Vectors

A vector is a 1-dimensional array of numbers. Each component of a vector is a real number. We will distinguish between an n-dimensional column vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

and a row vector

$$\mathbf{y} = [y_1 \ldots y_n]$$

Transposition of a vector turns a column vector in a row vector, and vice versa. Boldface letters are often used in the literature to denote vectors.

Operations with vectors

Let

$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$,

$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$. Then $\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$. Subtraction of vectors is defined analogously to addition.

If α is a scalar (real number) then $\alpha\mathbf{x} = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix}$.

One can create a linear combination of vectors using multiplication by scalars and addition. Let $\mathbf{x_1}, \cdots, \mathbf{x_k}$ are vectors, and $\alpha_1, \cdots \alpha_k$ are scalars. Then a vector $\mathbf{y} = \alpha_1\mathbf{x_1} + \cdots + \alpha_k\mathbf{x_k}$ is called a linear combination of vectors $\mathbf{x_1}, \cdots, \mathbf{x_k}$.

Definition (7.1)

A **vector norm** on $\mathbb{R}^n$ is a function, $\|\cdot\|$, from $\mathbb{R}^n$ into $\mathbb{R}$ with the following properties:

(i) $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$,

(ii) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$,

(iii) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$,

(iv) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}$

Definition (7.2)

The l_2 and l_∞ norms for the vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ are defined by

$$\|\mathbf{x}\|_2 = \left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2} \quad \text{and} \quad \|\mathbf{x}\|_\infty = \max_{1\leq i\leq n} |x_i|.$$

Theorem (7.3: Cauchy-Bunyakovsky-Schwarz inequality)

For each $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ *and* $\mathbf{y} = (y_1, y_2, \ldots, y_n)^t$ *in* $\mathbb{R}^n$,

$$\mathbf{x}^t\mathbf{y} = \sum_{i=1}^{n} x_i y_i \le \left\{\sum_{i=1}^{n} x_i^2\right\}^{1/2} \left\{\sum_{i=1}^{n} y_i^2\right\}^{1/2} = \|\mathbf{x}\|_2 \cdot \|\mathbf{y}\|_2.$$

Definition (7.4)

If $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)^t$ are vectors in $\mathbb{R}^n$, the l_2 and l_∞ distances between $\mathbf{x}$ and $\mathbf{y}$ are defined by

$$\|\mathbf{x} - \mathbf{y}\|_2 = \left\{\sum_{i=1}^{n} (x_i - y_i)^2\right\}^{1/2} \quad \text{and} \quad \|\mathbf{x} - \mathbf{y}\|_\infty = \max_{1 \le i \le n} |x_i - y_i|.$$

Definition (7.5)

A sequence $\{\mathbf{x}^{(k)}\}_{k=1}^{\infty}$ of vectors in $\mathbb{R}^n$ is said to **converge** to $\mathbf{x}$ with respect to the norm $\|\cdot\|$ if, given any $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that

$$\|\mathbf{x}^{(k)} - \mathbf{x}\| < \varepsilon, \quad \text{for all } k \geq N(\varepsilon).$$

Theorem (7.6)

The sequence of vectors $\{\mathbf{x}^{(k)}\}$ converges to $\mathbf{x}$ in $\mathbb{R}^n$ with respect to the l_∞ norm if and only if $\lim_{k\to\infty} x_i^{(k)} = x_i$, for each $i = 1, 2, \ldots, n$.

Theorem (7.7)

For each $\mathbf{x} \in \mathbb{R}^n$, $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \sqrt{n}\|\mathbf{x}\|_\infty$.

Definition (7.8)

A **matrix norm** on the set of all $n \times n$ matrices is a real-valued function, $\|\cdot\|$, defined on this set, satisfying for all $n \times n$ matrices A and B and all real numbers α:

(i) $\|A\| \geq 0$;

(ii) $\|A\| = 0$, if and only if A is O, the matrix with all 0 entries;

(iii) $\|\alpha A\| = |\alpha|\|A\|$;

(iv) $\|A + B\| \leq \|A\| + \|B\|$;

(v) $\|AB\| \leq \|A\|\|B\|$.

Theorem (7.9)

If $||\cdot||$ is a vector norm on $\mathbb{R}^n$, then $\|A\| = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$ is a matrix norm.

Corollary (7.10)

For any vector $\mathbf{z} \neq \mathbf{0}$, matrix A, and any natural norm $\|\cdot\|$, we have

$$\|A\mathbf{z}\| \leq \|A\| \cdot \|\mathbf{z}\|.$$

Theorem (7.11)

If $A = (a_{ij})$ is an $n \times n$ matrix, then $\|A\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|.$

Iterative techniques in matrix algebra.

7.1. Norms of vectors and matrices.

Matrix norms defined by vector norms are called natural, or induced, matrix norms associated with the vector norm.

#1(a). $\vec{x} = (3, -4, 0, \frac{3}{2})^t$

$$\|\vec{x}\|_\infty = \max(3, 4, 0, \tfrac{3}{2}) = 4$$

$$\|\vec{x}\|_2 = \sqrt{9+16+\tfrac{9}{4}} = \frac{\sqrt{36+64+9}}{2} = \frac{\sqrt{109}}{2}$$

#3 (a) $x^{(k)} = (\frac{1}{k}, e^{1-k}, -\frac{2}{k^2})^t$

$$\lim_{k\to 0} x^{(k)} = (0,0,0)^t.$$

#5(a).

$$A = \begin{bmatrix} 10 & 15 \\ 0 & 1 \end{bmatrix} \qquad \|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$$

$$\|A\|_\infty = \max((10+15),(0+1)) = 15+10 = 25$$

#7(b)

$$x_1 + 2x_2 + 3x_3 = 1$$
$$2x_1 + 3x_2 + 4x_3 = -1$$
$$3x_1 + 4x_2 + 6x_3 = 2$$

$\vec{x} = (0, -7, 5)^t$

$\tilde{\vec{x}} = (-0.33, -7.9, 5.8)^t$

$$\|x-\tilde{x}\|_\infty = \|(0.33, 0.9, -0.8)^t\|_\infty = 0.9$$

$$A\tilde{x} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix} \begin{bmatrix} -0.33 \\ -7.9 \\ 5.8 \end{bmatrix} = \begin{bmatrix} 1.27 \\ -1.16 \\ 2.21 \end{bmatrix}$$

$$A\tilde{x}-b = \begin{bmatrix} 0.27 \\ -0.16 \\ 0.21 \end{bmatrix}$$

$$\|A\tilde{x}-b\|_\infty = 0.27$$

Inner (aka dot) product of vectors

The inner product of two n-dimensional vectors, **x** and **y**, is calculated as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}'\mathbf{y}$$

Properties of the inner product:

(1) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$
(2) $\langle \alpha\mathbf{x}, \mathbf{y} \rangle = \alpha\langle \mathbf{x}, \mathbf{y} \rangle$
(3) $\langle \mathbf{x} + \mathbf{z}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle$
(4) $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$
(5) $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = \mathbf{0}.$

Here $\mathbf{x}, \mathbf{y}, \text{and } \mathbf{z}$ are any vectors and α is any real number.

Properties (4) and (5) allow to introduce

$||\mathbf{x}|| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ (A)

and check that all four properties of a vector norm given in Definition 7.1 are satisfied meaning that (A) defines a legitimate norm, namely, a norm associated with the inner product $\langle \, , \rangle$.

Triangle inequality for a norm.

As is well-known from geometry, in a triangle, length of each side doesn't exceed (strictly less in a non-degenerated triangle) the sum of the lengths of the other two sides. Additionally, length of each side is no less than the absolute value of the difference of the lengths of the other two sides. Since a norm is an extension of the concept of length to multiple dimensions, it is not surprising that these inequalities extend to norms of n-dimensional vectors:

$\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\|+\|\mathbf{y}\| \quad (tr1)$;

$\|\mathbf{x}-\mathbf{y}\| \geq |\|\mathbf{x}\|-\|\mathbf{y}\|| \quad (tr2)$.

The equality in these inequalities can be achieved only if $\mathbf{x}$ and **$\mathbf{y}$ are collinear meaning that $\mathbf{x}=\alpha\mathbf{y}$ with some constant scalar α**. For a norm associated with an inner product via (A), the proof of these inequalities can be obtained by raising both sides into the second power and using the Schwarz inequality.