Linear Transformation

A linear transformation $f: X \to Y$

1) $f(x_1 + x_2) = f(x_1) + f(x_2)$ $\quad x_1, x_2 \in X$
2) $f(a x_1) = a f(x_1)$ $\quad a \in \mathbb{R}$

Thm: let V & W be two vector spaces with $L: V \to W$ is a linear transformation. let $\underline{0}_V$ be the zero-vector in V & $\underline{0}_W$ be the zero-vector in W.

1) $L(\underline{0}_V) = \underline{0}_W$

2) $L(-\underline{v}) = -L(\underline{v})$ for all $\underline{v} \in V$

3) $L(a_1 \underline{v}_1 + a_2 \underline{v}_2 + \dots + a_n \underline{v}_n) = a_1 L(\underline{v}_1) + \dots + a_n L(\underline{v}_n)$ $\quad \underline{v}_i \in V$

Proofs:

1) $L(\underline{0}_V) = L(0\, \underline{0}_V) = 0\, L(\underline{0}_V) = \underline{0}_W$

2) $L(-\underline{v}) = L((-1)\underline{v}) = (-1) L(\underline{v}) = -L(\underline{v})$

3) $L(a_1 \underline{v}_1 + (a_2 \underline{v}_2 + \dots + a_n \underline{v}_n)) = L(a_1 \underline{v}_1) + L(a_2 \underline{v}_2 + \dots + a_n \underline{v}_n)$

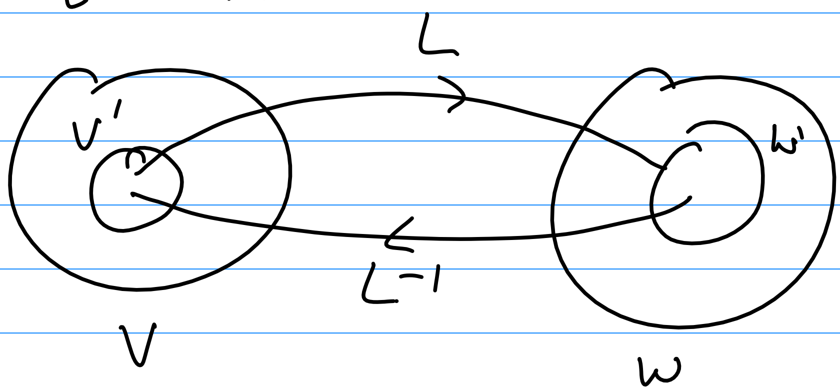
$= a_1 L(\underline{v}_1) + L(a_2 \underline{v}_2) + L(a_3 \underline{v}_3 + \dots + a_n \underline{v}_n)$

$= a_1 L(\underline{v}_1) + a_2 L(\underline{v}_2) + \dots + a_n L(\underline{v}_n)$

Thm: let $L: V \Rightarrow W$ be linear

1) If $V' \in V$ is a subspace of V, then $L(V') = W' \in W$ is a subspace of W

2) If $W' \in W$ is a subspace of W then $L^{-1}(W') = V' \in V$ is a subspace of V, if L^{-1} exists



Actions of Linear Transformations

The action of a linear transformation is captured by the effect of that linear operator on the basis for the domain vector space.

V = Domain w/ basis $= \{\underline{b}_1, \underline{b}_2, \dots, \underline{b}_n\}$

$\Rightarrow \underline{v} \in V$ must obey
$$\underline{v} = k_1 \underline{b}_1 + k_2 \underline{b}_2 + \dots + k_n \underline{b}_n$$

k_i = "coordinates" in the basis set

Look at $L(\underline{v}) = \underline{w} \in W$, $L: V \Rightarrow W$

$$L(\underline{v}) = L(k_1 \underline{b}_1 + \dots + k_n \underline{b}_n)$$

$$= k_1 L(\underline{b}_1) + k_2 L(\underline{b}_2) + \dots + k_n L(\underline{b}_n)$$

If I know $L(\underline{b}_i)$, I know $L(\underline{v})$

ex.) $\underline{B} = \left\{ \underset{\underline{b}_1}{\begin{bmatrix} 0 \\ 4 \\ 0 \\ 1 \end{bmatrix}}, \underset{\underline{b}_2}{\begin{bmatrix} -2 \\ 5 \\ 0 \\ 1 \end{bmatrix}}, \underset{\underline{b}_3}{\begin{bmatrix} -3 \\ 5 \\ 1 \\ 1 \end{bmatrix}}, \underset{\underline{b}_4}{\begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}} \right\}$

$\text{Span}(B) = \mathbb{R}^4$

let $L: \mathbb{R}^4 \to \mathbb{R}^3$

$L(\underline{b}_1) = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ $L(\underline{b}_2) = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ $L(\underline{b}_3) = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$ $L(\underline{b}_4) = \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix}$

What is $L(\underline{v})$ if $\underline{v} = \begin{bmatrix} -4 \\ 14 \\ 1 \\ 5 \end{bmatrix}$?

First: Determine $k_1 \to k_4$

Use $\text{rref}([\underline{b}_1 \ \underline{b}_2 \ \underline{b}_3 \ \underline{b}_4 : \underline{v}])$

$$\underline{B}\begin{bmatrix}k_1\\k_2\\k_3\\k_4\end{bmatrix} = \underline{v}$$

$$\Rightarrow \underline{v} = \underset{k_1}{(2)}\,\underline{b}_1 + \underset{k_2}{(-1)}\,\underline{b}_2 + \underset{k_3}{(1)}\,\underline{b}_3 + \underset{k_4}{(3)}\,\underline{b}_4$$

$$L(\underline{v}) = k_1 L(\underline{b}_1) + \cdots + k_4 L(\underline{b}_4)$$

$$= 2\begin{bmatrix}3\\1\\2\end{bmatrix} + (-1)\begin{bmatrix}2\\-1\\1\end{bmatrix} + (1)\begin{bmatrix}4\\3\\0\end{bmatrix} + 3\begin{bmatrix}6\\1\\1\end{bmatrix} = \underset{\displaystyle \overset{\|}{L(\underline{v})}}{\begin{bmatrix}18\\9\\0\end{bmatrix}}$$

$$L\left(\begin{bmatrix}\underline{b}_1 & \underline{b}_2 & \underline{b}_3 & \underline{b}_4\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\\k_4\end{bmatrix}\right) = L(\underline{v})$$

$$= \begin{bmatrix}L(\underline{b}_1) & L(\underline{b}_2) & L(\underline{b}_3) & L(\underline{b}_4)\end{bmatrix}\begin{bmatrix}k_1\\k_2\\k_3\\k_4\end{bmatrix} = L(\underline{v})$$

Thm: let $\underline{B} = \{\underline{b}_1, \underline{b}_2, \dots, \underline{b}_n\}$ form a basis for vector space V

let $\underline{w}_1, \underline{w}_2, \dots, \underline{w}_n$ be some n-vectors in W.

There will always be a unique linear transformation $L: V \Rightarrow W$ such that
$L(\underline{b}_1) = \underline{w}_1$,
$L(\underline{b}_2) = \underline{w}_2$
$\vdots$
$L(\underline{b}_n) = \underline{w}_n$

Now look at Matrix-vector products.

$$\underline{A}\,\underline{x} = [\underline{a}_1 \ \underline{a}_2 \ \cdots \ \underline{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \underline{a}_1 + x_2 \underline{a}_2 + \dots + x_n \underline{a}_n = \underline{b}$$

Takes in $\underline{x}$ & returns $\underline{b}$

Is it linear?

$\underline{A}(\underline{x} + \underline{y}) = \underline{A}\,\underline{x} + \underline{A}\,\underline{y}$ $\quad$ $\underline{A}(c\underline{x}) = c\,\underline{A}\,\underline{x}$

let $\underline{B} = \{\underline{b}_1, \underline{b}_2, \underline{b}_3, \underline{b}_4\}$ as before

$\underline{A} = [L(\underline{b}_1) \ \ L(\underline{b}_2) \ \ L(\underline{b}_3) \ \ L(\underline{b}_4)]$

$$\underline{A} = \begin{bmatrix} 3 & 2 & -4 & 6 \\ 1 & -1 & 3 & 1 \\ 2 & 1 & 0 & -1 \end{bmatrix} \qquad \underline{k} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 3 \end{bmatrix}$$

$$\underline{A}\,\underline{k} = \begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 18 \\ 9 \\ 0 \end{bmatrix} = L(\underline{v})$$

Thm: let B be an ordered basis for V & let C be an ordered basis for W. For <u>any</u> linear transformation $L: V \to W$ there exists a matrix such that

$$\underline{A}_{BC}\,[\underline{v}]_B = [L(\underline{v})]_C$$

$\underline{A}_{BC} = (L(\underline{b}_1) \cdots L(\underline{b}_n))$ is the linear operator between basis B & C

$[\underline{v}]_B =$ "coordinates" of $\underline{v}$ in B

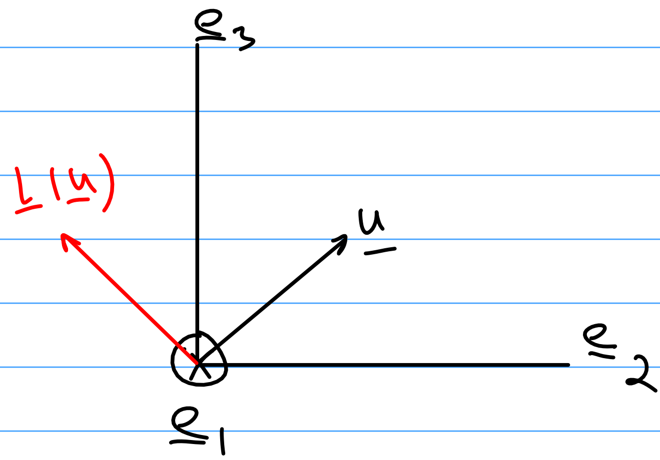
$[L(\underline{v})]_C =$ "coordinates" of $L(\underline{v})$ in C

This takes a vector $\underline{v} \in \mathbb{R}^4$ & uses $\underline{A}_{BC}$ to get $\underline{w} \in \mathbb{R}^3$

Geometric Operations in $\mathbb{R}^3$

let $\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\underline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

1) reflection about $\underline{e}_3$



$$\underline{L} = \left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right) = \begin{bmatrix} a_1 \\ a_2 \\ -a_3 \end{bmatrix} \qquad L : \mathbb{R}^3 \Rightarrow \mathbb{R}^3$$

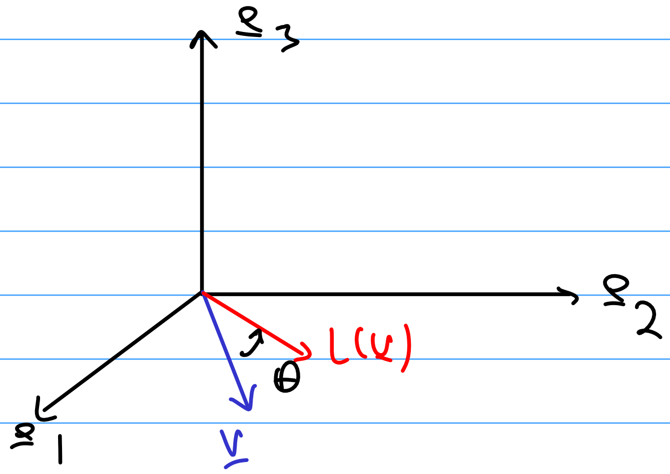
$$\underline{A}_{BC} = [\, L(\underline{e}_1) \;\; L(\underline{e}_2) \;\; L(\underline{e}_3) \,]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \underline{A}_{ref}$$

ex, 1) let $\underline{v} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$ $\quad k_1 = 2$, $k_2 = 4$, $k_3 = -1$ $\quad [\underline{v}]_B = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$

$$\underline{A}_{ref}\, \underline{v} = \begin{bmatrix} 2 \\ 4 \\ +1 \end{bmatrix}$$

2) Rotation about $\underline{e}_3$ (z-axis)

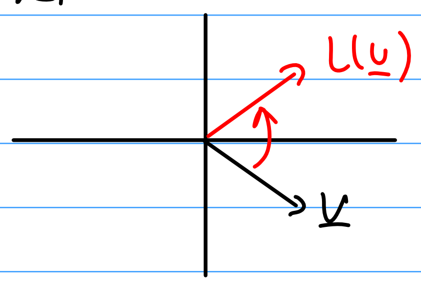


$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 \cos\theta - a_2 \sin\theta \\ a_1 \sin\theta + a_2 \cos\theta \\ a_3 \end{bmatrix}$$

$$\underline{A}_{BC} = \left[L(\underline{e}_1) \;\; L(\underline{e}_2) \;\; L(\underline{e}_3) \right] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \underline{A}_{rot}$$

ex1) $\underline{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ $\qquad \theta = \pi/2$



$$\underline{A} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{A}\,\underline{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(5) Scaling : $L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} c\ a_1 \\ c\ a_2 \\ c\ a_3 \end{bmatrix}$ $c \in \mathbb{R}$

$$\underline{A}_{scale} = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$$

Now, apply rotation, then scaling, then reflection.

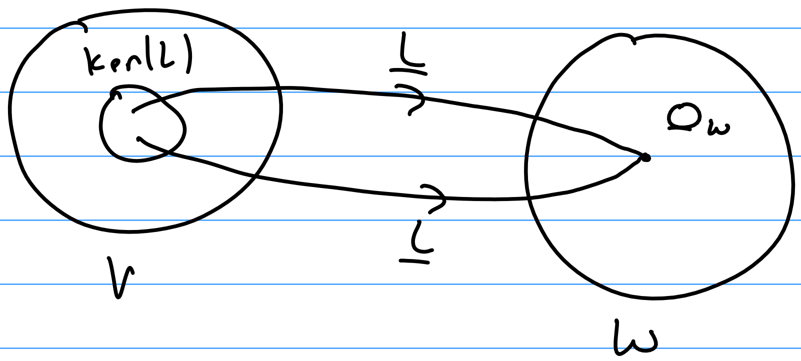
$$\underline{A}_{ref}\left(\underline{A}_{scale}\left(\underline{A}_{rot}\ \underline{v}\right)\right) = \underline{A}\ \underline{v}$$

$$\underline{A} = \begin{bmatrix} c\cos\theta & -c\sin\theta & 0 \\ c\sin\theta & c\cos\theta & 0 \\ 0 & 0 & c \end{bmatrix}$$

Rank, Nullity & Kernel

Let $L: V \Rightarrow W$ be a linear transformation

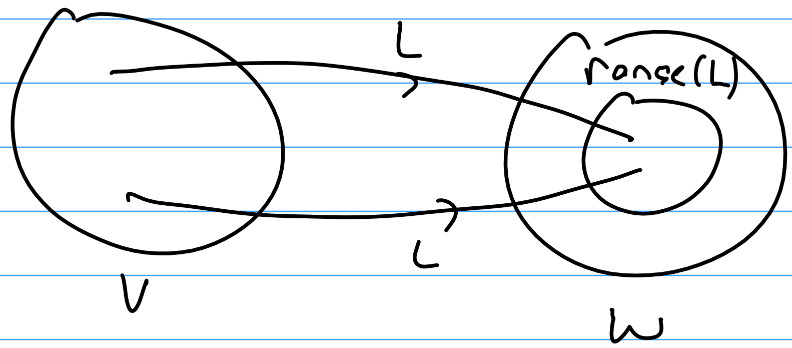
Kernel of L : $\ker(L)$ are all vectors in V that map to $\underline{O}_W$



Nullity of L : the dimension of ker(L)
(the # of vectors needed to span ker(L))

$$\text{nullity}(L) = \dim(\ker(L))$$

Rank of L : the rank of L is the dimension of the range of L



$$\text{rank}(L) = \dim(\text{range}(L)) = \dim(\text{image}(L))$$

Rank-Nullity Theorem
let $L: V \to W$ be a linear transformation

$$\text{rank}(L) + \text{nullity}(L) = \dim(V)$$

ex.) let $L: \mathbb{R}^8 \to \mathbb{R}^2$
nullity(L) = 3 ← need 3 vectors of length = 8 to describe the basis for ker(L)

$\Rightarrow \text{rank}(L) = \dim(V) - \text{nullity}(V) = 8 - 3 = 5$

Matrix Subspaces

4 types:

1) Column space: All possible linear combinations of the columns of $\underline{A}$

2) Nullspace: All vectors $\underline{v}$ such that $\underline{A}\,\underline{v} = \underline{0}$

3) Row space: All combos of the rows

4) Left-Nullspace: All vectors $\underline{v}$ such that $\underline{v}^T \underline{A} = \underline{0}^T$

Column-Space

Recall that $\underline{A}\,\underline{x} = x_1 \underline{a}_1 + x_2 \underline{a}_2 + \cdots + x_n \underline{a}_n = \underline{b}$

$\Rightarrow$ $\underline{b}$ is a linear combination of the columns of $\underline{A}$

$\Rightarrow$ $\underline{b}$ exists in the column (vector) space of $\underline{A}$: $\underline{b} \in C(\underline{A})$

If $\underline{A} \in M_{mn}$ (e.g. $\underline{A} \in \mathbb{R}^{m \times n}$)
$m = \#$ of rows = length of each column

$\Rightarrow$ $C(\underline{A})$ is a subspace of $\mathbb{R}^m$

Important Question: When does $\underline{x}$ exist such that $\underline{A}\,\underline{x} = \underline{b}$ if $\underline{A}$ & $\underline{b}$ are given?

For $\underline{x}$ to exist $\underline{b} \in C(\underline{A})$

If $\underline{b} \in C(\underline{A})$ then at least one solution $\underline{x}$ exists such that $\underline{A}\underline{x} = \underline{b}$

Two Cases:

#1 let $\underline{A} \in \mathbb{R}^{n \times n}$ w/ $\det(\underline{A}) \neq 0$ $\Rightarrow$ $\underline{A}^{-1}$ exists

then $\underline{A}\underline{x} = \underline{b}$ $\quad \underline{b} \in \mathbb{R}^n$

$$\underline{x} = \underline{A}^{-1}\underline{b}$$

$\Rightarrow$ (1) Any $\underline{b}$ is in $C(\underline{A})$

(2) $C(\underline{A})$ spans all of $\mathbb{R}^n$

(3) $\underline{x}$ are the coordinates of $\underline{b}$ in $C(\underline{A})$

#2 let $\underline{A} \in \mathbb{R}^{m \times n}$ w/ $m \neq n$ or $\det(\underline{A}) = 0$

$\Rightarrow$ $\underline{A}^{-1}$ does not exist

For there to be a solution to $\underline{A}\underline{x} = \underline{b}$, $\underline{b}$ must exist in $C(\underline{A})$

$\Rightarrow$ The existence of a solution to $\underline{A}\underline{x} = \underline{b}$ depends on $\underline{A}$ & $\underline{b}$

ex.) $\begin{bmatrix} 1 & 2 & 4 \\ -1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \underline{x} = \begin{bmatrix} 11 \\ -5 \\ 3 \end{bmatrix}$ $\det(\underline{A}) = 0$

$\underline{A} \quad \underline{x} = \underline{b}$

$\text{rref}([\underline{A} : \underline{b}]) = \left[\begin{array}{ccc:c} 1 & 0 & 2 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$ ∞ # of solutions

$\underline{b} = \begin{bmatrix} 11 \\ -5 \\ 0 \end{bmatrix}$, $\text{rref}([\underline{A} : \underline{b}]) = \left[\begin{array}{ccc:c} 1 & 0 & 2 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right]$

no solutions

recall that rref($\underline{A}$) gives the columns that form a basis.

rank(L) = dim of the range of L
= # of vectors in the basis for the range.

$\Rightarrow \dim(C(\underline{A}))$ = # of pivots in rref($\underline{A}$)
= rank($\underline{A}$)

If $\underline{A} \in \mathbb{R}^{m \times n}$ w/ $\underline{A} \neq \underline{0}$ then

$1 \leq \dim(C(\underline{A})) = \text{rank}(\underline{A}) \leq n$

technically $\min(m,n)$

Nullspace

Nullspace $= N(\underline{A}) =$ all vectors that map to $\underline{0}$

let $\underline{A} \in M_{mn}$ (e.g. $\underline{A} \in \mathbb{R}^{m \times n}$)
if $\underline{v} \in N(\underline{A})$ then
$\underline{A}\,\underline{v} = \underline{0}$

$\text{length}(\underline{v}) = n \Rightarrow \underline{v} \in \mathbb{R}^n$

Is it a subspace?
1) $\underline{A}(\underline{v}_1 + \underline{v}_2) = \underline{A}\,\underline{v}_1 + \underline{A}\,\underline{v}_2 = \underline{0}$ if $\underline{v}_1, \underline{v}_2 \in N(\underline{A})$

2) $\underline{A}(c\,\underline{v}_1) = c\,\underline{A}\,\underline{v}_1 = c\,\underline{0} = \underline{0}$ $c \in \mathbb{R}$

$\Rightarrow N(\underline{A})$ is a subspace of $\mathbb{R}^n$.

Note: the zero-vector, $\underline{0}$, always sits,
$\underline{A}\,\underline{0} = \underline{0}$, but it does not count.

$N(\underline{A})$ is only non-zero vectors.

If $\underline{0}$ is the ~~only~~ vector such that
$\underline{A}\,\underline{v} = \underline{0}$ then $N(\underline{A})$ is ~~empty~~

Next: when is $\dim(N(\underline{A})) > 0$

"Solve" $\underline{A}\,\underline{v} = \underline{0}$

Assume $\underline{A}^{-1}$ exists.

$\Rightarrow \underline{v} = \underline{A}^{-1}\underline{0} = \underline{0}$

$\Rightarrow$ If $\underline{A}^{-1}$ exists then $N(\underline{A}) = \{\}$
the empty set

$\Rightarrow \dim(N(\underline{A})) = 0$

$\Rightarrow$ For $\dim(N(\underline{A})) > 0$, $\underline{A}^{-1}$ must <u>not</u> exist

Non-trivial if $\underline{A} \in M_{mn}$ $m \neq n$ or if $\det(\underline{A}) = 0$

Why? Columns of $\underline{A}$ are not linearly independent

If $N(\underline{A}) \neq \{\}$ to set $N(\underline{A})$
solve $\mathrm{rref}([\underline{A} : 0])$