Problem 1. Prove the pasting lemma: Suppose $X = A \cup B$ is a topological space with A, B closed in X. If $f: X \to Y$ is a map such that the restrictions $f|_A$ and $f|_B$ are continuous, then f is continuous.

Solution. This follows from the definition of continuity. Let $U \subseteq Y$ be open. Then $f^{-1}(U) = (f|_A)^{-1}(U) \cup (f|_B)^{-1}(U)$. Since $f|_A$ and $f|_B$ are continuous, $(f|_A)^{-1}(U)$ and $(f|_B)^{-1}(U)$ are open in A and B, respectively. Because A and B are closed in X, these sets are also open in X. Hence, $f^{-1}(U)$ is open in X, and f is continuous.

Problem 2. In a connected space X, a point $x \in X$ is called a *cut point* if $X \setminus \{x\}$ is disconnected.

1. Suppose that $f : X \to Y$ is a homeomorphism of connected spaces. Show that $x \in X$ is a cut point if and only if $f(x) \in Y$ is a cut point.

2. Show that none of the spaces $(0, 1), (0, 1], [0, 1]$, and $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ are homeomorphic to each other.

3. Show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for any $n \geq 2$.

4. The bouquet B_n of n circles is the space obtained by gluing n disjoint copies of S^1 at a single point in each circle. Show that B_n and B_m are not homeomorphic for $n \neq m$.

Solution.

Problem 3. Define $\mathbb{RP}^n$ to be the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ by $(x_1, \ldots, x_{n+1}) \simeq (ax_1, \ldots, ax_{n+1})$ for all nonzero scalars a. Let $[x_1 : \ldots : x_{n+1}] \in \mathbb{RP}^n$ denote the image of $(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}$ under the quotient map. Show that $i\colon \mathbb{R}^n \to \mathbb{RP}^n$ given by

$$i(x_1, \ldots, x_n) = [1 : x_1 : \ldots : x_n]$$

is a topological embedding (i.e., a homeomorphism onto its image) and the complement of $i(\mathbb{R}^n)$ is homeomorphic to $\mathbb{RP}^{n-1}$.

Solution.

Problem 4. Suppose that $\mathcal{C}$ is a category, A, B, C are objects of $\mathcal{C}$, and $f \in \hom(A, B)$ and $g \in \hom(B, C)$ are isomorphisms.

1. Show that f has a unique inverse in $\hom(B, A)$.
2. Show that id_A is an isomorphism.
3. Show that the inverse of f is an isomorphism.
4. Show that $g \circ f$ is an isomorphism.
5. Show that $\mathrm{Aut}(A)$ and $\mathrm{Aut}(B)$ are isomorphic groups.
6. Show that if $F : \mathcal{C} \to \mathcal{D}$ is a functor, then $F(f) \in \hom_{\mathcal{D}}(F(A), F(B))$ is an isomorphism.

Solution.

Problem 5. For a space X, let $\pi_0(X)$ be the space of path components of X. Recall that the image of a path-connected space is path-connected. Thus, for a continuous map $f : X \to Y$, there is an induced map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ taking a path component A to the path component containing $f(A)$. Show that this makes π_0 a functor from the category of topological spaces to the category of sets.

Solution. First, we understand what the induced morphism $\pi_0(f)$ is. For a subset $A \subseteq X$, we let $[A] \in \pi_0(X)$ denote the path component of A. Then,

$$\pi_0(f)([A]) = [f(A)]$$

To show that π_0 is a functor, we need to show that it preserves identities and compositions.

- Let X be a topological space and $\mathrm{id}_X : X \to X$ be the identity map on X. Then, for any path component $A \subseteq X$, $\pi_0([\mathrm{id}_X])(A) = [\mathrm{id}_X(A)] = [A]$. Thus, $\pi_0(\mathrm{id}_X) = \mathrm{id}_{\pi_0(X)}$.
- Let X, Y, Z be topological spaces and $f : X \to Y$, $g : Y \to Z$ be continuous maps. Then, for any path component $A \subseteq X$, we have:

$$\begin{aligned} \pi_0(g) \circ \pi_0(f)([A]) &= \pi_0(g)([f(A)]) \\ &= [g(f(A))] \\ &= [g \circ f(A)] \\ &= \pi_0(g \circ f)([A]) \end{aligned}$$

  Thus,

$$\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$$

Thus, π_0 preserves identities and compositions, and hence is a functor.

Problem 6. (Munkres, *Topology*, 51.2) Given spaces X and Y, let $[X, Y]$ denote the set of homotopy classes of maps of X into Y.

1. Let $I = [0, 1]$. Show that for any X, the set $[X, I]$ has a single element.
2. Show that if Y is path connected, then $[I, Y]$ has a single element.

Solution. .

1. Let X be a topological space and let $A, B \in [X, I]$. We need to show that $A = B$, i.e. A and B are homotopic.

Problem 7. (Munkres, *Topology*, 51.3) A space X is said to be *contractible* if the identity map $i_X : X \to X$ is nullhomotopic.

1. Show that I and $\mathbb{R}$ are contractible.
2. Show that a contractible space is path connected.
3. Show that if Y is contractible, then for any X, the set $[X, Y]$ has a single element.
4. Show that if X is contractible and Y is path connected then $[X, Y]$ has a single element.