Matrix Subspaces

1) Column Space of $\underline{A}$: $C(\underline{A}) \Rightarrow$ All combinations of the columns of $\underline{A}$.

2) Nullspace of $\underline{A}$: $N(\underline{A}) \rightarrow$ All vectors $\underline{v}$ such that $\underline{A}\,\underline{v} = \underline{0}$

$N(\underline{A})$ is non-empty only if $\underline{A}^{-1}$ does not exist.

To find $N(\underline{A})$ Solve $\underline{A}\,\underline{v} = \underline{0}$

$$\text{rref}([A : \underline{0}]) = \text{rref}(\underline{A})$$

1) if $\text{rref}(\underline{A}) = \underline{I}$ $\Rightarrow$ $\left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$

$\underline{v} = \underline{0}$ is only solution

2) You have free-variables

$$\text{rref}(\underline{A}) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

free

let $v_3 = 1 \Rightarrow v_2 = 0$, $v_1 = -1$

$$N(\underline{A}) = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Dimension of $N(\underline{A})$ = # of free variables in rref($\underline{A}$)

ex$_1$) let $\underline{A} = \begin{bmatrix} 1 & 3 & 2 & 3 \\ 2 & 6 & 8 & 10 \\ 3 & 9 & 10 & 13 \end{bmatrix}$

$$\text{rref}(\underline{A}) = \begin{array}{c} \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} \quad & \uparrow & \quad & \uparrow \\ & v_2 & & v_4 \end{array} \end{array} \qquad \underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

2 free-variables $\Rightarrow$ $\dim(N(\underline{A})) = 2$

v_2	v_4	v_1	v_3
1	0	-3	0
0	1	-1	-1

$$N(\underline{A}) = \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(techinically the span)

③ Row-space: All linear combination of the rows of $\underline{A}$.

Denote as $C(\underline{A}^T)$

Subspace of $\mathbb{R}^n$ if $\underline{A} \in M_{mn}$

(4) Left-Null space: All vectors $\underline{v}^T$ such that
$\underline{v}^T \underline{A} = \underline{0}^T$

$(\underline{v}^T \underline{A})^T = (\underline{0}^T)^T$
$\underline{A}^T \underline{v} = \underline{0}$ => Denote as $N(\underline{A}^T)$

Subspace of $\mathbb{R}^m$

Summary
let $\underline{A} \in M_{mn}$

Subspace	Subspace of
$C(\underline{A})$	$\mathbb{R}^m$
$N(\underline{A})$	$\mathbb{R}^n$
$C(\underline{A}^T)$	$\mathbb{R}^n$
$N(\underline{A}^T)$	$\mathbb{R}^m$

Rank-Nullity Theorem

1) Kernel of L: ker(L) is all vectors in the domain that map to zero.

=> Nullspace of $\underline{A}$

=> $N(\underline{A}) = \ker(\underline{A})$

2) Nullity of L: Dimension of ker(L)

=> nullity $(\underline{A}) = \dim(N(\underline{A})) = \#$ of free variables in rref$(\underline{A})$

3) Rank of L: Dimension of the range

If $\underline{A} \in M_{mn}$, range is the output of $\underline{A}\underline{x} = \underline{b}$ = linear combo of columns

$$\Rightarrow \text{range}(\underline{A}) = C(\underline{A})$$

$$\Rightarrow \text{rank}(\underline{A}) = \dim(\text{range}(\underline{A})) = \dim(C(\underline{A})) = \# \text{ of pivots in rref}(\underline{A})$$

Apply rank - nullity theorem

$$\text{rank}(\underline{A}) + \text{nullity}(\underline{A}) = n \leftarrow \# \text{ of columns of } \underline{A}.$$

Apply to $\underline{A}^T$

$$\underbrace{\text{rank}(\underline{A}^T)}_{\dim(C(\underline{A}^T))} + \underbrace{\text{nullity}(\underline{A}^T)}_{\dim(N(\underline{A}^T))} = m \leftarrow \# \text{ of rows in } \underline{A}.$$

Thm: Let $\underline{A} \in M_{mn}$

1) $\text{Rank}(\underline{A}) = \dim(C(\underline{A})) = \dim(C(\underline{A}^T)) = r \leq \min(m,n)$

2) $\text{nullity}(\underline{A}) = \dim(N(\underline{A})) = n - r$

3) $\text{nullity}(\underline{A}^T) = \dim(N(\underline{A}^T)) = m - r$

ex$_1$) let $\underline{A} \in M_{8\times 10}$ w/ rank $= 4$

nullity $(\underline{A}) = n - r = 10 - 4 = 6$

6 vectors in basis for $N(\underline{A})$

nullity $(\underline{A}^T) = m - r = 8 - 4 = 4$

4 vectors in basis for $N(\underline{A}^T)$

ex$_1$) $\underline{A} = \begin{bmatrix} 8 & 2 & 1 & 23 \\ 4 & 2 & 9 & 19 \\ 10 & 1 & 6 & 35 \end{bmatrix}$

$\text{rref}(\underline{A}) = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

rank $= 3$

nullity $= 1$

$\Rightarrow$ rank + nullity $= 4$

$\text{rref}(\underline{A}^T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

rank $= 3$

nullity of $A^T = 0$

$3 + 0 = 3$

Let $\underline{A} \in M_{mn}$

(1) Matrix $\underline{A}$ has full column rank if $\text{rank}(\underline{A}) = n$

If $\underline{A}$ has full column rank

1) All columns of $\underline{A}$ are linearly independent
2) Only vector $\underline{v}$ such that $\underline{A}\underline{v} = \underline{0}$ is $\underline{v} = \underline{0}$
3) If a solution to $\underline{A}\underline{x} = \underline{b}$ exists it is unique.

(2) A matrix has full row rank if $\text{rank}(\underline{A}) = m$ then

1) All rows of $\underline{A}$ are independent
2) $C(\underline{A})$ spans all of $\mathbb{R}^m$
3) $\underline{A}\underline{x} = \underline{b}$ has at least one solution for all $\underline{b} \in \mathbb{R}^m$

(3) A matrix has full rank if it has both full column rank & full row rank.

$\Rightarrow$ $\underline{A}$ must be square

If $\underline{A}$ is full rank then

1) $\underline{A}\underline{x} = \underline{b}$ has a unique solution for all $\underline{b} \in \mathbb{R}^n$
2) $C(\underline{A})$ spans all of $\mathbb{R}^n$
3) $N(\underline{A})$ is empty
4) $\underline{A}^{-1}$ exists

A Very Important Theorem

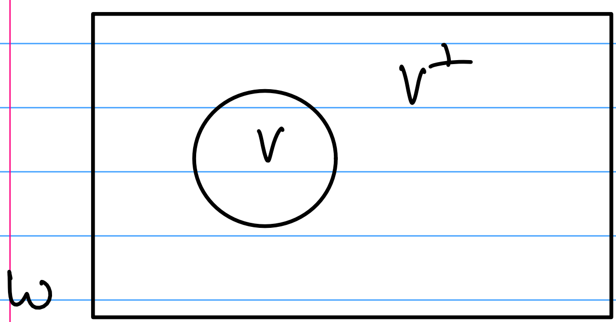
Let $\underline{A} \in M_{nn}$. All of these are equivalent

1. $\underline{A}$ is invertible ($\underline{A}^{-1}$ exists)
2. Columns of $\underline{A}$ are linearly independent
3. Rows of $\underline{A}$ are linearly independent
4. $\text{Det}(\underline{A}) \neq 0$
5. $\underline{A}\underline{x} = \underline{0}$ admits only $\underline{x} = \underline{0}$ as a solution
6. n-pivots in $\text{rref}(\underline{A})$
7. $\text{rref}(\underline{A}) = I$
8. $\underline{A}$ is full rank : $\text{rank}(\underline{A}) = n$
9. $C(\underline{A})$ spans all of $\mathbb{R}^n$
10. $C(\underline{A}^T)$ spans all of $\mathbb{R}^n$

Subspace Complements.

Two subspaces are orthogonal iff for any $\underline{v} \in V$ & $\underline{w} \in W$, $\underline{v} \cdot \underline{w} = 0$

An orthogonal complement of a subspace $V \in W$ are all vectors not in V but in W such that the inner product is zero.



$W = V \cup V^\perp$

if $v_1 \in V$ & $v_2 \in V^\perp$

if $\underline{v}_1 \cdot \underline{v}_2 = 0$ then

$\underline{V}^\perp$ & $\underline{V}$ are orthogonal complements

ex,) Let $W = \mathbb{R}^3$

let $\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\underline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$V = \text{span}(\{\underline{e}_1, \underline{e}_2\})$ $V^\perp = \text{span}(\{\underline{e}_3\})$

$\mathbb{R}^3 = V \cup V^\perp$

$\underline{v}_1 \in V \Rightarrow v_1 = a\underline{e}_1 + b\underline{e}_2$ $v_2 \in V^\perp \Rightarrow \underline{v}_2 = c\underline{e}_3$

$\underline{v}_1 \cdot \underline{v}_2 = (a\underline{e}_1 + b\underline{e}_2) \cdot (c\underline{e}_3)$

$= ac\, \underline{e}_1 \cdot \underline{e}_3 + bc\, \underline{e}_2 \cdot \underline{e}_3 = 0$

Thm: Let $\underline{A} \in M_{mn}$. The row space of $\underline{A}$ is the orthogonal complement to the null space of $\underline{A}$.

$$C(\underline{A}^T) \in \mathbb{R}^n \qquad N(\underline{A}) \in \mathbb{R}^n$$

$$\mathbb{R}^n = C(\underline{A}^T) \cup N(\underline{A})$$

Proof: Let $\underline{y} \in \mathbb{R}^m$, $\underline{x} \in N(\underline{A})$

$\underline{A}^T\underline{y}$ = Linear combination of rows of $\underline{A}$

$\Rightarrow \underline{A}^T\underline{y} \in C(\underline{A}^T)$

$$\underset{\substack{\in \\ N(\underline{A})}}{\underline{x}^T}(\underset{\substack{\in \\ C(\underline{A}^T)}}{\underline{A}^T\underline{y}}) = \underline{x}^T\underline{A}^T\underline{y} = (\underline{A}\underline{x})^T\underline{y} = \underline{0}^T\underline{y} = 0$$

Thm: Let $\underline{A} \in M_{mn}$. Column space of $\underline{A}$ is the orthogonal complement to the left null space of $\underline{A}$.

$$C(\underline{A}) \in \mathbb{R}^m \qquad N(\underline{A}^T) \in \mathbb{R}^m$$

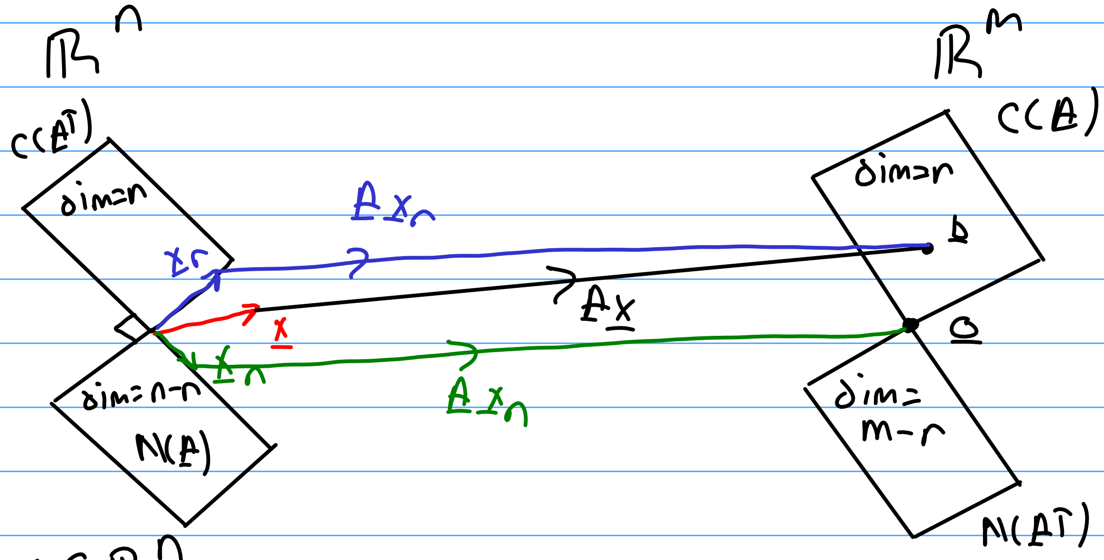
$$\mathbb{R}^m = C(\underline{A}) \cup N(\underline{A}^T)$$

Proof: Let $\underline{y} \in \mathbb{R}^n$, $\underline{x} \in N(\underline{A}^T)$

$\underline{A}\underline{y} \in C(\underline{A})$

$$\underset{N(\underline{A}^T)}{\underline{x}^T}(\underset{C(\underline{A})}{\underline{A}\underline{y}}) = \underline{x}^T\underline{A}\underline{y} = (\underline{x}^T\underline{A})\underline{y} = \underline{0}^T\underline{y} = 0$$

Graphical Representation



$\underline{x} \in \mathbb{R}^n$

$\underline{x} = \underline{x}_r + \underline{x}_n$

in $C(\underline{A}^T)$ in $N(\underline{A})$

$$\begin{aligned}\underline{A}\underline{x} &= \underline{A}(\underline{x}_r + \underline{x}_n) \\ &= \underline{A}\underline{x}_r + \underline{A}\underline{x}_n \\ &= \underline{A}\underline{x}_r + \underline{0} \\ &= \underline{b}\end{aligned}$$

$\underline{A}\underline{x}_n$ provides no useful information

Now solve $\underline{A}\underline{x} = \underline{b}$ w/ nullity$(\underline{A}) > 0$

$$\begin{aligned}\underline{A}\underline{x} &= \underline{b} \\ \underline{A}\underline{x} + \underline{0} &= \underline{b} + \underline{0} \\ \underline{A}\underline{x} + \underline{A}\underline{x}_n &= \underline{b} + \underline{0} \\ \underline{A}(\underline{x} + \underline{x}_n) &= \underline{b}\end{aligned}$$

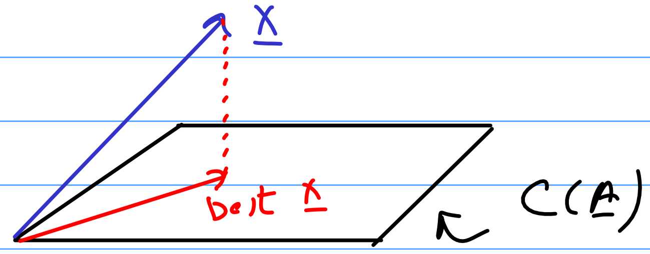
both $\underline{x}$ & $\underline{x} + \underline{x}_n$ solve $\underline{A}\underline{x} = \underline{b}$

$\underline{x} \neq \underline{x} + \underline{x}_n$

$\Rightarrow \infty$ # of solutions

Next week: what if $\underline{b} \notin C(\underline{A})$?
& we have $\underline{A}\,\underline{x} = \underline{b}$

Introduce Projection onto subspaces.



Normal Equations do this: $\underline{A}^T \underline{A}\,\underline{x} = \underline{A}^T \underline{b}$