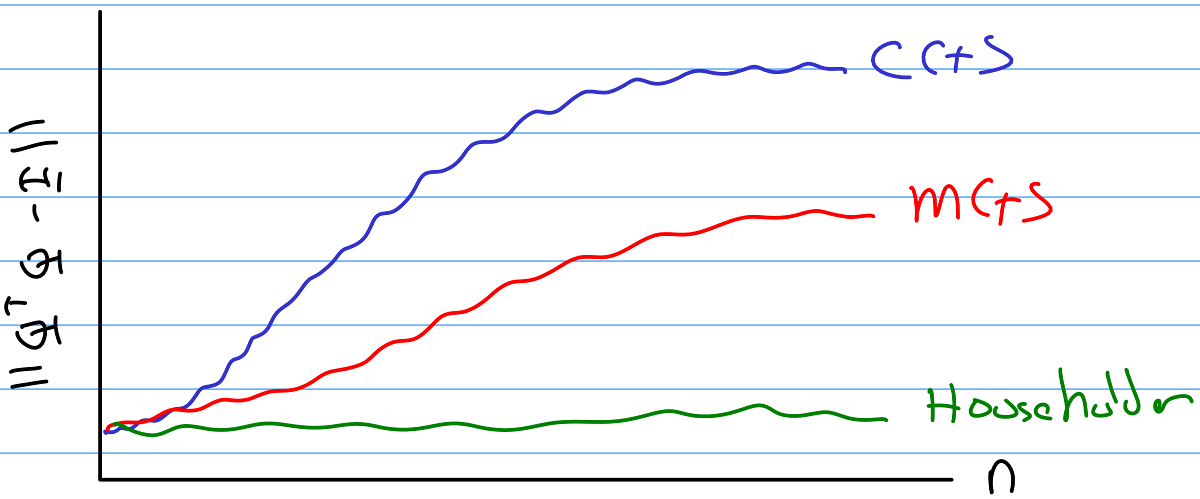
QR Decomp.

$\underline{A} = \underline{Q}\,\underline{R}$ w/ $\underline{Q}^{-1} = \underline{Q}^T$

$\underline{A}\,\underline{x} = \underline{b} \Rightarrow \underline{Q}\,\underline{R}\,\underline{x} = \underline{b} \Rightarrow \underline{x} = \underline{R}^{-1}\underline{Q}^T\underline{b}$

Solving $\underline{A}\,\underline{x} = \underline{b}$ via QR returns the least squares solution.
Even if $\underline{A} \in M_{mn}$ w/ $m \neq n$

3 methods: Classical Gram-Schmidt
Modified Gram-Schmidt
Householder Reflectors



Eigensystems

Motivation: look at $\frac{dy}{dt} = ay(t) \quad a \in \mathbb{R}$

Solution is $y(t) = Ce^{at} \quad C \in \mathbb{R}$

$$\frac{d}{dt}(Ce^{at}) = Cae^{at} = a(Ce^{at}) = a\, y(t)$$

What about coupled ODEs?

$$\frac{dy_1}{dt} = a_{11} y_1(t) + a_{12} y_2(t)$$
$$\frac{dy_2}{dt} = a_{21} y_1(t) + a_{22} y_2(t)$$

Arises in chemical reaction, springs, etc.

let $\underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad \underline{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$

$$\frac{d\underline{y}}{dt} = \begin{bmatrix} \partial y_1/\partial t \\ \partial y_2/\partial t \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \underline{A}\, \underline{y}(t)$$

$$\Rightarrow \frac{d\underline{y}}{dt} = \underline{A}\, \underline{y}(t)$$

As $\frac{dy}{dt} = a\, y(t)$ has a solution of $y(t) = Ce^{at}$,

make the ansatz that $\underline{y}(t) = e^{\lambda t}\underline{x}$ solves

$$\frac{\partial \underline{y}}{\partial t} = \underline{A}\, \underline{y}(t)$$

$$\frac{d\underline{y}}{dt} = \frac{d}{dt}\left(e^{\lambda t}\underline{x}\right) = \lambda e^{\lambda t}\,\underline{x}$$

$$\Updownarrow$$

$$\underline{A}\,\underline{y} = \underline{A}\left(e^{\lambda t}\underline{x}\right) = e^{\lambda t}\,\underline{A}\,\underline{x}$$

$$\Rightarrow \lambda \cancel{e^{\lambda t}}\,\underline{x} = \cancel{e^{\lambda t}}\,\underline{A}\,\underline{x}$$

$\Rightarrow \underline{A}\,\underline{x} = \lambda \underline{x}$ $\Rightarrow$ Given $\underline{A}$, if you can find a λ & $\underline{x}$ such that $\underline{A}\,\underline{x} = \lambda\underline{x}$, then $\underline{y}(t) = e^{\lambda t}\,\underline{x}$ solves $\frac{d\underline{y}}{dt} = \underline{A}\,\underline{y}$

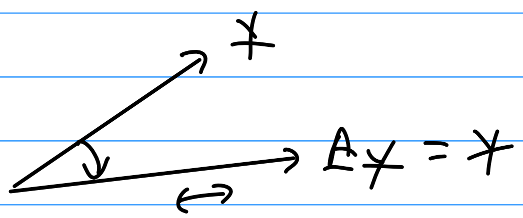
$\underline{A}\,\underline{x} = \lambda\underline{x}$ is the Eigensystem of $\underline{A}$

$\underline{x}$ = eigenvector

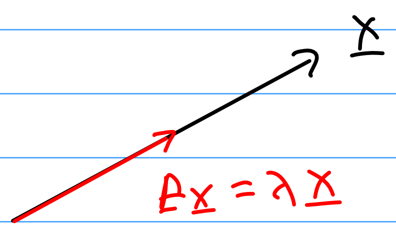
λ = eigenvalue

Why are λ & $\underline{x}$ special?

Look at $\underline{y} \neq$ eigenvector



Now look at $\underline{A}\,\underline{x} = \lambda\underline{x}$



$\underline{x}$ just scales by $\underline{A}$

ex.) $\underline{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $\quad y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\underline{A}y = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \neq \lambda y$$

try $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\underline{A}x = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\,\underline{x}$$

$$\Rightarrow \lambda = 3$$

Eigenvalues & eigenvectors always come as a pair.

How to compute λ

let $\underline{A} \in M_{nn}$ (only square)

$$\underline{A}x = \lambda \underline{x}$$
$$(n \times p)(p \times 1) = p \times 1$$
$$n \times 1 = p \times 1 \quad \Rightarrow \quad n = p$$

Given $\underline{A}$ find λ

$$\underline{A}\,\underline{x} = \lambda \underline{x}$$
$$\underline{A}\,\underline{x} - \lambda \underline{x} = \underline{0}$$
$$(\underline{A} - \lambda \underline{I})\,\underline{x} = \underline{0}$$

1) $\underline{x}$ must be in the Nullspace of $\underline{A} - \lambda \underline{I}$

2) $\det(\underline{A} - \lambda \underline{I}) = 0$ → Characteristic Eq. of $\underline{A}$

A polynomial

$\Rightarrow$ Procedure :

1) Solve $\det(\underline{A} - \lambda \underline{I}) = 0$ for all λ
(roots)

2) for each λ find the nullspace of
$\underline{A} - \lambda \underline{I}$

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ex.) $\underline{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$\underline{A} - \lambda \underline{I} = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) - 4 = 0$$

$$= (1-\lambda)^2 - 4 = 0$$

$$\Rightarrow (1-\lambda)^2 = 4 \quad \Rightarrow 1-\lambda = \pm 2$$

$$\Rightarrow \lambda_1 = -1 \qquad \lambda_2 = +3$$

$\lambda_1 = -1$: $\begin{bmatrix} 1-(-1) & 2 \\ 2 & 1-(-1) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$\text{rref}([\underline{A} - \lambda \underline{I} \vdots 0]) = \begin{bmatrix} 1 & 1 & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{bmatrix}$

Note: If $\text{rref}(\underline{A} - \lambda \underline{I}) = \underline{I}$, then λ is
$\underline{\underline{\text{not}}}$ an eigenvalue.

Set $x_2 = 1 \Rightarrow x_1 = -1 \Rightarrow \underline{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Check: $\underline{A}\underline{x}_1 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \underbrace{(-1)}_{\lambda_1} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 3$: $\underline{A} - \lambda_2 \underline{I} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$

$\text{rref}(\underline{A} - \lambda_2 \underline{I}) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

$x_2 = +1 \Rightarrow x_1 = +1 \Rightarrow \underline{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Note: Eigenvectors are unique up to a sign & scale.

If $\underline{x}_1$ is an eigenvector so is $\alpha \underline{x}_1$ $\alpha \in \mathbb{R}$, $\alpha \neq 0$

$\underline{A}(\alpha \underline{x}_1) = \lambda(\alpha \underline{x}_1) \Rightarrow \alpha(\underline{A}\underline{x}_1) = \alpha(\lambda \underline{x}_1)$

Comments

1) Eigenvalues of $\underline{A}^2$ are the square of those of $\underline{A}$.

let $\underline{A}\underline{x} = \lambda \underline{x}$

$\underline{A}^2 \underline{x} = \underline{A}(\underline{A}\underline{x}) = \underline{A}(\lambda \underline{x}) = \lambda \underline{A}\underline{x} = \lambda(\lambda \underline{x}) = \lambda^2 \underline{x}$

In general, if $\underline{A}\underline{x} = \lambda \underline{x}$ then

$\underline{A}^n \underline{x} = \lambda^n \underline{x}$

2) Row operations on $\underline{A}$ do $\underline{\underline{\text{not}}}$ preserve eigenvalues or eigenvectors

$$\begin{bmatrix} 4 & -1 & 0 \\ 0 & 1 & 0 \\ 2 & -3 & 6 \end{bmatrix} \qquad \begin{bmatrix} 2 & -3 & 6 \\ 0 & 1 & 0 \\ 4 & -1 & 0 \end{bmatrix}$$

$\lambda = 4, 1, 6$ $\qquad$ $\lambda = -4, 1, 6$

3) $tr(\underline{A}) = \sum_i \lambda_i$

4) $\det(\underline{A}) = \prod_i \lambda_i$

thus if $\det(\underline{A}) = 0$ then at least one λ is zero.

If $\lambda = 0$: $(\underline{A} - \cancel{\lambda}^{0} \underline{I})\underline{x} = \underline{0}$

$\underline{A}\,\underline{x} = \underline{0}$

$\Rightarrow \det(\underline{A}) = 0$ if $\underline{x}$ is not trivial

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Possibilities for $\det(\underline{A} - \lambda \underline{I}) = 0$

1) All roots of $\det(\underline{A} - \lambda \underline{I})$ are real & unique (Given n eigenvalues find n eigenvector.

2) Complex eigenvalues

ex. 1) $\underline{A} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 2 \\ 4 & -1 & 0 \end{bmatrix}$

$|\underline{A} - \lambda \underline{I}| = \begin{vmatrix} 6-\lambda & 0 & 0 \\ 0 & 2-\lambda & 2 \\ 4 & -1 & -\lambda \end{vmatrix} = (6-\lambda)(2-\lambda)(-\lambda) - (-1)(2)(6-\lambda) = 0$

$$(6-\lambda)(\lambda^2 - 2\lambda + 2) = 0$$

$\uparrow$ $\uparrow$

$\lambda_1 = 6$ $\lambda_{2,3} = 1 \pm i$

As $\lambda_{2,3}$ are complex, $\underline{x}_{2,3}$ is complex.

$\lambda_2 = 1+i$: $(\underline{A} - \lambda_2 \underline{I})\underline{x} = \begin{bmatrix} 6-(1+i) & 0 & 0 \\ 0 & 2-(1+i) & 0 \\ -4 & -1 & -(1+i) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\text{rref}(\underline{A} - \lambda_2 \underline{I}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-i & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\Rightarrow x_1 = 0$
$\Rightarrow (1-i)x_2 + 2x_3 = 0$

Set $x_2 = 1 \Rightarrow 1 - i + 2x_3 = 0$
$\Rightarrow x_3 = -1/2 + 1/2\, i$

$\Rightarrow \lambda_2 = 1+i$ $\underline{x}_2 = \begin{bmatrix} 0 \\ 1 \\ -1/2(1+i) \end{bmatrix}$

If $\underline{A}$ is real-valued, all eigenvectors & eigenvalues come in complex conjugate pairs.

If $\lambda = a + ib$ w/ $\underline{A}\,\underline{x} = \lambda \underline{x}$

$$\overline{(\underline{A}\,\underline{x})} = \overline{(\lambda \underline{x})} \Rightarrow \overline{\underline{A}}\;\overline{\underline{x}} = \overline{\lambda}\;\overline{\underline{x}}$$

$$\underline{A}\;\overline{\underline{x}} = \overline{\lambda}\;\overline{\underline{x}}$$

$\Rightarrow \lambda = a - ib$ also eigenvalue

$\Rightarrow \lambda_3 = 1 - i$ & $\underline{x}_3 = \begin{bmatrix} 0 \\ 1 \\ -\frac{1}{2}(1-i) \end{bmatrix}$

3) Repeated Eigenvalues

$\underline{A} = \begin{bmatrix} 3 & -1 & 2 \\ 3 & -1 & 6 \\ -2 & 2 & 2 \end{bmatrix}$ $\quad \lambda = -4, 2, 2$

Algebraic Multiplicity = # of times an eigenvalue shows up

$\lambda = -4$: Algebraic multiplicity of 1

$\lambda = 2$: " " " 2

$\lambda = 2$: $\underline{A} - \lambda \underline{I} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \\ -2 & 2 & 4 \end{bmatrix}$

$$\text{rref}(\underline{A} - \lambda \underline{I}) = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

nullity $(\underline{A} - \lambda \underline{I}) = 2 \Rightarrow 2$ unique eigenvectors

$x_1 - x_2 + 2x_3 = 0$ $\quad$ x_2 & x_3 are free

$x_2 = 1,\ x_3 = 0 \Rightarrow x_1 = 1$

$x_2 = 0,\ x_3 = 1 \Rightarrow x_1 = -2$

$$\Rightarrow \underline{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \underline{x}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \qquad \lambda = 2, 2$$

Geometric Multiplicity: the # of eigenvector for an eigenvalue.

$= \text{nullity}(\underline{A} - \lambda \underline{I})$

In this case it's 2

If algebraic & geometric multiplicity are equal that eigenvalue is complete

If not, then λ is defective

If any λ is defective, $\underline{A}$ is defective

ex1) $\underline{A} = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ $\quad \lambda = 3, 3$ $\quad \underline{A} - \lambda \underline{I} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

nullity $= 1 \Rightarrow$ only 1 eigenvector $\Rightarrow$ defective

All unique λ must be complete

Real, Symmetric Matrices

$$\underline{A}^T = \underline{A} \qquad \overline{\underline{A}} = \underline{A}$$

1) A real, symmetric matrix only has real eigenvalues

let $\underline{A}$ be real & symmetric
let λ be an eigenvalue, even complex.

$$\underline{A}\,\underline{x} = \lambda \underline{x}$$

$\underline{A}$ is real $\Rightarrow$ $\bar{\lambda}$ & $\bar{\underline{x}}$ are an eigenpair

$$\underline{A}\,\underline{x} = \lambda \underline{x}$$

$$\underbrace{\bar{\underline{x}}^T \underline{A}\,\underline{x}} = \lambda \bar{\underline{x}}^T \underline{x}$$

$$\underline{A}\,\bar{\underline{x}} = \bar{\lambda}\,\bar{\underline{x}}$$

$$(\underline{A}\,\bar{\underline{x}})^T = (\bar{\lambda}\,\bar{\underline{x}})^T$$

$$\bar{\underline{x}}^T \underline{A}^T = \bar{\lambda}\,\bar{\underline{x}}^T$$

$$\bar{\underline{x}}^T \underline{A} = \bar{\lambda}\,\bar{\underline{x}}^T$$

$$\underbrace{\bar{\underline{x}}^T \underline{A}\,\underline{x}} = \bar{\lambda}\,\bar{\underline{x}}^T \underline{x}$$

$$\lambda \bar{\underline{x}}^T \underline{x} = \bar{\underline{x}}^T \underline{A}\,\underline{x} = \bar{\lambda}\,\bar{\underline{x}}^T \underline{x}$$

$$\lambda \bar{\underline{x}}^T \underline{x} = \bar{\lambda}\,\bar{\underline{x}}^T \underline{x}$$

$$\bar{\underline{x}}^T \underline{x} = |x_1|^2 + |x_2|^2 + \ldots > 0$$

$\lambda = \bar{\lambda}$ $\leftarrow$ only true if λ is real

2) All eigenvectors of a real & symmetric are orthogonal for individual λ

let $\underline{A}\,\underline{x} = \lambda_1 \underline{x}$ $\quad \underline{A}\,\underline{y} = \lambda_2 \underline{y}$ $\quad \lambda_1 \neq \lambda_2$

λ_1 & λ_2 are real

$$(\lambda_1 \underline{x})^T \underline{y} = (\underline{A}\,\underline{x})^T \underline{y} = \underline{x}^T \underline{A}^T \underline{y} = \underline{x}^T \underline{A}\,\underline{y} = \underline{x}^T(\underline{A}\,\underline{y}) = \underline{x}^T(\lambda_2 \underline{y})$$

$$\Rightarrow \lambda_1 \underline{x}^T \underline{y} = \lambda_2 \underline{x}^T \underline{y}$$

$$\lambda_1 \neq \lambda_2$$

$\Rightarrow \underline{x}^T \underline{y} = 0$ must hold

$\Rightarrow \underline{x} \perp \underline{y}$

might not hold for algebraic multiplicity > 1

Matrix Diagonalization

Given $\underline{A}$ find a $\underline{P}$ such that

$\underline{P}^{-1} \underline{A}\,\underline{P} = \underline{\Lambda}$ $\leftarrow$ A Diagonal matrix

$\underline{A} = \underline{P}\,\underline{\Lambda}\,\underline{P}^{-1}$

look at $\underline{A}\,\underline{x} = \lambda \underline{x}$

let λ_i be an eigenvalue of $\underline{A}$

$\Rightarrow \underline{A}\,\underline{x}_1 = \lambda_1 \underline{x}_1 \rightarrow \underline{A}\,\underline{x}_p = \lambda_p \underline{x}_p$ if $\underline{A} \in M_{nn}$

max of $p = n$

Introduce $\underline{S} = [\underline{x}_1 \quad \underline{x}_2 \quad \cdots \quad \underline{x}_p]$

If $\underline{A}$ is complete, then $n = p$
& all eigenvectors are unique,
$\Rightarrow$ rank$(S) = n \Rightarrow \underline{S}^{-1}$ exists

$$\underline{A}\,\underline{x}_1 = \lambda_1 \underline{x}_1 \rightarrow \underline{A}\,\underline{x}_n = \lambda_n \underline{x}_n$$

$$\underline{A}\,\underline{S} = \underline{\Lambda}\,\underline{S}$$

$$\underline{A}[\underline{x}_1 \quad \underline{x}_2 \quad \cdots \quad \underline{x}_n] = \begin{bmatrix} \lambda_1 & & & \underline{0} \\ & \lambda_2 & & \\ & & \ddots & \\ \underline{0} & & & \lambda_n \end{bmatrix} [\underline{x}_1 \quad \underline{x}_2 \quad \cdots \quad \underline{x}_n]$$

$\Rightarrow$ Correct $\quad \underline{A}\,\underline{S} = \underline{\Lambda}\,\underline{S} \Rightarrow \underline{A}\,\underline{S} = \underline{S}\,\underline{\Lambda}$

$$\Rightarrow \underline{A} = \underline{S}\,\underline{\Lambda}\,\underline{S}^{-1}$$

Called an Eigendecomposition

Need $\underline{A}$ to be complete.

If $\underline{A}$ is defective, then rank$(\underline{S}) < n$
$\underline{S}^{-1}$ does not exist $\Rightarrow$
No eigendecomposition

Markov Chains

let $\underline{A}$ = transition matrix that is complete

$$\underline{A} = \underline{S}\,\underline{\Lambda}\,\underline{S}^{-1}$$

$\underline{p}_0$ = initial probability vector

$$\underline{p}_1 = \underline{A}\,\underline{p}_0 \Rightarrow \underline{p}_n = \underline{A}^n \underline{p}_0$$

$$\underline{A}^2 = \underline{A}\,\underline{A} = (\underline{S}\,\underline{\Lambda}\,\underline{S}^{-1})(\underline{S}\,\underline{\Lambda}\,\underline{S}^{-1})$$

$$= \underline{S}\,\underline{\Lambda}^2\,\underline{S}^{-1}$$

$$\underline{\Lambda}^2 = \begin{bmatrix} \lambda_1 & & \underline{0} \\ & \ddots & \\ \underline{0} & & \lambda_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \underline{0} \\ & \ddots & \\ \underline{0} & & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & & \underline{0} \\ & \ddots & \\ \underline{0} & & \lambda_n^2 \end{bmatrix}$$

$$\underline{A}^n = \underline{S}\,\underline{\Lambda}^n\,\underline{S}^{-1}$$

$$\Rightarrow \underline{p}_n = \underline{S}\,\underline{\Lambda}^n\,\underline{S}^{-1}$$

If $|\lambda_i| < 1$ for all eigenvalues, $\underline{p}_\infty = \underline{0}$

A stochastic Markov process will have exactly one eigenvalue of 1 with all others $|\lambda_i| < 1$

$$\Rightarrow \underline{A}^{\infty} = \underline{S}\,\underline{\Lambda}^{\infty}\,\underline{S}^{-1} = \underline{S}\begin{bmatrix} 1 & & & \underline{0} \\ & \lambda_2^{\infty} & & \\ & & \ddots & \\ \underline{0} & & & \lambda_n^{\infty} \end{bmatrix}\underline{S}^{-1}$$

$$= \underline{S}\begin{bmatrix} 1 & & & \underline{0} \\ & 0 & & \\ & & \ddots & \\ \underline{0} & & & 0 \end{bmatrix}\underline{S}^{-1}$$

$\Rightarrow P_{\infty}$ related to eigenvectors