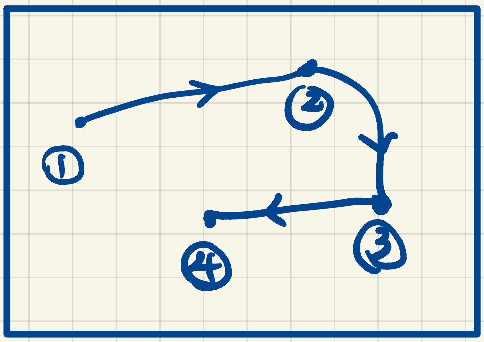
(Vector) Spaces

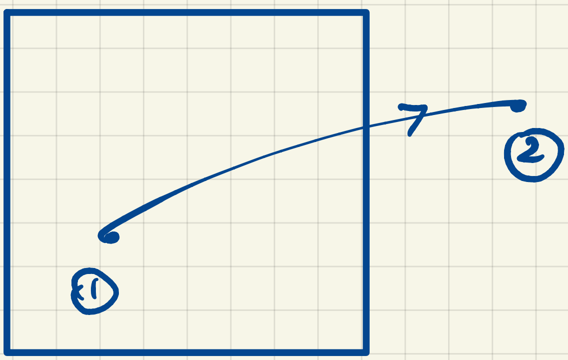
A space is a collection of similar objects that follow a set of rule.

The objects must obey "inside-remain-inside"

$\Rightarrow$ no operation on these objects may remore them from the space.



space



NOT a space

Vector space : A space of vectors.

ex.

polynomial space $\{1, x, x^2, x^3\}$

A vector $\underline{v}$ belongs to a space V

$\Rightarrow \underline{v} \in V$

Space can be finite $\{1, x, x^2\}$

or infinite (Fourier Series)

Rules. (Properties)

Let $\underline{x}$, $\underline{y}$, $\underline{z}$ all exist in the same space

$a, b \in \mathbb{R}^1$

A vector space is closed iff

1. $\underline{x} + \underline{y} = \underline{y} + \underline{x}$ must remain in space

(MRIS)

2. $\underline{x} + (\underline{y} + \underline{z}) = (\underline{x} + \underline{y}) + \underline{z}$ MRIS

3. A unique "zero vector" exist such that

$$\underline{0} + \underline{x} = \underline{x} = \underline{x} + \underline{0}$$

4. For every $\underline{x}$, there is a $-\underline{x}$ such that $\underline{x} + (-\underline{x}) = \underline{0}$

$$= (-\underline{x}) + \underline{x}$$

5. $a(\underline{x} + \underline{y}) = a\underline{x} + a\underline{y}$ MRIS

6. $(a+b)\underline{x} = a\underline{x} + b\underline{x}$ MRIS

7. $a(b\underline{x}) = b(a\underline{x})$ MRIS

8. $1\underline{x} = \underline{x}$ MRIS

ex.

Let V be all vectors of the form

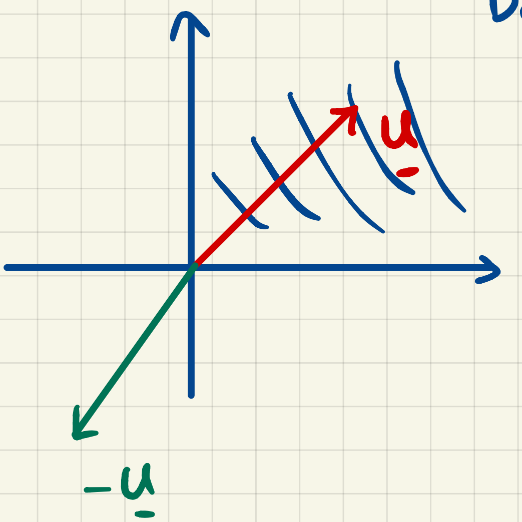
$$\underline{v} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad a, b \geq 0$$

Look $c(\underline{u} + \underline{w})$

w $c = -1$

$\underline{w} = \underline{0}$

$\Rightarrow c\,\underline{u} = -\underline{u}$



Do not remain in the space

$\Rightarrow$ not closed.

thm. Let V be a closed vector space

for any $\underline{v} \in V$ with $a \in \mathbb{R}$

#1. $a\,\underline{0} = \underline{0}$

#2 $0\,\underline{v} = \underline{0}$

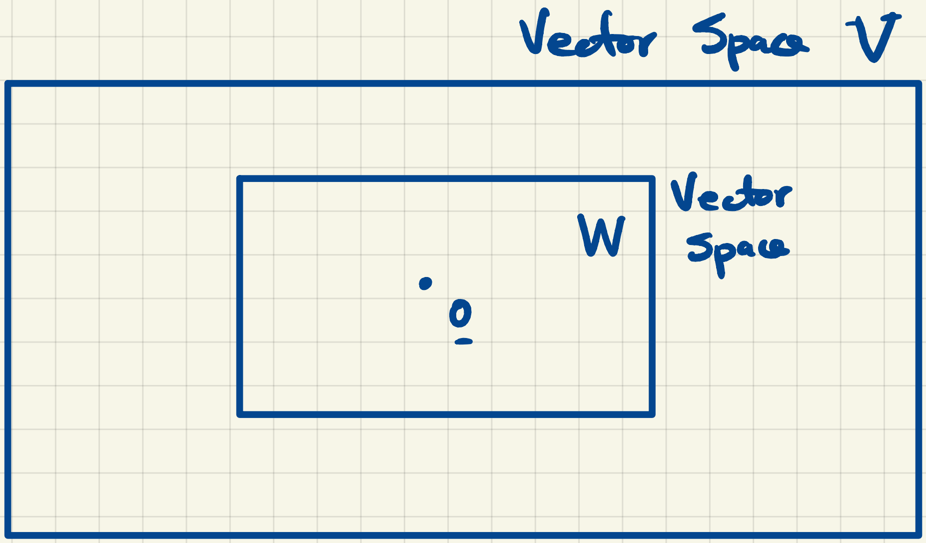
#3 $(-1)\,\underline{v} = -\underline{v}$

#4. if $a\,\underline{v} = \underline{0}$

then ① $a=0$ or ② $\underline{v} = \underline{0}$

Subspaces.

A subspace is a subset of a vector space that is also a vector space.



ex). All vectors in $\mathbb{R}^2$ can be thought of as a subspace of $\mathbb{R}^3$

Formal Def. If W is a subset of vector space V and if W is closed under addition and multiplication then W is a subspace of V.

Closed under addition : if $\underline{u}, \underline{v} \in W$

$\underline{u} + \underline{v} \in W$

" multiplication : if $a \in \mathbb{R}^1$ and

$\underline{u} \in W$

then $a\underline{u} \in W$

If both hold, the subset is

a subspace.

Span.

Let S be a non-empty set of vectors in vector space V.

$$S = \{ \underline{v_1}, \underline{v_2}, \underline{v_3}, \ldots, v_n \} \in V$$

$$\underline{v_i} \neq 0$$

Span of S (Span (S)) is all linear combination of $\underline{v_i}$

$$\text{Span}(S) = a_1 \underline{v_1} + a_2 \underline{v_2} + \cdots + a_n \underline{v_n}$$

$$a_i \in \mathbb{R}^1$$

ex).

Let $S = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$$\text{Span}(S) = a \begin{bmatrix} 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad a, b \in \mathbb{R}^1$$

$\Rightarrow$ All of $\mathbb{R}^2$

all vector in $\mathbb{R}^2$

ex.

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\text{span}(S) = \begin{bmatrix} a \\ 0 \\ 0 \\ b \end{bmatrix} \quad \text{subset of } \mathbb{R}^4$$

Span holds for other objects.

ex.

$$S = \{1, x, x^2\}$$

$$\text{Span}(S) = a(1) + b(x) + c(x^2)$$

Thm. Let S be a non-empty subset of vector space V.

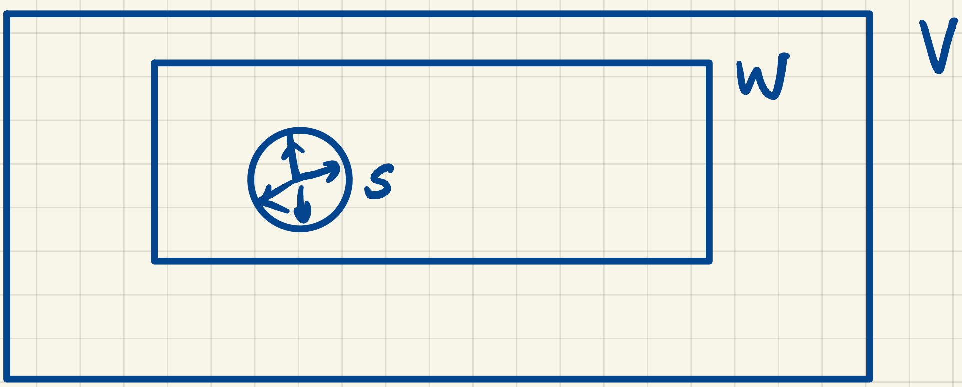
$$S = \{ \underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n \} , \quad \underline{v}_i \in V$$

$$i \in [1, n]$$

① $S \in \text{Span}(S)$

② $\text{Span}(S)$ is a subspace of V.

③ If W is a subspace of V w/ $S \in W$, then $\text{span}(S) \in W$ is a subspace of W.



④ $\text{Span}(S)$ is the smallest subset of V that contains S.

Vector Independence.

Let $\underline{v}_1, \underline{v}_2, \underline{v}_3 \in V$

A set of vectors are (linearly) independent iff the only linear combination of these vectors that results in $\underline{0}$ is the trivial solution.

$$0\underline{v}_1 + 0\underline{v}_2 + 0\underline{v}_3 = \underline{0}$$

If a non-trivial solution exist then the set is dependent.

Orthogonal set

Vectors are orthogonal iff $\underline{u} \cdot \underline{v} = 0$

$$\underline{u} \cdot \underline{v} = 0 \Rightarrow \underline{u} \perp \underline{v}$$

Two vector spaces are orthogonal iff any vector in one subspace is orthogonal to all vectors in the other.

Thm. Let subspace V be the span of vectors $\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}\}$, $\underline{v_i} \neq 0$

Span (S) is an orthogonal set iff $\underline{v_i} \cdot \underline{v_j} = 0$ for $i \neq j$.

Note.

Linear Independence $\neq$ orthogonality

Orthogonality $\Rightarrow$ Linear Inde.

Basis (Basis set)

A set of vectors $B = \{ \underline{b_1}, \underline{b_2}, \cdots, \underline{b_n} \}$ form a basis for vector space V

iff

① B span all of V.

$$a_1 \underline{b_1} + a_2 \underline{b_2} + \cdots + a_n \underline{b_n} = V$$

② B must be linear independent

↑

B contains the minimum # of vectors necessary to span all of V.

ex. $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \Rightarrow$ basis of $\mathbb{R}^3$

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \cancel{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} \right\}$$

not a basis

Note. coordinate.

$$\left\{ \overset{b_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \overset{b_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right\}$$

Different

$$\left\{ \overset{b_3}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}, \overset{b_4}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \right\}$$

Order matters.

$$\begin{bmatrix} 9 \\ 2 \end{bmatrix} = 9\,\underline{b}_1 + 2\,\underline{b}_2 \Rightarrow (9,2)$$

$$\begin{bmatrix} 9 \\ 2 \end{bmatrix} = 2\,\underline{b}_3 + 9\,\underline{b}_4 \Rightarrow (2,9)$$

Preview: the basis set for the columns of matrix is tied to inverse of the matrix.