Matrix Norms

Typically look at vector-induced norms
- How a matrix changes a vector

let $\underline{A} \in \mathbb{R}^{m\times n}$
induced p-norm of $\underline{A}$ is $C \in \mathbb{R}$
such that
$\|\underline{A}\underline{x}\|_p \leq C\|\underline{x}\|_p$ for all $\underline{x} \in \mathbb{R}^{n\times 1}$
Typically use $\|\underline{x}\|_p = 1$

1-norm : let $\|\underline{x}\|_1 \leq 1$

$$\|\underline{A}\underline{x}\|_1 = \|\sum_{i=1}^{n} x_i \underline{a}_i\|_1 \leq \sum_{i=1}^{n} |x_i| \|a_i\|_1$$

$$\leq \max_{1\leq i\leq n} \|\underline{a}_i\|_1$$

$\|\underline{A}\underline{x}\|_1$ = largest column 1-norm

$\|\underline{A}\underline{x}\|_\infty$ = largest row 1-norm

$$= \max_{1\leq j\leq m} \|\underline{a}_j\|_1$$

Also: Pointwise Norms

Frobenius Norm: $\|\underline{A}\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2}$

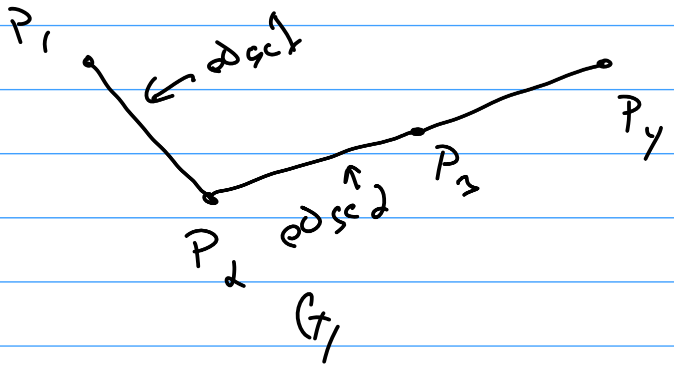
2-norm: $\|\underline{A}\|_2 \leq \|\underline{A}\|_F$
Matlab: norm(A,2) norm(A, 'fro')

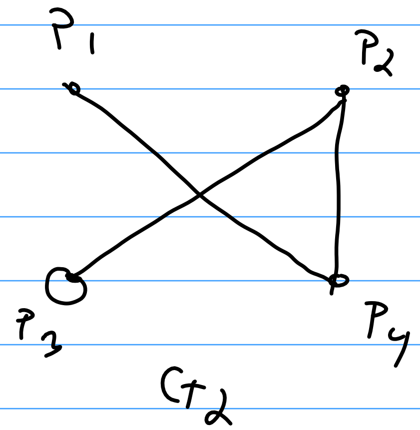
All norms obey:

1) $\|\underline{A}\|_p \geq 0$
2) $\|\underline{A}\|_p = 0$ iff $\underline{A} = \underline{\phi}$
3) $\|\underline{A} + \underline{B}\|_p \leq \|\underline{A}\|_p + \|\underline{B}\|_p$
4) $\|\alpha \underline{A}\|_p = |\alpha| \, \|\underline{A}\|_p$

Intro to Graphs

A **graph** is a collection of **vertices** connected by **edges**



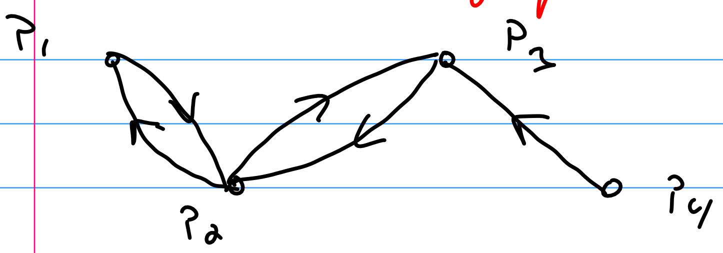


G_1: It is possible to "travel" from $P_1 \to P_2$ directly. To go from $P_1 \to P_4$ you need to pass through P_2 & P_3

Both G_1 & G_2 are **undirected graphs**

$$P_1 \to P_2 \quad \& \quad P_2 \to P_1$$

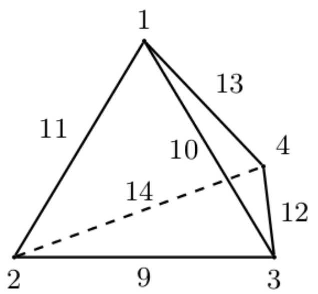
A **directed graph** has directions

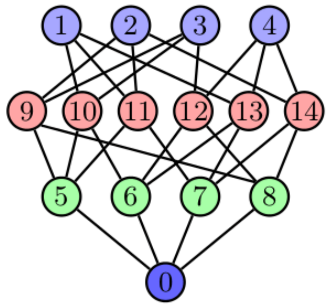


$P_2 \to P_3$
$P_3 \to P_2$
$P_4 \to P_3$
$P_3 \to P_4$ not possible

Applications:

- linguistics
- chemistry
- networking
- Scientific computing
- Machine learning





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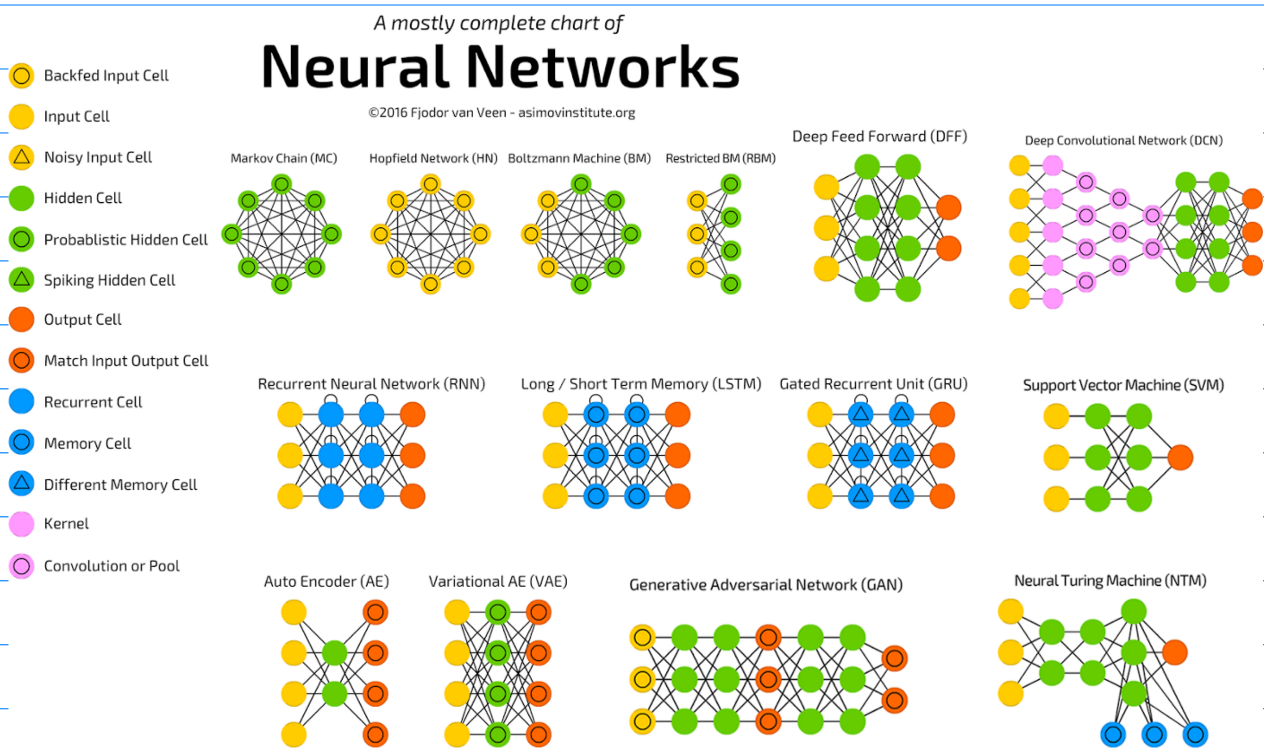
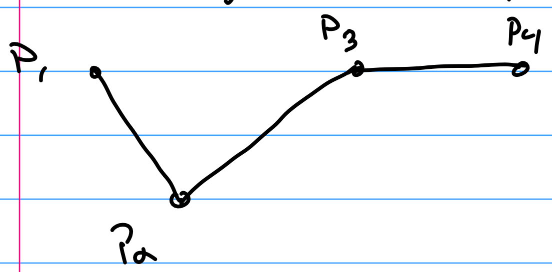


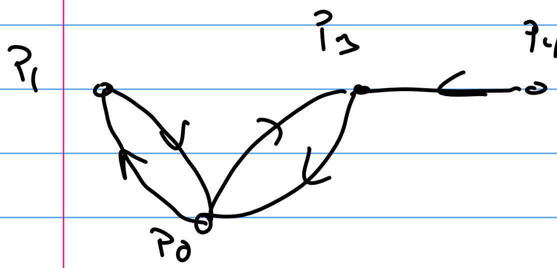
An adjacency matrix is a graph in matrix form

let there be n-vertices. The adjacency matrix is $n \times n$ w/ 1 if vertices $i \to j$ are connected ("travel") 0 otherwise



$$
G_1 = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}
\begin{array}{cccc|c}
P_1 & P_2 & P_3 & P_4 & \\
\end{array}
$$

$$
G_1 = \begin{array}{cc}
 & \begin{array}{cccc} P_1 & P_2 & P_3 & P_4 \end{array} \\
 & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\end{array}
\begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array}
$$



$$
D_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}
$$

Undirected graphs will be symmetric
Directed graphs will not be symmetric

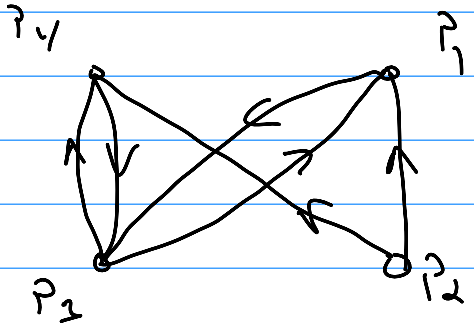
Path: A finite sequence of edges that connects 2 vertices

G_1: Path from $P_1 \to P_3$: $P_1 \to P_2 \to P_3$

length = 2

Length: # of edges in a path

ex.)



Path from $P_1 \to P_4$

$P_1 \to P_3 \to P_4$

length = 2

$P_2 \to P_4$: multiple paths

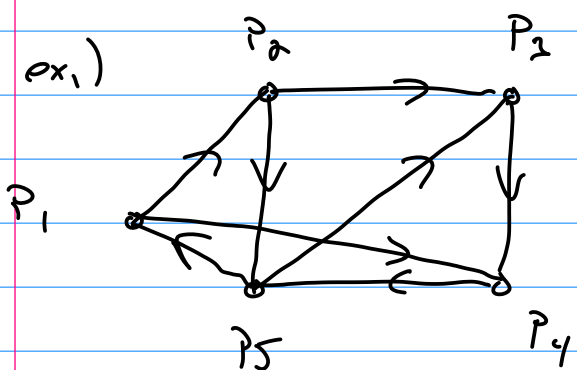
$P_2 \to P_4$ $\quad$ $P_2 \to P_1 \to P_3 \to P_4$

Thm: let A be the adjacency matrix for a graph

The total # of paths of length = k between vertices P_i & P_j is the value at the (i,j) location of A^k.

Collorary: total # of paths of length $\leq k$ is the value (i,j) of $\sum_{j=1}^{k} A^j$

ex.)



P_2 to P_4

$$D_2^1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

P_4 to P_2

$$D_2^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix}$$

1 path of length = 2 from P_2 to P_4

2 paths of length = 2 from P_1 to P_5

$$D_2^3 = \begin{bmatrix} 2 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 3 \end{bmatrix}$$

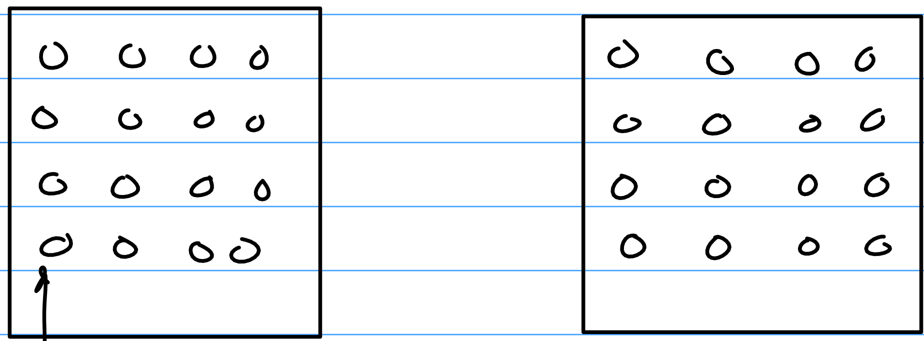
In Matlab: graph() & digraph()

Markov Chains

Used to describe a stochastic process where a sequence of events w/ a given & fixed probability

A state is a particular configuration.

ex1) A 2-state system A & B



A B

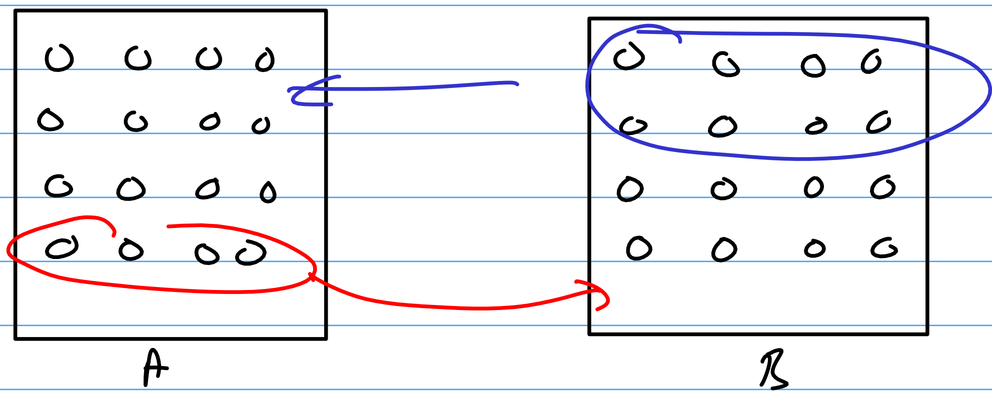
An element can be either in A or B

$n_A = 16$ $n_B = 16$ $n_t = n_A + n_B = 32 = \text{fixed}$

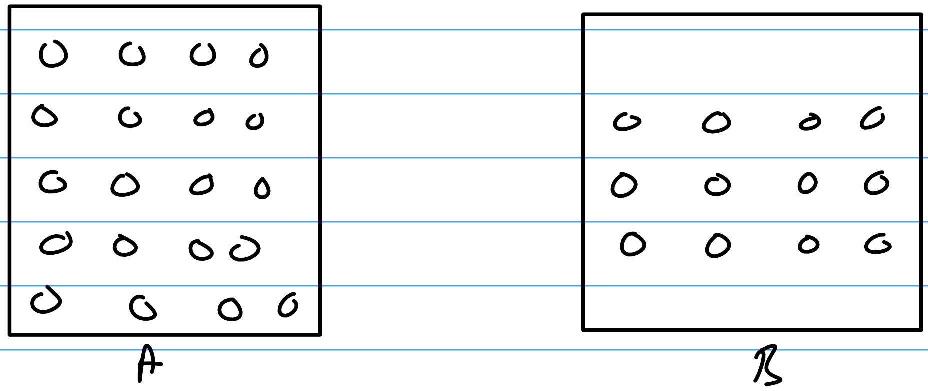
Probability vector : probability that a random element is in A or B

$$P_0 = \begin{bmatrix} n_A/n_t \\ n_B/n_t \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \leftarrow$$ Sum of P_0 must be 1

During an iteration: 75% of A remains in A
25% of A go to B
50% of B go to A , 50% of B stays in B



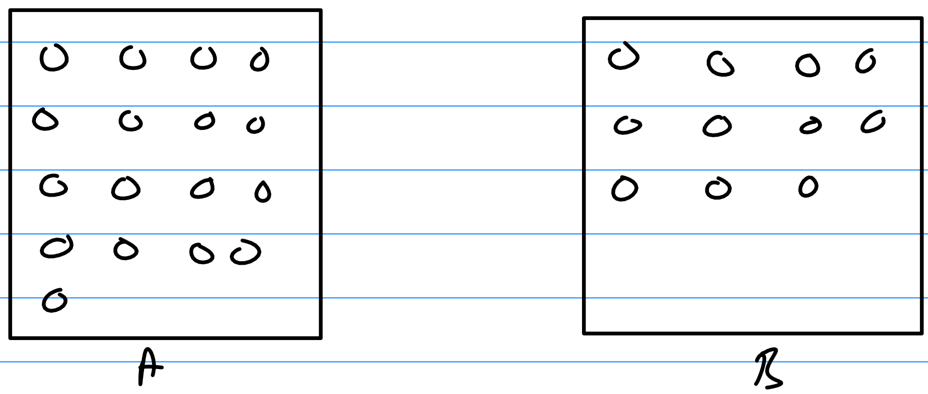
One iteration:



$n_A = 20$ $\quad$ $n_B = 12$

$$P_1 = \begin{bmatrix} 20/32 \\ 12/32 \end{bmatrix} = \begin{bmatrix} 0.625 \\ 0.375 \end{bmatrix}$$

Second iteration:



$$P_2 \simeq \begin{bmatrix} 0.6563 \\ 0.3437 \end{bmatrix}$$

$n_A = 21$ $\quad$ $n_B = 11$

Introduce the "Switching" vectors

$\underline{v}_A = \begin{bmatrix} A \to A \\ A \to B \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$ $\quad \underline{v}_B = \begin{bmatrix} B \to A \\ B \to B \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

One iteration: $\underline{v}_A n_A = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} 16 = \begin{bmatrix} 12 \\ 4 \end{bmatrix}$

$\underline{v}_B n_B = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} 16 = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$

$\underline{v}_A n_A + \underline{v}_B n_B = \begin{bmatrix} 20 \\ 12 \end{bmatrix} = [\underline{v}_A \ \underline{v}_B] \begin{bmatrix} n_A \\ n_B \end{bmatrix}$

Divide by n_t

$\underbrace{[\underline{v}_A \ \underline{v}_B]}_{\underline{M}} \underbrace{\begin{bmatrix} n_A^0/n_t \\ n_B^0/n_t \end{bmatrix}}_{\underline{p}_0} = \begin{bmatrix} n_A'/n_t \\ n_B'/n_t \end{bmatrix} = \underbrace{\begin{bmatrix} 0.625 \\ 0.375 \end{bmatrix}}_{\underline{p}_1}$

$\underline{M} = \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix} =$ Transition Matrix

A Markov Chain has a fixed transition matrix. Columns of $\underline{M}$ must sum to $\underline{1}$ unless you have growth/decay

Given $\underline{M}$, $\underline{p}_0$, $\underline{p}_1 = \underline{M}\,\underline{p}_0$

$\underline{p}_2 = \underline{M}\,\underline{p}_1 = \underline{M}(\underline{M}\,\underline{p}_0) = \underline{M}^2 \underline{p}_0$

In general $P_n = \underline{M}^n P_0$

In our example

$$\lim_{n\to\infty} \underline{M}^n P_0 = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = P_\infty \Rightarrow$$ equilibrium probability vector

Note: This does ~~not~~ mean elements are fixed in a state. If 1 element goes $A \to B$ then 1 goes $B \to A$

A Markov Chain is a set of distinct states S_1 to S_N
1) Each element is in a state
2) Fixed transition between states
3) No difference between elements.

Stochastic (transition) Matrix : A matrix that is
1) Square
2) All entries are non-negative (≥ 0)
3) All columns sum to 1.

A regular matrix is a stochastic matrix such that for some $\underline{M}^k$ w/ $k \geq 1$ all entries are strictly positive (> 0)

Thm: If $\underline{M}$ is a regular, stochastic matrix then

1) $\lim_{n\to\infty} \underline{M}^n = \underline{M}_\infty$ a set matrix

2) All values of $\underline{M}_\infty$ are strictly positive

3) All columns of $\underline{M}_\infty$ are the same

4) $\vec{p}_\infty$ is a column of $\underline{M}_\infty$ no matter $\vec{p}_0$

5) $\vec{p}_\infty$ is a fixed-point: $\vec{p}_\infty = \underline{M}_\infty \vec{p}_0$
$= \underline{M}\,\vec{p}_\infty$

ex.) $\underline{M} = \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix}$

$\underline{M}^3 = \begin{bmatrix} 0.6719 & 0.6562 \\ 0.3281 & 0.3438 \end{bmatrix}$

$\underline{M}^{10} = \begin{bmatrix} 0.6667\ldots & 0.6667\ldots \\ 0.3333\ldots & 0.3333\ldots \end{bmatrix}$

$\underline{M}^{50} = \begin{bmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{bmatrix} = \underline{M}^{100} = \underline{M}^{1000} = \underline{M}_\infty$

$\quad\ \vec{p}_\infty \quad \vec{p}_\infty$

let $\vec{p}_0 = \begin{bmatrix} a \\ 1-a \end{bmatrix}$ $\quad a \in [0, 1]$

$\underline{M}_\infty \vec{p}_0 = [\vec{p}_\infty\ \ \vec{p}_\infty]\begin{bmatrix} a \\ 1-a \end{bmatrix} = a\vec{p}_\infty + (1-a)\vec{p}_\infty = \vec{p}_\infty$

ex.) 3- bank problem

Initially Bank- A has 40% share

Bank-B has 10%

Bank -C has 50%

$$P_0 = \begin{bmatrix} 0.4 \\ 0.1 \\ 0.5 \end{bmatrix}$$

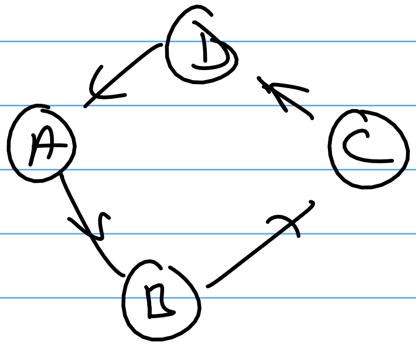
A to A

$$\begin{array}{c c} & \\ \underline{M} = & \\ \text{A to B} \rightarrow & \end{array} \begin{bmatrix} 0.5 & 0.1665 & 0.25 \\ 0.25 & 0.667 & 0.25 \\ 0.25 & 0.1665 & 0.5 \end{bmatrix}$$

$$P_1 = \underline{M}\, P_0 = \begin{bmatrix} 0.3417 \\ 0.2917 \\ 0.3667 \end{bmatrix}$$

$$\underline{M}_{\infty} = \begin{bmatrix} 0.2856 & 0.2856 & 0.2856 \\ 0.4288 & 0.4288 & 0.4288 \\ 0.2856 & 0.2856 & 0.2856 \end{bmatrix}$$

ex.) Stochastic but not regular



4-stations (trains)

$$\underline{M} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\underline{P}_0 = \begin{bmatrix} 1/2 \\ 1/6 \\ 1/4 \\ 1/12 \end{bmatrix} \qquad \underline{P}_1 = \underline{M}\,\underline{P}_0 = \begin{bmatrix} 1/12 \\ 1/2 \\ 1/6 \\ 1/4 \end{bmatrix}$$

$$\underline{P}_2 = \begin{bmatrix} 1/4 \\ 1/12 \\ 1/2 \\ 1/6 \end{bmatrix} \qquad \underline{P}_3 = \begin{bmatrix} 1/6 \\ 1/4 \\ 1/12 \\ 1/2 \end{bmatrix} \qquad \underline{P}_4 = \begin{bmatrix} 1/2 \\ 1/6 \\ 1/4 \\ 1/12 \end{bmatrix} = \underline{P}_0$$

no equilibrium $\underline{P}_\infty$