \leq X) \approx 0.10$

Solution:

(a) $P(X = 6) = \Pi(6) - \Pi(5) \approx 0.4497 - 0.3007 \approx 0.1490$

(b) $P(5 \leq X \leq 12) = \Pi(12) - \Pi(4) \approx 0.9730 - 0.1730 \approx 0.8000$

(c) In order for $P(k \leq X) \approx 0.10$, we need $P(X \leq k-1) \approx 0.90$. Since $P(X \leq 10) \approx 0.9015 \approx 0.90$, we see that $k - 1 = 10$ so that $k = 11$.

□

Problem 94. Suppose that the random variable X follows a Poisson distribution with parameter $\lambda = 3.5$. Determine the following:

(a) $P(X = 2)$

(b) $P(X = 3)$

(c) $P(X = 4)$

(d) $P(X = 4.5)$

Problem 95. Suppose that the random variable X follows a Poisson distribution with parameter $\lambda = 3.1$. Determine the following:

(a) $P(X = 0)$

(b) $P(X = 3)$

(c) $P(X \leq 3)$

(d) $\mathrm{E}[X]$

(e) $\mathrm{Var}[X]$

Problem 96. Suppose that the random variable X follows a Poisson distribution with parameter $\lambda = 12$. Determine the following:

(a) $P(X = 10)$

(b) $P(X \geq 10)$

(c) $P(X < 10)$

(d) $\mathrm{E}[X]$

(e) $\mathrm{Var}[X]$

Problem 97. Suppose that the random variable X follows a Poisson distribution with parameter $\lambda = 12$. Determine the following:

(a) $P(X = 8)$

(b) $P(6 \leq X \leq 11)$

(c) the value of k such that $P(k \leq X) \approx 0.10$

(d) $P(X = 15)$

(e) $P(6 < X < 11)$

(f) the value of k such that $P(X < k) \approx 0.15$

Problem 98. Suppose that the random variable X follows a Poisson distribution with parameter $\lambda = 15$. Determine the following:

(a) $P(X = 10)$

(b) $P(9 \leq X \leq 17)$

(c) the value of k such that $P(k \leq X) \approx 0.05$

(d) $P(X = 15)$

(e) $P(9 < X < 17)$

(f) the value of k such that $P(X < k) \approx 0.07$

Problem 99. Suppose that the random variable X follows a Poisson distribution with parameter $\lambda = 18$. Determine the following:

(a) $P(X = 12)$

(b) $P(12 \leq X \leq 20)$

(c) the value of k such that $P(k \leq X) \approx 0.10$

(d) $P(X = 19)$

(e) $P(12 < X < 20)$

(f) the value of k such that $P(X < k) \approx 0.05$

Problem 100. The number of phone calls coming into a call center is well-modeled by a Poisson distribution with an arrival rate of 10 calls per 20-minute period. Let X denote the number of calls coming in the next 20 minutes. Determine the following:

(a) $P(X = 12)$

(b) $P(12 \leq X \leq 20)$

(c) Let Y count the number of phone calls in the next 24 minutes. Determine $P(Y = 15)$.

(d) Let Z count the number of phone calls in the next 15 minutes. Determine $P(Z = 15)$.

Solution:

(a) $P(X = 12) = \Pi(12) - \Pi(11) \approx 0.7916 - 0.6968 \approx 0.0948$

(b) $P(12 \leq X \leq 20) = \Pi(20) - \Pi(11) \approx 0.9984 - 0.6968 \approx 0.3016$

(c) Note that a Poisson rate of 10 calls per 20-minute period implies a rate of 12 calls per 24-minute period. Using $\lambda = 12$, we get $P(Y = 15) = \Pi(15) - \Pi(14) \approx 0.8444 - 0.7720 \approx 0.0724$.

(d) Note that a Poisson rate of 10 calls per 20-minute period implies a rate of 7.5 calls per 15-minute period. Using $\lambda = 7.5$, we get $P(Z = 15) = \Pi(15) - \Pi(14) \approx 0.9954 - 0.9897 \approx 0.0057$.

$\square$

Problem 101. The number of phone calls coming into a call center is well-modeled by a Poisson distribution with an arrival rate of 9 calls per 15-minute period. Let X denote the number of calls coming in the next 15 minutes. Determine the following:

(a) $P(X = 8)$

(b) $P(6 \leq X \leq 10)$

(c) $P(6 < X < 10)$

(d) Let Y count the number of phone calls in the next 20 minutes. Determine $P(Y = 11)$.

(e) Let Z count the number of phone calls in the next 7.5 minutes. Determine $P(Z = 11)$.

Problem 102. The number of phone calls coming into a call center is well-modeled by a Poisson distribution with an arrival rate of 12 calls per 30-minute period. Let X denote the number of calls coming in the next 30 minutes. Determine the following:

(a) $P(X = 8)$

(b) $P(6 \leq X \leq 10)$

(c) $P(6 < X < 10)$

(d) Let Y count the number of phone calls in the next 20 minutes. Determine $P(Y = 20)$.

(e) Let Z count the number of phone calls in the next 60 minutes. Determine $P(Z = 20)$.

Problem 103. The number of phone calls coming into a call center is well-modeled by a Poisson distribution with an arrival rate of 15 calls per 25-minute period. Let X denote the number of calls coming in the next 25 minutes. Determine the following:

(a) $P(X = 10)$

(b) $P(10 \leq X \leq 18)$

(c) $P(10 < X < 18)$

(d) Let Y count the number of phone calls in the next 20 minutes. Determine $P(Y = 11)$.

(e) Let Z count the number of phone calls in the next 12 minutes. Determine $P(Z = 11)$.

Problem 104. The number of patients requiring a certain type of treatment in a given day is well-modeled by a Poisson distribution with a mean rate of $\lambda = 6$. The facility currently has 4 machines that are required for the treatment. Each machine can be used on 2 patients per day.

(a) Determine the probability that they have enough machines to treat all patients on a given day.

(b) How many machines would they need if they wanted the probability that they would have enough machines to treat all patients on a given day to exceed 0.99?

Solution:

(a) Note that the facility can handle $4 \cdot 2 = 8$ or fewer patients per day. Let X be the number of patients that need treatment on a given day. The facility would be able to accommodate all patients if $X \leq 8$. Thus, the probability that they have enough machines to treat all patients on a given day is $P(X \leq 8) = \Pi(8) \approx 0.8472$.

(b) Since $P(X \leq 12) = \Pi(12) \approx 0.9912 > 0.99$, we see that 12 or fewer patients needing treatment on a given day occurs with probability just over 99%. Thus, if the facility had 6 machines, the probability that the 6 machines would be enough to treat all patients on a given day would be $0.9912 > 0.99$.

$\square$

Problem 105. The number of patients requiring a certain type of treatment in a given day is well-modeled by a Poisson distribution with a mean rate of $\lambda = 5$. The facility currently has 2 machines that are required for the treatment. Each machine can be used on 3 patients per day.

(a) Determine the probability that they have enough machines to treat all patients on a given day.

(b) How many machines would they need if they wanted the probability that they would have enough machines to treat all patients on a given day to exceed 0.99?

Problem 106. The number of patients requiring a certain type of treatment in a given day is well-modeled by a Poisson distribution with a mean rate of $\lambda = 14$. The facility currently has 5 machines that are required for the treatment. Each machine can be used on 3 patients per day.

(a) Determine the probability that they have enough machines to treat all patients on a given day.

(b) How many machines would they need if they wanted the probability that they would have enough machines to treat all patients on a given day to exceed 0.999?

Problem 107. The number of patients requiring a certain type of treatment in a given day is well-modeled by a Poisson distribution with a mean rate of $\lambda = 8$. The facility currently has 3 machines that are required for the treatment. Each machine can be used on 3 patients per day.

(a) Determine the probability that they have enough machines to treat all patients on a given day.

(b) How many machines would they need if they wanted the probability that they would have enough machines to treat all patients on a given day to exceed 0.999?

Problem 108. The number of patients requiring a certain type of treatment in a given day is well-modeled by a Poisson distribution with a mean rate of $\lambda = 8$. The facility currently has 5 machines that are required for the treatment. Each machine can be used on 2 patients per day.

(a) Determine the probability that they have enough machines to treat all patients on a given day.

(b) How many machines would they need if they wanted the probability that they would have enough machines to treat all patients on a given day to exceed 0.99?

Problem 109. Suppose a life insurance company sold 10,000 policies ($100,000 payout value on each policy) this year. The probability of death for each person has been determined to be 0.001. The company charged $130 for each policy. Use the Poisson approximation to the binomial to determine the following probabilities:

(a) P(the company breaks even)

(b) P(the company profits $200,000 or more)

(c) P(the company loses $300,000 or more)

Solution: Since $n = 10{,}000$ and $p = 0.001$, we should use a Poisson rate of $\lambda = np = 10{,}000(0.001) = 10$. Also note that the company's total revenue is $10{,}000(\$130) = \$1{,}300{,}000$. Let X be the number of policyholders that die during the year. The amount paid out by the insurance company is $\$100{,}000X$.

(a) The company breaks even if $X = 13$, which happens with probability $P(X = 13) = \Pi(13) - \Pi(12) \approx 0.8645 - 0.7916 \approx 0.0729$.

(b) The company profits $200,000 or more if 11 or fewer policyholders die, i.e., $X \leq 11$. This occurs with probability $P(X \leq 11) = \Pi(11) \approx 0.6968$.

(c) The company loses $300,000 or more if 16 or more policyholders die, i.e., $X \geq 16$. This occurs with probability $P(X \geq 16) = 1 - P(X \leq 15) = 1 - \Pi(15) \approx 1 - 0.9513 \approx 0.0487$.

$\square$

Problem 110. Suppose a life insurance company sold 8,000 policies (\$100,000 payout value on each policy) this year. The probability of death for each person has been determined to be 0.002. The company charged \$225 for each policy. Use the Poisson approximation to the binomial to determine the following probabilities:

(a) P(the company breaks even)

(b) P(the company profits \$200,000 or more)

(c) P(the company loses \$300,000 or more)

Problem 111. Suppose a life insurance company sold 4,000 policies (\$200,000 payout value on each policy) this year. The probability of death for each person has been determined to be 0.003. The company charged \$800 for each policy. Use the Poisson approximation to the binomial to determine the following probabilities:

(a) P(the company breaks even)

(b) P(the company profits \$400,000 or more)

(c) P(the company loses \$600,000 or more)

Problem 112. Suppose a life insurance company sold 16,000 policies (\$100,000 payout value on each policy) this year. The probability of death for each person has been determined to be 0.00125. The company charged \$150 for each policy. Use the Poisson approximation to the binomial to determine the following probabilities:

(a) P(the company breaks even)

(b) P(the company profits \$400,000 or more)

(c) P(the company loses \$200,000 or more)

Problem 113. Nationwide, a certain disease occurs in only a small proportion of people (0.00008). A certain town has 150,000 residents. Use the Poisson approximation to the binomial to answer the questions below.

(a) What is the expected number of residents with the disease?

(b) Determine the probability that the town has exactly 10 residents with the disease.

(c) Suppose the town has 15 residents with the disease. Is this more than expected?

(d) Suppose the town has 15 residents with the disease. Should we be alarmed?

(e) Determine the probability that the town has 15 or more residents with the disease.

(f) Suppose that we wanted to create a 98% to 2% split for the number of cases for a town of this size. In other words, we need to determine the "cut-off" value k such that $P(X \leq k) \approx 0.98$ and $P(X > k) \approx 0.02$. Determine k.

(g) Suppose the town had 27 or more residents with the disease. Should we be alarmed?

Solution:

(a) Since $n = 150{,}000$ and $p = 0.00008$, we should use a Poisson rate of $\lambda = np = 150{,}000(0.00008) = 12$. The expected number of residents with the disease is 12.

(b) $P(X = 10) = \pi(10) = \frac{e^{-12}(12^{10})}{10!} \approx 0.1048$ or $P(X = 10) = \Pi(10) - \Pi(9) \approx 0.3472 - 0.2424 \approx 0.1048$

(c) Yes, 15 is more than expected since the expected number of residents with the disease is 12 from part (a).

(d) It's hard to say without calculating a probability. While the expected number of residents with the disease is 12, this doesn't mean that we should definitely observe exactly 12 residents with the disease. So we have a few more cases than expected. The question is, how rare or unusual is it to have 3 cases (or more) than expected?

(e) $P(X \geq 15) = 1 - P(X \leq 14) = 1 - \Pi(14) \approx 1 - 0.7720 \approx 0.2280$; note that 15 or more cases occurs with probability 22.8%, which would not be considered rare or unusual, so we should not be alarmed.

(f) Since $P(X \leq 19) = \Pi(19) \approx 0.9787 \approx 0.98$, we see that $k = 19$. This means there is about a 98% chance that the number of cases is in the interval $[0, 19]$ and about a 2% chance that the number of cases is in the interval $[20, 150{,}000]$ for a town of this size with disease rate 0.00008. In other words, if the number of cases is 20 or more, this would be considered rare or unusual.

(g) From part (f), 27 cases would be considered rare or unusual. And since 27 is substantially more than the "cut-off" for "rare", we should be alarmed. How rare is 27 (or more) cases? Since $P(X \geq 27) = 1 - P(X \leq 26) = 1 - \Pi(26) \approx 1 - 0.99987 \approx 0.00013$, we would expect to see a number like 27 or more only about 1 in every 10,000 towns of this size. So this is incredibly rare. What is one feasible explanation for why we are observing something so unusual? Note that this probability was calculated using a disease rate of $p = 0.00008$. Perhaps the actual disease rate in this town is higher than 0.00008, which is why we're observing an unusually large number of cases.

$\square$

Note that in the previous problem, we used probability to assess the “rareness” of some event (27 or more cases). Based on that rareness, we then attempted to make some sort of decision, such as “should we be alarmed if we observe 27 cases?” This is an example of inferential statistics, a branch of statistics in which we try to make decisions about a population based on a random sample from the population. Below is a summary of what was done in this problem:

- We started by formulating a hypothesis. In this case, we assumed this was a typical town with a disease rate of $p = 0.00008$.
- We collected data from the population, which led to our observation that the town had 27 residents with the disease.
- We then determined if what we observed was unusual, under the assumption that this was a typical town. In this case, what we observed was unusual since $P(X \geq 27) \approx 0.00013$.
- We then made a decision based on how unusual the observed results were. If this were a typical town, we should not have 27 cases of the disease. What we're observing here is not consistent with a typical town that has a low disease rate of $p = 0.00008$. The most reasonable explanation is that the reason we're observing an usually large number of cases is because this town is not a “typical” town and actually has a disease rate that is higher than 0.00008.

Problem 114. Nationwide, a certain disease occurs in only a small proportion of people (0.00015). A certain town has 20,000 residents. Use the Poisson approximation to the binomial to answer the questions below.

(a) What is the expected number of residents with the disease?

(b) Determine the probability that the town has 3 or fewer residents with the disease.

(c) Determine the number of cases k such that $P(X \leq k) \approx 0.05$.

(d) Determine the number of cases k such that $P(k \leq X) \approx 0.05$.

Problem 115. Nationwide, a certain disease occurs in only a small proportion of people (0.0002). A certain town has 70,000 residents. Use the Poisson approximation to the binomial to answer the questions below.

(a) What is the expected number of residents with the disease?

(b) Determine the probability that the town has exactly 10 residents with the disease.

(c) Suppose that we wanted to create a 98% to 2% split (as close to 98% as we can get) for the number of cases for a town of this size. Determine the split.

(d) If the town had 28 or more residents with the disease, what would you think based on the answer to part (c)?

(e) Determine the probability that the town has 28 or more residents with the disease.

Problem 116. Nationwide, a certain disease occurs in only a small proportion of people (0.00002). A certain town has 150,000 residents. Use the Poisson approximation to the binomial to answer the questions below.

(a) What is the expected number of residents with the disease?

(b) Determine the probability that the town has exactly 4 residents with the disease.

(c) Suppose that we wanted to create a 99% to 1% split (as close to 99% as we can get) for the number of cases for a town of this size. Determine the split.

(d) If the town had 10 or more residents with the disease, what would you think based on the answer to part (c)?

(e) Determine the probability that the town has 10 or more residents with the disease.

Problem 117. Nationwide, a certain disease occurs in only a small proportion of people (0.0006). A certain town has 20,000 residents. Use the Poisson approximation to the binomial to answer the questions below.

(a) What is the expected number of residents with the disease?

(b) Determine the probability that the town has exactly 10 residents with the disease.

(c) Suppose that we wanted to create a 98% to 2% split (as close to 98% as we can get) for the number of cases for a town of this size. Determine the split.

(d) If the town had 16 or more residents with the disease, what would you think based on the answer to part (c)?

(e) Determine the probability that the town has 16 or more residents with the disease.