Math 113: Functional Analysis

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Problem 30. Suppose that X and Y are normed vector spaces.

1. Show that $\mathcal{L}(X, Y)$ is a normed vector space with respect to the the operator norm defined in lecture such that:

$$||T(x)|| \leq ||T|| ||x||$$

2. Show that if $S \in \mathcal{L}(Y, Z)$. Then,

$$||ST|| \leq ||S|| ||T||$$

3. Show that:

$$||T|| = \inf\{a \geq 0 : ||T(x)|| \leq a||x|| \quad \forall x \in X\}$$

Solution.

1. First, note that $\mathcal{L}(X, Y)$ is a vector space, through pointwise addition and scalar multiplication defined in Y, i.e. for $T, S \in \mathcal{L}(X, Y)$ and $\alpha \in \mathbb{F}$ we let:

$$(\alpha T + S)(x) := \alpha T(x) + S(x)$$

for all $x \in X$. Thus, we only need to show that the operator norm is a norm and satisfies the given property. Recall the definition of the operator norm for $T \in \mathcal{L}(X, Y)$:

$$||T|| = \sup_{||x|| \leq 1} ||T(x)||$$

Non-Neg. Let $T \in \mathcal{L}(X, Y)$ be arbitrary. Then, for all $x \in X$ with $||x|| \leq 1$, $||T(x)|| \geq 0$ by the non-negativity of the norm on Y. Thus, $||T|| \geq 0$.

Homogeneity. Let $T \in \mathcal{L}(X, Y)$ and $\alpha \in \mathbb{F}$. Then, for every $x \in X$, with $||x|| \leq 1$, we have that:

$$\begin{aligned} ||(\alpha T)(x)|| &= ||\alpha T(x)|| \\ &= |\alpha| \cdot ||T(x)|| \\ &\leq |\alpha| \cdot ||T|| \end{aligned}$$

using the homogeneity of the norm on Y. Thus,

$$||\alpha T|| \leq |\alpha| \cdot ||T(x)||$$

Similarly, we also have:

$$\begin{aligned} |\alpha| \cdot ||T(x)|| &= ||\alpha T(x)|| \\ &= ||(\alpha T)(x)|| \\ &\leq ||\alpha T|| \end{aligned}$$

Thus, we have:

$$|\alpha| \cdot ||T|| \leq ||\alpha T||$$

Thus, we have shown that:

$$||\alpha T|| = |\alpha| \cdot ||T||$$

$\triangle$ ineq. Let $T, S \in \mathcal{L}(X, Y)$. For every $x \in X$, with $||x|| \leq 1$, we have:

$$\begin{aligned} ||(T+S)(x)|| &= ||T(x) + S(x)|| \\ &\leq ||T(x)|| + ||S(x)|| \\ &\leq ||T|| + ||S|| \end{aligned}$$

using the triangle inequality for the norm in Y. Thus,

$$||T + S|| \leq ||T|| + ||S||$$

Pos. Def. To prove this, we first will show that $||T(x)|| \leq ||T||||x||$ for all $x \in X$. For $x \in X$, by homogeneity (and non-negativity) of the norm on X:

$$\left|\left|\frac{1}{||x||}x\right|\right| = \frac{1}{||x||} \cdot ||x|| = 1$$

Thus, we note that:

$$\left|\left|T\left(\frac{1}{||x||}x\right)\right|\right| \leq ||T||$$

By the linearity of T and the homogeneity of the norm in Y, this implies:

$$\begin{aligned} \frac{1}{||x||}||T(x)|| &= \left|\left|\frac{1}{||x||}T(x)\right|\right| \\ &= \left|\left|T\left(\frac{1}{||x||}x\right)\right|\right| \\ &\leq ||T|| \end{aligned}$$

Thus,

$$||T(x)|| \leq ||T|| \cdot ||x||$$

Now, let $T \in \mathcal{L}(X,Y)$ be such that $||T|| = 0$. Then, for all $x \in X$, we have that:

$$\begin{aligned} ||T(x)|| &\leq ||T|| \cdot ||x|| \\ &= 0 \end{aligned}$$

However, by the non-negativity of the norm in Y, we must have that $||T(x)|| = 0$ for all $x \in X$. Then, by positive definiteness of the norm in Y, we have that $T(x) = 0$ for all $x \in X$. Thus, T is the zero map.

Conversely, if T is the zero map, then for all $x \in X$ with $||x|| \leq 1$, we have that:

$$||T|| = ||0 \cdot T|| = 0 \cdot ||T|| = 0$$

by homogeneity. Thus, the norm is positive definite.

Therefore, we have shown that the operator norm is a norm on $\mathcal{L}(X,Y)$ and satisfies for all $T \in \mathcal{L}(X,Y)$ and $x \in X$:

$$||T(x)|| \leq ||T|| \cdot ||x||$$

2. For any $x \in X$, with $||x|| = 1$, by applying the property from part 1 twice, we have that:

$$\begin{aligned} ||ST(x)|| &= ||S(T(x))|| \\ &\leq ||S|| \cdot ||T(x)|| \\ &\leq ||S|| \cdot ||T|| \cdot ||x|| \\ &= ||S|| \cdot ||T|| \end{aligned}$$

Thus,

$$||ST|| \leq ||S|| \cdot ||T||$$

3. Let $\alpha(T)$ be the defined infimum.

Since we have that $||T(x)|| \leq ||T|| \cdot ||x||$ for all $x \in X$, $||T||$ is in the set we are taking the infimum over. Thus, $\alpha(T) \geq ||T||$.

Moreover, by the definition of $\alpha(T)$, we have that for all $x \in X$ with $||x|| \leq 1$,

$$||T(x)|| \leq \alpha(T)||x|| = \alpha(T)$$

Thus, $\alpha(T)$ is an upperbound on $||T(x)||$ with $||x|| \leq 1$. Therefore, by the definition of the supremum, we have that:

$$||T|| \leq \alpha(T)$$

Thus, we have shown that:

$$||T|| = \alpha(T) = \inf\{a \geq 0 : ||T(x)|| \leq a||x|| \quad \forall x \in X\}$$

Problem 31. Suppose that X and Y are Banach spaces with $T \in \mathcal{L}(X, Y)$. Suppose that E is a closed proper subspace of X such that $E \subset \ker(T)$. Show that there is a unique operator $\overline{T} \in \mathcal{L}(X/E, Y)$ such that $\overline{T}(q(x)) = T(x)$ for all $x \in X$ where $q : X \to X/E$ is the quotient map. Moreover, $||\overline{T}|| = ||T||$.

Solution. We claim the map $\overline{T} : X/E \to Y$ given by:

$$\overline{T}([x]) = T(x)$$

satisfies the desired properties. We need to show that $\overline{T}$ is well-defined, linear, satisfies $\overline{T}(q(x)) = T(x)$ for all $x \in X$, and satisfies $||\overline{T}|| = ||T||$ (hence is bounded/continous).

Let $x, y \in X$ be such that $[x] = [y]$. Then, note that $x - y \in E \subset \ker(T)$. Thus, $T(x) - T(y) = T(x - y) = 0$. Thus, $T(x) = T(y)$. Then, by the definition of $\overline{T}$, we have $\overline{T}([x]) = \overline{T}([y])$. Thus, $\overline{T}$ is well-defined.

Next, let $[x], [y] \in X/E$ and $\alpha \in \mathbb{F}$ be arbitrary. Then, noting the linearity of q and T, we have that:

$$\begin{aligned}
\overline{T}(\alpha[x] + [y]) &= \overline{T}([\alpha x + y]) \\
&= T(\alpha x + y) \\
&= \alpha T(x) + T(y) \\
&= \alpha \overline{T}([x]) + \overline{T}([y])
\end{aligned}$$

Thus, $\overline{T}$ is linear.

Problem 33. Let E and X be Banach spaces with E finite dimensional.

1. Show that every linear map $S : E \to X$ is bounded.
2. Show that a linear map $T : X \to E$ is bounded if and only if $\ker(T)$ is closed.

Solution.

Problem 34. Supposed that E and M are closed subspaces of a Banach space X. If E is finite dimensional, show that $E + M = \{x + y : x \in E\, y \in M\}$ is closed.

Solution.

Problem 35. Suppose that X and Y are Banach spaces for $T \in \mathcal{L}(X, Y)$. Show that T is injective with closed range if and only if:

$$\inf\{||T(x)|| : ||x|| = 1\} > 0$$

Solution.

Problem 38. Let X be a normed vector space. A Banach space $\tilde{X}$ is called a completion of X is there is an isometric isomorphism $\iota : X \to \tilde{X}$ onto a dense subspace of $\tilde{X}$. Show that any two completions $(tildeX_1, \iota_1)$ and $(\tilde{X}_2, \iota_2)$ are isometrically isomorphic by an isomorphism:

$$\Phi : \tilde{X}_1 \to \tilde{X}_2$$

such that $\Phi(\iota_1(x)) = \iota_2(x)$ for all $x \in X$.

Solution.

Problem 39. Lets find a use for a genuine Minkowski functional. In this problem, we'll let $l^\infty_\mathbb{R}$ be the real Banach space of bounded sequences in $\mathbb{R}$. Define m on $l^\infty_\mathbb{R}$:

$$m(x) = \text{limsup}_n\, x_n$$

We clearly have $m(tx) = tm(x)$ if $t \geq 0$ and it is not hard to check that $m(x+y) \leq m(x) + m(y)$ for all $x, y \in l^\infty_\mathbb{R}$. We want to show that there are Banach limits or what I prefer to call a generalized limit on $l^\infty_\mathbb{R}$. This is we want to show that there is a functional $L \in {l^\infty_\mathbb{R}}^*$ such that:

$$L(S(x)) = L(x)$$

where $S \in \mathcal{L}(l^\infty_\mathbb{R})$ is given by $S(x)_n = x_{n+1}$ and such that $\liminf_n x_n \leq L(x) \leq \text{limsup}_n\, x_n$. (Hint provided).

Solution.

Problem 40. Prove the following Lemma from lecture. Let X be a complex vector space. Every real linear functional of X is the real part of a complex linear functional on X. In fact, if $\phi = \Re(\psi)$ then $\psi(x) = \phi(x) - i\phi(ix)$.

Solution.

Problem 41. Suppose that X is a normed vector space such that X^* is seperable. Show that X is seperable. (Hint provided).

Solution.