Vectors.

A **vector** is an **organized** collection of number,
↓
components

\# of components ⇒ **length**

$$\underline{a} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \qquad \text{length}(\underline{a}) = 3$$

$$\underline{b} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \qquad \text{length}(\underline{b}) = 3$$

$$\underline{a} \neq \underline{b}$$

Component of $\underline{a}$ ⇒ i-th component of $\underline{a}$

⇒ a_i

$a_1 = 1$

$a_3 = 5$

Vector

row $\leftrightarrow$

Column $\updownarrow$

$\underline{a}$: column

vectors. 3×1

$\underline{b}$: column

A vector has a dimension of

$$m \times n$$

$m \downarrow$ row

$n \downarrow$ column

$\underline{c} = [\,3 \quad 2 \quad 5\,]$ row

1×3

$\underline{a} = \underline{b}$ iff ① same dimension

② $a_i = b_i$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$ zero counts

All real vectors of length - n exist in $\mathbb{R}^n$ - space

$n \rightarrow$ length

$\mathbb{R} \rightarrow$ real

$$\begin{bmatrix} 1 \\ 2 \\ \pi \end{bmatrix} \in \mathbb{R}^3$$

Complex # $\Rightarrow \mathbb{C}^n$

Scalar $\mathbb{R}^1$ or $\mathbb{R}$

Vector Operation.

$\underline{u}, \underline{v} \in \mathbb{R}^3$

$u_i, v_i \in \mathbb{R}$

$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

① Addition

$\underline{u} + \underline{v} = \underline{w}$

$\underline{u}$ & $\underline{v}$ must have same dimension

$\Rightarrow u_i + v_i = w_i$

$$\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \underline{w}$$

② Subtraction

③ Scalar Multiple let $a \in \mathbb{R}$

$$\underline{w} = a\,\underline{u} = a\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} au_1 \\ au_2 \\ au_3 \end{bmatrix}$$

$$w_i = a\,u_i$$

④ Dot Product

(in a bit)

⑤ Cross-Product

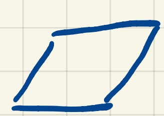
$$\underline{u} \times \underline{v} = \underline{w}$$

$$u_i v_j e_{ijk} = w_k$$

Permutation symbol.

$\underline{w}$ is a vector perpendicular to $\underline{u}$ & $\underline{v}$

the direction is decided by RH rule

the magnitude is the area of the parallelogram 

defined by $\underline{u}$ & $\underline{v}$

⑥ Division

NOT DEFINED

⑦ Associativity

$$\underline{u} + \underline{v} = \underline{v} + \underline{u}$$

$$a\,\underline{u} = \underline{u}\,a$$

⑧ Transpose "rotate" 90°

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\cancel{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T = [\,- \sim m\,]}$$

$$\underline{u}^T = [u_1 \;\; u_2 \;\; u_3]$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T = [1 \quad 2 \quad 3]$$

⑨ Special Vector

$$\underline{\phi} = \left.\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\} n \qquad \underline{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Vector Dot Product

Inner Product

Operation on 2 vectors of equal length that returns a scalar

$$\underline{u} \cdot \underline{v} = \langle \underline{u}, \underline{v} \rangle = \underline{u}^T \underline{v}$$

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\underline{u}^T \underline{v} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= u_1 v_1 + u_2 v_2$$

$$\underline{u} \cdot \underline{v} = \sum_{i=1}^{n} u_i v_i$$

summation

$$\begin{bmatrix} 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

$$= 1 \times 3 + 2 \times 2 + (-5) \times 4$$

$$= -13_{\#}$$

Operation

① $(\underline{u} + \underline{v}) \cdot (\underline{w} + \underline{x}) = \underline{u} \cdot \underline{w} + \underline{u} \cdot \underline{x} + \underline{v} \cdot \underline{w} + \underline{v} \cdot \underline{x}$

② $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$

③ $a \in \mathbb{R}$

$$a(\underline{u} \cdot \underline{v}) = (a\underline{u}) \cdot \underline{v} = \underline{u} \cdot (a\underline{v})$$

$$\cancel{a^2(\underline{u} \cdot \underline{v})}$$

Vector Norm

A vector norm tells us how "big" a vector is. How far from $\underline{\phi}$

Scalar : Absolute Value

2 - norm

$$\|\underline{u}\|_2 = (\underline{u}\cdot\underline{u})^{\frac{1}{2}}$$

$$= \left(\sum_{i=1}^{n} u_i^2\right)^{\frac{1}{2}}$$

$$= (u_1^2 + u_2^2 + \cdots + u_n^2)^{\frac{1}{2}}$$

ex).

$$\underline{u} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix} \qquad \|\underline{u}\|_2 = \sqrt{15}$$

A vector w/ $\|u\|_2 = 1 \Rightarrow$ unit vector

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\cancel{\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ unit vector ?}}$$

$$\underline{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\|\underline{u}\|_2 = \sqrt{2}$$

$$\underline{\hat{u}} = \frac{\underline{u}}{\|\underline{u}\|_2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

P- norm

$$\|\underline{u}\|_p = \left(\sum_{i=1}^{n} |u_i|^p\right)^{\frac{1}{p}}$$

Most common, $p = 2 \Rightarrow$ 2 norm

1 norm $\quad \|\underline{u}\|_1 = \sum_{i=1}^{n} |u_i|$

$$\lim_{p\to\infty} \|\underline{u}\|_p = \|\underline{u}\|_\infty = \max_i |u_i|$$

$$\underline{w} = \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}, \quad \|\underline{w}\|_\infty = |-5| = 5$$

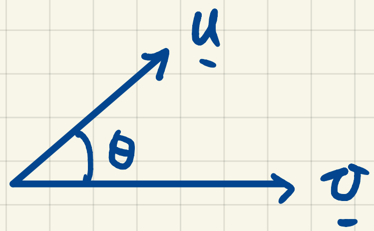
① $\|\underline{u}\|_p \geq \phi$

② $\|\underline{u}\|_p = \phi$ iff $\underline{u} = \underline{\phi}$

③ $\|\underline{u} + \underline{v}\|_p \leq \|u\|_p + \|\underline{v}\|_p$: triangle inequality

④ $\|a\,\underline{u}\|_p = |a|\,\|\underline{u}\|_p$

$$\underline{u} \cdot \underline{v} = \|\underline{u}\| \, \|\underline{v}\| \cos\theta$$



what if $\underline{u} \cdot \underline{v} = 0$

let $\|\underline{u}\| \neq 0$ $\|\underline{v}\| \neq 0$

$\Rightarrow \cos\theta = 0$

$\Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$\underline{u} \cdot \underline{v} = 0$ if $\underline{u} \perp \underline{v}$

$\Rightarrow$ $\underline{u}$ and $\underline{v}$ are orthogonal

Cauchy - Schwarz Inequality.

Upper Bound on $|\underline{u} \cdot \underline{v}|$

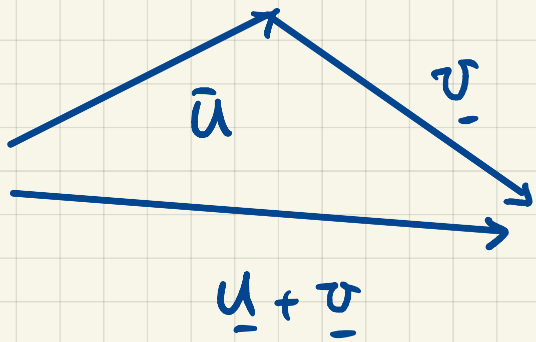
$$|\underline{u} \cdot \underline{v}| = \left| \|\underline{u}\| \, \|\underline{v}\| \cos\theta \right|$$

$$= \|\underline{u}\| \, \|\underline{v}\| \, |\cos\theta|$$

$$|\cos\theta| \leq 1$$

$$|\underline{u} \cdot \underline{v}| \leq \|\underline{u}\| \, \|\underline{v}\|$$

Triangle Inequality.



$$\|\underline{u} + \underline{v}\|_p \leq \|\underline{u}\|_p + \|\underline{v}\|_p$$

holds for all p-norm

Linear Combination of Vector

A linear combination is a weighted sum.

Let $\underline{u}, \underline{v}, \underline{w} \in \mathbb{R}^{n \times 1}$

$a, b, c \in \mathbb{R}$

$$\boxed{\underline{z} = a\underline{u} + b\underline{v} + c\underline{w}}$$

Matrix