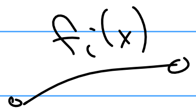
Cubic Splines

Splines that are C^2- continuous

$f_i(x) = a_i + b_i x + c_i x^2 + d_i x^3$



Goal: Get $4(n-1)$ equations for n-points

Condition #1: Each spline must return y @ the left point

$f_i(x_i) = y_i \qquad i = 1, \dots, n-1$

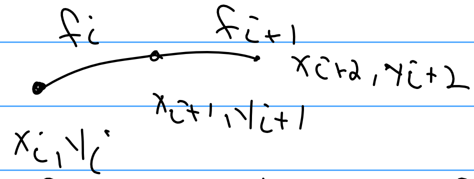
$n-1$ constraints

$\Rightarrow a_i = y_i$

Condition 1a : $f_{n-1}(x_n) = y_n$

+1 constraint

Condition #2: Splines must be C^0



$f_i(x_{i+1}) = f_{i+1}(x_{i+1}) \qquad i = 2, \dots, n-1$

$n-2$ constraints

$$f_i(x_{i+1}) = a_i + b_i(\underbrace{x_{i+1} - x_i}_{h_i}) + c_i(x_{i+1} - x_i)^2 + d_i(x_{i+1} - x_i)^3$$

$$= a_i + b_i h_i + c_i h_i^2 + d_i h_i^3$$

$$f_{i+1}(x_{i+1}) = a_{i+1} = y_{i+1}$$

$$\Rightarrow a_i + b_i h_i + c_i h_i^2 + d_i h_i^3 = y_{i+1}$$

Condition #3: Must be C^1 @ knots

$$f_i'(x_{i+1}) = f_{i+1}'(x_{i+1}) \qquad i = 2, \dots, n-1$$

$n-2$ constraints

$$\Rightarrow b_i + 2c_i h_i + 3d_i h_i^2 = b_{i+1}$$

Condition: C^2 @ knots

$$f_i''(x_{i+1}) = f_{i+1}''(x_{i+1}) \qquad i = 2, \dots, n-1$$

$n-2$ constraints

$$\Rightarrow 2c_i + 6d_i h_i = 2c_{i+1}$$

Add up equations

$$\underset{\#1}{(n-1)} + \underset{\#1a}{(1)} + \underset{\substack{\#2 \\ C^0}}{(n-2)} + \underset{\substack{\#3 \\ C^1}}{(n-2)} + \underset{\substack{\#4 \\ C^2}}{(n-2)} = 4n-6$$

$$= 4(n-1) - 2$$

missing 2 constraints

Various Options

1) Natural Cubic Splines : $f_1''(x_1) = 0$
$f_{n-1}''(x_n) = 0$

2) Clamped Cubic Splines : $f_1'(x_1) = a \in \mathbb{R}$
$f_{n-1}'(x_n) = b \in \mathbb{R}$

3) "not-a-knot" $f_1'''(x_2) = f_2'''(x_2)$

$f_{n-2}'''(x_{n-1}) = f_{n-1}'''(x_{n-1})$

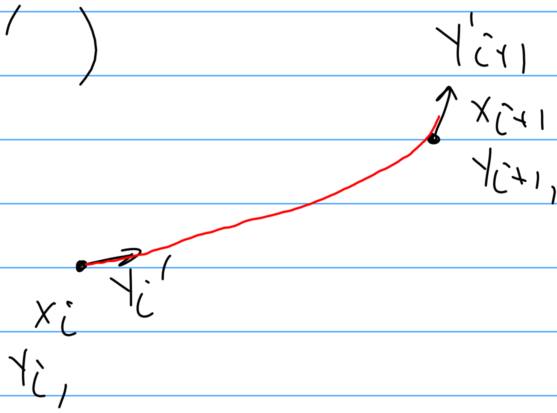
Will result in a $4(n-1)$ by $4(n-1)$ linear system

Cubic Hermite Splines

A related method where you know function values & derivatives

Given (x_i, y_i, y_i')

Let $x \in [x_i, x_{i+1}]$



Let $t(x) = \dfrac{x - x_i}{x_{i+1} - x_i} = \dfrac{x - x_i}{h_i}$, $t(x) \in [0, 1]$

Define the spline as

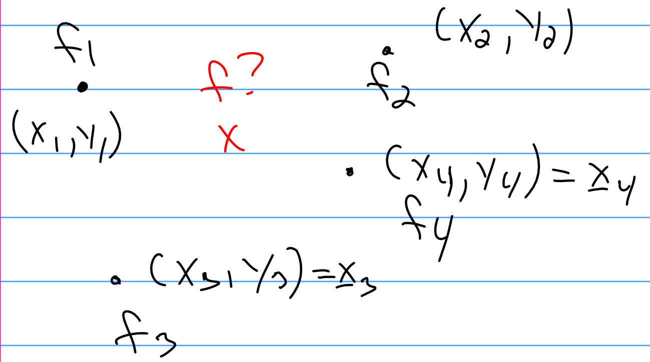
$$p_i(t) = (2t^3 - 2t^2 + 1)\, y_i + (t^3 - 2t^2 + t)\, h_i\, y_i' + (-2t^3 + 3t^2)\, y_{i+1} + (t^3 - t^2)\, h_i\, y_{i+1}'$$

$\Rightarrow$ $f(x) = p\left(\frac{x - x_i}{h_i}\right)$ gives value

$$f'(x) = \frac{1}{h_i}\, p'\left(\frac{x - x_i}{h_i}\right)$$

Radial Basis Interpolation

Scattered Data in 1D/2D/3D



RBF Interpolant : $S(\underline{x}) = \sum_{i=1}^{n} \underbrace{\omega_i}_{\text{weight}} \phi(\underbrace{\|\underline{x} - \underline{x}_i\|}_{\text{Distance between } \underline{x} \text{ \& } \underline{x}_i})$

$r = \|\underline{x} - \underline{y}\|$ ← 2-norm

$\phi(r) = e^{-(\varepsilon r)^2}$ Gaussian

$\phi(r) = (1 + (\varepsilon r)^2)^{1/2}$ Multiquadric

$\phi(r) = r^{2m+1}$ $m \geq 1$ integer

$r^{3,5,7}$

Require that $S(\underline{x}_i) = f_i$

Let $r_{ij} = \|\underline{x}_i - \underline{x}_j\|$

$$\phi(r_{11}) w_1 + \phi(r_{12}) w_2 + \dots + \phi(r_{1n}) w_n = f_1$$
$$\phi(r_{21}) w_1 + \phi(r_{22}) w_2 + \dots + \phi(r_{2n}) w_n = f_2$$
$$\vdots$$
$$\phi(r_{n1}) w_1 + \phi(r_{n2}) w_2 + \dots + \phi(r_{nn}) w_n = f_n$$

$$\begin{bmatrix} \phi(r_{11}) & \phi(r_{12}) & \cdots & \phi(r_{1n}) \\ \phi(r_{12}) & \phi(r_{22}) & \cdots & \phi(r_{2n}) \\ & & & \vdots \\ & & & \phi(r_{nn}) \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$$

Due to $r_{12} = r_{21}$

Numeric Integration

Let $f(x)$ be given for $x \in [a, b]$

Approximate $\int_a^b f(x)\,dx$

Let $x_i \in [a,b]$ be discrete points for $i=1,\dots,n$
such that $x_{i-1} < x_i < x_{i+1}$

Let $w_i \in \mathbb{R}$ be **weights** such that

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

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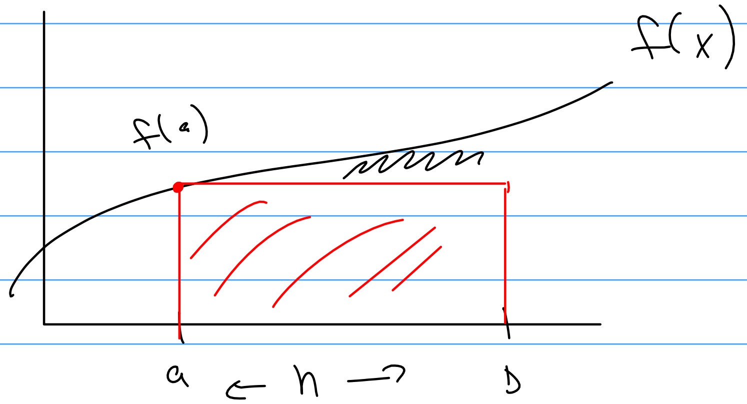
1) Left / Right Point Approximation



$h = b - a$

$$\Rightarrow \int_a^b f(x)\,dx \approx f(a)\,h = f(a)(b-a)$$

Look at the **Order of Accuracy**

Assume $x \in [0, h]$ $\quad (a=0,\ b=h)$

Taylor Series of $f(x)$ @ $x=0$

$$f(x) = f(0) + hf'(0) + \tfrac{1}{2}h^2 f''(0) + \tfrac{1}{6}h^3 f'''(0) + \dots$$

Plug into $\int_0^h f(x)\,dx = \int_0^h \left(f(0) + hf'(0) + \dots\right) dx$

$$= f(0)h + \tfrac{1}{2}h^2 f'(0) + \tfrac{1}{6}h^3 f''(0) + \dots$$

↑ exact

Error is exact minus approximation

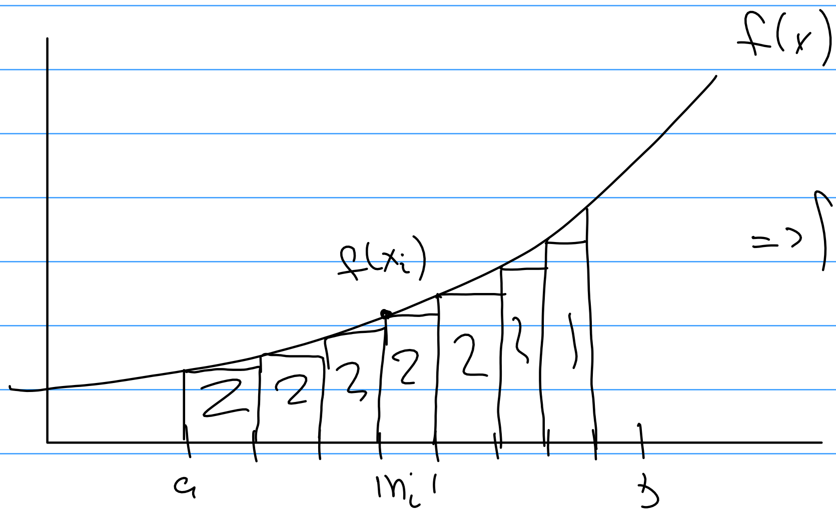
$$\left[\cancel{f(0)h} + \tfrac{1}{2}h^2 f'(0) + \tfrac{1}{6}h^3 f''(0) + \dots\right] - \cancel{f(0)h}$$

$$= \tfrac{1}{2}h^2 f'(0) + \tfrac{1}{6}h^3 f''(0) + \dots$$

$$= \tfrac{1}{2}h^2 f'(0) + H.O.T. = O(h^2)$$

$\Rightarrow$ left point rule has a local truncation error of $O(h^2)$

Now look at $\int_a^b f(x)\,dx$ by breaking $[a,b]$ into sub-intervals



$$\Rightarrow \int_a^b f(x)\,dx = \sum_{i=1}^{n-1} f(x_i)\,h_i$$

Called a Composite or Reimann Sum

Look at the Global Error.

n - intervals, each w/ $O(h^2)$

$n \sim \frac{1}{h} \Rightarrow$ Global Error $\sim \frac{1}{h} O(h^2) = O(h)$

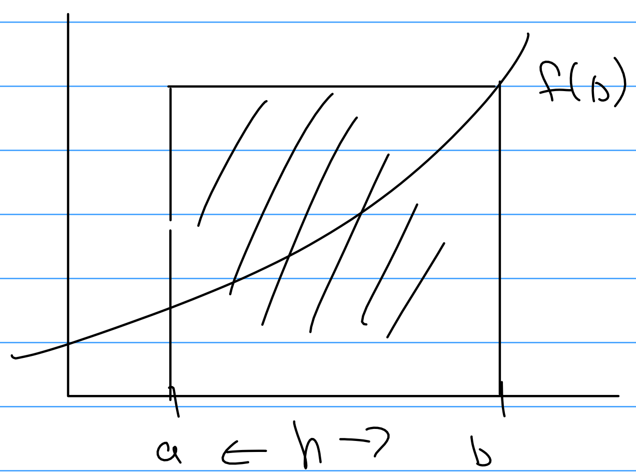
ex.) $\int_0^1 x^2 \, dx = 1/3$

n	h	E	$E(n)/E(0.5n)$
2	0.5	0.2917	
4	0.25	0.1354	2.15
8	0.125	0.0654	2.08
16	0.0625	0.0319	2.04

$\Rightarrow O(h)$ means if h is cut by $1/2$ then so is the error.

$O(h^2)$: h cut by $1/2$ error goes down by 4 (to $1/4$ it's original value)

Right-Point:

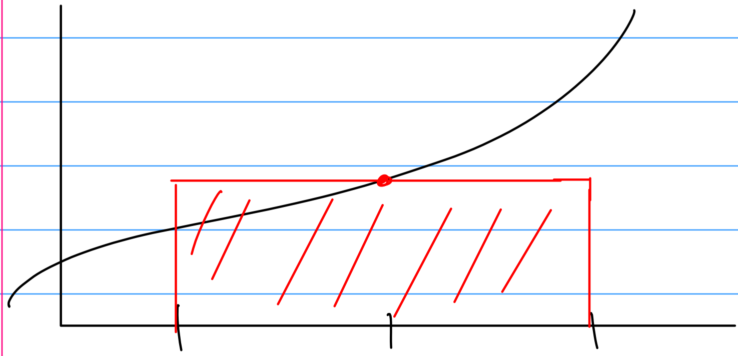


$\int_a^b f(x)\, dx \approx f(b)\, h$

local: $O(h^2)$

Global: $O(h)$

Midpoint Rule

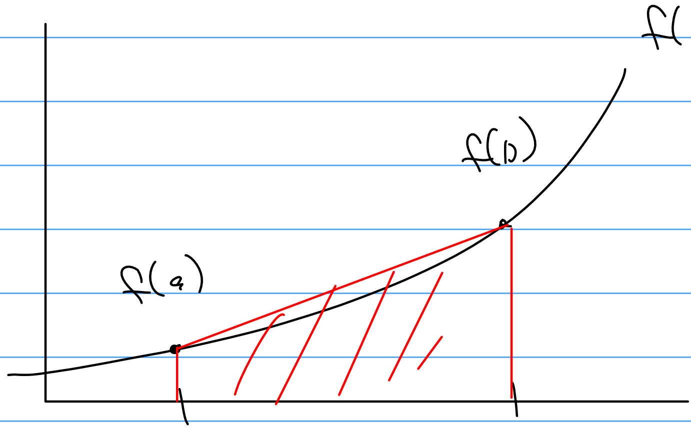


$$\int_a^b f(x)\,dx \approx h\, f\left(\frac{a+b}{2}\right)$$

Local Error is $O(h^3)$
Global is $O(h^2)$

Trapezoid Rule

Evaluate a Linear Interpolant



$h = b - a$

$$\int_a^b f(x)\,dx \approx \int_a^b \left(f(a) + \frac{f(b)-f(a)}{h}(x-a) \right) dx$$

$$= f(a)h + \frac{f(b)-f(a)}{2} h$$

$$= \frac{h}{2} f(a) + \frac{h}{2} f(b) = \sum_{i=1}^{2} w_i f(x_i)$$

$\Rightarrow \quad x_1 = a \qquad w_1 = h/2$

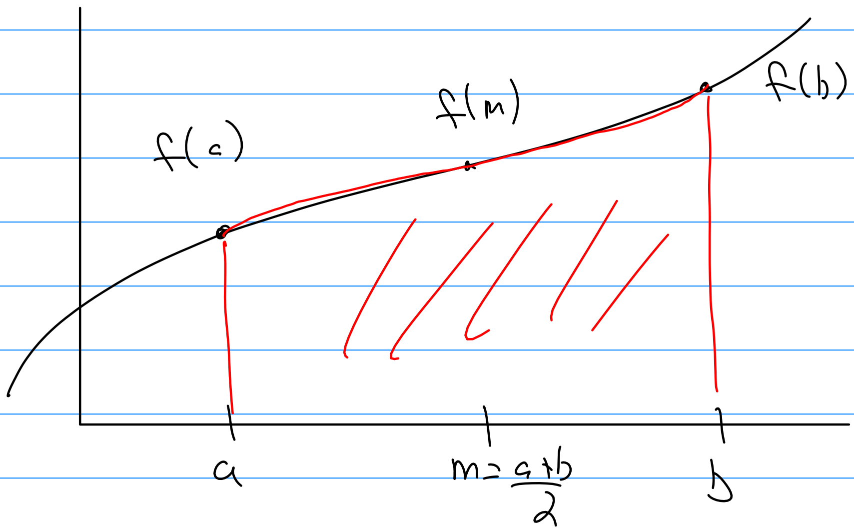
$\qquad x_2 = b \qquad w_2 = h/2$

local is $O(h^3)$

Global is $O(h^2)$

Simpson's Rule

fit a quadratic interpolant over $[a, b]$



$$\int_a^b f(x)\,dx \simeq \int_a^b (a_0 + a_1 x + a_2 x^2)\,dx = \frac{b-a}{6}\left[f(a) + 4f(m) + f(b)\right]$$

$\Rightarrow \quad x_1 = a \qquad w_1 = h/6$

$\qquad x_2 = m \qquad w_2 = 2h/3$

$\qquad x_3 = b \qquad w_3 = h/6$

local $= O(h^5)$ $\qquad$ Global $= O(h^4)$

Gaussian Quadrature

Do not use a uniform h

Instead, choose x_i & w_i to maximize accuracy.

Construct a 2-point rule that integrates that integrates all order-3 polynomials exactly over $x \in [-1, 1]$

$$\int_{-1}^{1} f(x)\,dx = w_1 f(x_1) + w_2 f(x_2)$$

Let $f(x) = 1$: $\int_{-1}^{1} f(x)dx = \int_{-1}^{1} 1\,dx = 2 = w_1 + w_2$

$f(x) = x$: $\int_{-1}^{1} x\,dx = 0 = w_1 x_1 + w_2 x_2$

$f(x) = x^2$: $\int_{-1}^{1} x^2 dx = 2/3 = w_1 x_1^2 + w_2 x_2^2$

$f(x) = x^3$: $\int_{-1}^{1} x^3 dx = 0 = w_1 x_1^3 + w_2 x_2^3$

from $f(x) = x$ you get $w_2 = -w_1 \dfrac{x_1}{x_2}$

Use in $f(x) = x^3$: $w_1 x_1^3 - \left(w_1 \dfrac{x_1}{x_2}\right) x_2^3 = 0$

$\Rightarrow x_1^2 = x_2^2$

$x_1 \neq x_2 \Rightarrow x_1 = -x_2$

Then $w_2 = -w_1 \dfrac{x_1}{x_2} = -w_1 \dfrac{(-x_2)}{x_2} = w_1 \Rightarrow w_1 = w_2$

In $f(x)=1$: $2 = w_1 + w_2 = 2w_1 \Rightarrow w_1 = w_2 = 1$

$f(x) = x^2$: $\dfrac{2}{3} = w_1 x_1^2 + w_2 x_2^2 = x_1^2 + x_2^2 = 2x_1^2$

$\Rightarrow x_1 = \dfrac{1}{\sqrt{3}}, \quad x_2 = -\dfrac{1}{\sqrt{3}}$

$\Rightarrow \displaystyle\int_{-1}^{1} f(x)\,dx = f\left(\frac{1}{\sqrt{3}}\right) + f\left(\frac{-1}{\sqrt{3}}\right)$