Linear Systems

→ A linear equation is one where all unknowns are polynomials of order-1

$$a_1 x_1^1 + a_2 x_2^1 + \dots + a_n x_n^1 = b$$

$b, a_1, a_2, \dots, a_n$ known

$x_1, x_2, \dots, x_n$ unknown

A nonlinear equation is one where at least one unknown is not polynomial of order-1

$$a_1 x_1 + a_2 \sin(y_2) + \dots + a_n x_n = b$$

A system of linear equations are many linear equations sharing unknowns

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 &= b_1 \\ 0 x_1 + a_{22} x_2 + a_{23} x_3 &= b_2 \\ 0 x_1 + 0 x_2 + a_{33} x_3 &= b_3 \end{aligned}$$

a_{ij}, b_i known, x_i unknown

Write as

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}}_{\underline{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\underline{b}} \leftarrow \text{A linear system}$$

Recall that if $\underline{A}^{-1}$ exists $(\underline{A}\,\underline{A}^{-1} = \underline{A}^{-1}\underline{A} = \underline{I})$

then given $\underline{A}\,\underline{x} = \underline{b}$, $\underline{x} = \underline{A}^{-1}\underline{b}$

But A^{-1} not exiting does **not** mean there is no $\underline{x}$ such that $A\underline{x} = \underline{b}$

Goal: Given $\underline{A}$ & $\underline{b}$ is there an $\underline{x}$ such that $A\underline{x} = \underline{b}$

Need: 1 Existence: Does $\underline{x}$ exist?

2 Uniqueness: If $\underline{x}$ exists is it unique?

3 Methods to obtain $\underline{x}$.

Existence & Uniqueness

Given the columns of $\underline{A}$, can I take a linear combination that gives $\underline{b}$?

$$\underline{a}_1 x_1 + \underline{a}_2 x_2 + \dots + \underline{a}_n x_n \overset{?}{=} \underline{b}$$

Look at
$$\begin{aligned} a_{11} x_1 + a_{12} x_2 &= b_1 \\ a_{21} x_1 + a_{22} x_2 &= b_2 \end{aligned} \iff \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

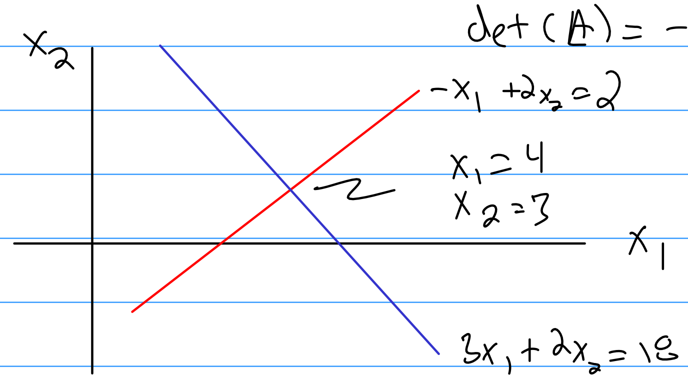
Linear means straight lines!

① $-x_1 + 2x_2 = 2$

$3x_1 + 2x_2 = 18$

$$\iff \begin{bmatrix} -1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 18 \end{bmatrix}$$

$\det(\underline{A}) = -8 \Rightarrow \underline{A}^{-1}$ exists, $\underline{x} = \underline{A}^{-1}\underline{b}$



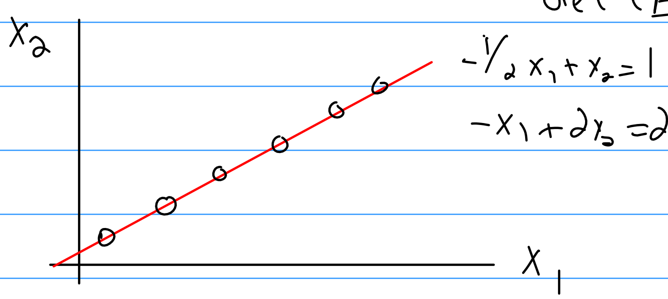
$$\underline{x} = \underline{A}^{-1}\underline{b} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

② $-\frac{1}{2}x_1 + x_2 = 1$

$-x_1 + 2x_2 = 2$

$$\iff \begin{bmatrix} -1/2 & 1 \\ -1 & 2 \end{bmatrix} \underline{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\det(\underline{A}) = 0 \Rightarrow \underline{A}^{-1}$ does not exist



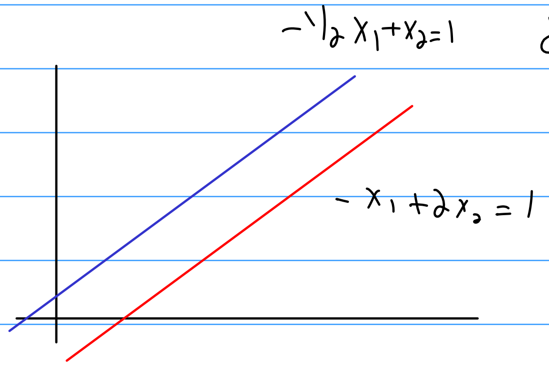
Infinite # of solutions!

③ $-\frac{1}{2}x_1 + x_2 = 1$

$-x_1 + 2x_2 = 1$

$$\iff \begin{bmatrix} -1/2 & 1 \\ -1 & 2 \end{bmatrix} \underline{x} = \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{b}$$

$\det(\underline{A}) = 0 \Rightarrow \underline{A}^{-1}$ does not exist



No crossing $\Rightarrow$ no solutions

$\Rightarrow$ Only possibilities for $\underline{A}\underline{x} = \underline{b}$

① $\underline{x}$ exists & is unique

② $\underline{x}$ exists & is not unique (infinite # of $\underline{x}$)

③ $\underline{x}$ Does not exist

Gaussian Elimination

Gaussian Elimination: A method to solve linear systems

Also called row reduction

Write $\underline{A}\underline{x} = \underline{b}$ as an augmented matrix: $[\underline{A} \vdots \underline{b}]$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \underline{x} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \Rightarrow \left[\begin{array}{cc:c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right]$$

You can:
1) Swap rows
2) multiply a row by a non-zero Scalar
3) Add one row to another

2) & 3) are usually combined.

Goal: Get $\left[\begin{array}{cc:c} 1 & c & d \\ 0 & 1 & e \end{array}\right]$ or $\left[\begin{array}{cc:c} 1 & 0 & f \\ 0 & 1 & g \end{array}\right]$

$x_2 = e \quad x_1 = d - ec$ $\qquad$ $x_1 = f \quad x_2 = g$

ex.)
$$\begin{bmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -11 \\ -3 \end{bmatrix}$$

$$\begin{array}{c} L_1 \\ L_2 \\ L_3 \end{array} \left[\begin{array}{ccc:c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array}\right]$$

$$L_1 + L_3 \to L_3 : \left[\begin{array}{ccc:c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ 0 & 2 & 1 & 5 \end{array}\right]$$

$$3/2\, L_1 + L_2 \to L_2 : \left[\begin{array}{ccc:c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 2 & 1 & 5 \end{array}\right]$$

$$-4\, L_2 + L_3 \to L_3 : \left[\begin{array}{ccc:c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right] \leftarrow$$

$$\begin{array}{l} 1/2\, L_1 \to L_1 \\ 2\, L_2 \to L_2 \\ -1\, L_3 \to L_3 \end{array} : \left[\begin{array}{ccc:c} 1 & 1/2 & -1/2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$\Rightarrow x_3 = -1$

$x_2 + x_3 = 2 \Rightarrow x_2 = 3$ $\quad\Rightarrow \underline{x} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

$x_1 + 1/2\, x_2 - 1/2\, x_3 = 4 \Rightarrow x_1 = 2$

This matrix is in row echlon form

$$\left[\begin{array}{ccc:c} a & b & c & g \\ 0 & d & e & h \\ 0 & 0 & f & i \end{array}\right]$$

The leading # in each row is a pivot

pivots

$$\left[\begin{array}{ccc:c} a & b & c & g \\ 0 & 0 & d & h \\ 0 & 0 & 0 & i \end{array}\right]$$

pivots

A matrix is in Reduced Row Echlon Form (rref) if:

1) It is in row echlon form
2) each pivot is equal to one
3) The only non-zero in columns containing a pivot is that pivot.

$$\left[\begin{array}{ccc:c} 1 & 4 & 9 & 4 \\ 0 & 2 & 2 & 10 \\ 0 & -4 & -4 & -40 \end{array}\right]$$

not in row echlon form

$$\left[\begin{array}{ccc:c} 1 & 4 & 9 & 4 \\ 0 & 2 & 2 & 10 \\ 0 & 0 & 0 & -20 \end{array}\right]$$

2 pivots in row echlon form

$$\left[\begin{array}{ccc:c} 1 & 4 & 9 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & -20 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & -16 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & -20 \end{array}\right]$$ 2 pivots

rref

The rref tells how many solutions there are to $\underline{A}\,\underline{x} = \underline{b}$

One Solution: The # of pivots equals the number of rows & unknowns (columns)

$\Rightarrow$ $\underline{A}$ $\underline{\text{must}}$ be Square

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right] \Rightarrow \underline{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

ex.)
$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 2 & 4 & 8 & 8 \\ 3 & 5 & 12 & 10 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

∞-Solutions: The # of pivots is less than the # of rows $\underline{\text{or}}$ columns and each row is consitent / possible

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & a & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \end{array}$$

2 pivots

$x_3 = c$

$x_1 = b - a x_2$

$x_2 = ?$

The last row is true no matter the value of x_2!

$$0x_1 + 0x_2 + 0x_3 = 0$$

x_2 is a free variable $\Rightarrow$ it can be any number!

Once x_2 is chosen then you have x_1 & x_3, which are fixed variables

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 5 \\ 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1, x_2 \Rightarrow$ fixed

$x_3 \Rightarrow$ free

let $x_3 = 1 \Rightarrow x_2 + 2x_3 = 2 \Rightarrow x_2 = 0$

$x_1 + x_3 = 3 \Rightarrow x_1 = 2$

$$\Rightarrow \underline{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

check: $\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+3 \\ 2 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$

Note: $x_3 = -4$ is also valid

$x_3 = 0$

Zero Solutions: # of pivots is less than # of rows or columns with at least one inconsistent row.

$$\begin{bmatrix} 1 & a & 0 & \vdots & b \\ 0 & 0 & 1 & \vdots & c \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix} \rightarrow 0x_1 + 0x_2 + 0x_3 = 1$$

not possible!

$\Rightarrow$ $\underline{A}\,\underline{x} = \underline{b}$ has no solution!

RREF, Inverse, Determinants

let $\underline{A} \in \mathbb{R}^{m \times n}$

You can get the rref of any matrix.

Recall that if $m = n$ & $\det(\underline{A}) \neq 0$, then $\underline{A}^{-1}$ exists

If $\underline{A}^{-1}$ exists then there is a unique $\underline{x}$ such that $\underline{A}\,\underline{x} = \underline{b} \Rightarrow \underline{x} = \underline{A}^{-1}\underline{b}$

You can not have 2 inverses!

Now recall pivots

$\rightarrow$ # pivots = # rows & # columns (square) $\rightarrow$ one solution

pivots $\neq$ # rows or # columns $\rightarrow$ ∞ or no solution

Start building the following theorem:

Let $\underline{A} \in \mathbb{R}^{n \times n}$. All of these statements are equivalent

1. $\underline{A}$ is invertible ($\underline{A}^{-1}$ exists)
2. $\det(\underline{A}) \neq 0$
3. $\underline{A}$ has n-pivots in rref($\underline{A}$)
4. $\text{rref}(\underline{A}) = \underline{I}$
5. $\underline{A}\underline{x} = \underline{b}$ has a unique solution for <u>any</u> $\underline{b}$

If <u>any</u> of these are true, then <u>all</u> are true.
If <u>any</u> is false, then all are false.

You do <u>not</u> know if you have 0 or ∞ solutions if $\det(\underline{A}) = 0$ w/o knowing what $\underline{b}$ is.

Gaussian Elimination Algorithm

Look at Gaussian elimination of

$$\begin{bmatrix} a_0 & b_0 & c_0 \\ d_0 & e_0 & f_0 \\ g_0 & h_0 & i_0 \end{bmatrix}$$

$a_0 \to i_0$ all non-zero

Step 1:

$$\begin{bmatrix} a_0 & b_0 & c_0 \\ d_0 & e_0 & f_0 \\ g_0 & h_0 & i_0 \end{bmatrix} \to \begin{array}{c} \begin{array}{ccc} & b_1 & c_1 \\ & \Downarrow & \Downarrow \end{array} \\ \begin{bmatrix} 1 & b_0/a_0 & c_0/a_0 \\ d_0 & e_0 & f_0 \\ g_0 & h_0 & i_0 \end{bmatrix} \end{array}$$

Step 2:

$$\begin{bmatrix} 1 & b_1 & c_1 \\ d_0 & e_0 & f_0 \\ g_0 & h_0 & i_0 \end{bmatrix} \to \begin{bmatrix} 1 & b_1 & c_1 \\ 0 & (e_0 - b_1 d_0) & (f_0 - c_1 d_0) \\ g_0 & h_0 & i_0 \end{bmatrix}$$

Step 3:

$$\begin{bmatrix} 1 & b_1 & c_1 \\ 0 & e_1 & f_1 \\ g_0 & h_0 & i_0 \end{bmatrix} \to \begin{bmatrix} 1 & b_1 & c_1 \\ 0 & e_1 & f_1 \\ 0 & h_0 - g_0 b_1 & i_0 - g_0 c_1 \end{bmatrix}$$

Step 4:

$$\begin{bmatrix} 1 & b_1 & c_1 \\ 0 & e_1 & f_1 \\ 0 & h_1 & i_1 \end{bmatrix} \to \begin{bmatrix} 1 & b_1 & c_1 \\ 0 & 1 & f_1/e_1 \\ 0 & h_1 & i_1 \end{bmatrix}$$

5:

$$\begin{bmatrix} 1 & b_1 & c_1 \\ 0 & 1 & f_2 \\ 0 & h_1 & i_1 \end{bmatrix} \to \begin{bmatrix} 1 & b_1 & c_1 \\ 0 & 1 & f_2 \\ 0 & 0 & i_1 - h_1 f_2 \end{bmatrix}$$

6:

$$\begin{bmatrix} 1 & b_1 & c_1 \\ 0 & 1 & f_2 \\ 0 & 0 & i_2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & c_1 - b_1 f_2 \\ 0 & 1 & f_2 \\ 0 & 0 & \underline{i_2} \end{bmatrix}$$

$$7: \begin{bmatrix} 1 & 0 & c_2 \\ 0 & 1 & f_2 \\ 0 & 0 & i_2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & c_2 \\ 0 & 1 & f_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$8 \,\&\, 9: \begin{bmatrix} 1 & 0 & c_2 \\ 0 & 1 & f_2 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This will fail if the leading element of a row is zero.

$$\begin{bmatrix} 1 & 2 & 4 \\ -2 & -4 & -9 \\ 3 & 4 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & -1 \\ 0 & -2 & -7 \end{bmatrix}$$

To fix this introduce pivoting

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -2 & -7 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 2 & 9 \end{bmatrix} \qquad i=2, \; j=2$$

$$\hookrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 9 \\ 0 & 1 & 4 \end{bmatrix}$$

General Algorithm

let $A \in \mathbb{R}^{m \times n}$

$i=1,\ j=1$ % Start at first row, first column

while $i \le m$ & $j \le n$ % iterate until either i or j exceed size

    Select $k \ge i$ that maximizes $|A_{kj}|$ % Row with largest value
% in same Column

    Swap rows k & i % make pivot location largest value

    if $|A_{ij}| > 10^{-16}$ % Pivot might be zero

        $a = A_{ij}$ % Store original pivot value

        for $k = [j:n]$ % normalize row i

            $A_{ik} = A_{ik}/a$

        end

        for $k = [1:i-1\ \ i+1:m]$ % other rows

            $a = A_{kj}$ % original value in row

            for $l = [j:n]$ % other columns

                $A_{kl} = A_{kl} - A_{il}\, a$

            end

        end

$$\begin{bmatrix} 1 & 0 & a & b & c \\ 0 & 0 & 1 & d & e \\ 0 & 0 & f & g & h \end{bmatrix}$$

        $i = i+1,\ j = j+1$

    else

        $j = j+1$ % Same row, next column

    end

end