Consider the Trapezoidal rule applied to determine the integrals of the functions whose graphs are shown in Figure 4.15.
The Trapezoidal rule approximates the integral of the function by integrating the linear function that joins the endpoints of the graph of the function.

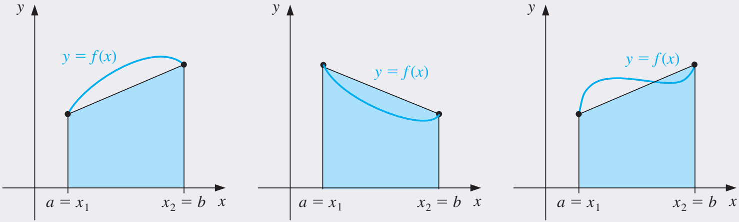

Figure: Figure 4.15

But this is not likely the best line for approximating the integral. Lines such as those shown in Figure 4.16 would likely give much better approximations in most cases.

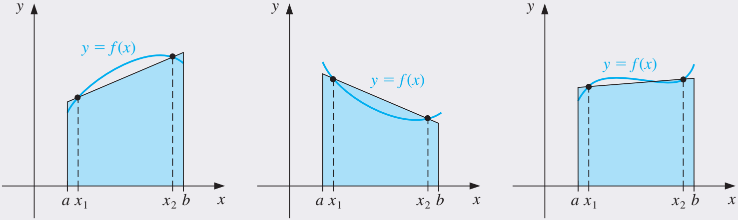

Figure: Figure 4.16

In Gaussian quadrature the points for evaluation are chosen in an optimal, rather than equally-spaced, way. The nodes $x_1, x_2, \ldots, x_n$ in the interval $[a, b]$ and coefficients $c_1, c_2, \ldots, c_n$, are chosen to minimize the expected error obtained in the approximation

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} c_i f(x_i).$$

The YouTube video developed by Wen Shen can serve as a good illustration of the introduction to Gaussian Quadrature for students. ▸ Illustration Introducing Gaussian Quadrature

4.7. Gaussian Quadrature

The Newton-Cotes formulas use interpolating polynomials of degree n and are exact when approximating $\int_a^b Q_n(x)$ where Q_n is a polynomial of degree n. These formulas use evenly spaced nodes.

In Gaussian quadrature, the nodes are chosen in an optimal rather than an equally spaced way. The nodes $x_1, \dots, x_n$ in the interval $[a, b]$ and coefficients $c_1, \dots, c_n$ are chosen to minimize the expected error of approximation $\int_a^b f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i)$. There are $2n$ parameters. Consequently, we will consider the class of polynomials of degree $2n - 1$.

Example: $n = 2;\ [a, b] = [-1,1]$.

$\int_{-1}^{1} f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$.

We would like the approximation to be exact on the polynomials of degree

$2 * 2 - 1 = 3$, that is, when $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$.

$$\int_{-1}^{1} 1dx = 2 = c_1 + c_2$$

$$\int_{-1}^{1} xdx = 0 = c_1 x_1 + c_2 x_2$$

$$\int_{-1}^{1} x^2 dx = \frac{2}{3} = c_1 {x_1}^2 + c_2 {x_2}^2$$

$$\int_{-1}^{1} x^3 dx = 0 = c_1 {x_1}^3 + c_2 {x_2}^3$$

The solution to this system of equations is $c_1 = 1,\ c_2 = 1,\ x_1 = -\frac{\sqrt{3}}{3}$,

$x_2 = \frac{\sqrt{3}}{3}$.

So

$$\int_{-1}^{1} f(x)dx \approx f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right) \tag{4.40}$$

Legendre polynomials:

(1) $P_n(x)$ is a monic polynomial of degree n

(2) $\int_{-1}^{1} P(x)P_n(x)dx = 0$ for every polynomial $P(x)$ of degree less than n.

$$P_0(x) = 1, \quad P_0(x) = x, \qquad P_2(x) = x^2 - \frac{1}{3},$$

$P_3(x) = x^3 - \frac{3}{5}x$, $P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$.

The roots of Legendre polynomials are distinct, lie in $(-1,1)$, and produce the nodes for the quadrature problem.

Gaussian Quadrature on arbitrary intervals.

$$\int_a^b f(x)dx$$

Substitute $t = \frac{2x-a-b}{b-a}$, which is equivalent to $x = \frac{1}{2}[(b-a)t + a + b]$

$$\int_a^b f(x)dx = \int_{-1}^{1} f\left(\frac{(b-a)t + a + b}{2}\right)\frac{b-a}{2}dt$$

The nodes $x_1, x_2, \ldots, x_n$ needed to produce an integral approximation formula that gives exact results for any polynomial of degree less than $2n$ are the roots of the nth-degree Legendre polynomial. (See Theorem 4.7.)

Theorem (4.7)

Suppose that $x_1, x_2, \ldots, x_n$ are the roots of the nth Legendre polynomial $P_n(x)$ and that for each $i = 1, 2, \ldots, n$, the numbers c_i are defined by

$$c_i = \int_{-1}^{1} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}\, dx.$$

If $P(x)$ is any polynomial of degree less than $2n$, then

$$\int_{-1}^{1} P(x)\, dx = \sum_{i=1}^{n} c_i P(x_i).$$

#11. Determine constants $a, b, c, and\ d$ that will produce a quadrature formula

$\int_{-1}^{1} f(x)dx = af(-1) + b\,f(1) + cf'(-1) + df'(1)$

that has degree of precision three.

$$1:\ 2 = a + b$$

$$x:\ 0 = -a + b + c + d$$

$$x^2:\ \frac{2}{3} = a + b - 2c + 2d$$

$$x^3:\ 0 = -a + b + 3c + 3d$$

$a = 1, b = 1, c = \frac{1}{3}, d = -\frac{1}{3}.$

#12. Determine constants $a, b, c, d\ and\ e$ that will produce a quadrature formula

$$\int_{-1}^{1} f(x)dx = af(-1) + b\,f(0) + c\,f(1) + df'(-1) + ef'(1)$$

that has degree of precision four.

$$1:\ 2 = a + b + c$$

$$x:\ 0 = -a + \ \ c + d + e$$

$$x^2:\ \frac{2}{3} = a + \quad c - 2d + 2e$$

$$x^3:\ 0 = -a + \quad c + 3d + 3e$$

$$x^4:\ \frac{2}{5} = a + \quad c - 4d + 4e$$

$$a = \frac{7}{15}, b = \frac{16}{15}, c = \frac{7}{15}, d = \frac{1}{15}, e = \frac{1}{15}$$

#1(a). Approximate the following integrals using Gaussian quadrature with $n = 2$ and compare your results to the exact values of the integrals.

$\int_1^{1.5} x^2 \ln x \, dx$

$$\int_1^{1.5} x^2 \ln x \, dx = \frac{1}{3}\int_1^{1.5} \ln x \, d(x^3) = \frac{1}{3}\left[\ln x * x^3|_1^{1.5} - \int_1^{1.5} \frac{1}{x} * x^3 \, dx\right] =$$

$$\frac{1}{3}\left[\ln 1.5 * 1.5^3 - \frac{1}{3}(1.5^3 - 1)\right] = 0.1922594$$

$n = 2$

$r_{2,i}$	$c_{2,i}$
0.5773503	1
-0.5773503	1

Substitute $t = \frac{2x-a-b}{b-a} = \frac{2x-2.5}{0.5}$; $x = \frac{1}{2}(0.5t + 2.5)$

$$\int_1^{1.5} x^2 \ln x \, dx = \int_{-1}^{1} \frac{1}{4}(0.5t + 2.5)^2 \left(\ln \frac{0.5t + 2.5}{2}\right)(0.25) \, dt \approx$$

0.25*0.25$\left((0.5 * 0.5773503 + 2.5)^2 * \ln \frac{0.5*0.5773503+2.5}{2} + (-0.5 * 0.5773503 + 2.5)^2 * \ln \frac{-0.5*0.5773503+2.5}{2}\right) =$

$$0.0625 * (7.776709 * 0.3324194 + 4.889958 * 0.1004447) =$$

$0.0625(2.585129 + 0.4911704) = 0.1922687$

#3(a) with $n = 3$, $\int_1^{1.5} x^2 \ln x \, dx \approx 0.1922594$

For comparison, other methods introduced in 4.3 result in the following (see 4.3 #1(c), 5(c), 9(c)):

Trapezoidal Rule: 0.2280740

Simpson’s Rule: 0.1922453

Midpoint Rule: 0.1743309