Vectors

A vector is an organized collection of numbers called components. Also called a tuple. The # of components is the the length.

Denote as a lower case letter w/ underline.

$$\underline{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \leftarrow \end{matrix} \qquad \underline{b} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$\text{length}(\underline{a}) = 3$ $\qquad$ $\text{length}(\underline{b}) = 3$

$$\underline{a} \neq \underline{b}$$

let a_i be the i^{th} component of $\underline{a}$.

$a_1 = 1$ $\qquad$ $b_1 = 2$

$a_2 = 2$ $\qquad$ $b_2 = 1$

You can have row & column vectors

$\longleftrightarrow$ $\qquad$ $\updownarrow$

$\underline{a}$ & $\underline{b}$ are column vectors

A vector has a dimension of $m \times n$

$\uparrow$ # of rows $\qquad$ $\uparrow$ # of columns

$\underline{a}$ & $\underline{b}$ are both 3×1

$c = [4 \ 0 \ 2]$ is 1×3 is a row vector

$\underline{a} = \underline{b}$ iff they have the same dimension &
$a_i = b_i$

Note: Zeros count!

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

All real vectors of length-n exist in
in $\mathbb{R}^n$-space

real length

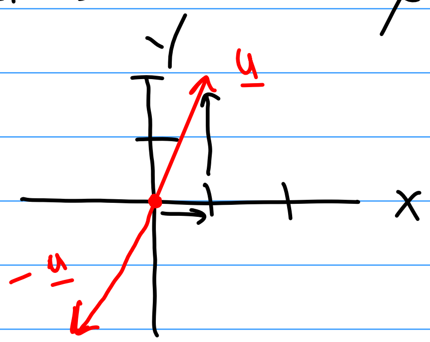
$$\begin{bmatrix} 1 \\ 2 \\ \pi \end{bmatrix} \in \mathbb{R}^{3(\times 1)}$$

Complex #s in $\mathbb{C}^n$

Scalar is $\mathbb{R}^1$ or $\mathbb{R}$

In 2D & 3D vectors have a physical

$$\begin{array}{c} \\ \underline{u} = \end{array} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{array}{c} x \\ y \end{array}$$



Vector Operations

let $\underline{u}, \underline{v} \in \mathbb{R}^3$ $\quad \underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$u_i, v_i \in \mathbb{R}$

1) Addition: $\underline{u} + \underline{v} = \underline{w}$ $\qquad u_i + v_i = w_i$

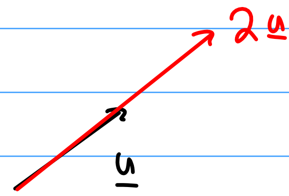
($\underline{u}$ & $\underline{v}$ must have the same dimension)

$$\underline{u} + \underline{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \underline{w}$$

2) Subtraction: $\underline{u} - \underline{v} = \underline{w}$ $\qquad w_i = u_i - v_i$

3) Scalar multiple, let $a \in \mathbb{R}$

$$\underline{w} = a\underline{u} = a \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} au_1 \\ au_2 \\ au_3 \end{bmatrix}$$



4) "Multiplication" of Vectors: Dot product (in a bit)

5) Cross-product: $\underline{u} \times \underline{v} = \underline{w}$: $\underline{w}$ is a vector perpendicular to $\underline{u}$ & $\underline{v}$, the direction of $\underline{w}$ is given by the right-hand rule, magnitude of the area of the parallelogram defined by $\underline{u}$ & $\underline{v}$.

6) Division : Not Defined

$\underline{u}^{-1}$ is not defined.

7) Associativity : $\underline{u} + \underline{v} = \underline{v} + \underline{u}$

$a\underline{u} = \underline{u}\,a$

8) Transpose : "Rotate" 90°

row → column & column → row

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad \underline{u}^{T} = [u_1 \quad u_2 \quad u_3]$$

$$(\underline{u}^T)^T = \underline{u}$$

In Matlab: transpose(u) or u'

9) Special Vectors :

$$\underline{\emptyset} = \left.\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\} n \qquad \underline{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Vector Dot Product

Also called the Inner Product

Operation on 2 vectors of equal length that returns a scalar.

Notation: $\underline{u} \cdot \underline{v} = \langle \underline{u}, \underline{v} \rangle = \underbrace{\underline{u}^T \underline{v}}_{}$

If $\underline{u}$ & $\underline{v}$ are $m \times 1$

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\underline{u}^T \underline{v} = [u_1 \quad u_2] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_1 v_1 + u_2 v_2$$

In general $\underline{u} \cdot \underline{v} = \sum_{i=1}^{n} u_i v_i$

ex.) $\underset{\underline{x}^T}{[1 \quad 2 \quad 5]} \underset{\underline{y}}{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} = 1(-1) + 2(1) + 5(0) = 1$

Operations

1) $(\underline{u} + \underline{v}) \cdot (\underline{w} + \underline{x}) = \underline{u} \cdot \underline{w} + \underline{u} \cdot \underline{x} + \underline{v} \cdot \underline{w} + \underline{v} \cdot \underline{x}$

2) $\underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u}$

3) let $a \in \mathbb{R}$, $a(\underline{u} \cdot \underline{v}) = (a\underline{u}) \cdot \underline{v} = \underline{u} \cdot (a\underline{v})$

Vector Norms

A vector norm tells us how "big" a vector is: How far from $\underline{0}$.

Scalar: Absolute Value

Introduce via $\underline{u} \cdot \underline{u}$

$$\underline{u} \cdot \underline{u} = \sum_{i=1}^{n} u_i^2 = u_1^2 + u_2^2 + \dots + u_n^2$$ ← Like a distance

Use to introduce the 2-norm of $\underline{u}$:

$$\text{2-norm: } \|\underline{u}\|_2 = (\underline{u} \cdot \underline{u})^{1/2} = (u_1^2 + u_2^2 + \dots + u_n^2)^{1/2}$$

↑ 2-norm

ex.) $\underline{u} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix}$

$$\|\underline{u}\|_2 = (1^2 + (-1)^2 + (3)^2 + (2)^2)^{1/2} = \sqrt{15}$$

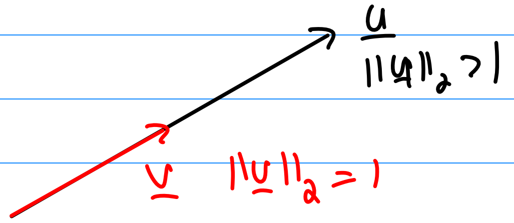
Note: $\underline{u} \cdot \underline{u} = \|\underline{u}\|_2^2 = 15$

A vector w/ $\|\underline{u}\|_2 = 1$ are called unit vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is <u>not</u> a unit vector

If $\underline{u}$ is not a unit vector then a unit vector $\underline{v}$ in the same direction as $\underline{u}$ via $\underline{v} = \frac{\underline{u}}{\|\underline{u}\|_2} = \left(\frac{1}{\|\underline{u}\|_2}\right)\underline{u}$



Proof: $\|\underline{v}\|_2 = (\underline{v} \cdot \underline{v})^{1/2} = \left(\frac{\underline{u}}{\|\underline{u}\|_2} \cdot \frac{\underline{u}}{\|\underline{u}\|_2}\right)^{1/2}$

$$= \left(\frac{\underline{u} \cdot \underline{u}}{\|\underline{u}\|_2^2}\right)^{1/2} = \left(\frac{\|\underline{u}\|_2^2}{\|\underline{u}\|_2^2}\right)^{1/2} = 1$$

P-norm

All norms given by $\|\underline{u}\|_p = \left(\sum_{i=1}^{n} |u_i|^p\right)^{1/p}$

Most common: $\|\underline{u}\|_2$

1-norm: $\|\underline{u}\|_1 = \sum_{i=1}^{n} |u_i|$

$\lim_{p \to \infty} \|\underline{u}\|_p = \|\underline{u}\|_\infty = \max_i |u_i|$

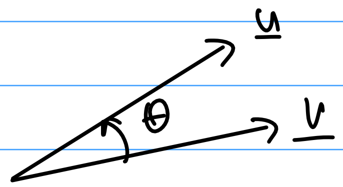
If no norm is specified, it's the 2-norm

$$\|\underline{u}\| = \|\underline{u}\|_2$$

All norms obey:

1) $\|\underline{u}\|_p \geq 0$
2) $\|\underline{u}\|_p = 0$ iff $\underline{u} = \emptyset$
3) $\|\underline{u} + \underline{v}\|_p \leq \|\underline{u}\|_p + \|\underline{v}\|_p$ Triangle Inequality
4) $\|a\,\underline{u}\|_p = |a|\,\|\underline{u}\|_p$

Geometric Interpretation of the Dot Product



$$\underline{u} \cdot \underline{v} = \|\underline{u}\|\,\|\underline{v}\| \cos\theta$$

Prove via Law of Cosines

$$\|\underline{u} - \underline{v}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2 - 2\|\underline{u}\|\,\|\underline{v}\| \cos\theta$$

$$\|\underline{u} - \underline{v}\|^2 = (\underline{u} - \underline{v}) \cdot (\underline{u} - \underline{v}) = \underline{u} \cdot \underline{u} - \underline{u} \cdot \underline{v} - \underline{v} \cdot \underline{u} + \underline{v} \cdot \underline{v}$$
$$= \|\underline{u}\|^2 - 2\,\underline{u} \cdot \underline{v} + \|\underline{v}\|^2$$

$$\Rightarrow \cancel{\|\underline{u}\|^2} + \cancel{\|\underline{v}\|^2} \cancel{-2}\,\underline{u} \cdot \underline{v} = \cancel{\|\underline{u}\|^2} + \cancel{\|\underline{v}\|^2} \cancel{-2}\,\|\underline{u}\|\,\|\underline{v}\| \cos\theta$$

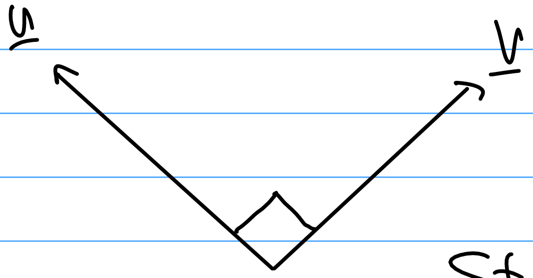
$$\boxed{\underline{u} \cdot \underline{v} = \|\underline{u}\|\,\|\underline{v}\| \cos\theta}$$

What if $\underline{u} \cdot \underline{v} = 0$?

let $\|\underline{u}\| \neq 0$ & $\|\underline{v}\| \neq 0$ (non-trivial case)

$\|\underline{u}\| \, \|\underline{v}\| \cos\theta = 0 = \underline{u} \cdot \underline{v}$

$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2} \; \left(\frac{3\pi}{2}, \text{ etc.}\right)$



$\underline{u} \cdot \underline{v} = 0$ if $\underline{u} \perp \underline{v}$

State that $\underline{u}$ & $\underline{v}$ are orthogonal

Show that $\underline{u} \cdot \underline{v} = 0$ iff $\underline{u} \perp \underline{v}$

Proof: 1) Show that if $\underline{u} \cdot \underline{v} = 0$ then $\underline{u} \perp \underline{v}$ $\to$ Done above

2) If $\underline{u} \perp \underline{v}$ then $\underline{u} \cdot \underline{v} = 0$

$\underline{u} \cdot \underline{v} = \|\underline{u}\| \, \|\underline{v}\| \cos\left(\frac{\pi}{2}\right) = 0$

Cauchy-Schwartz Inequality

Bound on $|\underline{u} \cdot \underline{v}|$

$$|\underline{u} \cdot \underline{v}| = \big| \, \|\underline{u}\| \, \|\underline{v}\| \cos\theta \big| = \|\underline{u}\| \, \|\underline{v}\| \, |\cos\theta|$$

As $|\cos\theta| \leq 1$

$$\Rightarrow |\underline{u} \cdot \underline{v}| = \|\underline{u}\| \, \|\underline{v}\| \, |\cos\theta| \leq \|\underline{u}\| \, \|\underline{v}\|$$

$$\boxed{|\underline{u} \cdot \underline{v}| \leq \|\underline{u}\| \, \|\underline{v}\|}$$

Triangle Inequality

$$\|\underline{u} + \underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$$

$$\|\underline{u} + \underline{v}\|^2 \leq (\|\underline{u}\| + \|\underline{v}\|)^2$$

a) $\|\underline{u} + \underline{v}\|^2 = (\underline{u} + \underline{v}) \cdot (\underline{u} + \underline{v}) = \underline{u} \cdot \underline{u} + \underline{u} \cdot \underline{v} + \underline{v} \cdot \underline{u} + \underline{v} \cdot \underline{v}$

$= \|\underline{u}\|^2 + \|\underline{v}\|^2 + 2 \, \underline{u} \cdot \underline{v}$

b) $(\|\underline{u}\| + \|\underline{v}\|)^2 = (\|\underline{u}\| + \|\underline{v}\|)(\|\underline{u}\| + \|\underline{v}\|)$

$= \|\underline{u}\|^2 + \|\underline{v}\|^2 + 2 \, \|\underline{u}\| \, \|\underline{v}\|$

Does, a) $\leq$ b)

$$\cancel{\|\underline{u}\|^2} + \cancel{\|\underline{v}\|^2} + \cancel{2} \, \underline{u} \cdot \underline{v} \;?\; \cancel{\|\underline{u}\|^2} + \cancel{\|\underline{v}\|^2} + \cancel{2} \, \|\underline{u}\| \, \|\underline{v}\|$$

$\underline{u} \cdot \underline{v} \leq \|\underline{u}\| \, \|\underline{v}\|$ via Cauchy-Schwartz

Triangle Inequality holds for all p-norms

$\|\underline{u} + \underline{v}\|_p \le \|\underline{u}\|_p + \|\underline{v}\|_p$

Matlab

norm(u) $\leftarrow$ 2-norm of $\underline{u}$

norm(u,p) $\leftarrow$ p-norm p = a number

norm(u, 'inf') $\leftarrow$ ∞-norm

Unit vector: u./norm(u)

./ : pointwise division

Dot product: dot(u, v)

Linear Combination of Vectors

A linear combination is a weighted sum

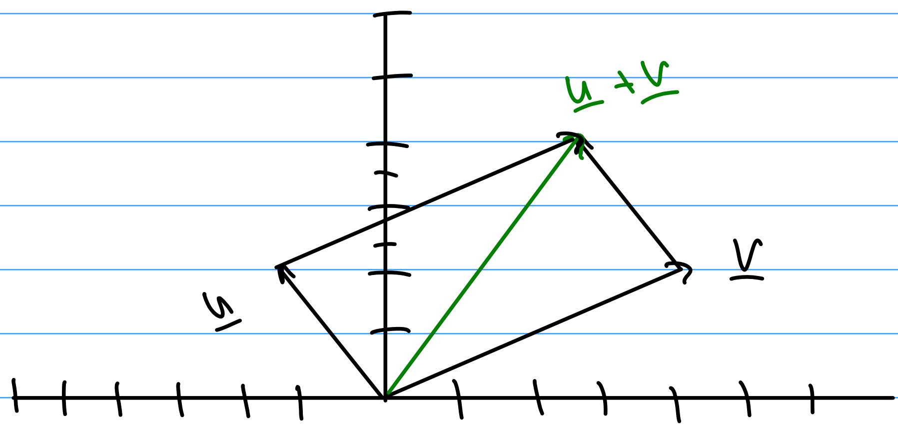
let $\underline{u}, \underline{v}, \underline{w} \in \mathbb{R}^{n \times 1}$ $\quad a, b, c \in \mathbb{R}$

$\underline{z} = a\underline{u} + b\underline{v} + c\underline{w} \in \mathbb{R}^{n \times 1}$

ex.) $\underline{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\quad \underline{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ $\quad a = 2$, $b = 1$

$$a\underline{u} + b\underline{v} = 2\begin{bmatrix} -1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

Graphically "head-to-toe"



Identities

1) $\phi\,\underline{u} + \phi\,\underline{v} = \underline{\phi}$
2) $\phi\,\underline{u} + b\,\underline{v} = b\,\underline{v}$
3) $\underline{u} + \underline{u} = 2\,\underline{u}$
4) $\underline{u} - \underline{u} = \underline{0}$

ex.) let $\underline{u} = \begin{bmatrix} 9 \\ 0 \\ 3 \end{bmatrix}$ $\underline{v} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

find a, b, c such that

$$a\underline{u} + b\underline{v} = \begin{bmatrix} 7 \\ 2 \\ c \end{bmatrix}$$

$$a \begin{bmatrix} 9 \\ 0 \\ 3 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ c \end{bmatrix} \begin{matrix} \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \\ \leftarrow \text{row } 3 \end{matrix}$$

1: $9a - b = 7 \Rightarrow 9a - 2 = 7 \Rightarrow a = 1$

2: $0a + b = 2 \Rightarrow b = 2$

3: $3a + 2b = c \Rightarrow c = 7$

$$\begin{bmatrix} 9 \\ 0 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 7 \end{bmatrix}$$

ex.) find a, b, c such that

$$a \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$\underline{a}$	$\underline{b}$	$\underline{c}$
0	0	1
1	1	0
-3	-3	4
$\vdots$	$\vdots$	$\vdots$

Infinite # of a, b, c

Next time:

Show that a matrix : $\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} = \underline{A}$

is simply $[\underset{\substack{\uparrow \\ \text{a column} \\ \text{vector}}}{\underline{u}} \quad \underline{v} \quad \underline{w}]$

$$\underset{\substack{\uparrow \\ \text{a column vector}}}{\underline{A} \; \underline{x}} = [\underline{u} \quad \underline{v} \quad \underline{w}] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \underline{u} + x_2 \underline{v} + x_3 \underline{w}$$

when Does $\underline{x}$ exist such that $\underline{A}\,\underline{x} = \underline{b}$?