Basis

A basis for a vector space is the minimum # of vectors needed to span that space.

Terminology:

let $B = \{ \underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n \}$ be a basis for vector space V

1) If any $\underline{b}_i \cdot \underline{b}_j \neq 0$ if $i \neq j$ then this is simply called a basis

2) If $\underline{b}_i \cdot \underline{b}_j = 0$ if $i \neq j$ and at least one $\underline{b}_i \cdot \underline{b}_i \neq 1$, this is an orthogonal basis

3) If $\underline{b}_i \cdot \underline{b}_j = 0$ if $i \neq j$ & all $\underline{b}_i \cdot \underline{b}_i = 1$, this is an orthonormal basis.

The dimension of a vector space is the # of vectors in it's basis.

$V = \text{span}(B)$ $\quad \dim(V) = \dim(\text{span}(B)) = n$

$B = \{ \underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n \}$

ex.) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ $\quad \dim(\text{span}(B)) = 2$

Subspace of $\mathbb{R}^4$

ex) Taylor Series

$B = \{ (x-a)^0, (x-a)^1, (x-a)^2, \dots \}$

$f(x) = \alpha (x-a)^0 + \beta (x-a)^1 + \dots$

$\dim(\text{span}(B)) = \infty$

Next, tie $\dim(\text{span}(B)) = \dim(V)$ to rref

For $B = \{ \underline{b}_1, \underline{b}_2, \underline{b}_3, \underline{b}_4 \}$ to be a basis we need $x_1 \underline{b}_1 + x_2 \underline{b}_2 + x_3 \underline{b}_3 + x_4 \underline{b}_4 = \underline{0}$ iff $x_i = 0$

let $\underline{B} = [\underline{b}_1 \;\; \underline{b}_2 \;\; \underline{b}_3 \;\; \underline{b}_4]$ $\quad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$\underline{B}\,\underline{x} = x_1 \underline{b}_1 + x_2 \underline{b}_2 + x_3 \underline{b}_3 + x_4 \underline{b}_4$

The columns of $\underline{B}$ form a basis iff $\underline{x} = \underline{0}$ is the only solution to $\underline{B}\,\underline{x} = \underline{0}$

Look at $\text{rref}([\underline{B} \,\vdots\, \cancel{\underline{0}}]) = \text{rref}(\underline{B})$

If $\text{rref}(\underline{B}) = \underline{I}$, all of the columns of $\underline{B}$ are independent & thus form a basis.

ex.) $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

Is this a basis?

$$\text{rref}\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right) = \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$x_1 = 0 \quad x_2 = 0 \quad x_3 = 0$

$\Rightarrow$ A basis for $\mathbb{R}^3$

ex.) $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$\underline{s}_1 \quad \underline{s}_2 \quad \underline{s}_3 \quad \underline{s}_4$

$$\text{rref}([\underline{s}_1 \ \underline{s}_2 \ \underline{s}_3 \ \underline{s}_4]) = \begin{array}{cccc} & & & \\ \end{array}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\underline{s}_1 \quad \underline{s}_2 \quad \underline{s}_3 \quad \underline{s}_4$

Columns which have a pivot correspond to the basis for the original matrix.

ex.) $S = \left\{ \begin{bmatrix} 5 \\ -2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 1 \\ 10 \end{bmatrix}, \begin{bmatrix} 10 \\ 5 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} -5 \\ -7 \\ -3 \\ -3 \end{bmatrix} \right\}$

$\text{rref}([\underline{s}_1 \ \underline{s}_2 \ \underline{s}_3 \ \underline{s}_4 \ \underline{s}_5 \ \underline{s}_6])$

$$= \begin{bmatrix} 1 & 0 & -1/2 & 0 & 0 & 0 \\ 0 & 1 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Basis is $\{\underline{s}_1, \underline{s}_2, \underline{s}_4, \underline{s}_5\}$

Comments:

1) the basis vectors obtained using rref are <u>not</u> unique.

$\text{rref}([\underline{s}_6 \ \underline{s}_2 \ \underline{s}_1 \ \underline{s}_4 \ \underline{s}_3 \ \underline{s}_5])$ will return a different set but the # of vectors will always be 4.

2) the # of vectors in the basis might be smaller than the length.

$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\quad \dim(\text{span}(S)) = 2$

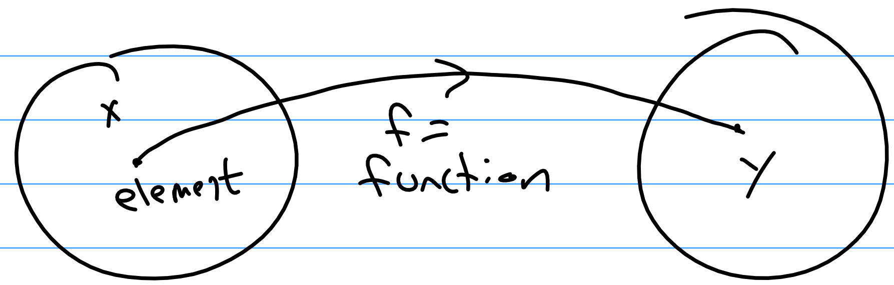
3) the maximum dimension of a vector space is the # of elements in a vector.

$S \in \mathbb{R}^6$, basis will have max of 6 vectors

Actual # will be between 1 and n

Functions

A function is a transformation of an element in the Domain into a single element in the codomain



Domain X

$x \in X$

Codomain Y

$y \in Y$

f = function

write as $f: X \to Y$ or $f(x) = y$

All functions will do:

1) Assign each element in X to a single element in Y

2) Multiple elements in X may be assigned to the same element Y.

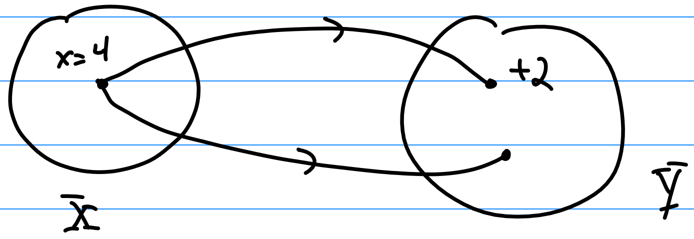
3) Not all elements in Y need to be mapped to.

ex.) let $f: X \to Y$ be the solution to $y^2 = x$

$\Rightarrow f(x) = \pm\sqrt{x}$

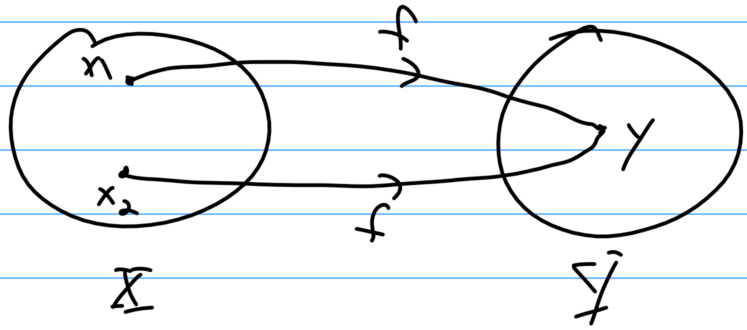
$\Rightarrow$ $x = 4$ gives both $+2$ & -2

$\Rightarrow$ Not a function



ex.) $X = \mathbb{R}^1$ $\quad Y = \mathbb{R}^1$

Is $\sin(x)$ a function? Yes



Comment: if f is a function depends not only on f but also the codomain.

ex.) let $f(x) = x^2$, $X = \mathbb{R}^1$

let $Y = \mathbb{R}^1 \Rightarrow f(x)$ is a function

$f(1.5) = (1.5)^2 \in Y$

let $Y = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$f(x)$ is <u>not</u> a function

$f(1.5) = (1.5)^2 \notin \mathbb{Z}$

Terms

Image : the image of a domain element is the (one) codomain element

$f(x)=x^2$, image of 2 is 4

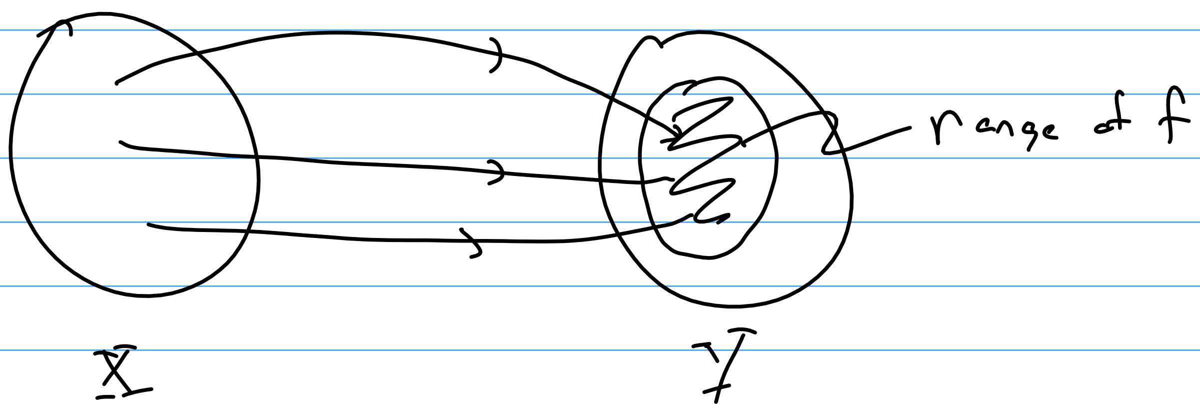
Pre-image : the domain element(s) that give a codomain element

$f(x)=x^2$, pre-image of 4 is -2 & $+2$

All elements in the domain have an image

Not all elements in the codomain must have a pre-image.

Range : the subspace of the codomain that all elements in the domain will map to.



ex.) $X=\mathbb{R}^1$ $Y=\mathbb{R}^1$ $f(x)=\sin(x)$

range$(f(x))$ = All real # w/ absolute value ≤ 1

All elements in range(f) have a pre-image one-to-one : A function is one-to-one iff every element in the domain has a unique element in the codomain.

To prove for $f(x)$ you need to show that $f(x_1)=f(x_2)$ only holds if $x_1=x_2$

ex.) Is $f(x)=x-1$ one-to-one?

let $x_1, x_2 \in \mathbb{R} = \underline{X}$

$$f(x_1) = f(x_2)$$
$$x_1 - 1 = x_2 - 1$$
$$x_1 = x_2 \Rightarrow \text{one-to-one}$$

ex.) Is $f(x)=x^2$ $x_1, x_2 \in X$ one-to-one?

$$f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2$$
$$\pm x_1 = \pm x_2$$
$$\Rightarrow -x_1 = +x_2, \quad +x_1 = -x_2 \Rightarrow x_1 \neq x$$

Not one-to-one

Onto : A function is onto iff every element in the co-domain has a pre-image. $\Rightarrow$ range $(f) = Y =$ Co-domain

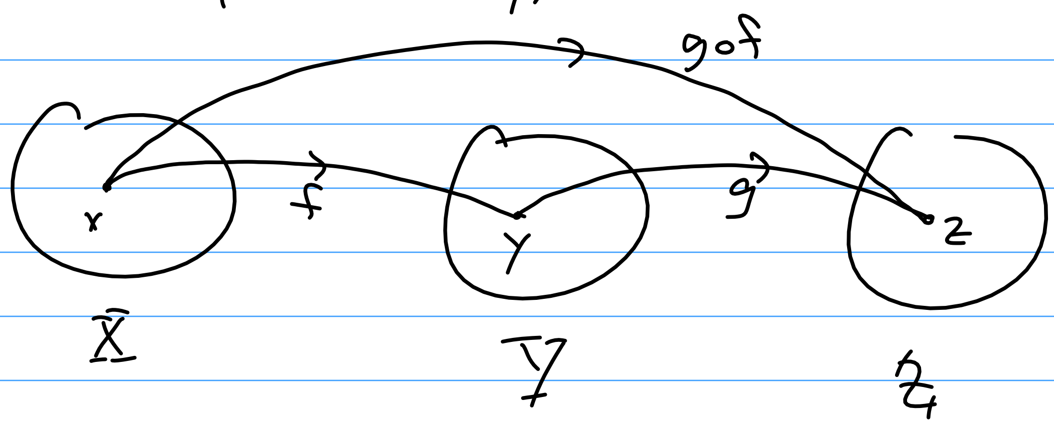
ex.) $f(x) = 2x$ $X, Y \in \mathbb{R}$ is onto

but, $X = \mathbb{Z}$, $Y = \mathbb{R}$, $f(x)$ is not onto

$f(x) = 4.5$? what $x \in \mathbb{Z}$

Function Composition

A function composition is a sequential application of functions.



$f: X \to Y$ $\qquad$ $g: Y \to Z$

Composition: $g \circ f: X \to Z$

$$g \circ f \Rightarrow g(f(x)) = z$$

ex) $f(x) = x+1$ $\qquad$ $g(y) = y^2$, $X, Y, Z \in \mathbb{R}$

$$(g \circ f)(x) = g(f(x)) = g(x+1) = (x+1)^2 = z$$

Thm: let $f: X \to Y$ & $g: Y \to Z$

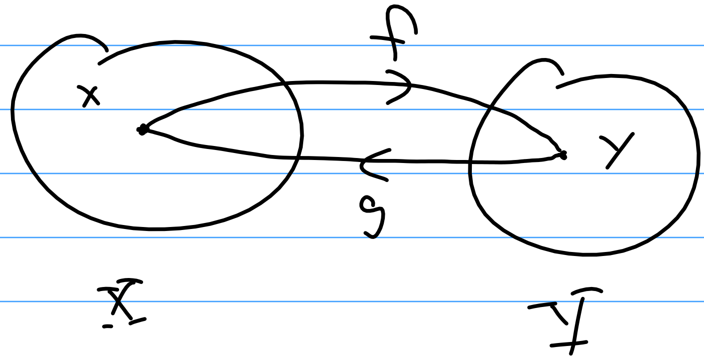
1) If f & g are onto $g \circ f$ is onto

2) If f & g are one-to-one, $g \circ f$ is one-to-one

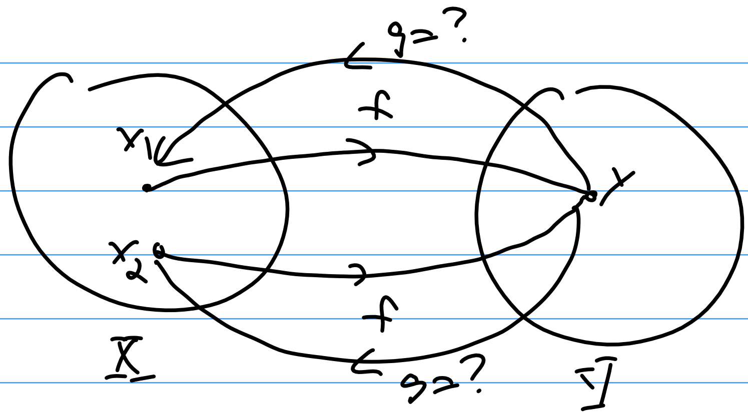
Function Inverses

functions $f: X \to Y$ & $g: Y \to X$ are inverses iff for any $x \in X$ & $y \in Y$ we have $(g \circ f)(x) = x$

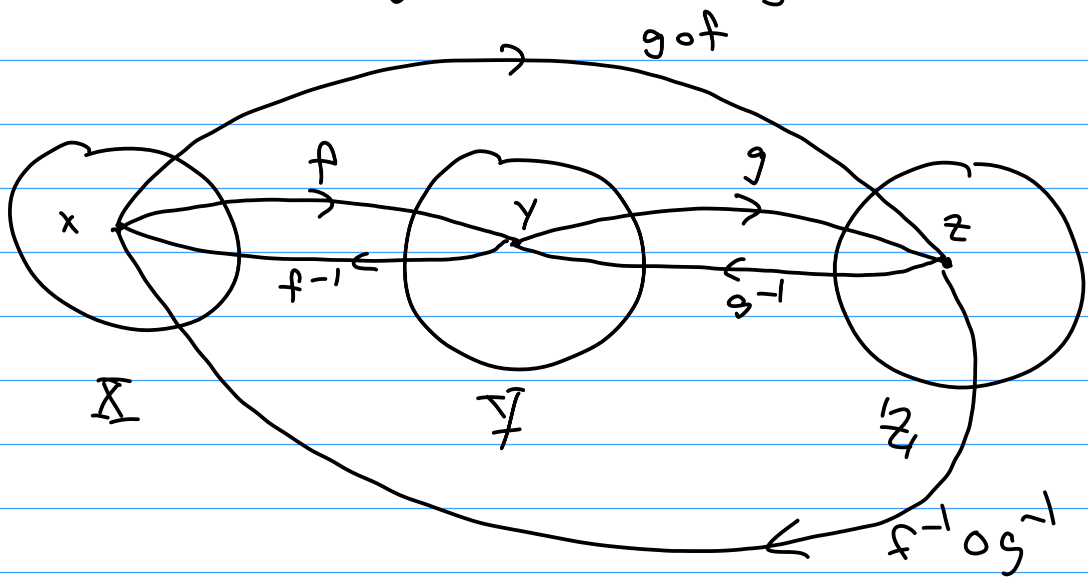
$(f \circ g)(y) = y$



Thm: $f: X \to Y$ & $g: Y \to X$ can be inverses iff both f & g are onto & one-to-one



Thm: If $f: X \to Y$ & $g: Y \to Z$ have inverses of $f^{-1}: Y \to X$ & $g^{-1}: Z \to Y$ the inverse of $g \circ f$ is $f^{-1} \circ g^{-1}$



ex.) let $f(x) = x+1 = y$ $\quad$ $g(y) = 10y = z$

$f^{-1}(x) = y-1 = x$ $\quad$ $g^{-1}(z) = z/10 = y$

Check: $(f^{-1} \circ f)(x) = (x+1) - 1 = x$

$(g^{-1} \circ g)(y) = 1/10(10y) = y$

$\to (g \circ f)(x) = g(f(x)) = 10(x+1) = z$

$(f^{-1} \circ g^{-1})(z) = \left(\frac{z}{10}\right) - 1$

$(f^{-1} \circ g^{-1}) \circ (g \circ f) = (f^{-1} \circ g^{-1})(10(x+1))$

$= \left(\frac{\cancel{10}(x+1)}{\cancel{10}}\right) - 1 = x+1-1 = x$

Future: $(\underline{A}\,\underline{B})^{-1} = \underline{B}^{-1}\underline{A}^{-1}$

$$(\underline{A}\,\underline{B})^{-1}(\underline{A}\,\underline{B})$$
$$\underline{B}^{-1}\underline{A}^{-1}\underline{A}\,\underline{B}$$
$$\underline{B}^{-1}\,\underline{I}\,\underline{B}$$
$$\underline{B}^{-1}\,\underline{B}$$
$$\underline{I}$$

Linear transformations / functions

A linear function is a function that obeys:

1) $f(u+v) = f(u) + f(v)$ for $u, v \in \underline{X}$

2) $f(a\,u) = a\,f(u)$ $\quad u \in \underline{X} \quad a \in \mathbb{R}$

ex.) Matrix transpose $\quad f(\underline{A}) = \underline{A}^T$

$f: M_{mn} \to M_{nm}$

1) $f(\underline{A}+\underline{B}) = (\underline{A}+\underline{B})^T = \underline{A}^T + \underline{B}^T = f(\underline{A}) + f(\underline{B})$

2) $f(a\,\underline{A}) = (a\,\underline{A})^T = a^T\,\underline{A}^T = a\,\underline{A}^T = a\,f(\underline{A})$

$\Rightarrow$ Linear

ex.) Dot product : $f(\underline{v}) = \underline{v}\cdot\underline{v}$ $f: \mathbb{R}^n \to \mathbb{R}^1$

1) $f(\underline{u}+\underline{v}) = (\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v}) = \underline{u}\cdot\underline{u} + \underline{u}\cdot\underline{v} + \underline{v}\cdot\underline{u} + \underline{v}\cdot\underline{v}$
$= f(\underline{u}) + f(\underline{v}) + 2\,\underline{u}\cdot\underline{v} \neq f(\underline{u}) + f(\underline{v})$

2) $f(c\underline{u}) = (c\underline{u})\cdot(c\underline{u}) = c^2\,\underline{u}\cdot\underline{u} = c^2 f(\underline{u}) \neq f(\underline{u})$
$\Rightarrow$ Not linear.

Tie to matrix-vector product.

$\underline{A}\,\underline{x} = \underline{y}$ $\underline{x} \in \mathbb{R}^{n\times 1}$ $\underline{y} \in \mathbb{R}^{m\times 1}$

$\underline{A} \in \mathbb{R}^{m\times n}$

Matrix $\underline{A}$ transforms $\underline{x}$ in the domain $\mathbb{R}^n$ into vector $\underline{y}$ in the codomain $\mathbb{R}^m$

Linear?
$\underline{A}(\underline{x}+\underline{z}) = \underline{A}\,\underline{x} + \underline{A}\,\underline{z} = f(\underline{x}) + f(\underline{z})$
$\underline{A}(c\,\underline{x}) = c\,\underline{A}\,\underline{x} = c\,f(\underline{x})$
Linear.

If $\underline{A}$ is a linear function, when does $\underline{A}^{-1}$ exist?

$\underline{A}$ must be one-to-one & onto
Only time $\underline{A}$ can be one to one & onto is a) $\underline{A}$ is square
b) columns of $\underline{A}$ must be independent