Math 113: Functional Analysis

Sair Shaikh

April 19, 2025

Problem 1. Show that X is compact if and only if given any family $\mathcal{F}$ of closed sets of X with the finite intersection property, we have $\bigcap_{F\in\mathcal{F}} F \neq \emptyset$.

Solution.

($\implies$) Assume X is compact. We will prove that contrapositive. Let $\{F_\alpha\}$ be a family of closed sets such that:

$$\bigcap_{\alpha} F_\alpha = \emptyset$$

Let $\{U_\alpha\}$ be a family of open sets such that $U_\alpha = X \setminus F_\alpha$. Then, we have:

$$\bigcup_{\alpha} U_\alpha = X \setminus \bigcap_{\alpha} F_\alpha = X$$

Thus, $\{U_\alpha\}$ is an open cover of X. Since X is compact, there exists a finite subcover of $\{U_\alpha\}$. Let $\{U_1, \dots, U_n\}$ be the finite subcover. Then, we have:

$$\bigcap_{i=1}^{n} F_i = X \setminus \bigcap_{i=1}^{n} U_i = \emptyset$$

Thus, we have found a finite subset of $\{F_\alpha\}$ with empty intersection. Thus, $\{F_\alpha\}$ does not have the finite intersection property. Therefore, by the contrapositive, any family of closed sets with the finite intersection property has non-empty intersection.

($\impliedby$) Assume any family of closed sets with the finite intersection property has non-empty intersection. By the contrapositive, this implies that if the family has empty intersection, it cannot have the finite intersection property.

Let $\{U_\alpha\}$ be an open cover of X. To show that X is compact it suffices to show that there exists a finite subcover of $\{U_\alpha\}$. Let $\{F_\alpha\}$ be the family of closed sets such that $F_\alpha = X \setminus U_\alpha$. Then,

$$\bigcap_\alpha F_\alpha = X \setminus \bigcup_\alpha U_\alpha = \emptyset$$

Thus, $\{F_\alpha\}$ does not have the finite intersection property. Therefore, there exists a finite subset of $\{F_\alpha\}$, call it $\{F_1, \ldots, F_n\}$, such that:

$$\bigcap_{i=1}^n F_i = \emptyset$$

However, we know that:

$$\bigcap_{i=1}^n F_i = X \setminus \bigcup_{i=1}^n U_i$$

Thus, we conclude that:

$$X \setminus \bigcup_{i=1}^n U_i = \emptyset$$

Thus, $\bigcup_{i=1}^n U_i = X$. Therefore, $\{U_1, \ldots, U_n\}$ is a finite subcover of $\{U_\alpha\}$. Therefore, X is compact.

Problem 2. Let X be a metric space.

1. Show that if E is a compact subspace of X, then E is closed.
2. Show that if X is compact and E is closed in X, then E is compact.

Solution.

1. Let E be a compact subspace of X. To show that E is closed, it suffices to show that $X \setminus E$ is open. Let $x \in X \setminus E$. Then, we define to families of open sets, $\{U_e\}_{e\in E}$ and $\{V_e\}_{e\in E}$, such that:

$$U_e = B_\epsilon(e) \text{ where } \epsilon < \rho(x,e)/2$$
$$V_e = B_\epsilon(x) \text{ where } \epsilon < \rho(x,e)/2$$

Note that by definition, $x \in V_e$ for all $e \in E$ and that $U_e \cap V_e = \emptyset$. Moreover, by definition, $\{U_e\}_{e\in E}$ is an open cover of E. Since E is compact, there exists a finite subcover of $\{U_e\}_{e\in E}$. Let $\{U_{e_1}, \ldots, U_{e_n}\}$ be the finite subcover. Then, since $\{V_{e_1}, \ldots, V_{e_n}\}$ is a finite collection of open sets, $V = \bigcap_{i=1}^n V_{e_n}$ is an open set that contains x. However, since $U_{e_i} \cap V_{e_i} = \emptyset$, and $V \subseteq V_{e_i}$, we have that $V \cap U_{e_i} = \emptyset$ for all i. Thus,

$$V \cap E = V \cap \bigcup_{i=1}^n U_{e_i} = \emptyset$$

Thus, there exists an open set V containing x disjoint from E, thus contained in $X \setminus E$. Therefore, $X \setminus E$ is open. Thus, E is closed.

2. Assume X is compact and E is closed in X. If $E = X$, then we are done. Thus, assume $X \neq E$. Let $\{U_\alpha\}$ be an open cover of E in E. Then, there exists a family of open sets $\{V_\alpha\}$ in X where, for all α,

$$U_\alpha = V_\alpha \cap E$$

Note that since $U_\alpha \subseteq V_\alpha$ for all α, $\{V_\alpha\}$ is an open cover for E in X. Since E is closed, $X \setminus E$ is open. Thus, $\{V_\alpha\} \cup \{X \setminus E\}$ is an open cover of X. Since X is compact, there exists a finite subcover of $\{V_\alpha\} \cup \{X \setminus E\}$.

Let $\{V_1, \ldots, V_n, X \setminus E\}$, be the finite subcover. If the provided finite subcover does not contain $X \setminus E$, we can just add it while maintaining a finite subcover, we assume it does. Then, $V_1, \cdots, V_n$ is an open cover for E in X. Then, we have:

$$\bigcup_{i=1}^n U_i = \bigcup_{i=1}^n (V_i \cap E) = \left(\bigcup_{i=1}^n V_i\right) \cap E = E$$

Thus, $U_1, \ldots, U_n$ is a finite subcover of U_α in E. Therefore, E is compact.

Problem 3. We say that D is dense in X if $\overline{D} = X$. Show that D is dense if and only if D meets every non-empty open set in X.

Solution. We use the fact that $x \in \overline{D}$ (is a point of closure) if and only if every open neighborhood of x intersects D. This is in Royden-Fitzpatrick as the definition of closure, with Ch9.2, Proposition 3 and 4 proving that the closure is closed and the complement of an open.

($\implies$) Assume D is dense in X. Let U be a non-empty open set in X. Then, there exists a point $x \in U$. Since $\overline{D} = X$, $x \in \overline{D}$. Thus, every open neighborhood of x intersects D. Thus, U intersects D.

($\impliedby$) Assume D meets every non-empty open set in X. Let $x \in X$ be arbitrary. Then, D meets every non-empty open set containing x. Thus, x is a point of closure of D. Thus, $x \in \overline{D}$. Since x is arbitrary, we have that $\overline{D} = X$. Thus, D is dense in X.

Problem 4. Show that a compact metric space has a countable dense subset. It is enough for the space to be totally bounded.

Solution. Let (X, ρ) be a compact metric space. We define a countable set of open covers, indexed by $\mathbb{N}$, where $\mathcal{U}_n$ is the open cover:

$$\mathcal{U}_n = \{B_{1/n}(x) : x \in X\}$$

Then, for each n, the cover $\mathcal{U}_n$ has a finite subcover. Let C_n be the centers of the balls in the finite subcover, for each n. The union of these sets over n is countable, as it is a countable union of finite sets. Call this set C. We will show that C is dense in X.

Let U be a non-empty open set in X. Then, there exists a point $x \in U$. Since U is open, there exists an ϵ, such that $B_\epsilon(x) \subseteq U$. Pick an n such that $1/n < \epsilon$. Then, there exists a center $c \in C_n$ such that $x \in B_{1/n}(c)$ as C_n is the set of centers of a finite cover made of $1/n$-balls. Thus, we have:

$$\rho(x, c) < 1/n < \epsilon$$

Thus, $c \in B_\epsilon(x) \subseteq U$. Therefore, U intersects C. Since U is arbitrary, every non-empty open set in X intersects C. Thus, C is dense in X. Therefore, C is a countable dense subset of X.

Problem 5. Show that an equicontinous family of functions on a compact metric space is uniformly equicontinous as in lecture. (Some texts do not define equicontinous at a point. Instead, whether X is compact or not, equicontinuity is what we have called uniformly equicontinuous. Fortunately, there is no distinction for compact spaces.)

Solution.

Problem 6. Show that if X a metric space which is not totally bounded, then there is an unbounded continous function $f : X \to \mathbb{R}$. (Hints provided).

Solution.

Problem 7. Let X be a metric space such that every continous function $f : X \to \mathbb{R}$ attains its minimum value. Show that X is complete. (Hints provided).

Solution.

Problem 8. Show that a metric space is compact if and only if every continous real-valued function on X attains its maximum value. (Note that every real-valued function attains its maximum if and only if every real-valued function attains its minimum. Consider $-f$.)