Math 113: Functional Analysis

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Problem 1. Recall that two metrics ρ_1 and ρ_2 are *equivalent* on X if they generate the same topology on X, and *strongly equivalent* on X if there are strictly positive numbers c and d such that

$$c\rho_1(x, y) \leq \rho_2(x, y) \leq d\rho_1(x, y) \quad \text{for all } x, y \in X.$$

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on a vector space X. Let $\rho_1(x, y) = \|x - y\|_1$ and $\rho_2(x, y) = \|x - y\|_2$ be the corresponding metrics. Show that ρ_1 and ρ_2 are equivalent if and only if they are strongly equivalent.

Solution. First assume that ρ_1 and ρ_2 are strongly equivalent. Then, there exist positive constants c and d such that:

$$c\|x - y\|_1 \leq \|x - y\|_2 \leq d\|x - y\|_1 \quad \text{for all } x, y \in X$$

Using Problem 5, we note that ρ_1 and ρ_2 are equivalent if and only if they have the same convergent sequences. Thus, let $(x_n) \subset (X, \rho_2)$ converge to x. Let $\epsilon > 0$. Then, there exists N such that for all $n \geq N$, we have:

$$\rho_2(x_n, x) < c \cdot \epsilon$$

Then, since $c > 0$, we have for all $n \geq N$:

$$\rho_1(x_n, x) \leq \frac{1}{c}\rho_2(x_n, x) < \epsilon$$

Thus, $(x_n) \to x$ in (X, ρ_1). Moreover, manipulating the inequalities above, we also have:

$$\frac{1}{d}\|x - y\|_2 \leq \|x - y\|_1 \leq \frac{1}{c}\|x - y\|_2 \quad \text{for all } x, y \in X$$

Then, as $\frac{1}{d} > 0$, we can use the same argument to show that a sequence converging in (X, ρ_1) also converges in (X, ρ_2). Thus, ρ_1 and ρ_2 have the same convergent sequences and thus, by Problem 5, they are equivalent.

Next, assume that ρ_1 and ρ_2 are equivalent, i.e. they generate the same topology on X. Then, by Problem 5, they have the same convergent sequences. Assume to the contrary that there is no such $c > 0$ such that $||x-y||_1 \leq c \cdot ||x-y||_2$ for all $x, y \in X$. Thus, for each $n \in \mathbb{N}$, there exists $x_n, y_n \in X$ such that:

$$||x_n - y_n||_1 > n \cdot ||x_n - y_n||_2$$

Thus, noting that $||x_n - y_n||_1 > n > 0$ by the previous inequality, we get:

$$\frac{||x_n - y_n||_2}{||x_n - y_n||_1} < \frac{1}{n}$$

Let $z_n := \frac{x_n - y_n}{||x_n - y_n||_1}$. Then, we have $||z_n||_1 = 1$ for all $n \in \mathbb{N}$. Moreover, by homogeneity, we have:

$$||z_n||_2 = \frac{1}{||x_n - y_n||_1} \, ||x_n - y_n||_2 < \frac{1}{n}$$

Thus, consider the sequence $(z_n) \subset (X, \rho_2)$. For $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then, for all $n \geq N$, we have:

$$||z_n||_2 < \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

Thus, $(z_n) \to 0$ in (X, ρ_2). However, since $||z_n||_1 = 1$ for all n, we have that (z_n) does not converge to 0 in (X, ρ_1). This is a contradiction, thus there exists a positive constant c such that:

$$||x-y||_1 \leq c \cdot ||x-y||_2$$

for all $x, y \in X$.

Using the same argument, swapping ρ_1 and ρ_2, we can show that there exists a positive constant d such that:

$$||x-y||_2 \leq d \cdot ||x-y||_1$$

for all $x, y \in X$. Then, we have:

$$\frac{1}{c} \, ||x-y||_1 \leq ||x-y||_2 \leq d \cdot ||x-y||_1$$

for all $x, y \in X$. Thus, ρ_1 and ρ_2 are strongly equivalent.

Problem 2. Let X be a Banach space and Y a closed subspace. We say that Y is *complemented* in X if there is a closed subspace Z such that $X = Y \oplus Z$. Show that Y is complemented in X if and only if there is a bounded linear map $P : X \to X$ such that $P = P^2$ and $P(X) = Y$. That is, Y is complemented if and only if there is a continuous projection P with range Y. (To show P is continuous, I used the Closed Graph Theorem.)

Solution. First, assume that Y is complemented in X, i.e. there exists a subspace Z such that $X = Y \oplus Z$. Then, every $x \in X$ can be written uniquely as $x = y + z$ for $y \in Y$ and $z \in Z$. Define the map $P : X \to X$ by $P(x) = y$. We show that P is a bounded linear map satisfying $P = P^2$ and $P(X) = Y$.

First, we show linearity. Let $x_1, x_2 \in X$ and $\alpha \in \mathbb{F}$. Let $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$ for $y_1, y_2 \in Y$ and $z_1, z_2 \in Z$. Then, we have:

$$\begin{aligned} P(\alpha x_1 + x_2) &= P(\alpha(y_1 + z_1) + (y_2 + z_2)) \\ &= \alpha P(y_1) + \alpha P(z_1) + P(y_2) + P(z_2) \\ &= \alpha y_1 + y_2 \\ &= \alpha P(x_1) + P(x_2) \end{aligned}$$

Hence, P is linear.

Next, we use the Closed Graph Theorem to show that P is bounded. Let $x_n \to x$ in X and $P(x_n) \to y'$ in X. Let $x = y + z$ where $y \in Y, z \in Z$. We need to show that $y' = P(x) = y$. Since $x_n \to x$, we can write $x_n = y_n + z_n$ for $y_n \in Y$ and $z_n \in Z$. Then, note that:

$$P(x_n) = y_n$$

Thus, by uniqueness of limits (Hausdorff space), it suffices to show that $y_n \to y$. Notice that since $X = Y \oplus Z$, we have an (vector space) isomorphism from $X \to Y \times Z$ given by $x' \mapsto (y', z')$ where $x' = y' + z'$, $y' \in Y, z' \in Z$ is the unique decomposition. Bijectivity and linearity are both immediate. **TODO:** What? Since Y and Z are closed, they are Banach spaces with the subspace topology. Using the results of Problem 27, we conclude that $Y \times Z$ is a Banach space under the product metric $\delta((y_1, z_1), (y_2, z_2)) = ||y_1 - y_2|| + ||z_1 - z_2||$. Moreover, since $(x_n) \to x$ in X, we must have $(y_n) \to y$ in Y and $(z_n) \to z$ in Z (Problem 27). Thus, we have shown that:

$$(P(x_n)) = (y_n) \to y = P(x)$$

Thus, P is continuous and bounded by the Closed Graph Theorem.
TODO: Check this argument!

By definition of P, we have that $P(x) \in Y$ for all $x \in X$. Thus, $P(X) \subset Y$. Moreover, also by definition, we have $P(y) = y$ for $y \in Y$. This implies $Y = P(Y) \subset P(X)$. Thus, $P(X) = Y$.

Finally, for any $x \in X$, as $P(x) \in Y$, we have that P acts as the identity on Y. Thus, we have:

$$P^2(x) = P(x)$$

Hence, $P = P^2$.

Conversely, assume that there exists a bounded linear map $P : X \to X$ with $P(X) = Y$ and $P = P^2$. We need to show that there exists a closed subspace Z such that $X = Y \oplus Z$.

First, we show that P is the identity on Y. Let $y \in Y$. Then, there exists $x \in X$ such that $P(x) = y$ as $P(X) = Y$. Then,

$$P(y) = P^2(x) = P(x) = y$$

Thus, P acts as the identity on Y.

Let $Z = \ker(P)$. Since $\{0\}$ is closed, $Z = P^{-1}(0)$ is closed as P is continous. Moreover, since P is the identity on Y, for $y \in Y$, $P(y) = 0$ implies $y = 0$. Thus, $Y \cap Z = \{0\}$. Hence, we only need to show that $X = Y \oplus Z$.

Consider the map $I - P \in \mathcal{L}(X)$, where I is the identity map. We claim that that $(I-P)^2 = I - P$ and $(I-P)(X) = Z$. To see this, note that for $x = y + z \in X$, with $y \in Y$ and $z \in Z$, we have:

$$(I-P)(x) = (I-P)(y+z) = (y - P(y)) + (z - P(z)) = (y-y) + (z-0) = z$$

Thus, $(I-P)(X) = Z$. Moreover, if $z \in Z$, we have:

$$(I-P)(z) = z - P(z) = z$$

Thus, $(I-P)$ acts as the identity on Z. Thus, $Z = (I-P)(Z) \subset (I-P)(X)$. Additionally, for $x \in X$, as $(I-P)(x) \in Z$, as $(I-P)$ acts as the identity on Z, we have:

$$(I-P)^2(x) = (I-P)(x)$$

Finally, as we can write the identity map as $I = P + (I-P)$, we have a unique way to write, for every $x \in X$,

$$x = P(x) + (I-P)(x)$$

with $P(x) \in Y$ and $(I-P)(x) \in Z$. Thus, we have $X = Y \oplus Z$ and Y is complemented by $\ker(P)$.

Problem 3. Suppose that X is a Banach space and that $X = Y \oplus Z$ for closed subspaces Y and Z. Let X/Y be the quotient Banach space and $q : X \to X/Y$ the quotient map. Show that there is a continuous isomorphism $\Psi : X/Y \to Z$. (Therefore the Open Mapping Theorem implies that $\Psi^{-1}(z) = q(z)$ is a continuous isomorphism of Z onto X/Y.)

Solution. We will write $[x]$ for $q(x)$ throughout this solution.

First assume that $Y = 0$, then $X = X/Y = Z$. In this case, let Ψ be the identity map, which is clearly a continous isomorphism. Next, assume $Z = \{0\}$. Then, $X = Y$ and thus, $X/Y = \{0\}$. Thus, let Ψ be the identity map. This is also a continous isomorphism.

Now, we assume that both Y and Z are proper subspaces of X. By the previous problem, as Z is closed and complemented, we know that there exists a continous projection $P : X \to Z$. Additionally, we know that $\ker(P) = Y$ from the proof of the previous problem. **TODO:** Check this!

Then, using Problem 31, since $Y \subset \ker(P)$ is a proper closed subspace, there exists a unique bounded linear map $\Psi : X/Y \to X$ satisfying $\Psi([x]) = P(x)$ for all $x \in X$ and satisfying $||\Psi|| = ||P||$. Since $P(X) = Z$, we actually have $\Psi : X/Y \to Z$. Thus, we only need to show that Ψ is a bijection.

To show injectivity, let $[x] \in X/Y$ be such that $\Psi([x]) = 0$. Then, $P(x) = 0$, which implies that $x \in Y = \ker(P)$. Thus, $[x] = [0]$. Hence, Ψ is injective.

To show surjectivity, let $z \in Z \subset X$. Then, note that:

$$\Psi([z]) = P(z) = z$$

as P acts as the identity on Z (previous problem). Thus, Ψ is surjective.

Thus, we have constructed Ψ as a bounded (hence continuous) isomorphism $X/Y \to Z$.

Problem 4. Let c_0 be the subspace of ℓ^∞ of sequences (x_n) such that $\lim_n x_n = 0$.

1. Show that c_0 is closed in ℓ^∞.
2. Let $q : \ell^\infty \to \ell^\infty / c_0$ be the quotient map. Show that the quotient norm is given by
$$\|q(x)\| = \text{limsup}_n |x_n|.$$

Solution.

1. Let $(x^{(k)}) \subset c_0$ be a sequence converging to $x \in \ell^\infty$. We need to show that $x \in c_0$.

   Let $\epsilon > 0$. Since $x^{(k)} \to x$, there exists N such that for all $k \geq N$, we have:
$$\left|\left|x^{(k)} - x\right|\right|_\infty < \frac{\epsilon}{2}$$
   Moreover, since $x^{(N)} \in c_0$, there exists M such that for all $n \geq M$, we have:
$$|x_n^{(N)}| < \frac{\epsilon}{2}$$
   Thus, for all $n \geq M$, we have:
$$\begin{aligned}
|x_n| &\leq |x_n - x_n^{(N)}| + |x_n^{(N)}| \\
&\leq \left|\left|x - x^{(N)}\right|\right|_\infty + |x_n^{(N)}| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$
   Thus, $\lim_n |x_n| = 0$. Therefore, $x \in c_0$, and hence c_0 is closed in ℓ^∞.

2. Let $x \in \ell^\infty$. Recall the quotient norm is defined as:
$$||q(x)|| = \inf\{||x - y||_\infty : y \in c_0\} = \inf\{||x + y||_\infty : y \in c_0\}$$
   Pick $(y^{(k)}) \subset c_0$ such that $y_i^{(k)} = x_i$ for $i < k$ and $y_i^{(k)} = 0$ for $i \geq k$. Note that by definition, $\lim_n y_n^{(k)} = 0$. Notice that since the first k entries of $x - y^{(k)}$ are 0, we have:
$$\left|\left|x - y^{(k)}\right|\right| = \sup_{n \geq k} |x_n - y_n| = \sup_{n \geq k} |x_n|$$
   Using this, we compute:
$$\begin{aligned}
||q(x)|| &= \inf\{||x - y||_\infty : y \in c_0\} \\
&\leq \inf_k \left|\left|x - y^{(k)}\right|\right|_\infty \\
&= \inf_k \sup_{n \geq k} |x_n| \\
&= \text{limsup}_n |x_n|
\end{aligned}$$

Where the last equality follows as $\sup_{n \geq k'} |x_n| \leq \sup_{n \geq k} |x_n|$ for all $k' \geq k$, i.e., since it is non-increasing, the infimum is the limit. **TODO:** Check this!

To show the other direction, we need to show that $||q(x)||$ gets arbitrarily close to $\text{limsup}_n |x_n|$, i.e. for $\epsilon > 0$, we have:

$$||q(x)|| \geq \text{limsup}_n |x_n| - \epsilon$$

Let $y \in c_0$ be arbitrary. Then, there exists N such that $|y_n| < \epsilon$. Then, we have:

$$\begin{aligned} ||x + y||_\infty &\geq \text{limsup}_n |x_n + y_n| \\ &\geq \text{limsup}_n |x_n| - \text{limsup}_n |y_n| \\ &\geq \text{limsup}_n |x_n| - \epsilon \end{aligned}$$

Thus,

$$||q(x)|| \geq \text{limsup}_n |x_n| - \epsilon$$

Therefore, we conclude:

$$||q(x)|| = \text{limsup}_n |x_n|$$

Problem 5. Let E and F be closed subspaces of a Hilbert space H with $\dim E < \infty$ and $\dim E < \dim F$. Show that $E^\perp \cap F \neq \{0\}$.

Solution. Assume for the sake of contradiction that $E^\perp \cap F = \{0\}$. Since E is closed, we can write $H = E \oplus E^\perp$. Then, consider the projection map $P : H \to E$. For $f \neq 0 \in F$, we have $f \notin E^\perp$. Thus, $P(f) \neq 0$. Taking the contrapositive, this implies for $f \in F$, $P(f) = 0$ if and only if $f = 0$. Thus, $P|_F$ is injective. Since $\dim E < \infty$ and $\dim E < \dim F$, this is a contradiction. Thus, $E^\perp \cap F \neq \{0\}$.

Problem 6. Suppose that H is a Hilbert space and that $T : H \to H$ is linear and norm-weak continuous. Show that T is bounded.

Solution. Let ω be the weak topology on H. We are given that:

$$T : (H, ||\cdot||) \to (H, \omega)$$

is continuous. For $h \in H$, let $\phi_h : (H, \omega) \to \mathbb{F}$ be the linear functional defined by:

$$\phi_h(x) = (x \mid h)$$

Note that by the definition of the weak topology, ϕ_h is continuous for all $h \in H$. Thus, $\phi_h \circ T : (H, ||\cdot||) \to \mathbb{F}$ is a composition of continuous maps, and hence is continuous. Thus, $\phi_h \circ T$ is bounded for all $h \in H$.

Now, let $x_\lambda \to x \subset (H, ||\cdot||)$ be a convergent net. Then, since $\phi_h \circ T$ is continuous, by Problem 48, we have that:

$$\phi_h(T(x_\lambda)) \to \phi_h(T(x))$$

Then, note that $T(x_\lambda)$ is a net, such that for each ϕ_h, we have $\phi_h(T(x_\lambda)) \to \phi_h(T(x))$. Moreover, as we proved in class, every functional in H^* is of the form ϕ_h. Thus, by Problem 49, we have that $T(x_\lambda) \to T(x)$ in the weak topology.

Thus, we have shown that T takes a convergent net to a convergent net. By Problem 48, this implies that T is continuous. Thus, T is bounded. **TODO:** Check this whole thing!