Linear Transformation

A linear transformation

$$f : \underline{X} \to \underline{Y}$$

① $f(x_1 + x_2) = f(x_1) + f(x_2)$ $\quad x_1, x_2 \in \underline{X}$

② $f(ax_1) = a f(x_1)$ $\quad a \in R$

thm: Let V & W be two vector space such that $L : V \to W$ is a linear transformation. Let $\underline{0}_v$ be the zero vector in V & $\underline{0}_w$ the zero vector in W.

① $L(\underline{0}_v) = \underline{0}_w$

② $L(-\underline{v}) = -L(\underline{v})$ for $\underline{v} \in V$

③ $L(a_1 \underline{v}_1 + a_2 \underline{v}_2 + \cdots + a_n \underline{v}_n)$

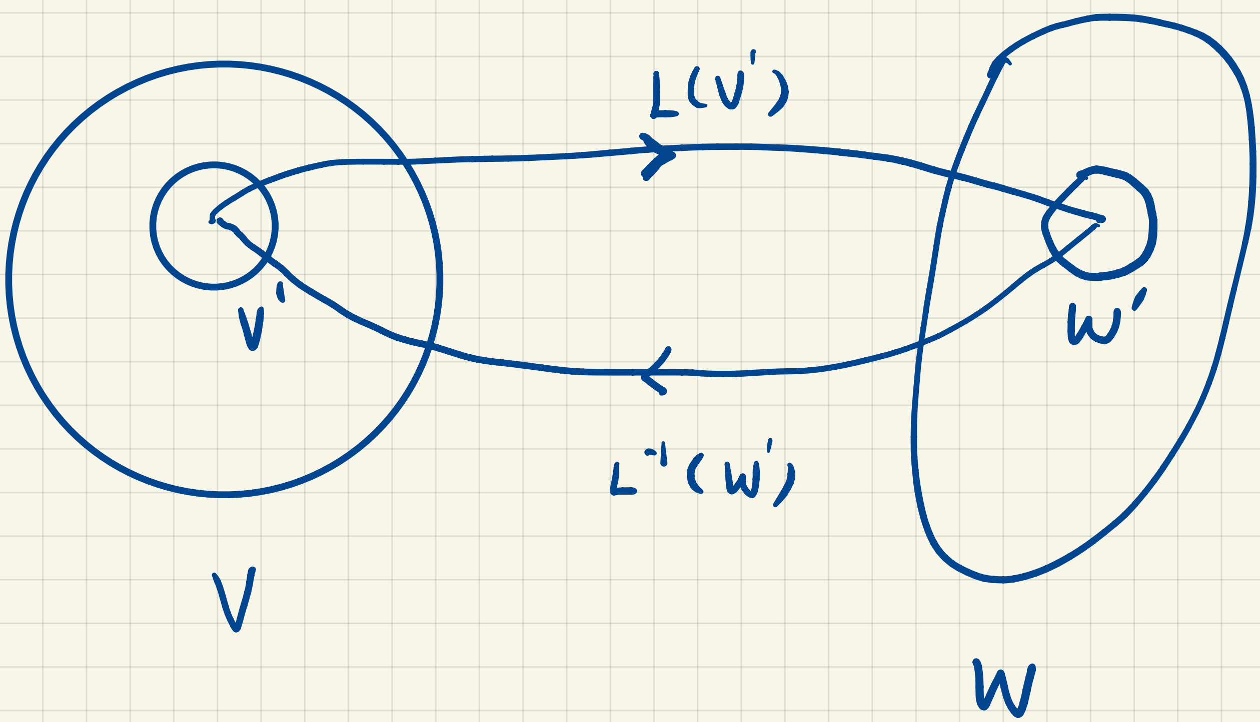
$= a_1 L(\underline{v}_1) + a_2 L(\underline{v}_2) + \cdots + a_n L(\underline{v}_n)$

for all $\underline{v}_i \in V$ and $a_i \in R$

Thm: Let $L: V \to W$ be a linear transformation

① If $V' \in V$ is a subspace of V, then $L(V') = W' \in W$ is a subspace of W.

② If W' is a subspace of W and L^{-1} exists, then $L^{-1}(W') = V' \in V$ is a subspace of V



The Action of a Linear Transformation is completely defined by the action of said transformation on the basis of the domain.

V = Domain w/ basis set $\{\underline{b}_1, \underline{b}_2, \cdots, \underline{b}_n\}$

Any $\underline{v} \in V$ has $\underline{v} = k_1 \underline{b}_1 + k_2 \underline{b}_2 + \cdots + k_n \underline{b}_n$

k_i : coordinate of $\underline{v}$ in V.

Look at $L(\underline{v}) = \underline{w} \in W$

$L: V \to W$

$$L(\underline{v}) = L(k_1 \underline{b}_1 + k_2 \underline{b}_2 + \cdots + k_n \underline{b}_n)$$

$$= k_1 L(\underline{b}_1) + k_2 L(\underline{b}_2) + \cdots + k_n L(\underline{b}_n)$$

If I know $L(\underline{b_1})$, $L(\underline{b_2})$ etc., I can determine $L(\underline{v})$

Thm:

Let $\underline{B} = \{\underline{b_1}, \underline{b_2}, \ldots, \underline{b_n}\}$ form a basis for vector space V.

Let $\underline{w_1}, \underline{w_2}, \ldots, \underline{w_n}$ be some n-vectors in vector space W.

There will always be a unique linear transformation $L: V \to W$ such that

$$L(\underline{b_1}) = \underline{w_1}$$
$$L(\underline{b_2}) = \underline{w_2}$$
$$\vdots$$
$$L(\underline{b_n}) = \underline{w_n}$$

Now, tie to matrix-vector product.

$$\underline{A}\,\underline{x} = [\underline{a}_1 \quad \underline{a}_2 \quad \cdots \quad \underline{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

$$= x_1 \underline{a}_1 + x_2 \underline{a}_2 + \cdots + x_m \underline{a}_m$$

This takes $\underline{x}$ and transform it into $\underline{b}$

Is this linear ?

① $\underline{A}(\underline{x} + \underline{y}) = \underline{A}\,\underline{x} + \underline{A}\,\underline{y}$

② $\underline{A}(c\,\underline{x}) = c\,A(\underline{x})$

Let $B = \{ \underline{b}_1 \ \underline{b}_2 \ \underline{b}_3 \ \underline{b}_4 \}$ as in an example.

If $\quad \underline{v} = \begin{bmatrix} -4 \\ 14 \\ 1 \\ 5 \end{bmatrix}, \quad L(\underline{v}) = ?$

$$\underline{x} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 3 \end{bmatrix} \qquad [b_1 \quad b_2 \quad b_3 \quad b_4]\,\underline{x} = \underline{v}$$

$$\underline{\underline{A}} = [L(\underline{b_1}) \quad L(\underline{b_2}) \quad L(\underline{b_3}) \quad L(\underline{b_4})]$$

$$= \begin{bmatrix} 3 & 2 & -4 & 6 \\ 1 & -1 & 3 & 1 \\ 2 & 1 & 0 & -1 \end{bmatrix}$$

$$\underline{\underline{A}}\,\underline{x} = \begin{bmatrix} 3 & 2 & -4 & 6 \\ 1 & -1 & 3 & 1 \\ 2 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 18 \\ 9 \\ 0 \end{bmatrix}$$

$\underline{\text{Thm}}$: Let $\underline{B}$ be an ordered basis set for V and Let C be an ordered basis set for W. For any linear transformation $L: V \to W$. there exist a matrix such that

$$\underline{\underline{A}}_{BC}\,[\underline{v}]_{\underline{B}} = [L(\underline{v})]_C$$

$\underline{A}_{BC} = [L(\underline{b_1}) \quad L(\underline{b_2}) \quad \cdots \quad L(b_n)]$

is the linear operator

$[\underline{V}]_B$: coordinate of $\underline{V}$ using $\underline{B}$

$\underline{V} \in V$

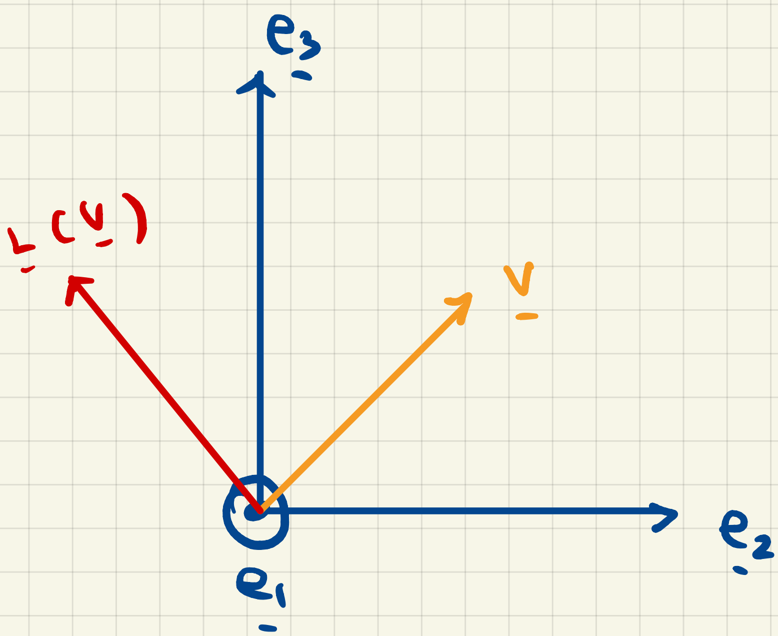
$[L(\underline{V})]_C$: coordinate of $L(\underline{V})$ usig $\underline{C}$

$L(\underline{V}) \in W$

Geometric Operator in $\mathbb{R}^3$

Let $\underline{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

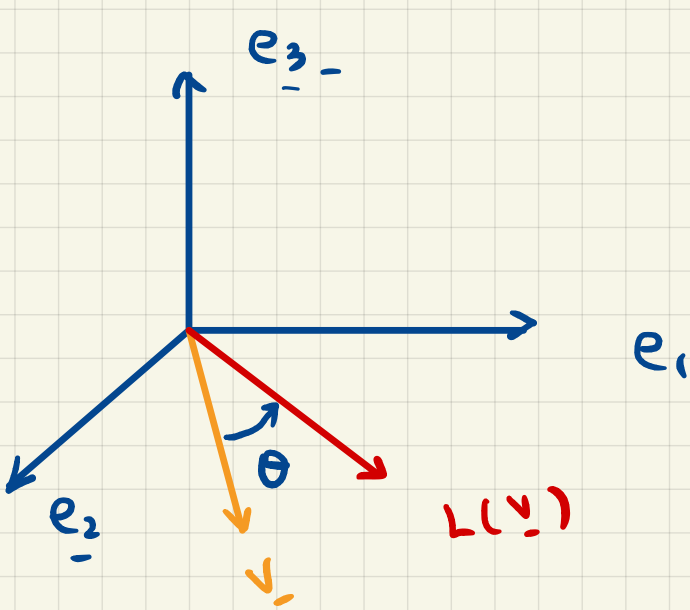
① Reflection about $\underline{e}_3$



$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_2 \\ -a_3 \end{bmatrix}$$

$$\underline{A} = \left[L(\underline{e}_1) \;\; L(\underline{e}_2) \;\; L(\underline{e}_3) \right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

② Rotation about z-axis.



$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 \cos\theta - a_2 \sin\theta \\ a_1 \sin\theta + a_2 \cos\theta \\ a_3 \end{bmatrix}$$

$$\underset{=}{A} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

③ Scaling

$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} c\ a_1 \\ c\ a_2 \\ c\ a_3 \end{bmatrix}$$

$$\underset{=}{A} = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$$

Now, apply rotation → scaling → reflection

$$L_{ref}\left(L_{scale} \left(L_{rot} \left(\underline{v} \right) \right) \right)$$

$$= A_{ref} \, A_{scale} \, A_{rot} \, \underline{v}$$

$$\Rightarrow A_{ref} \, A_{scale} \, A_{rot} = \begin{bmatrix} c\cos\theta & -c\sin\theta & 0 \\ c\sin\theta & c\cos\theta & 0 \\ 0 & 0 & -c \end{bmatrix}$$

Rank, Nullity, Kernel

Let $L : V \to W$ be a linear transformation

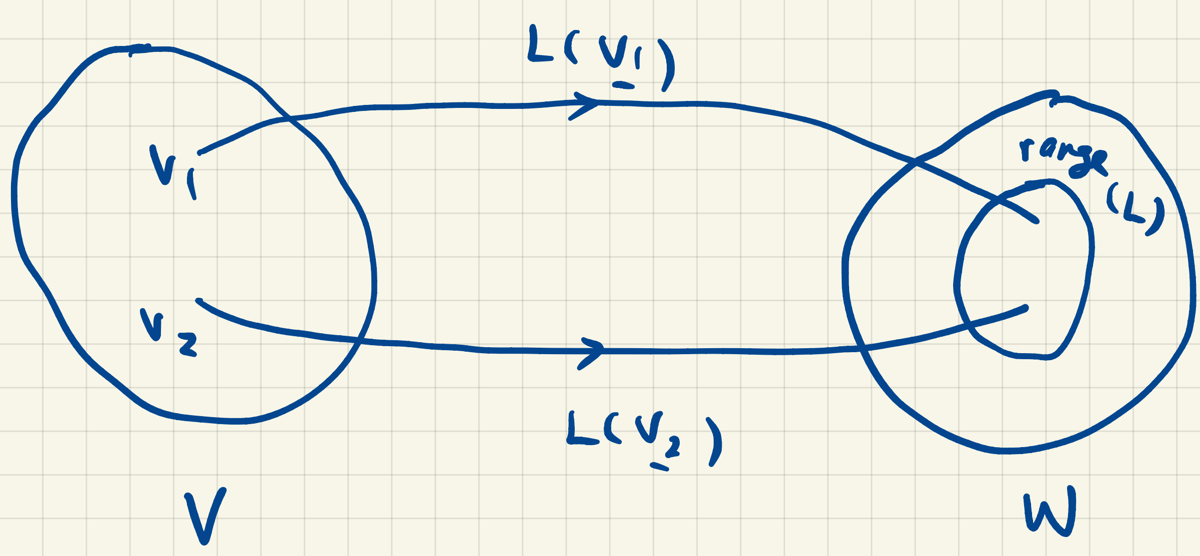
Kernel of L : ker(L) are all vectors in V that maps to $\underline{0}_w$ in W

Nullity of L : the dimension of ker(L)

the # of vectors in a basis for ker(L)

$$\text{Nullity}(L) = \text{Dim}(\ker(L))$$

Rank of L : the rank of L is the dimension of the range of L.



rank (L) = Dim (range (L)) = Dim (image (L))

Rank- Nullity Theorem

Let $L: V \to W$ be a linear transformation

$$\boxed{\text{rank}(L) + \text{nullity}(L) = \text{Dim}(V)}$$

Matrix Subspace

① Column Space : All linear combination of columns.

② Nullspace : All vector $\underline{v}$ such that $\underline{\underline{A}}\,\underline{v} = 0$

③ Row Space : All possible linear combination of Rows.

④ Left - Null Space : All vectors $\underline{v}$ such that $\underline{v}^T \underline{\underline{A}} = 0^T$

Column Space.

Recall that $\underline{A}\,\underline{x} = x_1 \underline{a}_1 + x_2 \underline{a}_2 + \dots + x_n \underline{a}_n$

$= \underline{b}$

$\Rightarrow$ $\underline{b}$ is linear comb. of the columns of $\underline{\underline{A}}$

$\Rightarrow$ $\underline{b}$ exists in the column space of $\underline{\underline{A}}$: $b \in C(\underline{\underline{A}})$

If $\underline{\underline{A}} \in \mathbb{M}_{mn}$, then each column has a length of m.

$\Rightarrow$ $C(\underline{\underline{A}})$ is a subspace of $\mathbb{R}^m$

Important Question : When does $\underline{x}$ exist for $\underline{\underline{A}}\,\underline{x} = \underline{b}$ when $\underline{b} \in \mathbb{R}^m$

For $\underline{x}$ to exist, $\underline{b} \in (\underline{\underline{A}})$

If $\underline{b} \in C(\underline{\underline{A}})$ then at least one # of solution to $\underline{\underline{A}}\,\underline{x} = \underline{b}$ exist.

Two Cases.

#1. Let $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$ w/ $\det(\underline{\underline{A}}) \neq 0 \Rightarrow \underline{\underline{A}}^{-1}$ exist

then $\underline{\underline{A}}\,\underline{x} = \underline{b}$ with $\underline{b} \in \mathbb{R}^n$

$$\Rightarrow \underline{x} = \underline{\underline{A}}^{-1}\,\underline{b}$$

Thus.

1. $\underline{b}$ is in $C(\underline{\underline{A}})$
2. $C(\underline{\underline{A}})$ spans all of $\mathbb{R}^n$
3. $\underline{x}$ are the coor. of $\underline{b}$ in $C(\underline{\underline{A}})$

#2 let $\underline{\underline{A}} \in \mathbb{R}^{m\times n}$ w/ $m \neq n$ $\det(\underline{\underline{A}}) = 0$

$\Rightarrow \underline{\underline{A}}^{-1}$ does not exist

For a sol. to $\underline{\underline{A}}\,\underline{x} = \underline{b}$ to exist

$\underline{b}$ must be in $C(\underline{\underline{A}})$

$\Rightarrow$ Any possible sol. to $\underline{\underline{A}}\,\underline{x} = \underline{b}$

depends on $\underline{\underline{A}}$ and $\underline{b}$

Recall that rref$(\underline{\underline{A}})$ gives the columns which form a basis.

$$\text{Dim}(C(\underline{\underline{A}})) = \#\text{ of pivots in rref}(\underline{\underline{A}})$$

$$= \text{rank}(\underline{\underline{A}})$$

If $\underline{\underline{A}} \in \mathbb{R}^{m\times n}$ w/ $\underline{\underline{A}} \neq 0$ then

$$1 \leq \dim(C(\underline{\underline{A}})) = \text{rank}(\underline{\underline{A}}) \leq \min(m,n)$$

Nullspace: $N(\underline{\underline{A}})$ = All vectors such that $\underline{\underline{A}}\,\underline{v} = \underline{0}$ $\quad \underline{v} \in \mathbb{R}^n$

If $\underline{\underline{A}} \in \mathbb{R}^{m \times n}$ then $N(\underline{\underline{A}})$ is a subspace of $\mathbb{R}^n$

verify $N(\underline{\underline{A}})$ is a subspace.

Let $\underline{v}_1, \underline{v}_2 \in N(\underline{\underline{A}})$, $\underline{v}_1 \neq \underline{v}_2$, $c \in \mathbb{R}^1$

① $\underline{\underline{A}}(\underline{v}_1 + \underline{v}_2) = \underline{\underline{A}}\,\underline{v}_1 + \underline{\underline{A}}\,\underline{v}_2 = \underline{0} + \underline{0} = \underline{0}$

② $\underline{\underline{A}}(c\,\underline{v}_1) = c\,\underline{\underline{A}}\,\underline{v}_1 = \underline{0}$

The zero vector always gives $\underline{\underline{A}}\,\underline{0} = \underline{0}$ but it's the trivial sol. and does not count.

When is $\mathrm{Dim}\,(N(\underline{\underline{A}})) > 0$?

Sol. $\underline{\underline{A}}\,\underline{v} = \underline{0}$ w/ $\underline{\underline{A}}^{-1}$ existing.

$$\Rightarrow \underline{v} = \underline{\underline{A}}^{-1}\,\underline{0} = \underline{0}$$

empty set.

$\Rightarrow$ If $\underline{\underline{A}}^{-1}$ exists, the $N(\underline{\underline{A}}) = \{\ \}$

$\Rightarrow$ Need $\underline{\underline{A}}^{-1}$ to NOT exist

Non-trivial $N(\underline{\underline{A}})$ if $\underline{\underline{A}} \in R^{m \times n}$ w/ $m \neq n$

and $\mathrm{Det}\,(\underline{\underline{A}}) = 0$

To get $N(\underset{=}{A})$ "solve" $\underset{=}{A}\,\underline{v} = \underline{0}$

$$\operatorname{rref}([\underset{=}{A} \mid \underline{0}]) = \operatorname{rref}(\underset{=}{A})$$