Linear functions / transformations

A function f is linear iff for $\underline{v}_1, \underline{v}_2 \in V$ w/ $a \in \mathbb{R}$

1) $f(\underline{v}_1 + \underline{v}_2) = f(\underline{v}_1) + f(\underline{v}_2)$
2) $f(a\underline{v}_1) = a f(\underline{v}_1)$

ex$_n$) $f(\underline{A}) = \underline{A}^T$ is linear
ex$_1$) $f(\underline{v}) = \underline{v} \cdot \underline{v}$ not linear

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Theorems

Let V & W be two vector spaces such that $L: V \to W$ is a linear transformation.
Let $\emptyset_V$ be the zero vector in V & $\emptyset_W$ be the zero vector in W.

1) $L(\emptyset_V) = \emptyset_W$

2) $L(-\underline{v}) = -L(\underline{v}) \qquad \underline{v} \in V$

3) $\underline{L}(a_1\underline{v}_1 + a_2\underline{v}_2 + \dots + a_n\underline{v}_n) = a_1 L(\underline{v}_1) + a_2 L(\underline{v}_2) + \dots + a_n L(\underline{v}_n)$

Proof:

1) $L(\underline{\emptyset}_V) = L(\emptyset\,\underline{\emptyset}_V) = \emptyset\, L(\underline{\emptyset}_V) = \underline{\emptyset}_W$

2) $L(-\underline{v}) = L(-1(\underline{v})) = -1\, L(\underline{v}) = -L(\underline{v})$

3) $L(a_1 \underline{v}_1 + a_2 \underline{v}_2 + \dots + a_n \underline{v}_n) = L(a_1 \underline{v}_1) + L(a_2 \underline{v}_2 + \dots + a_n \underline{v}_n)$

$= a_1 L(\underline{v}_1) + L(a_2 \underline{v}_2) + L(a_3 \underline{v}_3 + \dots + a_n \underline{v}_n)$

$\vdots$

$= a_1 L(\underline{v}_1) + a_2 L(\underline{v}_2) + \dots + a_n L(\underline{v}_n)$

Thm: the composition of linear operators is linear.

ex.) let $L_1(\underline{A}) = \underline{A}^T$ $\quad L_2(\underline{A}) = c\,\underline{A}$ $\quad c \in \mathbb{R}$

$(L_1 \circ L_2)(\underline{A}) = L_1(L_2(\underline{A})) = L_1(c\underline{A}) = (c\underline{A})^T$
$= c\,\underline{A}^T$

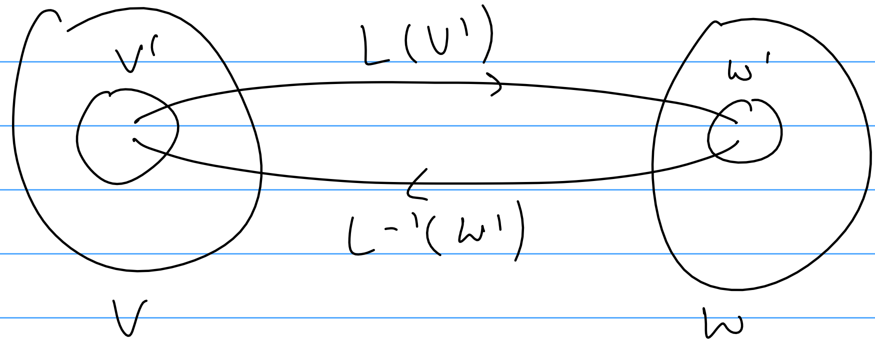
Check: $(L_1 \circ L_2)(\underline{A} + \underline{B}) = c(\underline{A} + \underline{B})^T = c(\underline{A}^T + \underline{B}^T)$
$= c\,\underline{A}^T + c\,\underline{B}^T$
$= (L_1 \circ L_2)(\underline{A}) + (L_1 \circ L_2)(\underline{B})$

$(L_1 \circ L_2)(a\underline{A}) = c(a\underline{A})^T = ac\,\underline{A}^T = a(L_1 \circ L_2)(\underline{A})$

Thm: let $L: V \to W$ be a linear transformation.

1) If V' is a subspace of V then $L(V') = W'$ is a subspace of W.

2) If W' is a subspace of W & L^{-1} exists then $L^{-1}(W') = V'$ is a subspace of V.



Action of a Linear Transformation.

The effect of a linear transformation (the action of it) on a vector space is completely defined by the action of that transformation on basis of the vector space.

Let $L: V \to W$ be linear w/ V having a basis of $\{\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n\}$

If $\underline{v} \in V$ then $\underline{v} = k_1 \underline{b}_1 + k_2 \underline{b}_2 + \ldots + k_n \underline{b}_n$ for $k_1 \to k_n$

$$L(\underline{v}) = L(k_1 \underline{b}_1 + k_2 \underline{b}_2 + \cdots + k_n \underline{b}_n)$$

$$= k_1 L(\underline{b}_1) + k_2 L(\underline{b}_2) + \ldots + k_n L(\underline{b}_n)$$

Note: $k_1 \to k_n$ are the "coordinates" of $\underline{v}$ in the basis.

ex1) Let
$$\begin{array}{ccccc} & \underline{b}_1 & \underline{b}_2 & \underline{b}_3 & \underline{b}_4 \\ B = \Bigg\{ & \begin{bmatrix} 0 \\ 4 \\ 0 \\ 1 \end{bmatrix}, & \begin{bmatrix} -2 \\ 5 \\ 0 \\ 1 \end{bmatrix}, & \begin{bmatrix} -3 \\ 5 \\ 1 \\ 1 \end{bmatrix}, & \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \Bigg\} \end{array}$$
is a basis for $\mathbb{R}^4$.

let $L: \mathbb{R}^4 \to \mathbb{R}^3$ such that

$$L(\underline{b}_1) = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad L(\underline{b}_2) = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad L(\underline{b}_3) = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} \quad L(\underline{b}_4) = \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix}$$

What is $L(\underline{v})$ if $\underline{v} = \begin{bmatrix} -4 \\ 14 \\ 1 \\ 5 \end{bmatrix}$?

First, determine $k_1 \to k_4$ in B.

$$\underline{v} = \underset{\substack{\uparrow \\ k_1}}{2}\, \underline{b}_1 + \underset{\substack{\uparrow \\ k_2}}{(-1)}\, \underline{b}_2 + \underset{\substack{\uparrow \\ k_3}}{(1)}\, \underline{b}_3 + \underset{\substack{\uparrow \\ k_4}}{3}\, \underline{b}_4$$

Then $L(\underline{v}) = k_1 L(\underline{b}_1) + \dots + k_4 L(\underline{b}_4)$

$$= 2 \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + (1) \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 18 \\ 9 \\ 0 \end{bmatrix}$$

Thm: Let $B = \{\underline{b}_1, \dots, \underline{b}_n\}$ form a basis for vector space V.

Let $\underline{w}_1, \underline{w}_2, \dots, \underline{w}_n$ be the n-vectors in vector space W.

There is always a unique linear transformation $L : V \to W$ such that $L(\underline{b}_1) = \underline{w}_1$

$L(\underline{b}_2) = \underline{w}_2$

$\vdots$

$L(\underline{b}_n) = \underline{w}_n$

Now tie to matrix-vector products.

A matrix-vector product is nothing but a linear combination of columns

$$\underline{A}\,\underline{x} = [\underline{a}_1\ \underline{a}_2\ \cdots\ \underline{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \underline{a}_1 + x_2 \underline{a}_2 + \cdots + x_n \underline{a}_n = \underline{b}$$

Takes a vector $\underline{x}$ & maps it to vector $\underline{b}$,

$\underline{b}$ is the image of $\underline{x}$ due to function $\underline{A}$,

Is this linear?

$\underline{A}(\underline{x} + \underline{y}) = \underline{A}\,\underline{x} + \underline{A}\,\underline{y}$ $\qquad$ $\underline{A}(c\underline{x}) = c\,\underline{A}\,\underline{x}$

Compare to the action of a linear operator.

$\underline{x}$ are the coordinates of a vector $\underline{u} \in V$

$\underline{b}$ is the vector $\underline{w}$ in $L(V) = W$

Columns of $\underline{A}$ are $L(\underline{b}_1)$ $L(\underline{b}_2)$ etc.

Let $B = \{\underline{b}_1, \underline{b}_2, \underline{b}_3, \underline{b}_4\}$ as above.

Let $L(\underline{b}_1)$, $L(\underline{b}_2)$, $L(\underline{b}_3)$ $L(\underline{b}_4)$ be as above.

If $\underline{v} = \begin{bmatrix} -4 \\ 14 \\ 1 \\ 5 \end{bmatrix}$ then $L(\underline{u}) = ?$

$$\underline{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 3 \end{bmatrix} \begin{matrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{matrix} \qquad L(B) = \begin{array}{cccc} L(b_1) & & & L(b_4) \end{array} \begin{bmatrix} 3 & 2 & -4 & 6 \\ 1 & -1 & 3 & 1 \\ 2 & 1 & 0 & -1 \end{bmatrix} = \underline{A}$$

$$L(\underline{u}) = \underline{A}\underline{x} = \begin{bmatrix} 18 \\ 9 \\ 0 \end{bmatrix}$$

Thm: Let B be an ordered basis for vector space V & let C be an ordered basis for vector space W.

For any linear transformation $L: V \to W$ there exists a matrix such that

$$\underline{A}_{BC}[\underline{v}]_B = [L(\underline{v})]_C$$

$\underline{A}_{BC} = [L(\underline{b}_1) \;\; L(\underline{b}_2) \ldots L(\underline{b}_n)]$ is the linear operator.

$[\underline{v}]_B$ = coordinates of $\underline{v}$ in basis B

$[L(\underline{v})]_C$ = coordinates of $L(\underline{v})$ in basis C

Ordered basis: the order of the set of vectors is fixed.

$$\underbrace{\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}}_{B_1} \quad \& \quad \underbrace{\left\{\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}}_{B_2}$$

are basis for $\mathbb{R}^2$,

$\begin{bmatrix}5\\3\end{bmatrix} \to$ coordinates of $\begin{bmatrix}5\\3\end{bmatrix}$ in B_1

" " $\begin{bmatrix}3\\5\end{bmatrix}$ in B_2

Geometric Operators in $\mathbb{R}^3$.

Let $\underline{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $\quad \underline{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $\quad \underline{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

be the ordered basis for $\mathbb{R}^3$.

1) reflection about $\underline{e}_3$

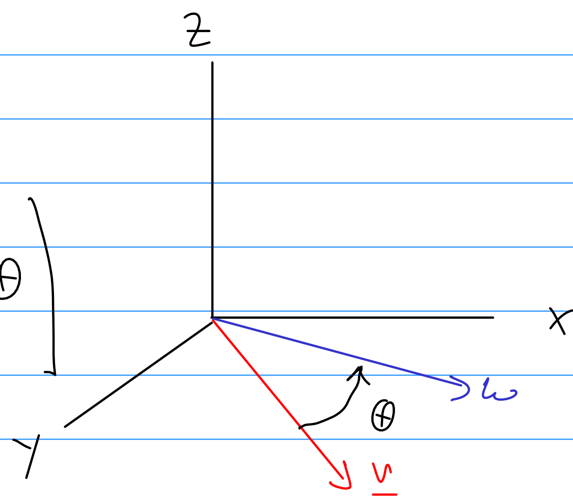
$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_2 \\ -a_3 \end{bmatrix}$$

$$L(\underline{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad L(\underline{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad L(\underline{e}_3) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\underline{A}_{BC} = [L(\underline{e}_1) \;\; L(\underline{e}_2) \;\; L(\underline{e}_3)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \underline{A}_{ref}$$

2) Rotation about z-axis

$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 \cos\theta - a_2 \sin\theta \\ a_1 \sin\theta + a_2 \cos\theta \\ a_3 \end{bmatrix}$$



$$\underline{A}_{BC} = [L(\underline{e}_1) \;\; L(\underline{e}_2) \;\; L(\underline{e}_3)] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{A}_{rot}$$

3) Scaling : $L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} c\,a_1 \\ c\,a_2 \\ c\,a_3 \end{bmatrix}$ $\quad c \in \mathbb{R}$

$$\underline{A}_{BC} = [\,L(\underline{e}_1) \;\; L(\underline{e}_2) \;\; L(\underline{e}_3)\,] = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix} = \underline{A}_{scale}$$

Apply rotation, then scaling, then reflection to $\underline{x}$,

$$\underline{A}_{ref}\,\underline{A}_{scale}\,\underline{A}_{rot}\,\underline{x}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 0 \end{bmatrix}\overbrace{\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

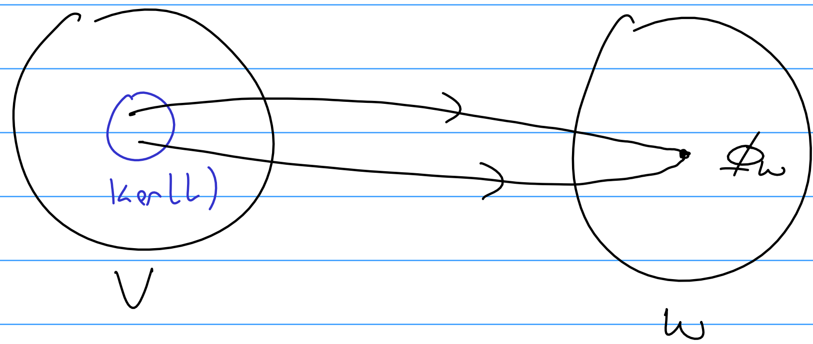
$$= \begin{bmatrix} c\cos\theta & -c\sin\theta & 0 \\ c\sin\theta & c\cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

As $\underline{A}_{rot}$, $\underline{A}_{scale}$, $\underline{A}_{ref}$ are linear so is this

Rank, nullity, Kernel

Let $L: V \to W$ be a linear transformation.

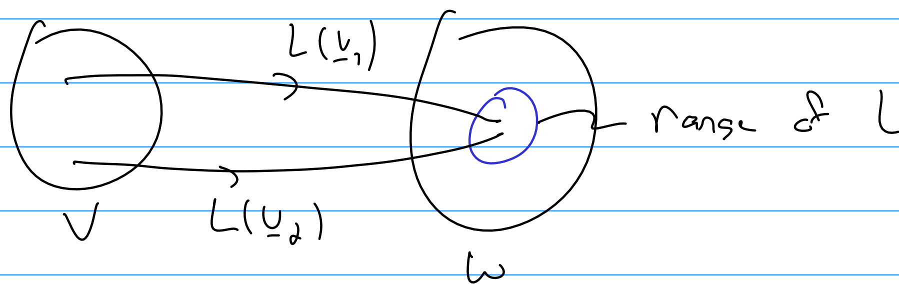
Kernel of L: $\ker(L)$ are all vectors in V that map to $\emptyset_W$ in W



nullity of L: Dimension of $\ker(L)$
For vector, the # of vectors in the basis of $\ker(L)$.

$$\text{nullity}(L) = \dim(\ker(L))$$

Rank of L: the rank of L is the dimension of the range of L (image space)



$$\text{rank}(L) = \dim(\text{image}(L)) = \dim(\text{range}(L))$$

Rank-Nullity Theorem

Let $L: V \to W$ be a linear transformation for vector spaces V & W.

The Rank-Nullity Theorem states that

$$\text{rank}(L) + \text{nullity}(L) = \dim(V)$$

Very important for matrices.

Matrix Subspaces

All matrix-vector products are linear transformations

$$\underline{A}\,\underline{x} = \underline{y} \qquad \underline{x} \in \underline{X} \qquad \underline{y} \in \underline{Y}$$

Four Subspaces

1. Column Space
2. Null space
3. Row space
4. Left-null space

Column Space

Recall that $\underline{A}\,\underline{x} = \underline{b}$ is

$$\underline{b} = x_1 \underline{a}_1 + x_2 \underline{a}_2 + \dots + a_n x_n$$

$\Rightarrow$ $\underline{b}$ is in the column space of $\underline{A}$: $\underline{b} \in C(\underline{A})$

If $\underline{A} \in \mathbb{R}^{m \times n}$ then each column has length - m.

$C(\underline{A})$ is a subspace of $\mathbb{R}^m$

Now, the question is when does $\underline{x}$ exist such that $\underline{A}\underline{x} = \underline{b}$

For $\underline{x}$ to exist then $\underline{b}$ must be in $C(\underline{A})$

If $\underline{b} \in C(\underline{A})$ then at least 1 $\underline{x}$ exists such that $\underline{A}\underline{x} = \underline{b}$.

Two cases:

#1 let $\underline{A} \in \mathbb{R}^{n \times n}$ with $\det(\underline{A}) \neq 0 \Rightarrow \underline{A}^{-1}$ exists

then $\underline{A}\underline{x} = \underline{b}$ gives $\underline{x} = \underline{A}^{-1}\underline{b}$

Therefore:
1. $\underline{x}$ is the coordinates of $\underline{b}$ in $C(\underline{A})$
2. Any $\underline{b} \in \mathbb{R}^n$ is in $C(\underline{A})$
3. $C(\underline{A})$ spans all of $\mathbb{R}^n$

#2 let $\underline{A} \in \mathbb{R}^{m \times n}$ w/ $m \neq n$ or $\det(\underline{A}) = 0$ if $m = n$
$\underline{A}^{-1}$ does not exist

For a solution to $\underline{A}\underline{x} = \underline{b}$ to exist then $\underline{b}$ must be in $C(\underline{A})$
$\Rightarrow$ if $\underline{x}$ exists depends on $\underline{A}$ & $\underline{b}$.

ex.) let $\underline{A} = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ $\det(\underline{A}) = 0$

but does $\underline{b} = \begin{bmatrix} 11 \\ -5 \\ 3 \end{bmatrix}$ have an $\underline{x}$ such that $\underline{A}\underline{x} = \underline{b}$?

$\text{rref}([\underline{A} : \underline{b}]) = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \infty$ # of solutions
$\underline{b} \in C(\underline{A})$

try $\begin{bmatrix} 11 \\ -5 \\ 0 \end{bmatrix}$: $\text{rref}([\underline{A} : \underline{b}]) = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ 0 Solutions

$\underline{b} \notin C(\underline{A})$

Recall that the dimension of a vector space is given by the # of pivots of rref.

$\Rightarrow \dim(C(\underline{A})) = \#$ of pivots in $\text{rref}(\underline{A})$
$= \text{rank}(\underline{A})$

$\Rightarrow$ If $\underline{A} \in \mathbb{R}^{m \times n}$ w/ $\underline{A} \neq \underline{0}$ then

$$1 \leq \dim(C(\underline{A})) = \text{rank}(\underline{A}) \leq n \leftarrow \text{the \# of columns}$$

technically $\min(m, n)$