OPERATIONS RESEARCH

Chapter 2

Transportation and Assignment Problems

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MODULE-3: Assignment Problem and Its Solution by Hungarian Method, and Travelling Salesman Problem

2.1 Assignment Problem

The assignment problem is a special type of transportation problem where the objective is to minimize the cost of allocating a number of jobs to a number of persons or facilities so that one person or facility is assigned to only one job. The assignment model is useful in solving problems such as assignment of machines to jobs, assignment of salesmen to sales territories, travelling salesman problem, etc. It may be noted that with n facilities and n jobs, there are $n!$ possible assignments. One way of finding an optimal assignment is to consider all the $n!$ possible arrangements, evaluate their total costs, and select the assignment with minimum cost. However, due to heavy computational burden, this method is not suitable. For solving assignment problem, we will discuss an efficient method which was developed by a Hungarian Mathematician D. Konig.

2.1.1 Mathematical Formulation of Assignment Problem

Consider the problem of assignment of a company which has n machines of different capacities for performing n different jobs and one machine can only be assigned to only one job. The objective is to minimize the total cost of assignment. The cost matrix for this problem is given in Table 2.1. This cost matrix is same as that of a transportation problem except that the availability at each of the machines is unity and the requirement at each of the destinations is also unity. In Table 2.1, c_{ij} is the cost

Machines	Jobs				Availability
	J_1	J_2	$\cdots$	J_n	
M_1	c_{11}	c_{12}	$\cdots$	c_{1n}	1
M_2	c_{21}	c_{22}	$\cdots$	c_{2n}	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
M_n	c_{n1}	c_{n2}	$\cdots$	c_{nn}	1
Requirement	1	1	$\cdots$	1	

Table 2.1: Cost matrix for assignment problem

associated with assigning the ith machine to the jth job. To formulate the assignment problem in mathematical programming terms, we define the activity variables $x_{ij} =$

$$\begin{cases} 1, & \text{if machine } i \text{ is assigned to job } j \\ 0, & \text{otherwise} \end{cases}$$

Then the mathematical model for the assignment problem can be stated as

$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{subject to } \sum_{j=1}^{n} x_{ij} = 1 \text{ for all } i \text{ (availability)}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \text{ for all } j \text{ (requirement)}$$

$$\text{and } x_{ij} = 0 \text{ or } 1 \text{ for all } i \text{ and } j$$

Theorem 2.1 (Reduction Theorem): *In an assignment problem, if we add (or subtract) a constant to every element of a row (or column) of the cost matrix (c_{ij}), then an assignment plan that minimizes the total cost for the new cost matrix also minimizes the total cost for the original cost matrix. In other words, if $x_{ij} = x_{ij}^*$ minimizes $Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$ with $\sum_{i=1}^{n} x_{ij} = 1$, $\sum_{j=1}^{n} x_{ij} = 1$; $x_{ij} = 0$ or 1 then x_{ij}^* also minimizes $Z^* = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}^* x_{ij}$ where $c_{ij}^* = c_{ij} - u_i - v_j$ for all i, j =1, 2, ..., n and u_i, v_j are real numbers.*

Proof: We write

$$
\begin{aligned}
Z^* &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^* x_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n}(c_{ij}-u_i-v_j)x_{ij} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} - \sum_{i=1}^{n} u_i \sum_{j=1}^{n} x_{ij} - \sum_{i=1}^{n} x_{ij} \sum_{j=1}^{n} v_j \\
&= Z - \sum_{i=1}^{n} u_i - \sum_{j=1}^{n} v_j \quad \left(\text{since } \sum_{i=1}^{n} x_{ij} = \sum_{j=1}^{n} x_{ij} = 1\right)
\end{aligned}
$$

This shows that the minimization of the new objective function Z^* yields the same solution as the minimization of the origin objective function Z, because $\sum u_i$ and $\sum v_j$ are independent of x_{ij}.

Corollary 2.1: *If (x_{ij}), $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$ is an optimal solution of an assignment problem with cost (c_{ij}) then it is also optimal solution of the problem with cost (c'_{ij}) where*

$$
\begin{aligned}
&c'_{ij} = c_{ij} \textit{ for all } i, j = 1, 2, \cdots, n;\ j \neq k \\
&c'_{ik} = c_{ik} - \alpha, \textit{ where } \alpha = a \textit{ constant.}
\end{aligned}
$$

Proof: The proof is left as an exercise to the reader.

Theorem 2.2: *If an assignment problem with cost $(c_{ij} > 0)$ is such that minimum $\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} = 0$ then (x_{ij}) provides an optimal assignment.*

Proof: The proof is left as an exercise.

2.2 Hungarian Method for Solving Assignment Problem

The Hungarian method is an efficient method for finding the optimal solution of an assignment problem. The method works on the principle of reducing the given cost matrix to a matrix of opportunity costs. Opportunity costs show the relative penalties associated with assigning a machine to a job. Hungarian method reduces the cost matrix to the extent of having at least one zero in each row and column so as to make optimal assignment. In the following, an algorithm for finding the optimal solution of an assignment problem is given.

Step 1 Identify the minimum element in each row and subtract it from each element of that row.

Step 2 Identify the minimum element in each column and subtract it from each element of that column.

Step 3 Draw the least possible number of horizontal and vertical lines to cover all the zeros. If the number of lines equals the order of the cost matrix then go to step 5; otherwise, go to step 4.

Step 4 Identify the smallest element among the uncovered elements left after drawing the horizontal and vertical lines in step 3. Subtract this element from all the uncovered elements and add the same element to the elements lying at the intersections of the horizontal and vertical lines. Then go to step 3.

Step 5 For each row or column with a single zero, box that zero as an assigned cell. For every zero that becomes assigned, cross out all other zeros in the same column or row. If for a row or a column, there are two or more zeros then choose one cell arbitrarily for assignment. The process is to be continued until every zero is either assigned with box or crossed out. Cells having a box indicate optimal assignment. In case a zero cell is chosen arbitrarily then there may be alternate optimal solution.

Example 2.1: A Company has four men available for work on four separate jobs. Only one man can work on any one job. The cost of assigning each man to each job is given in Table 2.2. Assign men to jobs in such a way that the total cost of assignment is minimum.

		Job			
		I	II	III	IV
Person	A	20	25	22	28
	B	15	18	23	17
	C	19	17	21	24
	D	25	23	24	24

Table 2.2: Table for Example 2.1

Solution: We will use the Hungarian algorithm to solve this problem. Following steps 1 and 2, the cost matrix reduces to the form as shown in Table 2.3.

Now, we draw minimum number of horizontal and vertical lines to cover all the zeros. The least number of lines is 3 (see Table 2.4) which less than 4, the order of the matrix.

	I	II	III	IV
A	0	5	1	7
B	0	3	7	1
C	2	0	3	6
D	2	0	0	0

Table 2.3: Reduced cost matrix

The smallest of the uncovered elements is 1. So, we subtract 1 from all the uncovered elements and add 1 to the elements lying at the intersections of the horizontal and vertical lines. Table 2.5 shows the revised matrix. Next, we draw the least number of horizontal and vertical lines covering all the zeros, see Table 2.6. Since the number of lines = 4 = the order of the matrix, optimality condition is satisfied.

	I	II	III	IV
A	0	5	1	7
B	0	3	7	1
C	2	0	3	6
D	2	0	0	0

Table 2.4

$\Rightarrow$

	I	II	III	IV
A	0	4	0	6
B	0	2	6	0
C	3	0	3	6
D	3	0	0	0

Table 2.5

	I	II	III	IV
A	0	4	0	6
B	0	2	6	0
C	3	0	3	6
D	3	0	0	0

Table 2.6

$\Rightarrow$

	I	II	III	IV
A	$\boxed{0}$	4	$\cancel{0}$	6
B	$\cancel{0}$	2	6	$\boxed{0}$
C	3	$\boxed{0}$	3	6
D	3	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$

Table 2.7

The optimal assignment is obtained as $A \to I$, $B \to IV$, $C \to II$, $D \to III$, see Table 2.7. The minimum cost of assignment is $Rs.(20+17+17+24) = Rs.78$.

2.2.1 Variation of Assignment Problem

Unbalanced Assignment Problem - If the number of facilities is not equal to the number of jobs then the problem is called an unbalanced assignment problem. In such a case, fictitious rows and/or columns are added in the cost matrix to make it a square matrix. Then, we apply Hungarian method to find the solution of the balanced assignment problem.

Maximization Problem - When the facilities are to be assigned to a number of jobs so as to maximize the overall performance or profit of the assignment, the Hungarian method can be applied to solve such a maximization problem by converting every element of the matrix to an opportunity loss. The conversion is accomplished by subtracting all the elements of the given matrix from the highest element. It turns out that minimizing opportunity loss produces the same assignment solution as the original maximization problem.

Restrictions on Assignments - If a particular resource (say, man or machine) cannot be assigned to perform a particular activity then the cost of performing that particular activity by that particular resource is considered to be very large (taken usually a large positive M or ∞) so as to prohibit the entry of this pair of resource-activity into the final solution.

Negative Cost - If the cost matrix of an assignment problem contains some negative costs then we add to each element of the rows or columns a quantity sufficient to make all the cell elements non-negative.

2.2.2 The Travelling Salesman Problem

Suppose that a salesman plans to visit each of n cities once and only once and return back to the city from where he started. The distance between each pair of cities is known. The objective is to determine the shortest route which the salesman can take.

Since there are n cities, there are $(n-1)!$ possible ways for his tour. Let c_{ij} be the distance between city i to city j and

$$x_{ij} = \begin{cases} 1, & \text{if the salesman goes directly from city } i \text{ to city } j \ (i \neq j) \\ 0, & \text{otherwise} \end{cases}$$

Since each city can be visited only once, we have

$$\sum_{i=1}^{n-1} x_{ij} = 1, \quad j = 1, 2, \ldots, n; \quad i \neq j$$

Again, since the salesman has to leave each city except city n, we have

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2, \ldots, n-1; \quad i \neq j$$

The objective function is then

$$\text{Minimize } Z = \sum_{i=1}^{n-1} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Since $c_{ij} = c_{ji}$ is not required, therefore, $c_{ij} = \infty$ for $i = j$. However, all c_{ij}'s must be non-negative, i.e., $c_{ij} \geq 0$ and $c_{ij} + c_{jk} \geq c_{jk}$ for all i, j, k.

Example 2.2: A travelling salesman plans to visit five cities I, II, III, IV and V. The travel time (in hours) between these cities is shown in Table 2.8. How should he schedule his touring plan in order to minimize the total travel time, if he visits each city once a week?

From \ To	I	II	III	IV	V
I	∞	2	5	7	1
II	6	∞	3	8	2
III	8	7	∞	4	7
IV	12	4	6	∞	5
V	1	3	2	8	∞

Table 2.8

Solution: We apply the Hungarian method to solve the given problem treating it as an assignment problem. The solution is obtained as given in Table 2.9.

	I	II	III	IV	V
I	∞	1	3	6	$\boxed{0}$
II	4	∞	$\boxed{0}$	6	0
III	4	3	∞	$\boxed{0}$	3
IV	8	$\boxed{0}$	1	∞	1
V	$\boxed{0}$	2	0	7	∞

Table 2.9

We see that the optimal assignment forms two subloops: $I \to V \to I$ and $II \to III \to IV \to II$. This is not the solution of the traveling salesman problem as the sequence of cities visited is not in cyclical order. We will now try for the next best solution which satisfies the restriction of the traveling salesman problem. The next minimum (non-zero) element in the Table 2.9 is 1 which occurs in the cells (I, II), (IV, III) and (IV, V). We try make an assignment in any one of these cells. Let us make an assignment in the cell (I, II) instead of zero assignment in the cell (I,V). Then no other assignment is possible in the first row and second column. Next, we make an assignment in the cell (IV, V) instead of the cell (IV, II). Thus we get a feasible solution as $I \to II \to III \to IV \to V \to I$. The assignment is shown in Table 2.10.

	I	II	III	IV	V
I	∞	$\boxed{1}$	3	6	0
II	4	∞	$\boxed{0}$	6	0
III	4	3	∞	$\boxed{0}$	3
IV	8	0	1	∞	$\boxed{1}$
V	$\boxed{0}$	2	0	7	∞

Table 2.10

The cost corresponding to the feasible solution is 15. If, instead of making assignment in the cell (IV, V), we make an assignment in the cell (IV, III) then we do not get a feasible solution as in that case both II and IV are followed by III.