Matrix Norm

Typically look at vector induced norms

$\Rightarrow$ how does a matrix affect a vector

Let $\underline{\underline{A}} \in \mathbb{R}^{m \times n}$

induced p-norm of $\underline{\underline{A}}$ is

$C \in \mathbb{R}$

such that

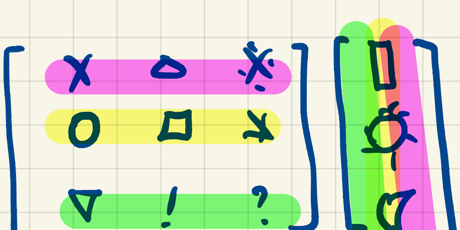
$$\|\underline{\underline{A}}\,\underline{x}\|_p \le C\,\|x\|_p \quad \text{for all } \underline{x} \in \mathbb{R}^{n \times 1}$$

typically set $\|\underline{x}\|_p = 1$

1 - norm, let $\|\underline{x}\|_1 \le 1$

$$\|\underline{\underline{A}}\,\underline{x}\|_1 = \left\|\sum_{i=1}^{n} x_i \underline{a}_i\right\|_1 \le \sum_{i=1}^{n} \|x_i\|\,\|a_i\|_1$$

$$\le \max_{1 \le i \le n} \|a_i\|_1$$



$\underline{\underline{A}}$

$\|\underline{\underline{A}}\,\underline{x}\|_1$ = longest column 1-norm

∞ - norm = longest row 1 - norm

$$\|\underline{\underline{A}}\,\underline{x}\|_\infty > \max_{1 \le j \le m} \|a_j\|_1$$

<u>Pointwise Norm</u>

Frobenius Norm $\quad \|\underline{\underline{A}}\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}}$

2- norm $\quad \|\underline{\underline{A}}\|_2 \le \|A\|_F$

norm (A, 'fro') - matlab

All norms obey

① $\|\underline{\underline{A}}\|_p \geq 0$

② $\|\underline{\underline{A}}\|_p = 0$ iff $\underline{\underline{A}} = \underline{\phi}$

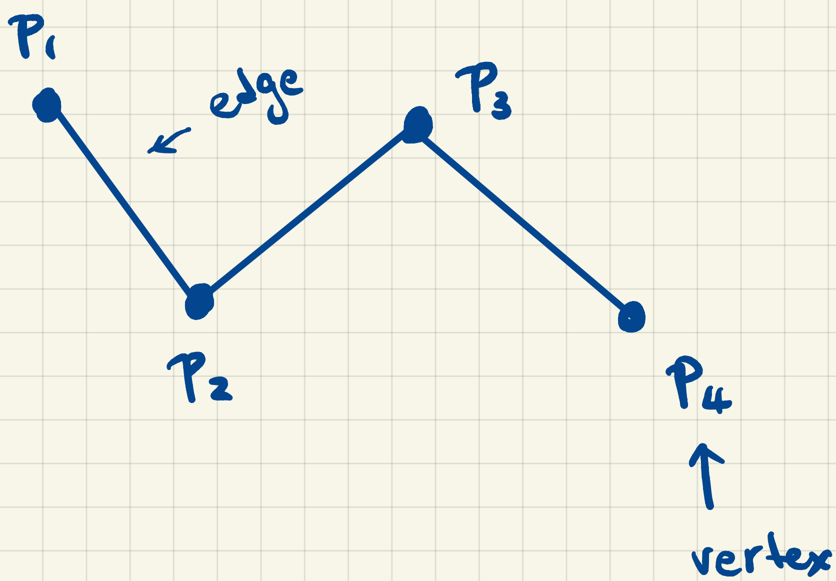
③ $\|\underline{\underline{A}} + \underline{\underline{B}}\|_p \le \|A\|_p + \|B\|_p$

④ $\|\alpha \underline{\underline{A}}\|_p = |\alpha| \, \|A\|_p$

Intro to Graph.

A graph is a collection of vertices

connected by edges



undirected graph

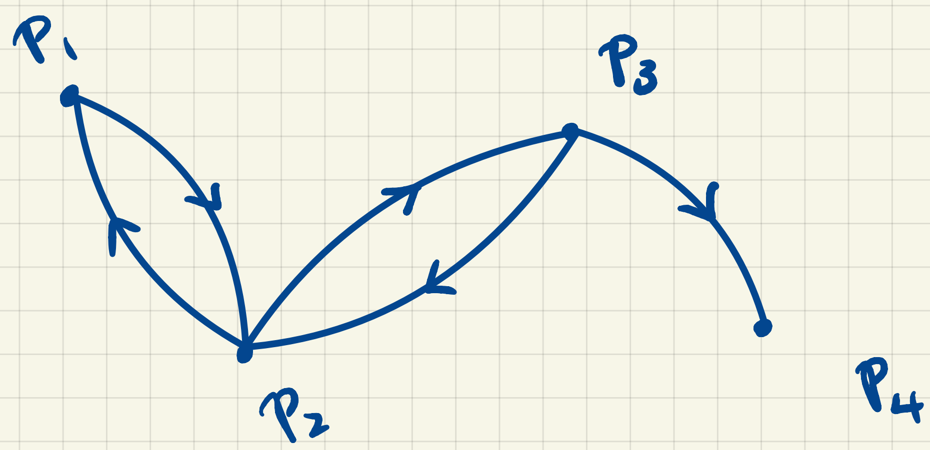
$P_2 \to P_1$

$P_1 \to P_2$

G_1

G_1 : It is possible to travel from P_1 directly to P_2. but if you need to get from P_1 to P_4, you need to pass through P_2 & P_3 first.

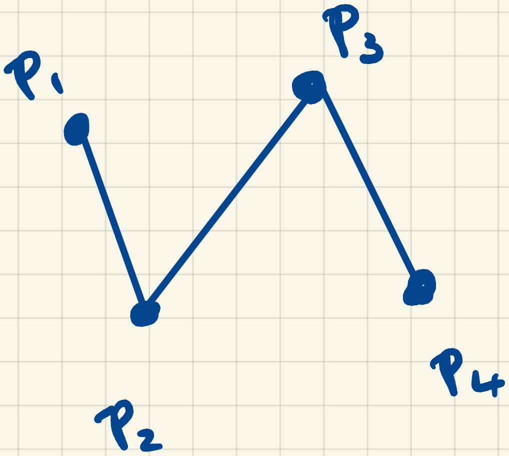
Directed Graph limits motion



An adjecency matrix is a graph in matrix form

Let there be n-vertices, The adjecency matrix is nxn matrix with $\emptyset$ or 1

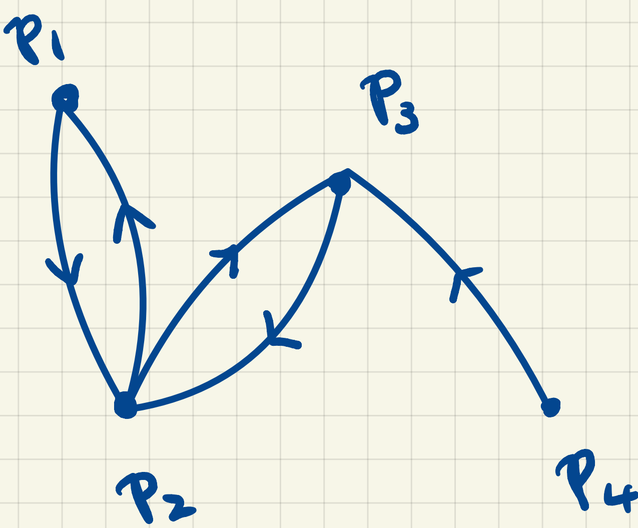
(simple matrices
weighted ones)



$$G_1 = \begin{array}{cc} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} & \begin{array}{c} P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \\ \begin{array}{cccc} P_1 & P_2 & P_3 & P_4 \end{array} & 4\times4 \end{array}$$

undirected graph

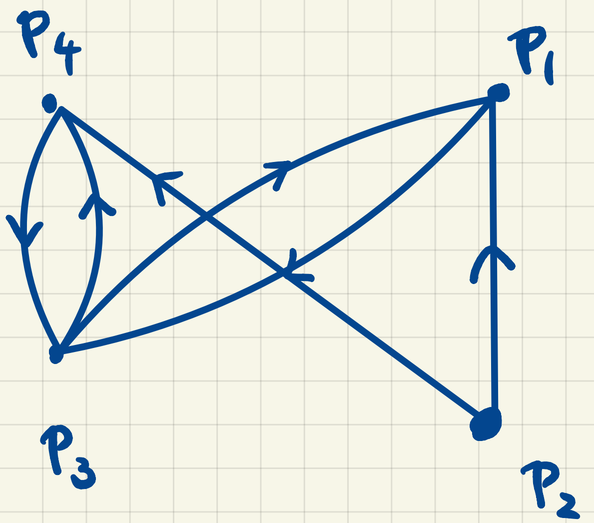
⇒ symmetric adjacency matrix



$$D_1 = \begin{array}{cc} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} & \begin{array}{c} P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \\ \begin{array}{cccc} P_1 & P_2 & P_3 & P_4 \end{array} & \end{array}$$

Path A finite sequence of edges that connect two vertices

length # of edges in a path



Path $P_1 \to P_4$

$P_1 \to P_3 \to P_4$

length = 2

Multiple Path $P_2 \to P_4$

$P_2 \to P_4$

length = 1

$P_2 \to P_1 \to P_3 \to P_4$

length = 3

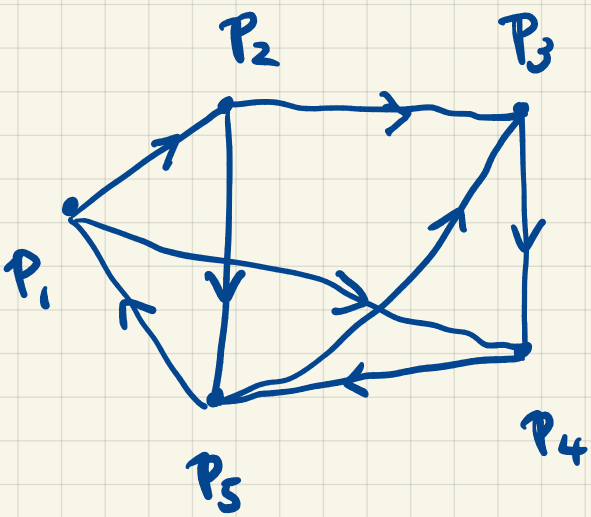
Thm : let $\underset{=}{A}$ be the adjerency matrix for any graph

The # & paths of length $= k$ between P_i and P_j is the # at location (i, j) of $\underset{=}{A}^k$

⇒ The total # ε path of length $\leq k$

is the value of $c_{i,j}$)

$$\sum_{p=1}^{k} \underline{A}^{k} = \underline{A}^{1} + \underline{A}^{2} + \underline{A}^{3} + \cdots \underline{A}^{k}$$



$$\underline{D_2} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\underline{D_2}^{2} = \begin{bmatrix} 0 & 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix}$$

$\underline{2}$ path of length $\underline{2}$

$P_1 \to P_5$

$$D_2^{3} = \begin{bmatrix} 2 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 3 \end{bmatrix}$$

2 path of length 3
from $P_2 \to P_4$

Markov Chain

Describe a stochastic process

where a sequence of events
occurs with a given probability

A state is a particular configuration

2 - state system A and B

A

16

B

16

An element can be in either stat A or stat B.

$n_A = 16 \qquad n_B = 16 \qquad n_t = n_A + n_B = 32$

the probability vector gives the probability that a random element is in a given state.

$$P_0 = \begin{bmatrix} n_A / n_t \\ n_A / n_t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

sum of P_0 must be 1.

During an iteration

75 % A remains in A 12

25 % A go to B 4

50% B remain in B 8

50 % B go to A 8

$12 = 4 + 8$, $\quad 12 + 8 = 20$

$$P_1 = \begin{bmatrix} n_A / n_t \\ \\ n_B / n_t \end{bmatrix} = \begin{bmatrix} 20/32 \\ \\ 12/32 \end{bmatrix} = \begin{bmatrix} 0.625 \\ \\ 0.375 \end{bmatrix}$$

$$P_2 \approx \begin{bmatrix} 0.6563 \\ \\ 0.3437 \end{bmatrix}$$

"switching" vector

$$\underline{v}_A = \begin{bmatrix} A \to A \\ A \to B \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$$

$$\underline{v}_B = \begin{bmatrix} B \to A \\ B \to B \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

$$\underline{v}_A n_A = \begin{bmatrix} 12 \\ 4 \end{bmatrix}$$

$$\underline{v}_B n_B = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$$

$$\Rightarrow \underline{v}_A n_A + \underline{v}_B n_B = \begin{bmatrix} 20 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} \underline{v}_A & \underline{v}_B \end{bmatrix} \begin{bmatrix} n_A \\ n_B \end{bmatrix}$$

$$= \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} n_A \\ n_B \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{array}{cc} \underline{v}_A & \underline{v}_B \\ \left[\begin{array}{c} 0.75 \\ 0.25 \end{array}\right. & \left.\begin{array}{c} 0.5 \\ 0.5 \end{array}\right] \end{array}}_{\underline{M}} \underbrace{\begin{bmatrix} n_A/n_t \\ n_B/n_t \end{bmatrix}}_{\underline{P}_0}$$

transition matrix

$$= \underline{P}_1$$

A markov chain has a fixed transition matrix.

Columns of $\underline{M}$ must sum to 1 unless you have growth / decay.

Given $\underline{\underline{M}}$, $\underline{P_0}$

$$\underline{P_1} = \underline{\underline{M}}\ \underline{P_0}$$

$$\underline{P_2} = \underline{\underline{M}}\ \underline{P_1} = \underline{\underline{M}}\ (\underline{\underline{M}}\ \underline{P_0}) = \underline{\underline{M}}^2\ \underline{P_0}$$

$$\Rightarrow \underline{P_n} = \underline{\underline{M}}^n\ \underline{P_0}$$

ratio

$$\underline{P_\infty} = \lim_{n\to\infty} \underline{M}^n\ \underline{P_0} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

equilibrium probability vector

Note: This does not mean elements are fixed in a state.

A markov chain is a set of distinct state S_1 to S_n

① Each element resides in a state

② Elements transition between states with a fixed probability

③ No difference between elements

Stochastic $\overset{\text{transition}}{\vee}$ Matrix : A transition matrix that

① square

② all entries are non-negative

③ all columns sum to 1.

Product of stochastic matrix are stochastic.

Regular Matrix : A stochastic matrix such that for some $\underline{\underline{M}}^k$ w/ $k \geq 1$ all entrices are strickly positive (>0)

If $\underline{\underline{M}}$ is a regular stochastic matrix.

① $\lim_{n \to \infty} \underline{\underline{M}}^n = \underline{\underline{M}}_\infty$ ← A set matrix

② All values in $\underline{\underline{M}}_\infty > 0$

③ All columns of $\underline{\underline{M}}_\infty$ are the same

④ P_∞ is the column of $\underline{\underline{M}}_\infty$ no matter the $\underline{P}_0$

⑤ $\underline{P}_\infty$ is a fixed point.

$$\underline{P}_\infty = \underline{\underline{M}}_\infty \underline{P}_\infty$$

$$\underline{\underline{M}} = \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix}$$

$$\underline{\underline{M}}^3 \quad , \quad \underline{\underline{M}}^{10}$$

$$\vdots$$

$$\underline{\underline{M}}^{50} = \begin{bmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{bmatrix}$$

$$\qquad\qquad \underline{P}_\infty \quad \underline{P}_\infty$$

let $\underline{P}_0 = \begin{bmatrix} a \\ 1-a \end{bmatrix}$

$$\underline{M}_\infty \, \underline{P}_0 = \begin{bmatrix} \underline{P}_\infty & \underline{P}_\infty \end{bmatrix} \begin{bmatrix} a \\ 1-a \end{bmatrix} = \underline{P}_\infty$$