Math 74: Algebraic Topology

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Problem 1. Let $p : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ be the quotient map from HW1. For $n \geq 2$, show that $p|_{S^n}$ is a degree two cover and deduce that $\pi_1(\mathbb{RP}^n) \simeq \mathbb{Z}/2\mathbb{Z}$. (The fundamental group is $\mathbb{Z}$ for $n = 1$ as $S^1 \cong \mathbb{RP}^1$.)

Solution. Recall the quotient map identifies points in $\mathbb{R}^{n+1} \setminus \{0\}$ as follows:

$$\forall \lambda \neq 0 \in \mathbb{R} : (x_0, \cdots, x_n) \sim \lambda(x_0, \cdots, x_n)$$

Let $(x_0, \cdots, x_n) \in S^n$ be a point on the sphere. Then, we additionally know that:

$$x_0^2 + x_1^2 + \cdots + x_n^2 = 1$$

The equivalence class of this point on the sphere contains all points such that $\lambda(x_0, \cdots, x_n)$ such that:

$$(\lambda x_0)^2 + \cdots + (\lambda x_n)^2 = \lambda^2(x_0^2 + \cdots + x_n^2) = 1$$

Thus, we have $\lambda = \pm 1$. Thus, restricting p to S^n, we get a quotient map that identifies antipodal points on the sphere, i.e. $x \sim -x$.
Thus, each $[p] \in \mathbb{RP}^n$ has exactly two pre-images in S^n, i.e. $p|_{S^n}$ is a two to one mapping.

Moreover, let $[x] \in \mathbb{RP}^n$ be a point. Then, $p|_{S^n}^{-1}([x]) = \{x, -x\}$. Let V and V' be two disjoint open sets in S^n around x and $-x$ (S^n is Hausdorff). Let $W = V \cap -V'$ and $W' = V' \cap -V$. Then, W and W' are still disjoint. Moreover, W and W' are such that they contain antipodal points (by definition).
Let $U = p(W)$. By the definition of p, we note that:

$$\rho|_{S^n}^{-1}(U) = W \bigsqcup W'$$

Since W and W' are open, U is open in $\mathbb{RP}^n$. Moreover, $p|_W(W) = U$ is a continous bijection of compact Hausdroff spaces, and is thus a homeomorphism. Similarly for W'. Thus, as $[x]$ was arbitrary, we have found an open set U around x whose pre-image is a disjoint union of two isomorphic copies of U. Thus, $p|_{S^n}$ is a degree 2 covering map.

Moreover, we note that for $n \geq 2$, S^n is simply connected. Thus, for any point $x \in \mathbb{RP}^n$, the map:

$$\pi_1(\mathbb{RP}^n, x) \to p|_{S^n}^{-1}(x)$$

is a bijection. Thus, $|\pi_1(\mathbb{RP}^n, x)| = 2$. Thus,

$$\pi_1(\mathbb{RP}^n, x) = \mathbb{Z}/2\mathbb{Z}$$

Problem 2. Let $g\colon S^1 \to S^1$ be the covering map $g(z) = z^n$ for $n \in \mathbb{Z}$. Under the isomorphism $\pi_1(S^1, (1,0)) \cong \mathbb{Z}$ that we proved, compute $g_*\colon \mathbb{Z} \to \mathbb{Z}$ and the map $\phi_g : \mathbb{Z} \to g^{-1}(1,0)$ defined by lifting loops so that they still start at $(1,0)$.

Solution. Let $\gamma_k(s) = e^{2\pi iks} \in \pi_1(S^1, (1,0))$ be the loop that goes around the circle k times. Recall the isomprhism $f : \pi_1(S^1, (1,0)) \to \mathbb{Z}$, which maps $f(\gamma_k) = k$.

Next, we calculate $g_*([\gamma_k])$. We have:

$$\begin{aligned} g_*([\gamma_k]) &= [g \circ \gamma_k] \\ &= [\gamma_k^n] \\ &= [s \mapsto e^{2\pi inks}] \\ &= [\gamma_{nk}] \end{aligned}$$

Thus, under the identification through the isomorphism, we have:

$$g_*(k) = nk$$

That is, g_* is just multiplication by n.

Next, we calculate $\phi_g : \pi_1(S^1, (1,0)) \to g^{-1}((1,0))$. Notice that $(1,0)$ is identified to $z = 1 = e^0$. Thus, we have:

$$g^{-1}(1) = \{z \in S^1 : z^n = 1\} = \{e^{2\pi ik/n} : k = 0, \cdots, n-1\}$$

For $\gamma_k \in \pi_1(S^1, (1,0))$, we have the lift:

$$\tilde{\gamma}_k = s \mapsto e^{2\pi ik/n}$$

as we can verify:

$$g \circ \tilde{\gamma}_k = s \mapsto (e^{2\pi ik/n})^n = \gamma_k$$

Moroever, as $\tilde{\gamma}_k(0) = \gamma_k(0) = 1$, this is unique. Thus, we can evaluate ϕ_g as:

$$\begin{aligned} \phi_g(\gamma_k) &= \tilde{\gamma}_k(1) \\ &= e^{2\pi ik/n} \end{aligned}$$

Beyond the requirements of the problem, it is also interesting to show that $g^{-1}(1)$ is the cyclic group of order n under multiplication, since it contains the nth roots of unity. If we identify $e^{2i\pi k/n} \to k$, then we get an isomorphism from $C_n \to \mathbb{Z}/n\mathbb{Z}$ (written additively). This is a standard exercise. Under this identification, as well as the identification of $\pi_1(S^1, (1,0))$ with $\mathbb{Z}$, we have that: $\phi_g : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is given by:

$$\phi_g(k) = k \pmod{n}$$

Problem 3. Show that there are no retractions $r\colon X \to A$ in the following cases:

1. $X = \mathbb{R}^3$ and A is any subspace homeomorphic to S^1.
2. $X = S^1 \times D^2$ and A is its boundary torus $S^1 \times S^1$.
3. X is the Möbius band and A is its boundary circle.

Solution.

1. If we had such a retraction r, the push-forwards on the fundamental group will yield the following commutative diagram:

$$\begin{tikzcd} \pi_1(S^1) \arrow[rr, "\mathrm{id}_*"] \arrow[dr, "\iota_*"'] & & \pi_1(S^1) \\ & \pi_1(\mathbb{R}^3) \arrow[ur, "r_*"'] & \end{tikzcd}$$

   that is:

$$\begin{tikzcd} \mathbb{Z} \arrow[rr, "\mathrm{id}_*"] \arrow[dr, "\iota_*"'] & & \mathbb{Z} \\ & \{1\} \arrow[ur, "r_*"'] & \end{tikzcd}$$

   This implies that ι_* is injective, as $r_* \circ \iota_*$ is bijective. However, this is clearly impossible.

2. Similar to the case above, noting that taking the product commutes with taking the fundamental group, a retraction r would yield the following commutative diagram:

$$\begin{tikzcd} \mathbb{Z}\times\mathbb{Z} \arrow[rr, "\mathrm{id}_*"] \arrow[dr, "\iota_*"'] & & \mathbb{Z}\times\mathbb{Z} \\ & \mathbb{Z} \arrow[ur, "r_*"'] & \end{tikzcd}$$

   Similar to before, there is no injective homomorphism from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$, since $\mathbb{Z} \times \mathbb{Z}$ is not cyclic and $\mathbb{Z}$ (and all its subgroups) are cyclic. Thus, r cannot exist.

3. Define the Mobius strip as:

$$M = [0,1] \times [0,1]/\sim$$

   with the equivalence relation:

$$(0,y) \sim (1, 1-y)$$

   Next, let $C = \{[(s, 1/2)] : s \in [0,1]\}$ be the central circle and $\iota_C : C \to M$ be the inclusion map. We define a retraction $r_C : M \to C$ as:

$$r_C([(s,t)]) = [(s, 1/2)]$$

This is well-defined as:

$$r_C([(0,y)]) = [(0,1/2)] = [(1,1/2)] = r_C([1,1-y])$$

as is clearly continous as (topologically) its a projection onto the first coordinate. Thus,

$$r_C \circ \iota_C = \mathrm{id}_C$$

Moreover, we show that $\iota_C \circ r_C$ is homotopic to id_M. We define the straight-line homotopy $H : M \times [0,1] \to M$ as:

$$H([(x,y)],t) = [(x,(1-t)y + t/2)]$$

Similarly notice that this is well-defined as:

$$\begin{aligned}
H([(0,y)],t) &= [(0,(1-t)y + t/2)] \\
&= [(1, 1-(t/2 + (1-t)y))] \\
&= [(1, 1-t-(1-t)y + t/2)] \\
&= [(1, (1-t)(1-y) + t/2)] \\
&= H([(1,1-y)],t)
\end{aligned}$$

and is clearly continous as it is a sum of products of continous functions. Moroever,

$$H([x,y],0) = [x,y] = \mathrm{id}_M([x,y]) \qquad H([x,y]) = [x,1/2] = r_C([x,y]))$$

Thus, we have:

$$[\iota_c \circ r_C] = [\mathrm{id}_M]$$

Since π_1 is functorial and homotopy invariant, this implies that:

$$r_{C*} \circ \iota_{C*} = \mathrm{id}_{\pi_1(C)} \qquad \iota_{C*} \circ r_{C*} = \mathrm{id}_{\pi_1(M)}$$

Thus, ι_{C*} is an isomorphism.

Let $B = \{[(s,t)] : t \in \{0,1\}, s \in [0,1]\}$ be the boundary circle of the Mobius strip and $\iota_B : B \to M$ be the inclusion map. Then, note that we have the continous map: $r_C \circ \iota_B : B \to C$. Taking the push-forward, if there was a retraction $r_B : M \to B$, we have the following commutative diagram:

$$\begin{tikzcd}
\pi_1(B) \arrow[r, "\mathrm{id}_*"] \arrow[d, "r_{C*}"] & \pi_1(B) \\
\pi_1(C) \arrow[r, "\iota_{C*}"] & \pi_1(M) \arrow[u, "r_{B*}"']
\end{tikzcd}$$

Finally, let $\gamma : I \to B$ be the generator of $\pi_1(B)$, i.e.:

$$\gamma(t) = \begin{cases} [(2t,0)] & t \in [0,1/2] \\ [(2t-1,1)] & t \in [1/2,1] \end{cases}$$

Note that γ is well-defined loop as $\gamma(1/2) = [(1,0)] = [(0,1)]$ and $\gamma(0) = [(0,0)] = [(1,1)] = \gamma(1)$. Moreover, it is easy to see that γ loops around the boundary circle once, and thus is a generator.

Finally, we calculate $\iota_{*C} \circ r_{*C}(\gamma)$. We have:

$$\begin{aligned}\iota_{*C} \circ r_{*C}(\gamma) &= [\iota_C \circ r_C \circ \gamma(t)] \\ &= \begin{cases} [(2t, 1/2)] & t \in [0, 1/2] \\ [(2t-1, 1/2)] & t \in [1/2, 1] \end{cases}\end{aligned}$$

Thus, we notice that $\iota_{*C} \circ r_{*C}(\gamma)$ is a loop that goes around the center circle twice. In terms of the identification of the fundamental group with $\mathbb{Z}$, $\iota_{*C} \circ r_{*C}(1) = 2$. Thus, by our commutative diagram, if r_B existed, we would have:

$$r_{B*}(2) = 1$$

However, r_{B*} is a group homomorphism, this implies:

$$1 = r_{B*}(2) = r_{B*}(1+1) = r_{B*}(1) + r_{B*}(1)$$

However, there is no integer whose sum with itself is 1. Thus, r_B cannot exist.

Problem 4. Use the intermediate value theorem to prove the 1-dimensional version of the Brouwer fixed point theorem: If $f\colon I \to I$ is continuous, there is a point $x \in I$ such that $f(x) = x$.

Solution. Notice first that if $f(1) = 1$ or $f(0) = 0$ we are done. Thus, we can assume that $f(0) > 0$ and $f(1) < 1$. Define $g : I \to [-1, 1]$ as $g(x) = f(x) - x$. g is continous as it is the difference of two continous functions. Then, notice:

$$g(0) = f(0) - 0 > 0 \qquad g(1) = f(1) - 1 < 0$$

Thus, by the intermediate value theorem, there exists a point $c \in (0, 1)$ such that $g(c) = 0$. Thus, we have:

$$f(c) - c = 0 \implies f(c) = c$$

and we are done.

Problem 5. Use the intermediate value theorem to prove the 1-dimensional version of the Borsuk-Ulam theorem: If $f : S^1 \to \mathbb{R}$ is continuous, there is a point $x \in S^1$ such that $f(x) = f(-x)$.

Solution. Define $g : S^1 \to \mathbb{R}$ as:

$$g(x) = f(x) - f(-x)$$

Notice that g is continous as it is the sum of two continous functions. Moreover, notice that:

$$g(-x) = f(-x) - f(x) = -g(x)$$

We claim that $g(c) = 0$ for some $c \in S^1$. First, if g is identically 0, we pick any point to be c. Otherwise, we can assume that there exists a point $x \in S^1$ such that $g(x) \neq 0$. Then, since $g(-x) = -g(y)$, we have that g takes both a positive and a negative value. Thus, there exists a point $c \in S^1$ (more specifically, on either arc from x to $-x$) such that $g(c) = 0$. Thus, we have:

$$g(c) = 0 \implies f(c) = f(-c)$$

and we are done.