Algorithm 2.3: NEWTON'S METHOD

To find a solution to $f(x) = 0$ given an initial approximation p_0:

INPUT initial approximation p_0; tolerance TOL; maximum number of iterations N_0.

OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 1$.

Step 2 While $i \leq N_0$ do Steps 3–6.

    Step 3 Set $p = p_0 - f(p_0)/f'(p_0)$. (*Compute* p_i.)

    Step 4 If $|p - p_0| < TOL$ then

        OUTPUT (p); (*The procedure was successful.*)

        STOP.

    Step 5 Set $i = i + 1$.

    Step 6 Set $p_0 = p$. (*Update* p_0.)

Step 7 OUTPUT ('The method failed after N_0 iterations, $N_0 =$', N_0);

    (*The procedure was unsuccessful.*)

    STOP.

Newton's Illustration

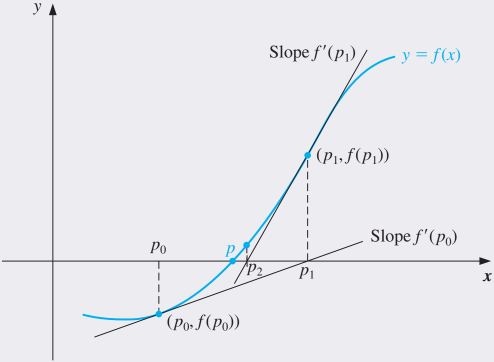

Figure: Figure 2.7

This YouTube video developed by MIT Open Courseware can serve as a good illustration of the Newton's Method for students. Newton's Method Video

Theorem (2.6)

Let $f \in C^2[a, b]$. If $p \in (a, b)$ is such that $f(p) = 0$ and $f'(p) \neq 0$, then there exists a $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any initial approximation $p_0 \in [p - \delta, p + \delta]$.

Algorithm 2.4: SECANT METHOD

To find a solution to $f(x) = 0$ given initial approximations p_0 and p_1:

INPUT initial approximations p_0, p_1; tolerance TOL; maximum number of iterations N_0.
OUTPUT approximate solution p or message of failure.

Step 1 Set $i = 2$;
$q_0 = f(p_0)$;
$q_1 = f(p_1)$.

Step 2 While $i \le N_0$ do Steps 3–6.

Step 3 Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$. (*Compute p_i.*)

Step 4 If $|p - p_1| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.

Step 5 Set $i = i + 1$.

Step 6 Set $p_0 = p_1$; (*Update p_0, q_0, p_1, q_1.*)
$q_0 = q_1$;
$p_1 = p$;
$q_1 = f(p)$.

Step 7 OUTPUT ('The method failed after N_0 iterations, $N_0 =$', N_0);
(*The procedure was unsuccessful.*)
STOP.

Secant Illustration

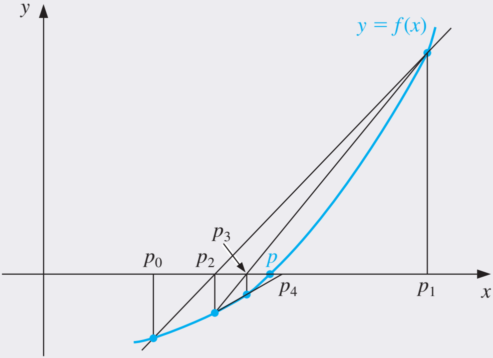

Figure: Figure 2.9

Chapter 2.3: Solutions: Secant Method

This YouTube video developed by Oscar Veliz can serve as a good illustration of the Secant Method for students.

Secant Method Video

Algorithm 2.5: FALSE POSITION

To find a solution to $f(x) = 0$ given the continuous function f on the interval $[p_0, p_1]$ where $f(p_0)$ and $f(p_1)$ have opposite signs:

INPUT initial approximations p_0, p_1; tolerance TOL; maximum number of iterations N_0.
OUTPUT approximate solution p or message of failure.
Step 1 Set $i = 2$;
$q_0 = f(p_0)$;
$q_1 = f(p_1)$.
Step 2 While $i \leq N_0$ do Steps 3–7.
Step 3 Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$. (*Compute* p_i.)
Step 4 If $|p - p_1| < TOL$ then
OUTPUT (p); (*The procedure was successful.*)
STOP.
Step 5 Set $i = i + 1$;
$q = f(p)$.
Step 6 If $q \cdot q_1 < 0$ then set $p_0 = p_1$;
$q_0 = q_1$.
Step 7 Set $p_1 = p$;
$q_1 = q$.
Step 8 OUTPUT ('Method failed after N_0 iterations, $N_0 =$', N_0);
(*The procedure unsuccessful.*)
STOP.

This YouTube video developed by Jacob Bishop can serve as a good illustration of the False Position Method for students.

▸ False Position Method Video

Secant - Method of False Position Illustration

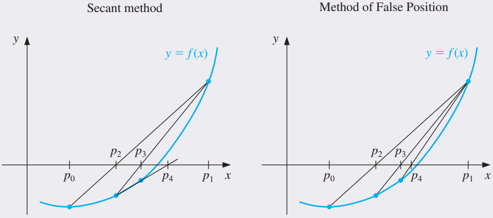

Figure: Figure 2.10

2.3 Newton's method and its extensions.

Newton's method.

Remind that the objective is to approximate solution of

$$f(x) = 0.$$

Let p is a root meaning that $f(p) = 0$. Assume that $f \in C^2[a, b]$ and let $p_0 \in [a, b]$ be an approximation to p such that $|p - p_0|$ is "small". Consider the Taylor expansion for $f(x)$ in the vicinity p_0 that uses the first Taylor polynomial:

$$f(x) = f(p_0) + f'(p_0)(x - p_0) + \frac{f''(\theta(x))}{2}(x - p_0)^2. \qquad (1)$$

Here $\theta(x)$ lies between x and p_0. Let $x = p$. Then (1) turns into

$$f(p) = f(p_0) + f'(p_0)(p - p_0) + \frac{f''(\theta(p))}{2}(p - p_0)^2 \qquad (1a).$$

Since $f(p) = 0$, (1a) becomes

$$0 = f(p_0) + f'(p_0)(p - p_0) + \frac{f''(\theta(p))}{2}(p - p_0)^2 \qquad (2)$$

If the third (quadratic) term in (2) is treated as negligible, then (2) can be written as

$$0 \approx f(p_0) + f'(p_0)(p - p_0) \qquad (2a).$$

Solve (2a) for :

$$p \approx p_0 - \frac{f(p_0)}{f'(p_0)} \equiv p_1 \qquad (3).$$

Equation (3) sets the stage for the Newton's method: start at some p_0 and generate the sequence $\{p_n\}_{n=0}^{\infty}$ by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})},\ n \geq 1\ . \qquad (4).$$

An algorithm implementing iterative procedure (4) needs a stopping rule.

Typically, one of the three following inequalities serves as a stopping rule:

$$|p_n - p_{n-1}| < \varepsilon \qquad (5a)$$

$$\left|\frac{p_n - p_{n-1}}{p_n}\right| < \varepsilon \qquad (5b)$$

$$|f(p_n)| < \varepsilon \qquad (5c).$$

Q. Does the Newton's method always converge to the root?

A. If p_0 is not sufficiently close to p, the Newton's method may not converge to the root.

Theorem 2.6 presents sufficient conditions for convergence.

#5(b). Use Newton's method to find solution accurate to within 10^{-4}.

$$x^3 + 3x^2 - 1 = 0 \quad [-3, -2]$$

$f(x) = x^3 + 3x^2 - 1.$

$$f(-3) = -27 + 27 - 1 < 0; f(-2) = -8 + 12 - 1 > 0$$

$$p_0 = -3; \; p_3 = -2.87939$$

#6(d). Use Newton's method to find solution accurate to within 10^{-5}.

$(x-2)^2 - \ln x = 0$ for $1 \le x \le 2$ and $e \le x \le 4$.

$$p_0 = 1; \; p_4 = 1.412391$$

$$p_0 = 4; \; p_5 = 3.057104$$

With the secant method (#8(d)) using the endpoints of the interval

$$p_8 = 1.412391; \quad p_7 = 3.057104.$$

#15. The equation $4x^2$-$e^x - e^{-x} = 0$ has two positive solutions x_1 and x_2 . Use Newton's method to approximate the solutions to within 10^{-5} with the following values of p_0.

(a) $p_0 = -10 \quad p_{11} = -4.30624527$
(b) $p_0 = -5 \quad p_5 = -4.30624527$

(f) $p_0 = 1 \quad p_4 = 0.824498585$

(g) $p_0 = 3 \quad p_5 = -0.824498585$

$(h)\ \ p_0 = 5 \quad p_5 = 4.30624527.$

The secant method.

By definition of the derivative,

$f'(p_{n-1}) = \lim_{x \to p_{n-1}} \frac{f(x) - f(p_{n-1})}{x - p_{n-1}}.$

If p_{n-2} is close to p_{n-1}, then $f'(p_{n-1}) \approx \frac{f(p_{n-2}) - f(p_{n-1})}{p_{n-2} - p_{n-1}}.$

Using this approximation for $f'(p_{n-1})$ in Newton's formula

$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$ leads to

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-2} - p_{n-1})}{f(p_{n-2}) - f(p_{n-1})} = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}$$

$Start\ with\ p_0\ and\ p_{1.}$ Unlike Newton's method, the secant method requires a single function evaluation at each step.

Tolerance is the bound on $|p_n - p_{n-1}|$ (this is $|p - p_1|$ in Step 4 of Algorithm 2.4).

Newton's method and secant method are often used to refine an answer obtained with another technique, such as bisection method. Both methods, Newton's and secant, require good first approximation but generally give rapid convergence. Generally, the convergence of the secant method is faster than functional iteration but slower than Newton's method.

Examples illustrating the method of false position in a greater detail could be discussed in recitation.