Linear System

A linear system is one where there are multiple unknowns of polynomial of order-1

let a_i & b known

Linear eq.: $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$

A nonlinear equation is everything else.

$$a_1 x_1 + a_2 x_2^2 + a_3 \sin(x_3) = b$$
$$a_1 x_1 x_2 + a_3 x_3 = b$$

A system of linear equations: multiple coupled linear equations

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$$

$$a_{22} x_2 + a_{23} x_3 = b_2$$

$$a_{33} x_3 = b_3$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\underset{\substack{\Downarrow \\ \text{system matrix}}}{\underline{\underline{A}}} \quad \underset{\substack{\Downarrow \\ \text{unknown}}}{\underline{x}} = \underset{\substack{\Downarrow \\ \text{RHS}}}{\underline{b}}$$

$$\Rightarrow \underline{\underline{A}}^{-1} \underline{\underline{A}} \, \underline{x} = \underline{\underline{A}}^{-1} \underline{b}$$

$$\Rightarrow \underline{x} = \boxed{\underline{\underline{A}}^{-1}} \underline{b}$$

If $\underline{x}$ uniquely exist, $\underline{\underline{A}}^{-1}$ exists

$\Rightarrow \det(\underline{\underline{A}}) \neq 0$

If $\det(\underline{\underline{A}}) = 0$, $\underline{\underline{A}}^{-1}$ does not exist

but there might still be a $\underline{x}$ such that $\underline{\underline{A}}\,\underline{x} = \underline{b}$

Goal: Given $\underline{\underline{A}}$ and $\underline{b}$, is there an $\underline{x}$ such that $\underline{\underline{A}}\,\underline{x} = \underline{b}$ and is this $\underline{x}$ unique?

Existness and Uniqueness

Look at
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\left.\begin{aligned} a_{11} x_1 + a_{12} x_1 &= b_1 \\ a_{21} x_1 + a_{22} x_2 &= b_2 \end{aligned}\right\} \text{ linear in 2 D}$$

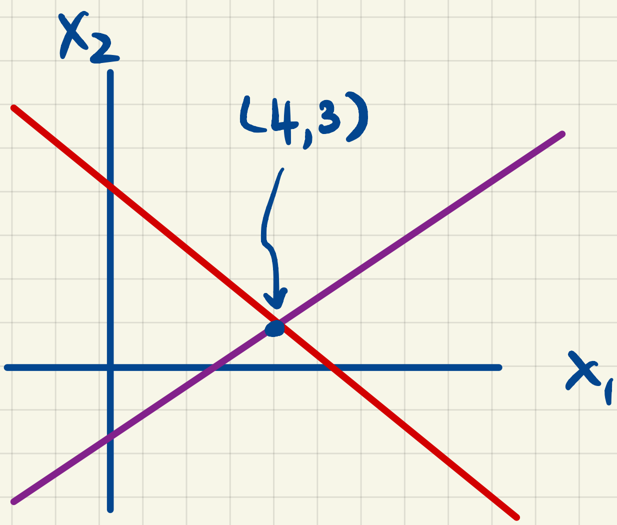
①

$-x_1 + 2x_2 = 2$

$3x_1 + 2x_2 = 18$

$\det(\underline{\underline{A}}) = -8$

$\Rightarrow \underline{\underline{A}}^{-1}$ exists



$\underline{\underline{A}}\,\underline{x} = \underline{b} \Rightarrow \underline{x} = \underline{\underline{A}}^{-1}\underline{b}$

$= \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

one sol.

②

$-\frac{1}{2}x_1 + x_2 = 1$

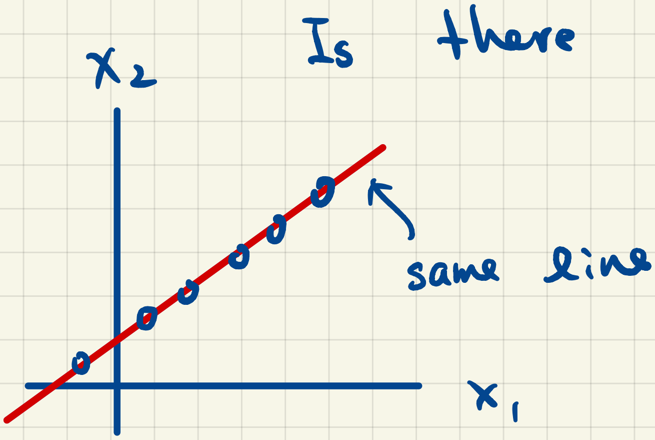
$-x_1 + 2x_2 = 2$

$\Rightarrow \begin{vmatrix} -\frac{1}{2} & 1 \\ -1 & 2 \end{vmatrix} = 0$

$\underline{\underline{A}}^{-1}$ does not exist

Is there any $\underline{x}$ such that

$\underline{\underline{A}}\,\underline{x} = \underline{b}$?



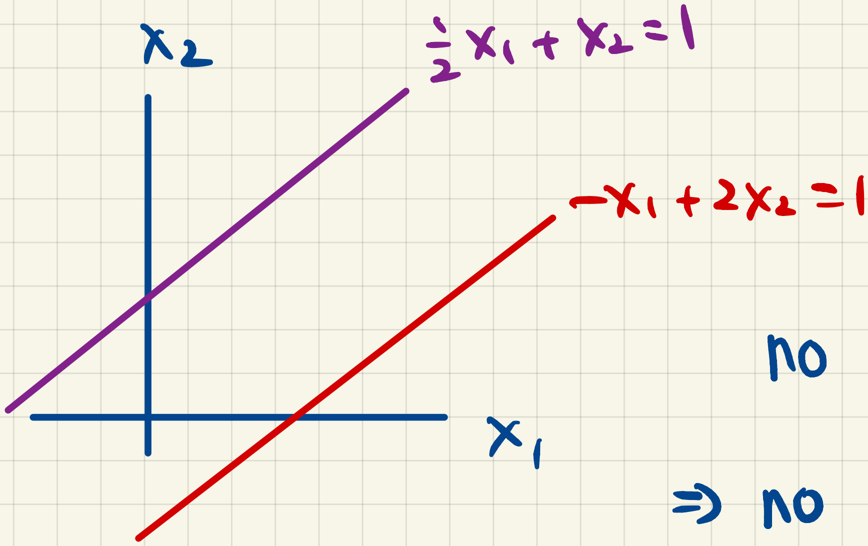
∞ # of sol.

③

$$-\frac{1}{2}x_1 + x_2 = 1$$

$$\Rightarrow \underline{\underline{A}}^{-1} \text{ does not exist}$$

$$-x_1 + 2x_2 = 1$$



no crossing

$\Rightarrow$ no solutions.

Only Possibilities

① one unique sol.

② infinite # of sol.

③ no sol.

Gaussian Elimination

(whatever we talked about is clas is NOT an AI violation)

Also called row reduction

write $\underline{\underline{A}}\,\underline{x} = \underline{b}$ as an augmented matrix

$$\underline{\underline{A}}\,\underline{x} = \underline{b} \;:\; [\, A \;\vdots\; \underline{b} \,]$$

$$\left[\begin{array}{cc:c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right]$$

You can

1. swap rows
2. multiply a row by a non-zero #
3. add rows

Goal : Get $\left[\begin{array}{cc|c} 1 & c & d \\ 0 & 1 & e \end{array}\right]$ $\quad x_2 = e$, $\; x_1 = d - ce$

or $\left[\begin{array}{cc|c} 1 & 0 & f \\ 0 & 1 & g \end{array}\right]$ $\quad x_2 = g$, $\; x_1 = f$

ex).

$$\begin{array}{c c} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array}\right] & \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \end{array}$$

$$\underbrace{}_{\underline{A}} \qquad \underbrace{}_{\underline{b}}$$

$R_1 + R_3 \to R_3$ $\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ 0 & 2 & 1 & 5 \end{array}\right]$

$\frac{3}{2} R_1 + R_2 \rightarrow R_2$
$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 2 & 1 & 3 \end{array}\right]$$

$-4 R_2 + R_3 \rightarrow R_3$
$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 8 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

pivots

$\frac{1}{2} R_1 \rightarrow R_1$

$2 R_2 \rightarrow R_2$

$-1 R_3 \rightarrow R_3$

$$\underbrace{\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]}_{\text{row echlon form}}$$

backward substitution

$x_3 = -1$

$x_2 = 2 - 1 \times (-1) = 3$

$x_1 = 2$

$, \quad \underline{x} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

A form of $\left[\begin{array}{ccc:c} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \end{array}\right]$

is called row echlon form

The leading # in each row is called a pivot

$$\left[\begin{array}{ccc:c} a & b & c & g \\ 0 & d & e & h \\ 0 & 0 & f & i \end{array}\right] \quad a - i \text{ non-zero} \quad \Rightarrow 3 \text{ pivots}$$

$$\left[\begin{array}{ccc:c} a & b & c & g \\ 0 & 0 & d & h \\ 0 & 0 & 0 & i \end{array}\right] \quad 2 \text{ pivots}$$

Introduce Reduced Row Echlon Form

(rref)

① in echelon form

② each pivot is one

③ Each column with a pivot can only have a non-zero value at that pivot location

$$\left[\begin{array}{ccc|c} 1 & 4 & 9 & 4 \\ 0 & 2 & 2 & 10 \\ 0 & -4 & -4 & -40 \end{array}\right]$$

Need to be in echelon form

$$\left[\begin{array}{ccc|c} 1 & 4 & 9 & 4 \\ 0 & 2 & 2 & 10 \\ 0 & 0 & 0 & -20 \end{array}\right]$$

Echelon form with 2 pivots

$$\left[\begin{array}{ccc:c} 1 & 4 & 9 & 4 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & -20 \end{array}\right]$$

$$\left[\begin{array}{ccc:c} 1 & 0 & 5 & -16 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & -20 \end{array}\right]$$ 2 pivots in rref

rref tells you how many sol.

to $\underline{\underline{A}}\ \underline{x} = \underline{b}$

$$x_1 + 5x_3 = -16$$

$$x_2 + x_3 = 5$$

$$0x_1 + 0x_2 + 0x_3 = -20$$

① One sol. : the # of pivots in rref equals to the # of rows and the # of columns ⇒ $\underline{\underline{A}}$ must be square.

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$$

② Infinite # of sol. : # of pivots is less than # of rows & columns

$\underline{\underline{\text{AND}}}$ all rows are consistent.

$$\left[\begin{array}{ccc:c} 1 & a & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 0 \end{array}\right]$$

③ No solution : An inconsistent row

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & -16 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 20 \end{array}\right]$$

Note: Look at $\text{rref}([\underset{=}{A} \mid \underset{=}{I}])$

$\Rightarrow$ get $[\underset{=}{I} \mid \underset{=}{A}^{-1}]$ if $\underset{=}{A}^{-1}$ exists.

RREF, Inverse, Determinant.

let $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$

You can get $\text{rref}(\underline{\underline{A}})$ for any $\underline{\underline{A}}$

If $\det(\underline{\underline{A}}) \neq 0$, $\underline{\underline{A}}^{-1}$ exists

$\Rightarrow \underline{x} = \underline{\underline{A}}^{-1} \underline{b}$ is unique

If $\text{rref}(\underline{\underline{A}})$ has the # of pivots equal to the # of rows and columns, $\underline{x}$ is unique

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Let's start building a Theorem

let $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$, All of these statements are equivalent.

1. $\underline{\underline{A}}$ is invertible ($\underline{\underline{A}}^{-1}$ exists)

2. $\det(\underline{\underline{A}}) \neq 0$

3. rref $(\underline{\underline{A}})$ has n pivots

4. rref $(\underline{\underline{A}})$ is $\underline{\underline{I}}$

5. $\underline{\underline{A}}\,\underline{x} = \underline{b}$ has a unique sol. for all $\underline{x} \in \mathbb{R}^{n \times 1}$

If any one is false,

solution to $\underline{\underline{A}}\,\underline{x} = \underline{b}$

depends on $\underline{b}$

Gaussian Elimination Algorithm

$$\begin{bmatrix} a_0 & b_0 & c_0 \\ d_0 & e_0 & f_0 \\ g_0 & h_0 & i_0 \end{bmatrix} \qquad a_0 \rightarrow i_0 \text{ all non-zero}$$

Step 1

$$\begin{bmatrix} a_0 & b_0 & c_0 \\ d_0 & e_0 & f_0 \\ g_0 & h_0 & i_0 \end{bmatrix} \rightarrow \begin{array}{c} \begin{array}{ccc} & b_1 & c_1 \end{array} \\ \begin{bmatrix} 1 & b_0/a_0 & c_0/a_0 \\ d_0 & e_0 & f_0 \\ g_0 & h_0 & i_0 \end{bmatrix} \end{array}$$

Step 2

$$\begin{bmatrix} 1 & b_1 & c_1 \\ d_0 & e_0 & f_0 \\ g_0 & h_0 & i_0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & b_1 & c_1 \\ 0 & \overset{e_1}{e_0 - b_1 d_0} & \overset{f_1}{f_0 - c_1 d_0} \\ g_0 & h_0 & i_0 \end{bmatrix}$$

Step 3
$$\begin{bmatrix} 1 & b_1 & c_1 \\ 0 & e_1 & f_1 \\ g_0 & h_0 & i_0 \end{bmatrix} \times g_0 \rightarrow \begin{bmatrix} 1 & h_1 & c_1 \\ 0 & e_1 & f_1 \\ 0 & h_1 & i_1 \end{bmatrix}$$

Step 4
$$\begin{bmatrix} 1 & b_1 & c_1 \\ 0 & 1 & f_2 \\ 0 & h_1 & i_1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & c_2 \\ 0 & 1 & f_2 \\ 0 & h_1 & i_1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & c_2 \\ 0 & 1 & f_2 \\ 0 & 0 & i_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & c_2 \\ 0 & 1 & f_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & c_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This might fail.

$$\begin{bmatrix} 1 & 2 & 4 \\ -2 & -4 & -9 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & -1 \\ 0 & -2 & -7 \end{bmatrix}$$

Introduce Pivoting.

Idea: Before eliminating values, swap a the current row with the one with th that has the largest absolute value below it.

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 5 & 3 & 2 \\ 0 & -9 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & -9 & 1 & 1 \\ 0 & 5 & 3 & 2 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

RREF Algorithm w/ Pivoting

```
let A ∈ R^{m×n}          % input
i=1 & j=1               % start at column=1 & row=1

while i ≤ m & j ≤ n     % must be in the matrix

    Select k ≥ i that maximizes |a_kj|

    Swap rows k & i

    If |a_ij| > 10^-16      % check for zero
        a = a_ij
        for k=[j:n]     % normalize row-i
            a_ik = a_ik / a
        end

        for k=[1:i-1 i+1:m]  % all other rows
            a = a_kj
            for l=[j:n]
                a_kj = a_kj - a_il a
            end
        end
        i = i+1     % next row
    end
    j = j+1         % next column
end
```

Matlab Commands

rref(A) → rref of $\underline{A}$

A(:,j) → j^{th} column of $\underline{A}$
A(i,:) → i^{th} row of $\underline{A}$

A([1:3 5:9],:) → return all columns of rows 1→3 & 5→9

A([1 2],:) = A([2 1],:) swaps rows 1 & 2

[m,k] = max(abs(A(:,2)))
k will store the row w/ the largest value in column 2

[m,k] = max(abs(A(4:end,2)))
k will be the # of rows below 4 (plus 1) that has the largest value

$$\underline{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 9 \\ 0 & -9 & 0 \end{bmatrix} \qquad \texttt{A([4:end, 2])} = \begin{bmatrix} 1 \\ 2 \\ -9 \end{bmatrix}$$

9

[m,k] = max(abs(A([4:end],2)))
k = 3