

Math 113: Functional Analysis

Sair Shaikh

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Problem 54. Suppose that X is a reflexive Banach space. Show that the unit ball $B = \{x \in X : \|x\| \leq 1\}$ is weakly compact. (Hint provided).

Solution.

Problem 59. Let E be a nonempty subset of a Hilbert space H . Let Y be a subspace spanned by E . Then $E^{\perp\perp}$ is the closure of Y in H .

Solution. We show both inclusions separately.

Let $y \in Y$ be arbitrary. Then, we have that $y = \sum_{i=1}^n a_i e_i$ for some $e_i \in E$ and $a_i \in \mathbb{F}$ (\mathbb{R} or \mathbb{C}). Then, for all $w \in E^\perp$, we have that:

$$\begin{aligned} (y \mid w) &= \left(\sum_{i=1}^n a_i e_i \mid w \right) \\ &= \sum_{i=1}^n a_i (e_i \mid w) \\ &= 0 \end{aligned}$$

Thus, $y \in E^{\perp\perp}$. Therefore, $Y \subseteq E^{\perp\perp}$ (as y was arbitrary). As noted in class, since the inner-product is continuous, $E^{\perp\perp}$ is closed. Thus, we have that $\overline{Y} \subseteq E^{\perp\perp}$.

Next, let $z \in E^{\perp\perp}$. Since \overline{Y} is closed, we have that $H = \overline{Y} \oplus \overline{Y}^\perp$. Thus, we can write $z = y + y'$ where $y \in \overline{Y}$ and $y' \in \overline{Y}^\perp$. It suffices to show $y' = 0$ and thus, $z = y \in \overline{Y}$, implying $E^{\perp\perp} \subseteq \overline{Y}$ (as z was arbitrary).

Let $w \in \overline{Y}^\perp$. Then, as $E \subset \overline{Y}$, we have that:

$$(w \mid e) = 0 \quad \forall e \in E$$

Thus, we have $w \in E^\perp$, thus, $\overline{Y}^\perp \subseteq E^\perp$. In particular, $y' \in E^\perp$. Moreover, $(y \mid y') = 0$ as $y \in \overline{Y}$ and $y' \in \overline{Y}^\perp$. Since $z \in E^{\perp\perp}$, we have:

$$\begin{aligned} 0 &= (z \mid y') \\ &= (y \mid y') + (y' \mid y') \\ &= (y' \mid y') \end{aligned}$$

Thus, $y' = 0$ by positive definiteness of the inner product. Therefore, $z = y \in \overline{Y}$. Thus, we have that $E^{\perp\perp} \subseteq \overline{Y}$.

Therefore, we have that $E^{\perp\perp} = \overline{Y}$.

Problem 60. Let $X = l^2$. Show that the sequence $\{e_n\}$ of standard basis vectors converges weakly to 0.

Solution. Let $\phi \in (l^2)^*$ and $\epsilon > 0$. We need to show that there exists an N such that for all $n \geq N$, $e_n \in U := \{x \in l^2 : |\phi(x)| < \epsilon\}$. Since $(l^2) \cong (l^2)^*$ (isometric isomorphism) via the map $y \rightarrow \Phi_y$ defined by:

$$\Phi_y(x) = \sum_{n=1}^{\infty} x_n y_n$$

Thus, there exists a $y \in l^2$ such that $\phi = \Phi_y$. Then, we have:

$$\phi(e_n) = y_n$$

Then, since $y \in l^2$, we have that $\sum_{n=1}^{\infty} |y_n|^2 < \infty$. Thus, there exists an N , such that $\sum_{n=N+1}^{\infty} |y_n|^2 < \epsilon$. Thus, for all $n \geq N$, we have that:

$$|\phi(e_n)| = |y_n| < \epsilon$$

Thus, $e_n \in U$ for all $n \geq N$. Therefore, $\{e_n\}$ converges weakly to 0.

Problem 61. Let H be a hilbert space. If $x, y \in H$, define $\Theta_{x,y} : H \rightarrow H$ by:

$$\Theta_{x,y}(z) = (z \mid y)x$$

Compute the norm of $\Theta_{x,y}$ and its adjoint $\Theta_{x,y}^*$.

Solution. First, note that if $x = 0$ or $y = 0$, then $\Theta_{x,y}(z) = (z \mid y)x = 0$ for all $z \in H$. Thus, $\|\Theta_{x,y}\| = 0$. Thus, assume $x \neq 0$ and $y \neq 0$. For $z \in H$ by homogeneity of the norm and Cauchy-Schwarz we have:

$$\begin{aligned} \|\Theta_{x,y}(z)\| &= \|(z \mid y)x\| \\ &= |(z \mid y)| \|x\| \\ &\leq \|z\| \|y\| \|x\| \end{aligned}$$

Thus, $\|\Theta_{x,y}\| \leq \|y\| \|x\|$. Let $z = \frac{y}{\|y\|}$. Then, $\|z\| = 1$. Moreover, we have:

$$\begin{aligned} \left\| \Theta_{x,y} \left(\frac{y}{\|y\|} \right) \right\| &= \left\| \left(\frac{y}{\|y\|} \mid y \right) x \right\| \\ &= \frac{1}{\|y\|} \|y\|^2 \|x\| \\ &= \|y\| \|x\| \end{aligned}$$

Thus, the bound is achieved. Therefore, $\|\Theta_{x,y}\| = \|y\| \|x\|$. This also captures the $x = 0$ or $y = 0$ case.

Next, let $z, w \in H$. Then, we have that:

$$\begin{aligned} (\Theta_{x,y}z \mid w) &= ((z \mid y)x \mid w) \\ &= (z \mid y)(x \mid w) \\ &= (z \mid y)\overline{(w \mid x)} \\ &= (z \mid (w \mid x)y) \end{aligned}$$

Thus, we can define $\Theta_{x,y}^*(w) = (w \mid x)y$. Notice that:

$$\Theta_{x,y}^* = \Theta_{y,x}$$

Thus, by the previous argument, we have that:

$$\|\Theta_{x,y}^*\| = \|\Theta_{y,x}\| = \|y\| \|x\|$$

Problem 64. Let $P \in \mathcal{L}(H)$ be the orthogonal projection onto a nonzero subspace W . Show that $P = P^* = P^2$ and that $\|P\| = 1$. Conversely, show that if $P \in \mathcal{L}(H)$ and $P = P^* = P^2$, then P is an orthogonal projection onto its range.

Solution. We assume W is closed as the definition of P requires it. Let P^* be the adjoint of P . Note that $x - Px \in W^\perp$ for all $x \in H$. Let $h, k \in H$. Note that:

$$\begin{aligned} (Ph \mid k) &= (Ph \mid Pk + (k - Pk)) \\ &= (Ph \mid Pk) + (Ph \mid k - Pk) \\ &= (Ph \mid Pk) + 0 \\ &= (Ph \mid Pk) + (h - Ph \mid Pk) \\ &= (Ph + (h - Ph) \mid Pk) \\ &= (h \mid Pk) \end{aligned}$$

Thus, P is self-adjoint. Moreover, note that:

$$\begin{aligned} (P^2h \mid k) &= (Ph \mid Pk) \\ &= (Ph \mid Pk) + (Ph \mid k - Pk) \\ &= (Ph \mid k) \end{aligned}$$

for all $h, k \in H$. Thus, $P^2 = P$.

First, note that by the Pythagorean theorem, for all $h \in H$,

$$\begin{aligned} \|Ph\|^2 &\leq \|Ph\|^2 + \|h - Ph\|^2 \\ &= \|h\|^2 \end{aligned}$$

Thus, $\|Ph\| \leq \|h\|$. Therefore, $\|P\| \leq 1$.

Next, pick $h \in W$ with $\|h\| = 1$ (we can do this as W is a subspace and we can divide any non-zero vector by its norm to get norm 1). Then, $Ph = h$, thus $\|Ph\| = \|h\|$. Thus, $\|P\| \geq 1$ (by the sup definition of the operator norm). Therefore, $\|P\| = 1$.

Conversely, let $P \in \mathcal{L}(H)$ such that $P = P^* = P^2$. We need to show that P is an orthogonal projection onto $V := \text{img}(P)$. Thus, we need to show that V is closed and $H = V \oplus V^\perp$.

Let $x \in H$ and $Py \in V$ be arbitrary. Then note:

$$\begin{aligned} (x - Px \mid Py) &= (x \mid Py) - (Px \mid Py) \\ &= (x \mid Py) - (x \mid P^*Py) \\ &= (x \mid Py) - (x \mid P^2y) \\ &= (x \mid Py) - (x \mid Py) \\ &= 0 \end{aligned}$$

Thus, as Py was arbitrary, $x - Px \in V^\perp$. Therefore, we have that $H = V + V^\perp$. Next, let $v \in V \cap V^\perp$. Then, we have $\langle v, v \rangle = 0$ thus $v = 0$. Thus, $V \cap V^\perp = \{0\}$. Therefore, we have that $H = V \oplus V^\perp$.

Then, note that V^\perp is closed as the inner product is continuous (as we noted in class). Then, we note that $H = V^\perp \oplus V^{\perp\perp}$ by the orthogonal projections theorem we proved in class. Moreover, $V^{\perp\perp} = \overline{V}$ (Problem 59). Thus, we have:

$$H = V \oplus V^\perp = \overline{V} \oplus V^\perp$$

Then, let $x \in \overline{V} \setminus V$. Then, as $H = V \oplus V^\perp$, we must have $x \in V^\perp$. However, $\overline{V} \cap V^\perp = \{0\}$, thus $x = 0$. But $0 \in V$ as V is a subspace. Thus, we have a contradiction, and $\overline{V} \subseteq V$ is empty. Thus, $V = \overline{V}$, i.e. V is closed.

Thus, we have shown that P is an orthogonal projection onto its range $V = \text{img}(P)$.

NOTE: Is this what the definition of orthogonal projection is?

Problem 65.(Dini's Theorem) Suppose that X is a compact metric space and that $C(X)$ is the Banach space of real-valued functions on X . Show that if $(f_n) \subset C(X)$ is such that there is a $f \in C(X)$ such that $f_n(x) \nearrow f(x)$ for all $x \in X$, then $f_n \rightarrow f$ in $C(X)$. Equivalently, show that $f_n \rightarrow f$ uniformly on X . (Hint provided).

Solution.

Problem 66. A linear map $V : H \rightarrow H$ is called an isometry if $\|V(x)\| = \|x\|$ for all $x \in H$. Show that the following are equivalent:

1. V is an isometry.
2. $(V(x) \mid V(y)) = (x \mid y)$ for all $x, y \in H$.
3. $V^*V = I$.

Solution. We show $(1) \implies (2)$, $(2) \implies (3)$, and $(3) \implies (1)$.

$((1) \implies (2))$. Let $x, y \in H$ be arbitrary. Then, by the polarization identity (for $\mathbb{F} = \mathbb{C}$), we have:

$$\begin{aligned} (Vx \mid Vy) &= \frac{1}{4} (\|Vx + Vy\|^2 - \|Vx - Vy\|^2 + i\|Vx + iVy\|^2 - i\|Vx - iVy\|^2) \\ &= \frac{1}{4} (\|V(x + y)\|^2 - \|V(x - y)\|^2 + i\|V(x + iy)\|^2 - i\|V(x - iy)\|^2) \\ &= \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2) \\ &= (x \mid y) \end{aligned}$$

and for $\mathbb{F} = \mathbb{R}$, we have:

$$\begin{aligned} (Vx \mid Vy) &= \frac{1}{4} (\|Vx + Vy\|^2 - \|Vx - Vy\|^2) \\ &= \frac{1}{4} (\|V(x + y)\|^2 - \|V(x - y)\|^2) \\ &= \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2) \\ &= (x \mid y) \end{aligned}$$

Thus, we have that $(Vx \mid Vy) = (x \mid y)$ for all $x, y \in H$.

$((2) \implies (3))$. Let $y \in H$ be arbitrary. Then, for all $x \in H$, we have:

$$\begin{aligned} (x \mid y) &= (Vx \mid Vy) \\ &= (x \mid V^*Vy) \end{aligned}$$

Thus, $(x \mid (y - V^*Vy)) = 0$ for all $x \in H$. Letting $x = y - V^*Vy$, we have that $y - V^*Vy = 0$ by the positive definiteness of the inner product. Thus, $V^*Vy = y$. Since y was arbitrary, $V^*V = I$.

((3) \implies (1)). Let $x \in H$ be arbitrary. Then, we have:

$$\begin{aligned}
 \|Vx\|^2 &= (Vx \mid Vx) \\
 &= (x \mid V^*Vx) \\
 &= (x \mid Ix) \\
 &= (x \mid x) \\
 &= \|x\|^2
 \end{aligned}$$

Thus, $\|Vx\| = \|x\|$ for all $x \in H$. Therefore, V is an isometry.

Thus, we have shown that the three statements are equivalent.

Problem 67. A surjective isometry $U : H \rightarrow H$ is called a unitary. Show that the following are equivalent for $U \in \mathcal{H}$.

1. U is a unitary.
2. U is invertible with $U^{-1} = U^*$.
3. If $\{e_n\}$ an orthonormal basis for H , then $\{U(e_n)\}$ is an orthonormal basis for H .

(Remark: (c) implies (a) is not true unless U is both linear and bounded.)

Solution. We show (1) \implies (2), (2) \implies (3), and (3) \implies (1).

((1) \implies (2)). Let U be a unitary. Then, as U is an isometry, we have that:

$$\|Ux\| = 0 \iff \|x\| = 0 \iff x = 0$$

Thus, U is injective. Since U is surjective, we have that U is a bijection. By a direct corollary of the Open Mapping theorem, we have that U is invertible (proved in class). Then, as $U^*U = I$ (Problem 66), we have that $U^* = U^{-1}$ by the uniqueness of the inverse.

((2) \implies (3)). Let $\{e_n\}$ be an orthonormal basis for H . Let $h \in H$ be such that:

$$(h \mid U(e_n)) = 0$$

for all n . It suffices to show that this implies $h = 0$ by one of the alternative characterizations of an orthonormal basis (proved in class).

As U is invertible, hence surjective, there exists $h' \in H$ such that $U(h') = h$. Moreover, as $U^* = U^{-1}$ implies $U^*U = I$, we use the results of Problem 66 to get:

$$(U(h') \mid U(e_n)) = (h' \mid e_n) = 0$$

for all n . Then, as $\{e_n\}$ is an orthonormal basis for H , we have that $h' = 0$. Then, $h = U(h') = 0$. Thus, $\{U(e_n)\}$ is an orthonormal basis.

((3) \implies (1)). Assume that if $\{e_n\}$ is an orthonormal basis for H , then $\{U(e_n)\}$ is an orthonormal basis for H . Since $\{Ue_n\}$ is an orthonormal basis, we can write $x \in H$ as:

$$x = \sum_{n=1}^{\infty} (x \mid Ue_n) Ue_n$$

Then, as U is linear and continuous, we can write:

$$\begin{aligned}
x &= \sum_{n=1}^{\infty} (x | Ue_n) Ue_n \\
&= \lim_{N \rightarrow \infty} \sum_{n=1}^N (x | Ue_n) Ue_n \\
&= U \left(\lim_{N \rightarrow \infty} \sum_{n=1}^N (x | Ue_n) e_n \right) \\
&= U \left(\sum_{n=1}^{\infty} (x | Ue_n) e_n \right)
\end{aligned}$$

Thus, U is surjective. Now, let $x, y \in H$ be arbitrary such that:

$$x = \sum_{n=1}^{\infty} (x | e_n) e_n \quad y = \sum_{n=1}^{\infty} (y | e_n) e_n$$

Then, we have that:

$$Ux = \sum_{n=1}^{\infty} (x | e_n) Ue_n \quad Uy = \sum_{n=1}^{\infty} (y | e_n) Ue_n$$

using the linearity and boundedness of U (as above). Thus, we have:

$$(Ux | Ue_n) = (x | e_n)$$

and similarly for Uy . Then, using alternative characterizations of an orthonormal basis, we have:

$$\begin{aligned}
(Ux | Uy) &= \sum_{n=1}^{\infty} (Ux | Ue_n) (Ue_n | Uy) \\
&= \sum_{n=1}^{\infty} (x | e_n) (e_n | y) \\
&= (x | y)
\end{aligned}$$

Thus, by Problem 66, we have that U is an isometry. Thus, U is a unitary.

Thus, we have shown that the three statements are equivalent.