Weierstrass Theorem Illustration

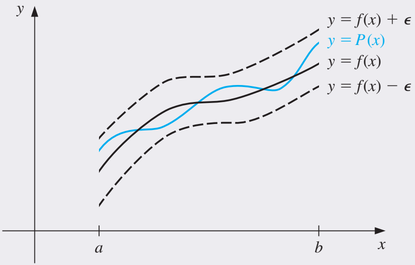

Figure: Figure 3.1

3.1 Interpolation and Lagrange Polynomial

Lagrange polynomials.

Motivation: While the Taylor polynomials do a good job approximating a function in the vicinity of a specified point x_0, they typically do not provide desired accuracy over an entire interval as they concentrate their accuracy near x_0. Usually, the approximation error gets worse at x values farther away from x_0 . Note that while increasing the degree of the polynomial helps to reduce the approximation error in some cases (see example of $f(x) = e^x, x_0 = 0; top\ of\ p.105$), this doesn't help in the other instances (example of $f(x) = \frac{1}{x}, x_0 = 1; bottom\ of\ p.105$). This motivates development of interpolation/approximation methods utilizing information at various points vs. a single point.

Let us start with most basic problem of determining a polynomial of degree one that passes through a pair of points $\{(x_0, y_0), (x_1, y_1)\}$. This is a polynomial interpolating function $f(x)$ that satisfies $f(x_0) = y_0$ and $f(x_1) = y_1$.

Interpolation means that the values of the polynomial agree with the values of the function at the given points.

Define $L_0(x) = \frac{x - x_1}{x_0 - x_1}$, $L_1(x) = \frac{x - x_0}{x_1 - x_0}$.

Note that $L_0(x_0) = 1, L_0(x_1) = 0$, $L_1(x_0) = 0$, $L_1(x_1) = 1$.

Then the well-known linear function passing through two specified points can be written as

$P(x) = L_0(x) * f(x_0) + L_1(x) * f(x_1)$.

Let us now generalize this idea.

Given the points $\{(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)\}$,

introduce the elementary polynomials $L_{n,k}(x) = \frac{(x-x_0)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$

Clearly, $L_{n,k}(x_i) = 0, i \neq k, and\ L_{n,k}(x_k) = 1$. This is the background leading to Thm. 3.2.

Theorem (3.2: n^{th} Lagrange Interpolating Polynomial)

If $x_0, x_1, \ldots, x_n$ are $n+1$ distinct numbers and f is a function whose values are given at these numbers, then a unique polynomial $P(x)$ of degree at most n exists with

$$f(x_k) = P(x_k), \quad \text{for each } k = 0, 1, \ldots, n.$$

This polynomial is given by

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x),$$

where, for each $k = 0, 1, \ldots, n$,

$$L_{n,k}(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)} = \prod_{\substack{i=0\\ i\neq k}}^{n} \frac{(x-x_i)}{(x_k-x_i)}.$$

Theorem (3.3)

Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each x in $[a, b]$, a number $\xi(x)$ (generally unknown) between $\min\{x_0, x_1, \ldots, x_n\}$, and the $\max\{x_0, x_1, \ldots, x_n\}$ and hence in (a, b), exists with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n),$$

where $P(x)$ is the interpolating polynomial given in Theorem 3.2.

3.1 #1(a). For the given function $f(x) = \cos x$, let $x_0 = 0, x_1 = 0.6, x_2 = 0.9$. Construct interpolation polynomials of degree at most one and at most two to approximate $f(0.45)$ and find the absolute error.

$L_0(x) = \frac{x-0.6}{0-0.6} = -\frac{x-0.6}{0.6}$, $L_1(x) = \frac{x-0}{0.6-0} = \frac{x}{0.6}$.

$P_1(x) = -\frac{x-0.6}{0.6} * 1 + \frac{x}{0.6} * \cos 0.6 \ = 0.148878x + 1.$

$$P_1(0.45) = 0.933005$$

Abs. error = $|\cos 0.45 - 0.933005\,| = 0.032558.$

The calculation of $P_2(x)$ (details are omitted here) results in

$P_2(x) = -0.452592x^2 - 0.0131009x + 1; so\ P_2(0.45) = 0.902455$

Abs. error = $|\cos 0.45 - 0.933005\,| = 0.002008.$

#7(a). Let $f(x) = x \ln x$; $x_0 = 8.1, x_1 = 8.3, x_2 = 8.6, x_3 = 8.7.$ Exercise #5(a) asked to approximate $f(8.4)$ using interpolating Lagrange polynomials of degrees one, two, and three. Use the error formula (given in Thm. 3.3) to find a bound for the error and compare the bound to the actual error for the cases n=1 and n=2.

We will need the derivatives of $f(x) = x \ln x$:

$f'(x) = \ln x + 1;\ f''(x) = \frac{1}{x}\ \ f'''(x) = -\frac{1}{x^2}$.

n=1: $E = \frac{|f''(\xi)|}{2!}|(8.4 - 8.3)(8.4 - 8.6)| \leq \frac{1}{2*8.3}$ $*0.02 = 1.2 * 10^{-3}$.

Note that $8.3 < \xi < 8.6$. Actual error = $1.18 * 10^{-3}$.

n=2: $E(8.4 - 8.6) = \frac{|f'''(\xi)|}{3!}|(8.4 - 8.3)(8.4 - 8.6)(8.4 - 8.7)| \leq \frac{1}{6*8.3^2} * (0.1)(0.2)(0.3) = 1.452 * 10^{-5}$.

Note that $8.3 < \xi < 8.7$. Actual error = $1.367 * 10^{-5}$.

#13(c). Construct the Lagrange interpolating polynomial (skip) and find a bound for the absolute error on the interval $[x_0, x_n]$.

$$f(x) = \ln x; \;\; x_0 = 1, x_1 = 1.1, x_2 = 1.3, x_3 = 1.4$$

$$f'(x) = \frac{1}{x} \quad f''(x) = -\frac{1}{x^2}$$

$$f'''(x) = 2x^{-3} \quad f^{(4)}(x) = -6x^{-4}\,; \quad n = 3$$

$$\max_{x\epsilon[x_0,x_3]} E(x) \le \frac{1}{4!} * \max_{x\epsilon[1,1.4]} \left|f^{(4)}\big(\xi(x)\big)\right|$$
$$* \max_{x\epsilon[1,1.4]} |(x-1)(x-1.1)(x-1.3)(x-1.4)| =$$

$\frac{6*1^{-4}}{4!}(0.1)^2(0.2)^2 = 10^{-4}$.

Note that $1 < \xi(x) < 1.4$ when $x\epsilon[1,1.4]$.