Definition (2.2)

The number p is a **fixed point** for a given function g if $g(p) = p$.

NOTES:

- Given a root-finding problem $f(p) = 0$, we can define functions g with a fixed point at p in a number of ways, for example, as

$$g(x) = x - f(x) \quad \text{or as} \quad g(x) = x + 3f(x).$$

- Conversely, if the function g has a fixed point at p, then the function defined by

$$f(x) = x - g(x)$$

has a zero at p.

Theorem (2.3)

(i) *If $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then g has at least one fixed point in $[a, b]$.*

(ii) *If, in addition, $g'(x)$ exists on (a, b) and a positive constant $k < 1$ exists with $|g'(x)| \leq k$, for all $x \in (a, b)$, then there is exactly one fixed point in $[a, b]$. (See Figure 2.3.)* □

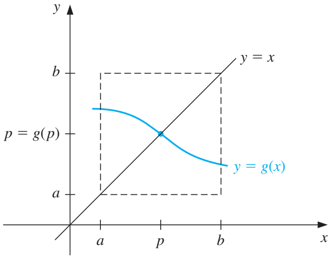

Figure: Figure 2.3

Algorithm 2.2: FIXED-POINT ITERATION

To find a solution to $p = g(p)$ given an initial approximation p_0:

INPUT initial approximation p_0; tolerance TOL; max # of iterations N_0.
OUTPUT approximate solution p or message of failure.
Step 1 Set $i = 1$.
Step 2 While $i \leq N_0$ do Steps 3–6.
  Step 3 Set $p = g(p_0)$. (*Compute* p_i.)
  Step 4 If $|p - p_0| < TOL$ then
    OUTPUT (p); (*The procedure was successful.*)
    STOP.
  Step 5 Set $i = i + 1$.
  Step 6 Set $p_0 = p$. (*Update* p_0.)
Step 7 OUTPUT ('The method failed after N_0 iterations, $N_0 =$', N_0);
  (*The procedure was unsuccessful.*)
  STOP.

Fixed-Point Illustration

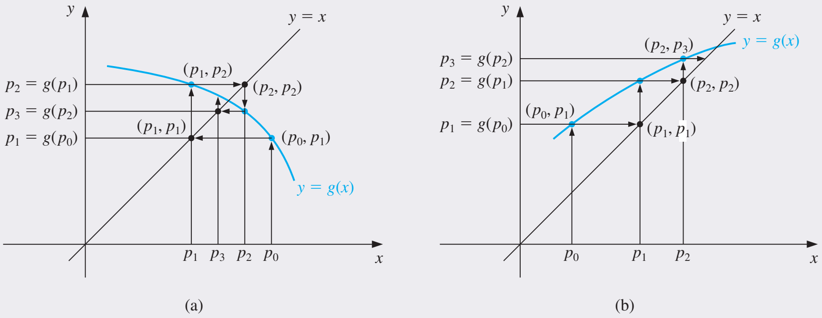

Figure: Figure 2.6

Chapter 2.2: Solutions: Fixed-Point

This YouTube video developed by Oscar Veliz can serve as a good illustration of the Fixed-Point Method for students.

Fixed-Point Video

Theorem (2.4: Fixed-Point Theorem)

Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all x in $[a, b]$. Suppose, in addition, that g' exists on (a, b) and that a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k, \quad \text{for all } x \in (a, b).$$

Then for any number p_0 in $[a, b]$, the sequence defined by

$$p_n = g(p_{n-1}), \quad n \geq 1,$$

converges to the unique fixed point p in $[a, b]$.

Corollary (2.5)

If g satisfies the hypotheses of Theorem 2.4, then bounds for the error involved in using p_n to approximate p are given by

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\} \tag{1}$$

and

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|, \quad \text{for all} \quad n \geq 1. \tag{2}$$

□

2.2 Fixed-point iteration

#3. Let $f(x) = x^3 - 2x + 1$. To solve $f(x) = 0$, the following four fixed-point problems are proposed. Derive each fixed-point method and compute p_1, p_2, p_3, p_4. Which method seem to be appropriate?

(a) $x = \frac{1}{2}(x^3 + 1)$, $p_0 = \frac{1}{2}$.

Start with $x^3 - 2x + 1 = 0$. Move $2x$ to the left and then divide both sides by 2. Denote $g(x) = \frac{1}{2}(x^3 + 1)$.

$p_1 = 0.5625$, $p_2 = .58898926$, $p_3 = 0.60216264$, $p_4 = 0.60917204$.

(b) $x = \frac{2}{x} - \frac{1}{x^2}$, $p_0 = \frac{1}{2}$.

Start with $x^3 - 2x + 1 = 0$.

$x^3 = 2x - 1$. Divide both sides by x^2.

$x = \frac{2}{x} - \frac{1}{x^2}$.

$p_1 = 0$, p_2 $undefined$.

(c) $x = \sqrt[2]{2 - \frac{1}{x}}$, $p_0 = \frac{1}{2}$.

Solve for x^3. $x^3 = 2x - 1$. Divide both sides by x, then take positive square root. $p_1 = 0$, p_2 $undefined$.

(d) $x = -\sqrt[3]{1 - 2x}$, $p_0 = \frac{1}{2}$.

$x^3 = 2x - 1$. Take cubic root of both sides.

$p_1 = 0$, $p_2 = -1$, $p_3 = -1.4422496$, $p_4 = -1.57197274$.

(a), (d) seem promising but require additional calculations.

#13. Determine an interval $[a, b]$ on which fixed-point iteration will converge.

Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-5} and perform the calculations.

$(d)\ \ x = 5^{-x}$.

$g(x) = 5^{-x}$.

Use Th. 2.4. Solve $|g'(x)| = 5^{-x} \ln 5 < 1$.

$$\left(\frac{1}{5}\right)^x < \frac{1}{\ln 5}$$

$$-x \ln 5 < -\ln \ln 5$$

$$x > \frac{\ln \ln 5}{\ln 5} \approx 0.295684$$

Let $[a, b] = [0.3,0.7]$ $(check\ that\ p \in [0.3,0.7])$, $p_0 = 0.3$. We will use Corollary 2.5 to estimate the number of iterations needed to obtain the specified accuracy.

$$k = 5^{-0.3} * \ln 5 \approx 0.993078$$

By inequality (2.5),

$|p_n - p| \le k^n * 0.4.$

Thus, any solution to $k^n * 0.4 < 10^{-5}$ will satisfy the specified accuracy.

This leads to $n \ln k + \ln 0.4 < -5 \ln 10$ meaning that

$n > -\frac{\ln 0.4 + 5 \ln 10}{\ln k}$, which results in $n \ge 1526$. However, in fact, the desired accuracy is obtained with n as small as 39. Then $p_{39} = 0.469625$

satisfies the accuracy. Note that $p = 0.469622$.

$$f(x) = 0.5(sinx + cosx)$$

$g'(x) = 0.5(cosx - sinx)$ so $|g'(x)| \leq 1$ for any x. One can show that there exists a fixed point $p \in [0,1]$. Thus, we will use $[a,b] = [0,1]\ and\ p_0 = 0$. Then $k = \max_{x\in[0,1]} |g'(x)| = 0.5$. Then Corollary 2.5 guarantees that solution to $k^n * 0.4 < 10^{-5}$ will satisfy the accuracy requirement. This inequality means that $n \geq 17$.

However, in fact, $n = 6$ will suffice. Then $p_6 = 0.704812$ which is the same as $p = 0.704812$ (first 6 decimals).

#14. Use the given interval or determine an interval $[a,b]$ on which fixed-point iteration will converge. Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-5} and perform the calculations.

(a) $$2 + sinx - x = 0, \quad [2,3]$$

$$x = 2 + sinx$$

$$g(x) = 2 + sinx$$

$$g'(x) = cosx$$

$$k = \max_{x\in[2,3]} |g'(x)| = \max_{x\in[2,3]} |cosx| = cos3 = 0.9899924966$$

Corollary 2.5 (inequality (2.5)) leads to $n > \frac{-5\ln 10}{lnk} \approx 1144.6632$.

So, the smallest such value is $= 1145$. However, it can be shown that $n = 63$ is sufficient; $p_{63} = 2.5541998$.

(c) $3x^2 - e^x = 0.$ Solve for $x^2, then\ take\ square\ root$:

$x = \sqrt{\frac{e^x}{3}} = \frac{e^{\frac{x}{2}}}{\sqrt{3}}$ so $g(x) = \frac{e^{\frac{x}{2}}}{\sqrt{3}}$. One can show that there exists a fixed point p on $[a,b] = [0,1]$.

Compute $g'(x) = \frac{e^{\frac{x}{2}}}{2\sqrt{3}}$. This is a monotone increasing function on $[0,1]$. Then

$$k = \max_{x\in[0,1]} |g'(x)| = \frac{\sqrt{e}}{2\sqrt{3}} = 0.4759448347$$

Inequality (2.5) from Corollary 2.5 gives

$$n > \frac{\ln(10^{-5})}{\ln k} = 15.50659829$$

However, $n = 12\ results\ in\ p_{12} = 0.91001496$ that meets tolerance.

#20. Let A be a positive constant and $g(x) = 2x - Ax^2$.

(a). Show that if fixed-point iteration converges to a nonzero limit, then the limit is $p = \frac{1}{A}$.

$p = \lim_{n\to\infty} p_n = \lim_{n\to\infty} 2p_{n-1} - Ap_{n-1}^2 = 2p - Ap^2$. Thus,

$p = Ap^2; p = \frac{1}{A}$.

(b). Find an interval about $\frac{1}{A}$ for which fixed-point iteration converges, provided p_0 is in that interval.

Compute $g'(x) = 2 - 2Ax$.

Solve $|2 - 2Ax| < 1$; $\frac{1}{2A} < x < \frac{3}{2A}$. Take, for example, $\left(\frac{1}{2A} + \varepsilon, \frac{3}{2A} - \varepsilon\right)$.