Math 113: Functional Analysis

Sair Shaikh

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Problem 30. Suppose that X and Y are normed vector spaces.

1. Show that $\mathcal{L}(X, Y)$ is a normed vector space with respect to the the operator norm defined in lecture such that:

$$||T(x)|| \leq ||T||||x||$$

2. Show that if $S \in \mathcal{L}(Y, Z)$. Then,

$$||ST|| \leq ||S||||T||$$

3. Show that:

$$||T|| = \inf\{a \geq 0 : ||T(x)|| \leq \alpha||x|| \quad \forall x \in X\}$$

Solution.

Problem 31. Suppose that X and Y are Banach spaces with $T \in \mathcal{L}(X, Y)$. Suppose that E is a closed proper subspace of X such that $E \subset \ker(T)$. Show that there is a unique operator $\overline{T} \in \mathcal{L}(X/E, Y)$ such that $\overline{T}(q(x)) = T(x)$ for all $x \in X$ where $q : X \to X/E$ is the quotient map. Moreover, $||\overline{T}|| = ||T||$.

Solution.

Problem 33. Let E and X be Banach spaces with E finite dimensional.

1. Show that every linear map $S : E \to X$ is bounded.
2. Show that a linear map $T : X \to E$ is bounded if and only if $\ker(T)$ is closed.

Solution.

Problem 34. Supposed that E and M are closed subspaces of a Banach space X. If E is finite dimensional, show that $E + M = \{x + y : x \in E\, y \in M\}$ is closed.

Solution.

Problem 35. Suppose that X and Y are Banach spaces for $T \in \mathcal{L}(X, Y)$. Show that T is injective with closed range if and only if:

$$\inf\{||T(x)|| : ||x|| = 1\} > 0$$

Solution.

Problem 38. Let X be a normed vector space. A Banach space $\tilde{X}$ is called a completion of X is there is an isometric isomorphism $\iota : X \to \tilde{X}$ onto a dense subspace of $\tilde{X}$. Show that any two completions $(tildeX_1, \iota_1)$ and $(\tilde{X}_2, \iota_2)$ are isometrically isomorphic by an isomorphism:

$$\Phi : \tilde{X}_1 \to \tilde{X}_2$$

such that $\Phi(\iota_1(x)) = \iota_2(x)$ for all $x \in X$.

Solution.

Problem 39. Lets find a use for a genuine Minkowski functional. In this problem, we'll let $l^\infty_\mathbb{R}$ be the real Banach space of bounded sequences in $\mathbb{R}$. Define m on $l^\infty_\mathbb{R}$:

$$m(x) = \text{limsup}_n\, x_n$$

We clearly have $m(tx) = tm(x)$ if $t \geq 0$ and it is not hard to check that $m(x+y) \leq m(x) + m(y)$ for all $x, y \in l^\infty_\mathbb{R}$. We want to show that there are Banach limits or what I prefer to call a generalized limit on $l^\infty_\mathbb{R}$. This is we want to show that there is a functional $L \in {l^\infty_\mathbb{R}}^*$ such that:

$$L(S(x)) = L(x)$$

where $S \in \mathcal{L}(l^\infty_\mathbb{R})$ is given by $S(x)_n = x_{n+1}$ and such that $\liminf_n x_n \leq L(x) \leq \text{limsup}_n\, x_n$. (Hint provided).

Solution.

Problem 40. Prove the following Lemma from lecture. Let X be a complex vector space. Every real linear functional of X is the real part of a complex linear functional on X. In fact, if $\phi = \Re(\psi)$ then $\psi(x) = \phi(x) - i\phi(ix)$.

Solution.

Problem 41. Suppose that X is a normed vector space such that X^* is seperable. Show that X is seperable. (Hint provided).

Solution.