Problem 1. Prove the pasting lemma: Suppose $X = A \cup B$ is a topological space with A, B closed in X. If $f: X \to Y$ is a map such that the restrictions $f|_A$ and $f|_B$ are continuous, then f is continuous.

Solution. This follows from the definition of continuity. Let $U \subseteq Y$ be open. Then $f^{-1}(U) = (f|_A)^{-1}(U) \cup (f|_B)^{-1}(U)$. Since $f|_A$ and $f|_B$ are continuous, $(f|_A)^{-1}(U)$ and $(f|_B)^{-1}(U)$ are open in A and B, respectively. Because A and B are closed in X, these sets are also open in X. Hence, $f^{-1}(U)$ is open in X, and f is continuous.

Problem 2. In a connected space X, a point $x \in X$ is called a *cut point* if $X \setminus \{x\}$ is disconnected.

- 1. Suppose that $f: X \to Y$ is a homeomorphism of connected spaces. Show that $x \in X$ is a cut point if and only if $f(x) \in Y$ is a cut point.
- 2. Show that none of the spaces (0,1),(0,1],[0,1], and $S^1=\{(x,y)\in\mathbb{R}^2:x^2+y^2=1\}$ are homeomorphic to each other.
- 3. Show that \mathbb{R} is not homeomorphic to \mathbb{R}^n for any $n \geq 2$.
- 4. The bouquet B_n of n circles is the space obtained by gluing n disjoint copies of S^1 at a single point in each circle. Show that B_n and B_m are not homeomorphic for $n \neq m$.

Solution. HELLO

Problem 3. Define \mathbb{RP}^n to be the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ by $(x_1, \ldots, x_{n+1}) \simeq (ax_1, \ldots, ax_{n+1})$ for all nonzero scalars a. Let $[x_1 : \ldots : x_{n+1}] \in \mathbb{RP}^n$ denote the image of $(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}$ under the quotient map. Show that $i : \mathbb{R}^n \to \mathbb{RP}^n$ given by

$$i(x_1,\ldots,x_n)=[1:x_1:\ldots:x_n]$$

is a topological embedding (i.e., a homeomorphism onto its image) and the complement of $i(\mathbb{R}^n)$ is homeomorphic to \mathbb{RP}^{n-1} .

Solution.

Problem 4. Suppose that C is a category, A, B, C are objects of C, and $f \in \text{hom}(A, B)$ and $g \in \text{hom}(B, C)$ are isomorphisms.

- 1. Show that f has a unique inverse in hom(B, A).
- 2. Show that id_A is an isomorphism.
- 3. Show that the inverse of f is an isomorphism.
- 4. Show that $g \circ f$ is an isomorphism.
- 5. Show that Aut(A) and Aut(B) are isomorphic groups.
- 6. Show that if $F: \mathcal{C} \to \mathcal{D}$ is a functor, then $F(f) \in \hom_{\mathcal{D}}(F(A), F(B))$ is an isomorphism.

Solution.

Problem 5. For a space X, let $\pi_0(X)$ be the space of path components of X. Recall that the image of a path-connected space is path-connected. Thus, for a continuous map $f: X \to Y$, there is an induced map $\pi_0(f): \pi_0(X) \to \pi_0(Y)$ taking a path component A to the path component containing f(A). Show that this makes π_0 a functor from the category of topological spaces to the category of sets.

Solution.

Problem 6. (Munkres, *Topology*, 51.2) Given spaces X and Y, let [X, Y] denote the set of homotopy classes of maps of X into Y.

- 1. Let I = [0, 1]. Show that for any X, the set [X, I] has a single element.
- 2. Show that if Y is path connected, then [I, Y] has a single element.

Solution. .

Problem 7. (Munkres, *Topology*, 51.3) A space X is said to be *contractible* if the identity map $i_X: X \to X$ is nullhomotopic.

- 1. Show that I and \mathbb{R} are contractible.
- 2. Show that a contractible space is path connected.
- 3. Show that if Y is contractible, then for any X, the set [X, Y] has a single element.
- 4. Show that if X is contractible and Y is path connected then [X,Y] has a single element.