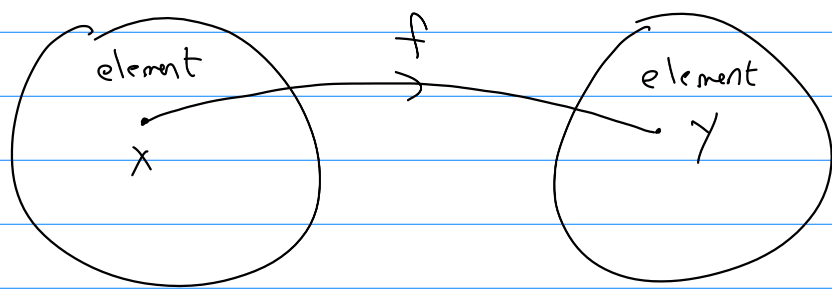
Functions

A **function** is a generic term which describes the transformation of an element in the **domain** X into an element in the **codomain** Y.

Not a mathematical function as typically understood.



Doman X codomain Y

$x =$ element in X
$y =$ element in Y
$f =$ function

Write $f : X \to Y$ or $f(x) = y$

1) the function will assign each element in X to a single element in Y.
2) Multiple elements in X can be assigned to a single element in Y.
3) Some elements in Y might not be accessible.

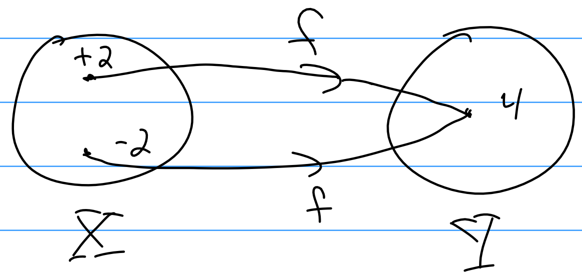
ex.) let $X = \mathbb{Z}^n = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$Y = \mathbb{Z}^n$

Define $f(x) = x^2 = y$

$f(2) = 4$ $\qquad\qquad$ $f(-2) = 4$

$\uparrow$ in X $\quad$ $\uparrow$ in Y



Not all elements in Y can be obtained.

ex.) No element in X will return -4.

ex.) let $X = \mathbb{Z}^n$ $\qquad$ $Y = \mathbb{Z}^n$

$f(x) = \sqrt{x}$ is <u>not</u> a function

1) Not all elements in X map to Y.

$$f(-4) = \sqrt{-4} \notin \mathbb{Z}^n$$

2) Single element in X returns multiple values in Y.

$$f(9) = \pm 3$$

but if $X = \mathbb{R}^1$ & $Y = \mathbb{C}^1$ then pt #1 above no longer holds, but #2 is still true

Function Terms

- Image : the image of a domain element is the unique codomain element.

  $f(x) = x^2$ Image of 2 is 4.

- Pre-image : the domain element(s) that give a codomain element.

  $f(x) = x^2$ pre-image of 4 is $-2, +2$

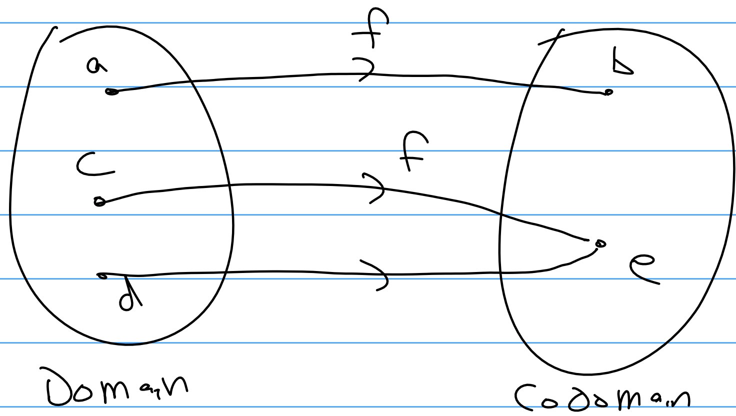


Image of a is b

Pre-image of b is a

Image of c & d is e

Pre-image of e is c & d

All elements in the domain have an image.

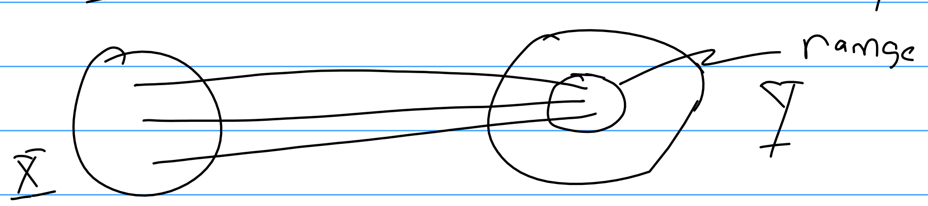
Not all elements in co-domain have a pre-image

ex.) let $X = \mathbb{Z}^n$ $Y = \mathbb{Z}^n$ w/ $f(x) = x^2$

5 in Y does not have a pre-image

$(5)^{1/2}$ not in $\mathbb{Z}^n$

- Range : the range of a function is the subspace of Y that all elements in X map to.



$f(x)$ can never return something outside of it's range.

- One-to-one: A function is one-to-one iff every element in X maps to a unique element in Y.

To show one-to-one you need to show that $f(x_1) = f(x_2)$ only if $x_1 = x_2$

ex.) Is $f(x) = x - 1$ one-to-one?

let $x_1, x_2 \in X$ w/ $f(x_1) = f(x_2)$

$$x_1 - 1 = x_2 - 1 \Rightarrow x_1 = x_2 \Rightarrow \text{one-to-one}$$

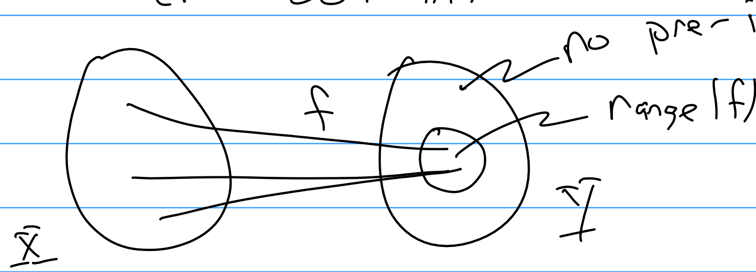
ex.) Is $f(x) = x^2$ one-to-one?

let $x_1, x_2 \in X$ w/ $f(x_1) = f(x_2)$

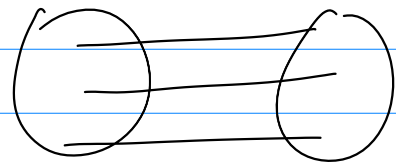
$$x_1^2 = x_2^2 \Rightarrow \pm x_1 = \pm x_2$$

$$\underbrace{x_1 = -x_2 \qquad x_1 = +x_2}_{\text{not one-to-one}}$$

- Onto: A function is onto iff every element in the codomain has a pre-image in the domain.



not onto



X $\qquad Y = \text{range}(f)$ $\qquad$ is onto.

These depend not only on the function but also the domain/codomain.

ex.) $f(x) = 2x$ for $X, Y \in \mathbb{R}$ is onto

but if $X \in \mathbb{Z}$ & $Y \in \mathbb{R}$ then $f(x)$ is <u>not</u> onto

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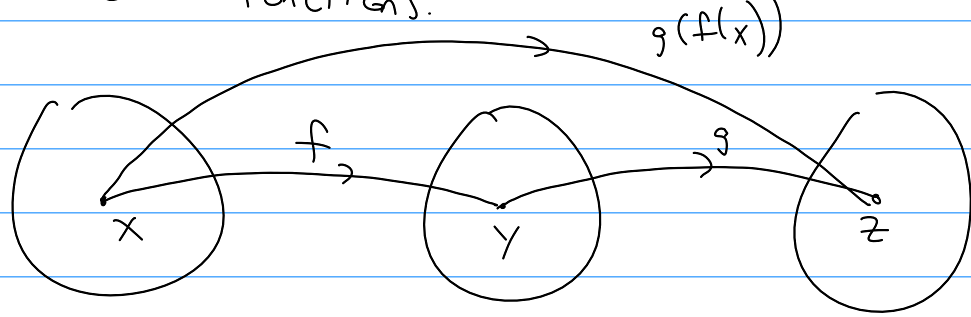
Function Composition

Function Composition: A sequential application of functions.



X $\qquad\qquad Y$ $\qquad\qquad Z$

$\qquad y = f(x)$ $\qquad z = g(y)$

A composition is $z = g(y) = g(\underline{f(x)}) = g\circ f(x)$
write as $g \circ f : X \to Z$

ex.) $f(x) = x+1$ $\quad g(y) = y^2$ $\quad X, Y, Z \in \mathbb{R}$

$$(g \circ f)(x) = g(f(x)) = g(x+1) = (x+1)^2 = Z$$

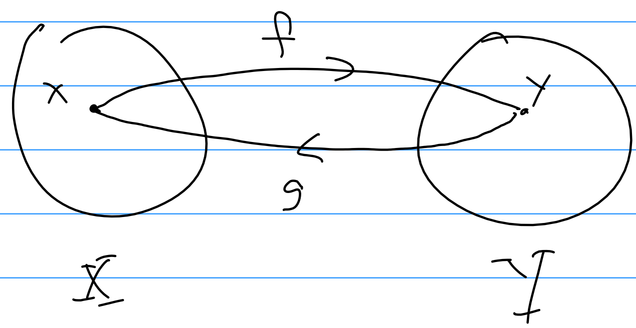
Thm: Let $f: X \to Y$ & $g: Y \to Z$

1) if f & g are both one-to-one then $g \circ f$ is one-to-one

2) if f & g are onto then $g \circ f$ is onto

Function Inverse

Functions $f: X \to Y$ & $g: Y \to X$ are inverses if for any $x \in X$ & $y \in Y$ we have $(g \circ f)(x) = x$ & $(f \circ g)(y) = y$



Thm: $f: X \to Y$ & $g: Y \to X$ are inverses iff both f & g are one-to-one & onto.

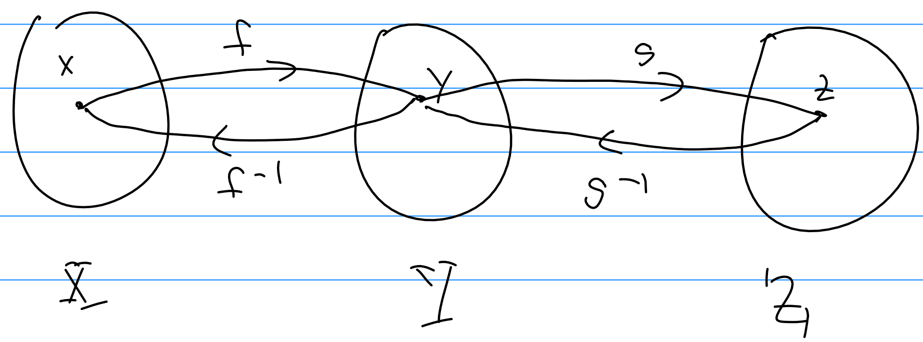
ex.) $f(x) = x^2$ $\quad g(y) = y^{1/2}$ $\quad x, y, \in \mathbb{R}$

$(g \circ f)(x) = (x^2)^{1/2} = \pm x$

$\Rightarrow (g \circ f)(+x) = -x$ $\leftarrow$ not inverses

Thm: If $f: X \to Y$ & $g: Y \to Z$ both have inverses $f^{-1}: Y : X$ & $g^{-1}: Z \to Y$ then the inverse of $g \circ f$ is $f^{-1} \circ g^{-1}$

$\Rightarrow (g \circ f)^{-1} = f^{-1} \circ g^{-1}$



Note: $(AB)^{-1} = B^{-1} A^{-1}$

ex.) let $f(x) = x+1$ $\quad g(y) = 10y$ $\quad x, y, z \in \mathbb{R}$

$f^{-1}(x) = y - 1$ $\quad g^{-1}(z) = z/10$

Check: $(f^{-1} \circ f)(x) = \overbrace{(x+1)}^{f(x)} - 1 = x$

$(g^{-1} \circ g)(y) = 1/10(10y) = y$

$(g \circ f)(x) = g(f(x)) = 10\overbrace{(x+1)}^{f(x)} = z$

$(f^{-1} \circ g^{-1})(z) = \underbrace{\frac{z}{10}}_{g^{-1}(z)} - 1$

now $(f^{-1} \circ g^{-1}) \circ (g \circ f) = (f^{-1} \circ g^{-1})(10(x+1))$

$$= \frac{\cancel{10}(x+\cancel{1})}{\cancel{10}} - \cancel{1} = x$$

Linear Transformations

A linear transformation are functions that follow:

1) $f(u+v) = f(u) + f(v)$ for $u, v \in X$
2) $f(au) = a f(u)$ $a \in \mathbb{R}$

For vector:

$f(\underline{u} + \underline{v}) = f(\underline{u}) + f(\underline{v})$

$f(a\underline{u}) = a f(\underline{u})$

Also called linear operators

ex.) Matrix Transpose : $f(\underline{A}) = \underline{A}^T$ $f: M_{mn} \rightarrow M_{nm}$

1) $f(\underline{A} + \underline{B}) = (\underline{A} + \underline{B})^T = \underline{A}^T + \underline{B}^T = f(\underline{A}) + f(\underline{B})$
2) $f(c\underline{A}) = (c\underline{A})^T = c\underline{A}^T = c f(\underline{A})$

$\Rightarrow$ Linear function

ex.) Dot product : $f(\underline{v}) = \underline{v} \cdot \underline{v}$ $\underline{v} \in \mathbb{R}^n$ $f(\underline{v}) \in \mathbb{R}$

1) $f(\underline{v} + \underline{w}) = (\underline{v} + \underline{w}) \cdot (\underline{v} + \underline{w}) = \underline{v} \cdot \underline{v} + \underline{w} \cdot \underline{w} + \underline{v} \cdot \underline{w} + \underline{w} \cdot \underline{v}$

$= f(\underline{v}) + f(\underline{w}) + \underbrace{2\underline{v} \cdot \underline{w}}$

$\neq f(\underline{v}) + f(\underline{w})$

2) $f(c\underline{v}) = (c\underline{v})\cdot(c\underline{v}) = c^2\,\underline{v}\cdot\underline{v} = c^2 f(\underline{v}) \neq c\, f(\underline{v})$

$\Rightarrow$ Not a linear function.