Math 74: Algebraic Topology

Sair Shaikh

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Problem 1. Compute the fundamental group of the complement of $k \geq 1$ points in the orientable surface M_g of genus g.

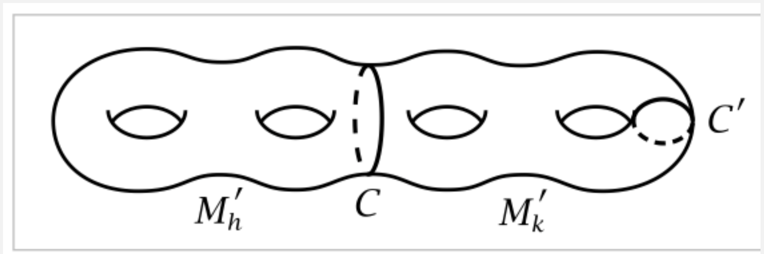
Solution. The orientable surface M_g of genus g can be realized as a $4g$-gon with pairs of edges identified becoming a $\bigwedge^{2g} S^1$, glued along the boundary with a disk. Let x_0 be the vertex of the polygon (all vertices are identified together).The fundamental group of M_g is given by the presentation:

$$\pi_1(M_g, x_0) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$$

We puncture this at k points in the interior of the polygon to get our space.

Let x_0 be the vertex of the CW complex. Draw $\alpha_1, \cdots, \alpha_k - 1$ around $k - 1$ of the holes. Each of these loops bounds a disk with a single hole, that can be contracted to this loop. The remaining center of the polygon is a 2-cell with one hole that can be contracted to the boundary. Notably, this removes the 2-cell and all the relations on the generators of the fundamental group. Thus, we have $2g$ free generators $a_1, b_1, \cdots, a_g, b_g$ and $k - 1$ additional generators $\alpha_1, \cdots, \alpha_k$ corresponding to the loops around the punctures. Thus, we get the free group on $2g + k - 1$ generators.

Problem 2. In the orientable surface M_g of genus g, let C be a circle that separates M_g into two compact subsurfaces M'_h and M'_k obtained from the closed surfaces M_h and M_k by deleting an open disk from each. Show that M'_h does not retract onto its boundary circle C, and hence M_g does not retract onto C. (*Hint:* abelianize π_1.) On the other hand, show that M_g does retract onto the nonseparating circle C' in the figure.



Solution. Note that M'_h is equivalent to a disk removed from M_h. In the CW complex picture, M_h looks like $\bigwedge^2 hS^1$, with a 2-cell attached along the boundary. Puncturing M_h involves puncturing the 2-cell, which then deformation retracts to its boundary, $\bigwedge^{2h} S^1$ and thus has fundamental group:

$$\pi_1(M'_h) = \langle a_1, b_1 \cdots, a_h, b_h \rangle$$

The abelianization of this is $\mathbb{Z}^{2h}$. If there existed a retraction $r : M'_h \to C$, then we would have $\iota_* : \pi_1(C) \to \pi_1(M'_h)$ injective (as $r_* \circ \iota_* = \text{id}_{\pi_1(C)}$). Thus, as abelianization is a functor, we induce:

$$\iota_*^{ab} : \mathbb{Z} \to \mathbb{Z}^{2h}$$

Since $\mathbb{Z}$ is abelian, this map must also be injective (as the image of ι_* was abelian to begin with). However, note that the generator of $\pi_1(C)$ is mapped by ι_* to the commutator $[a_1, b_1] \cdots [a_h, b_h]$ in $\pi_1(M'_h)$ (as we can easily see from the CW complex picture). In the abelianization, this is trivial, thus ι_*^{ab} is the trivial map, which is not injective. This is a contradiction. Thus, M'_h does not retract onto C. This implies that M_g does not retract onto C as restricting this would give a retraction of M'_h onto C.

However, consider a non-separating circle C'. Note that the CW complex picture of M_g is a $4g$-gon (cyclically $a_1 \cdot b_1 \cdot a_1^{-1} \cdot b_1^{-1} \cdot a_2 \cdot \ldots \cdot b_n^{-1} a_1$) with pairs of edges identified. We can identify C' with one of the 1-cells, a_1, on the boundary. Note next that we can retract the CW complex onto the line joining the end of b^{-1} to the start of a_1, and then retract this further to the identify the end of b^{-1} with the start of a_1. This trivially does not violate any of the edge identifications on the $4g$-gon. Then, we are left with a square with opposite edges identified (a torus). We can vertically project this square down to a (this does not violate any identifications, as identified points on a go to their counterparts, whereas for b they just retract to a single identified point on both ends of a). Thus, we have obtained a retraction from M_g to C'.

Problem 3. Suppose that we construct a space M from gluing two copies of the solid torus $S^1 \times D^2$ along their boundary tori by a map $f : S^1 \times S^1 \to S^1 \times S^1$. Compute $\pi_1(M)$ when

1. f is the identity map.
2. f swaps the two circles, that is, $f(x, y) = (y, x)$.

Solution. This can be done using CW-complexes, however, I will attempt to do this via Seifert Van Kampen since that appears clearer to me.
Let X be the whole space. Let U be the first solid torus together with a contractible neighborhood of the second. Let V be the second solid torus together with a contractible neighborhood of the first. Then $U \cap V$ is the identified boundary of the two solid tori, alongside contractible neighborhoods of both tori, which is path-connected. Clearly, U and V are open, $X = U \cup V$. Thus, we can apply Seifert Van Kampen Theorem.

Note U and V are contractible to solid tori. Then, $\pi_1(U) = \langle a \rangle$ and $\pi_2(V) = \langle b \rangle$, where a and b are loops that go around the (not filled in) S^1 of the two solid tori respectively. Thus, we have that:

$$\pi_1(X) = \langle a, b \rangle / N$$

where N is the normal subgroup generated by identifying pushforwards of the generators of $U \cap V$, which depends on the map f. Let α and β be these generators for $\pi_1(U \cap V) = \pi_1(S^1 \times S^1)$, going around longitudinally and around the meridian. Then we have the following cases:

1. Let f be the identity map, then the pushforward of α using the inclusions goes along the meridian of both solid tori U and V and the pushforward of β goes along the longitude of both solid tori. Thus, one of these is the boundary of a disk in both U and V, hence trivial, while the other maps to a and b in U and V respectively. Thus,
$$N = \langle ab^{-1} \rangle$$
Thus, we have that:
$$\pi_1(X) = \langle a, b \mid ab^{-1} \rangle = \langle a \rangle$$
2. Let f be the map that swaps the two circles. Then the pushforward of α using the inclusions goes along the meridian of U and the longitude of V, while the pushforward of β goes along the longitude of V and the meridian of V. For each of these, their image is the boundary of a disk in one of the two solid tori, and corresponds to the generator in the other. Thus, we have:
$$N = \langle a, b \rangle$$
Thus,
$$\pi_1(X) = \{1\}$$

Problem 4. Show that if a path-connected, locally path-connected space X has finite $\pi_1(X)$ (e.g., S^n, $\mathbb{RP}^n$), then every map $X \to S^1$ is nullhomotopic. (*Hint:* Use the covering space $\mathbb{R} \to S^1$.)

Solution. Let $f : (X, x_0) \to (S^1, (1, 0))$ be arbitrary (assume wlog that $(1, 0)$ is in the image as some point must be). Let $p : (\mathbb{R}, 0) \to (S^1, (1, 0))$ be the usual covering map $p(t) \mapsto e^{2\pi i t}$. Note that since $\pi_1(\mathbb{R}) = \{1\}$, we have $p_*(\pi_1(\mathbb{R})) = \{1\}$. Moreover, since $\pi_1(S^1) = \mathbb{Z}$ and $\pi_1(X)$ is finite, $f_* : \pi_1(X) \to \pi_1(S^1)$ is forced to be trivial (the only finite subgroup of $\mathbb{Z}$ is trivial). Thus, we have $f_*(\pi_1(X)) \subseteq p_*(\pi_1(\mathbb{R}))$. Since X is path-connected and locally path-connected, by the general lifting theorem, we can lift f to a map $\tilde{f} : X \to \mathbb{R}$ such that $p \circ \tilde{f} = f$ and $\overline{f}(x_0) = 0$. From Problem Set 1 Problem 7, we know that $\mathbb{R}$ is contractible, and that that all maps into a contractible space are nullhomotopic. In this case, we can also explicitly construct a straight-line homotopy to the zero map. Thus, $\overline{f}$ is nullhomotopic. Thus, f is nullhomotopic as $[f] = [p \circ \tilde{f}] = [p \circ \lambda] \in [X, S^1]$, where λ is constant, so $p \circ \lambda$ is also constant.

Problem 5. Show that there is no covering map:

1. From $\mathbb{RP}^2$ to the torus.
2. From the torus to $\mathbb{RP}^2$.
3. From $\mathbb{R}^2$ to $\mathbb{RP}^2$.

Solution.

1. Note that if we have $p : \mathbb{RP}^2 \to T^2$ a covering map, then p_* is injective. However, $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$ and $\pi_1(T^2) = \mathbb{Z}^2$. Thus, p_* cannot be injective as the only finite subgroup of $\mathbb{Z}^2$ is trivial. Thus, there cannot be a covering map from $\mathbb{RP}^2$ to T^2.

2. Similarly, if $p : T^2 \to \mathbb{RP}^2$ was a covering map, then $p_* : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ would be injective. However, since $\mathbb{Z}/2\mathbb{Z}$ is finite and $\mathbb{Z} \times \mathbb{Z}$ is not, this is impossible. Thus, there cannot be a covering map from T^2 to $\mathbb{RP}^2$.

3. Note that we proved that S^2 is a cover for $\mathbb{RP}^2$. Since it is simply connected, it is the universal cover. Suppose $\mathbb{R}^2$ was also a cover for $\mathbb{RP}^2$. Then, by the uniqueness of the universal cover, we would have that $\mathbb{R}^2$ is homeomorphic to S^2 (equivalent coverings). But this is not possible as $\mathbb{R}^2$ is not compact while S^2 is.