Definition (6.15)

Suppose that A is a square matrix.

(i) If $A = [a]$ is a 1×1 matrix, then $\det A = a$.

(ii) If A is an $n \times n$ matrix, with $n > 1$ the **minor** M_{ij} is the determinant of the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the ith row and jth column of the matrix A.

(iii) The **cofactor** A_{ij} associated with M_{ij} is defined by $A_{ij} = (-1)^{i+j} M_{ij}$.

(iv) The **determinant** of the $n \times n$ matrix A, when $n > 1$, is given either by

$$\det A = \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}, \quad \text{for any } i = 1, 2, \ldots, n,$$

or by

$$\det A = \sum_{i=1}^{n} a_{ij} A_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}, \quad \text{for any } j = 1, 2, \ldots, n.$$

Theorem (6.16)

Suppose A is an $n \times n$ matrix:

(i) *If any row or column of A has only zero entries, then* $\det A = 0$.

(ii) *If A has two rows or two columns the same, then* $\det A = 0$.

(iii) *If $\tilde{A}$ is obtained from A by the operation $(E_i) \leftrightarrow (E_j)$, with $i \neq j$, then* $\det \tilde{A} = -\det A$.

(iv) *If $\tilde{A}$ is obtained from A by the operation $(\lambda E_i) \rightarrow (E_i)$, then* $\det \tilde{A} = \lambda \det A$.

(v) *If $\tilde{A}$ is obtained from A by the operation $(E_i + \lambda E_j) \rightarrow (E_i)$ with $i \neq j$, then* $\det \tilde{A} = \det A$.

(vi) *If B is also an $n \times n$ matrix, then* $\det AB = \det A \det B$.

(vii) $\det A^t = \det A$.

(viii) *When A^{-1} exists,* $\det A^{-1} = (\det A)^{-1}$.

(ix) *If A is an upper triangular, lower triangular, or diagonal matrix, then* $\det A = \prod_{i=1}^{n} a_{ii}$.

Theorem (6.17)

The following statements are equivalent for any $n \times n$ matrix A:

(i) *The equation $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$.*

(ii) *The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for any n-dimensional column vector $\mathbf{b}$.*

(iii) *The matrix A is nonsingular; that is, A^{-1} exists.*

(iv) *$\det A \neq 0$.*

(v) *Gaussian elimination with row interchanges can be performed on the system $A\mathbf{x} = \mathbf{b}$ for any n-dimensional column vector $\mathbf{b}$.*

Corollary (6.18)

Suppose that A and B are both $n \times n$ matrices with either $AB = I$ or $BA = I$. Then $B = A^{-1}$ (and $A = B^{-1}$).

Ex. $\underset{3\times 2}{A}=\begin{bmatrix} 1 & 2 \\ -3 & 5 \\ 6 & 9 \end{bmatrix}$ $\quad \underset{2\times 3}{A'}=\begin{bmatrix} 1 & -3 & 6 \\ 2 & 5 & 9 \end{bmatrix}$

6.4. The determinant of a matrix.

Def. 6.15. Suppose that A is a square matrix

(i) If $\underset{1\times 1}{A}=[a]$ then $\det A = a$

(ii) If A is $n\times n$ matrix with $n>1$, the minor M_{ij} is the determinant of the $(n-1)\times(n-1)$ matrix obtained by deleting the ith row and j-th column of the matrix A.

(iii) The cofactor A_{ij} associated with M_{ij} is $A_{ij}=(-1)^{i+j}M_{ij}$.

(iv) The determinant of the $n\times n$ matrix A $(n>1)$ is given either by

$$\det A = \sum_{j=1}^{n} a_{ij}A_{ij} = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}\cdot M_{ij} \quad \text{for any } i=1,2,\ldots,n$$

or by

$$\det A = \sum_{i=1}^{n} a_{ij}A_{ij} = \sum_{i=1}^{n}(-1)^{i+j}a_{ij}M_{ij} \quad \text{for any } j=1,2,\ldots,n.$$

The calculation of $\det A$ by def. 6.15 requires $O(n!)$ multiplications/divisions and additions/

subtractions.

The calculation results in the same value regardless of which row/column is chosen. Usually, selecting a row/column with most zeros is beneficial.

#1(a) Compute the determinant.

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}$$

Select first row:

$$\det A = 4 \cdot \det\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} + 1 \cdot \det\begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} =$$

$$4 \cdot 1 \cdot 3 + (2 \cdot 2 - 1 \cdot 2) = 12 + 4 - 2 = 14$$

Properties of determinants (A is an $n \times n$ matrix).

(i) If any row or column has only zero entries, then $\det A = 0$.

(ii) If two rows or two columns are the same, then $\det A = 0$

(iii) If $\tilde{A}$ is obtained from A by $E_i \leftrightarrow E_j$ with $i \neq j$, then $\det \tilde{A} = -\det A$

(iv) If $\tilde{A}$ is obtained from A by $\lambda E_i \to E_i$ then $\det \tilde{A} = \lambda \det A$

(v) If $\hat{A}$ is obtained from A by $E_i + \lambda E_j \to E_i$ with $i \neq j$, then $\det \hat{A} = \det A$.

(vi) If B is an $n \times n$ matrix, then $\det(AB) = \det A \cdot \det B$

(vii) $\det A' = \det A$

(viii) If A^{-1} exists, then $\det(A^{-1}) = (\det A)^{-1}$.

(ix) If A is an upper-triangular, lower-triangular, or diagonal matrix, then $\det A = \prod_{i=1}^{n} a_{ii}$.

Apply (ix) to #1(a), together with (iii), (iv), (v):

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix} \xrightarrow[(1)]{E_2 - \frac{1}{2}E_1 \to E_2} \begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 2 & 2 & 3 \end{bmatrix} \xrightarrow[(2)]{E_3 - \frac{1}{2}E_1 \to E_3} \begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 2 & \frac{5}{2} \end{bmatrix}$$

$$\xrightarrow[(3)]{E_3 - 2E_2 \to E_3} \begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{7}{2} \end{bmatrix}.$$

Each (1), (2), (3) didn't change $\det A$ (since this requires property (v) only).

$$\det \begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{7}{2} \end{bmatrix} = 4 \cdot 1 \cdot \frac{7}{2} = 14.$$

Thm 6.17. The following statements are equivalent for any $n\times n$ matrix A:

(i) The equation $A\vec{x}=0$ has a unique solution $\vec{x}=\vec{0}$.

(ii) The system $A\vec{x}=\vec{b}$ has a unique solution for any n-dimensional vector $\vec{b}$.

(iii) The matrix A is nonsingular, that is, A^{-1} exists.

(iv) $\det A \neq 0$.

(v) Gaussian elimination with row interchanges can be performed on the system $A\vec{x}=\vec{b}$ for any n-dimensional column vector $\vec{b}$.

Corollary 6.18. Suppose that both A and B are $n\times n$ matrices with either $AB=I$ or $BA=I$. Then $B=A^{-1}$ (and $A=B^{-1}$).

#5.

$$A=\begin{bmatrix} 1 & -1 & \alpha \\ 2 & 2 & 1 \\ 0 & \alpha & -\frac{3}{2} \end{bmatrix}$$

$$\det A = 1\cdot\det\begin{bmatrix} 2 & 1 \\ \alpha & -\frac{3}{2}\end{bmatrix} - 2\cdot\det\begin{bmatrix} -1 & \alpha \\ \alpha & -\frac{3}{2}\end{bmatrix} =$$

$$2\cdot\left(-\tfrac{3}{2}\right)-\alpha\cdot 1 - 2\left((-1)\left(-\tfrac{3}{2}\right)-\alpha^2\right) =$$

$$-3-\alpha-2\cdot\tfrac{3}{2}+2\alpha^2 = 2\alpha^2-\alpha-6$$

$$2\alpha^2-\alpha-6=0.$$

$\alpha_1=2 \quad \alpha_2=-\frac{3}{2}$

Ex. Find $\det A$

$$A = \begin{bmatrix} 1 & 2 & -4 \\ 0 & 2 & 3 \\ -5 & 1 & 4 \end{bmatrix}$$

$$M_{11} = \det \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$A_{11} = (-1)^2 \cdot M_{11} = M_{11}$$

$$M_{12} = \det \begin{bmatrix} 0 & 3 \\ -5 & 4 \end{bmatrix}$$

$$M_{13} = \det \begin{bmatrix} 0 & 2 \\ -5 & 1 \end{bmatrix}$$

$$\det A = 1 \cdot \det \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} + (-1) \cdot 2 \cdot \det \begin{bmatrix} 0 & 3 \\ -5 & 4 \end{bmatrix} + (-4) \cdot \det \begin{bmatrix} 0 & 2 \\ -5 & 1 \end{bmatrix} =$$

$$= 1 \cdot \underbrace{\left(1 \cdot 2 \cdot 4 + (-1)^{1+2} \cdot 3 \cdot 1\right)} - 2 \cdot (0 + 15) - 4 \cdot (0 + 10) =$$

$$5 - 30 - 40 = -65$$

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

$$\det A = 2 \cdot \det \begin{bmatrix} 1 & -4 \\ -5 & 4 \end{bmatrix} - 3 \cdot \det \begin{bmatrix} 1 & 2 \\ -5 & 1 \end{bmatrix} =$$

$$2 \cdot (4 - 20) - 3 \cdot (1 + 10) = -32 - 33 = -65$$