Math 121: Hodge Theory

Sair Shaikh

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Problem 1.

1. Let X be a compact hermitian manifold and let $E \to X$ be a holomorphic vector bundle of rank d endowed with a Hermitian metric h. Prove that the cohomology groups $H^q(X, E)$ are finite-dimensional vector spaces. In particular, the vector space of holomorphic sections of E is finite dimensional. (Hint provided).

2. Let M be a differentiable manifold, $E, F, G \to M$ differentiable vector bundle and $P : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(F)$ and $Q : \mathcal{C}^\infty(F) \to \mathcal{C}^\infty(G)$ be differential operators. Show that the symbol of $Q \circ P$ is equal to $\sigma_Q \circ \sigma_P$ and compute its degree in terms of the degrees of P and Q.

3. Show that a form α is privitive if and only if $\Lambda\alpha = 0$, where Λ is the adjoint of the Lefschetz operator.

4. Let ω_{FS} the Fubini-Study metric on $\mathbb{P}^n\mathbb{C}$. Show that $\int_{\mathbb{P}^n\mathbb{C}} \omega_{FS}^n = 1$. (Hint provided.)

5. Conclude that $H^{2k}(\mathbb{P}^n\mathbb{C}, \mathbb{Z}) = \mathbb{Z}[\omega_{FS}^k]$.

6. Solve Exercise 2, Chapter 5 of Voisin.

Solution. Noting down ideas here.

1. First note that we have the decomposition:

$$H^k(X, E) = \bigoplus_{p+q=k} H^{p,q}(X, E)$$

   Thus, it suffices to show that $H^{p,q}(X, E)$ is finite dimensional. This is the cohomology of the Dobault complex.

2. First, trivialize the bundles E, F, G over U. Then, locally for coordinates $(x_1, \ldots, x_n)$ on M, and $(\alpha_1, \cdots, \alpha_p)$ on $\Gamma(X, E)$, we have:

Problem 2. Let X be a Kahler compact manifold and ω a differential form which is ∂ and $\overline{\partial}$ closed. Prove that if ω is either d, ∂, or $\overline{\partial}$ exact, then there exists a differential form χ such that $\omega = \partial\overline{\partial}\chi$.

Solution.

Problem x..

Solution.