Operations

$$\begin{aligned}
E_1 &: \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\
E_2 &: \quad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
E_n &: \quad a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n.
\end{aligned}$$

is a linear system with given constants a_{ij}, for each $i, j = 1, 2, \ldots, n$, and b_i, for each $i = 1, 2, \ldots, n$, and we need to determine the unknowns $x_1, \ldots, x_n$.

1. Equation E_i can be multiplied by any nonzero constant λ with the resulting equation used in place of E_i. This operation is denoted $(\lambda E_i) \to (E_i)$.
2. Equation E_j can be multiplied by any constant λ and added to equation E_i with the resulting equation used in place of E_i. This operation is denoted $(E_i + \lambda E_j) \to (E_i)$.
3. Equations E_i and E_j can be transposed in order. This operation is denoted $(E_i) \leftrightarrow (E_j)$.

Definition (6.1)

An $\boldsymbol{n} \times \boldsymbol{m}$ ($\boldsymbol{n}$ by $\boldsymbol{m}$) **matrix** is a rectangular array of elements with n rows and m columns in which not only is the value of an element important, but also its position in the array.

The notation for an $n \times m$ matrix will be a capital letter such as A for the matrix and lowercase letters with double subscripts, such as a_{ij}, to refer to the entry at the intersection of the ith row and jth column; that is,

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}.$$

An $n \times (n+1)$ matrix can be used to represent the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ \vdots \qquad\qquad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n, \end{aligned}$$

by constructing the **augmented matrix**

$$[A, \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right].$$

.

Gaussian elimination with backward substitution

Through a sequential procedure for $i = 2, 3, \ldots, n-1$ we perform the operation

$$(E_j - (a_{ji}/a_{ii})E_i) \rightarrow (E_j) \quad \text{for each } j = i+1, i+2, \ldots, n,$$

provided $a_{ii} \neq 0$. This eliminates (changes the coefficient to zero) x_i in each row below the ith for all values of $i = 1, 2, \ldots, n-1$. The resulting matrix has the form:

$$\tilde{\tilde{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & a_{1,n+1} \\ 0 & a_{22} & \cdots & a_{2n} & a_{2,n+1} \\ \vdots & & & \vdots & \vdots \\ 0 & \cdots & 0 & a_{nn} & a_{n,n+1} \end{bmatrix},$$

where, except in the first row, the values of a_{ij} are not expected to agree with those in the original matrix $\tilde{A} = [A, b]$. The matrix $\tilde{\tilde{A}}$ represents a linear system with the same solution set as the original system .

Algorithm 6.1: GAUSSIAN ELIMINATION WITH BACKSUB

To solve the $n \times n$ linear system

$$\begin{array}{lccccccccc}
E_1: & a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & a_{1,n+1} \\
E_2: & a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & a_{2,n+1} \\
\vdots & \vdots & & \vdots & & & & \vdots & & \vdots \\
E_n: & a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & a_{n,n+1}
\end{array}$$

INPUT number of unknowns and equations n; augmented matrix $A = [a_{ij}]$, where $1 \le i \le n$ and $1 \le j \le n+1$.

OUTPUT solution $x_1, x_2, \ldots, x_n$ or message that the linear system has no unique solution.

Follow a sequential procedure

$$\left(E_j - \frac{a_{ji}}{a_{ii}} E_i\right) \to (E_j) \quad \text{for each } j = i+1, i+2, \dots, n.$$

This leads to

$$\tilde{\tilde{A}} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & a_{1,n+1} \\ 0 & a_{22} & \cdots & a_{2n} & a_{2,n+1} \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & a_{nn} & a_{n,n+1} \end{array}\right]$$

The new linear system is triangular:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = a_{1,n+1} \\ \qquad\quad a_{22}x_2 + \cdots + a_{2n}x_n = a_{2,n+1} \\ \qquad\quad \cdots \\ \qquad\qquad\qquad\qquad a_{nn}x_n = a_{n,n+1} \end{cases}$$

Backward substitution:

$$x_n = \frac{a_{n,n+1}}{a_{n,n}}$$

Eq(n-1): $\quad a_{n-1,n-1}x_{n-1} + a_{n-1,n}\dfrac{a_{n,n+1}}{a_{n,n}} = a_{n-1,n+1}$

$$x_{n-1} = \frac{a_{n-1,n+1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

Continuing this process, one obtains

$$x_i = \frac{a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij} x_j}{a_{ii}} \quad ; \quad i = n-1, n-2, \ldots, 2, 1$$

#6 (a)

$$\begin{array}{ll} E_1 & x_2 - 2x_3 = 4 \\ E_2 & x_1 - x_2 + x_3 = 6 \\ E_3 & x_1 - x_3 = 2 \end{array}$$

$$\tilde{A} = \left[\begin{array}{ccc|c} 0 & 1 & -2 & 4 \\ 1 & -1 & 1 & 6 \\ 1 & 0 & -1 & 2 \end{array}\right] \xrightarrow{(E_2) \leftrightarrow (E_1)} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 1 & -2 & 4 \\ 1 & 0 & -1 & 2 \end{array}\right] (-E_1 + E_3) \rightarrow (E_3)$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 6 \\ 0 & 1 & -2 & 4 \\ 0 & 1 & -2 & -4 \end{array}\right]$$

no solutions.

#5(a)

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 2 \\ 3x_1 - 3x_2 + x_3 &= -1 \\ x_1 + x_2 &= 3 \end{aligned}$$

$$\tilde{A} = \left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 3 & -3 & 1 & -1 \\ 1 & 1 & 0 & 3 \end{array}\right] \xrightarrow{(-3E_1+E_2) \rightarrow (E_2)} \left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 0 & 0 & -8 & -7 \\ 1 & 1 & 0 & 3 \end{array}\right] \xrightarrow{(-E_1+E_3) \rightarrow (E_3)}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 0 & 0 & -8 & -7 \\ 0 & 2 & -3 & 1 \end{array}\right] \xrightarrow{(E_2)\leftrightarrow(E_3)} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & -8 & -7 \end{array}\right]$$

$$x_3 = \frac{7}{8}$$

$$2x_2 - 3x_3 = 1$$

$$2x_2 = 1 + \frac{21}{8} = \frac{29}{8}$$

$$x_2 = \frac{29}{16}$$

$$x_1 + x_2 = 3$$

$$x_1 = 3 - \frac{29}{16} = \frac{48-29}{16} = \frac{19}{16}.$$

#10.

$$x_1 - x_2 + \alpha x_3 = -2$$
$$-x_1 + 2x_2 - \alpha x_3 = 3$$
$$\alpha x_1 + x_2 + x_1 = 2$$

$$\tilde{A} = \left[\begin{array}{ccc|c} 1 & -1 & \alpha & -2 \\ -1 & 2 & -\alpha & 3 \\ \alpha & 1 & 1 & 2 \end{array}\right] \xrightarrow{(+E_1+E_2)\to(E_2)} \left[\begin{array}{ccc|c} 1 & -1 & \alpha & -2 \\ 0 & 1 & 0 & 1 \\ \alpha & 1 & 1 & 2 \end{array}\right] \xrightarrow{(-\alpha E_1+E_3)\to(E_3)}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & \alpha & -2 \\ 0 & 1 & 0 & 1 \\ 0 & +\alpha+1 & -\alpha^2+1 & 2\alpha+2 \end{array}\right]$$

$$\xrightarrow{(-(\alpha+1)E_2+E_3)\to(E_3)} \left[\begin{array}{ccc|c} 1 & -1 & \alpha & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -\alpha^2+1 & \alpha+1 \end{array}\right]$$

$$(1-\alpha^2)x_3 = \alpha+1$$

$$(1-\alpha)(1+\alpha)\,x_3 = \alpha+1$$

(a) $\boxed{\alpha = -1}$ $\quad x_3$ - any real number

(b)

$$x_2 = 1$$

$$x_1 - x_2 + \alpha = -2 \Rightarrow x_1 - 1 - 1 = -2 \Rightarrow x_1 = 0$$

infinite number of solutions $\begin{pmatrix} 0 \\ 1 \\ x_3 \end{pmatrix}$

(b) $\alpha = 1$ $\quad 0 \cdot x_3 = 2 \Rightarrow$ no solutions

(a)

(c) $\alpha \neq -1, 1$

$$(1-\alpha)x_3 = 1$$

$$x_3 = \frac{1}{1-\alpha}$$

$$x_2 = 1$$

$$x_1 - 1 + \frac{\alpha}{1-\alpha} = -2$$

$$x_1 = -1 - \frac{\alpha}{1-\alpha} = \frac{-1+\alpha-\alpha}{1-\alpha} = -\frac{1}{1-\alpha}$$

Unique solution $\begin{pmatrix} \frac{1}{\alpha-1} \\ 1 \\ \dots \end{pmatrix}$

Algorithm 6.1: GAUSSIAN ELIMINATION WITH BACKSUB

Step 1 For $i = 1, \ldots, n-1$ do Steps 2–4. (*Elimination process.*)

  Step 2 Let p be the smallest integer with $i \leq p \leq n$ and $a_{pi} \neq 0$.

    If no integer p can be found

     then OUTPUT ('no unique solution exists'); STOP.

  Step 3 If $p \neq i$ then perform $(E_p) \leftrightarrow (E_i)$.

  Step 4 For $j = i+1, \ldots, n$ do Steps 5 and 6.

   Step 5 Set $m_{ji} = a_{ji}/a_{ii}$.

   Step 6 Perform $(E_j - m_{ji}E_i) \rightarrow (E_j)$;

Step 7 If $a_{nn} = 0$ then OUTPUT ('no unique solution exists'); STOP.

Step 8 Set $x_n = a_{n,n+1}/a_{nn}$. (*Start backward substitution.*)

Step 9 For $i = n-1, \ldots, 1$ set $x_i = \left[a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij}x_j\right] \Big/ a_{ii}$.

Step 10 OUTPUT $(x_1, \ldots, x_n)$; (*Procedure completed successfully.*)

  STOP.

Operation Counts

Both the amount of time required to complete calculations and the subsequent round-off error depend on the number of floating-point arithmetic operations needed to solve a routine problem.

Multiplications/divisions

The total number of multiplications and divisions in Algorithm 6.1

$$\frac{2n^3 + 3n^2 - 5n}{6} + \frac{n^2 + n}{2} = \frac{n^3}{3} + n^2 - \frac{n}{3}.$$

Additions/subtractions

The total number of additions and subtractions in Algorithm 6.1

$$\frac{n^3 - n}{3} + \frac{n^2 - n}{2} = \frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6}.$$