Matrix Decomposition

✓ LU

→ QR

→ Spectral (square matrices)

SVD

Householder Triangularization

Look at G-S again

In G-S, each operation to compute a column of $\hat{\underline{Q}}$ is an upper triangular matrix multiplication

$$\underline{A}\,\underbrace{\underline{R}_1\underline{R}_2\cdots\underline{R}_n}_{\hat{\underline{R}}^{-1}} = \hat{\underline{Q}} \Rightarrow \underline{A} = \hat{Q}\hat{R}$$

This is called Triangular Orthogonalization:

R gives Q

Difficulty maintaining orthogonality due to accummulation of round-off errors

Improves by using Modified Gram-Schmidt, rather than Classical Gram-Schmidt, but is still an issue

One can do the reverse: repeated applications of $\underset{\sim}{Q}$ give R

$$\underbrace{\underset{\sim}{Q}_n \underset{\sim}{Q}_{n-1} \cdots \underset{\sim}{Q}_2 \underset{\sim}{Q}_1}_{\hat{\underset{\sim}{Q}}^T} \underset{\sim}{A} = \hat{\underset{\sim}{R}} \Rightarrow \underset{\sim}{A} = \hat{\underset{\sim}{Q}} \hat{\underset{\sim}{R}}$$

where each $\underset{\sim}{Q}_k$ is orthogonal

This is called Orthogonal Triangularization

Q gives R

For this, we need to find the $\underset{\sim}{Q}_k$

The main idea is to find a matrix $\underset{\sim}{Q}_k$ that zeros out the values below a diagonal while preserving all prior zeros

$$\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} \xrightarrow{\underline{Q}_1} \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & x & x \\ 0 & x & x \end{bmatrix} \xrightarrow{\underline{Q}_2} \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix} \xrightarrow{\underline{Q}_3} \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{A} \qquad \underline{Q}_1\underline{A} \qquad \underline{Q}_2\underline{Q}_1\underline{A} \qquad \underline{Q}_3\underline{Q}_2\underline{Q}_1\underline{A} = \underline{R}$$

One more requirement: Each $\underline{Q}_k$ must be unitary

$$\underline{Q}_k^T \underline{Q}_k = \underline{Q}_k \underline{Q}_k^T = \underline{I}$$

Choose the following block matrix

$$\underline{Q}_k = \begin{bmatrix} \underline{I} & \underline{0} \\ \underline{0} & \underline{F} \end{bmatrix}$$

$\underline{I} \in (k-1) \times (k-1)$ identity matrix

$\underline{F} \in (m-k+1) \times (m-k+1)$

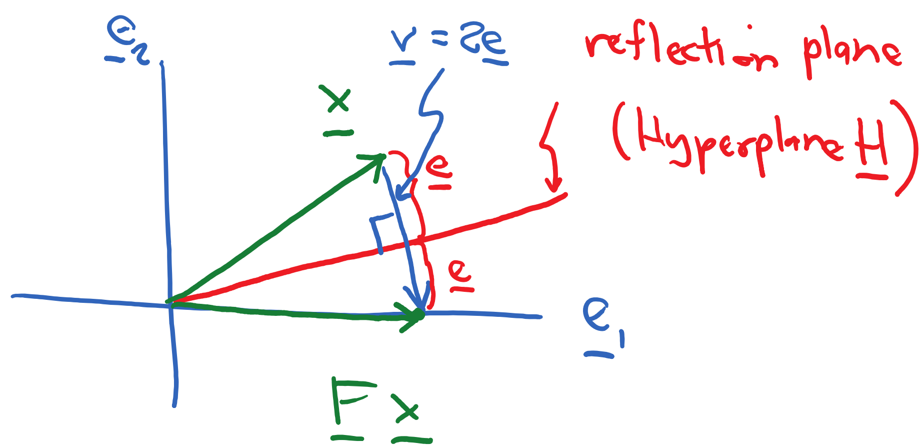
Householder reflector matrix

$\underline{F}$ is a specific type of operation, defined as follows:

$$\underline{x} = \begin{bmatrix} a \\ b \\ c \\ \vdots \end{bmatrix} \qquad \underline{F} \Rightarrow \underline{F}\,\underline{x} = \begin{bmatrix} \|\underline{x}\|_2 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \|\underline{x}\|_2\, \underline{e}_1$$

$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

How does this appear in 2d?



Hyperplane: A plane with a dimension one less than the embedding plane

(in 2d, H is 1d; in 3d, H is 2d, ...)

To determine this projection, look at the "error" vector between $\mathbf{F}\mathbf{x}$ and $\mathbf{x}$.

Let

$$\mathbf{v} = \mathbf{F}\mathbf{x} - \mathbf{x} = \|\mathbf{x}\|_2 \mathbf{e}_1 - \mathbf{x}$$

Is defined once $\mathbf{x}$ is defined

The key is that $\mathbf{v}$ is perpendicular to the hyperplane H (see diagram above in 2d)

To develop $\mathbf{F}$, project a vector $\mathbf{y}$ onto $\mathbf{H}$:

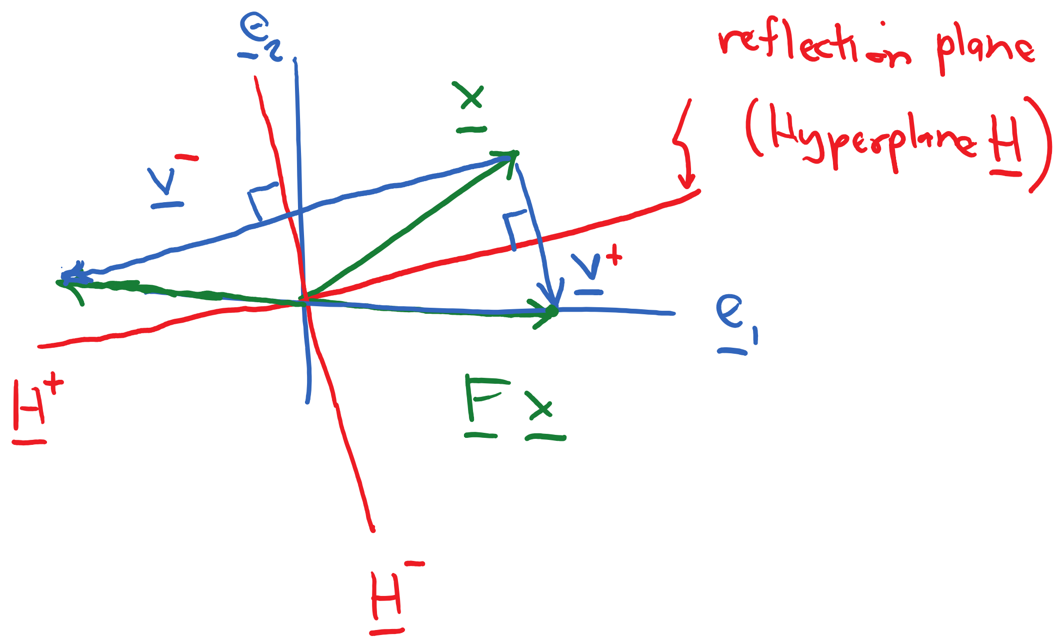
$$\mathbf{P}\mathbf{y} = \left(\mathbf{I} - \frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}\right)\mathbf{y} = \mathbf{y} - \mathbf{v}\left(\frac{\mathbf{v}^T\mathbf{y}}{\mathbf{v}^T\mathbf{v}}\right)$$

However, we actually need to go twice as far (see diagram above)

$$\mathbf{F}\mathbf{y} = \underbrace{\left(\mathbf{I} - \frac{2\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}}\right)}_{\mathbf{F}}\mathbf{y} = \mathbf{y} - 2\mathbf{v}\left(\frac{\mathbf{v}^T\mathbf{y}}{\mathbf{v}^T\mathbf{v}}\right)$$

Note: Householder reflectors are <u>not</u> unique

(see $\underline{H}^+$ and $\underline{H}^-$ below)



$$\underline{v}^+ = +\|\underline{x}\|_2\, \underline{e}_1 - \underline{x}$$

$$\underline{v}^- = -\|\underline{x}\|_2\, \underline{e}_1 - \underline{x}$$

Mathematically, choice does <u>not</u> matter

Numerically, choice does matter & we want large $\|\underline{v}\|$

$$\Rightarrow \text{Set } \underline{v} = -\operatorname{sign}(x_1)\, \|\underline{x}\|_2\, \underline{e}_1 - \underline{x}$$

w/ $\text{sign}(x) = 1$ if $x = 0$

and x_1 = first component of $\underline{x}$

After clearing the $\ominus$:

$$\underline{v} = \text{sign}(x_1)\, \|\underline{x}\|_2\, \underline{e}_1 + \underline{x}$$

Algorithm: Householder QR

for $k = 1:n$

$\underline{x} = \underline{A}(k:m, k)$

$\underline{v}_k = \text{sign}(x_1)\, \|\underline{x}\|_2\, \underline{e}_1 + \underline{x}$

$\underline{v}_k = \underline{v}_k / \|\underline{v}_k\|_2$

$\underline{A}(k:m, k:n) = \underline{A}(k:m, k:n) - 2\underline{v}_k\left(\underline{v}_k^T \underline{A}(k:m, k:n)\right)$

end

After these steps, $\underline{A}$ will be upper triangular

Note: Here $\hat{\underline{Q}}$ is never computed

To find $\hat{\underline{Q}}$ do the following, first define the operation of $\hat{\underline{Q}}\,\underline{x}$

for $k = n:-1:1$

$$\underline{x}(k:m) = \underline{x}(k:m) - 2\underline{v}_k\left(\underline{v}_k^T\underline{x}(k:m)\right)$$

end

To find $\hat{\underline{Q}}$ apply this operation to the identity matrix

$$\underline{Q}\,\underline{I} = \underline{Q} = \left[\underline{Q}\,\underline{e}_1 \;\; \underline{Q}\,\underline{e}_2 \;\; \cdots\right]$$

with $\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$, ...

QR Factorizations

QR Factorizations

Triangular Orthonormalization ($\hat{R} \to \hat{Q}$)

* Gram-Schmidt (Classical)

* Gram-Schmidt (Modified)

Orthogonal Triangularization ($\hat{Q} \to \hat{R}$)

* Householder Reflections

Givens Rotations (easier parallelization)

To motivate, consider the solutions to the following ordinary differential equation (ODE)

$$\frac{dy}{dt} = ay \quad \text{for } a = \text{constant}$$

Independent variable: t

Homogeneous: Yes

Dependent variable: $y \equiv y(t)$

Linear: Yes

Order: 1

Solution is $y(t) = C e^{at}$

Check: $\frac{dy}{dt} = \frac{d}{dt}\left(Ce^{at}\right) = aCe^{at} = ay(t)$

What happens if we have a set of two ODEs?

$$\frac{dy_1}{dt} = ay_1 \,, \quad \frac{dy_2}{dt} = by_2$$

Then $y_1(t) = C_1 e^{at}$, $y_2(t) = C_2 e^{bt}$

uncoupled

However, if instead we have

$$\frac{dy_1}{dt} = a y_2, \quad \frac{dy_2}{dt} = b y_1$$

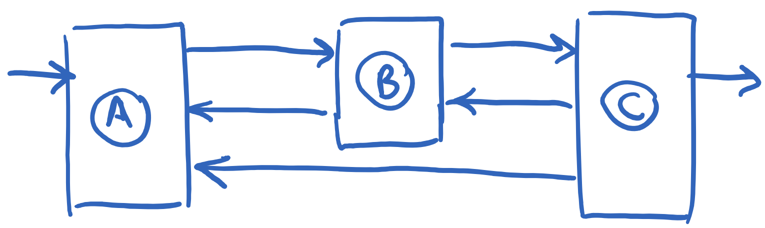
These ODEs are coupled. We need y_1 to solve for y_2 and vice versa

This is quite common in applications!

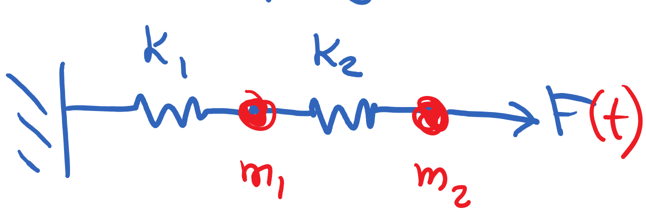
1. Chemical reactions

$$CO + H_2O \rightleftharpoons CO_2 + H_2$$

2. Flow between tanks



3. Series of springs



Generalize into the following for two dependent variables $y_1(t)$ + $y_2(t)$

$$\frac{dy_1}{dt} = a_{11} y_1 + a_{12} y_2$$

$$\frac{dy_2}{dt} = a_{21} y_1 + a_{22} y_2$$

with constants
$a_{11}, a_{12}, a_{21}, a_{22}$

Let $$\underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \underline{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\Rightarrow \frac{d\underline{y}}{dt} = \begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{bmatrix} \Rightarrow \frac{d\underline{y}}{dt} = \underline{A}\,\underline{y}$$

Assume a solution

$$y_1 = x_1 e^{\lambda t}, \quad y_2 = x_2 e^{\lambda t}$$

$$\frac{dy_1}{dt} = \lambda x_1 e^{\lambda t}, \quad \frac{dy_2}{dt} = \lambda x_2 e^{\lambda t}$$

$$\frac{d\underline{y}}{dt} = \underline{A}\,\underline{y}$$

$$\begin{bmatrix} \lambda x_1 \cancel{e^{\lambda t}} \\ \lambda x_2 \cancel{e^{\lambda t}} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a & \end{bmatrix} \begin{bmatrix} x_1 \cancel{e^{\lambda t}} \\ x_2 \cancel{e^{\lambda t}} \end{bmatrix}$$

$$\begin{bmatrix} \\ \lambda x_2 \cancel{e^{\lambda t}} \end{bmatrix} = \begin{bmatrix} & \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \\ x_2 \cancel{e^{\lambda t}} \end{bmatrix}$$

$$\underline{A}\,\underline{x} = \lambda \underline{x}$$

Eigensystem of $\underline{A}$

$$\underline{A}\,\underline{x} - \lambda \underline{x} = \underline{0}$$

$$(\underline{A} - \lambda \underline{I})\underline{x} = \underline{0}$$

λ : Eigenvalue

$\underline{x}$: Eigenvector

Frequently appears in applications

Eigensystems - Formulation (Part 1)

$$\underline{A}\,\underline{x} = \lambda \underline{x}$$ Eigensystem of $\underline{A}$

λ : Eigenvalue

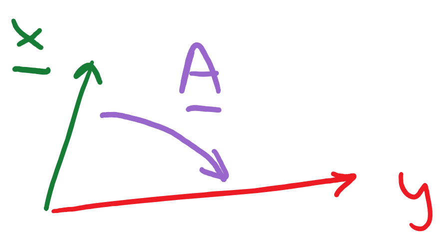
$\underline{x}$: Eigenvector

Eigensystems appear in many data science applications, e.g. networks & graphs, Markov chains, physical & social science systems

$$\underline{A}\,\underline{x} - \lambda \underline{x} = \underline{0}$$

$$(\underline{A} - \lambda \underline{I})\,\underline{x} = \underline{0}$$

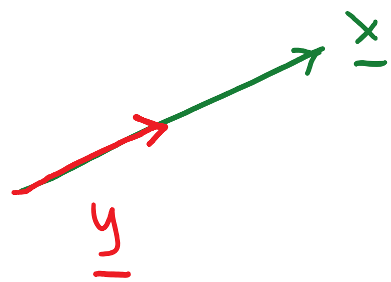
What, in general, does a matrix do?



$$\underline{A}\,\underline{x} = \underline{y}$$

Matrix $\underline{A}$ transforms vector $\underline{x}$ into $\underline{y}$

Now, consider $\underline{A}\,\underline{x} = \underbrace{\lambda\,\underline{x}}_{\underline{y}}$



$\underline{x}$ is a special set of vectors of $\underline{A}$ such that applying $\underline{A}$ to $\underline{x}$ does nothing but scale $\underline{x}$

Consider

$$\underline{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underline{A}\,\underline{x} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\updownarrow$ $\updownarrow$

λ $\underline{x}$

Eigenvalue $\lambda = 3$

Eigenvector $\underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Let $\underline{A} \in M_{n \times n}$ (Eigensystems such as these only valid for square matrices)

Given $\underline{A}$, find λ + $\underline{x}$

$$\underline{A}\,\underline{x} = \lambda\,\underline{x}$$

$$\underline{A}\,\underline{x} - \lambda\,\underline{x} = \underline{0}$$

$$(\underline{A} - \lambda\underline{I})\,\underline{x} = \underline{0}$$

Here $\underline{x}$ must be in the nullspace of $\underline{A} - \lambda\underline{I}$

We do <u>not</u> want the trivial solution. Thus,

$\Rightarrow \underline{A} - \lambda\underline{I}$ can not be of full rank

$\Rightarrow (\underline{A} - \lambda\underline{I})^{-1}$ does not exist

$$\Rightarrow \det(\underline{A} - \lambda \underline{I}) = 0$$

unknown eigenvalue

$\det(\underline{A} - \lambda \underline{I}) \Rightarrow$ Characteristic polynomial of matrix $\underline{A}$ (in terms of λ)

Eigensystem Procedure:

Given $\underline{A} \in M_{n \times n}$

① Solve for all λ's such that

$$\det(\underline{A} - \lambda \underline{I}) = 0$$

(Find roots of characteristic polynomial)

② For each λ_i, find the corresponding $\underline{x}_i$, such that

$$\underline{A}\,\underline{x}_i = \lambda_i \underline{x}_i$$

or

$$(\underline{A} - \lambda_i \underline{I})\underline{x}_i = \underline{0}$$

Example: Let $\underline{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$$\underline{A} - \lambda \underline{I} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$$

$$\det(\underline{A} - \lambda \underline{I}) = (1-\lambda)^2 - 4 = 0$$

Characteristic equation

$$(1-\lambda)^2 = 4, \quad 1-\lambda = \pm 2$$

$$\downarrow$$

$$\left.\begin{array}{l} \lambda_1 = -1 \\ \lambda_2 = 3 \end{array}\right\} \text{Eigenvalues of } \underline{A}$$

For $\lambda_1 = -1$

Solve $(\underline{A} - \lambda_1 \underline{I})\underline{x}_1 = \underline{0}$

$$\begin{bmatrix} 1-(-1) & 2 \\ 2 & 1-(-1) \end{bmatrix} \underline{x}_1 = \underline{0}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Note: Rows are linearly dependent, $\det(\underline{A} - \lambda_1 \underline{I}) = 0$

$\downarrow$ rref

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \underline{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Eigenvector associated with $\lambda_1 = -1$

Let $\lambda_2 = 3$

$$\begin{bmatrix} 1-\lambda_2 & 2 \\ 2 & 1-\lambda_2 \end{bmatrix} \underline{x}_2 = \underline{0}$$

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right] \xRightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\therefore \underline{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ Eigenvector associated with $\lambda_2 = 3$

Summary for $\underline{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Two eigenvalues & eigenvectors

$$\lambda_1 = -1, \quad \underline{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 3, \quad \underline{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Note: Eigenvectors can be multiplied by any non-zero constant & remain eigenvectors

Check:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\underline{A} \quad \underline{x}_1 \quad = \quad \lambda_1 \quad \underline{x}_1 \quad \checkmark$$

$$\underline{A}\,\underline{x}_1 = \lambda_1\,\underline{x}_1 \checkmark$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = (3)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underline{A}\,\underline{x}_2 = \lambda_2\,\underline{x}_2 \checkmark$$

Summary:

λ is an eigenvalue of square matrix $\underline{A}$ with eigenvector $\underline{x}$ iff

$$\underline{A}\,\underline{x} = \lambda\underline{x}$$

To determine λ, we need

$$(\underline{A} - \lambda\underline{I})\underline{x} = \underline{0}$$

For non-trivial solutions

$$\det(\underline{A} - \lambda\underline{I}) = |\underline{A} - \lambda\underline{I}| = 0$$

characteristic polynomial of $\underline{A}$

Find roots λ of characteristic polynomial

Example 2: Let $\underline{A} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

$$|\underline{A} - \lambda \underline{I}| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 4 = 0$$

$$\lambda^2 - 4\lambda - 1 = 0 \Rightarrow \lambda = \frac{-b \pm (b^2 - 4ac)^{1/2}}{2a}$$

$$\lambda = \frac{4 \pm (16+4)^{1/2}}{2} = 2 \pm \sqrt{5}$$

To find eigenvector, use λ for each root

1) $\lambda = 2 + \sqrt{5}$

$$\begin{bmatrix} 1-(2+\sqrt{5}) & 2 \\ 2 & 3-(2+\sqrt{5}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\det(\underbrace{\underline{A} - \lambda \underline{I}}_{\text{singular, rank} < 2}) = 0 \rightarrow$ rows must be linearly dependent

$$(-1-\sqrt{5})x_1 + 2x_2 = 0 \rightarrow \underline{x} = \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

2) $\lambda = 2 - \sqrt{5}$

$$\begin{bmatrix} 1 \end{bmatrix}$$

c) $\lambda = 2 - \sqrt{5}$

Similar procedure, $\underline{x} = \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$

Summary:

$$\lambda_1 = 2 + \sqrt{5}, \quad \underline{x}_1 = \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

$$\lambda_2 = 2 - \sqrt{5}, \quad \underline{x}_2 = \begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

Properties of Eigensystems:

1. Eigenvalues of $\underline{A}^2$ are the square of the eigenvalues of $\underline{A}$, but eigenvectors are exactly the same

Assume $\underline{A}\,\underline{x} = \lambda \underline{x}$ is known

Then,

$$\underline{A}^2 \underline{x} = \underline{A}(\underline{A}\underline{x}) = \underline{A}(\lambda \underline{x}) = \lambda \underline{A}\,\underline{x}$$

$$= \lambda(\lambda \underline{x}) = \lambda^2 \underline{x} \Rightarrow \underline{A}^2 \underline{x} = \lambda^2 \underline{x}$$

Higher powers:

$$\underline{A}^3 \underline{x} = \underline{A}(\underline{A}^2 \underline{x}) = \underline{A}(\lambda^2 \underline{x}) = \lambda^2 \underline{A}\underline{x} = \lambda^3 \underline{x}$$

$$\Rightarrow \underline{A}^3 \underline{x} = \lambda^3 \underline{x}$$

In general, eigenvalues of $\underline{A}^n$ are λ^n of

$$\underline{A}\underline{x} = \lambda \underline{x}$$

Recall Markov Chains:

$\underline{M}$ is transition matrix

Columns of $\underline{M}$ sum to one; all entries > 0

$\underline{P}_n$ probability of states after step n

$$\underline{P}_n = \underline{M}\,\underline{P}_{n-1}$$

$$\underline{P}_n = \underline{M}(\underline{M}\,\underline{P}_{n-2})$$

^{n}

$$\underline{P}_n = \underline{M}^n \underline{P}_0$$

may need to raise $\underline{M}$ to a large power; find eigensystem of $\underline{M}$

② Row reduction does not preserve eigenvalues

Row reduction involves the scaling & addition of the matrix rows

Example: $\underline{A} = \begin{bmatrix} 4 & -1 & 0 \\ 0 & 1 & 0 \\ 2 & -3 & 6 \end{bmatrix}$

$$\begin{vmatrix} 4-\lambda & -1 & 0 \\ 0 & 1-\lambda & 0 \\ 2 & 3 & 6-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda)(6-\lambda) = 0$$

$\therefore \lambda = 4, 1, 6$

Flip rows 1 & 3

$\underline{B} = \begin{bmatrix} 2 & -3 & 6 \\ 0 & 1 & 0 \\ 4 & -1 & 0 \end{bmatrix}$

$\begin{vmatrix} 2-\lambda & -3 & 6 \end{vmatrix}$

$$\begin{vmatrix} 2-\lambda & -3 & 6 \\ 0 & 1-\lambda & 0 \\ 4 & -1 & -\lambda \end{vmatrix} = (2-\lambda)(1-\lambda)(-\lambda) - 4(1-\lambda)6$$

$$= -(2-3\lambda+\lambda^2)\lambda - 24(1-\lambda)$$

$$= -\lambda^3 + 3\lambda^2 - 2\lambda + 24\lambda - 24$$

$$= -\left[\lambda^3 - 3\lambda^2 - 22\lambda + 24\right]$$

$$= -(\lambda-1)(\lambda+4)(\lambda-6)$$

$$\therefore \lambda = 1, \underline{\underline{-4}}, 6$$

$$\therefore \lambda \text{ of } \underline{B} \neq \lambda \text{ of } \underline{A}$$

③ The product of the eigenvalues of $\underline{A}$ equals $\det(\underline{A})$ and the sum of the eigenvalues equals $tr(\underline{A})$, where

$$tr(\underline{A}) = \text{trace of } \underline{A} = \text{sum of the diagonal}$$

Aside: ERO of swapping rows. What happens to $\det(\underline{A})$ after swap?