Problem 1. Prove the pasting lemma: Suppose $X = A \cup B$ is a topological space with A, B closed in X. If $f\colon X \to Y$ is a map such that the restrictions $f|_A$ and $f|_B$ are continuous, then f is continuous.

Solution. Let $V \subseteq Y$ be any closed set. To show that f is continous, we need to show that $f^{-1}(V) \subseteq X$ is closed (this definition is equivalent to the definition of continuity in terms of open sets, as taking the complement commutes with taking pre-images).

Note that $f^{-1}(V) \cap A$ and $f^{-1}(V) \cap B$ are closed, since they are pre-images of closed set V under continuous functions $f|_A$ and $f|_B$ respectively. However, since $X = A \cup B$, $f^{-1}(V) = (f^{-1}(V) \cap A) \cup (f^{-1}(V) \cap B)$. Thus, $f^{-1}(V)$ is closed.

Problem 2. In a connected space X, a point $x \in X$ is called a *cut point* if $X \setminus \{x\}$ is disconnected.

1. Suppose that $f : X \to Y$ is a homeomorphism of connected spaces. Show that $x \in X$ is a cut point if and only if $f(x) \in Y$ is a cut point.

2. Show that none of the spaces $(0, 1)$, $(0, 1]$, $[0, 1]$, and $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$ are homeomorphic to each other.

3. Show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for any $n \geq 2$.

4. The bouquet B_n of n circles is the space obtained by gluing n disjoint copies of S^1 at a single point in each circle. Show that B_n and B_m are not homeomorphic for $n \neq m$.

Solution.

1. Note that $f|_{X\setminus\{x\}}$ is a homeomorphism onto (its image) $Y \setminus \{f(x)\}$ as the restriction of a homeomorphism is a homeomorphism onto its image. Thus, $X\setminus\{x\}$ is homeomorphic to $Y \setminus \{f(x)\}$. Thus, noting that X and Y are both connected, we have:

$$\begin{aligned} &x \in X \text{ is a cut-point} \\ \iff &X \setminus \{x\} \text{ is disconnected} \\ \iff &Y \setminus \{f(x)\} \text{ is disconnected} \\ \iff &f(x) \in Y \text{ is a cut-point} \end{aligned}$$

2. Note that in $(0, 1)$ every point is a cut-point, in $(0, 1]$, there is only 1 point that is not a cut-point $\{1\}$, in $[0, 1]$ there are two points that are not cut-points $\{0, 1\}$, and in S^1 no point is a cut-point. Since cut-points are in bijection with cut-points under a homeomorphisms, points that are not cut-points are in bijection with points that are not cut-points. Since each of these spaces have a different number of points that are not cut-points, with 0, 1, 2, and an infinite number, respectively, none of them are homeomorphic.

3. Note that $0 \in \mathbb{R}$ is a cut-point, as $\mathbb{R} \setminus \{0\}$ is disconnected. For sake of contradiction, suppose there existed an homeomorphism from $\mathbb{R} \to \mathbb{R}^n$ for $n \geq 2$. Then, $f(0) \in \mathbb{R}^n$ would be a cut-point, as f is a homeomorphism (part a). However, $\mathbb{R}^n\setminus\{x\}$ is connected for all $x \in \mathbb{R}^n, n \geq 2$, as it is path-connected. Thus, $\mathbb{R}^n \setminus \{f(0)\}$ is also connected. Thus, $f(0)$ is not a cut-point and we have a contradiction.

4. Let $X = B^n$ and $Y = B^m$ for $n \neq m$. Suppose f is a homemorphism between them. Since homeomorphisms map cut-points to cut-points, f must map the unique cut-point in X to the unique cut-point in Y. Since restrictions of homeomorphism is a

homeomorphism, $g := f|_{X \setminus \{c\}}$ is an homeomorphism onto $Y \setminus \{f(c)\}$. From problem 5, we know that the π_0 is a functor, thus it takes a homeomorphism of spaces to a set isomorphism (bijection) $\pi_0(g) : \pi_0(X \setminus \{c\}) \to \pi_0(Y \setminus \{f(c)\})$. However, $\pi_0(X \setminus \{c\})$ has n elements while $\pi_0(Y \setminus \{f(c)\})$ has m elements. Thus, $\pi_0(g)$ is not a bijection and we have a contradiction. Thus, $X = B_n$ and $Y = B_m$ are not homeomorphic for $n \neq m$.

Problem 3. Define $\mathbb{RP}^n$ to be the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ by $(x_1, \ldots, x_{n+1}) \simeq (ax_1, \ldots, ax_{n+1})$ for all nonzero scalars a. Let $[x_1 : \ldots : x_{n+1}] \in \mathbb{RP}^n$ denote the image of $(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}$ under the quotient map. Show that $i\colon \mathbb{R}^n \to \mathbb{RP}^n$ given by

$$i(x_1, \ldots, x_n) = [1 : x_1 : \ldots : x_n]$$

is a topological embedding (i.e., a homeomorphism onto its image) and the complement of $i(\mathbb{R}^n)$ is homeomorphic to $\mathbb{RP}^{n-1}$.

Solution. Call the quotient map π. To show that i is a topological embedding, we will show that i is a continous map with a well-defined continuous inverse from its image.

Note that $\phi : \mathbb{R}^n \to \mathbb{R}^{n+1}$ defined by $\phi(x_1, \cdots, x_n) = (1, x_1, \cdots, x_n)$ is continous. Since π is continous, we have $i = \pi \circ \phi$ is also continous.

To show that i is injective, let $i(x_1, \cdots, x_n) = i(y_1, \cdots, y_n)$. Then, we have:

$$[1 : x_1 : \cdots : x_n] = [1 : y_1 : \cdots : y_n]$$

Thus, there exists a non-zero $\lambda \in \mathbb{R}$ such that:

$$(1, x_1, \cdots, x_n) = \lambda(1, y_1, \cdots, y_n)$$

Due to the first entry, we must have $\lambda = 1$. Thus, we have:

$$(x_1, \cdots, x_n) = (y_1, \cdots, y_n)$$

Thus, i is injective. Next, let $U = \{[x_0 : \cdots : x_n] \in \mathbb{RP}^n : x_0 \neq 0\}$. Define $\phi : U \to \mathbb{R}^n$ by:

$$\phi([x_0 : x_1 : x_2 : \ldots : x_n]) = (x_1/x_0, \cdots, x_n/x_0)$$

This map is well-defined, as for any $\lambda > 0 \in \mathbb{R}$,

$$\phi([\lambda x_0 : \cdots : \lambda x_n]) = (x_1/x_0, \cdots, x_n/x_0)$$

Moreover, note that the map ψ from $U' = \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} : x_0 = 1\}$ to $\mathbb{R}^n$ given by:

$$(1, x_1, \cdots, x_n) \to (x_1, \cdots, x_n)$$

is continous as it is projection onto the last n coordinates. Moreover, note that $\pi|_{U'}$ is bijective onto U as every class in U has a unique representative with $x_0 = 1$, thus, as it is also open, $\pi|_{U'}$ is a homeomorphism. Since $\psi = \phi \circ \pi|_{U'}$, ψ is continous, and $\pi|_{U'}$ is a homeomorphism, ϕ is also continous.

We observe that $\phi|_{i(\mathbb{R}^n)}$ is the inverse of i (easy to check). Thus, i is a homeomorphism onto its image.

To show that the complement of $i(\mathbb{R}^n)$ is homeomorphic to $\mathbb{RP}^{n-1}$, we note that the complement only contains elements that do not have a representative of the form $[1 : x_1 : \cdots : x_n]$. However, every $(x_0, x_1, \cdots, x_n) \in \mathbb{R}^{n+1}$ with $x_0 \neq 0$ projects to $[1 : x_1/x_0 : \cdots : x_n/x_0]$ under π. Thus, the complement must have $x_0 = 0$, and is of the form:

Problem 4. Suppose that $\mathcal{C}$ is a category, A, B, C are objects of $\mathcal{C}$, and $f \in \hom(A, B)$ and $g \in \hom(B, C)$ are isomorphisms.

1. Show that f has a unique inverse in $\hom(B, A)$.
2. Show that id_A is an isomorphism.
3. Show that the inverse of f is an isomorphism.
4. Show that $g \circ f$ is an isomorphism.
5. Show that $\mathrm{Aut}(A)$ and $\mathrm{Aut}(B)$ are isomorphic groups.
6. Show that if $F : \mathcal{C} \to \mathcal{D}$ is a functor, then $F(f) \in \hom_{\mathcal{D}}(F(A), F(B))$ is an isomorphism.

Solution.

1. Let $f : A \to B$ be an isomorphism, with $f', f'' \in \hom(B, A)$ inverses of f. By the associativity of composition, we have:

   $$f' = f' \circ \mathrm{id}_B = f' \circ (f \circ f'') = (f' \circ f) \circ f'' = \mathrm{id}_A \circ f'' = f''$$

   Thus, f' and f'' are equal. Hence, f has a unique inverse in $\hom(B, A)$.

2. We claim that $\mathrm{id}_A \in \hom(A, A)$ is its own inverse. Clearly, by the definition of the identity:

   $$\mathrm{id}_A \circ \mathrm{id}_A = \mathrm{id}_A$$

   Since id_A has an inverse, it is an isomorphism.

3. Let $f^{-1} \in \hom(B, A)$ be the unique inverse of $f \in \hom(A, B)$. We claim that f is the inverse of f^{-1}. By the definition of the inverse, we have:

   $$f \circ f^{-1} = \mathrm{id}_B \qquad f^{-1} \circ f = \mathrm{id}_A$$

   Thus, f^{-1} has an inverse f, and is thus an isomorphism.

4. Since $f \in \hom(A, B)$ and $g \in \hom(B, C)$ are isomorphisms, there exists inverses $f^{-1} \in \hom(B, A)$ and $g^{-1} \in \hom(C, B)$. We claim that $g \circ f$ is an isomorphism with

inverse $f^{-1} \circ g^{-1}$. Using associatiativity, we check:

$$\begin{aligned}(g \circ f) \circ (f^{-1} \circ g^{-1}) &= g \circ \mathrm{id}_B \circ g^{-1} \\ &= g \circ g^{-1} \\ &= \mathrm{id}_C \\ (f^{-1} \circ g^{-1}) \circ (g \circ f) &= f^{-1} \circ \mathrm{id}_B \circ f \\ &= f^{-1} \circ f \\ &= \mathrm{id}_A\end{aligned}$$

Thus, $g \circ f$ is an isomorphism with inverse $f^{-1} \circ g^{-1}$.

5. We define the map $\phi : \mathrm{Aut}(A) \to \mathrm{Aut}(B)$ by $\phi(\alpha) = f \circ \alpha \circ f^{-1}$. Since we proved that f and f^{-1} are isomorphisms, α is an isomorphism by definition, and that the composition of isomorphisms is an isomorphism, we conclude that $\phi(\alpha) \in \mathrm{Aut}(B)$. To show that ϕ is a group homomorphism, we check for any $\alpha, \beta \in \mathrm{Aut}(A)$:

$$\begin{aligned}\phi(\alpha) \circ \phi(\beta) &= (f \circ \alpha \circ f^{-1}) \circ (f \circ \beta \circ f^{-1}) \\ &= f \circ \alpha \circ \mathrm{id}_A \circ \beta \circ f^{-1} \\ &= f \circ \alpha \circ \beta \circ f^{-1} \\ &= \phi(\alpha \circ \beta)\end{aligned}$$

Moreover, we claim that ϕ is an isomorphism as it has an inverse $\phi^{-1} : \mathrm{Aut}(B) \to \mathrm{Aut}(A)$, given by:

$$\phi^{-1}(\gamma) = f^{-1} \circ \gamma \circ f$$

The proof that ϕ^{-1} is a well-defined homomorphism is analogous to the proof for ϕ. Thus, we check that these maps are inverses, for $\alpha \in \mathrm{Aut}(A)$ and $\gamma \in \mathrm{Aut}(B)$:

$$\begin{aligned}\phi \circ \phi^{-1}(\gamma) &= \phi(f^{-1} \circ \gamma \circ f) \\ &= f \circ (f^{-1} \circ \gamma \circ f) \circ f^{-1} \\ &= \mathrm{id}_B \circ \gamma \circ \mathrm{id}_A \\ &= \gamma \\ \phi^{-1} \circ \phi(\alpha) &= \phi^{-1}(f \circ \alpha \circ f^{-1}) \\ &= f^{-1} \circ (f \circ \alpha \circ f^{-1}) \circ f \\ &= \mathrm{id}_A \circ \alpha \circ \mathrm{id}_B \\ &= \alpha\end{aligned}$$

Thus,

$$\phi \circ \phi^{-1} = \mathrm{id}_{\mathrm{Aut}(B)} \qquad \phi^{-1} \circ \phi = \mathrm{id}_{\mathrm{Aut}(A)}$$

Thus, ϕ is a group isomorphism and $\mathrm{Aut}(A) \cong \mathrm{Aut}(B)$.

6. We claim $F(f)$ is an isomorhism, with inverse $F(f^{-1}) \in \hom_{\mathcal{D}}(F(B), F(A))$. Since F is a functor, it respects composition and identities. Thus, we check:

$$\begin{aligned} F(f) \circ F(f^{-1}) = F(f \circ f^{-1}) = F(\mathrm{id}_B) = \mathrm{id}_{F(B)} \\ F(f^{-1}) \circ F(f) = F(f^{-1} \circ f) = F(\mathrm{id}_A) = \mathrm{id}_{F(A)} \end{aligned}$$

Problem 5. For a space X, let $\pi_0(X)$ be the space of path components of X. Recall that the image of a path-connected space is path-connected. Thus, for a continuous map $f : X \to Y$, there is an induced map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ taking a path component A to the path component containing $f(A)$. Show that this makes π_0 a functor from the category of topological spaces to the category of sets.

Solution. First, we understand what the induced morphism $\pi_0(f)$ is. For a subset $A \subseteq X$, we let $[A] \in \pi_0(X)$ denote the path component of A. Then,

$$\pi_0(f)([A]) = [f(A)]$$

To show that π_0 is a functor, we need to show that it preserves identities and compositions.

- Let X be a topological space and $\mathrm{id}_X : X \to X$ be the identity map on X. Then, for any path component $A \subseteq X$, $\pi_0([\mathrm{id}_X])(A) = [\mathrm{id}_X(A)] = [A]$. Thus, $\pi_0(\mathrm{id}_X) = \mathrm{id}_{\pi_0(X)}$.
- Let X, Y, Z be topological spaces and $f : X \to Y$, $g : Y \to Z$ be continuous maps. Then, for any path component $A \subseteq X$, we have:

$$\begin{aligned} \pi_0(g) \circ \pi_0(f)([A]) &= \pi_0(g)([f(A)]) \\ &= [g(f(A))] \\ &= [g \circ f(A)] \\ &= \pi_0(g \circ f)([A]) \end{aligned}$$

  Thus,

$$\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$$

Thus, π_0 preserves identities and compositions, and hence is a functor.

Problem 6. (Munkres, *Topology*, 51.2) Given spaces X and Y, let $[X, Y]$ denote the set of homotopy classes of maps of X into Y.

1. Let $I = [0, 1]$. Show that for any X, the set $[X, I]$ has a single element.
2. Show that if Y is path connected, then $[I, Y]$ has a single element.

Solution. .

1. Let X be a topological space and let $f \in [X, I]$ be arbitrary. We will show that f is homotopic to the constant map $c : X \to I$ given by $c(x) = 0$ for all $x \in X$. We define a homotopy $H : X \times I \to I$ by:

   $$H(x, t) = (1 - t)f(x)$$

   Clearly, at $t = 0$, $H(x, 0) = f(x)$ and at $t = 1$, $H(x, 1) = 0$. Since $f(x)$ and $(1 - t)$ are continous functions, H is continous. Thus, H is a homotopy between f and c. Since f was arbitrary, by the trasitivity of homotopy, $[X, I]$ has a single element.

   Note that we can also do this by realizing that I is convex and using the straight-line homotopy between any two maps.

2. Let Y be a path connected space. Let $f, g \in [I, Y]$ be arbitrary. Let $\sigma : I \to Y$ be a path from $f(0)$ to $g(0)$. Define $H : I \times I \to Y$ by:

   $$H(x, t) = \begin{cases} f((1 - 3t)x) & \text{if } 0 \leq x \leq \frac{1}{3} \\ \sigma((3t - 1)x) & \text{if } \frac{1}{3} \leq x \leq \frac{2}{3} \\ g((3t - 2)x) & \text{if } \frac{2}{3} \leq x \leq 1 \end{cases}$$

   Note that H is well-defined, as $f(x, 1/3) = f(0) = \sigma(0)$ and $g(x, 2/3) = g(0) = \sigma(1)$. Moreover, as $(1 - 3t)x$, $(3t - 1)x$, and $(3t - 2)x$ are continous, each of the three cases are composition of continous functions, hence, continous. Thus, by the pasting lemma, H is continous.

   Finally, note that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$, hence f and g are homotopic. Since f and g were arbitrary, we conclude that $[I, Y]$ has a single element.

Problem 7. (Munkres, *Topology*, 51.3) A space X is said to be *contractible* if the identity map $i_X : X \to X$ is nullhomotopic.

1. Show that I and $\mathbb{R}$ are contractible.
2. Show that a contractible space is path connected.
3. Show that if Y is contractible, then for any X, the set $[X, Y]$ has a single element.
4. Show that if X is contractible and Y is path connected then $[X, Y]$ has a single element.

Solution.

1. Let $X = \mathbb{R}$. We define a homotopy $H : \mathbb{R} \times I \to \mathbb{R}$ by:
$$H(x, t) = \mathrm{id}_X((1 - t)x)$$
H is continous as it is a composition of continous functions. Note that $H(x, 0) = \mathrm{id}_X(x)$ and $H(x, 1) = \mathrm{id}_X(0) = 0$ for all $x \in X$. Thus, id_X is null-homotopic and $\mathbb{R}$ is contractible. Since $I \subset \mathbb{R}$, we can use the same homotopy to show that I is contractible.

2. Let X be a topological space. Let $a \in X$ be arbitrary. Since X is contractible, there exists a homotopy $H : X \times I \to X$ between the identity map and a constant map $\lambda_c : X \to X$ with $\lambda_c(x) = c \in X$ for all $x \in X$. Then, we can define a path from a to c as follows:
$$\sigma : I \to X \qquad \sigma(t) = H(a, t)$$
This map is continous as it is a restriction of a continous map H. Moreover,
$$\sigma(0) = H(a, 0) = \mathrm{id}_X(a) = a \qquad \sigma(1) = H(a, 1) = \lambda_c(a) = c$$
Thus, σ is a path from a to c. Since a was arbitrary, there is a path from x to c for all $x \in X$. Since being in the same path component defines an equivalence relation, X has only one path component. Hence, X is path connected.

3. Let X and Y be topological spaces with Y contractible. Since Y is contractible, there exists a homotopy $H : Y \times I \to Y$ between id_Y the identity on Y and the constant map $\lambda_c : Y \to Y$ with $\lambda_c(y) = c \in Y$ for all $y \in Y$. Let $f \in [X, Y]$ be arbitrary. We define a homotopy $H' : X \times I \to Y$ by:
$$H'(x, t) = H(f(x), t)$$
Since H and f are continous, H' is a composition of continous functions, hence, it is continous. Note that:
$$H'(x, 0) = H(f(x), 0) = \mathrm{id}_Y(f(x)) = f(x) \qquad H'(x, 1) = H(f(x), 1) = \lambda_c(f(x)) = c$$

Thus, f is homotopic to the constant map $\lambda_c \circ f$. Since f was arbitrary, and homotopy is transitive, we conclude that $[X, Y]$ has a single element.

4. Let X be a contractible space and Y be a path connected space. Since X is contractible, there exists a homotopy $H : X \times I \to X$ between id_X and the constant map $\lambda_c : X \to X$ with $\lambda_c(x) = c \in X$ for all $x \in X$. Since Y is path connected, there exists a path $\sigma_y : I \to Y$ from $f(c)$ to a fixed $y \in Y$. Let $f \in [X, Y]$ be arbitrary. We will define a homotopy from f to the constant function with value y.

   We define the homotopy $H' : X \times I \to Y$ by:

$$H'(x,t) = \begin{cases} f(H(x,2t)) & \text{if } 0 \le t < frac12 \\ \sigma_y((2t-1)) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

   Since H, f and σ_y are continous, each of the cases of H' is continous. Moreover, it agrees on the intersection, as:

$$H'(x, 1/2) = f(H(x,1)) = f(\lambda_c(x)) = f(c) = \sigma_y(0)$$

   Thus, by the pasting lemma, H' is continous.

   Note that:

$$\begin{aligned} H'(x,0) &= f(H(x,0)) = f(\mathrm{id}_X(x)) = f(x) \\ H'(x,1) &= \sigma(1) = y \end{aligned}$$

   Thus, as f was arbitrary, every map $f \in [X, Y]$ is homotopic to the constant map with value y. Thus, the the transitivity of homotopy, $[X, Y]$ has a single element.