Math 121: Hodge Theory

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Problem 1.

(a) Check the equivalence between the two definitions of the Hodge structure of weight k given in class.

(b) Check the a morphism Hodge structures is strict for the Hodge filtration.

(c) Show that the kernel, cokernel, and image of a morphism of Hodge structures are Hodge structures.

(d) Let $\phi : X \to Y$ a surjective holomorphic map of complex compact manifolds such that X is kählerian. Show that ϕ^* is injective.

Solution.

(a) First assume that we have filtration of $V_{\mathbb{C}}$:

$$0 = F^{k+1} \subset F^k V_{\mathbb{C}} \subset F^{n-1} V_{\mathbb{C}} \subset \cdots \subset F^0 V_{\mathbb{C}} = V_{\mathbb{C}}$$

that satisfies the condition:

$$F^p V_{\mathbb{C}} \oplus \overline{F^{k-p+1} V_{\mathbb{C}}} = V_{\mathbb{C}}$$

Then, we define:

$$H^{p,q} = F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}}$$

Clearly, with this definition, we have:

$$\begin{aligned}
\overline{H^{p,q}} &= \overline{F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}}} \\
&= \overline{F^p V_{\mathbb{C}}} \cap F^q V_{\mathbb{C}} \\
&= H^{q,p}
\end{aligned}$$

We will show that:

$$F^iV = \bigoplus_{p \geq i} H^{p,k-p}$$

Notice that:

$$\begin{aligned}
F^iV_{\mathbb{C}} &= F^iV_{\mathbb{C}} \cap V_{\mathbb{C}} \\
&= F^iV_{\mathbb{C}} \cap (F^{i+1}V_{\mathbb{C}} \oplus \overline{F^{k-(i+1)+1}V_{\mathbb{C}}}) \\
&= F^iV_{\mathbb{C}} \cap (F^{i+1}V_{\mathbb{C}} \oplus \overline{F^{k-i}V_{\mathbb{C}}}) \\
&= (F^iV_{\mathbb{C}} \cap F^{i+1}V_{\mathbb{C}}) \oplus (F^iV_{\mathbb{C}} \cap \overline{F^{k-i}V_{\mathbb{C}}}) \\
&= F^{i+1}V_{\mathbb{C}} \ \oplus (F^iV_{\mathbb{C}} \cap \overline{F^{k-i}V_{\mathbb{C}}}) \\
&= F^{i+1}V_{\mathbb{C}} \oplus H_{i,k-1} \\
&= \bigoplus_{p \geq i} H^{p,k-p}
\end{aligned}$$

Where the third equality uses the fact that $A \cap (B \oplus C) = A \cap B \oplus A \cap C$ if $B \subseteq A$ as if $a = b + c \in A \cap (B \oplus C)$ $(a \in A, b \in B, c \in C)$, then since $b \in B \subset A$, we have $a - b = c \in A$. Thus, $b \in A \cap B$ and $c \in A \cap C$ and $A \cap (B \oplus C) \subseteq A \cap B \oplus A \cap C$. The other inclusion is clear.

Thus, noting the result for $i = 0$, we have recovered the first definition of a Hodge structure of weight k.

Now, assume that we have a decomposition of $V_{\mathbb{C}}$ into the direct sum of the $H^{p,q}$:

$$V_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$$

with $H^{p,q} = \overline{H^{q,p}}$. Then, we define the filtration similarly:

$$F^pV_{\mathbb{C}} = \bigoplus_{i \geq p} H^{i,k-i}$$

Then, we have that:

$$\begin{aligned}
F^pV_{\mathbb{C}} \oplus \overline{F^{k-p+1}V_{\mathbb{C}}} &= \bigoplus_{i \geq p} H^{i,k-i} \oplus \overline{\bigoplus_{j \geq k-p+1} H^{j,k-j}} \\
&= \bigoplus_{i \geq p} H^{i,k-i} \oplus \bigoplus_{j \geq k-p+1} H^{k-j,j} \\
&= \bigoplus_{i \geq p} H^{i,k-i} \oplus \bigoplus_{j' \leq p-1} H^{j',k-j'} \\
&= \bigoplus_{i \geq 0} H^{i,k-i} \\
&= V_{\mathbb{C}}
\end{aligned}$$

Thus, we have recovered the second definition of a Hodge structure of weight k.

(b) Recall the defnition of a morphism of Hodge structures. Let $V_{\mathbb{Z}}$ and $V'_{\mathbb{Z}}$ be two hodge structures of weight k and $k+2r$. Then a morphism of Hodge structures of type (r,r) is a group morphism ϕ whose $\mathbb{C}$-linear extension (which we will also call ϕ for this part) satisfies:

$$\phi(V^{p,q}) \subseteq V'^{p+r,q+r} \iff \phi(F^pV) \subset F^{p+r}V'$$

We need to show that this is strict for the Hodge filtration, i.e.

$$\operatorname{img}(\phi) \cap F^{p+r}V' = \phi(F^pV)$$

Let $\alpha \in \operatorname{img}(\phi) \cap F^{p+r}V'$. Then, there exists $\beta \in V$ such that $\phi(\beta) = \alpha$. Then, using the decomposition of V, we can write:

$$\begin{aligned} \alpha &= \phi(\beta) \\ &= \phi\left(\sum_{i+j=k} \beta^{i,j}\right) \\ &= \sum_{i+j=k} \phi(\beta^{i,j}) \end{aligned}$$

Then, each $\phi(\beta^{i,j})$ is of type $(i+r, j+r)$. Then, if $i < p$, note that $\phi(\beta^{i,j}) \in V'^{i+r,j+r} \cap F^{p+r}$. But, as noted from the previous problem, we have:

$$F^{p+r}V' = \bigoplus_{i \geq p+r} H^{i,k-i}$$

Thus, we have that $\phi(\beta^{i,j}) = 0$ for $i < p$. Thus, we can write:

$$\alpha = \sum_{i \geq p} \phi(\beta^{i,k-i}) \in F^{p+r}V'$$

noting the decomposition for $F^{p+r}V'$ from the previous problem and that $\phi(\beta^{i,j}) \in V'^{i+r,j+r}$ (as noted before). Thus, we have that:

$$\operatorname{img}(\phi) \cap F^{p+r}V' \subseteq \phi(F^pV)$$

The other direction is implied in the definition of a Hodge structure morphism.

(c) Let $\phi : V_{\mathbb{Z}} \to V'_{\mathbb{Z}}$ be a morphism of Hodge structures of type (r,r), where V is of weight k and V' is of weight $k+2r$. We first prove that $\operatorname{img}(\phi)$ and $\ker(\phi)$ have natural Hodge structures.

For $\operatorname{img}(\phi)$, first note that:

$$\operatorname{img}(\phi_{\mathbb{C}}) = \operatorname{img}(\phi) \otimes_{\mathbb{Z}} \mathbb{C} \qquad \ker(\phi_{\mathbb{C}}) = \ker(\phi) \otimes_{\mathbb{Z}} \mathbb{C}$$

Thus, we can define the filtration on $\operatorname{img}(\phi_{\mathbb{C}})$ by:

$$F^p \operatorname{img}(\phi_{\mathbb{C}}) := \operatorname{img}(\phi) \cap F^p V'_{\mathbb{C}}$$

It is easy to see that these define a decreasing filtration on $\operatorname{img}(\phi)$ and $\ker(\phi)$ by the filtration structures on the codomain and domain. Then, note for $\operatorname{img}(\phi)$ that we have:

$$\begin{aligned} F^p \operatorname{img}(\phi_{\mathbb{C}}) \oplus \overline{F^{k+2r-p+1} \operatorname{img}(\phi_{\mathbb{C}})} &= \operatorname{img}(\phi) \cap F^p V'_{\mathbb{C}} \oplus \overline{\operatorname{img}(\phi) \cap F^{k+2r-p+1} V'_{\mathbb{C}}} \\ &= \phi(F^{p-r}V) \oplus \overline{\phi(F^{k+r-p+1}V_{\mathbb{C}})} \\ &= \phi(F^{p-r}V \oplus \overline{F^{k+r-p+1}V_{\mathbb{C}}}) \\ &= \phi(V_{\mathbb{C}}) \\ &= \operatorname{img}(\phi_{\mathbb{C}}) \end{aligned}$$

Thus, $\operatorname{img}(\phi)$ has an Hodge structure of weight $k + 2r$ and for $\ker(\phi)$, we note that:

$$\ker(\phi_{\mathbb{C}}) = \bigoplus_{p+q=k} \ker(\phi_{\mathbb{C}}) \cap V^{p,q}$$

as the $V^{p,q}$ are disjoint. Let $K^{p,q} = \ker(\phi) \cap V^{p,q}$. Then, we have that:

$$\overline{\ker(\phi_{\mathbb{C}}) \cap V^{p,q}} = \overline{\ker(\phi_{\mathbb{C}})} \cap V^{q,p}$$

But $\ker(\phi_{\mathbb{C}})$ is closed under complex conjugation as $\phi_{\mathbb{C}}$ is $\mathbb{C}$-linear. Thus, we have shown that:

$$\ker(\phi_{\mathbb{C}}) = \bigoplus_{p+q=k} K^{p,q} \qquad K^{p,q} = \overline{K^{q,p}}$$

Thus, we have that $\ker(\phi_{\mathbb{C}})$ is a Hodge structure of weight k.

Finally, for the cokernel, note that:

$$\begin{aligned} \operatorname{coker}(\phi_{\mathbb{C}}) &= V'_{\mathbb{C}} / \operatorname{img}(\phi_{\mathbb{C}}) \\ &= (V' \otimes_{\mathbb{Z}} \mathbb{C}) / (\operatorname{img}(\phi) \otimes_{\mathbb{Z}} \mathbb{C}) \\ &= (V'_{\mathbb{C}} / \operatorname{img}(\phi_{\mathbb{C}})) \otimes_{\mathbb{Z}} \mathbb{C} \\ &= \operatorname{coker}(\phi) \otimes_{\mathbb{Z}} \mathbb{C} \end{aligned}$$

Let

$$\operatorname{coker}(\phi_{\mathbb{C}})^{p,q} = V'^{p,q} / \operatorname{img}(\phi)^{p,q}$$

where $\operatorname{img}(\phi)^{p,q}$ is the (p,q)-part of $\operatorname{img}(\phi_{\mathbb{C}})$. Then, we have:

$$\begin{aligned}\operatorname{img}(\phi_{\mathbb{C}})^{p,q} &= (\operatorname{img}(\phi_{\mathbb{C}}) \cap F^p V'_{\mathbb{C}}) \cap \overline{(\operatorname{img}(\phi_{\mathbb{C}}) \cap F^q V'_{\mathbb{C}})} \\ &= \operatorname{img}(\phi_{\mathbb{C}}) \cap V'^{p,q} \subseteq V'^{p,q}\end{aligned}$$

as the image is closed under complex conjugation. Thus, using these compatible decompositions, we note:

$$\begin{aligned}\bigoplus_{p+q=k+2r} \operatorname{coker}(\phi_{\mathbb{C}})^{p,q} &= \bigoplus_{p+q=k+2r} V'^{p,q} / \operatorname{img}(\phi_{\mathbb{C}})^{p,q} \\ &= \left(\bigoplus_{p+q=k+2r} V'^{p,q} \right) / \left(\bigoplus_{p+q=k+2r} \operatorname{img}(\phi_{\mathbb{C}})^{p,q} \right) \\ &= {V'}_{\mathbb{C}} / \operatorname{img}(\phi_{\mathbb{C}}) \\ &= \operatorname{coker}(\phi_{\mathbb{C}})\end{aligned}$$

Moreover, we have that:

$$\begin{aligned}\overline{\operatorname{coker}(\phi_{\mathbb{C}})^{p,q}} &= \overline{V'^{p,q} / \operatorname{img}(\phi_{\mathbb{C}})^{p,q}} \\ &= V'^{q,p} / \operatorname{img}(\phi_{\mathbb{C}})^{q,p} \\ &= \operatorname{coker}(\phi_{\mathbb{C}})^{q,p}\end{aligned}$$

as the quotient map is $\mathbb{C}$-linear, thus respects the complex structure. Thus, we have that $\operatorname{coker}(\phi_{\mathbb{C}})$ is a Hodge structure of weight $k+2r$.

(d) (I took several hints from the proof in Voisin, and the proof gradually looked more and more like hers).

Note that the pullback is:

$$\phi^* : H^k(Y, \mathbb{Z}) \to H^k(X, \mathbb{Z})$$

It suffices to show that ϕ^* is injective with $\mathbb{R}$ coefficients. Let $\dim_{\mathbb{C}}(X) = n$ and $\dim_{\mathbb{C}}(Y) = m$. Let $\alpha \in H^{2m}(Y, \mathbb{R})$. Then, we have $\phi^*\alpha \in H^{2m}(X, \mathbb{R})$. Let $r = n - m > 0$ (as ϕ surjective). Then, we have that $\omega^r \wedge \phi^*\alpha \in H^{2n}(X)$ is a top-degree form on X, where ω is the Kähler form. Voisin claims that this map is always non-negative, and positive on at least an open set. Thus, the integral:

$$\int_X \omega^r \wedge \phi^*\alpha > 0$$

Thus, $\phi^*\alpha \geq 0$. We conclude that ϕ^* is injective on $H^k(Y, \mathbb{R})$ using Poincare duality to get a top-form on Y, and noticing that the pullback distributes over the cup product (and is, in particular, a ring homomorphism).

Problem 2. Let $H_{\mathbb{R}}$ be a $\mathbb{R}$-vector space, and $H_{\mathbb{C}} := H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

(a) Show that a decomposition

$$H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q} \qquad H^{p,q} = \overline{H^{q,p}}$$

determines a continous action $\rho : \mathbb{C}^{\times} \to \mathrm{GL}(H_{\mathbb{C}})$ of $\mathbb{C}^{\times}$ on $H_{\mathbb{C}}$ given by:

$$z \cdot \alpha^{p,q} = z^p \overline{z^q} \alpha^{p,q}$$

for $\alpha^{p,q} \in H^{p,q}$. Show that this action satisfies:

$$\rho(z) = \overline{\rho(z)}$$

where the conjugacy on $GL(H_{\mathbb{C}})$ is defined by:

$$\overline{g}(u) = \overline{g(\overline{u})}$$

Show that one also has $\rho(t) = t^k \,\mathrm{id}$ for $t \in \mathbb{R}^{\times}$.

Conversely, let $\rho : \mathbb{C}^{\times} \to GL(H_{\mathbb{C}})$ be a continous action of $\mathbb{C}^*$ on $H_{\mathbb{C}}$ satisfying $\rho(t) = t^k \,\mathrm{id}$ for $t \in \mathbb{R}^*$ and $\rho(z) = \overline{\rho(z)}$. Show that there exists a decomposition of $H_{\mathbb{C}}$ into the direct sum of the $H^{p,q}$ such that $\rho(z)$ acts as above.

(b) Applying the diagonalization theorem for the actions of torsion abelian groups to the torsion points of $\mathbb{C}^*$, show that there exists a decompositon into a direct sum:

$$H = \bigoplus_{\chi} H_{\chi}$$

where χ belongs to the set of characters of $\mathbb{C}^*$ and $\mathbb{C}^*$ acts by $z \to \chi(z)\,\mathrm{id}$ on H_{χ}.

(c) Show that only the characters $\chi_{p,q} : z \to z^p \overline{z}^q$ with $p + q = k$ appear in the decomposition of $H_{\mathbb{C}}$.

(d) Let $H^{p,q} := H_{\chi_{p,q}}$. Show that $H^{p,q} = \overline{H^{q,p}}$.

(e) Let $V_{\mathbb{Z}}$ be a Hodge structure of weight $2k$ and $\mathbb{C}^* \to GL(V_{\mathbb{R}})$ the corresponding group morphism defined in the previous question. Show that the group of Hodge classes is equal to the group of stable (i.e. fixed up to a scaler) vectors of $V_{\mathbb{Z}}$ under the action of $\mathbb{C}^*$.

Solution. Collaboration notice: For parts b) and c), I got hints from Henry.

1. First, note that the map $z \to z^p \overline{z}^q$ is continous as a map from $\mathbb{C}^\times$ to $\mathbb{C}^\times$. Thus, the map $z \to z^p \overline{z}^q \,\mathrm{id}$ is continous on each factor of the direct sum, thus it is continous by definition of the product/direct sum topology on vector spaces.

   Next, note that for $\alpha \in H_\mathbb{C}$, we have:

$$\begin{aligned}\overline{\rho(z)}(\alpha^{p,q}) &= \overline{\rho(z)(\overline{\alpha^{p,q}})}\\ &= \overline{z^p\overline{z}^q\alpha^{q,p}}\\ &= \overline{z}^p z^q \overline{\alpha^{q,p}}\\ &= \overline{z}^p z^q \alpha^{p,q}\\ &= \rho(\overline{z})(\alpha^{p,q})\end{aligned}$$

   Finally, if $t \in \mathbb{R}^\times$, we have $t = \overline{t}$, thus, for $\alpha \in H_\mathbb{C}$ with $\alpha \in H_\mathbb{C}$, we have:

$$\begin{aligned}\rho(t)(\alpha^{p,q}) &= t^p\overline{t}^q\alpha^{p,q}\\ &= t^k\alpha^{p,q}\end{aligned}$$

   for all $p+q=k$. Thus, we have that $\rho(t) = t^k \,\mathrm{id}$ for $t \in \mathbb{R}^\times$.

2. I did not know how to do this part.

3. First consider some primitive nth root of unity ζ_n (one of them generates the rest). Then, since χ is a group homomorphism $\mathbb{C}^\times \to \mathbb{C}^\times$, we have:

$$\chi(\zeta_n)^n = \chi(\zeta^n) = \chi(1) = 1$$

   Thus, $\chi(\zeta_n)$ is a nth root of unity, thus is some power of ζ_n, i.e.:

$$\chi(\zeta_n) = \zeta_n^m$$

   Since the action is continous and nth roots of unity are dense in the unit circle, we conclude that there is a fixed m such that for all $z \in \mathbb{C}^\times$ with $|z| = 1$, we have:

$$\chi(z) = z^m$$

   For $z \in \mathbb{C}^\times$, we can write $z = (z\overline{z})^{1/2}e^{i\theta}$. Then, we have:

$$\begin{aligned}\chi(z) &= \chi((z\overline{z})^{1/2}e^{i\theta})\\ &= \chi(z\overline{z})^{1/2}\chi(e^{i\theta})\\ &= (z^k\overline{z}^k)^{1/2}e^{im\theta}\end{aligned}$$

   Then (Henry helped with this part), we compute:

$$\begin{aligned}\chi(z) &= (z^{k-m}\overline{z}^{k-m})^{1/2}(z^m\overline{z}^m)^{1/2}e^{im\theta}\\ &= (z^{k-m}\overline{z}^{k-m})^{1/2}z^m\\ &= \overline{z}^{(k-m)/2}z^{(k+m)/2}\end{aligned}$$

Thus, we can set $p := (k+m)/2$ and $q := (k-m)/2$. We only need to show that $p, q \in \mathbb{Z}$. Note that, since $\chi(t) = t^k$ for $t \in \mathbb{R}^\times$,:

$$\chi(-1) = (-1)^k = 1^k e^{im\pi}$$

Thus, $m \equiv k \pmod 2$. Thus, $p, q \in \mathbb{Z}$ and we finally have:

$$\chi(z) = z^p \overline{z}^q$$

4. Note that, we are also given:

$$\rho(z) = \chi(\overline{z})\,\mathrm{id} = \overline{\chi(z)\,\mathrm{id}} = \overline{\rho(z)}$$

Thus, for $\alpha \in H_{\mathbb{C}}$, let $\alpha^{p,q} \in H^{p,q}$, then, we have:

$$\begin{aligned}
\rho(z)(\overline{\alpha^{p,q}}) &= \chi(z)\,\mathrm{id}(\overline{\alpha^{p,q}}) \\
&= \overline{\chi(z)}\,\mathrm{id}(\overline{\alpha^{p,q}}) \\
&= \overline{\chi(z)\,\mathrm{id}(\alpha^{p,q})} \\
&= \overline{z^p \overline{z}^q\,\mathrm{id}(\alpha^{p,q})} \\
&= z^q \overline{z}^p\,\mathrm{id}(\overline{\alpha^{p,q}})
\end{aligned}$$

Thus, $\overline{\alpha^{p,q}} \in H^{q,p}$. As complex conjugation is self-inverse, we get the other inclusion. Thus, we have that $H^{p,q} = \overline{H^{q,p}}$.

5. Let $\alpha \in V^{n,n} \cap V_{\mathbb{Z}}$ be a Hodge class. Then, for any $z \in \mathbb{C}^\times$, we have:

$$\begin{aligned}
\rho(z)(\alpha) &= z^n \overline{z}^n \alpha \\
&= |z|^{2n} \alpha
\end{aligned}$$

Thus, α is a stable vector under the action of $\mathbb{C}^\times$. Next, let $\alpha \in V_{\mathbb{Z}} \subset V_{\mathbb{C}}$ be a stable vector under the action of $\mathbb{C}^\times$. Let $\alpha^{p,q} \in V^{p,q}$. Then, we have:

$$\rho(z)(\alpha^{p,q}) = \sum_{p+q=2k} z^p \overline{z}^q \alpha^{p,q}$$

Then, if we want all components of α to be scaled by the same scaler, we want $\alpha^{p,q} \neq 0$ for at most one (p,q). Thus, $\alpha = \alpha^{p,q}$ for some p, q such that $p + q = 2k$. I do not understand why we also want $z^p \overline{z}^q$ to also be real, but assuming we do want that (why does the definition of the action includes $V_{\mathbb{R}}$ in place of $V_{\mathbb{C}}$ in Voisin?), it follows that we want $p = q = k$. Thus, α is a Hodge class.

In general, I used the correction you emailed Henry about, i.e. that $\rho(z) = \overline{\rho(z)}$. This makes part 4) work out. However, I still showed that $\rho(\overline{z}) = \overline{\rho(z)}$ in part 1). These do not agree. I'm wondering if the definition of conjugation for $GL(H_{\mathbb{C}})$ is incorrect.

Problem 3.(The Hodge Decomposition for Curves) Let X be a compact connected complex curve. We have the differential:

$$d : \mathcal{O}_X \to \Omega_X$$

between the sheaf of homolormphic functions and the sheaf of holomorphic differentials.

(a) Show that d is surjective with kernel equal to the constant sheaf $\mathbb{C}$. Hence, we have an exact sequence:

$$0 \to \mathbb{C} \to \mathcal{O}_X \to\to \Omega_X \to 0$$

(b) Deduce from Serre duality that $H^1(X, \Omega_X) \cong \mathbb{C}$. Deduce from Poincare duality that $H^2(X, \mathbb{C}) = \mathbb{C}$.

(c) Show that (6.15) induces a short exact sequence:

$$0 \to H^0(X, \Omega_X) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X) \to 0$$

(d) Show that the map which to a holomorphic form α associates the class of $\overline{\alpha}$ in $H^1(X, \mathcal{O}_X)$ is injective.

(e) Deduce from Serre duality that it is also surjective and that we have the decomposiiton:

$$H^1(X, \mathbb{C}) = H^0(X, \Omega_X) \oplus \overline{H^0(X, \Omega_X)}$$

with

$$\overline{H^0(X, \Omega_X)} \cong H^1(X, \mathcal{O}_X)$$

Solution.

1. Let U be an open set of X. Let $\omega \in \Omega_X(U)$ be a section. Then $\omega = f dz$ in local coordinates. Since holomorphic functions are analytic, they have a primitive locally. Thus, there exists $g \in \mathcal{O}_X(U)$ such that $dg/dz = f$. Then, we have that $d(g) = f dz = \omega$. Thus, d is surjective.

   Now let $f \in \mathcal{O}_X(U)$ such that $df = 0$. Thus, f is locally constant. Since X is compact and connected, f extends to a globally constant function in $\mathcal{O}_X(X)$. Thus, $\ker(d) = \mathbb{C}$, the constant sheaf.

   Thus, we have the exact sequence:

$$0 \to \mathbb{C} \to \mathcal{O}_X \xrightarrow{d} \Omega_X \to 0$$

2. Recall that Serre duality gives us the isomorphism:

$$H^1(X, \Omega_X) \cong H^0(X, \Omega_X^\vee \otimes \Omega_X)^\vee$$

However, $\Omega_X^\vee \otimes \Omega_X \cong \mathcal{O}_X$ (by contracting). Thus, we have:

$$H^1(X, \Omega_X) \cong H^0(X, \mathcal{O}_X)^\vee$$

However, $H^0(X, \mathcal{O}_X) \cong \mathbb{C}$ as global holomorphic functions on a compact connected surface are constant. Thus, $H^1(X, \Omega_X) \cong H^0(X, \mathcal{O}_X)^\vee \cong \mathbb{C}$.

Recall that Poincare duality gives us the isomorphism:

$$H^2(X, \mathbb{C}) \cong H^0(X, \mathbb{C})^\vee$$

Similar to before, as X is compact and connected, we have that $H^0(X, \mathbb{C}) \cong \mathbb{C}$. Thus, we have:

$$H^2(X, \mathbb{C}) \cong H^0(X, \mathbb{C})^\vee \cong \mathbb{C}$$

3. We obtain the long exact sequence in cohomology from the short exact sequence.

$$\to H^0(X, \mathbb{C}) \to H^0(X, \mathcal{O}_X) \xrightarrow{d} H^0(X, \Omega_X) \to H^1(X, \mathbb{C}) \to \cdots$$

Note that $H^0(X, \mathbb{C})$ are the globally constant functions, which are the same as the global holomorphic functions on a compact connected manifold. Thus, the first map is an isomorphism. Thus, $d : H^0(X, \mathcal{O}_X) \to H^0(X, \Omega_X)$ is 0. Thus, the sequence becomes:

$$0 \to H^0(X, \Omega_X) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \Omega_X) \to H^2(X, \mathbb{C}) \to H^2(X, \mathcal{O}_X)$$

Then, since $H^2(X, \mathcal{O}_X) = H^{0,2}(X) = 0$, the last map is 0. Thus, the map $H^1(X, \Omega_X) \to H^2(X, \mathbb{C})$ is an isomorphism. Thus, the map before it is 0, and we get:

$$0 \to H^0(X, \Omega_X) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X) \to 0$$

4. Let $\alpha \in H^0(X, \Omega_X)$ be a global holomorphic form. Then, $\overline{\alpha} \in H^1(X, \mathbb{C})$ is of type $(0, 1)$ (antiholomorphic). If $[\overline{\alpha}] = 0 \in H^1(X, \mathcal{O}_X)$, then $\overline{\alpha}$ is in $\operatorname{img}(H^0(X, \Omega_X) \to H^1(X, \mathbb{C}))$. But this is an inclusion map (the connecting morphism deals solely with holomorphic objects), thus, we would have $\overline{\alpha} = \beta$ for some global holomorphic form β. This implies $\alpha = 0$. Thus, the map is injective.

5. By Serre duality, we noted that

$$H^1(X, \Omega_X) \cong H^0(X, \mathcal{O}_X)^\vee$$

Moreover, since these are finite-dimensional vector spaces, the dual has the same dimension as the original space. Thus,

$$\dim(H^1(X, \Omega_X)) = \dim(H^0(X, \mathcal{O}_X))$$

Thus, an injective linear map is also surjective. Thus, the map given is an isomorphism. Since taking the complex conjugate is also an isomorphism, we must have that:

$$\overline{H^0(X, \Omega_X)} \cong H^1(X, \mathcal{O}_X)$$

Then, also noting that an SES of vector spaces splits, we get:

$$H^1(X, \mathbb{C}) = H^0(X, \Omega_X) \oplus \overline{H^0(X, \Omega_X)}$$