

Math 113: Functional Analysis

Sair Shaikh

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Defn. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of (X, ρ) . We say that $d > 0$ is a Lebesgue number for the cover if given any d -ball $B_d(x_0)$ with $x_0 \in X$, there exists $a_0 \in A$ such that $B_d(x_0) \subseteq U_{a_0}$.

Ex $X = \mathbb{R}$. $U_1 = (-\infty, 1)$, $U_2 = (0, 2)$, and $U_3 = (1, \infty)$. Here $d = 1/2$ is a LN for $\{U_1, U_2, U_3\}$. This is clear if $x_0 \in (1/2, 3/2)$.

Ex. (Hwk.) Given $x \in (0, 1)$, $\exists \delta_x > 0$ such that:

$$y \in B_{\delta_x}(x) = \{y \in (0, 1) : |y - x| \leq \delta_x\} \\ \implies |1/x - 1/y| < 1$$

Then,

$$(0, 1) = \bigcup_{x \in (0, 1)} B_{\delta_x}(x)$$

has no Lebesgue number.

Lemma (Lebesgue Covering Lemma) Every open cover of a compact metric space has a Lebesgue number.

Proof. Pictures. Apr 9.

Thm. Suppose (X, ρ) is compact, and $F : (X, \rho) \rightarrow (Y, \sigma)$ is continuous. Then F is uniformly continuous.

Proof. Let $\epsilon > 0$. We need to find $\delta > 0$ such that $\forall x, y \in X$:

$$\rho(x, y) < \delta \implies \sigma(F(x), F(y)) < \epsilon$$

Since F is continuous, $\forall z \in X$, $\exists \delta_z > 0$ such that:

$$\rho(x, z) < \delta_z \implies \sigma(F(x), F(z)) < \epsilon/2$$

That is,

$$F(B_{\delta_z}(z)) \subseteq B_{\epsilon/2}(F(z))$$

Let $\delta > 0$ be a Lebesgue number for the cover $\{B_{\delta_z}(z)\}_{z \in X}$.
Now supposed $\rho(x, y) < \delta$. Then $\exists z \in X$ such that:

$$B_\delta(x) \subseteq B_{\delta_z}(z)$$

and

$$\sigma(F(x), F(y)) \leq \sigma(F(x), F(y)) + \sigma(F(z), F(y)) < \epsilon/2 + \epsilon/2 = \epsilon$$

Defn. Let (X, ρ) be a metric space and $C(X)$ the \mathbb{C} -vector space of continuous functions on X . We say $\mathcal{J} \subset C$ is equicontinuous at $x \in X$ if $\forall \epsilon > 0, \exists \delta > 0$ such that:

$$\forall F \in \mathcal{J}, F(B_\delta(x)) \subseteq B_\epsilon(F(x))$$

We say \mathcal{J} is equicontinuous on X if $\forall x \in X$, \mathcal{J} is equicontinuous at x .

Ex. Let $X = [0, 1] \subset \mathbb{R}$. Let $F_n(x) = x^n \forall n \geq 1$. Let:

$$\mathcal{J} = \{F_n : n \in \mathbb{N}\}$$

Let $x_n = \frac{1}{2}^{1/n}$. Then, x_n arrow up to 1. Then,

$$|F_n(x_n) - F_n(1)| = |1/2 - 1| = 1/2$$

Thus, \mathcal{J} is not equicontinuous at 1.

Ex. (Hwk) Show that \mathcal{J} is equicontinuous on $[0, 1)$.

Defn. Let (F_n) be a sequence of (\mathbb{C} -valued) functions on X . Then, (F_n) is uniformly bounded if $\exists M > 0$ such that $\forall n \geq 1, \forall x \in X$:

$$|F_n(x)| < M$$

We say that (F_n) is pointwise bounded if $\forall x \in X, \exists M_x > 0$ such that:

$$|F_n(x)| < M_x$$

Defn. A metric space (top. space) is separable if there is a countable dense subset $D \subset X$.

Ex. Since $\mathbb{Q}^n \subset \mathbb{R}^n$ is dense, $(\mathbb{R}^n, \|\cdot\|_p)$ is separable.

Lemma. (Arzelà-Ascoli) Let (X, ρ) is a separable metric space and that (F_n) is pointwise bounded and equicontinuous in $C(X)$. Then, there is subsequence (F_{n_k}) such that:

$$\lim_{x \rightarrow \infty} F_{n_k}(x)$$

exists $\forall x \in X$.

Yap. Given a sequence (x_n) , we get subsequence by finding $n_k \in \mathbb{N}$ such that $n_{k+1} > n_k$ and $(x_{n_k})_{k=1}^\infty \rightarrow x$ is a sequence.

A subsubsequence is determined by finding $n_{k_1} < n_{k_2} < \dots$ and then we write:

$$(x_{n_{k_j}})_{j=1}^\infty$$

A subsequence is determined by an infinite subset $S_1 = \{n_1 \leq n_2 \leq \dots\} \subset \mathbb{N}$. A subsubsequence is determined by an infinite subset $S_2 \subset S_1$,

$$S_2 = \{n_{k_1} < n_{k_2} < \dots\} \subset S_1$$

Now, we write:

$$\lim_{n \in S_1} x_n = a \text{ instead of } \lim_{k \rightarrow \infty} x_{n_k}$$

Note that $\lim_{n \in S_1} x_n = a$ if:

$$\forall \epsilon > 0 \exists N : n \geq N, n \in S' \implies |x_n - a| < \epsilon$$

Rmk. Suppose $S_1 \subset \mathbb{N}$ determines a subsequence as above. Suppose $S' \subset \mathbb{N}$ is infinite and:

$$\{n \in S' : n \notin S_1\}$$

is finite. The $\lim_{n \in S_1} x_n = a$ then $\lim_{n \in S'} x_n = a$ as well.

Proof of the AA Lemma: Pictures. Apr 9 and 10.

Rmk. If X is compact, then $C(X) = C_b(X)$ is a complete metric space with respect to the uniform norm $\|\cdot\|_\infty$.

Thm. (Arzelà-Ascoli) Let (X, ρ) be a compact metric space and $(F_n) \subset C(X)$ be a sequence of functions that are point-wise bounded and equicontinuous. Then (F_n) has a subsequence converging uniformly to some function $F \in C(X)$. Proof. Pictures.

Lemma. Suppose X is compact and that $\mathcal{J} \subset C(X)$ is equicontinuous on X . Then, \mathcal{J} is uniformly equicontinuous on X , in that for all $\epsilon > 0 \exists \delta > 0$ such that for all $x, y \in X$ and all $F \in \mathcal{J}$,

$$\rho(x, y) < \delta \implies |F(x) - F(y)| < \epsilon$$

Rewriting,

$$F(B_\delta(x)) \subseteq B_\epsilon(F(x))$$

Proof left as homework.

Corr. Let X be a compact metric space. Let $\mathcal{J} \subset C(X)$ be a closed subset such that \mathcal{J} is equicontinuous and pointwise bounded. Then \mathcal{J} is compact and uniformly bounded.

Thm. Suppose X is a compact metric space. Then $\mathcal{J} \subset C(X)$ is compact if and only if \mathcal{J} is closed, uniformly bounded, and equicontinuous on X . Proof. Pictures.

Defn. A topological space is called a Baire space if the countable intersection of dense open sets is dense.

Rmk. If ρ and σ are equivalent metrics on X then (X, ρ) is a Baire space if and only if (X, σ) is a Baire space.

Defn. If $S \subset X$ then the interior of S is:

$$\text{Int}(S) = \bigcup \{U \subset S : U \text{ is open in } X\}$$

Rmk. One can write $\text{Int}_X(S)$ as interior of S in X . Where you take the interior matters.

Lemma. A space X is a Baire space if and only if given a countable $\{F_n\}_{n=1}^\infty$ of closed sets, such that:

$$\bigcup_{n=1}^\infty F_n$$

has non-empty interior, then at least one F_n has (non-empty) interior.

Hwk. Write $O_n = F_n^C = X \setminus F_n$. O_n dense $\iff \text{Int}(F_n) = \emptyset$.

Thm. Baire Category Thm. Every complete metric space is a Baire space.

Rmk. Note that $(0, 1)$ is homeomorphic to \mathbb{R} . Hence, $(0, 1)$ is a Baire space.

Hwk. More generally, every non-empty open subset V of a complete metric space admits an equivalent complete metric (Hwk). Hence V is a Baire space. Thus, if:

$$V = \bigcup_{n=1}^\infty V \cap F_n$$

of each F_n closed in X , thus $V \cap F_n$ is closed in V . Hence, at least one $V \cap F_n$ has non-empty interior in V . But V is open, so it has interior in X .

Thm. Suppose that X is a Baire space and that $(F_n) \subset C(X)$ such that $F_n \rightarrow F$ pointwise. Then,

$$A = \{x \in X : F \text{ is cts at } x\}$$

is dense in X .

Proof. Pictures.

Rmk. If $X = \mathbb{R}$, then we can assume:

$$A = \{x \in X : F \text{ is cts at } x\}$$

is uncountable.

Defn. A normed vector space $(V, \|\cdot\|)$ is called a Banach space if V is complete in the induced metric. Recall that \mathbb{F} is always \mathbb{C} or \mathbb{R} .

Yap. Normed vector spaces are special.

1. $B_r(x) = x + B_r(0)$. The topology is homogenous.
2. $|||v|| - ||w||| \leq ||v - w|| \implies v \rightarrow ||v||$ is continuous.
3. $\overline{B_r(v)} = \{x \in V : ||x - v|| \leq r\}$.
4. $\epsilon B_r(0) = B_{\epsilon r}(0)$.
5. $\overline{\epsilon B_r(0)} = \overline{B_{\epsilon r}(0)}$.

Thm. Supposed that X and Y are normed vector spaces and $T : X \rightarrow Y$ is linear. Then, the following are equivalent:

1. T is continuous.
2. T is continuous at a single point.
3. $\exists \alpha \geq 0$ such that $||T(x)|| \leq \alpha ||x||$ for all $v \in X$.

Proof.

(1) \implies (2) is trivial.

(2) \implies (3). Since T is continuous at x_0 , $\exists \delta > 0$ such that:

$$T(\overline{B_\delta(x_0)}) \subset B_1(T(x_0))$$

$$\text{LHS} = T(B_\delta(x_0)) + T(x_0).$$

$$\text{RHS} = B_1(0) + T(x_0).$$

This implies, $T(\overline{B_\delta(0)}) \subset B_1(0)$.

Now if $z \neq 0$,

$$||T(z)|| = \left| \left| \frac{||z||}{\delta} T\left(\delta \cdot \frac{z}{||z||}\right) \right| \right| \leq \frac{||z||}{\delta}$$

Let $\alpha = \frac{1}{\delta}$.

(3) \implies (1). Erased.

Rmk. Suppose that $||\cdot||_1$ and $||\cdot||_2$ (two different norms) induce equivalent metrics on V . Then, take the id : $(V, ||\cdot||_1) \rightarrow (V, ||\cdot||_2)$ is continuous (same topology). Hence, $\exists c \geq 0$ such that:

$$||x_2|| \leq c \cdot ||x_1||$$

Clearly, that means $c > 0$. By symmetry, there exists a $d > 0$, such that:

$$||x_1|| \leq d \cdot ||x_2||$$

Thus, the metrics and the norms are strongly equivalent.

Defn. If X and Y are normed vector spaces, then $\mathcal{L}(X, Y)$ is the vector space of continuous linear maps $T : X \rightarrow Y$. Define $\|T\| = \sup_{\|x\| \leq 1} \|T(x)\|$. If $X = Y$, then we write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$.

Lem. With $\|T\|$ as above, $\mathcal{L}(X, Y)$ is a normed vector space with:

$$\|T(x)\| \leq \|T\| \|x\| \forall x \in X$$

If $S \in \mathcal{L}(Y, Z)$, then we write ST in place of $S \circ T$ and:

$$\|ST\| \leq \|S\| \|T\|$$

Defn. An algebra over \mathbb{F} is a vector space A over \mathbb{F} with a ring structure, with $\lambda(xy) = (\lambda x)y = x(\lambda y)$ for all $\lambda \in \mathbb{F}$ and $x, y \in A$.

Ex. $M_n(\mathbb{F}), \mathbb{R}[x], \mathbb{C}[x], C(X)$.

Defn. If $\|\cdot\|$ is a norm on an algebra A , then we call $(A, \|\cdot\|)$ a normed algebra if $\forall x, y \in A$:

$$\|xy\| \leq \|x\| \|y\|$$

We call $(A, \|\cdot\|)$ a Banach algebra if $(A, \|\cdot\|)$ is a normed algebra and A is complete with respect to $\|\cdot\|$.

Prop. If X and Y are normed vector spaces, and Y is a Banach space, then $\mathcal{L}(X, Y)$ is a Banach space. If X is a Banach space, then $\mathcal{L}(X)$ is a Banach algebra.

Proof. Suppose that (T_n) is a Cauchy sequence in $\mathcal{L}(X, Y)$. Then, for each $z \in X$, $T_n(z)$ is a Cauchy sequence in Y . Hence $\exists T(z) \in Y$ such that:

$$T_n(z) \rightarrow T(z)$$

It is not hard to see that $T : X \rightarrow Y$ is linear.

Since (T_n) is Cauchy in norm, it's bounded i.e. $\exists M > 0$ such that:

$$\|T_n\| \leq M \forall n \geq 1$$

Now if $\|x\| \leq 1$, then:

$$\|T(x)\| = \lim_{n \rightarrow \infty} \|T_n(x)\| \leq \limsup_n \|T_n\| \|x\| \leq M \|x\|$$

This implies $T \in \mathcal{L}(X, Y)$.

Let $\epsilon > 0$. Let N be such that $m, n \geq N$. Thus,

$$\|T_n - T_m\| \leq \frac{\epsilon}{2}$$

Now if $\|x\| \leq 1$, and if $n \geq N$,

$$\begin{aligned} \|(T - T_n)(x)\| &= \|T(x) - T_n(x)\| \\ &= \lim_{m \rightarrow \infty} \|T_m(x) - T_n(x)\| \\ &\leq \limsup_m \|T_m - T_n\| \cdot \|x\| \\ &\leq \frac{\epsilon}{2} < \epsilon \end{aligned}$$

Yap. Let $Y \subset X$ be a subspace of a normed vector space X . Then we can form the quotient vector space $X/Y = \{x + Y : x \in X\}$ with $q : X \rightarrow X/Y$ the quotient map.

If $x \in X$, then:

$$\inf\{\|x - y\| : y \in Y\}$$

depends only on $q(x)$. We call:

$$\|q(x)\| := \inf\{\|x - y\| : y \in Y\}$$

the quotient norm on X/Y .

Rmk. As Y is a subspace,

$$\begin{aligned} \|q(x)\| &= \inf\{\|x + y\| : y \in Y\} \\ &= \inf\{\|x + \alpha y\| : y \in Y\} \forall \alpha \in \mathbb{F} \setminus \{0\} \end{aligned}$$

Thm. If Y be a subspace of $(X, \|\cdot\|)$. Then,

$$\|q(x)\| = \inf\{\|x - y\| : y \in Y\}$$

is a seminorm on X/Y which is a norm exactly when Y is closed. If X is a Banach space, and Y is closed in X , then X/Y is a Banach space.

Proof. Note that for $\alpha \neq 0$:

$$\|\alpha q(x)\| = \|q(\alpha x)\| = \inf\{\|\alpha x + \alpha y\| : y \in Y\} = |\alpha| \inf\{\|x + y\| : y \in Y\} = |\alpha| \|q(x)\|$$

Next, fix $x_1, x_2 \in X$. Then given $\epsilon > 0$, $\exists y_1, y_2 \in Y$ such that:

$$\begin{aligned} \|q(x_1)\| + \|q(x_2)\| + \epsilon &\geq \|x_1 - y_1\| + \|x_2 - y_2\| \\ &\geq \|x_1 - y_1 + x_2 - y_2\| \\ &\geq \|q(x_1 + x_2)\| \\ &= \|q(x_1)\| + \|q(x_2)\| \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\|q(x_1) + q(x_2)\| \leq \|q(x_1)\| + \|q(x_2)\|$. Thus, it is a semi-norm. Sketch for closed. If quotient norm is 0, there exists a sequence y_n converging to x . If

Y closed, $x \in Y$ and $g(x) = 0$. Conversely, if quotient norm is a norm and $y_n \rightarrow x$. Then, $\|q(x)\| = 0$. Then, $q(x) = 0$ and $x \in X$.

Now suppose X is a Banach space and Y is closed in X . Let $(q(z_0))$ be Cauchy in X/Y . Then we can pass to a subsequence and assume:

$$\|q(z_{n+1}) - q(z_0)\| \leq 1/2^n$$

Let $x_1 = z_1$. Since:

$$\|q(z_2) - q(x_1)\| < 1/2$$

Thus,

$$\inf\{\|z_2 - x_1 - y\| : y \in Y\} < 1/2$$

Hence, we can find x_2 such that $q(x_1) = q(z_2)$ and

$$\|x_2 - x_1\| < 1/2$$

Continuing, we get (x_n) such that $q(x_n) = q(z_n)$ and:

$$\|x_{n+1} - x_n\| < 1/2^n$$

(Hwk) (x_n) is Cauchy, thus, $x_n \rightarrow x$ in X .

But $\|q(z)\| \leq \|z\|$. Hence, q is continuous. Thus,

$$q(z_n) = q(x_n) \rightarrow q(x)$$

Rmk. Note that $q : X \rightarrow X/Y$ is a bounded linear map of norm at most 1.

Thm. Suppose that Y is a closed subspace of $(X, \|\cdot\|)$. Then X is a Banach space if and only if both Y and X/Y are Banach spaces.

Proof. Pictures. 4/16

Thm. Every finite-dimensional subspace Y of a normed vector space X is a Banach space and hence closed in X . If $\dim(Y) = n$, then every linear isomorphism $\Phi : \mathbb{F}^n \rightarrow Y$ is a homeomorphism.

Proof. Pictures. 4/16.

Corr. If Y is finite-dimensional vector space, then all norms $\|\cdot\|$ are strongly equivalent.

Proof. $\text{id} : (Y, \|\cdot\|_1) \rightarrow (Y, \|\cdot\|_2)$ is a homeomorphism and continuous linear maps are bounded.

Recall. Recall the topological definition of continuous functions and open maps. If $F : X \rightarrow Y$ is a bijection, then $F^{-1} : Y \rightarrow X$ is continuous if and only if F is open.

Thm. (The Big Three) (Open Mapping Theorem) Suppose that X and Y are Banach spaces and that $T \in \mathcal{L}(X, Y)$ is a surjection. Then T is an open map.

Lem. It will suffice to find $r > 0$ such that:

$$B_r(0) \subseteq T(B_1(0))$$

Proof. By homogeneity, $T(B_\delta(0))$ is a neighborhood of 0_Y for all $\delta > 0$. By linearity, $T(B_\delta(x))$ is a neighborhood of $T(x)$ for all $x \in X$ and $\delta > 0$. Thus, if V is open in X and $x \in V$, then $\exists \delta > 0$ such that:

$$B_\delta(x) \subseteq V$$

Then, $T(B_\delta(x))$ is a neighborhood of $T(x)$ in $T(Y)$. Thus, T is an open map. Thus, $T(V)$ is open.

Lem. It will suffice to find $r > 0$ such that:

$$B_r(0) \subseteq \overline{T(B_1(0))}$$

Proof. Assume $r > 0$ is such that the property holds. Let $y \in B_r(0)$. Then,

$$\exists y_1 \in T(B_1(0)) : \|y - y_1\| < r/2$$

Then, $y - y_1 \in B_{r/2}(0)$.

Hence, $\exists y_2 \in 1/2T(B_1(0)) = T(B_{1/2}(0))$ such that:

$$\|y - y_1 - y_2\| < r/2^2$$

Continue to get a sequence (y_n) such that:

$$y_n \in 2^{-n+1}T(B_1(0)) = T(B_{2^{-n+1}}(0))$$

and:

$$\|y - \sum_{i=1}^n y_i\| < 2^{-n}r$$

By construction, $\exists x_n \in X$ such that:

$$T(x_n) = y_n \quad \|x_n\| < 2^{-n+1}$$

Since X is a Banach space, $x = \sum_{n=1}^{\infty} x_n$ converges and since T is continuous (hence bounded), and also:

$$\|x\| < \sum_{n=1}^{\infty} 2^{-n+1} = 2$$

and

$$T(x) = y$$

Since y was arbitrary, we have shown that:

$$B_r(0) \subset T(B_2(0))$$

This implies:

$$B_{r/2}(0) \subset T(B_1(0))$$

This suffices by lemma 1.

Lem. 3 It suffices to see that:

$$\overline{T(B_n(0))}$$

has interior for some $n \geq 1$.

Proof. By homogeneity, we can assume that $\overline{T(B_1(0))}$ has interior.

Thus, $\exists \epsilon > 0$ such that:

$$B_\epsilon(y) \subset \overline{T(B_1(0))}$$

Let $z \in B_\epsilon(0)$. Write

$$z = z/2 + y - (y - z/2) \in B_{\epsilon/2}(y) - B_{\epsilon/2}(y) \subseteq \overline{T(B_{1/2}(0))} - \overline{T(B_{1/2}(0))}$$

Rest in pictures.

Proof. Of the Open Mapping Theorem. Pictures. 4/18.

Ex. Let $1 \leq p < \infty$. Then, let:

$$l_0^p = \text{span}\{e_n : n \geq 1\} = \{x \in l^p : x(n) = 0 \text{ for all but finitely many } n\}$$

Notte l_0^p is dense in l^p .

Define: $T_0 : l_0^p \rightarrow l_0^p$ by:

$$T_0(e_n) = 1/ne_n$$

You can check that $\|T\| = 1$. Also T_0 is a bijection.

$$T_0^{-1}(e_n) = ne_n$$

but $T_0^{-1} \notin \mathcal{L}(l_0^p)$

Thm. Suppose that X and Y are Banach spaces and $T \in \mathcal{L}(X, Y)$ is a bijection. Then $T^{-1} \in \mathcal{L}(Y, X)$.

Proof. T^{-1} is linear by general nonsense and T is open by the Open Mapping Theorem. Hence T^{-1} is continuous hence bounded.

Yap. If X and Y are Banach spaces, then I can give $X \times Y$ a norm by:

$$\|(x, y)\| = \max\{\|x\|, \|y\|\}$$

This makes $X \times Y$ a Banach space.

[Missed 3 classes due to Covid. Did Open Mapping Theorem, Closed Graph Theorem, Principle of Uniform Boundedness, defined norm-dual, and did Hahn-Banach with 3 important corollaries.]

Thm. The map:

$$\Phi : l^q \rightarrow (l^p)^\vee$$

given by $\Phi(y) = \phi_y$ is an isometric isomorphism. That is, Φ is a isomorphism such that:

$$\|\Phi(y)\| = \|y\|$$

Rmk. If $z \in \mathbb{C}$, then:

$$\text{sgn}(z) = \frac{z}{|z|} \text{ if } z \neq 0 \text{ else } 0$$

Then, $z = |z| \text{sgn}(z)$ and $|z| = \text{sgn}(z)z$.

Lemma. If $y \in l^q$, then $\|\phi_y\| = \|y\|_q$. Proof. Pictures. 4/28.

Lemma. If $\phi \in (l^p)^\vee$, then $\exists y \in l^q$ such that:

$$\phi = \phi_y$$

Proof. Pictures. 4/28. Let $y_n = \phi(e_n)$. Then, if $x \in l^p$, then:

$$x = \sum_{n=1}^{\infty} x_n e_n$$

Note,

$$\|x = \sum_{n=1}^N x_n e_n\|_p^p = \sum_{n=N+1}^{\infty} |x_n|^p$$

converges in the Banach space l^p .

Hence,

$$\phi(x) = \sum_{n=1}^{\infty} x_n y_n$$

Then, if we can show $y \in l^q$, then $\phi = \phi_y$, and we are done.

Let $y^N \in l^q$ be given by:

$$y^N(n) = \begin{cases} y_n & \text{if } 1 \leq n \leq N \\ 0 & \text{otherwise} \end{cases}$$

Then, $\phi_{y^N} \in (l^p)^\vee$ and $\|\phi_{y^N}\| \leq \|y^N\|_q$.

If $x \in l^p$, then,

$$\phi_{y^N}(x) = \sum_{n=1}^N x_n y_n \rightarrow \sum_{n=1}^{\infty} x_n y_n = \phi(x)$$

Hence, by the Principle of Uniform Boundedness, $\exists M > 0$ such that:

$$\|\phi_{y^N}\| \leq M \quad \forall N \geq 1$$

Thus, if $q < \infty$, then,

$$\sum_{n=1}^N |y_n|^q \leq M^q < \infty$$

Rmk. If X is a normed vector space, then we know that X^* is a Banach space. Then, we can form the bidual $(X^*)^* = X^{**}$. We get a natural map ι

$$\iota : X \rightarrow X^{**}$$

$\iota(x)(\phi) = \phi(x)$ is evaluation at x .

Now,

$$\|\iota(x)\| = \sup_{\|\phi\| \leq 1} |\iota(x)(\phi)| = \sup_{\|\phi\| \leq 1} |\phi(x)| (*)$$

Then, $(*) \leq \|x\|$ since $\|\phi\| \leq 1$.

Then, by Hahn-Banach, $\exists \phi \in X^*$ such that $\|\phi\| = 1$ and $\phi(x) = \|x\|$. Thus,

$$\|\iota(x)\| = \|x\|$$

We can identify X at $\iota(X)$ in X^{**} .

If X is a Banach space, then, $\iota(x)$ is complete, hence closed in X^{**} .

Otherwise, $\overline{\iota(X)}$ is a Banach space containing X as a dense subspace. Then, $\overline{\iota(X)}$ is the completion of X as a Banach space.

Defn. If X is a Banach space then X is reflexive if $\iota(X) = X^{**}$ is onto.

Rmk. If X is reflexive, then X^{**} are isometrically isomorphic. The converse can fail. It can be that they are isometrically isomorphic but ι is not the map. [Robert James 1951].

Ex. Let $1 < p < \infty$. Let $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$(l^p)^* = \{\phi_y^p : y \in l^q\}$$

and also:

$$(l^q)^* = \{\phi_x^q : x \in l^p\}$$

If $x \in l^p$, then, $\iota(x) \in (l^p)^{**}$, and:

$$\begin{aligned} \iota(x)(\phi_y^p) &= \phi_y^p(x) \\ &= \phi_x^q(y) \end{aligned}$$

Thus, $\iota : l^p \rightarrow (l^p)^{**}$ is surjective and l^p is reflexive for $1 < p < \infty$.

Prop. l^p is reflexive for $1 < p < \infty$.

Rmk. Let X be a normed vector space and $D = \{d_n\}_{n=1}^\infty \subset X$ be a countable subset of X . Then the rational space of D , $\text{span}_\mathbb{Q}(D)$ is the span of D viewing X as a rational vector space. Thus,

$$\text{span}_\mathbb{Q}(D) = \bigcup_{n=1}^\infty \left\{ \sum_{k=1}^n r_k d_k : r_k \in \mathbb{Q} d_k \in D \right\}$$

Then, $\text{span}_\mathbb{Q}(D)$ is countable. Then if $\text{span}(D)$ is dense, then since $\text{span}_\mathbb{Q}(D)$ is dense in $\text{span}(D)$. Thus, X is separable.

Same works over \mathbb{C} replacing \mathbb{Q} by $\mathbb{Q} + i\mathbb{Q}$.

Corr. l^p is separable for $1 \leq p < \infty$. Let $D = \{e_n\}_{n=1}^\infty$. But l^∞ is not separable.

Hwk. If X^* is separable, then X is.

Thm. If $1 \leq p \leq \infty$, and if q is the conjugate exponent, then:

$$\Phi : l^q \rightarrow (l^p)^*$$

is an isometric isomorphism when $\Phi(y) = \phi_y^p$ and

$$\phi_y^p = \sum_{n=1}^{\infty} x_n y_n$$

Rmk. If $(l^p)^* = \{\phi_y^p : y \in l^q\}$ and if $q \leq \infty$, then:

$$(l^q)^* = \{\phi_x^q : x \in l^p\}$$

Defn. Recall: If X is a Banach space, then X is reflexive if $\iota(X) = X^{**}$ is surjective.

Prop. If $1 < p < \infty$, then l^p is reflexive.

Proof. Consider $\iota : l^p \rightarrow (l^p)^{**}$. If $\phi_y^p \in (l^p)^*$, then:

$$\iota(x)(\phi_y^p) = \phi_y^p(x) = \phi_x^q(y)$$

Let $\psi \in (l^p)^{**}$. Then,

$$\psi(\phi_y^p) = \psi(\Phi(y))$$

Now $y \rightarrow \psi(\Phi(y))$ is a in $(l^q)^*$. Hence, $\exists x \in l^p$ such that:

$$\psi(\Phi(y)) = \phi_x^q(y)$$

Thus, $\iota(x) = \psi$. Thus, ι is onto.

Rmk. If X is a normed vector space over \mathbb{F} and $D \subset X$ is countable and $\overline{\text{span}(D)} = X$, then X is separable.

Corr. If $1 \leq p < \infty$, then l^p is separable.

Proof. $D = \{e_n : n \geq 1\}$.

Ex. l^∞ is not separable.

Proof. Let $A \subseteq \mathbb{N}$. Let:

$$x_A(n) = \mathbf{1}_{n \in A}$$

If $A \neq B$, then $\|x_A - x_B\|_\infty = 1$. But $\{x_A : A \subseteq \mathbb{N}\}$ is uncountable. You cannot find a countable dense subset anymore (cant fit all of them into countably many $1/4$ balls).

Hwk. $(l^\infty)^*$ is not separable. Thus, l^1 is not reflexive.

Defn. Suppose $T \in \mathcal{L}(X, Y)$. Then we define:

$$T^* : Y^* \rightarrow X^*$$

by:

$$T^*(\phi)(x) = \phi(T(x))$$

Prop. If X and Y are normed vector spaces, and $T \in \mathcal{L}(X, Y)$, then $T^* \in \mathcal{L}(Y^*, X^*)$ and $\|T^*\| = \|T\|$.

Proof. First, check that T^* is linear (easy).

Next, show the norm equality. Look at:

$$\begin{aligned} \|T^*(\phi)\| &= \sup_{\|x\| \leq 1} |T^*(\phi)(x)| \\ &= \sup_{\|x\| \leq 1} |\phi(T(x))| \\ &\leq \|\phi\| \|T\| \\ &\leq \|\phi\| \|T\| \end{aligned}$$

Thus,

$$\|T^*\| \leq \|T\|$$

Fix $\epsilon > 0$. Then, $\exists x_0 \in X$ such that $\|x_0\| = 1$ and:

$$\|T(x_0)\| > \|T\| - \epsilon$$

But $\exists \phi \in Y^*$ such that $\|\phi\| = 1$ and:

$$\phi(T(x_0)) = \|T(x_0)\|$$

Then,

$$\begin{aligned} \|T^*\| &\geq \|T^*(\phi)\| \\ &\geq |T^*(\phi)(x_0)| \\ &= |\phi(T(x_0))| \\ &= \|T(x_0)\| > \|T\| - \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have:

$$\|T^*\| \geq \|T\|$$

Thus, $\|T^*\| = \|T\|$.

Thm. Let X and Y be Banach spaces and suppose that:

$$T : X \rightarrow Y \quad S : Y^* \rightarrow X^*$$

are functions (not linear or bounded) such that $\forall \phi \in Y^*$ and $x \in X$:

$$S(\phi)(x) = \phi(T(x))$$

Then, $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y^*, X^*)$, with $S = T^*$.

Proof. Supposed that $x, y \in X$ and $\alpha \in \mathbb{F}$. Then, if $\phi \in Y^*$, we have:

$$\begin{aligned} \phi(T(\alpha x + y)) &= S(\phi)(\alpha x + y) \\ &= \alpha(S(\phi)(x)) + S(\phi)(y) \\ &= \alpha\phi(T(x)) + \phi(T(y)) \\ &= \phi(\alpha T(x) + T(y)) \end{aligned}$$

Since $\phi \in Y^*$ is arbitrary, we have $T(\alpha x + y) = \alpha T(x) + T(y)$. (something about separate points).

To see that T is bounded, use CGT. Suppose that $x_n \rightarrow x$ in X and $T(x_n) \rightarrow y$ in Y . But $\forall \phi \in Y^*$, we have:

$$\begin{aligned} \phi(y) &= \lim \phi(T(x_n)) \\ &= \lim S(\phi)(x_n) \\ &= S(\phi)(x) \\ &= \phi(T(x)) \end{aligned}$$

Thus, $y = T(x)$. Thus, T is bounded.

But $T^*(\phi) = S(\phi)$.

Yap. Let (X, τ) be a topological space. Then $\beta \subset \tau$ is a basis for τ if given any $U \in \tau$ and $x \in U$, then $\exists V \in \beta$ such that $x \in V \subset U$.

We say S is a neighborhood of $x \in X$ if $\exists U \in \tau$ such that $x \in U \subset S$.

Wrote $\mathcal{N}(x)$ for the set of all neighborhoods of x . We say that $\alpha \subseteq \mathcal{N}(x)$ is a neighborhood basis at x if $U \in \mathcal{N}(x)$, there exists $V \in \alpha$ such that:

$$x \in V \subseteq U$$

Ex. (1) In a metric space, the collection of all open balls is a basis for the metric topology.
 (2) In \mathbb{R}^n , every point has a neighborhood basis consisting of compact sets. Such spaces are called locally compact.

Lemma. Low Hanging Fruit: Let (X, τ) be a topological space and $\alpha(x)$ has a neighborhood basis at $x \in X$ consisting of open sets. Then:

$$\beta = \bigcup_{x \in X} \alpha(x)$$

is a basis for τ .

Lemma. Low Hanging Fruit: $\beta \in \tau$ is a bssis for τ if and only if given $U \in \tau$,

$$U = \bigcup_{V \in \beta \subset U} V$$

Defn. Let (X, τ) be a topological space. Then,

1. (X, τ) is seperable if it ihas a countable dense subset.
2. (X, τ) is 2nd countable if it has a countable basis.
3. (X, τ) 1st countable if every point has a countable neighborhood basis.

Rmk. 1. Every 2nd countable space is seperable. The converse holds in metric spaces.
Hwk 8.

2. Metric spaces are 1st countable.

- Ex. 1. Let X be a set. Then $\tau = \mathcal{P}(X)$ is the discrete topology. This is the metric topology coming from the the discrete metric.
2. $\tau = \{X, \emptyset\}$.

Lemma. Let $S \subset \mathcal{P}(X)$. Then, there is a smallest topology $\tau(S)$ that contains S .

Proof. Let $\tau(S) = \bigcap \{\tau' : \tau' \text{ is a top and } S \in \tau'\}$.

Prop. Let $\beta \subset \mathcal{P}(X)$ be a coer of X . Then β is a basis for $\tau(\beta)$ if and only if $U, V \in \beta$ and $x \in U \cap V$, then $\exists W \in \beta$ such that $x \in W \subset U \cap V$.

Proof. Hwk.

[One lecture on omitted. Defined weak topology.]

Defn. An ordered set (X, \leq) is directed if given $x, y \in X$, $\exists z \in X$ such that $x \leq z$ and $y \leq z$.

Ex. – $X = \mathbb{N}$.

– Let (Y, τ) be a topological space and $y_0 \in Y$. Then $X = \mathcal{N}(y_0)$ be the collection of neighborhoods of y_0 induced by reverse inclusion. Then, $U \leq V \iff V \subset U$. Then $\mathcal{N}(y_0)$ is directed. If $U, V \in \mathcal{N}(y_0)$, then so is $U \cap V$.

– We can do the same thing for open neighborhoods.

Defn. – A net x is a set X is a function $x : \Lambda \rightarrow X$ where Λ is a directed set. As with sequences, we usually write x_λ for $x(\lambda)$ and $(x_\lambda)_{\lambda \in \Lambda}$ in place of $x : \Lambda \rightarrow X$, or just x .

– If X is a topological space, then we say that a net $(x_\lambda)_{\lambda \in \Lambda}$ converges to $x_0 \in X$ if (x_λ) is eventually in every neighborhood of x_0 . That is, if $U \in \mathcal{N}(x_0)$, then $\exists \lambda_0 \in \Lambda$ such that:

$$\lambda \geq \lambda_0 \implies x_\lambda \in U$$

– We say that $x_0 \in X$ is an accumulation point of a net x_λ if (x_λ) is frequently in every neighborhood of x_0 . That is if U is a neighborhood of x_0 and $\lambda_0 \in \Lambda$, then $\exists \lambda \geq \lambda_0$ such that $x_\lambda \in U$.

Prop. Suppose X is a topological space and $E \subset X$. Then, $x_0 \in \overline{E}$ if and only if there exists a net $(x_\lambda) \subset E$ with $x_\lambda \rightarrow x_0$.

Proof. Suppose we have $x_\lambda \subset E$ with $x_\lambda \rightarrow x_0$. If $x_0 \notin \overline{E}$, then $\exists U \in \mathcal{O}(x_0)$ such that $U \cap E = \emptyset$. But then we would eventually have (x_λ) in U . Contradiction.

Now suppose that $x_0 \in \overline{E}$. Let $\Lambda = \mathcal{O}(x_0)$.

If $W \in \Lambda$, then $W \cap E \neq \emptyset$. Then, we let $x_W \in W \cap E$. Then, $(x_W)_{W \in \Lambda}$ converges to x_0 .

If $U_0 \in \mathcal{O}(x_0)$, then $U \geq U_0$, $x_U \in U \subset U_0$.

Ex. $S = \{\sqrt{n}e_n \in l^2 : n \geq 1\}$, then $0 \in \overline{S}^w$ (weak topology).

Defn. A subset C of a vector space V over \mathcal{F} is convex if $x, y \in C$ and $t \in [0, 1]$, then $tx + (1 - t)y \in C$.

Ex. In a normed vector space X , every open ball $B_r(x_0)$ is convex if $x_0 \in X$ and $r > 0$. To see this, supposed $x, y \in B_r(x_0)$, and $t \in [0, 1]$. Then,

$$\begin{aligned} \|tx + (1 - t)y - x_0\| &\leq \|t(x - x_0)\| + \|(1 - t)(y - x_0)\| \\ &< tr + (1 - t)r \\ &= r \end{aligned}$$

Lemma. Let C be an open convex neighborhood of 0 in a normed vector space X . Then,

$$m(x) = \inf\{s > 0, s^{-1}x \in C\}$$

Then m is a Minkowski functional on X such that:

$$C = \{x \in X : m(x) < 1\}$$

Proof. If $x \in X$, then $\frac{1}{n}x \rightarrow 0$. So $\frac{1}{n}x$ is eventually in C . So $m(x) < \infty$ for all $x \in X$.

Note that if $t \geq 0$, then $m(tx) = tm(x)$. (This is one of the axioms for a Minkowski functional).

Also let $s^{-1}x, t^{-1}y \in C$. Then:

$$(s+t)^{-1}(x+y) = \frac{s}{s+t}s^{-1}x + \frac{t}{s+t}t^{-1}y \in C$$

Thus, $m(x+y) \leq s+t$.

Since $s^{-1}x, t^{-1}y \in C$ is arbitrary, we have:

$$m(x+y) \leq m(x) + m(y)$$

This is the second axiom for a Minkowski functional.

Thus, $m : X \rightarrow \mathbb{R}$ is a Minkowski functional. Now if $x \in C$, then $(1 + 1/n)x \rightarrow x$.

Thus, for some n ,

$$m(x) = \frac{1}{1 + 1/n} < 1$$

If $m(x) < 1$, then $\exists s < 1$ such that $s^{-1}x \in C$.

But $0 \in C$ and $x = (1-s)0 + s(s^{-1}x) \in C$.

Thm. (Hahn-Banach Separation Theorem). Let A and B be disjoint non-empty convex subsets of some normed vector space X . If A is also open, then there exists $\phi \in X^*$ and $t \in \mathbb{R}$ such that:

$$\Re(\phi(x)) < t \leq \Re(\phi(y))$$

for all $x \in A$ and $y \in B$.

Proof. We start with $\mathbb{F} = \mathbb{R}$. Then, we can use the Basic Extension Lemma. Fix $x_0 \in A$ and $y_0 \in B$. Let $z_0 = x_0 - y_0$. Let:

$$C = A - B + z_0$$

We can check that C is convex.

Since $C = \bigcup_{y \in B} A - y + z_0$. Thus, C is union of open sets, hence open neighborhood of 0.

Let $m : X \rightarrow \mathbb{R}$ be the corresponding Minkowski functional.
 We claim $z_0 \notin C$. If $x - y + z_0 = z_0$, then $x = y \in A \cap B = \emptyset$.
 Thus, $m(z_0) \geq 1$.
 Define:

$$\phi_0 : \mathbb{R}z_0 \rightarrow \mathbb{R}$$

by $\phi_0(\alpha z_0) = \alpha$.
 Since $m(x) \geq 0$, then for all x if $\alpha \leq 0$, then:

$$\phi_0(\alpha z_0) \leq m(\alpha z_0)$$

If $\alpha \geq 0$, then:

$$\phi_0(\alpha z_0) = \alpha \leq \alpha m(z_0) = m(\alpha z_0)$$

Thus, $\phi_0(z) \leq m(z)$ for all $z \in \mathbb{R}z_0$.

Thus, the Basic Extension Lemma gives us $\phi : X \rightarrow \mathbb{R}$ such that, $\phi(x) \leq m(x)$.

Thus, $\phi(x) < 1$ if $x \in C$.

Also, $-\phi(x) = \phi(-x) < 1$ if $x \in -C$.

Thus, $\forall \epsilon > 0$,

$$|\phi(x)| < \epsilon \text{ if } x \in \epsilon C \cap -\epsilon C$$

Since $\epsilon C \cap -\epsilon C$ is a neighborhood of 0 for all $\epsilon > 0$, this means ϕ is continuous at 0.

Thus, $\phi \in X^*$. If $x \in A$ and $y \in B$, then:

$$x - y + z_0 \in C$$

Then,

$$m(x - y + z_0) < 1$$

Thus,

$$\phi(x - y + z_0) < 1$$

Since $\phi(z_0) = \phi_0(z_0) = 1$, we have:

$$\phi(x) < \phi(y)$$

Since ϕ is linear and since A and B are convex, $\phi(A)$ and $\phi(B)$ are intervals. Since A is open and ϕ is linear, $\phi(A)$ must be open.

Then, we can let t be the right-hand endpoint of $\phi(A)$.

Now if $\mathbb{F} = \mathbb{C}$, then we can treat X as a real space and produce a real linear functional $\psi : X \rightarrow \mathbb{R}$ such that:

$$\psi(A) < t \leq \psi(B)$$

Now let $\phi(x) = \psi(x) - i\psi(ix)$. ϕ is continuous since ψ is and continuity implies boundedness.

[Skipped a class of notes. Took pictures. 5/9]

Ex. The sequence $(\sqrt{n}e_n) \subset l^2$ has no weakly convergent sequence.

Ex. For each $\alpha \in l^\infty$, let:

$$D_\alpha = \{z \in \mathbb{C} : |z| \leq \|\alpha\|_\infty\}$$

Then if $\alpha = (\alpha_n)$ then $\alpha_n \in D_\alpha$ for all $n \geq 1$.

We now know that:

$$Z = \prod_{\alpha \in l^\infty} D_\alpha$$

is compact in the product topology. Let $(z_n) \subset Z$ be the sequence given by:

$$z_n(\alpha) = \alpha_n$$

But we claim (z_n) has no convergent subsequence.

Proof. Suppose (z_{n_k}) converges to $z_0 \in Z$. Then it converges pointwise. Thus for all $\alpha \in l^\infty$, we have:

$$z_{n_k}(\alpha) \rightarrow z_0(\alpha)$$

Let $\alpha_0 \in l^\infty$ be given by:

$$\alpha_0(n) = \begin{cases} (-1)^k & \text{if } n = n_k \\ 0 & \text{otherwise} \end{cases}$$

But now:

$$z_{n_k}(\alpha_0) = (-1)^k$$

But

$$\lim_{k \rightarrow \infty} z_{n_k}(\alpha_0)$$

does not exist.

Defn. Let $x : \Lambda \rightarrow X$ be a net in a set x . Then a net $y : M \rightarrow X$ is a subnet of x if $\exists h : M \rightarrow \Lambda$ such that $y = x \circ h$ and for all $\lambda_0 \in \Lambda$, there exists $m(\lambda_0)$ such that:

$$m \geq m(\lambda_0) \implies h(m) \geq \lambda_0$$

Rmk. If $h : M \rightarrow \Lambda$ is monotone in that $m_1 \geq m_2$ then $h(m_1) \geq h(m_2)$, then given $\lambda_0 \in \Lambda$ it suffices to find $m(\lambda_0)$ such that $h(m(\lambda_0)) \geq \lambda_0$.

Yap. We usually write:

$$(y_m)_{m \in M} \text{ as } (x_{h(m)})_{m \in M} \text{ or } (x_{\lambda_m})_{m \in M}$$

Example: If you have a subsequence $(x_n)_{n \in \mathbb{N}}$ and $h : \mathbb{N} \rightarrow \mathbb{N}$ strictly increasing, we can write:

$$x_k = h(k) \quad n_{k+1} > n_k$$

Lemma. Let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in a topological space. Then x_0 is an accumulation point of (x_λ) if and only if there is a subnet $(x_{h(m)})_{m \in M}$ converging to x_0 .

Proof. Suppose that $(x_{h(m)})_{m \in M}$ converges to x_0 .

Let $U \in \mathcal{O}(x_0)$ and $\lambda_0 \in \Lambda$.

Since the subnet converges to x_0 , there is a $m_0 \in M$ such that $m \geq m_0$ implies $x_{h(m)} \in U$.

But there exists $m_1 = m(\lambda_0)$ such that $m \geq m_1$ implies $h(m) \geq \lambda_0$.

Let m_2 be such that $m_2 \geq m_1$ and $M_2 \geq m_0$. Then,

$$h(m_2) \geq \lambda_0 \text{ and } x_{h(m_2)} \in U$$

Thus, x_0 is an accumulation point of (x_λ) .

Now suppose that x_0 is an accumulation point of (x_λ) . Let:

$$\mathcal{M} \{ (U, \lambda) \in \mathcal{O}(x_0) \times \Lambda : x_\lambda \in U \}$$

and $(U_1, \lambda_1) \leq (U_2, \lambda_2)$ if $U_2 \subset U_1$ and $\lambda_1 \geq \lambda_2$.

Claim: \mathcal{M} is a directed set. To see this, let $(U_1, \lambda_1), (U_2, \lambda_2) \in \mathcal{M}$. Let $\lambda_3 \geq \lambda_1$ and $\lambda_3 \geq \lambda_2$. Let $\lambda_4 \geq \lambda_3$ such that $x_{\lambda_4} \in U_1 \cap U_2$. Then, $(U_1 \cap U_2, \lambda_4) \geq (U_k, \lambda_k)$ for $k = 1, 2$.

Now define $h : M \rightarrow \Lambda$ by $h(U, \lambda) = \lambda$. Clearly,

$$(x_{(U, \lambda)})_{(U, \lambda) \in \mathcal{M}} \rightarrow x_0$$

Corr. A topological space is compact if and only if every net has a convergent subnet.

Not. If X is a normed vector space, and Y a collection of linear functional on X . Then we write $\sigma(X, Y)$ for the initial topology on X induced by $\mathcal{F} = Y$ (smallest topology that makes all of these functional continuous, likely want separation so you get Hausdorff etc).

Ex. $\sigma(X, X^*)$ is the weak topology on X .

$\sigma(X^*, X^{**})$ is the weak topology on X^* .

Yap. But now we want to consider:

$$\sigma(X^*, X) := \sigma(X^*, \iota(X))$$

Defn. If X is a normed vector space, then:

$$\sigma(X^*, X)$$

is called the weak-* topology on X^* .

Lemma. Let X be a normed vector space. Then a sub-basis for the weak-* topology on X^* consists of all sets of the form:

$$\mathcal{U}(x, \phi_0, \epsilon) = \{\phi \in X^* : |\phi(x) - \phi_0(x)| \leq \epsilon\}$$

where $x \in X, \phi_0 \in X^*$ and $\epsilon > 0$.

Then we get a neighborhood basis for $\phi_0 \in X^*$ by taking:

$$U(\{x_1, \dots, x_n\}, \phi_0, \epsilon) = \{\phi \in X^* : |\phi(x_k) - \phi_0(x_k)| < \epsilon \forall k\}$$

Lemma. A net (ϕ_λ) in X^* converges in weak-* to ϕ_0 if $\phi_\lambda(x) \rightarrow \phi_0(x)$ for all $x \in X$.

Thm. (Alaoglu's Theorem.) If X is a normed vector space and:

$$B^* = \{\phi \in X^* : \|\phi\| \leq 1\}$$

is the closed unit ball, then B^* is compact in the weak-* topology.

Proof. Let $D_r = \{z \in \mathcal{F} : |z| \leq r\}$. Let:

$$Z = \prod_{x \in X} D_{\|x\|}$$

Then we know that Z is compact in the product topology by Tychonoff's theorem.

Define:

$$j : B^* \rightarrow Z$$

by $j(\phi)(x) = \phi(x)$. Note that j is injective.

If $\phi_\lambda \rightarrow \phi_0$ in the weak-* topology, then $\phi_\lambda(x) \rightarrow \phi_0(x)$ for all $x \in X$. Thus, $j(\phi_\lambda)(x) \rightarrow j(\phi_0)(x)$ for all $x \in X$. Thus, $j(\phi_\lambda) \rightarrow j(\phi_0)$ in Z .

Now, claim that j has closed range in Z . Suppose $j(\phi_\lambda) \rightarrow z_0$ in Z . Then,

$$j(\phi_\lambda)(x) \rightarrow z_0(x) \quad \forall x \in X$$

That is:

$$\phi_\lambda(x) \rightarrow z_0(x)$$

Then,

$$\begin{aligned} z_0(x+y) &= \lim_{\lambda} \phi_\lambda(x+y) \\ &= \lim_{\lambda} \phi_\lambda(x) + \phi_\lambda(y) \\ &= z_0(x) + z_0(y) \end{aligned}$$

We also clearly have:

$$|z_0(x)| \leq \|x\|$$

Thus, $z_0 \in j(B^*)$. Thus, $j(B^*)$ is compact.

If $j(\phi_\lambda) \rightarrow j(\phi)$ in Z , then:

$$j(\phi_\lambda)(x) \rightarrow j(\phi)(x) \quad \forall x \in X$$

Then, $\phi_\lambda(x) \rightarrow \phi(x)$ for all $x \in X$. Thus, $\phi_\lambda \rightarrow \phi$ in the weak-* topology (thus inverse is continuous). Thus, j is a homeomorphism onto $j(B^*)$ and then B^* is compact.

Defn. We defined the weak-* topology on X^* , $\sigma(X^*, X)$, to be the initial topology on X^* induced by \mathcal{J} a family of linear functional on X .

Defn. A subbases for $\sigma(X^*, X)$ is given by the sets:

$$\mathcal{U}(x, \phi_0, \epsilon) = \{\phi \in X^* : |\phi(x) - \phi_0(x)| < \epsilon\}$$

for all $x \in X$, $\phi_0 \in X^*$ and $\epsilon > 0$.

$$\mathcal{U}(x, \phi_0, \epsilon) = \omega(\iota(x), \phi_0, \epsilon) = \{\phi \in X^* : |\iota(x)(\phi) - \iota(x)(\phi_0)| < \epsilon\}$$

Defn. If V is a vecto space over \mathcal{F} , then we call $(\cdot, \cdot) : V \times V \rightarrow \mathcal{F}$ a sesquilinear form on V if for all $x, y, z \in V$ and $\alpha \in \mathcal{F}$, we have:

$$- (x + \alpha y, z) = (x, z) + \alpha(y, z)$$

$$- (x, \alpha y + z) = \bar{\alpha}(x, y) + (x, z)$$

If we also have:

$$(x, y) = \overline{(y, x)}$$

then we say that (\cdot, \cdot) is a self-adjoint and if:

$$(x, x) \geq 0$$

then we say (\cdot, \cdot) is positive.

Lem. (Polarization Identity). If $\mathbb{F} = \mathbb{C}$ and (\cdot, \cdot) is a sesquilinear form on V , then:

$$(x, y) = \frac{1}{4} \sum_{k=0}^3 i^k (x + i^k y, x + i^k y)$$

Proof. Check that:

$$4(x, y) = (x + y, x + y) + i(x + iy, x + iy) - (x - y, x - y) - i(x - iy, x - iy) \quad (1)$$

$$(2)$$

Corr. If $\mathbb{F} = \mathbb{C}$, then (\cdot, \cdot) is self-adjoint if and only if $(x, x) \in \mathbb{R}$ for all $x \in V$. Hence on a complex vector space, a positive sesquilinear form is always self-adjoint.

Defn. A positive self-adjoint sesquilinear form on V is called a pre-inner product. If $(x, x) = 0$ if and only if $x = 0$, then we call it an inner product.

Defn. If (\cdot, \cdot) is a pre-inner product on V , then we define:

$$||v|| = (v, v)^{frac{1}{2}}$$

for all $v \in V$.

Rmk. The polarization identity then becomes for \mathbb{C} :

$$(x, y) = \frac{1}{4} \sum_{k=0}^3 i^k \|x + i^k y\|^2$$

for \mathbb{R} ,

$$(x, y) = \frac{1}{3} (\|x + y\|^2 - \|x - y\|^2)$$

Lemm. (Cauchy-Schwarz Inequality). If (\cdot, \cdot) is a pre-inner product on V , then $\forall x, y \in V$:

$$|(x, y)| \leq \|x\| \|y\|$$

Proof. For all $\alpha \in \mathcal{F}$,

$$\begin{aligned} 0 &\leq \|\alpha x + y\|^2 = (\alpha x + y, \alpha x + y) \\ &= |\alpha|^2 \|x\|^2 + \alpha(x, y) + \overline{\alpha}(x, y) + \|y\|^2 \\ &= |\alpha|^2 \|x\|^2 + 2\Re(\alpha(x, y)) + \|y\|^2 \end{aligned}$$

Let $\tau \in \mathcal{F}$ be such that $\tau(x, y) = |(x, y)|$. If $\alpha = t\tau$ with $t \in \mathbb{R}$, then:

$$0 \leq t^2 \|x\|^2 + 2t |(x, y)| + \|y\|^2$$

Then, by the quadratic formula,

$$\begin{aligned} 4|(x, y)|^2 - 4\|x\|^2 \|y\|^2 &\leq 0 \\ \implies |(x, y)| &\leq \|x\| \|y\| \end{aligned}$$

Corr. Let (\cdot, \cdot) be a pre-inner product on V . Then:

$$\|x\| = (x, x)^{\frac{1}{2}}$$

is a semi-norm on V which is a norm exactly when (\cdot, \cdot) is an inner product.

Proof. The homogeneity is clear. Use Cauchy Schwarz to see:

$$\|x + y\|^2 \leq (\|x\| + \|y\|)^2$$

Defn. A vector space H with an inner product is called a Hilbert space if the normed vector space $(H, \|\cdot\|)$ is complete with respect to $\|x\| = (x, x)^{\frac{1}{2}}$.

Ex. (1) If $H = \mathbb{F}^n$ and

$$(x, y) = \sum_{k=1}^n x_k \overline{y_k}$$

Here,

$$\|x\| = \|x\|_2$$

(2) Let $H = l^2$. Now,

$$(x, y) = \sum_{n=1}^{\infty} x_n \overline{y_n}$$

Again,

$$||x|| = ||x||_2$$

(3) Let $H = L^2(x, \mathcal{M}, \mu)$ and:

$$(f, g) = \int_X (f(x) \overline{g(x)}) d\mu(x)$$

Thm. (Parallelogram Law). If V is a pre-inner product space, then for all $x, y \in H$,

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2$$

Thm. (Jordan von Neumann Thm.) Let X be a normed vector space where norm satisfies the parallelogram law. Then X admits an inner product such that $||x|| = (x, x)^{\frac{1}{2}}$.

Prop. If H is an inner product space and if $x_n \rightarrow x$ and $y_n \rightarrow y$ in H . Then,

$$(x_n, y_n) \rightarrow (x, y)$$

Proof. Since H is a normed vector space, the norm is continuous. Thus,

$$||x_n|| \rightarrow ||x|| \quad ||y_n|| \rightarrow ||y||$$

Now, using Cauchy-Schwarz,

$$\begin{aligned} |(x_n, y_n) - (x, y)| &\leq |(x_n - x, y_n)| + |(x, y_n - y)| \\ &\leq ||x_n - x|| \cdot ||y_n|| + ||x|| \cdot ||y_n - y|| \\ &\rightarrow 0 \end{aligned}$$

Defn. Let H be an inner product space. We say $x \in H$ is orthogonal to $g \in H$ or just $x \perp y$ if $(x, y) = 0$. We say $E \subset H$ is orthogonal to $F \subset H$ if $\forall x \in E, y \in F, x \perp y$. We would write $E \perp F$.

Thm. (Pythagorean Theorem). Suppose H is an inner product space and that $x_1, \dots, x_n \in H$ are pairwise orthogonal. Then,

$$||\sum_{k=1}^n x_k||^2 = \sum_{k=1}^n ||x_k||^2$$

Proof. Simple.

Lemm. Let C be a nonempty closed, convex subset of a Hilbert space H . Then $\forall y \in H$, $\exists! x \in C$ such that:

$$d(y, C) = \|y - x\|$$

Proof. We can replace C by $C - \{y\}$ and assume $y = 0$.

Let $\alpha = \inf\{\|x\| : x \in C\} = d(0, C)$.

Let $(x_n) \subset C$ be a sequence such that $\|x_n\| \rightarrow \alpha$.

Then,

$$\begin{aligned} 2(\|x_n\|^2 + \|x_m\|^2) &= \|x_n + x_m\|^2 + \|x_n - x_m\|^2 \\ &= 4 \left\| \frac{x_n + x_m}{2} \right\|^2 + \|x_n - x_m\|^2 \\ &\geq 4\alpha^2 + \|x_n - x_m\|^2 \end{aligned}$$

But LHS converges to $4\alpha^2$. Thus, (x_n) is Cauchy. Then,

$$x_n \rightarrow x \quad \|x\| = \alpha$$

Suppose $y \in C$ and $\|y\| = \alpha$. Then,

$$\begin{aligned} 4\alpha^2 &= 2(\|x\|^2 + \|y\|^2) \\ &= \|x + y\|^2 + \|x - y\|^2 \\ &\geq 4\alpha^2 + \|x - y\|^2 \end{aligned}$$

Thus, $\|x - y\|^2 = 0$, thus $x = y$.