Project 1: Cosmological Models

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November 18, 2025

1 Introduction

Type Ia supernovae (SNe Ia) act as standard candles, so their observed fluxes and redshifts provide a direct probe of the expansion history of the Universe. In particular, the apparent magnitude of a supernova can be expressed in terms of the distance modulus

$$\mu = 5\log_{10}\left(\frac{d_L}{\mathrm{Mpc}}\right) + 25, \tag{1}$$

where $d_L(z)$ is the luminosity distance. The Hubble diagram, i.e. μ as a function of redshift z, can then be used to constrain cosmological parameters and to test whether the expansion is accelerating or decelerating.

In a homogeneous and isotropic Friedmann–Lemaître–Robertson–Walker universe, the luminosity distance is related to the Hubble parameter via

$$d_L(z) = \frac{c\,(1+z)}{H_0}\int_0^z \frac{\mathrm{d}z'}{E(z')}, \tag{2}$$

where H_0 is the present–day Hubble constant and $E(z) \equiv H(z)/H_0$ encodes the cosmological model. In this project we focus on two flat models: the ΛCDM model, for which

$$E_\Lambda^2(z) = \Omega_{m,0}(1+z)^3 + (1-\Omega_{m,0}), \tag{3}$$

and its one–parameter extension wCDM with a constant dark–energy equation of state w,

$$E_w^2(z) = \Omega_{m,0}(1+z)^3 + (1-\Omega_{m,0})(1+z)^{3(1+w)}. \tag{4}$$

For $w = -1$ the wCDM model reduces exactly to ΛCDM. In the low–redshift regime $z \ll 1$, the luminosity distance can also be written as a Taylor series in z involving the deceleration parameter q_0, which provides a purely kinematic way to test for accelerated expansion ($q_0 < 0$) or deceleration ($q_0 > 0$).

We base our analysis on the SCP Union2.1 compilation of SNe Ia, which contains 580 objects spanning a wide redshift range. The project is divided into two main parts. In the first part, we restrict attention to the small–z regime ($z < 0.5$) and use the kinematic Taylor expansion to infer the joint posterior distribution of H_0 and q_0, including an intrinsic scatter term, and to perform posterior predictive checks of the resulting model. In the second part, we analyse the full redshift range under the assumption of a flat ΛCDM or wCDM cosmology with fixed $H_0 = 70\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$. We obtain maximum–likelihood estimates for $\Omega_{m,0}$ (and w), compare the two models using the Akaike and Bayesian information criteria (AIC and BIC), and derive a one–dimensional posterior for $\Omega_{m,0}$ in the ΛCDM case. Together, these steps allow us to test for cosmic acceleration, assess whether the data justify introducing a free w, and compare our constraints on $\Omega_{m,0}$ and q_0 to results in the literature.

2 Methods

We use the SCP Union2.1 Type Ia supernova compilation, which provides for each SN its redshift z_i, distance modulus μ_i and measurement uncertainty σ_i. In all cases we assume independent Gaussian errors and write the likelihood as

$$\ln\mathcal{L}(\theta) = -\tfrac{1}{2}\sum_i \left[\frac{\big(\mu_i - \mu_{\mathrm{model}}(z_i;\theta)\big)^2}{\sigma_{i,\mathrm{tot}}^2} + \ln\!\big(2\pi\sigma_{i,\mathrm{tot}}^2\big)\right], \tag{5}$$

where θ denotes the model parameters and $\sigma_{i,\mathrm{tot}}^2 = \sigma_i^2 + \sigma_{\mathrm{int}}^2$ includes a possible intrinsic scatter term in the small-z analysis.

2.1 Task 1: Small-z expansion

For the low-redshift subset ($z < 0.5$) we use the second–order kinematic expansion of the luminosity distance,

$$d_L(z) = \frac{c}{H_0}\left[z + \tfrac{1}{2}(1 - q_0)z^2\right], \tag{6}$$

and convert to distance modulus via $\mu_{\rm model}(z) = 5\log_{10}(d_L/{\rm Mpc}) + 25$. The free parameters are $(H_0, q_0, \sigma^2_{\rm int})$. We adopt broad uniform priors $H_0 \sim \mathcal{U}(50, 100)$, $q_0 \sim \mathcal{U}(-2, 2)$ and an inverse–gamma prior for the intrinsic variance $\sigma^2_{\rm int}$.

First, we obtain maximum–a–posteriori (MAP) estimates by minimizing the negative log–posterior with `scipy.optimize.minimize` (Nelder-Mead). We then sample the full posterior with the EMCEE ensemble MCMC sampler, using 32 walkers and 2000 steps. These samples are used to compute credible intervals, one– and two–dimensional posterior plots, and posterior predictive bands for $\mu(z)$.

2.2 Task 2: ΛCDM–wCDM comparison

For the full redshift range we assume a flat universe and consider two models. In flat ΛCDM,

$$E^2(z) = \Omega_{m,0}(1 + z)^3 + (1 - \Omega_{m,0}), \tag{7}$$

while in flat wCDM with constant equation of state w,

$$E^2(z) = \Omega_{m,0}(1 + z)^3 + (1 - \Omega_{m,0})(1 + z)^{3(1+w)}. \tag{8}$$

In both cases we fix $H_0 = 70\ {\rm km\,s^{-1}\,Mpc^{-1}}$ and compute the luminosity distance as

$$d_L(z) = \frac{c(1 + z)}{H_0}\int_0^z \frac{{\rm d}z'}{E(z')}, \tag{9}$$

evaluated numerically, and then convert to $\mu_{\rm model}(z)$ as above. The free parameters are $\Omega_{m,0}$ for ΛCDM and $(\Omega_{m,0}, w)$ for wCDM. We maximize the Gaussian likelihood with `scipy.optimize.minimize`, using box constraints $0 < \Omega_{m,0} < 1$ and $-3 < w < 1$.

To compare the models we use the Akaike and Bayesian information criteria,

$$\text{AIC} = 2k - 2\ln\mathcal{L}_{\max}, \qquad \text{BIC} = k\ln N - 2\ln\mathcal{L}_{\max}, \tag{10}$$

where k is the number of free parameters, N the number of data points, and $\mathcal{L}_{\max}$ the maximum likelihood. Differences ΔAIC and ΔBIC between the two models quantify the evidence for or against introducing the extra parameter w.

Finally, under the assumption of flat ΛCDM we perform a one–parameter Bayesian inference for $\Omega_{m,0}$, adopting a uniform prior $\Omega_{m,0} \sim \mathcal{U}(0, 1)$ and sampling the posterior with EMCEE (32 walkers, 3000 steps). These samples are used to quote credible intervals and to construct the posterior for $\Omega_{m,0}$.

3 Results

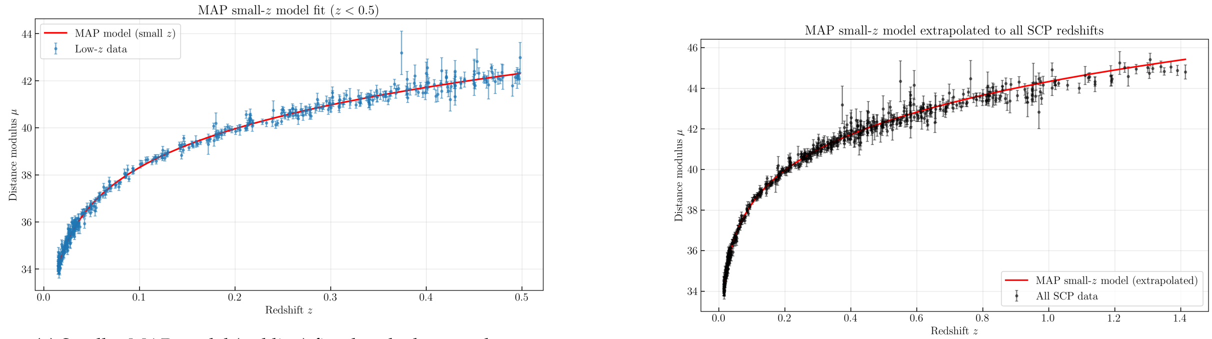
3.1 Task 1: Small-z expansion

The distance–modulus–redshift relation for the SCP 2.1 supernovae is clearly non–linear, so in the low–redshift regime $z < 0.5$ we model $\mu(z)$ with the second–order expansion given in 6. Using this small-z model and the likelihood described in Sec. 2, we obtain the maximum–a–posteriori (MAP) estimates

$$H_0^{\rm MAP} = 69.71\ {\rm km\,s^{-1}\,Mpc^{-1}}, \qquad q_0^{\rm MAP} = -0.415, \qquad \sigma^2_{\rm MAP} \simeq 7 \times 10^{-3}.$$

The corresponding model curve provides an excellent fit in the low–z regime and still describes the full SCP 2.1 dataset reasonably well (Figs. 1a and 1b).

As an internal consistency check of our H_0 determination, we performed a separate fit restricted to a very low–redshift subset ($z < 0.1$), where the luminosity distance can be approximated by the linear Hubble law and the q_0 term can be neglected. Fitting a purely linear model in this regime yields $H_0 \simeq 67.9\ {\rm km\,s^{-1}\,Mpc^{-1}}$, only a few per cent lower than the value obtained from the full second–order model. As shown in Fig. 2, the linear and quadratic models are almost indistinguishable over $z < 0.1$, which supports the robustness of our H_0 estimate and illustrates that the quadratic correction mainly becomes important at slightly higher redshift.



(a) Small-z MAP model (red line) fitted to the low-z subset ($z < 0.5$), with error bars showing the observed distance moduli.

(b) Small-z MAP model extrapolated to all SCP redshifts.

Figure 1: Comparison of small-z MAP model fit on low-z data (left) and full dataset (right).

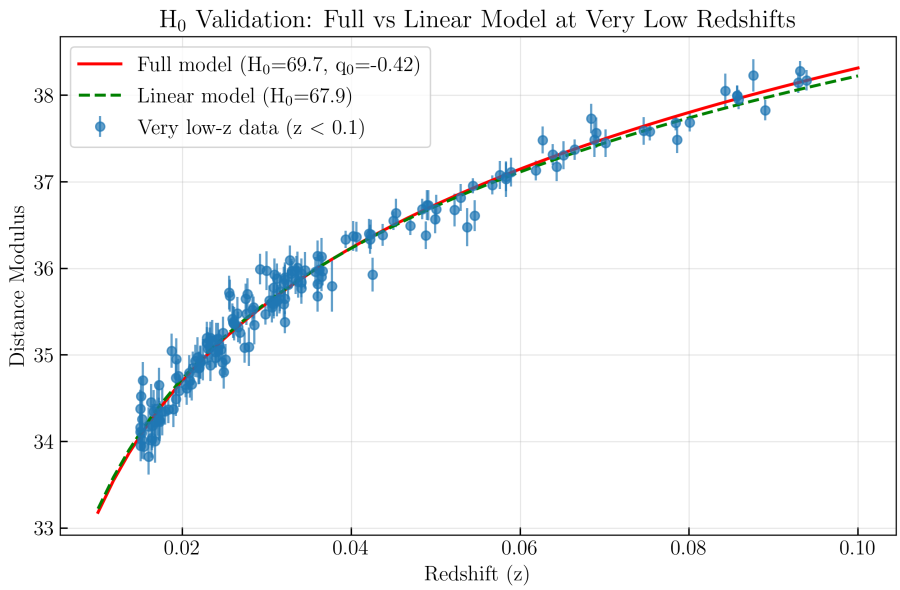


Figure 2: Very low–z validation of H_0. The red line shows the full small-z model with the MAP parameters, while the green dashed line shows a purely linear Hubble–law fit to the $z < 0.1$ subset. Blue points with error bars are the very low–z SCP supernovae.

To quantify the uncertainties and parameter degeneracies we sample the full posterior with the EMCEE ensemble sampler, using 32 walkers and 2000 steps. The chains for (H_0, q_0, σ^2) are well mixed, and the marginalized posteriors yield

$$H_0 = 69.7^{+0.5}_{-0.5}\,\mathrm{km\,s^{-1}\,Mpc^{-1}}, \qquad q_0 = -0.419^{+0.081}_{-0.080}, \qquad \sigma^2 \simeq 7 \times 10^{-3},$$

where the uncertainties denote central 68% credible intervals. The posterior for q_0 lies entirely below zero, which constitutes very strong evidence that the present–day expansion of the Universe is accelerating (Fig. 3).

Finally, we perform a posterior predictive check (Fig. 4). For each of 200 posterior draws of (H_0, q_0, σ^2) we compute the corresponding model $\mu(z)$ and its intrinsic scatter. The mean of these realizations defines the red curve in Fig. 4, while the blue shaded region marks the central 68% posterior predictive band. Most of the data points lie within this band at low redshift; at higher z the number of outliers increases, as expected given that the model is only a second–order expansion. Overall, however, the small-z model provides a reasonably good phenomenological description of the full SCP 2.1 Hubble diagram.

3.2 Task 2: Cosmological model comparison

In the second task we compare two flat cosmological models fitted to the full SCP 2.1 dataset: the standard ΛCDM model and a wCDM model with a constant dark–energy equation–of–state parameter w. Following the project instructions we fix the Hubble parameter to $H_0 = 70\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$ and treat

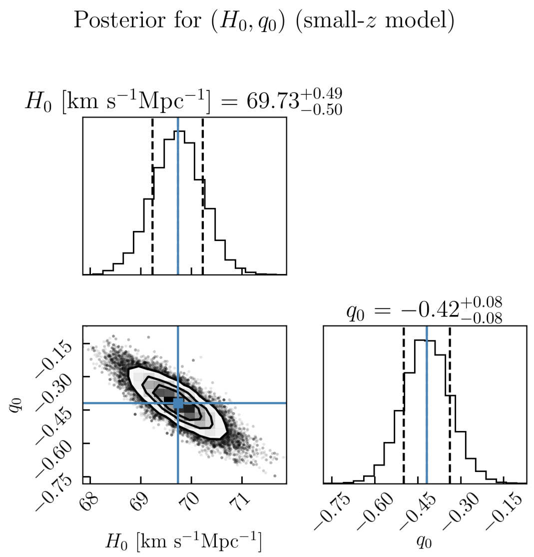


Figure 3: Posterior distribution of (H_0, q_0) for the small-z model. The blue lines mark the posterior medians, and the dashed lines indicate the central 68% credible intervals.

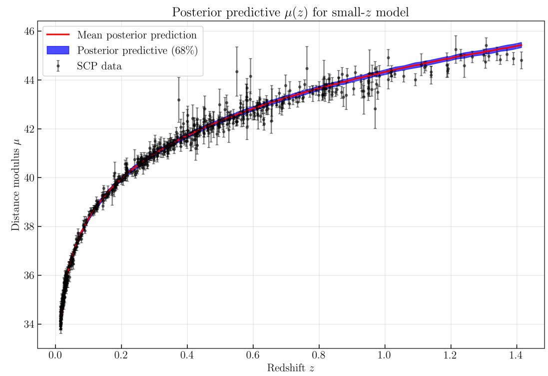


Figure 4: Posterior predictive check for the small-z model. The red curve shows the mean posterior prediction, and the blue band the central 68% posterior predictive interval.

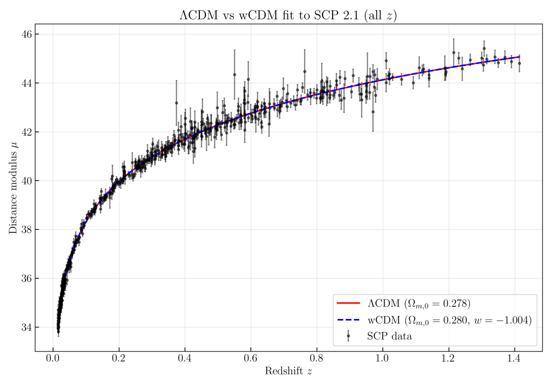


Figure 5: ΛCDM and wCDM model fits to the full SCP 2.1 dataset. The two model curves are nearly indistinguishable on the scale of the plot.

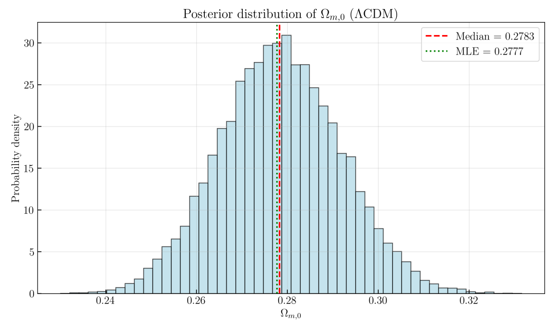


Figure 6: Posterior distribution of $\Omega_{m,0}$ in the flat ΛCDM model with a uniform prior. The vertical lines indicate the posterior median and the central 68% credible interval.

only the matter density $\Omega_{m,0}$ (and w in the wCDM case) as free parameters. The luminosity distance is computed as in eqn 9. The likelihood assumes Gaussian measurement errors with known variances.

Maximizing the likelihood yields the maximum–likelihood estimates

$$\Lambda\text{CDM}: \qquad \Omega_{m,0}^{\text{MLE}} = 0.2777, \tag{11}$$

$$w\text{CDM}: \qquad \Omega_{m,0}^{\text{MLE}} = 0.2796, \qquad w^{\text{MLE}} = -1.0045. \tag{12}$$

Both models therefore give very similar best–fit parameters. The resulting Hubble diagrams (Fig. 5) are almost indistinguishable over the redshift range of the SCP 2.1 sample.

To quantify model preference we compute the maximized log–likelihood and the Akaike and Bayesian information criteria (AIC and BIC). For ΛCDM (one free parameter) we obtain

$$\log L_{\max}^{\Lambda\text{CDM}} = 118.74, \qquad \text{AIC}_{\Lambda\text{CDM}} = -235.48, \qquad \text{BIC}_{\Lambda\text{CDM}} = -231.12,$$

while for wCDM (two free parameters)

$$\log L_{\max}^{w\text{CDM}} = 118.74, \qquad \text{AIC}_{w\text{CDM}} = -233.48, \qquad \text{BIC}_{w\text{CDM}} = -224.76.$$

The likelihood values are essentially identical, so the differences in AIC and BIC are entirely due to the penalty for the additional parameter w. Defining $\Delta\text{IC} \equiv \text{IC}_{w\text{CDM}} - \text{IC}_{\Lambda\text{CDM}}$, we find

$$\Delta\text{AIC} = 2.0, \qquad \Delta\text{BIC} = 6.36.$$

Both quantities are positive, indicating that the extra complexity of the wCDM model is not warranted by the data. The AIC difference corresponds to only weak evidence against wCDM, whereas the BIC difference is often interpreted as moderate evidence in favour of ΛCDM because BIC imposes a stronger penalty on additional parameters. Consistently, the best–fit value $w^{\mathrm{MLE}} = -1.0045$ lies extremely close to $w = -1$, at which point wCDM reduces exactly to ΛCDM.

Finally, we also perform a one–parameter Bayesian inference for $\Omega_{m,0}$ under ΛCDM, assuming a uniform prior $\Omega_{m,0} \sim \mathcal{U}(0, 1)$. Using EMCEE with 32 walkers, 3000 steps, we obtain the posterior

$$\Omega_{m,0} = 0.278^{+0.014}_{-0.013},$$

corresponding to a central 68% credible interval. The posterior is very close to a Gaussian centered on the MLE (Fig. 6).

4 Conclusions

In this project we have used the SCP Union2.1 Type Ia supernova compilation to infer properties of the late–time expansion and to compare simple cosmological models. From the small–z analysis (Sec. 3.1) we inferred a negative deceleration parameter, providing direct evidence for accelerated expansion. Our best–fit q_0 is somewhat less negative than the estimates $q_0 \simeq -0.55$ obtained from SCP data by Bochner et al. [1] and the constraints $q_0 \simeq -0.58$ reported by Rahman [2]. A likely reason is that these studies include higher–order terms in the Taylor expansion and combine several cosmological probes, whereas we use a second–order expansion and SNe alone. Nevertheless, our result is statistically compatible with theirs and leads to the same qualitative conclusion. The inferred Hubble constant in our small–z fit is also very close to the reference value $H_0 = 70\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$ adopted elsewhere in the assignment.

In our second task we used the SCP 2.1 sample to compare ΛCDM and wCDM. The best–fit w in the wCDM model is extremely close to -1, so that the preferred wCDM cosmology is effectively indistinguishable from ΛCDM, and our posterior for $\Omega_{m,0}$ in ΛCDM peaks near $\Omega_{m,0} \simeq 0.28$. This is fully consistent with the matter and dark–energy densities inferred from Union2.1 and related SN compilations in the literature, e.g. the flat ΛCDM constraints $\Omega_m \simeq 0.28$, $\Omega_\Lambda \simeq 0.72$ reported by Suzuki et al. [3], and the combined SN+BAO+CMB fits $\Omega_m = 0.292 \pm 0.015$, $w = -0.99 \pm 0.04$ found by Shi et al. [4] and by Arévalo et al. [5]. Our AIC and BIC values differ between wCDM and ΛCDM only through the penalty for the extra parameter w, since the maximum likelihoods are essentially identical. The resulting differences, $\Delta\mathrm{AIC} \approx 2$ and $\Delta\mathrm{BIC} \approx 6$ in favor of ΛCDM, indicate that the current SN data do not provide compelling evidence for freeing w from -1. These numbers are remarkably close to the values reported by Shi et al. [4], who find $\Delta\mathrm{AIC} \simeq 1.98$ and $\Delta\mathrm{BIC} \simeq 6.38$ for wCDM relative to ΛCDM when combining Union2.1 with BAO, CMB and $H(z)$ data, and they are qualitatively in line with the AIC/BIC comparisons of interacting dark–energy models in Arévalo et al. [5].

From a practical perspective, the most challenging part of the project was the wCDM fit and the MCMC implementation. Early on, the optimizer tended to drive $\Omega_{m,0}$ towards zero or back to its initial guess, and we had to identify and cure numerical issues in the luminosity–distance calculation and in the treatment of very small redshifts. The extremely close agreement between the ΛCDM and wCDM fits was also initially suspicious until we compared our results to the literature and found that this degeneracy is not too bad. Running the MCMC chains, especially for the larger models with many variables, took a lot of time and computer power(It took more 30 min for each run in the Task 2).

Looking ahead, the inferences for H_0 and q_0 could be improved by including higher order terms in the Taylor expansion or by extending the analysis with external data such as CMB or BAO constraints, as in the published studies. In addition, replacing a single inverse–gamma prior on a global intrinsic variance σ^2 with a hierarchical model allowing for supernova–to–supernova or redshift–dependent scatter, and using more informative priors in the cosmological fits, would bring our simplified framework closer to state–of–the–art analyses while retaining the Bayesian structure developed in this work.

References

[1] Brett Bochner, Daniel Pappas, and Jiangang Dong. Testing Λ and the limits of cosmography with the union2.1 supernova compilation. *Journal of Cosmology and Astroparticle Physics*, 06:036, 2013. doi: 10.1088/1475-7516/2013/06/036.

[2] M. Rahman. Constraints on deceleration, jerk and transition redshift in the late-time universe. *Monthly Notices of the Royal Astronomical Society*, 507(3):3754–3765, 2021. doi: 10.1093/mnras/stab2356.

[3] N. Suzuki, D. Rubin, C. Lidman, G. Aldering, R. Amanullah, et al. The hubble space telescope cluster supernova survey. v. improving the dark-energy constraints above $z > 1$ and building an early-type-hosted supernova sample. *Astrophysical Journal*, 746(1):85, 2012. doi: 10.1088/0004-637X/746/1/85.

[4] Ke Shi, Q.-G. Huang, and T. Lu. A comprehensive comparison of cosmological models from the latest observational data. *Monthly Notices of the Royal Astronomical Society*, 426(3):2452–2461, 2012. doi: 10.1111/j.1365-2966.2012.21899.x.

[5] F. Arévalo, A. Cid, and J. Moya. Aic and bic for cosmological interacting scenarios. *European Physical Journal C*, 77(8):565, 2017. doi: 10.1140/epjc/s10052-017-5128-7.