Matrix
(Matrices)

A matrix is a 2D collection of numbers

⇒ collection of vectors.

$$\underline{\underline{A}} = \begin{bmatrix} 9 & 4 \\ 3 & 5 \\ 4 & 7 \end{bmatrix} \quad \begin{array}{l} \longleftrightarrow \text{ row} \\ \updownarrow \text{ column} \end{array}$$

column

size of $\underline{\underline{A}}$ is 3×2

row

Components given by a_{ij}

i : row #

j : column #

$a_{12} = 4 \qquad a_{13} = \text{undefined}$

$$\underline{a} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} 9 \\ -1 \\ 0 \end{bmatrix}$$

$$\underline{\underline{C}} = \begin{bmatrix} \underline{a} & \underline{b} \end{bmatrix}$$

$$c = \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 9 \\ 2 & -1 \\ 4 & 0 \end{bmatrix}$$

$$d = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\underline{\underline{E}} = \begin{bmatrix} \underline{a} & \underline{d} \end{bmatrix} = \cancel{\begin{bmatrix} \quad & \quad \\ & \end{bmatrix}}$$

undefined

You can also use row vectors

(less common)

$$\underset{=}{F} = \begin{bmatrix} \underset{-}{a}^T \\ \underset{-}{c}^T \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \end{bmatrix}$$

Transpose

$$\underset{=}{C} = \begin{bmatrix} 1 & 9 \\ 2 & -1 \\ 4 & 0 \end{bmatrix}$$

$$\underset{=}{C}^T = \begin{bmatrix} 1 & 9 \\ 2 & -1 \\ 4 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 9 & -1 & 0 \end{bmatrix}$$

A matrix is symmetric if

$$\underline{\underline{A}} = \underline{\underline{A}}^T$$

$$\Rightarrow a_{ij} = a_{ji}$$

All symmetric matrices are square

(row # = column #)

A rectangular matrix is NEVER symmetric

Identity Matrix $\underline{\underline{I}}$

$\underline{\underline{I}}$ is a square matrix with 1 on the diagonal and $\emptyset$ else where

$$\underline{\underline{I}} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\underline{\underline{A}}\,\underline{\underline{I}} = \underline{\underline{A}}$$

$$\underline{\underline{I}}\,\underline{\underline{A}} = \underline{\underline{A}}$$

$$\underline{\underline{I}}\,\underline{x} = \underline{x}$$

Operation I.

① Addition / Subtraction at equaled sized matrices.

$$\underline{\underline{A}} \;\&\; \underline{B} \in \mathbb{R}^{n\times n}$$

$$\underline{\underline{A}} + \underline{\underline{B}} = \underline{\underline{C}} \qquad \underline{\underline{A}} - \underline{\underline{B}} = \underline{\underline{D}}$$

$$a_{ij} + b_{ij} = c_{ij} \qquad a_{ij} - b_{ij} = d_{ij}$$

② Matrix-Vector Product.

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\underline{\underline{A}}\,\underline{x} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = \underline{b}$$

Index Notation : $a_{ij}x_j = b_i$

$$\underset{3\times 2}{\begin{bmatrix}1 & 2\\ -1 & 0\\ 9 & 1\end{bmatrix}}\underset{3\times 1}{\begin{bmatrix}2\\ 1\\ 0\end{bmatrix}} = \text{Undefined}$$

③ Matrix - Matrix Product

$$\underline{\underline{A}} \in \mathbb{R}^{m\times n}$$

$$\underline{\underline{B}} \in \mathbb{R}^{p\times q}$$

$\underline{\underline{A}}\ \underline{\underline{B}}$ is defined if $(m\times n)(p\times q)$ with $n=p$

$= m\times q$

Comments.

- $\underline{\underline{A}} + \underline{\underline{B}} = \underline{\underline{B}} + \underline{\underline{A}}$
- $c \in \mathbb{R}$: $c(\underline{\underline{A}}+\underline{\underline{B}}) = c\underline{\underline{A}} + c\underline{\underline{B}}$
- $(\underline{\underline{A}}\ \underline{\underline{B}})\underline{\underline{C}} = \underline{\underline{A}}(\underline{\underline{B}}\ \underline{\underline{C}})$
- In general : $\underline{\underline{A}}\ \underline{\underline{B}} \neq \underline{\underline{B}}\ \underline{\underline{A}}$

- $\underline{\underline{A}}(\underline{\underline{B}}+\underline{\underline{C}}) = \underline{\underline{A}}\,\underline{\underline{B}} + \underline{\underline{A}}\,\underline{\underline{C}}$

  $(\underline{\underline{A}}+\underline{\underline{B}})\underline{\underline{C}} = \underline{\underline{A}}\,\underline{\underline{C}} + \underline{\underline{B}}\,\underline{\underline{C}}$

  $(\underline{\underline{A}}+\underline{\underline{B}})\underline{x} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{x}$

  $\underline{\underline{A}}(\underline{x}+\underline{y}) = \underline{\underline{A}}\,\underline{x} + \underline{\underline{A}}\,\underline{y}$

- $(\underline{\underline{A}}+\underline{\underline{B}})^T = \underline{\underline{A}}^T + \underline{\underline{B}}^T$

  $(\underline{\underline{A}}\,\underline{\underline{B}})^T = \underline{\underline{B}}^T\underline{\underline{A}}^T$

  $(\underline{\underline{A}}\,\underline{x})^T = \underline{x}^T\underline{\underline{A}}^T$

  $(c\,\underline{\underline{A}})^T = c\,\underline{\underline{A}}^T$

Operation Part 2

④ Powers.

$$\underline{\underline{A}}^{p} = \underbrace{(\underline{\underline{A}}\,\underline{\underline{A}} \cdots \underline{\underline{A}})}_{p \text{ times}}$$

$$\underline{\underline{A}}^{p}\,\underline{\underline{A}}^{q} = \underline{\underline{A}}^{p+q}$$

$$(\underline{\underline{A}}^{p})^{q} = \underline{\underline{A}}^{pq}$$

⑤ Block Matrices

matrix

$$\underline{\underline{A}} = \begin{bmatrix} \boxed{\underline{\underline{B}}} & \underline{\underline{C}} & \underline{\underline{D}} \\ \underline{\underline{E}} & \underline{\underline{F}} & \underline{\underline{G}} \end{bmatrix}$$

$\underline{\underline{B}}$ $\quad\quad$ $\underline{\underline{C}}$

$$\underline{\underline{A}} = \begin{bmatrix} 1 & 9 & 2 & 9 \\ 2 & 8 & 4 & 7 \\ 3 & 7 & 6 & 5 \\ 4 & 6 & 8 & 3 \\ 5 & 5 & 10 & 1 \end{bmatrix}$$

$\underline{\underline{D}}$ $\quad\quad$ $\underline{\underline{E}}$

⑥ trace : sum of the matrix diagonal

$$tr(\underline{\underline{A}}) = \sum_{i=1}^{n} a_{ii}$$

$$tr\left(\begin{bmatrix} 1 & 0 & 9 \\ 0 & -1 & 2 \\ -1 & 0 & 8 \end{bmatrix}\right) = 1 - 1 + 8 = 8$$

let $\underline{\underline{A}}, \underline{\underline{B}} \in \mathbb{R}^{n \times n}$, $a \in \mathbb{R}$

$$tr(a\underline{\underline{A}}) = a\, tr(\underline{\underline{A}})$$

$$tr(\underline{\underline{A}}^T) = tr(\underline{\underline{A}})$$

$$\mathrm{tr}(\underline{\underline{A}}\,\underline{\underline{B}}) = \mathrm{tr}(\underline{\underline{B}}\,\underline{\underline{A}})$$

$$\mathrm{tr}(\underline{\underline{A}}\,\underline{\underline{B}}) \cancel{=} \mathrm{tr}(\underline{\underline{A}})\,\mathrm{tr}(\underline{\underline{B}})$$

$$\mathrm{tr}(\underline{\underline{A}}^T \underline{\underline{B}}) = \mathrm{tr}(\underline{\underline{A}}\,\underline{\underline{B}}^T) = \mathrm{tr}(\underline{\underline{B}}^T \underline{\underline{A}})$$

$$= \mathrm{tr}(\underline{\underline{B}}\,\underline{\underline{A}}^T)$$

⑦ **Outer Product** : Operation on 2 vectors that produced a matrix

$$\underline{a} \in \mathbb{R}^{m\times 1}, \quad \underline{b} \in \mathbb{R}^{n\times 1}$$

$$\underline{a}\,\underline{b}^T = \underline{a} \otimes \underline{b}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_{3\times 1} \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}_{1\times 3}$$

$$= \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}_{3\times 3} = \underline{\underline{C}}$$

$$a_i b_j = c_{ij}$$

Inner Product.

$$\underline{a}^T \underline{b} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$= \underline{a} \cdot \underline{b}$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$

Matrix Determinant.

The determinant is an operation on a square matrix that results in a scalar that encodes information about that matrix.

1×1 : $\det([a_{11}]) = a_{11}$

2×2 : $\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11} a_{22} - a_{12} a_{21}$

3×3 : Practice

In general : $\det(\underline{\underline{A}}) = \sum_{j=1}^{n} a_{ij} (-1)^{i+j} M_{ij}$

Det. of sub matrix

Properties

① $\mathrm{Det}(\underline{\underline{A}}\,\underline{\underline{B}}) = \mathrm{Det}(\underline{\underline{A}})\,\mathrm{Det}(\underline{\underline{B}})$

② $\mathrm{Det}(\alpha \underline{\underline{I}}_n) = \alpha^n$

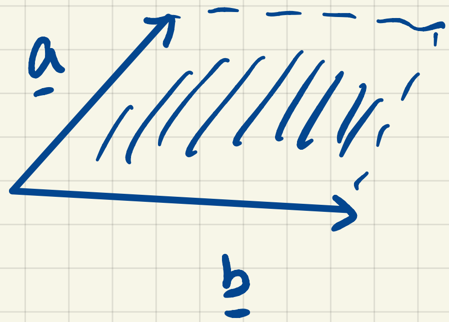
③ $\mathrm{Det}(\alpha \underline{\underline{A}}) = \mathrm{Det}(\alpha \underline{\underline{I}}_n \underline{\underline{A}})$

$= \mathrm{Det}(\alpha \underline{\underline{I}}_n)\,\mathrm{Det}(\underline{\underline{A}})$

$= \alpha^n\,\mathrm{Det}(\underline{\underline{A}})$

④ $\text{Det}(\underline{\underline{A}}^T) = \text{Det}(\underline{\underline{A}})$

⑤ If you define a shape in 2D & 3D via vectors, $\det(\underline{\underline{A}})$ gives the area or volume.



$\det([\underline{a} \;\; \underline{b}]) = \text{Area}$

⑥ If $\underline{\underline{A}}$ exists, so does $\det(\underline{\underline{A}})$ vice versa

$\Rightarrow$ If $\det(\underline{\underline{A}})$ is undefined, $\underline{\underline{A}}$ does not exist.

Matrix Inverse

Only for <u>square</u> matrix

let $\underline{\underline{I}}$ = $n \times n$ identity matrix

$\underline{\underline{A}}$ & $\underline{\underline{B}}$ are inverse of each other

iff $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{I}}$

or $\underline{\underline{B}}\,\underline{\underline{A}} = \underline{\underline{I}}$ $\Rightarrow$ $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{I}} = \underline{\underline{B}}\,\underline{\underline{A}}$

Denote inverse as $\underline{\underline{A}}^{-1}$

$$\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{I}} = \underline{\underline{A}}\,\underline{\underline{A}}^{-1}$$

Closed form for 2×2 & 3×3 only

2×2 let $\underline{\underline{A}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

show that $\underline{\underline{A}}^{-1} = \dfrac{1}{\det(\underline{\underline{A}})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\underline{\underline{A}}^{-1} \underline{\underline{A}} = \frac{1}{\det(\underline{\underline{A}})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \frac{1}{\det(\underline{\underline{A}})} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

* $\det(\underline{\underline{A}}^{-1}) = \dfrac{1}{\det(\underline{\underline{A}})}$

* $\det(\underline{\underline{A}}) \neq 0$, $\det(\underline{\underline{A}}^{-1})$ is defined

$\Rightarrow$ $\underline{\underline{A}}^{-1}$ exists

If $\det(\underline{\underline{A}}) = 0$, $\det(\underline{\underline{A}}^{-1})$ is not defined.

$\Rightarrow$ $\underline{\underline{A}}^{-1}$ does not exist.