Math 113: Functional Analysis

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April 21, 2025

Problem 1. Show that X is compact if and only if given any family $\mathcal{F}$ of closed sets of X with the finite intersection property, we have $\bigcap_{F\in\mathcal{F}} F \neq \emptyset$.

Solution.

$(\implies)$ Assume X is compact. We will prove that contrapositive. Let $\{F_\alpha\}$ be a family of closed sets such that:

$$\bigcap_\alpha F_\alpha = \emptyset$$

Let $\{U_\alpha\}$ be a family of open sets such that $U_\alpha = X \setminus F_\alpha$. Then, we have:

$$\bigcup_\alpha U_\alpha = X \setminus \bigcap_\alpha F_\alpha = X$$

Thus, $\{U_\alpha\}$ is an open cover of X. Since X is compact, there exists a finite subcover of $\{U_\alpha\}$. Let $\{U_1, \dots, U_n\}$ be the finite subcover. Then, we have:

$$\bigcap_{i=1}^n F_i = X \setminus \bigcap_{i=1}^n U_i = \emptyset$$

Thus, we have found a finite subset of $\{F_\alpha\}$ with empty intersection. Thus, $\{F_\alpha\}$ does not have the finite intersection property. Therefore, by the contrapositive, any family of closed sets with the finite intersection property has non-empty intersection.

$(\impliedby)$ Assume any family of closed sets with the finite intersection property has non-empty intersection. By the contrapositive, this implies that if the family has empty intersection, it cannot have the finite intersection property.

Let $\{U_\alpha\}$ be an open cover of X. To show that X is compact it suffices to show that there exists a finite subcover of $\{U_\alpha\}$. Let $\{F_\alpha\}$ be the family of closed sets such that $F_\alpha = X \setminus U_\alpha$. Then,

$$\bigcap_\alpha F_\alpha = X \setminus \bigcup_\alpha U_\alpha = \emptyset$$

Thus, $\{F_\alpha\}$ does not have the finite intersection property. Therefore, there exists a finite subset of $\{F_\alpha\}$, call it $\{F_1, \ldots, F_n\}$, such that:

$$\bigcap_{i=1}^n F_i = \emptyset$$

However, we know that:

$$\bigcap_{i=1}^n F_i = X \setminus \bigcup_{i=1}^n U_i$$

Thus, we conclude that:

$$X \setminus \bigcup_{i=1}^n U_i = \emptyset$$

Thus, $\bigcup_{i=1}^n U_i = X$. Therefore, $\{U_1, \ldots, U_n\}$ is a finite subcover of $\{U_\alpha\}$. Therefore, X is compact.

Problem 2. Let X be a metric space.

1. Show that if E is a compact subspace of X, then E is closed.
2. Show that if X is compact and E is closed in X, then E is compact.

Solution.

1. Let E be a compact subspace of X. To show that E is closed, it suffices to show that $X \setminus E$ is open. Let $x \in X \setminus E$. Then, we define to families of open sets, $\{U_e\}_{e\in E}$ and $\{V_e\}_{e\in E}$, such that:

$$U_e = B_\epsilon(e) \text{ where } \epsilon < \rho(x,e)/2$$
$$V_e = B_\epsilon(x) \text{ where } \epsilon < \rho(x,e)/2$$

Note that by definition, $x \in V_e$ for all $e \in E$ and that $U_e \cap V_e = \emptyset$. Moreover, by definition, $\{U_e\}_{e\in E}$ is an open cover of E. Since E is compact, there exists a finite subcover of $\{U_e\}_{e\in E}$. Let $\{U_{e_1}, \dots, U_{e_n}\}$ be the finite subcover. Then, since $\{V_{e_1}, \dots, V_{e_n}\}$ is a finite collection of open sets, $V = \bigcap_{i=1}^n V_{e_n}$ is an open set that contains x. However, since $U_{e_i} \cap V_{e_i} = \emptyset$, and $V \subseteq V_{e_i}$, we have that $V \cap U_{e_i} = \emptyset$ for all i. Thus,

$$V \cap E = V \cap \bigcup_{i=1}^n U_{e_i} = \emptyset$$

Thus, there exists an open set V containing x disjoint from E, thus contained in $X \setminus E$. Therefore, $X \setminus E$ is open. Thus, E is closed.

2. Assume X is compact and E is closed in X. If $E = X$, then we are done. Thus, assume $X \neq E$. Let $\{U_\alpha\}$ be an open cover of E in E. Then, there exists a family of open sets $\{V_\alpha\}$ in X where, for all α,

$$U_\alpha = V_\alpha \cap E$$

Note that since $U_\alpha \subseteq V_\alpha$ for all α, $\{V_\alpha\}$ is an open cover for E in X. Since E is closed, $X \setminus E$ is open. Thus, $\{V_\alpha\} \cup \{X \setminus E\}$ is an open cover of X. Since X is compact, there exists a finite subcover of $\{V_\alpha\} \cup \{X \setminus E\}$.

Let $\{V_1, \dots, V_n, X \setminus E\}$, be the finite subcover. If the provided finite subcover does not contain $X \setminus E$, we can just add it while maintaining a finite subcover, we assume it does. Then, $V_1, \cdots, V_n$ is an open cover for E in X. Then, we have:

$$\bigcup_{i=1}^n U_i = \bigcup_{i=1}^n (V_i \cap E) = \left(\bigcup_{i=1}^n V_i\right) \cap E = E$$

Thus, $U_1, \dots, U_n$ is a finite subcover of U_α in E. Therefore, E is compact.

Problem 3. We say that D is dense in X if $\overline{D} = X$. Show that D is dense if and only if D meets every non-empty open set in X.

Solution. We use the fact that $x \in \overline{D}$ (is a point of closure) if and only if every open neighborhood of x intersects D. This is in Royden-Fitzpatrick as the definition of closure, with Ch9.2, Proposition 3 and 4 proving that the closure is closed and the complement of an open.

($\Longrightarrow$) Assume D is dense in X. Let U be a non-empty open set in X. Then, there exists a point $x \in U$. Since $\overline{D} = X$, $x \in \overline{D}$. Thus, every open neighborhood of x intersects D. Thus, U intersects D.

($\Longleftarrow$) Assume D meets every non-empty open set in X. Let $x \in X$ be arbitrary. Then, D meets every non-empty open set containing x. Thus, x is a point of closure of D. Thus, $x \in \overline{D}$. Since x is arbitrary, we have that $\overline{D} = X$. Thus, D is dense in X.

Problem 4. Show that a compact metric space has a countable dense subset. It is enough for the space to be totally bounded.

Solution. Let (X, ρ) be a compact metric space. We define a countable set of open covers, indexed by $\mathbb{N}$, where $\mathcal{U}_n$ is the open cover:

$$\mathcal{U}_n = \{B_{1/n}(x) : x \in X\}$$

Then, for each n, the cover $\mathcal{U}_n$ has a finite subcover. Let C_n be the centers of the balls in the finite subcover, for each n. The union of these sets over n is countable, as it is a countable union of finite sets. Call this set C. We will show that C is dense in X.

Let U be a non-empty open set in X. Then, there exists a point $x \in U$. Since U is open, there exists an ϵ, such that $B_\epsilon(x) \subseteq U$. Pick an n such that $1/n < \epsilon$. Then, there exists a center $c \in C_n$ such that $x \in B_{1/n}(c)$ as C_n is the set of centers of a finite cover made of $1/n$-balls. Thus, we have:

$$\rho(x, c) < 1/n < \epsilon$$

Thus, $c \in B_\epsilon(x) \subseteq U$. Therefore, U intersects C. Since U is arbitrary, every non-empty open set in X intersects C. Thus, C is dense in X. Therefore, C is a countable dense subset of X.

Problem 5. Show that an equicontinous family of functions on a compact metric space is uniformly equicontinous as in lecture. (Some texts do not define equicontinous at a point. Instead, whether X is compact or not, equicontinuity is what we have called uniformly equicontinuous. Fortunately, there is no distinction for compact spaces.)

Solution. Let (X, ρ) be a compact metric space. Let $\mathcal{J}$ be an equicontinous family of functions from (X, ρ) to (Y, σ). Let $\epsilon > 0$. We need to show that there exists a $\delta > 0$, such that for all $F \in \mathcal{J}$, $x, y \in X$:

$$\rho(x, y) < \delta \implies \sigma(F(x), F(y)) < \epsilon$$

Since $\mathcal{J}$ is equicontinous, for every $x \in X$, there exists a $\delta_x > 0$ such that for all $F \in \mathcal{J}$, $y \in X$:

$$\rho(x, y) < \delta_x \implies \sigma(F(x), F(y)) < \epsilon/2$$

Let $U_x = B_{\delta_x}(x)$. Then, $\{U_x\}_{x \in X}$ is an open cover of X. Since X is compact, this open cover has a Lebesgue number $\delta > 0$. Thus, for all $F \in \mathcal{J}$, $x, y \in X$, if $\rho(x, y) < \delta$, there exists a z such that:

$$B_\delta(x) \subseteq B_{\delta_z}(z)$$

Thus, we have:

$$\rho(x, z) < \delta_z \qquad \rho(y, z) < \delta_z$$

Finally, we compute:

$$\begin{aligned}
\sigma(F(x), F(y)) &\leq \sigma(F(x), F(z)) + \sigma(F(z), F(y)) \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon
\end{aligned}$$

Thus, $\mathcal{J}$ is uniformly equicontinous.

Problem 6. Show that if X a metric space which is not totally bounded, then there is an unbounded continous function $f : X \to \mathbb{R}$. (Hints provided).

Solution. Assume (X, ρ) is not totally bounded. Then, there exists a $r > 0$ such that there is no finite collection of open balls of radius r that covers X. We will use this to construct a sequence (x_n) where $\rho(x_i, x_j) > r$ for all $i \neq j$.

Since X is non-empty, we can pick a point $x_1 \in X$. Moreover, if $x_1, \cdots, x_k$ are already picked points with $\rho(x_i, x_j) > r$ for all $i \neq j$ and $1 \leq i, j \leq k$, we can pick a point $x_{k+1} \in X \setminus \bigcup_{i=1}^{k} B_r(x_i)$, as the finite collection of open balls of radius r centered at $x_1, \cdots, x_k$ does not cover X. Moreover, this means that $\rho(x_{k+1}, x_i) > r$ for all $i = 1, \cdots, k$. Thus, $\rho(x_i, x_j) > r$ for all $i \neq j$ where $1 \leq i, j \leq k+1$. Thus, by induction, we can construct a sequence (x_n) such that $\rho(x_i, x_j) > r$ for all $i \neq j$.

Secondly, we claim that for $i \neq j$, we have $B_{\frac{r}{2}}(x_i) \cap B_{\frac{r}{2}}(x_j) = \emptyset$. To see this, assume that there exists a $y \in B_{\frac{r}{2}}(x_i) \cap B_{\frac{r}{2}}(x_j)$. Then, we have:

$$\rho(x_i, x_j) \leq \rho(x_i, y) + \rho(y, x_j) < \frac{r}{2} + \frac{r}{2} < r$$

which is a contradiction.

Next, we define a sequence of functions $f_n : X \to \mathbb{R}$ such that $f_n(x_n) = 1$ and $f_n(x) = 0$ for all $x \in X \setminus B_{\frac{r}{2}}(x_n)$. Consider the following:

$$x \mapsto \begin{cases} e^{1 - \frac{1}{1 - \left(\frac{2\rho(x, x_n)}{r}\right)^2}} & \text{if } x \in B_{\frac{r}{2}}(x_n) \\ 0 & \text{if } x \notin B_{\frac{r}{2}}(x_n) \end{cases}$$

Notice that if $x = x_n$ then $\rho(x, x_n) = 0$ and thus $f_n(x) = e^0 = 1$. Moreover f_n is evidently continous in $B_{\frac{r}{2}}(x_n)$, as it is a composition of continous functions, as well as on $X \setminus \overline{B_{\frac{r}{2}}(x_n)}$. Thus, we only need to check the boundary.
Let x be a point on the boundary. Then, notice that f_n is a monotonically decreasing function of $|\rho(x, x_n)|$ in $B_{\frac{r}{2}}(x_n)$ and 0 outside. Moreover, notice that $\lim_{y \to x} \rho(y, x_n) = \frac{r}{2}$. Thus, we have that $\lim_{y \to x} 1 - \left(\frac{2\rho(y, x_n)}{r}\right)^2 \to 0$. Thus, $\lim_{y \to x} f_n(y) \to 0$ from inside the ball. Since the function is identically 0 outside, the limit is well-defined. Moreover, since for $x \notin B_{\frac{r}{2}}(x_n)$, $f_n(x) = 0$. Thus, the limit equals the value and f_n is continous on X.

Next, we consider the function:

$$f(x) = \sum_{n \in \mathbb{N}} n f_n(x)$$

since the sets $B_{\frac{r}{2}}(x_i)$ are disjoint, we have that at most one $f_n(x)$ is non-zero for every X. Thus, the function is continous. Moreover, the function is unbounded, as for any $M > 0$, we can pick $n > M$, and then $f(x_n) = n f_n(x_n) > M$. Thus, we are done.

Problem 7. Let X be a metric space such that every continous function $f : X \to \mathbb{R}$ attains its minimum value. Show that X is complete. (Hints provided).

Solution. Let (x_n) be a Cauchy sequence in X. We will show that (x_n) converges to a point in X.

Since every function attains its minimum value, every function also attains its maximum value, as this is the minimum value for $-f$. Thus, every function $X \to \mathbb{R}$ is bounded. By the contrapositive of the previous question, this implies that X is totally bounded.

For any $x \in X$, we claim that $(\rho(x, x_n))_n$ is Cauchy in $\mathbb{R}$. Let $\epsilon > 0$. Since (x_n) is Cauchy, there exists an N such that for all $m, n > N$, we have:

$$\rho(x_n, x_m) < \epsilon$$

Then, we have:

$$\begin{aligned} \rho(x, x_n) &\leq \rho(x, x_m) + \rho(x_m, x_n) \\ \implies \rho(x, x_n) - \rho(x, x_m) &\leq \rho(x_n, x_m) < \epsilon \end{aligned}$$

Symmetrically, we have:

$$\rho(x, x_m) - \rho(x, x_n) < \epsilon$$

Thus, for all $n, m > N$, we have:

$$|\rho(x, x_n) - \rho(x, x_m)| < \epsilon$$

Thus, $(\rho(x, x_n))_n$ is Cauchy in $\mathbb{R}$.

Let $f(x) = \lim_{n\to\infty} \rho(x, x_n)$. We show that f is continous. Let $\epsilon > 0$. We need to find a $\delta > 0$ such that for all $x, y \in X$:

$$\rho(x, y) < \delta \implies |f(x) - f(y)| < \epsilon$$

Pick $\delta = \epsilon$, and notice:

$$\begin{aligned} \rho(x, x_n) &\leq (\rho(x, y) + \rho(y, x_n)) \\ \rho(x, x_n) - \rho(y, x_n) &\leq \rho(x, y) < \delta = \epsilon \end{aligned}$$

Symmetrically, we have:

$$\rho(y, x_n) - \rho(x, x_n) < \epsilon$$

Thus, we have:

$$|\rho(x, x_n) - \rho(y, x_n)| < \epsilon$$

Taking the limit as $n \to \infty$, we have:

$$|f(x) - f(y)| < \epsilon$$

Thus, f is continous.

Moreover, notice that f is bounded below by 0, as it is a limit of a sequence of non-negative functions. We prove that its infimum is 0. Let $\epsilon > 0$. Since (x_n) is Cauchy, there exists an N such that for all $m, n > N$, we have:

$$\rho(x_m, x_n) < \epsilon$$

Letting $n \to \infty$, we have:

$$|f(x_m) - 0| < \epsilon$$

Thus, the infimum of f is 0. Since f attains its minimum value, there exists a $x_0 \in X$ such that $f(x_0) = 0$.

Lastly, we show that $(x_n) \to x_0$. Since $f(x_0) = 0$, we have:

$$\lim_{n\to\infty} \rho(x_n, x_0) = 0$$

which is precisely equivalent to the definition of convergence in metric spaces.

Problem 8. Show that a metric space is compact if and only if every continous real-valued function on X attains its maximum value. (Note that every real-valued function attains its maximum if and only if every real-valued function attains its minimum. Consider $-f$.)

Solution.

($\Longrightarrow$) We did this in class. Here's a reproduction of the argument. Let X be a compact metric space and $f : X \to \mathbb{R}$ be a continous function. As X is Since X is compact, f is bounded. Let $M := \sup f(x) \in (-\infty, \infty]$. Thus, for $M \neq \infty$, there exists a sequence (x_n) such that:

$$|f(x_n) - M| < 1/n$$

and for $M = \infty$, there exists a sequence (x_n) such that:

$$|f(x_n)| > n$$

That is, in both cases, there exists a sequence such that:

$$\lim_{n\to\infty} f(x_n) = M$$

Since X is compact, it is sequentially compact. Thus, there exists a subsequence (x_{n_k}) that converges to a point $x_0 \in X$.

Then, since f is continous, we have:

$$f(x_0) = \lim_{n\to\infty} f(x_n) = M$$

Thus, f attains its maximum value. Considering $-f$, the same argument shows that f attains its minimum value.

($\Longleftarrow$) Assume every real-valued continous function on X attains its maximum value. By the previous problem (Problem 7), we know that X is complete. Moreover, as every real-valued function attains its maximum value, there is no unbounded continous function on X. Thus, by the contrapositive of the problem before the previous problem (Problem 6), we have that X is totally bounded. Finally, we claimed in class that a metric space is compact if and only if it is complete and totally bounded. Thus, X is compact.

Problem 9. Show that X is a Baire space if and only if whenever a countable union $\bigcup F_n$ of closed sets F_n has interior in X, then at least one of the F_n has interior in X.

Solution. First, we introduce a lemma:

Lemma 1. *If F is a closed set with empty interior in X, if and only if $X \setminus F$ is dense in X.*

Proof. First assume that F has empty interior. Let U be a non-empty open set in X. Clearly, $U \not\subseteq F$ as the interior of F is empty. Thus, there exists a point $x \in U$ such that $x \notin F$. Thus, $x \in X \setminus F$. Since U is an arbitrary open, we have $X \setminus F$ is dense in X.
Next, assume that $X \setminus F$ is dense in X. Let U be a non-empty open set in X. Then, by an earlier problem, we know that $U \cap (X \setminus F) \neq \emptyset$. Thus, there exists a point $x \in U$ such that $x \notin F$. Thus, $U \not\subseteq F$. As U was arbitrary, no open set is contained in F. Thus, F has empty interior. □

Now, let F_n be closed sets. Let $O_n = X \setminus F_n$ for all $n \in \mathbb{N}$. Then, using De Morgan's Laws:

$$(*) \qquad \bigcup_{n \in \mathbb{N}} F_n = X \setminus \bigcap_{n \in \mathbb{N}} O_n$$

($\Longrightarrow$) Assume that X is a Baire space and all F_n have empty interior. Then, by the lemma, we have that O_n is dense in X for all $n \in \mathbb{N}$. As X is a Baire space, we note that:

$$\bigcap_{n \in \mathbb{N}} O_n \text{ is dense in } X$$

is also dense in X. Then, by $(*)$, and our lemma above, we note that $\bigcup_{n \in \mathbb{N}} F_n$ has empty interior. Thus, by contrapositive, we have shown that if X is a Baire space, then if $\bigcup_{n \in \mathbb{N}} F_n$ has interior in X, at least one of the F_n must have interior in X.

($\Longleftarrow$) Assume that for any collection F_n of closed sets, if $\bigcup_{n \in \mathbb{N}} F_n$ has interior in X, then at least one of the F_n has interior in X. Let O_n be a countable collection of dense open sets in X. Then $F_n = X \setminus O_n$ is a countable collection of closed sets. Moreover, by the lemma, F_n has empty interior for all $n \in \mathbb{N}$. Thus, by the contrapositive of our assumption, we have that $\bigcup_{n \in \mathbb{N}} F_n$ has empty interior. Thus, by $(*)$ and using the lemma again, we have that:

$$\bigcap_{n \in \mathbb{N}} O_n \text{ is dense in } X$$

is dense in X. Thus, X is a Baire space.

Problem 10. Let U be an non-empty open subset of a metric space (X, ρ). Show that U admits a complete metric equivalent to the metric inherited from X.

Solution. First, we claim that if there exists a homeomorphism $\phi : (U, \rho) \to (Y, \sigma)$, where (Y, σ) is a complete metric space, then there exists a complete metric δ on U that is equivalent to $\rho|_U$. Define: $\delta(x, y) = \sigma(\phi(x), \phi(y))$. Then, for any $x \in X$ and $\epsilon > 0$, there exists ϵ' such that:

$$\phi(B_{\epsilon'}(x)) \cong B_\epsilon(\phi(x))$$

via the homeomorphism. Thus, we have:

$$\begin{aligned} y \in B_\epsilon^\delta(x) &\iff \delta(x, y) < \epsilon \\ &\iff \sigma(\phi(x), \phi(y)) < \epsilon \\ &\iff \phi(y) \in B_\epsilon^\sigma(\phi(x)) \\ &\iff y \in B_{\epsilon'}^\rho(x) \text{ as } \phi \text{ is a homeomorphism} \end{aligned}$$

Thus, δ is equivalent to $\rho|_U$ as they both generate the same basic opens. Moreover, we claim that δ is complete. Indeed, if (x_n) is Cauchy in (U, δ), then, $f(x_n)$ is Cauchy in (Y, σ). Siince Y is complete, this converges to $f(x)$. Then, (x_n) converges to $f^{-1}(x) \in U$ (as the topologiees match).

Let $A = X \setminus U$. We define the following map, $f : (U, \rho) \to ((X, \mathbb{R}), \sigma)$, where σ is the product metric on $X \times \mathbb{R}$:

$$f(x) \to (x, \rho(x, A)^{-1})$$

Since $x \notin A$ and A is closed, we have that $\rho(x, A) > 0$, thus, f is well-defined. Moreover, since it is continous in each entry, it is continous. The inverse map, defined by projection onto the first coordinate is also continous. Moroever, f is clearly injective, as if $f(x) = f(y)$, then $x = y$ by matching the first coordinate of the output. Thus, f is a homeomorphism onto its image.

Thus, it suffices to show that $f(U)$ is closed, hence complete. Let $(x_n, y_n) \to (x, y)$ with $(x_n, y_n) \in f(U)$ be a convergent sequence. Since σ is the product metric, we know that $x_n \to x \in X$ and $y_n \to y \in \mathbb{R}$. Since $x_n \in U$ (via the projection map f^{-1}), we have that $x \in \overline{U}$.

Assume $x \notin U$. Then, $(x_n) \to x \in A$. However, as $(x_n) \to x$, we have $\rho(x_n, A) \to 0$. Thus, $(y_n) \to \infty$ is unbounded and does not converge. This is a contradiction. Thus, we have $x \in U$.

Thus, $(x, t) = (x, \rho(x, A)^{-1})$ is in the image of f. Thus, we have that $(x_n, y_n) \to (x, y)$ in $f(U)$. Thus, $f(U)$ is closed and hence complete.

Finally, we note that since f is a homeomorphism onto its image, we have found an equivalent metric δ to $\rho|_U$ such that (U, δ) is complete.