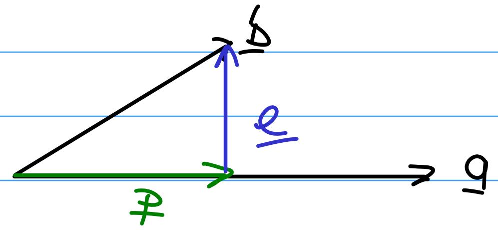
Projections

If $\underline{A}\,\underline{x} = \underline{b}$ w/ $\underline{b} \in C(\underline{A})$, then at least one $\underline{x}$ exists

What if $\underline{b} \notin C(\underline{A})$? What is the "best" possible solution?

Introduce projection

Look at $\underline{a}$ & $\underline{b}$:



Find the portion of $\underline{b}$ aligned w/ $\underline{a}$

Find $\underline{p}$ & $\underline{e}$.

let $\underline{p} = \hat{x}\,\underline{a}$ $\qquad \hat{x}$ = some scalar

then $\underline{e} = \underline{b} - \underline{p} = \underline{b} - \hat{x}\,\underline{a}$

as $\underline{e} \perp \underline{a} \Rightarrow \underline{e} \cdot \underline{a} = \underline{e}^T \underline{a} = 0$

$$\underline{e}^T \underline{a} = (\underline{b} - \hat{x}\,\underline{a})^T \underline{a} = \underline{b}^T \underline{a} - \hat{x}\,\underline{a}^T \underline{a} = 0$$

$$\hat{x} = \frac{\underline{b}^T \underline{a}}{\underline{a}^T \underline{a}} \Rightarrow \underline{p} = \hat{x}\,\underline{a} = \left(\frac{\underline{b}^T \underline{a}}{\underline{a}^T \underline{a}}\right) \underline{a} = (\underline{b}^T \underline{a}) \frac{\underline{a}}{\underline{a}^T \underline{a}}$$

$\underline{b}^T \underline{a}$ = portion of $\underline{b}$ in $\underline{a}$ direction

Issue: this depends on both $\underline{a}$ & $\underline{b}$.

To project $\underline{c}$ onto $\underline{a}$ must redo the calculation.

Find a projection matrix such that $\underline{p} = \underline{A}\,\underline{b}$ will work for any $\underline{b}$.

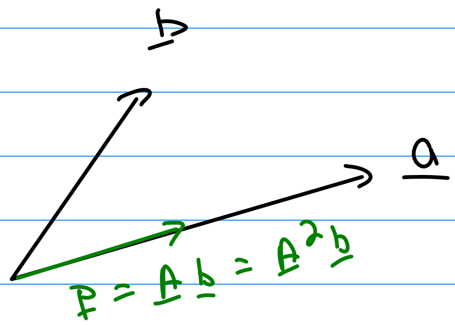
$$\underline{p} = \left(\frac{\underline{a}^T\underline{b}}{\underline{a}^T\underline{a}}\right)\underline{a} = \underline{a}\left(\frac{\underline{a}^T\underline{b}}{\underline{a}^T\underline{a}}\right) = \left(\frac{\underline{a}\,\underline{a}^T}{\underline{a}^T\underline{a}}\right)\underline{b}$$

Define $\underline{A} = \dfrac{\underline{a}\,\underline{a}^T}{\underline{a}^T\underline{a}}$ ← Projection matrix

The projection matrix is idempotent
→ repeated applications does not change anything

$$\underline{A} = \underline{A}^2 = \underline{A}^3 = \ldots$$

$$\underline{A}^2 = \underline{A}\,\underline{A} = \left(\frac{\underline{a}\,\underline{a}^T}{\underline{a}^T\underline{a}}\right)\left(\frac{\underline{a}\,\underline{a}^T}{\underline{a}^T\underline{a}}\right) = \frac{\underline{a}(\cancel{\underline{a}^T\underline{a}})\underline{a}^T}{(\cancel{\underline{a}^T\underline{a}})(\underline{a}^T\underline{a})} = \frac{\underline{a}\,\underline{a}^T}{\underline{a}^T\underline{a}} = \underline{A}$$



Now get $\underline{e}$:

$\underline{e} = \underline{b} - \underline{p} = \underline{b} - \underline{A}\,\underline{b} = (\underline{I} - \underline{A})\underline{b}$

$\underline{I} - \underline{A}$ will project any vector onto the $\perp$ of $\underline{a}$,

$$(\underline{I} - \underline{A})^2 = (\underline{I} - \underline{A})(\underline{I} - \underline{A}) = \underline{I} - \underline{A} - \underline{A} + \underline{A}^2$$
$$= \underline{I} - \underline{A} \cancel{- \underline{A} + \underline{A}}$$
$$= \underline{I} - \underline{A}$$

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ex.) let $\underline{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ Project $\underline{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ onto $\underline{a}$ & find $\underline{e}$

$\underline{a}^T\underline{a} = 1^2 + 2^2 = 5$

$\underline{a}\,\underline{a}^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$\underline{A} = \dfrac{\underline{a}\,\underline{a}^T}{\underline{a}^T\underline{a}} = \dfrac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}$

$\underline{p} = \underline{A}\,\underline{b} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 6/5 \end{bmatrix}$

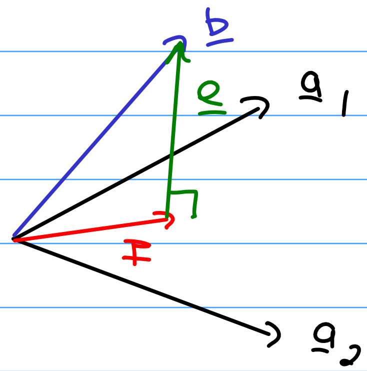
$\underline{e} = \underline{b} - \underline{p} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3/5 \\ 6/5 \end{bmatrix} = \begin{bmatrix} 2/5 \\ -1/5 \end{bmatrix}$

check: $\underline{e} \cdot \underline{a} = (2/5)(1) + (-1/5)(2) = 0$

Projection onto Subspaces

Given a vector $\underline{v} \in V$, find the portion of $\underline{v}$ in another subspace.

ex$_1$) let $\underline{b} \in \mathbb{R}^3$, $S = \text{Span}(\underline{a}_1, \underline{a}_2)$



$\underline{e} \cdot \underline{a}_1 = 0$

$\underline{e} \cdot \underline{a}_2 = 0$

Generalize to any $\text{span}(\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\}) = S$

Determine a matrix $\underline{A}$ such that $\underline{p} = \underline{A}\,\hat{\underline{x}}$ is the projection of $\underline{b}$ onto S where $\hat{\underline{x}}$ are the "coordinates" of $\underline{p}$ in S.

Need all $\underline{a}_i$ to be independent.

Also, $\underline{e}^T \underline{a}_i = 0$ w/ $\underline{e} = \underline{b} - \underline{A}\,\hat{\underline{x}}$

$\underline{a}_i^T \underline{e} = 0$

$$\begin{aligned} \underline{a}_1^T \underline{e} &= 0 \\ \underline{a}_2^T \underline{e} &= 0 \\ &\vdots \\ \underline{a}_n^T \underline{e} &= 0 \end{aligned} \Rightarrow \begin{bmatrix} - \underline{a}_1^T - \\ - \underline{a}_2^T - \\ \vdots \\ - \underline{a}_n^T - \end{bmatrix} \underline{e} = \underline{0} \Rightarrow \underline{A}^T \underline{e} = \underline{0}$$

$\underline{a}_1 \to \underline{a}_n$ independent $\Rightarrow \underline{A}^T$ is full row rank.

$\Rightarrow \underline{A}$ is full column rank.

$$\underline{A}^T \underline{e} = \underline{A}^T(\underline{b} - \underline{A}\,\underline{\hat{x}}) = \underline{A}^T\underline{b} - \underline{A}^T\underline{A}\,\underline{\hat{x}} = \underline{0}$$

$$\boxed{\underline{A}^T\underline{A}\,\underline{\hat{x}} = \underline{A}^T\underline{b}} \leftarrow \text{the Normal Equations}$$

The normal equations find the best possible solution to $\underline{A}\,\underline{x} = \underline{b}$ in $C(\underline{A})$

Only true if $(\underline{A}^T\underline{A})^{-1}$ exists.

1) let $\underline{A} \in M_{mn}$, $\underline{A}^T\underline{A} \in M_{nn} \Rightarrow$ Square

2) Show $N(\underline{A}^T\underline{A})$ is empty.

let $\underline{x} \in N(\underline{A}^T\underline{A})$ (any vector in null space)

$\Rightarrow (\underline{A}^T\underline{A})\underline{x} = \underline{0}$

$$\underline{x}^T(\underline{A}^T\underline{A})\underline{x} = \underline{x}^T\underline{0} = 0$$

$$\underline{x}^T(\underline{A}^T\underline{A})\underline{x} = (\underline{x}^T\underline{A}^T)(\underline{A}\,\underline{x}) = (\underline{A}\,\underline{x})^T(\underline{A}\,\underline{x}) = 0$$

$$\Rightarrow \|\underline{A}\,\underline{x}\|_2^2 = 0$$

$\underline{A}$ is full column rank $\Rightarrow$ all columns are independent.

If $\underline{x} \neq \underline{0}$ then $\underline{A}\,\underline{x} \neq \underline{0}$

Thus, if $\underline{x} \neq \underline{0}$ then $\|\underline{A}\,\underline{x}\|_2^2 > 0$

$\Rightarrow$ For $\|\underline{A}\,\underline{x}\|_2^2 = 0$, $\underline{x} = \underline{0}$ only

$\Rightarrow$ Only $\underline{0}$ in $N(\underline{A}^T\underline{A}) \Rightarrow N(\underline{A}^T\underline{A})$ is empty

$\Rightarrow (\underline{A}^T\underline{A})^{-1}$ will always exist.

$(\underline{A}^T\underline{A})\hat{\underline{x}} = \underline{A}^T\underline{b} \quad \Rightarrow \hat{\underline{x}} = (\underline{A}^T\underline{A})^{-1}\underline{A}^T\underline{b}$

$$\underline{p} = \underline{A}\hat{\underline{x}} = \underbrace{\underline{A}(\underline{A}^T\underline{A})^{-1}\underline{A}^T}_{\text{projection matrix}}\underline{b}$$

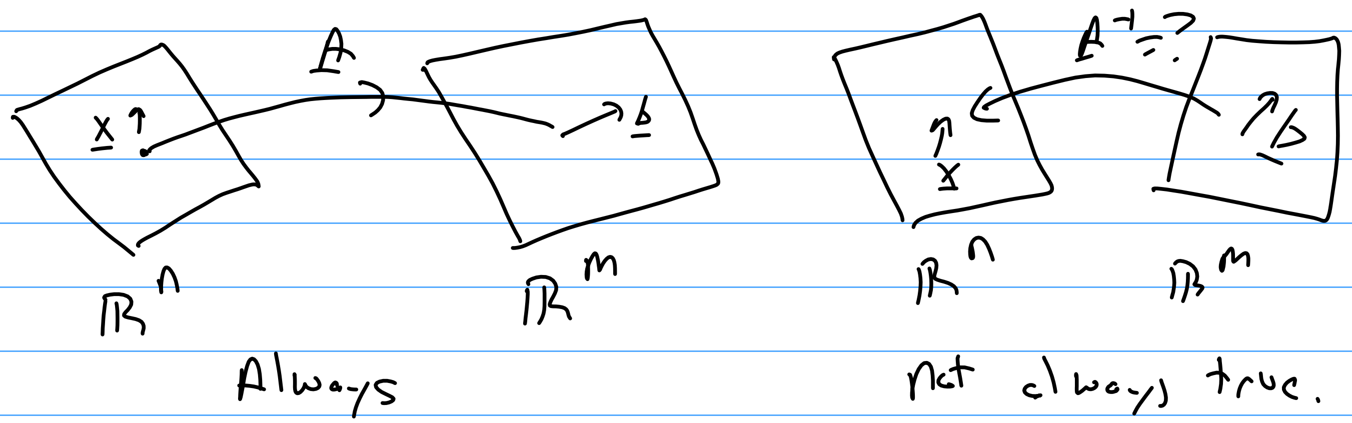
What is wrong here?

$$\underline{A}(\underline{A}^T\underline{A})^{-1}\underline{A}^T \underset{\uparrow}{=} \underline{A}\,\underline{A}^{-1}\underline{A}^{-T}\underline{A}^T = \underline{I}\,\underline{I} = \underline{I}$$

$\underline{A}^{-1}$ must exist for this to hold.

$\Rightarrow$ If $\underline{A}^{-1}$ exists then no projection needed.

Least-Squares Solution.

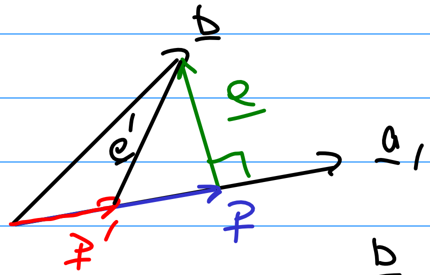


If no $\underline{x}$ exists such that $\underline{A}\underline{x} = \underline{b}$, find $\hat{\underline{x}} \in C(\underline{A})$ that minimizes the residual: $\underline{r} = \underline{b} - \underline{A}\hat{\underline{x}}$

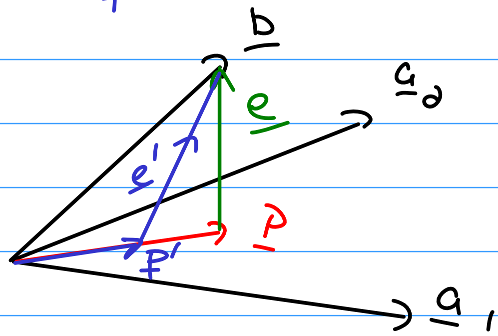
From before $\hat{\underline{x}} = (\underline{A}^T \underline{A})^{-1} \underline{A}^T \underline{b}$

$$\Rightarrow \boxed{(\underline{A}^T \underline{A}) \hat{\underline{x}} = \underline{A}^T \underline{b}}$$

The least-squares solution, $\hat{\underline{x}}$, minimizes $\|\underline{e}\|_2 = \|\underline{b} - \underline{A}\hat{\underline{x}}\|_2$ for all $\hat{\underline{x}} \in \mathbb{R}^n$



$\|\underline{e}\| < \|\underline{e}'\|$



ex.) $\underline{A} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$ $\quad \underline{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

$\underline{A}^T \underline{A} \hat{\underline{x}} = \underline{A}^T \underline{b} \Rightarrow \underline{A}^T \underline{A} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$

$$(\underline{A}^T \underline{A})^{-1} = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$\underline{A}^T \underline{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix} \Rightarrow \hat{\underline{x}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

residual: $\underline{e} = \underline{b} - \underline{A}\,\hat{\underline{x}} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

residual gives the portion of $\underline{b}$ not in $C(\underline{A})$

look at $\hat{\underline{x}}' = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}$, $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$

$$\underline{e}' = \underline{b} - \underline{A}\,\hat{\underline{x}}' = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -\varepsilon_1 - 2\varepsilon_2 \\ 1 \\ -\varepsilon_1 \end{bmatrix}$$

$$\|\underline{e}'\|_2 = 1 + \varepsilon_1^2 + (\varepsilon_1^2 + 2\varepsilon_2^2)^2 > \|\underline{e}\|_2 = 1$$

for any $\varepsilon_1 \neq 0$ or $\varepsilon_2 \neq 0$

Notes:

① If $\underline{A}^{-1}$ exists then

$$\hat{\underline{x}} = (\underline{A}^T\underline{A})^{-1}\underline{A}^T\underline{b} = \underline{A}^{-1}\underline{A}^{-T}\underline{A}^T\underline{b} = \underline{A}^{-1}\underline{b} = \underline{x}$$

② If $\underline{b} \in N(\underline{A}^T)$, then $\hat{\underline{x}} = \underline{0}$ as no portion of $\underline{b}$ is in $C(\underline{A})$

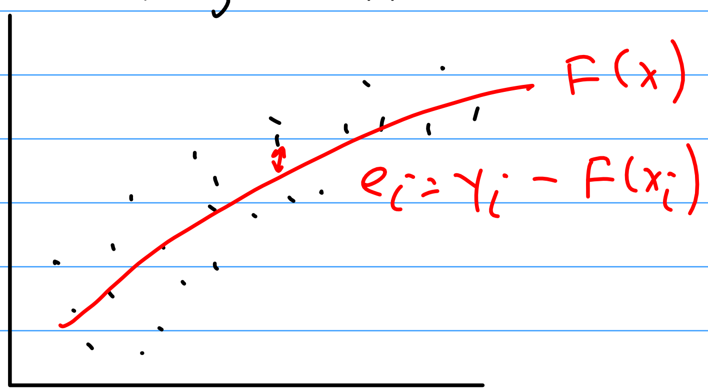
$N(\underline{A}^T)$ & $C(\underline{A})$ are orthogonal complements.

ex 1) $\underline{A} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$ $\qquad \underline{b} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$

$$\underline{A}^T \underline{b} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \underline{\hat{x}} = (\underline{A}^T \underline{A})^{-1} \underline{A}^T \underline{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Regression



minimize $\|\underline{e}\|_2^2 = e_1^2 + e_2^2 + \dots + e_m^2$

Start w/ $F(x) = a + bx$

Ideally:
$$\begin{aligned} a + bx_1 &= f_1 \\ a + bx_2 &= f_2 \\ &\vdots \\ a + bx_m &= f_m \end{aligned} \Rightarrow \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

$$\underline{A} \quad \underline{\alpha} = \underline{f}$$

Options:

1. One point: (x_1, f_1)

$$\begin{bmatrix} 1 & x_1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = f_1 \quad \Rightarrow a + bx_1 = f_1$$

$\Rightarrow$ no unique solution

2. 2 points:

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$\det(\underline{A}) = x_2 - x_1 \Rightarrow$ if $x_1 \neq x_2$, $\underline{A}^{-1}$ exists & unique solution.

3. More than 2 points w/ at least 2 unique x_i values.

Solve $\underline{A}^T \underline{A}\, \underline{\alpha} = \underline{A}^T \underline{f}$

$$\underline{A}^T \underline{A} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_m \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_m \end{bmatrix} = \begin{bmatrix} m & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix}$$

$$\underline{A}^T \underline{f} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_m \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ \vdots \\ f_m \end{bmatrix} = \begin{bmatrix} \Sigma f_i \\ \Sigma x_i f_i \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} m & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \Sigma f_i \\ \Sigma x_i f_i \end{bmatrix}$$ minimizes $\|\underline{e}\|_2$

let the residual be

$$e = \tfrac{1}{2}\|\underline{e}\|_2^2 = \frac{1}{2}\sum_{i=1}^{m}(f_i - F(x_i))^2$$

w/ $F(x_i) = a + bx_i$

To be at the minimum $\frac{\partial e}{\partial a} = 0$ & $\frac{\partial e}{\partial b} = 0$

$$e = \frac{1}{2}\sum_{i=1}^{m}[f_i - (a + bx_i)]^2$$

$$\frac{\partial e}{\partial a} = -\sum_{i=1}^{m}(f_i - a - bx_i) = 0$$

$$\Rightarrow \sum_{i=1}^{m}(f_i - a - bx_i) = 0$$

$$\Rightarrow \sum_{i=1}^{m} f_i = \sum_{i=1}^{m} a + \sum_{i=1}^{m} bx_i$$

$$\Rightarrow \sum^{m} f_i = a\sum^{m} 1 + b\sum^{m} x_i = ma + b\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} f_i$$

$$\frac{\partial e}{\partial b} = -\sum_{i=1}^{m} x_i(f_i - a - bx_i) = 0$$

$$\Rightarrow \sum_{i=1}^{m} x_i f_i = \sum_{i=1}^{m} ax_i + \sum_{i=1}^{m} bx_i^2$$

$$\Rightarrow a\sum_{i=1}^{m} x_i + b\sum_{i=1}^{m} x_i^2 = \sum_{i=1}^{m} x_i f_i$$

$$\Rightarrow \begin{bmatrix} m & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum f_i \\ \sum x_i f_i \end{bmatrix}$$ $\leftarrow$ the normal equations!

This holds for any regression.

$$F(x) = a_1 g_1(x) + a_2 g_2(x) + \dots + a_n g_n(x)$$

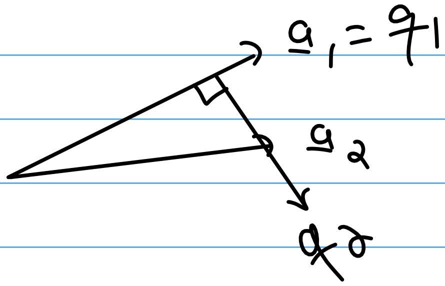
$$\underline{A} = \begin{bmatrix} g_1(x_1) & g_2(x_1) & \cdots & g_n(x_1) \\ \vdots & & & \\ g_1(x_m) & & \cdots & g_n(x_m) \end{bmatrix}$$

$$\underline{A}\,\underline{\alpha} = \underline{f} \Rightarrow \underline{\alpha} = (\underline{A}^T\underline{A})^{-1}\underline{A}^T\underline{f}$$

Next time: Matrix Decompositions

$$\underline{A} = \underline{L}\,\underline{U}$$

$$\underline{A} = \underline{Q}\,\underline{R} \; : \; \underline{Q}^{-1} = \underline{Q}^T \Rightarrow \underline{Q}^T\underline{Q} = \underline{I}$$



$$\underline{Q} = [q_1 \;\; q_2]$$

$$\begin{bmatrix} q_1^T \\ q_2^T \end{bmatrix}[q_1 \;\; q_2] = \begin{bmatrix} q_1^T q_1 & q_1^T q_2 \\ q_2^T q_1 & q_2^T q_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{A} = \underline{U}\,\underline{\Lambda}\,\underline{V}$$