Matrices

A matrix is a 2D collection of numbers

$$\underline{A} = \begin{bmatrix} 9 & 4 \\ -1 & 0 \\ 2 & 3 \end{bmatrix}$$

← rows ↔

columns ↕

Size is $m \times n$

$m = \#$ of rows

$n = \#$ of columns

$\underline{A}$ is 3×2 , $\underline{A} \in \mathbb{R}^{3\times 2}$

$$\underline{B} = \begin{bmatrix} 4 & 2 & 0 \\ 8 & 3 & -1 \end{bmatrix} \qquad \underline{B} \in \mathbb{R}^{2\times 3}$$

Components are a_{ij}

$i =$ row

$j =$ column

$a_{12} = 4$ $\qquad a_{31} = 3$

$b_{11} = 4$ $\qquad a_{13} =$ undefined

Look at matrices as a collection of vectors

let $\underline{a} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ $\quad \underline{b} = \begin{bmatrix} 9 \\ -1 \\ 0 \end{bmatrix}$ $\quad \underline{c} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad \underline{d} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Define $\underline{C} = [\underline{a} \quad \underline{b}] = \begin{bmatrix} 1 & 9 \\ 2 & -1 \\ 4 & 0 \end{bmatrix}$

$$\underline{D} = [\underline{a} \quad \underline{b} \quad \underline{c}] = \begin{array}{ccc} \left[\begin{matrix} 1 & 9 & 0 \\ 2 & -1 & 1 \\ 4 & 0 & 0 \end{matrix}\right] \\ \underline{a} \quad\quad \underline{b} \quad\quad \underline{c} \end{array}$$

$$\underline{E} = [\underset{\substack{\uparrow \\ 3\times 1}}{\underline{a}} \quad \underset{\substack{\uparrow \\ 2\times 1}}{\underline{d}}] = \text{undefined}$$

$$\underline{F} = \begin{bmatrix} \underline{a}^T \\ \underline{b}^T \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 9 & -1 & 0 \end{bmatrix}$$

Transpose applied to matrices

$$\underline{C}^T = \begin{bmatrix} 1 & 9 \\ 2 & -1 \\ 4 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 4 \\ 9 & -1 & 0 \end{bmatrix}$$

$$[\underline{a} \quad \underline{b}]^T = \begin{bmatrix} \underline{a}^T \\ \underline{b}^T \end{bmatrix}^T \implies \underline{C}^T = \underline{F}$$

A matrix is square if the # of rows = # of columns.

$\underline{C}$ is not square. $\underline{D}$ is.

Any matrix that is not square is rectangular.

A matrix is symmetric is $\underline{A} = \underline{A}^T$

$$\underline{G} = \begin{bmatrix} 1 & 9 & 4 \\ 9 & 3 & -1 \\ 4 & -1 & 2 \end{bmatrix} \qquad \underline{G}^T = \begin{bmatrix} 1 & 9 & 4 \\ 9 & 3 & -1 \\ 4 & -1 & 2 \end{bmatrix} = \underline{G}$$

Symmetric as $g_{ij} = g_{ji}$

<u>If</u> $\underline{G} = -\underline{G}^T$, $\underline{G}$ is anti-symmetric

Identity Matrix: A square matrix w/ 1 on the Diagonal & ϕ elsewhere

$$\underline{I} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & & & 1 \end{bmatrix}$$

Diagonal is a_{ii}

$\underline{A}\,\underline{I} = \underline{A} \qquad \underline{I}\,\underline{A} = \underline{A}$

$\underline{I}\,\underline{x} = \underline{x}$

Operations Pt 1

1) Addition / Subtraction of equally sized matrices

$\underline{A}, \underline{B} \in \mathbb{R}^{m \times n}$

$\underline{A} + \underline{B} = \underline{C}$ $\quad\quad$ $\underline{A} - \underline{B} = \underline{D}$

$a_{ij} + b_{ij} = c_{ij}$ $\quad\quad$ $a_{ij} - b_{ij} = d_{ij}$

2) Matrix-Vector products

$\underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\quad$ $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\underline{A}\,\underline{x} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = \underline{b}$

Index Notation $\quad a_{ij}\, x_j = b_i$

repeated indicies means sum over

$\underline{A}$ & $\underline{x}$ must be compatible

$\underline{A} \in \mathbb{R}^{m \times n}$ $\quad$ $\underline{x} \in \mathbb{R}^{p \times 1}$

$\underline{A}\,\underline{x} \Rightarrow \overset{A}{(m \times n)}\overset{x}{(p \times 1)} = m \times 1$

$n = p$

ex.) $(10 \times 4)(4 \times 1) = 10 \times 1$

ex.)
$$\underset{3\times 2}{\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 9 & 1 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \begin{bmatrix} 1(2)+2(1) \\ -1(2)+0(1) \\ 9(2)+1(1) \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 19 \end{bmatrix}$$

ex.)
$$\begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1(2)+2(1)+?(0) \\ \\ \\ \end{bmatrix} = \text{undefined}$$

Now, look at $\underline{A} = [\underline{a} \;\; \underline{b} \;\; \underline{c}]$ $\quad \underline{x} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$

$$\underline{A}\,\underline{x} = [\underline{a} \;\; \underline{b} \;\; \underline{c}] \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \underbrace{d\underline{a} + e\underline{b} + c\underline{f}}_{\text{Linear Combo of vectors}}$$

Note: Most of the time you have $\underline{A}\,\underline{x}$.

It is possible to do $\underline{y}\,\underline{A}$

$\underline{y} \in \mathbb{R}^{p\times r} \quad \underline{A} \in \mathbb{R}^{m\times n}$

$$(p\times r)(m\times n) = p\times n \quad w/\ p=1$$
$$r=m$$

$\underline{y}\,\underline{A}$ if $\underline{y}$ is row vector

Common way: let $\underline{x}$ = column vector, $\underline{x}^T\underline{A}$

(5) Matrix-Matrix products

$\underline{A} \in \mathbb{R}^{m \times n}$ $\quad \underline{B} \in \mathbb{R}^{p \times q}$

$\underline{A}\,\underline{B}$ is defined if $(m \times n)(p \times q) = m \times q$
if $n = p$

$\underline{B}\,\underline{A}$ is defined if $(p \times q)(m \times n) = p \times n$
if $m = q$

Mechanism: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$

$a_{ik} b_{kj} = c_{ij}$ $\qquad \underline{A}\,\underline{B} = \underline{C}$

Comments:

- $\underline{A} + \underline{B} = \underline{B} + \underline{A}$
- $c \in \mathbb{R}$ $\quad c(\underline{A} + \underline{B}) = c\underline{A} + c\underline{B}$
- $(\underline{A}\,\underline{B})\underline{C} = \underline{A}(\underline{B}\,\underline{C})$
- In general $\underline{A}\,\underline{B} \neq \underline{B}\,\underline{A}$

$\underline{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $\qquad \underline{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\underline{A}\,\underline{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq \underline{B}\,\underline{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

- $\underline{A}(\underline{B}+\underline{C}) = \underline{A}\,\underline{B} + \underline{A}\,\underline{C}$

  $(\underline{A}+\underline{B})\,\underline{C} = \underline{A}\,\underline{C} + \underline{B}\,\underline{C}$

  $(\underline{A}+\underline{B})\,\underline{x} = \underline{A}\,\underline{x} + \underline{B}\,\underline{x}$

  $\underline{A}(\underline{x}+\underline{y}) = \underline{A}\,\underline{x} + \underline{A}\,\underline{y}$

  $\underline{x}^T(\underline{A}+\underline{B}) = \underline{x}^T\underline{A} + \underline{x}^T\underline{B}$

- $(\underline{A}+\underline{B})^T = \underline{A}^T + \underline{B}^T$

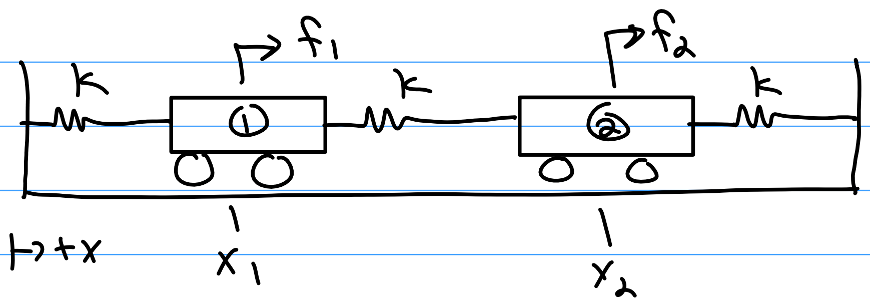
  $(\underline{A}\,\underline{B})^T = \underline{B}^T\underline{A}^T$

  $(\underline{A}\,\underline{x})^T = \underline{x}^T\underline{A}^T$

  $(c\underline{A})^T = \underline{A}^T c^T = \underline{A}^T c = c\,\underline{A}^T$

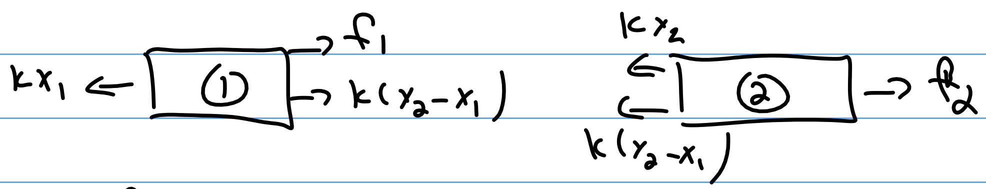
Matrix-Vector Product Example

Mass-Spring System



Given k, x_1, x_2, what f_1 & f_2 needed for equilibrium?

At equilibrium $\Sigma F = 0$



① : $f_1 + k(x_2 - x_1) - kx_1 = 0$

② : $f_2 - k(x_2) - k(x_2 - x_1) = 0$

$$\Rightarrow \begin{aligned} 2kx_1 - kx_2 &= f_1 \\ -kx_1 + 2kx_2 &= f_2 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}}_{\underline{K}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}}_{\underline{f}}$$

Given $k \to \underline{K}$, $x_1, x_2 \to \underline{x}$ $\to \underline{f}$

Later: Given $\underline{K}$ & $\underline{f}$ find $\underline{x}$

Operations Pt 2

4) Powers: $\underline{A}^p = (\underbrace{\underline{A}\,\underline{A} \cdots \underline{A}}_{p\text{-times}})$

$$\underline{A}^p \underline{A}^q = \underline{A}^{(p+q)} \qquad (\underline{A}^p)^q = \underline{A}^{pq}$$

5) Block Matrices

$$\begin{array}{c} \\ \underline{A} = \end{array} \begin{array}{c} \text{4 rows} \quad\quad \text{4 rows} \\ \begin{bmatrix} \underline{B} & \underline{C} & \underline{D} \\ \underline{E} & \underline{F} & \underline{G} \end{bmatrix} \\ \text{10 columns} \end{array}$$

must be compatible

10 columns

Same Rules:

$$\begin{bmatrix} \underline{A}_{11} & \underline{A}_{12} \\ \underline{A}_{21} & \underline{A}_{22} \end{bmatrix} \begin{bmatrix} \underline{B}_{11} \\ \underline{B}_{21} \end{bmatrix} = \begin{bmatrix} \underline{A}_{11}\underline{B}_{11} + \underline{A}_{12}\underline{B}_{21} \\ \underline{A}_{21}\underline{B}_{11} + \underline{A}_{22}\underline{B}_{21} \end{bmatrix}$$

ex.) $\underline{A} = \begin{bmatrix} \underline{A}_0 & \underline{P} \\ \underline{P}^T & \underline{0} \end{bmatrix}$

6) Trace: Sum of the Diagonal

$$tr(\underline{A}) = \sum_{i=1}^{n} a_{ii}$$

$$tr\left(\begin{bmatrix} 1 & 0 & 9 \\ 0 & -1 & 2 \\ -1 & 0 & 8 \end{bmatrix}\right) = 1 - 1 + 8 = 8$$

let $\underline{A}, \underline{B} \in \mathbb{R}^{n\times n}$ $\quad a \in \mathbb{R}$

- $tr(a\underline{A}) = a\, tr(\underline{A})$
- $tr(\underline{A}^T) = tr(\underline{A})$
- $tr(\underline{B}\underline{A}) = tr(\underline{A}\underline{B})$
- $tr(\underline{A}\underline{B}) \neq tr(\underline{A})\, tr(\underline{B})$
- $tr(\underline{A}^T\underline{B}) = tr(\underline{A}\underline{B}^T) = tr(\underline{B}^T\underline{A}) = tr(\underline{B}\underline{A}^T)$

7) Outer product: Given 2 vectors get a matrix

let $\underline{a} \in \mathbb{R}^{m\times 1}$ $\quad \underline{b} \in \mathbb{R}^{n\times 1}$

$$\underline{a}\,\underline{b}^T = \underline{a} \otimes \underline{b}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \\ a_3 b_1 & a_3 b_2 \end{bmatrix} = \underline{C} \quad c_{ij} = a_i b_j$$

Matrix Determinant

The determinant is an operation on a square matrix that results in a scalar that encodes information about said matrix

let $\underline{A} \in \mathbb{R}^{n \times n}$ $\det(\underline{A}) = |\underline{A}|$ ← not absolute value

Define by recursion

1×1: $\det([a_{11}]) = a_{11}$

2×2: $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\det([a_{22}]) - a_{12}\det([a_{21}])$
$= a_{11}a_{22} - a_{12}a_{21}$

3×3: $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$

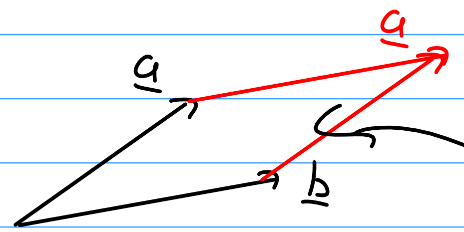
In General: $\det(\underline{A}) = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} M_{ij}$

Pick a row i

↑ Det of sub-matrix

Properties:

1) $\det(\underline{A}\,\underline{B}) = \det(\underline{A})\det(\underline{B})$
2) $\det(\alpha \underline{I}_n) = \alpha^n$ $\quad \alpha \in \mathbb{R} \quad \underline{I}_n = n \times n$ identity
3) $\det(\alpha \underline{A}) = \det(\alpha \underline{I}_n \underline{A}) = \det(\alpha \underline{I}_n)\det(\underline{A}) = \alpha^n \det(\underline{A})$
4) $\det(\underline{A}^T) = \det(\underline{A})$
5) If you define a shape as the parallelogram defined by the columns of a matrix, $\det(\underline{A})$ gives the area / volume



$\det([\underline{a}\ \ \underline{b}]) =$ area of

6) If $\underline{A}$ exists then so does $\det(\underline{A})$
If $\det(\underline{A})$ is defined/exist then $\underline{A}$ exists
If $\det(\underline{A})$ is undefined then $\underline{A}$ can <u>not</u> exist.

Matrix Inverse

Only for square matrices

let $\underline{A}, \underline{B} \in \mathbb{R}^{n \times n}$, $\underline{I}_n = n \times n$ identity matrix

$\underline{A}$ & $\underline{B}$ are inverses of each other iff

$\underline{A}\,\underline{B} = \underline{I}_n$ or $\underline{B}\,\underline{A} = \underline{I}_n$

Denote inverse as $\underline{A}^{-1}$: $\underline{A}\,\underline{A}^{-1} = \underline{I} = \underline{A}^{-1}\underline{A}$

Closed form solutions only exist for
2×2 & 3×3 matrices

2×2: let $\underline{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Show that $\underline{A}^{-1} = \frac{1}{\det(\underline{A})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\underline{A}^{-1}\underline{A} = \frac{1}{\det(\underline{A})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{\det(\underline{A})} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similar for $\underline{A}\,\underline{A}^{-1}$

It can be shown that $\det(\underline{A}^{-1}) = \frac{1}{\det(\underline{A})}$

If $\det(\underline{A}) \neq 0$, $\frac{1}{\det(\underline{A})} = \det(\underline{A}^{-1})$ is defined

$\Rightarrow$ $\underline{A}^{-1}$ must exist

If $\det(\underline{A}) = 0$ then $\frac{1}{\det(\underline{A})} = \det(\underline{A}^{-1})$ is undefined

$\Rightarrow$ $\underline{A}^{-1}$ can <u>not</u> exist

Back to Mass-Spring

$$\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$\underline{K} \quad \underline{x} = \underline{f}$$

Given $\underline{K}$ & $\underline{f}$, find $\underline{x}$?

Any real spring has $k > 0$

$\det(\underline{K}) > ?$

$$\begin{vmatrix} 2k & -k \\ -k & 2k \end{vmatrix} = 4k^2 - k^2 = 3k^2 \neq 0 \implies \underline{K}^{-1} \text{ exists}$$

$$\underline{K} \, \underline{x} = \underline{f}$$
$$\underline{K}^{-1} \underline{K} \, \underline{x} = \underline{K}^{-1} \underline{f}$$
$$\underline{I} \, \underline{x} = \underline{K}^{-1} \underline{f}$$
$$\underline{x} = \underline{K}^{-1} \underline{f}$$

In reality, <u>never</u> compute $\underline{K}^{-1}$

Instead turn to: 1) Iterative methods (not covered)
Goal is to find $\underline{x}$ such that
$\underline{K} \, \underline{x} = \underline{f}$

2) Decomposition of the matrix
(covered)

$\underline{K} = \underline{A} \, \underline{B}$ w/ solving $\underline{A} \, \underline{y} = \underline{f}$
$\underline{B} \, \underline{x} = \underline{y}$ is cheap

Aside: $(\alpha \underline{A} + \beta \underline{B} + \gamma \underline{C})^{-1}$ is not easy to get

$$\underline{K}\,\underline{x} = \underline{f}$$
$$\underline{A}\,\underline{B}\,\underline{x} = \underline{f}$$
$$\underline{B}\,\underline{x} = \underline{A}^{-1}\underline{f}$$
$$\underline{x} = \underline{B}^{-1}\underline{A}^{-1}\underline{f}$$

ex1) $\underline{A} = \underline{L}\,\underline{U}$ ← LU Decomposition

Solu $\underline{U}\,\underline{x} = \underline{b}$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = b_1 \Rightarrow x_1 = \cdots$$
$$u_{22}x_2 + u_{23}x_3 = b_2 \Rightarrow x_2 = (b_2 - u_{23}x_3)/u_{22}$$
$$u_{33}x_3 = b_3 \Rightarrow x_3 = b_3/u_{33}$$