Math 74: Algebraic Topology

Sair Shaikh

May 28, 2025

Problem 1.(2.2.9) Compute the homology of the following 2-complexes:

1. The quotient of S^2 by identifying the north and south poles to a point.
2. $S^1 \times (S^1 \vee S^1)$.
3. The space obtained from D^2 by first deleting the interiors of two disjoint subdisks in the interior of D^2 and then identifying all three resulting boundary circles via homeomorphisms preserving clockwise orientations.

Solution.

1. We build the CW complex as follows:
   (a) Start with one 0-cell, representing the identified north and south poles.
   (b) Attach 2 1-cells, a and b. Think of these are loops from the north to south and south to north poles, respectively (they are loops as the points are identified).
   (c) Attach 2 2-cells, along ab.

   Then, the CW chain complex is as follows:

$$0 \to \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^2 \xrightarrow{\partial_1} \mathbb{Z}^1 \to 0$$

   Then, we compute the homology groups as follows:

   (a) For $i = 0$, we have that $\text{img}(\partial_1)$ is trivial as all the generators are loops and thus have trivial boundary. Thus, $H_0(X) \cong \mathbb{Z}$. The space has one connected component, so this is expected.
   (b) For $i = 1$, we have that $\text{img}(\partial_2)$ is generated by $a+b$, the boundary of both 2-cells and $\ker(\partial_1) = \mathbb{Z}^2$ as noted before. Thus, $H_1(X) \cong \mathbb{Z}^2/\mathbb{Z} = \mathbb{Z}$.

(c) For $i = 2$, we have that $\ker(\partial_2) \cong \mathbb{Z}$ as $\text{img}(\partial_2) \cong \mathbb{Z}$, by rank-nullity. Thus, $H_2(X) = \mathbb{Z}$.

(d) Since this is a 2-dimensional CW complex, we have that $H_i(X) = 0$ for $i > 2$.

Overall, we have that:

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} & i = 1 \\ \mathbb{Z} & i = 2 \\ 0 & i > 2 \end{cases}$$

2. This space is two tori glued together along a circle. We can build this space as follows:

(a) Start with one 0-cell.

(b) Attach 3 1-cells, a, b, and c to the 0-cell.

(c) Attach 2 2-cells, e_1 and e_2 where e_1 is attached to $aca^{-1}c^{-1}$ and e_2 is attached to $bcb^{-1}c^{-1}$.

Then, the CW chain complex is as follows:

$$0 \to \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z}^1 \to 0$$

Then, we compute the homology groups as follows:

(a) For $i = 0$, we have that $\text{img}(\partial_1)$ is trivial as all generators are loops, thus have trivial boundary. Thus, $H_0(X) \cong \mathbb{Z}$. The space has one connected component, so this is expected.

(b) For $i = 1$, we have that $\ker(\partial_1) \cong \mathbb{Z}^3$ as noted before. Moreover, the 2-cells have boundary $a + c - a - c = 0$ and $b + c - b - c = 0$ respectively, thus the map is also the 0 map (we can says this in terms of all deg being 0 if one wishes). Thus, $H_1(X) \cong \mathbb{Z}^3/\{0\} = \mathbb{Z}^3$.

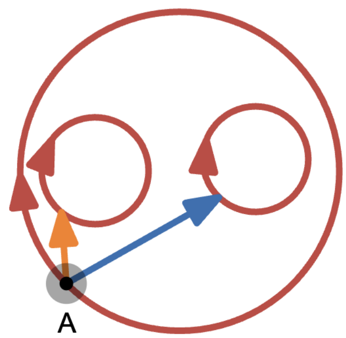
(c) For $i = 2$, we have that $\ker(\partial_2) \cong \mathbb{Z}^2$, as the two 2-cells have trivial boundary. Thus, $H_2(X) = \mathbb{Z}^2$.

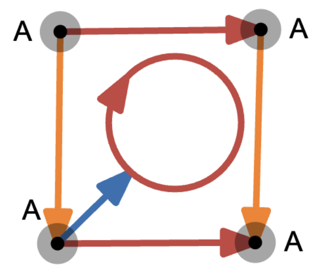
(d) Since this is a 2-dimensional CW complex, we have that $H_i(X) = 0$ for $i > 2$.

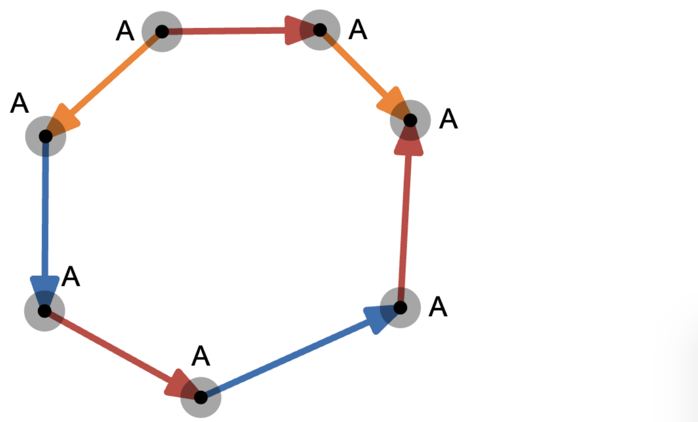
Overall, we have that:

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^3 & i = 1 \\ \mathbb{Z}^2 & i = 2 \\ 0 & i > 2 \end{cases}$$

3. Notice that the CW complex structure is as follows:







Thus, X is obtained from taking 1 0-cell, attaching 3 1-cells to get a wedge of 3 circles, and then attaching a 2-cell in the manner in the diagram (colors indicated identified edges). Thus, the CW complex looks like:

$$0 \to \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z} \to 0$$

Then, we compute the homology groups as follows:

1. For $i = 0$, note that $\operatorname{img}(\partial_1)$ is trivial as all generators are loops, thus have trivial boundary. Thus, $H_0(X) \cong \mathbb{Z}$. The space has one connected component, so this is expected.

2. For $i = 1$, we have that $\ker(\partial_1) \cong \mathbb{Z}^2$ as noted before. Moreover, the boundary of the 2-cell generates $\operatorname{img}(\partial_2)$. Call the red edge a, the blue edge b, and the orange edge c. Then, we have

$$\operatorname{img}(\partial_2) = \mathbb{Z}(a + b + a - c - a + c + b) = \mathbb{Z}(a + 2b) \cong \mathbb{Z}$$

Then, noting that $a+2b, b, c$ are linearly independent generators for (X^1, X_0), we note that there are two generators left in the quotient. Thus, $H_1(X) \cong \mathbb{Z}^2$.

3. For $i = 2$, we have that $\ker(\partial_2)$ is trivial, as the generator maps to a non-zero element. Thus, $H_2(X) = 0$.

4. For $i > 2$, we have that $H_i(X) = 0$ as this is a 2-dimensional CW complex. Overall, we have that:

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^2 & i = 1 \\ 0 & i \geq 2 \end{cases}$$

Problem 2. Compute the homology of the torus with $n \geq 1$ vertical disks filled in, that is,

$$X = (S^1 \times S^1) \bigcup \left(\bigcup_{k=1}^{n} \{e^{2\pi ik/n}\} \times D^2 \right).$$

Solution. First, to simply, we slide all the disks so that they share a boundary circle homotopically (contract the open cylinders along the torus joining two disks to circles). Then, the CW structures of X is as follows:

1. 1 0-cell.
2. 2 1-cells, call them a and b.
3. One 2-cell, call in e attached in the usual way to $aba^{-1}b^{-1}$ and n 2-cells attached to b.

The CW chain complex is then as follows:

$$0 \to \mathbb{Z}^n \oplus \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^2 \xrightarrow{\partial_1} \mathbb{Z}^1 \to 0$$

Then, we compute the homology groups as follows:

1. For $i = 0$, note that as every generate of $\mathbb{Z}^n$ is a loop, they have trivial boundary, thus ∂_1 is the 0 map and has trivial image. Thus, $H_0(X) \cong \mathbb{Z}$. We can also see this directly, as X is path-connected.
2. For $i = 1$, we have that $\ker(\partial_1) \cong \mathbb{Z}^2$, as noted above. Moreover, each of the n disks has b as a boundary, thus $b \in \mathrm{img}(\partial_2)$. Moreover, $\partial_2(e)$ has boundary $a + b - a - b = 0$ (i.e. has deg 0 for both a and b, if we wanna say it that way), thus, $\mathrm{img}(\partial_2) \cong \mathbb{Z}$. Thus, $H_1(X) \cong \mathbb{Z}^2/\mathbb{Z} \cong \mathbb{Z}$. We can also see this directly by abelianizing the fundamental group ($\pi_1(X) = \mathbb{Z}$) from Quiz 2.
3. For $i = 2$, we know that $\mathrm{img}(\partial_2) \cong \mathbb{Z}$, as noted above. Thus, $\ker(\partial_2) \cong \mathbb{Z}^n$ by rank-nullity. Thus, we have that $H_2(X) = \ker(\partial_3) \cong \mathbb{Z}^2$.

Thus, we have:

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} & i = 1 \\ \mathbb{Z}^n & i = 2 \\ 0 & i > 2 \end{cases}$$

Problem 3.(2.2.21) If a finite CW complex X is a union of subcomplexes A and B, show that
$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B).$$

Solution. Recall the definition of the Euler characteristic:
$$\chi(X) = \sum_n (-1)^n |I_n|$$
where I_n is the set of n-cells of X. Since A and B are subcomplexes, any cell that intersects with A or B must lie fully within A or B, respectively. Since $X = A \cup B$, each cell of X either lies just in A (i.e. in $X \setminus B$), just in B (i.e. in $X \setminus A$), or in both A and B (i.e. in $A \cap B$). For n-cells, call these respective sets $I_{n,A}$, $I_{n,B}$, and $I_{n,A\cap B}$. Clearly, by the inclusion-exclusion principle, we have:
$$|I_n| = |I_{n,A}| + |I_{n,B}| - |I_{n,A\cap B}|$$
Thus, we can write the Euler characteristic of X as:
$$\begin{aligned}
\chi(X) &= \sum_n (-1)^n |I_n| \\
&= \sum_n (-1)^n \left(|I_{n,A}| + |I_{n,B}| - |I_{n,A\cap B}|\right) \\
&= \sum_n (-1)^n |I_{n,A}| + \sum_n (-1)^n |I_{n,B}| - \sum_n (-1)^n |I_{n,A\cap B}| \\
&= \chi(A) + \chi(B) - \chi(A \cap B).
\end{aligned}$$
Thus,
$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B).$$

Problem 4.(2.2.22) If X is a finite CW complex and $p\colon \widetilde{X} \to X$ is a degree n covering, show that $\chi(\widetilde{X}) = n \cdot \chi(X)$.

Solution. Let X be m-dimensional. It suffices to show that $|I_j(\widetilde{X})| = n|I_j(X)|$ for each $j \leq m$, as:

$$\begin{aligned}\chi(\widetilde{X}) &= \sum_{k=0}^{m} (-1)^k |I_k(\widetilde{X})| \\ &= \sum_{k=0}^{m} (-1)^k n \cdot |I_k(X)| \\ &= n \sum_{k=0}^{m} (-1)^k |I_k(X)| \\ &= n \cdot \chi(X)\end{aligned}$$

To show this, we claim that $\rho^{-1}(X^k)$ is a k-dimensional CW complex in $\widetilde{X}$ for each $k \geq 0$, with $|I_j(\widetilde{X})| = n \cdot |I_j(X)|$ for all $j \leq k$. We proceed by induction.

For $k = 0$, we know that for every 0-cell $x \in X$, there are n distinct preimages under ρ in $\widetilde{X}$, as p is a covering map of degree n. Thus, $\widetilde{X}^0 = \rho^{-1}(X^0)$ is a 0-dimensional CW complex with $|I_0(\widetilde{X})| = n|I_0(X)|$.

For $k > 0$, let e^k be a k-cell of X with map $\phi : D^k \to X$. Since $\pi_1(D^k, d_0)$ is trivial (for $d_0 \in \text{int}(D^k)$), we have that $\phi_*(\pi_1(D^n, d_0))$ is also trivial. Thus, as D^n is path-connected and locally path-connected we can use the universal lifting property to get a unique lift for each pre-image under ρ of $\phi(d_0)$, call these $\phi_1, \ldots, \phi_n : D^k \to \widetilde{X}$. We claim that $\text{int}(\text{img}(\phi_i))$ are disjoint k-cells of $\widetilde{X}$ that map homeomorphically to e^k under p.

For $1 \leq i \leq n$, note that we have:

$$\rho \circ \phi_i = \phi$$

Then note the following:

1. As $\phi|_{\text{int}(D^k)}$ is a homeomorphism onto e^k, so is $\rho \circ \phi_i|_{\text{int}(D^k)}$. Thus, $\rho|_{\phi_i(\text{int}(D^k))}$ is a homeomorphism onto e^k, i.e. $\phi_i(\text{int}(D^k)) \cong e^k$. Thus, we have $\phi_i(\text{int}(D^k)) \cong e^k \cong \text{int}(D^k)$.

2. Note $\rho \circ \phi_i(\delta D^n) = \phi(\delta D^n) \subseteq X^{k-1}$. Then, by the induction hypothesis, we have that $\rho^{-1}(X^{k-1}) = \widetilde{X}^{k-1}$. Thus, $\phi_i(\delta D^n) \subseteq \widetilde{X}^{k-1}$.

Thus, $\phi_i(\text{int}(D^k))$ are k-cells of $\widetilde{X}$ for each i. Finally, as these cells contain a distinct pre-image of the $\phi(d_0)$, they must be disjoint by uniqueness of the lift. Thus, we have that:

$$\rho^{-1}(e^k) = \bigsqcup_i \phi_i(\text{int}(D^k))$$

since we have n distinct homeomorphic copies of e^k. Thus, considering these as the k-cells of $\widetilde{X}$, we have that $\rho^{-1}(X^k) = \widetilde{X}^k$ is a k-dimensional CW complex, with $|I_k(\widetilde{X})| = n \cdot |I_k(X)|$. By the induction hypothesis, we have that $|I_j(\widetilde{X})| = n \cdot |I_j(X)|$ for each $j \leq k$.

Thus, we note that since X is a finite CW complex of some dimension m, so is $\widetilde{X}$ and we have that $|I_j(\widetilde{X})| = n \cdot |I_j(X)|$ for each $j \leq m$. As noted before, this concludes the proof.

Problem 5. Use the previous problem to show that if $\rho\colon \mathbb{RP}^{2n} \to X$ is a covering map where X is a finite CW complex, then p is a homeomorphism.

Solution. Note that from the previous question, we have that:

$$\chi(\mathbb{RP}^{2n}) = \deg(\rho) \cdot \chi(X)$$

where $\deg(\rho)$ is the degree of the covering map ρ. Moreover, note that we showed:

$$H_i(\mathbb{RP}^{2n}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2\mathbb{Z} & 1 < i < 2n, i \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Thus, we can calculate $\chi(\mathbb{RP}^{2n})$ as follows:

$$\chi(\mathbb{RP}^{2n}) = \sum_{i=0}^{2n} (-1)^i \operatorname{rk}(H_i(\mathbb{RP}^{2n})) = 1$$

as the free rank of $\mathbb{Z}/2\mathbb{Z}$ is 0. Thus, we have that:

$$\deg(\rho) \cdot \chi(X) = 1$$

Since both $\deg(\rho)$ and $\chi(X)$ are integers, we have $\deg(\rho) = \chi(X) = \pm 1$. However, as the degree of a covering map cannot be negative, we have that $\deg(\rho) = \chi(X) = 1$. In particular, this means that ρ is a homeomorphism.