Linear Systems

A linear system is one where there are multiple linear equations

A linear equation is a weighted sum of order-1 polynomials:

$$a_1 x_1^1 + a_2 x_2^1 + \dots + a_{n-1} x_{n-1}^1 + a_n x_n^1$$

a_i: coefficients

A nonlinear equation is everything else

$$a_1 x_1 + a_2 \underline{x_2^2} + a_3 \underline{\sin(x_3)} = b$$

A system of linear equations:

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 &= b_1 \\ a_{22} x_2 + a_{23} x_3 &= b_2 \\ a_{33} x_3 &= b_3 \end{aligned}$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\underline{b}}$$

$\det(\underline{A}) \neq 0 \Rightarrow \underline{A}^{-1}$ exists $\Rightarrow \underline{x} = \underline{A}^{-1}\underline{b}$

What if $\det(\underline{A}) = 0 \Rightarrow \underline{A}^{-1}$ does not exist

$\underline{\text{but}}$, there still might be an $\underline{x}$

Such that $\underline{A}\,\underline{x} = \underline{b}$

Goal: Given $\underline{A}$ & $\underline{b}$ is there an $\underline{x}$ such that $\underline{A}\,\underline{x} = \underline{b}$ and if so, is it unique?

Existence & Uniqueness

look at
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

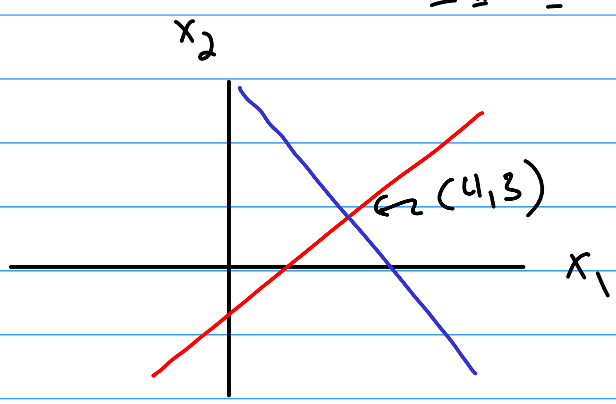
$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}\right\} \text{Lines in 2D}$$

1) $-x_1 + 2x_2 = 2$

$3x_1 + 2x_2 = 18$

$\det(\underline{A}) = -8 \Rightarrow \underline{A}^{-1}$ exists

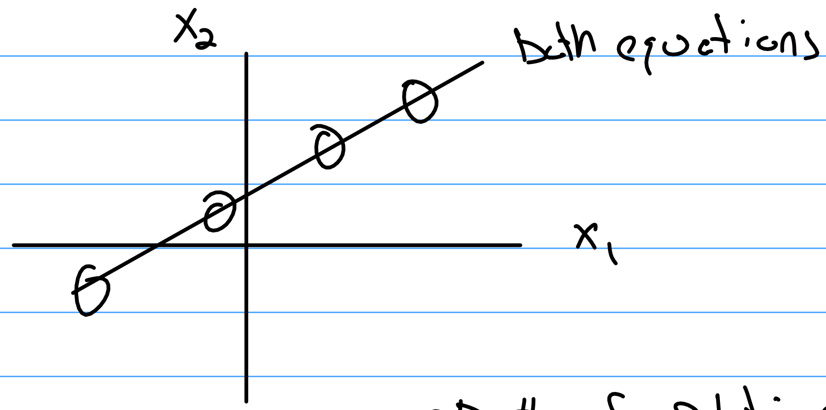
$\underline{A}\,\underline{x} = \underline{b} \Rightarrow \underline{x} = \underline{A}^{-1}\underline{b} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$



One Solution

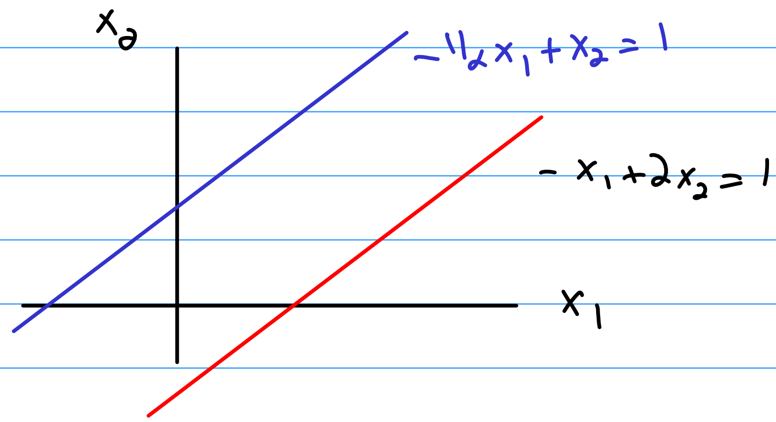
2) $-\frac{1}{2}x_1 + x_2 = 1$
$-x_1 + 2x_2 = 2$ $\Rightarrow \begin{vmatrix} -\frac{1}{2} & 1 \\ -1 & 2 \end{vmatrix} = 0$

A^{-1} does not exist.



∞ # of solutions $\Rightarrow$ not unique

3) $-\frac{1}{2}x_1 + x_2 = 1$
$-x_1 + 2x_2 = 1$ $\begin{vmatrix} -\frac{1}{2} & 1 \\ -1 & 2 \end{vmatrix} = 0 \Rightarrow$ No A^{-1}



no crossing $\Rightarrow$
no solutions

Only 3 possibilities:
1) One, unique solution
2) Infinite # of solutions
3) No solutions

Gaussian Elimination: Solve $\underline{A}\,\underline{x} = \underline{b}$

Also called row reduction

Introduce the Augmented Matrix

$$\underline{A}\,\underline{x} = \underline{b} \Rightarrow [\,\underline{A} \,\vdots\, \underline{b}\,]$$

$$\begin{array}{cc}
\left[\begin{array}{cc:c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right] \\
\begin{array}{cc} \uparrow & \uparrow \\ x_1 & x_2 \end{array}
\end{array}$$

You can: 1) Swap rows
2) Multiply a row by a non-zero number
3) Add rows

Goal: Get $\left[\begin{array}{cc:c} 1 & c & d \\ 0 & 1 & e \end{array}\right]$ or $\left[\begin{array}{cc:c} 1 & 0 & f \\ 0 & 1 & g \end{array}\right]$

$x_2 = e$ $\qquad$ $x_1 = f$

$x_1 = d - ce$ $\qquad$ $x_2 = g$

ex) $\left[\begin{array}{ccc:c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$

$R_1 + R_3 \rightarrow R_3$: $\left[\begin{array}{ccc:c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ 0 & 2 & 1 & 5 \end{array}\right]$

$\frac{3}{2}R_1 + R_2 \to R_2$: $\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 2 & 1 & 5 \end{array}\right]$

$-4R_2 + R_3 \to R_3$ $\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 0 & -1 & 1 \end{array}\right]$

$\frac{1}{2}R_1 \to R_1$
$2R_2 \to R_2$
$-1R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$x_3 = -1$
$x_2 + x_3 = 2 \Rightarrow x_2 = 3$
$x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 = 4 \Rightarrow x_1 = 2$

$$\underline{x} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

pivots

A form of $\left[\begin{array}{ccc|c} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \end{array}\right]$ is called row echlon form

pivot : leading column in each row w/ non-zero value

This has 3 pivots

$\left[\begin{array}{ccc|c} a & b & c & g \\ 0 & 0 & d & h \\ 0 & 0 & 0 & i \end{array}\right]$ all are non-zero
2 pivots

Introduce Reduced Row Echlon Form (RREF)

1) Is in echlon form
2) Each pivot is one
3) Each column w/ a pivot only has a non-zero # at that pivot.

$$\left[\begin{array}{ccc|c} 1 & 4 & 9 & 4 \\ 0 & 2 & 2 & 10 \\ 0 & -4 & -4 & -40 \end{array}\right]$$

Not in echlon form

$$\left[\begin{array}{ccc|c} 1 & 4 & 9 & 4 \\ 0 & 2 & 2 & 10 \\ 0 & 0 & 0 & -20 \end{array}\right]$$

Echlon form w/ 2 pivots

$$\left[\begin{array}{ccc|c} 1 & 4 & 9 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & -20 \end{array}\right]$$

$\uparrow \quad \uparrow$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & -16 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & -20 \end{array}\right]$$

2 pivots in rref

Now, look at rref([A : b]) & get the # of solutions

1) One, unique solution : the # of pivots equals the # of rows & the # of columns => matrix must be square

$$\text{rref}([\underline{A} : \underline{b}]) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\underline{I}} \begin{matrix} \vdots \\ \vdots \\ \vdots \end{matrix} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\underline{x}}$$

2) Infinite # of solutions : # of pivots is less than the # of rows <u>or</u> columns <u>and</u> all rows are consistent

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & a & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ \text{fixed} & \text{free} & \text{fixed} \end{matrix} \end{array}$$

columns w/ pivots give fixed variables
columns w/o pivots give free variables

To obtain a solution set the free variables & then compute fixed variables

ex1) $$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & -2 & -4 & -4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

let $x_3 = 1 \Rightarrow x_1 + x_3 = 3 \Rightarrow x_1 = 2$
$\qquad\qquad\quad x_1 + 2x_3 = 2 \Rightarrow x_2 = 0$

$$\Rightarrow \underline{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

If multiple free variables set one to a value w/ others zero then do same for all other free variables

3) No Solution: Inconsistent row

$$\left[\begin{array}{ccc:c} 1 & 0 & 5 & -16 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 20 \end{array}\right]$$

Note: If you find rref($[\underline{A} : \underline{I}]$) then $[\underline{I} : \underline{A}^{-1}]$ if $\underline{A}^{-1}$ exists

Solving $\underline{A}\,\underline{X} = \underline{I} \Rightarrow \underline{X} = \underline{A}^{-1}\underline{I} = \underline{A}^{-1}$

RREF, Inverse & Determinant

Start building a theorem

Let $\underline{A} \in \mathbb{R}^{n \times n}$

rref($\underline{A}$) exists for all matrices

If det($\underline{A}$) $\neq 0$, $\underline{A}^{-1}$ exists

recall that rref($\underline{A}$) gives pivots of $\underline{A}$.

All statements are equivalent
If one is true all are
If one is false, all are

① $\underline{A}$ is invertible ($\underline{A}^{-1}$ exists)
② $\det(\underline{A}) \neq 0$
③ $\text{rref}(\underline{A})$ has n-pivots
④ $\text{rref}(\underline{A}) = \underline{I}$
⑤ $\underline{A}\underline{x} = \underline{b}$ has a unique solution for all $\underline{b} \in \mathbb{R}^{n \times 1}$

If any are false the # of solutions to $\underline{A}\underline{x} = \underline{b}$ depends on $\underline{b}$

Gaussian Elimination Algorithm

$$\begin{bmatrix} a_0 & b_0 & c_0 \\ d_0 & e_0 & f_0 \\ g_0 & h_0 & i_0 \end{bmatrix}$$

$a_0 \to i_0$ all non-zeros

Step 1:
$$\begin{bmatrix} a_0 & b_0 & c_0 \\ d_0 & e_0 & f_0 \\ g_0 & h_0 & i_0 \end{bmatrix} \Rightarrow \begin{array}{ccc} & b_1 & c_1 \\ & \swarrow & \swarrow \end{array} \begin{bmatrix} 1 & b_0/a_0 & c_0/a_0 \\ d_0 & e_0 & f_0 \\ g_0 & h_0 & i_0 \end{bmatrix}$$

Step 2:
$$\begin{bmatrix} 1 & b_1 & c_1 \\ d_0 & e_0 & f_0 \\ g_0 & h_0 & i_0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & b_1 & c_1 \\ 0 & e_0 - b_1 d_0 & f_0 - c_1 d_0 \\ g_0 & h_0 & i_0 \end{bmatrix}$$

Step 3: $\begin{bmatrix} 1 & b_1 & c_1 \\ 0 & e_1 & f_1 \\ g_0 & h_0 & i_0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & b_1 & c_1 \\ 0 & e_1 & f_1 \\ 0 & h_0 - b_1 g_0 & i_0 - c_1 g_0 \end{bmatrix}$

Step 4: $\begin{bmatrix} 1 & b_1 & c_1 \\ 0 & e_1 & f_1 \\ 0 & h_1 & i_1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & b_1 & c_1 \\ 0 & 1 & f_2 \\ 0 & h_1 & i_1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & c_2 \\ 0 & 1 & f_2 \\ 0 & 0 & i_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & c_2 \\ 0 & 1 & f_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This might fail:

$$\begin{bmatrix} 1 & 2 & 4 \\ -2 & -4 & -9 \\ 3 & 4 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & -1 \\ 0 & -2 & -7 \end{bmatrix}$$

Can't divide by 0

Introduce Pivoting

Idea: before eliminating values swap a current row with the one that has the largest absolute value below.

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 5 & 3 & 2 \\ 0 & -9 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & -9 & 1 & 1 \\ 0 & 5 & 3 & 2 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

RREF Algorithm w/ Pivoting

```
let A ∈ R^(m×n)            % input
i=1 & j=1                  % start at column=1 & row=1

while i ≤ m & j ≤ n        % must be in the matrix

    Select k ≥ i that maximizes |a_kj|

    Swap rows k & i

    If |a_ij| > 10^-16      % check for zero
        a = a_ij
        for k=[j:n]          % normalize row-i
            a_ik = a_ik / a
        end

        for k=[1:i-1 i+1:m]  % all other rows
            a = a_kj
            for l=[j:n]
                a_kj = a_kj - a_il a
            end
        end
        i=i+1                % next row
    end
    j=j+1                    % next column
end
```

Possibilities:

$$\begin{bmatrix} 1 & 0 & a & b & c \\ 0 & 0 & 1 & d & e \\ 0 & 0 & f & g & h \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & a & b & c \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & \cancel{1} \\ 0 & 1 & \cancel{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Matlab Commands

rref(A) → rref of $\underline{A}$

A(:,j) → j^{th} column of $\underline{A}$
A(i,:) → i^{th} row of $\underline{A}$

A([1:3 5:9],:) → return all columns of rows 1→3 & 5→9

A([1 2],:) = A([2 1],:) Swaps rows 1 & 2

[m,k] = max(abs(A(:,2)))
k will store the row w/ the largest value in column 2

[m,k] = max(abs(A(4:end,2)))
k will be the # of rows below 4 (plus $\underline{1}$) that has the largest value

$$\underline{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 9 \\ 0 & -9 & 0 \end{bmatrix}$$

9

A([$\underline{4:end}$, 2]) = $\begin{bmatrix} 1 \\ 2 \\ -9 \end{bmatrix}$

[m,k] = max(abs(A([4:end],2)))
k = 3