Definition (6.2)

Two matrices A and B are **equal** if they have the same number of rows and columns, say $n \times m$, and if $a_{ij} = b_{ij}$, for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Definition (6.3)

If A and B are both $n \times m$ matrices, then the **sum** of A and B, denoted $A + B$, is the $n \times m$ matrix whose entries are $a_{ij} + b_{ij}$, for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Definition (6.4)

If A is an $n \times m$ matrix and λ is a real number, then the **scalar multiplication** of λ and A, denoted λA, is the $n \times m$ matrix whose entries are λa_{ij}, for each $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

We let O denote a matrix all of whose entries are 0.

Theorem (6.5)

Let A, B, and C be $n \times m$ matrices and λ and μ be real numbers. The following properties of addition and scalar multiplication hold:

(i) $A + B = B + A$,

(ii) $(A + B) + C = A + (B + C)$,

(iii) $A + O = O + A = A$,

(iv) $A + (-A) = -A + A = 0$,

(v) $\lambda(A + B) = \lambda A + \lambda B$,

(vi) $(\lambda + \mu)A = \lambda A + \mu A$,

(vii) $\lambda(\mu A) = (\lambda\mu)A$,

(viii) $1A = A$.

All these properties follow from similar results concerning the real numbers.

Definition (6.6)

Let A be an $n \times m$ matrix and $\mathbf{b}$ an m-dimensional column vector. The **matrix-vector product** of A and $\mathbf{b}$, denoted $A\mathbf{b}$, is an n-dimensional column vector given by

$$A\mathbf{b} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} a_{1i} b_i \\ \sum_{i=1}^{m} a_{2i} b_i \\ \vdots \\ \sum_{i=1}^{m} a_{ni} b_i \end{bmatrix}.$$

NOTE: For this product to be defined the number of columns of the matrix A must match the number of rows of the vector $\mathbf{b}$, and the result is another column vector with the number of rows matching the number of rows in the matrix.

Definition (6.7)

Let A be an $n \times m$ matrix and B an $m \times p$ matrix. The **matrix product** of A and B, denoted AB, is an $n \times p$ matrix C whose entries c_{ij} are

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{im} b_{mj},$$

for each $i = 1, 2, \ldots n$, and $j = 1, 2, \ldots, p$.

Theorem (6.8)

Let A be an $n \times m$ matrix, B be an $m \times k$ matrix, C be a $k \times p$ matrix, D be an $m \times k$ matrix, and λ be a real number. The following properties hold:

(a) $A(BC) = (AB)C$;

(b) $A(B + D) = AB + AD$;

(c) $\lambda(AB) = (\lambda A)B = A(\lambda B)$.

Definition (6.9)

(i) A **square** matrix has the same number of rows as columns.

(ii) A **diagonal** matrix $D = [d_{ij}]$ is a square matrix with $d_{ij} = 0$ whenever $i \neq j$.

(iii) The **identity matrix of order** n, $I_n = [\delta_{ij}]$, is a diagonal matrix whose diagonal entries are all 1s. When the size of I_n is clear, this matrix is generally written simply as I. □

Definition (6.10)

An **upper-triangular** $n \times n$ matrix $U = [u_{ij}]$ has, for each $j = 1, 2, \ldots, n$, the entries

$$u_{ij} = 0, \quad \text{for each } i = j+1, j+2, \ldots, n;$$

and a **lower-triangular** matrix $L = [l_{ij}]$ has, for each $j = 1, 2, \ldots, n$, the entries

$$l_{ij} = 0, \quad \text{for each } i = 1, 2, \ldots, j-1.$$

Definition (6.11)

An $n \times n$ matrix A is said to be **nonsingular** (or *invertible*) if an $n \times n$ matrix A^{-1} exists with $AA^{-1} = A^{-1}A = I$. The matrix A^{-1} is called the **inverse** of A. A matrix without an inverse is called **singular** (or *noninvertible*).

Theorem (6.12)

For any nonsingular $n \times n$ matrix A:

(i) *A^{-1} is unique.*

(ii) *A^{-1} is nonsingular and $(A^{-1})^{-1} = A$.*

(iii) *If B is also a nonsingular $n \times n$ matrix, then $(AB)^{-1} = B^{-1}A^{-1}$.* □

Definition (6.13)

The **transpose** of an $n \times m$ matrix $A = [a_{ij}]$ is the $m \times n$ matrix $A^t = [a_{ji}]$, where for each i, the ith column of A^t is the same as the ith row of A. A square matrix A is called **symmetric** if $A = A^t$.

Theorem (6.14)

The following operations involving the transpose of a matrix hold whenever the operation is possible:

(i) $(A^t)^t = A$,

(ii) $(A + B)^t = A^t + B^t$,

(iii) $(AB)^t = B^t A^t$,

(iv) *if A^{-1} exists, then* $(A^{-1})^t = (A^t)^{-1}$.

6.3. Linear algebra and matrix inversion.

Def. 6.2. Two matrices A and B are equal if they have the same number of rows and columns, say, $n \times m$, and if $a_{ij} = b_{ij}$ for all $i=1,\dots,n$, and $j=1,\dots,m$.

Def. 6.3. $A+B$ is the matrix $\{a_{ij}+b_{ij}\}_{\substack{i=1,\dots,n\\ j=1,\dots,m}}$

Def. 6.4 λA is the matrix $\{\lambda a_{ij}\}_{\substack{i=1,\dots,n\\ j=1,\dots,m}}$

Let $O = \{0\}_{\substack{i=1,\dots,n\\ j=1,\dots,m}}$, $-A = \{-a_{ij}\}_{\substack{i=1,\dots,n\\ j=1,\dots,m}}$

Matrix arithmetic.

Thm. 6.5 Let A, B, C are $n \times m$ matrices and λ and μ are real numbers. The following properties hold:

(i) $A+B = B+A$ (ii) $(A+B)+C = A+(B+C)$

(iii) $A+O = O+A = A$ (iv) $A+(-A) = -A+A = O$

(v) $\lambda(A+B) = \lambda A + \lambda B$ (vi) $(\lambda+\mu)A = \lambda A + \mu A$

(vii) $\lambda(\mu A) = (\lambda\mu)A$ (viii) $1 \cdot A = A$

Def. 6.6. Let A be an $n \times m$ matrix and $\vec{b}$ an m-dimensional column vector. The matrix-vector product of A and $\vec{b}$, denoted $A\vec{b}$, is a n-dimensional column vector given by

$$A\vec{b} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ & & \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} a_{1i} b_i \\ \cdots \\ \sum_{i=1}^{m} a_{ni} b_i \end{bmatrix}$$

The linear system

$$a_{11}x_1 + \dots + a_{1n}x_n = b_1$$
$$\text{— — — — —}$$
$$a_{n1}x_1 + \dots + a_{nn}x_n = b_n$$

can be expressed as

$$A\vec{x} = \vec{b}$$

where

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ - & & — \\ a_{n1} & \dots & a_{nn} \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Def. 6.7. Let A be $n \times \underline{\underline{m}}$ matrix and B an $\underline{\underline{m}} \times p$ matrix. The matrix product of A and B, denoted AB is an $n \times p$ matrix $C = \{c_{ij}\}_{\substack{i=1,\dots,n \\ j=1,\dots,p}}$

where $c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{im} b_{mj}$,

which is the same as

$$\begin{bmatrix} a_{i1} & \dots & a_{im} \end{bmatrix} \begin{bmatrix} b_{1j} \\ \vdots \\ b_{mj} \end{bmatrix}$$

Note: Generally, AB is not the same as BA.

Thm 6.8. Let A be $n \times m$ matrix, B be an $m \times k$ matrix, C be a $k \times p$ matrix, D be an $m \times k$ matrix, and λ be a real number. The following properties hold:

(a) $A(BC) = (AB)C$ (b) $A(B+D) = AB + AD$

(c) $\lambda(AB) = (\lambda A)B = A(\lambda B)$.

Square matrices

Def. 6.9. (i) A square matrix has the same number of rows and columns

(ii) A diagonal matrix $D = \{d_{ij}\}$ is a square matrix with $d_{ij} = 0$ if $i \neq j$.

(iii) The identity matrix of order n, $I_n = \{\delta_{ij}\}$ is a diagonal matrix with $\delta_{ii} = 1$ for all $i = 1, \ldots, n$. When the size is clear, I_n is written as I.

Def. 6.10. An upper-triangular matrix $U = \{u_{ij}\}_{\substack{i=1,\ldots,n \\ j=1,\ldots,n}}$ has $u_{ij} = 0$ for each $j = 1, \ldots, n$ and $i = j+1, \ldots, n$

$$\begin{bmatrix} a_{11} & \cdots & & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & 0 & & a_{nn} \end{bmatrix}$$

Lower-triangular L:

$l_{ij} = 0$ for each $j = 1, \ldots, n$, and $i = 1, \ldots, j-1$

$$\begin{bmatrix} a_{11} & & & 0 \\ & \ddots & & \\ a_{n-1,1} & \cdots & a_{n-1,n-1} & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

$I_n A = A I_n = A$ for any $n \times n$ matrix A.

Inverse matrices.

Def. 6.11. An $n \times n$ matrix A is called nonsingular (or invertible) if an $n \times n$ matrix A^{-1} ~~such~~ exists such that $A^{-1} A = A A^{-1} = I$.

The matrix A^{-1} is called inverse of A.

A matrix is called singular if it doesn't have an inverse.

Thm 6.12. For any nonsingular $n \times n$ matrix A:

(i) A^{-1} is unique

(ii) A^{-1} is nonsingular and $(A^{-1})^{-1} = A$

(iii) If B is also a nonsingular $n \times n$ matrix, then $(AB)^{-1} = B^{-1} A^{-1}$.

Computing the inverse matrix.

Let $B_j = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$, $\underset{n\times n}{B} = [B_1 \dots B_n]$

Assume $AB = C$. The j-th column of C is given by

$$\begin{bmatrix} c_{1j} \\ \vdots \\ c_{nj} \end{bmatrix} = C_j = AB_j = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^{n} a_{1k} b_{kj} \\ \dots \\ \sum_{k=1}^{n} a_{nk} b_{kj} \end{bmatrix}$$

If A^{-1} exists then $A \cdot A^{-1} = I$ meaning that

$$[A \cdot A^{-1}]_j = A \cdot A_j^{-1} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{-th row}$$

#5(a)

$$A = \begin{bmatrix} 4 & 2 & 6 \\ 3 & 0 & 7 \\ -2 & -1 & -3 \end{bmatrix}$$

① Set up an augmented matrix:

$$\left[\begin{array}{ccc|ccc} 4 & 2 & 6 & 1 & 0 & 0 \\ 3 & 0 & 7 & 0 & 1 & 0 \\ -2 & -1 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{E_2 - \frac{3}{4}E_1 \to E_2}$$

and solve 3 systems of equations simultaneously:

$$\left[\begin{array}{ccc|ccc} 4 & 2 & 6 & 1 & 0 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & -\frac{3}{4} & 1 & 0 \\ -2 & -1 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{E_3 + \frac{2}{4}E_1 \to E_3}$$

$$\left[\begin{array}{ccc|ccc} 4 & 2 & 6 & 1 & 0 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & -\frac{3}{4} & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 1 \end{array}\right] \leftarrow \text{no solutions!}$$

The matrix A is singular!

#5(b).

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{E_2 - 2E_1 \to E_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -3 & -1 & -2 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{E_3 - 3E_1 \to E_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -3 & -1 & -2 & 1 & 0 \\ 0 & -5 & 1 & -3 & 0 & 1 \end{array}\right] \xrightarrow{E_3 - \frac{5}{3}E_2 \to E_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -3 & -1 & -2 & 1 & 0 \\ 0 & 0 & \frac{8}{3} & \frac{1}{3} & -\frac{5}{3} & 1 \end{array}\right]$$

$\frac{8}{3}x_3 = \frac{1}{3}$	$\frac{8}{3}x_3 = -\frac{5}{3}$	$\frac{8}{3}x_3 = 1$
$x_3 = \frac{1}{8}$	$x_3 = -\frac{5}{8}$	$x_3 = \frac{3}{8}$
$-3x_2 - \frac{1}{8} = -2$	$-3x_2 + \frac{5}{8} = 1$	$-3x_2 - \frac{3}{8} = 0$
$-3x_2 = -\frac{15}{8}$	$-3x_2 = \frac{3}{8}$	$x_2 = -\frac{1}{8}$
$x_2 = \frac{15}{24} = \frac{5}{8}$	$x_2 = -\frac{1}{8}$	
$x_1 + \frac{2 \cdot 15}{24} = 1$	$x_1 - \frac{2}{8} = 0$	$x_1 - \frac{2}{8} = 0$
$x_1 = 1 - \frac{15}{12} = -\frac{3}{12} = -\frac{1}{4}$	$x_1 = \frac{2}{8}$	$x_1 = \frac{2}{8}$

$$\begin{bmatrix} -\frac{3}{12} \\ \frac{15}{24} \\ \frac{1}{8} \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ \frac{5}{8} \\ \frac{1}{8} \end{bmatrix} \quad \begin{bmatrix} \frac{2}{8} \\ -\frac{1}{8} \\ -\frac{5}{8} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{8} \\ -\frac{5}{8} \end{bmatrix} \quad \begin{bmatrix} \frac{2}{8} \\ -\frac{1}{8} \\ \frac{3}{8} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{8} \\ \frac{3}{8} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{5}{8} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & -\frac{5}{8} & \frac{3}{8} \end{bmatrix}$$

Check:

$$AA^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{5}{8} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & -\frac{5}{8} & \frac{3}{8} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{5}{8} & -\frac{1}{8} & -\frac{1}{8} \\ \frac{1}{8} & -\frac{5}{8} & \frac{3}{8} \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

° Note: Gaussian elimination with backward substitution requires $\frac{4}{3}n^3 - \frac{1}{3}n$ multiplications/divisions and $\frac{4}{3}n^3 - \frac{3}{2}n^2 + \frac{n}{6}$ additions/subtractions to solve the n linear systems.

Transpose of a matrix.

Def. 6.13. The transpose of an $n \times m$ matrix $A = [a_{ij}]$ is the $m \times n$ matrix $A' = [a_{ji}]$.

Thus, the columns of matrix A become rows of A', and vice versa.