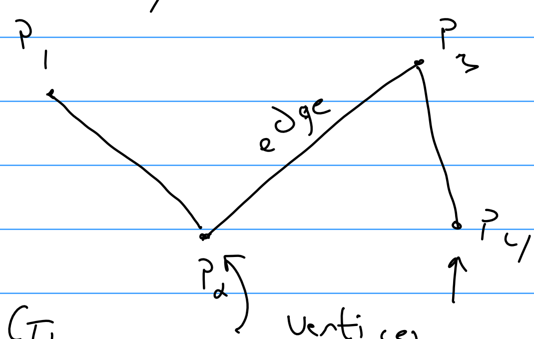
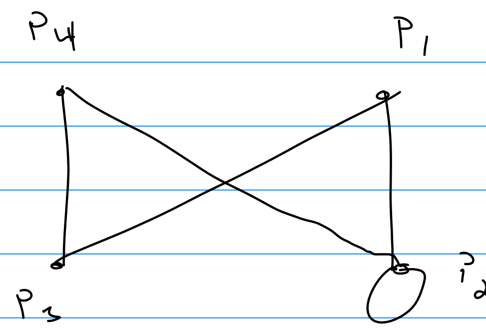
Graphs / Intro to Graph Theory

A graph is a collection of vertices connected by edges.



G_1

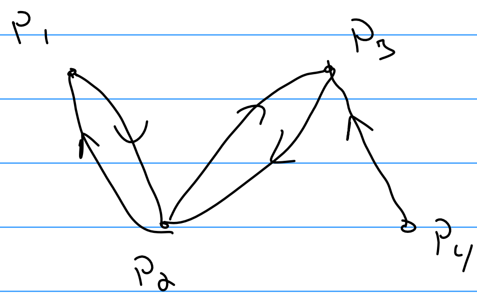


G_2

G_1: It is possible to go from $P_1 \to P_2$ directly, but $P_1 \to P_3$ requires $P_1 \to P_2 \to P_3$

G_1 & G_2 are called undirected graphs
G_2: $P_3 \Rightarrow P_4$ $P_4 \Rightarrow P_3$

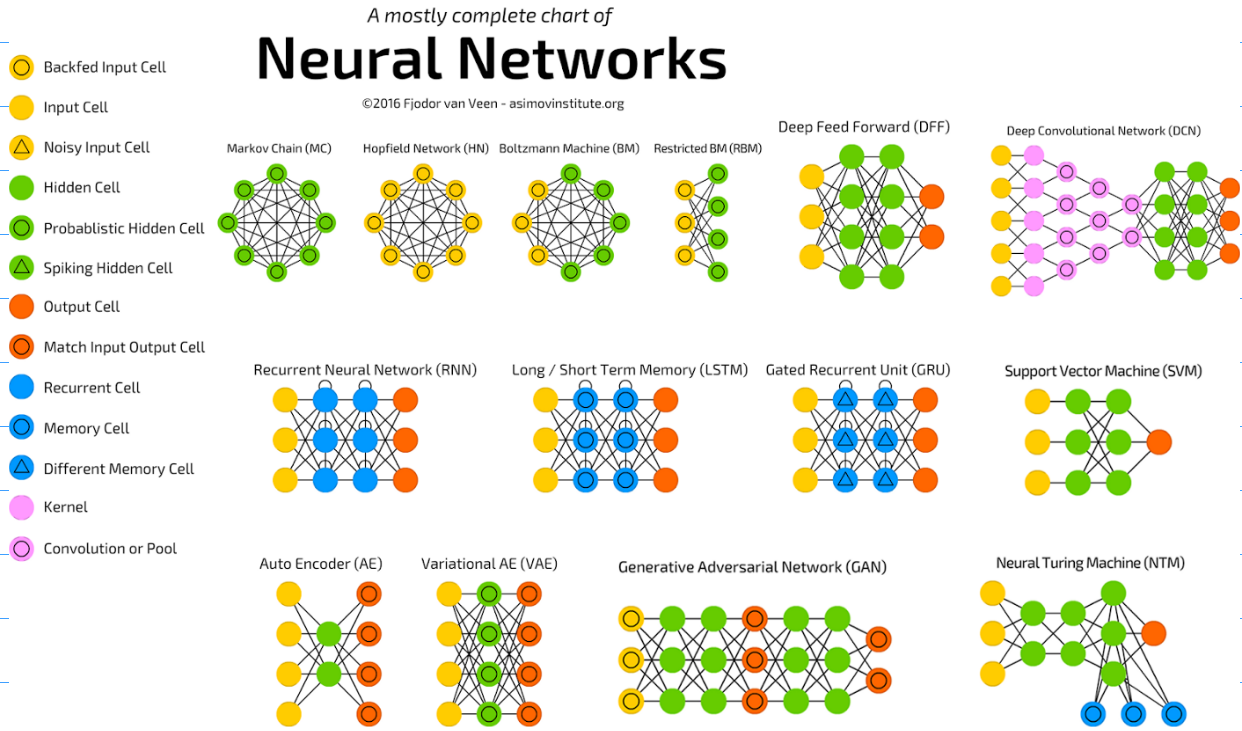
A directed graph (digraph) has directions



D_1

$P_2 \to P_3$ but not $P_3 \Rightarrow P_4$

Applications
- Linguistics
- Chemistry
- networking
- Scientific Computing
- machine learning.

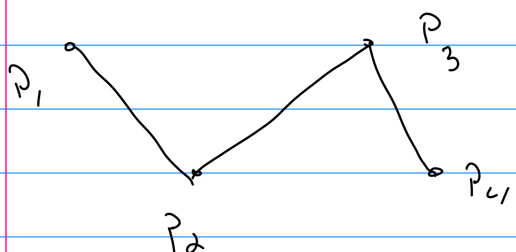


Adjacency Matrix

An adjacency matrix encodes a (di)graph

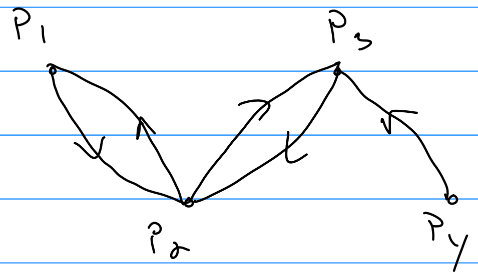
Let there be n-total vertices. The matrix is a n×n square matrix full of either $\emptyset$ or 1,

If $P_i \to P_j$ is possible put 1 in location (i,j) Otherwise zero



G_1

$$G_1 = \begin{array}{cc} & \begin{array}{cccc} P_1 & P_2 & P_3 & P_4 \end{array} \\ & \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{c} P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \end{array}$$



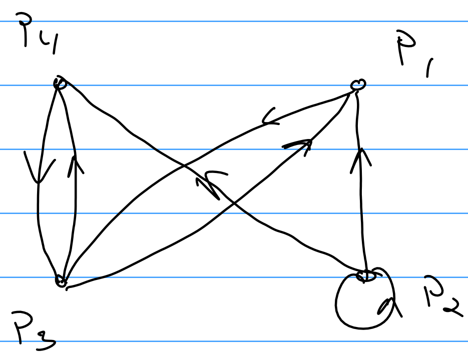
$$D_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Undirected Graphs are Symmetric
Directed Graphs are not symmetric

Path: A finite sequence of edges starting at P_i and ending at P_j.

Length: the # of edges in that path.

ex.)



Path for $P_1 \to P_4$
$P_1 \to P_3 \to P_4$
Length of 2

Paths might not be unique: P_2 to P_4

$P_2 \to P_4$ $\quad$ $P_2 \to P_2 \to P_4$ $\quad$ $P_2 \to P_1 \to P_3 \to P_4$

$\ell = 1$ $\quad$ $\ell = 2$ $\quad$ $\ell = 3$

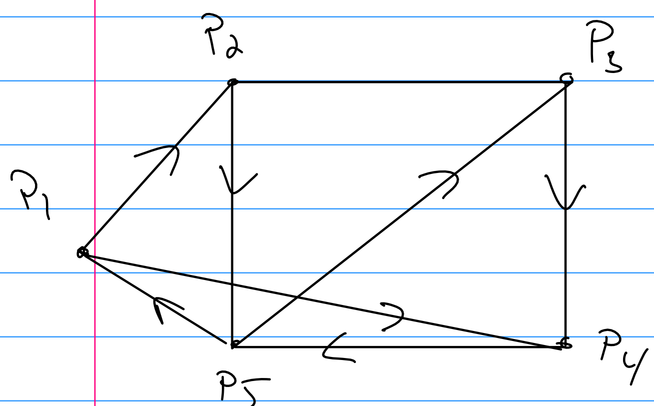
Thm: Let $\underline{A}$ be the adjacency matrix of a graph w/ n-vertices.

The # of paths between P_i & P_j of length k equals the value in (i,j) of $\underline{A}^k$

Collorary: the # of paths of length $\leq k$ from P_i to P_j is the value in (i,j) of

$$\sum_{p=1}^{k} \underline{A}^{p} = \underline{A}^{1} + \underline{A}^{2} + \dots + \underline{A}^{k}$$

Can be used to ID minimum path length.



$P_2 \to P_4$

$$\underline{D}_2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$P_4 \to P_2$

$$\underline{D}_2^{2} = \begin{bmatrix} 0 & 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix}$$

minimum path length for $P_2 \to P_4$ is 2

$P_2 \to P_3 \to P_4$

$$\underline{D}_2^{3} = \begin{bmatrix} 2 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 3 \end{bmatrix}$$

2 paths for $P_2 \to P_4$

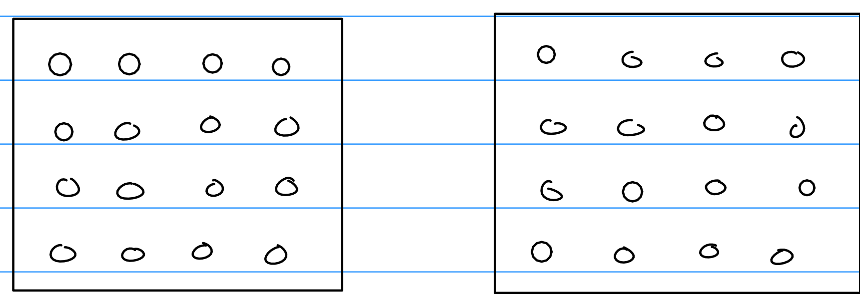
1 path for $P_4 \to P_2$ (min length)

Markov Chains

Used in stochastic modeling to describe a sequence of probable events w/ probabilities

ex.) Look at a 2-state system: A & B

State: A particular configuration



A B

$n_A = 16 \quad n_B = 16$

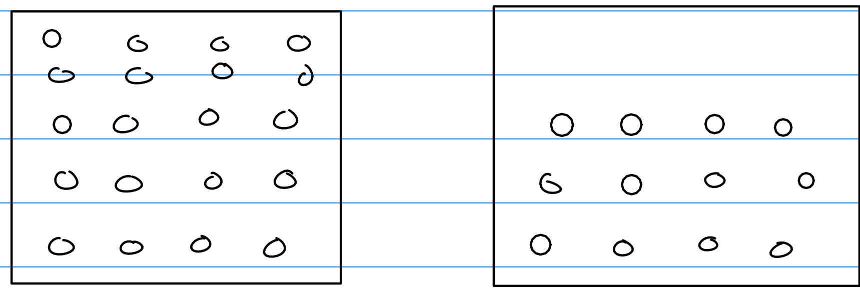
$n_t = n_A + n_B = 32$

Introduce the probability vector $\underline{P}$ which show the probability of a random element (the circles) is in state A or B

$$\underline{P}_0 = \begin{bmatrix} n_A / n_t \\ n_B / n_t \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

During this iteration: 75% of A remains in A
25% of A go to B
50% of B go to A
50% of B remains in B

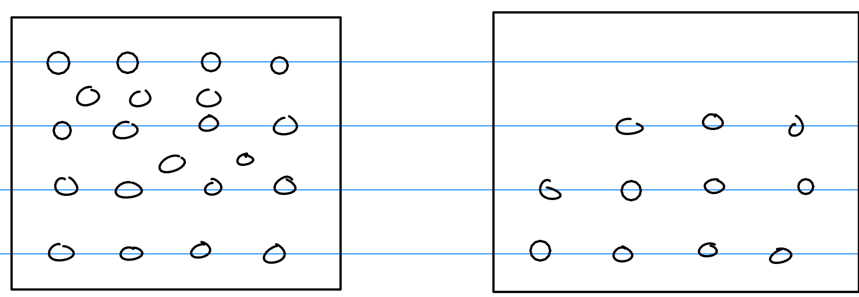
One iteration



A B

$$P_1 = \begin{bmatrix} 20/32 \\ 12/32 \end{bmatrix} = \begin{bmatrix} 0{,}625 \\ 0{,}375 \end{bmatrix}$$

Second Iteration



A B

$$P_2 = \begin{bmatrix} 21/32 \\ 11/32 \end{bmatrix} \sim \begin{bmatrix} 0{,}6563 \\ 0{,}3437 \end{bmatrix}$$

Introduce the "switching" vector:

$$\underline{v}_A = \begin{bmatrix} A \to A \\ A \to B \end{bmatrix} = \begin{bmatrix} 0{,}75 \\ 0{,}25 \end{bmatrix}$$

$$\underline{v}_B = \begin{bmatrix} B \to A \\ B \to B \end{bmatrix} = \begin{bmatrix} 0{,}5 \\ 0{,}5 \end{bmatrix}$$

After 1 iteration the # in A is $\underline{v}_A n_A = \begin{bmatrix} 0{,}75 \\ 0{,}25 \end{bmatrix} 16 = \begin{bmatrix} 12 \\ 4 \end{bmatrix}$

in B: $\underline{v}_B n_B = \begin{bmatrix} 0{,}5 \\ 0{,}5 \end{bmatrix} 16 = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$

Total is $\underbrace{\underline{v}_A n_A + \underline{v}_B n_B} = \begin{bmatrix} 20 \\ 12 \end{bmatrix}$

$$\begin{bmatrix} \underline{v}_A & \underline{v}_B \end{bmatrix} \begin{bmatrix} n_A^0 \\ n_B^0 \end{bmatrix} = \begin{bmatrix} n_A^1 \\ n_B^1 \end{bmatrix}$$

Divide by n_t

$$\begin{bmatrix} \underline{v}_A & \underline{v}_B \end{bmatrix} \underset{\underline{P}_0}{\begin{bmatrix} n_A^0 / n_t \\ n_B^0 / n_t \end{bmatrix}} = \underset{\underline{P}_1}{\begin{bmatrix} n_A^1 / n_t \\ n_B^1 / n_t \end{bmatrix}}$$

$$\underline{M} = \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix}$$ is the transition matrix

All columns of $\underline{M}$ must sum to 1 unless there is growth / death.

Given $\underline{P}_0$, $\underline{P}_1 = \underline{M}\,\underline{P}_0$

$$\underline{P}_2 = \underline{M}\,\underline{P}_1 = \underline{M}(\underline{M}\,\underline{P}_0) = \underline{M}^2\,\underline{P}_0$$

$$\Rightarrow \underline{P}_n = \underline{M}^n\,\underline{P}_0$$

In this example $\lim_{n \to \infty} \underline{M}^n \underline{P}_0 = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$

Note! This does not mean elements are fixed!
If 20 elements move from $A \to B$ then 20 move from $B \to A$

A **Markov Chain** is simply the set of distinct states S_1 to S_n where:

1) each element resides in a state
2) elements move between states w/ a fixed probability
3) no difference between elements

Stochastic & Regular Matrices

Stochastic Matrix: A transition matrix that

1) is square
2) All entries are non-negative (≥ 0)
3) All columns sum to one.

Products of stochastic matrices are also stochastic.

$\underline{M}^2$ is stochastic if $\underline{M}$ is.

Regular Matrix: A stochastic matrix such that for some $k \geq 1$ all entries of $\underline{M}^k$ are strictly positive (>0)

Thm: If the transition matrix is regular then

1) $\lim_{n \to \infty} \underline{M}^n = \underline{M}_\infty$

2) $\underline{M}_\infty$ has strictly positive values

3) All columns of $\underline{M}_\infty$ are the same,

4) P_∞ is a column of $\underline{M}_\infty$ no matter P_0

5) P_∞ is a fixed-point of $\underline{M}_\infty$: $P_\infty = \underline{M}_\infty P_\infty$

$$(f(a) = a)$$

ex) $\underline{M} = \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix}$

$$\underline{M}^3 = \begin{bmatrix} 0.6719 & 0.6562 \\ 0.3281 & 0.3438 \end{bmatrix}$$

$$\underline{M}^{10} = \begin{bmatrix} 0.6667 & 0.6667 \\ 0.3333 & 0.3333 \end{bmatrix}$$

$$\underline{M}^{50} = \begin{bmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{bmatrix} = \underline{M}^{100} = \underline{M}^{1000} = \cdots =$$

Now look at $\underline{M}^\infty P_0$

$$\underline{M}^\infty \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \qquad \underline{M}^\infty \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

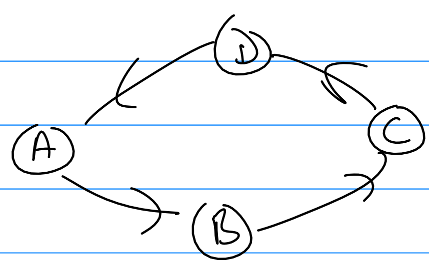
$$\underline{M}^{\infty} = [\underline{m}_1 \;\; \underline{m}_2] = [\underline{m}_1 \;\; \underline{m}_1]$$

$$\text{w/} \quad \underline{p}_0 = \begin{bmatrix} a \\ 1-a \end{bmatrix} \qquad 0 \le a \le 1$$

$$[\underline{m}_1 \;\; \underline{m}_1]\begin{bmatrix} a \\ 1-a \end{bmatrix} = a\,\underline{m}_1 + (1-a)\,\underline{m}_1$$

$$= a\,\underline{m}_1 + \underline{m}_1 - a\,\underline{m}_1 = \underline{m}_1$$

Ex of non-regular stochastic matrix



$$\underline{M} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\uparrow$ All in A go to B $\qquad$ $\uparrow$ All elements in D go to A

Initially $n_A = 6$, $n_B = 2$, $n_C = 3$, $n_D = 1$

$$\underline{p}_0 = \begin{bmatrix} 1/2 \\ 1/6 \\ 1/4 \\ 1/12 \end{bmatrix} \qquad \underline{p}_1 = \underline{M}\,\underline{p}_0 = \begin{bmatrix} 1/12 \\ 1/2 \\ 1/6 \\ 1/4 \end{bmatrix} \qquad \underline{p}_2 = \begin{bmatrix} 1/4 \\ 1/12 \\ 1/2 \\ 1/6 \end{bmatrix}$$

$$P_4 = \begin{bmatrix} 1/2 \\ 1/6 \\ 1/4 \\ 1/12 \end{bmatrix} = P_0$$

No equilibrium P vector.

3-bank example

Initially A has 40% share, B has 10%, C has 50%

$$\underline{M} = \begin{bmatrix} 0.5 & 0.1665 & 0.25 \\ 0.25 & 0.667 & 0.25 \\ 0.25 & 0.1665 & 0.5 \end{bmatrix} \qquad P_0 = \begin{bmatrix} 0.4 \\ 0.1 \\ 0.5 \end{bmatrix}$$

25% of A go to B

16.65% of B go to C

$$P_1 = \underline{M} P_0 = \begin{bmatrix} 0.3417 \\ 0.2917 \\ 0.3667 \end{bmatrix}$$

$$\underline{M}_\infty = \lim_{n\to\infty} \underline{M}^n = \begin{bmatrix} 0.2856 & 0.2856 & 0.2856 \\ 0.4288 & 0.4288 & 0.4288 \\ 0.2856 & 0.2856 & 0.2856 \end{bmatrix}$$