Math 74: Algebraic Topology

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Problem 1. Consider the chain complexes C and D given by

$$C_i = \begin{cases} \mathbb{Z}/2\mathbb{Z} & i = 0 \\ 0 & i \neq 0 \end{cases} \qquad D_i = \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & i \neq 0, 1 \end{cases}$$

where all the differentials on C is zero and $\partial_1(n) = 2n$ on D. Show that C and D are quasi-isomorphic (have isomorphic homology) but are not chain homotopy equivalent.

Solution. We write down the chain complexes explicitly:

$$\begin{aligned} C &: 0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} 0 \to 0 \\ D &: 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} 0 \end{aligned}$$

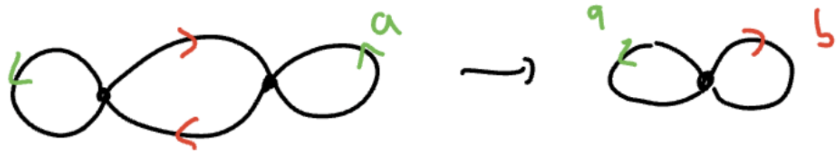
Then, we compute the homology of C and D:

$$\begin{aligned} &H_0(C) = \ker(\mathbb{Z}/2\mathbb{Z} \to 0)/\operatorname{img}(0 \to \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \\ &H_k(C) = 0 \text{ for } k \neq 0 \\ &H_1(D) = \ker(\mathbb{Z} \xrightarrow{2} \mathbb{Z})/\operatorname{img}(0 \to \mathbb{Z}) = 0 \\ &H_0(D) = \ker(\mathbb{Z} \to 0)/\operatorname{img}(\mathbb{Z} \xrightarrow{2} \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \\ &H_k(D) = 0 \text{ for } k \neq 0, 1 \end{aligned}$$

Thus, the homology groups of C and D are isomorphic. Thus, C and D are quasi-isomorphic. However, note that if we had a chain homotopy equivalence, we would have maps $f : C \to D$ and $g : D \to C$ such that $f \circ g$ is chain homotopic to id_D. However, this would imply that $f_0 \circ g_0$ is chain homotopic to $\mathrm{id}_{D_0} = \mathrm{id}_{\mathbb{Z}}$. However, $f_0 : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ must be the zero map. Thus, $f_0 \circ g_0$ is also zero, which is not chain homotopic to $\mathrm{id}_{\mathbb{Z}}$.

Problem 2. If $p\colon E \to B$ is a covering map, we know it induces an injective map on fundamental groups. Is the same true for the map $H_1(E) \to H_1(B)$?

Solution. Let $B = S^1 \vee S^1$ and E be the following covering space from class.



Then, note that $\pi_1(E) = F_3$ and $\pi_1(B) = F_2$, where F_n is the free group on n generators. Moreover, as E and B are path-connected, H_1 is the abelianization of the fundamental group. Thus, we have $H_1(E) = \mathbb{Z}^3$ and $H_1(B) = \mathbb{Z}^2$. Thus, the induced map is:

$$p : \mathbb{Z}^3 \to \mathbb{Z}^2$$

This map cannot be injective, as $\mathbb{Z}^3$ has a larger rank than $\mathbb{Z}^2$.

Problem 3.(2.1.16)

1. Show that $H_0(X,A)=0$ if and only if A meets every path component of X.
2. Show that $H_1(X,A)=0$ if and only if $H_1(A)\to H_1(X)$ is surjective and every path component of X contains at most one path component of A.

Solution.

1. Note that we have the long exact sequence, ending in:

$$H_1(X,A)\to H_0(A)\xrightarrow{\iota_*}H_0(X)\to H_0(X,A)\to 0$$

Moreover, if X_i for $i\in I$ are the path components of X, then we also have:

$$H_0(X)=\bigoplus_{i\in I}H_0(X_i)$$

Where $H_0(X_i)\cong\mathbb{Z}$ as they are path-connected. Finally, note that if $[\sigma]=[\sigma']$ in $H_0(\cdot)$, $\sigma'-\sigma$ is a boundary if and only if the images of σ and σ' (single points) have a path connecting them, i.e. are in the same path component.

Assume $H_0(X,A)=0$. Then, we have that $\iota_*:H_0(A)\to H_0(X)$ is surjective. Thus, if $\sigma_i:\Delta_0\to X_i$ is a representative of the generator of $H_0(X_i)$, we have a $\tau_i:\Delta_0\to A$ such that $\iota_*([\tau_i])=[\sigma_i]$. Thus, we have that $[\iota\circ\tau_i]=[\sigma_i]$. Thus, the point picked out by $\iota\circ\tau_i$ is in the same path-component as the image of σ_i, i.e. X_i. However, as ι is the inclusion map, this means the image of τ_i is in $A\cap X_i$. Thus, A meets every path component of X.

Conversely, assume that A meets every path component of X. We show that ι_* is surjective. Let $[\sigma_i]$ be a generator of $H_0(X_i)$. Since A meets every path component of X, there exists a point $x_i\in A\cap X_i$. Then, we can take $\tau_i:\Delta_0\to A$ to be the map picking out x_i. Then, we have that:

$$\iota_*([\tau_i])=[\iota\circ\tau_i]=[\sigma_i]$$

as the image of $\iota\circ\tau_i$ and σ_i are in the same path component X_i. Thus, ι_* maps to each generator of $H_0(X)$ and is thus surjective. Thus, the long exact sequence above implies that $H_0(X,A)=0$.

2. Consider the end of same long exact sequence as above:

$$\cdots\to H_1(A)\xrightarrow{\iota_*}H_1(X)\to H_1(X,A)\to H_0(A)\xrightarrow{\iota_*}H_0(X)\to H_0(X,A)\to 0$$

Assume $H_1(X, A) = 0$. Then, the sequence breaks into two exact sequences:

$$\cdots \to H_1(A) \xrightarrow{\iota_*} H_1(X) \to 0$$
$$0 \to H_0(A) \xrightarrow{\iota_*} H_0(X) \to H_0(X, A) \to 0$$

Then, we have that $\iota_* : H_1(A) \to H_1(X)$ is surjective and that $\iota_* : H_0(A) \to H_0(X)$ is injective. Now let X_i be a path-component of X. Assume X_i contains at least one path-component of A. Then let A_i and A_j be path-components of A contained in X_i. Then, let $\tau_i : \Delta_0 \to A_i$ and $\tau_j : \Delta_0 \to A_j$ be two maps picking out a point in A_i and A_j respectively. Then, note that $\iota_*([\tau_i]) = [\iota \circ \tau_i]$ and $\iota_*([\tau_j]) = [\iota \circ \tau_j]$ both pick out a point in X_i. Since X_i is path-connected, this implies:

$$\iota_*([\tau_i]) = \iota_*([\tau_j])$$

As ι_* is injective, this implies that $[\tau_i] = [\tau_j]$. Thus, A_i and A_j are in the same path-component, i.e. are equal.

On the other hand, assume that $\iota_* : H_1(A) \to H_1(X)$ is surjective and that every path component of X contains at most one path component of A. Then, we first show that $\iota_* : H_0(A) \to H_0(X)$ is injective.

Let $\tau_i : \Delta_0 \to A$ and $\tau_j : \Delta_0 \to A$ be two maps picking out points in A such that:

$$\iota_*([\tau_i]) = \tau_*([\tau_j])$$

Then, we have that $[\iota \circ \tau_i] = [\iota \circ \tau_j]$. Thus, the points picked out by $\iota \circ \tau_i$ and $\iota \circ \tau_j$ are in the same path component of X, call it X_i. However, as τ_i and τ_j pick out points in path-components of A contained in X_i. By assumption, there is at most one such path-component (in this case exactly one, as we know there exists one). Thus, τ_i and τ_j pick points in the same path-component of A. Thus, $[\tau_i] = [\tau_j]$ and ι_* is injective.

Consider the long exact sequence again. We have that:

$$\cdots \to H_1(A) \xrightarrow{\iota_*} H_1(X) \to H_1(X, A) \to H_0(A) \xrightarrow{\iota_*} H_0(X) \to H_0(X, A) \to 0$$

Then, as $\iota_* : H_1(A) \to H_1(X)$ is surjective, we have that:

$$\ker(H_1(X) \to H_1(X, A)) = \operatorname{img}(H_1(A) \to H_1(X)) = H_1(X)$$

Thus, the map $H_1(X) \to H_1(X, A)$ is the zero map. Thus, we have:

$$\cdots \to H_1(A) \xrightarrow{\iota_*} H_1(X) \xrightarrow{0} H_1(X, A) \to H_0(A) \xrightarrow{\iota_*} H_0(X) \to H_0(X, A) \to 0$$

Continuing on, this implies:

$$\ker(H_1(X, A) \to H_0(A)) = \operatorname{img}(H_1(X) \to H_1(X, A)) = 0$$

Thus, the map $H_1(X, A) \to H_0(A)$ is injective. However, finally, as $\iota_* : H_0(A) \to H_0(X)$ is injective, we have that:

$$\operatorname{img}(H_1(X, A) \to H_0(A)) = \ker(H_0(A) \to H_0(X)) = 0$$

Thus, the map $H_1(X, A) \to H_0(A)$ is an injective map with 0 as its image, and we must have that $H_1(X, A) = 0$.

Problem 4. Use the Mayer-Vietoris sequence to compute

1. all homology groups of a bouquet of circles $\vee^n S^1$.
2. all homology groups of the oriented surface of genus g, M_g.
3. all homology groups of the non-oriented surface of genus k, N_k
4. $H_1(S^3 \setminus K)$ where K is a knot in S^3. (*Note:* A knot is the homeomorphic image of a circle. You can use the fact that every knot has a 'tubular neighborhood' homeomorphic to the interior of a solid torus.)

Solution.

1. We do this via induction. Let $X = \bigvee^n S^1$. We claim that the homology groups for X are:
$$H_k(X) \cong \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^n & k = 1 \\ 0 & k \geq 2 \end{cases}$$
For $n = 1$, we have that $X = S^1$, and we know the homology groups are:
$$H_k(X) = \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & k \geq 2 \end{cases}$$
Next, let $n > 1$. Let $X = \bigvee^n S^1$. Let $A = \bigvee^{n-1} S^1$ and $B = S^1$ (the final circle), . Then, we have that $X = \operatorname{int}(A) \cup \operatorname{int}(B)$ and $A \cap B = \{*\}$ is a single point. We have the Mayer-Vietoris sequence:
$$\cdots \to H_k(A) \oplus H_k(B) \to H_k(X) \to H_{k-1}(A \cap B) \to H_{k-1}(A) \oplus H_{k-1}(B) \to \cdots$$
For $k \geq 2$, we have that $H_k(A) = H_k(B) = 0$. Moreover, as $k - 1 \geq 1$, we have that $H_{k-1}(A \cap B) = 0$. Thus, we reduce to:
$$0 \to H_k(X) \to 0$$
Thus, $H_k(X) = 0$ for $k \geq 2$.

For $k = 1$, we use the induction hypothesis to note $H_1(A) = \mathbb{Z}^{n-1}$ and $H_0(A) = \mathbb{Z}$. Moreover, $H_1(B) = H_0(B) = \mathbb{Z}$, $H_0(A \cap B) = \mathbb{Z}$, and $H_1(A \cap B) = 0$. Thus, we have:
$$\cdots \to 0 \to \mathbb{Z}^{n-1} \oplus \mathbb{Z} \to H_1(X) \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \cdots$$
The final map is the diagonal inclusion map between H_0, i.e. it maps $1 \to (1, 1)$. Thus, the kernel of this map is trivial. Thus, the image of the previous map is trivial, i.e. the map $H_1(X) \to \mathbb{Z}$ is zero. Thus, the map $\mathbb{Z}^{n-1} \oplus \mathbb{Z} \to H_1(X)$ is surjective. Moreover, the kernel of this map is trivial as the image of the previous map is trivial. Thus, we have that $H_1(X) \cong \mathbb{Z}^n$.

For $k=0$, we have that $H_0(A) = H_0(B) = H_0(A\cap B) = \mathbb{Z}$, thus we have the sequence:

$$\cdots \to \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z} \to H_0(X) \to 0$$

The first map is the diagonal inclusion map $1 \to (1,1)$. Thus, the image of this map is isomorphic to $\mathbb{Z}$. Thus, the kernel of the map $\mathbb{Z}\oplus\mathbb{Z} \to H_0(X)$ is isomorphic to $\mathbb{Z}$. Moreover, this map is surjective, as the image of the next map is trivial. By rank-nullity, we have that $H_0(X) \cong \mathbb{Z}$.

This finishes the induction.

2. Let M_g be the oriented surface of genus g. We claim that the homology groups are:

$$H_k(M_g) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}^{2g} & k=1 \\ \mathbb{Z} & k=2 \\ 0 & k\geq 3 \end{cases}$$

Let U be all but a small disk on the left-most side of the surface. Let V be a slightly larger than the complement of U. Then, clearly $\mathrm{int}(U)\cup\mathrm{int}(V) = M_g$. Moreover, we have that $U\cap V$ is homotopic to S^1 and U is homotopic to $\bigvee^{2g} S^1$ (CW complex picture), and V is contractible to a point. Next, consider the Mayer-Vietoris sequence:

$$\cdots \to H_k(U\cap V) \to H_k(U)\oplus H_k(V) \to H_k(X) \to H_{k-1}(U\cap V) \to \cdots$$

For $k=0$, we have that $H_0(U) = H_0(V) = H_0(U\cap V) = \mathbb{Z}$. Thus, the sequence reduces to:

$$\cdots \to \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z} \to H_0(X) \to 0$$

Then the map $\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}$ is the diagonal inclusion map $1\to(1,1)$ (similar to the first part). Thus, the image of this map is isomorphic to $\mathbb{Z}$, thus the kernel of the next map is isomorphic to $\mathbb{Z}$. By rank-nullity, the image of the next map is isomorphic to $\mathbb{Z}$. However, the map $\mathbb{Z}\oplus\mathbb{Z}\to H_0(X)$ is surjective, thus $H_0(X)\cong\mathbb{Z}$.

For $k=1$, we have that $H_1(U) = \mathbb{Z}^{2g}$, $H_1(V) = 0$ (abelinization of 0), $H_0(U\cap V) = H_1(U\cap V) = \mathbb{Z}$, and $H_0(U)\oplus H_0(V) = \mathbb{Z}^2$. Thus, we have the sequence:

$$\cdots \to \mathbb{Z} \to \mathbb{Z}^{2g}\oplus 0 \to H_1(X) \to \mathbb{Z} \to \mathbb{Z}^2 \to \cdots$$

Similar to before, the map $\mathbb{Z}\to\mathbb{Z}^2$ is injective, thus has trivial kernel. Thus the previous map is 0. Thus, the map $\mathbb{Z}^{2g}\to H_1(X)$ is surjective.

We consider carefully the map $H_1(U \cap V) \to H_1(U) \oplus H_1(V)$, i.e. $\mathbb{Z} \to \mathbb{Z}^{2g}$. In the CW picture, X is obtained from attaching a 2-cell, along $U \cap V$ to U. The generator of $H_1(U \cap V)$ then maps to the boundary of U in the CW-complex picture, which contains a path around each circle twice, with opposite signs. Thus, the image of this map is 0. Thus, the map $\mathbb{Z}^{2g} \to H_1(X)$ is also injective. Thus, we have that:

$$H_1(X) \cong \mathbb{Z}^{2g}$$

For $k = 2$, we have that $H_2(U) = 0$ (part a), $H_2(V) = 0$ (contractible), $H_1(U \cap V) = \mathbb{Z}$. Thus, we have the sequence:

$$\cdots \to 0 \to H_2(X) \to \mathbb{Z} \to \mathbb{Z}^2$$

As we previously argued, the final map $\mathbb{Z} \to \mathbb{Z}^2$ is the zero map, thus the map $H_2(X) \to \mathbb{Z}$ is surjective. It is also injective as the image of the previous map is trivial. Thus, we have that $H_2(X) \cong \mathbb{Z}$.

For $k \geq 3$, we have that $H_k(U) = H_k(V) = 0$. Moreover, as $k - 1 \geq 2$, we have that $H_{k-1}(U \cap V) = 0$. Thus, the sequence reduces to:

$$\cdots \to 0 \to H_k(X) \to 0 \to \cdots$$

Thus, $H_k(X) = 0$ for $k \geq 3$.

$$H_k(M_g) \cong \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^{2g} & k = 1 \\ \mathbb{Z} & k = 2 \\ 0 & k \geq 3 \end{cases}$$

3. Consider the CW-complex picture. Let U be all but a disk in the middle of the 2-cell. Let V be a slightly larger disk than the complement of U. Then, we have that $\operatorname{int}(U) \cup \operatorname{int}(V) = N_k$. Moreover, we have that $U \cap V$ is homotopic to S^1 and U is homotopic to $\bigvee^k S^1$, and V is contractible to a point. Next, consider the Mayer-Vietoris sequence:

$$\cdots \to H_k(U \cap V) \to H_k(U) \oplus H_k(V) \to H_k(X) \to H_{k-1}(U \cap B) \to \cdots$$

For $k = 0$, we have that $H_0(U) = H_0(V) = H_0(U \cap V) = \mathbb{Z}$. Moreover, the map $H_0(U \cap V) \to H_0(U) \oplus H_0(V)$ is the diagonal inclusion map $1 \to (1, 1)$. Thus, we have:

$$\cdots \to \mathbb{Z} \to \mathbb{Z}^2 \to H_0(X) \to 0$$

and the map $\mathbb{Z}^2 \to H_0(X)$ has image isomorphic to $\mathbb{Z}$ and is surjective, thus, $H_0(X) \cong \mathbb{Z}$.

For $k = 1$, we have that $H_1(U) = \mathbb{Z}^k$, $H_1(V) = 0$, $H_1(U \cap V) = \mathbb{Z}$, and H_0 of everything is $\mathbb{Z}$. Thus, we have the sequence:

$$\cdots \to \mathbb{Z} \to \mathbb{Z}^k \oplus 0 \to H_1(X) \to \mathbb{Z} \to \mathbb{Z}^2$$

Similar to before, consider the map $H_1(U \cap V) \to H_1(U)$ i.e. $\mathbb{Z} \to \mathbb{Z}^k$. The generator of $H_1(U \cap V)$ maps to the boundary of U in the CW-complex picture, which contains a copy of each of the k circles twice each with the same signs. Thus, this maps:

$$1 \to (2, \cdots, 2)$$

Thus, the image of this map is isomorphic to $2\mathbb{Z}$. Thus, the kernel of the map $\mathbb{Z}^k \to H_1(X)$ is isomorphic to $2\mathbb{Z}$. Using the first isomorphism theorem, the image of the map is isomorphic to $\mathbb{Z}^k/2\mathbb{Z} \cong \mathbb{Z}^{k-1} \oplus \mathbb{Z}/2\mathbb{Z}$.
Moreover, as noted before, the map $H_1(U \cap V) \to H_1(U) \oplus H_1(V)$ is the diagonal inclusion map, which is injective, thus has 0 kernel. Thus, the image of $H_1(X) \to \mathbb{Z}$ is 0. Thus, the map $\mathbb{Z}^k \to H_1(X)$ is surjective. Thus, we have that $H_1(X) \cong \mathbb{Z}^{k-1} \oplus \mathbb{Z}/2\mathbb{Z}$.

For $k = 2$, we have that $H_2(U) = 0$ (part a), $H_2(V) = 0$ (contractible), $H_1(U \cap V) = \mathbb{Z}$, $H_1(U) \oplus H_1(V) = \mathbb{Z}^k$. Thus, we have the sequence:

$$\cdots \to 0 \to H_2(X) \to \mathbb{Z} \to \mathbb{Z}^k$$

By previous discussion, the map $\mathbb{Z} \to \mathbb{Z}^k$ is injective, thus has 0 kernel. Thus, the map $H_2(X) \to \mathbb{Z}$ is 0. However, this map is injective as the image of the previous map is trivial. Thus, we have that $H_2(X) = 0$.

For $k \geq 3$, we have that $H_k(U) = H_k(V) = 0$ and $H_{k-1}(U \cap V) = 0$ (as $k - 1 \geq 2$). Thus, the sequence reduces to:

$$\cdots \to 0 \to H_k(X) \to 0$$

Thus, $H_k(X) = 0$ for $k \geq 3$.

Overall, the homology groups are:

$$H_k(N_g) \cong \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z} & k = 1 \\ 0 & k = 2 \\ 0 & k \geq 3 \end{cases}$$

4. Let V be the tubular neighborhood of the knot K (homeomorphic to a circle) and $U = H_1(S^3 \setminus K)$. Then, $\operatorname{int}(U) \cup \operatorname{int}(V) = S^3$ and $U \cap V$ is contractible to T^2. Thus, we can use the Mayer-Vietoris sequence.

$$\cdots \to H_k(U \cap V) \to H_k(U) \oplus H_k(V) \to H_k(S^3) \to H_{k-1}(U \cap V) \to \cdots$$

For $k = 0$, we have that $H_0(V) = H_0(U \cap V) = H_0(S^3) = \mathbb{Z}$ and $H_1(S^3) = 0$. Thus, the sequence becomes:

$$0 \to \mathbb{Z} \to H_0(U) \oplus \mathbb{Z} \to \mathbb{Z} \to 0$$

The map $\mathbb{Z} \to H_0(U) \oplus \mathbb{Z}$ is injective and the map $H_0(U) \oplus \mathbb{Z} \to \mathbb{Z}$ is surjective. Thus, by rank-nullity considerations, we have $H_0(U) \oplus \mathbb{Z} \cong \mathbb{Z}^2$, thus $H_0(U) \cong \mathbb{Z}$.

For $k = 1$, we have that $H_2(S^3) = 0$, $H_1(U \cap V) = \mathbb{Z}^2$ (homeomorphic to a torus), $H_1(V) = \mathbb{Z}$ (a circle), and $H_1(S^3) = 0$. Thus, we have the sequence:

$$0 \to \mathbb{Z}^2 \to H_1(U) \oplus \mathbb{Z} \to 0$$

Thus, the map $\mathbb{Z}^2 \to H_1(U) \otimes \mathbb{Z}$ is both injective and surjective. Thus, $H_1(U) \cong \mathbb{Z}$.

For $k = 2$ and 3, we write down the sequence:

$$H_3(U \cap V) \to H_3(U) \oplus H_3(V) \to H_3(S^3) \to H_2(U \cap V) \to H_2(U) \oplus H_2(V) \to H_2(S^3)$$

Note that $H_3(U \cap V) = 0$ (torus), $H_3(V) = 0$ (a circle), $H_3(S^3) = \mathbb{Z}$, $H_2(U \cap V) = \mathbb{Z}$ (torus), $H_2(V) = 0$ (a circle), and $H_2(S^3) = 0$. Thus, we have the sequence:

$$0 \to H_3(U) \oplus 0 \to \mathbb{Z} \to \mathbb{Z} \to H_2(U) \oplus 0 \to 0$$

Consider carefully the map $\delta : H_3(S^3) \to H_2(U \cap V)$, i.e. $\delta : \mathbb{Z} \to \mathbb{Z}$. Let $[\alpha] \in H_3(S^3)$ be the generator. Then, we can subdivide S^3 such that $[\alpha] = [\beta + \gamma]$ where β and γ are entirely within U and V, respectively. Then, $\delta([\alpha]) = [\partial\beta] = [\partial\gamma]$ which is in $U \cap V$. If we triangulate S^3, the generator is the sum of the 2-simplices representing the triangles. We can choose a triangulation that includes a triangulation of the torus, by refining. Then, β and γ are then sum of the triangles in U and V. $\partial\beta$ then is the sum of all the triangles on the torus $U \cap V$. Thus, δ maps generator to generator and $\mathbb{Z} \to \mathbb{Z}$ is the identity.
Thus, the map $\mathbb{Z} \to H_2(U)$ is zero. Thus, $H_2(U)$ is 0 (as this map is surjective). Similarly, $H_3(U) \to \mathbb{Z}$ is 0, thus $H_3(U) \cong \mathbb{Z}$ (as this map is injective.)

For $k \geq 4$, we have that $H^k(U \cap V) = 0$ (torus), $H_k(V) = 0$ (a circle), and $H_k(S^3) = 0$. Thus, we have the sequence:

$$0 \to H_k(U) \to 0$$

Thus, $H_k(U) = 0$ for $k \geq 4$.

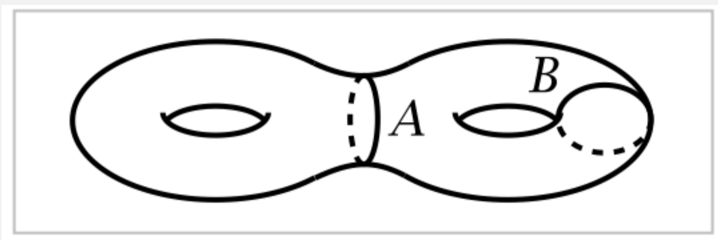
Thus, we have:

$$H_k(S^3 \setminus K) \cong \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z} & k = 1 \\ 0 & k \geq 2 \end{cases}$$

Problem 5.(2.1.17)

1. Compute the homology groups $H_n(X, A)$ when X is S^2 or $S^1 \times S^1$ and A is a finite set of points in X.

2. Compute the homology groups $H_n(X, A)$ and $H_n(X, B)$ for X a closed orientable surface of genus 2, A is separating circle, and B is a non-separating circle as depicted below. (*Note:* You can use that these are good pairs and hence there is an isomorphism with homology of the quotient away from degree 0.)



Solution.

1. Note that previous results:

$$H_k(S^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & k \neq 0, 2 \end{cases} \qquad\qquad H_k(S^1 \times S^1) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ \mathbb{Z}^2 & k = 1 \\ 0 & k \geq 2 \end{cases}$$

Moreover, since A is a set of n points, we have that:

$$H_k(A) = \begin{cases} \mathbb{Z}^n & k = 0 \\ 0 & k \geq 1 \end{cases}$$

as it has n connected components, and no non-boundary k-chains. Note the long exact sequence:

$$\cdots \to H_k(A) \xrightarrow{\iota_*} H_k(X) \to H_k(X, A) \to H_{k-1}(A) \xrightarrow{\iota_*} H_{k-1}(X) \to 0$$

For $k = 0$, we get:

$$\cdots \to H_0(A) \xrightarrow{\iota_*} H_0(X) \to H_0(X, A) \to 0$$

which gives:

$$\cdots \to \mathbb{Z}^n \xrightarrow{\iota_*} \mathbb{Z} \to H_0(X, A) \to 0$$

Then, note that the map $H_0(A) \xrightarrow{\iota_*} H_0(X)$ is surjective, as $H_0(X)$ is path-connected (every 0-chain on A maps to some 0-chain on X, and all 0-chains on X differ by boundaries). Thus, the kernel of $\mathbb{Z} \to H_0(X, A)$ is trivial. Thus, this map is injective. It is followed by a 0 map, and is thus also surjective. Thus, we have that $H_0(X, A) \cong \mathbb{Z}$.

For $k = 1$, we have that:

$$\cdots \to H_1(A) \xrightarrow{\iota_*} H_1(X) \to H_1(X, A) \to H_0(A) \to H_0(X)$$

which gives:

$$0 \xrightarrow{\iota_*} H_1(X) \to H_1(X, A) \to \mathbb{Z}^n \to \mathbb{Z}$$

Since $\mathbb{Z}^n \to \mathbb{Z}$ is surjective, by rank-nullity / first isomorphism theorem, we have that the kernel of $\mathbb{Z}^n \to \mathbb{Z}$ and hence the image of $H_1(X, A) \to \mathbb{Z}^n$ is isomorphic to $\mathbb{Z}^{n-1}$. Moreover, $H_1(X) \to H_1(X, A)$ is injective, thus its image is isomorphic to $H_1(X)$. Note that for the case of S^2, the image is 0 and for the case of $S^1 \times S^1$, the image is $\mathbb{Z}^2$. Thus, by rank-nullity, we again have:

$$H_1(X, A) \cong \begin{cases} \mathbb{Z}^{n-1} & X = S^2 \\ \mathbb{Z}^{n-1} \oplus \mathbb{Z}^2 = \mathbb{Z}^{n+1} & X = S^1 \times S^1 \end{cases}$$

For $k = 2$, we have that:

$$\cdots \to H_2(A) \xrightarrow{\iota_*} H_2(X) \to H_2(X, A) \to H_1(A)$$

which gives:

$$0 \xrightarrow{\iota_*} \mathbb{Z} \to H_2(X, A) \to 0$$

Thus, the map $\mathbb{Z} \to H_2(X, A)$ is injective and surjective. Thus, $H_2(X, A) \cong \mathbb{Z}$.

For $k \geq 3$, we have that both $H_k(X) = 0$ and $H_{k-1}(A) = 0$. Thus, we get that $H_k(X, A) = 0$ for $k \geq 3$.

Overall, we summarize the results:

$$H_k(S^2, A) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^{n-1} & k = 1 \\ \mathbb{Z} & k = 2 \\ 0 & k \geq 3 \end{cases} \qquad H_k(S^1 \times S^1, A) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^{n+1} & k = 1 \\ \mathbb{Z} & k = 2 \\ 0 & k \geq 3 \end{cases}$$

2. Note that since (X, A) and (X, B) are good pairs, we have that for $k > 0$:

$$H_k(X, A) \cong \tilde{H}_k(X/A) \qquad H_k(X, B) \cong \tilde{H}_k(X/B)$$

For the separating circle A, we have that $X/A = T^2 \vee T^2$. We can divide this into each open torus, U and V, with $U \cap V$ a point. The union is clearly the whole space. Thus, we can use the Mayer-Vietoris sequence.

$$H_k(U \cap V) \to H_k(U) \oplus H_k(V) \to H_k(X/A) \to H_{k-1}(U \cap V)$$

Note that for $k \geq 3$, we have that $H_k(U) = H_k(V) = 0$ and $H_{k-1}(U \cap V) = 0$. Thus, we have:

$$0 \to H_k(X/A) \to 0$$

Thus, $H_k(X/A) = 0$ for $k \geq 3$.

For $k = 2$, we have the following sequence:

$$H_2(U \cap V) \to H_2(U) \oplus H_2(V) \to H_2(X/A) \to H_1(U \cap V)$$

which gives:

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \to H_2(X/A) \to 0$$

Thus, $H_2(X/A) = \tilde{H}_2(X/A) \cong \mathbb{Z}^2$.

For $k = 1$, we have the following sequence:

$$H_1(U \cap V) \to H_1(U) \oplus H_1(V) \to H_1(X/A) \to H_0(U \cap V) \to H_0(U) \oplus H_0(V)$$

which gives:

$$0 \to \mathbb{Z}^2 \oplus \mathbb{Z}^2 \to H_1(X/A) \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$

as we've noted before, the final map $\mathbb{Z} \to \mathbb{Z}^2$ is the diagonal inclusion map $1 \to (1, 1)$ as U and V are both path-connected. Thus, the map is injective. Thus, the map $H_1(X/A) \to \mathbb{Z}$ is the 0 map. Moreover, the map $\mathbb{Z}^2 \oplus \mathbb{Z}^2 \to H_1(X/A)$ is injective, thus the kernel of $H_1(X/A) \to \mathbb{Z}$ is the image of $\mathbb{Z}^2 \oplus \mathbb{Z}^2 \to H_1(X/A)$ is isomorphic to Z^4. Thus, $H_1(X/A) = \tilde{H}_1(X/A) \cong \mathbb{Z}^4$.

Next, we consider the space X/B. This space is homotopic to $T^2 \vee S^1$. Let U be the torus and V be the circle. Then, similarly, $U \cap V$ is a point and the union of the interiors is the whole space. Thus, we can use the Mayer-Vietoris sequence.

$$H_k(U \cap V) \to H_k(U) \oplus H_k(V) \to H_k(X/B) \to H_{k-1}(U \cap V)$$

Note that for $k \geq 3$, we have that $H_k(U) = H_k(V) = 0$ and $H_{k-1}(U \cap V) = 0$. Thus, we have:

$$0 \to H_k(X/B) \to 0$$

Thus, $H_k(X/B) = 0$ for $k \geq 3$.

For $k = 2$, we have the following sequence:

$$H_2(U \cap V) \to H_2(U) \oplus H_2(V) \to H_2(X/B) \to H_1(U \cap V)$$

which gives:

$$0 \to \mathbb{Z} \oplus 0 \to H_2(X/B) \to 0$$

Thus, $H_2(X/B) = \tilde{H}_2(X/B) \cong \mathbb{Z}$.

For $k = 1$, we have the following sequence:

$$H_1(U \cap V) \to H_1(U) \oplus H_1(V) \to H_1(X/B) \to H_0(U \cap V) \to H_0(U) \oplus H_0(V)$$

which gives:

$$0 \to \mathbb{Z}^2 \oplus \mathbb{Z} \to H_1(X/B) \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$

as we've noted before, the final map $\mathbb{Z} \to \mathbb{Z}^2$ is the diagonal inclusion map $1 \to (1,1)$ as U and V are both path-connected. Thus, the map is injective. Thus, the map $H_1(X/B) \to \mathbb{Z}$ is the 0 map. Moreover, the map $\mathbb{Z}^2 \oplus \mathbb{Z} \to H_1(X/B)$ is injective, thus the kernel of $H_1(X/B) \to \mathbb{Z}$ is the image of $\mathbb{Z}^2 \oplus \mathbb{Z} \to H_1(X/B)$ is isomorphic to Z^3. Thus, $H_1(X/B) = \tilde{H}_1(X/B) \cong \mathbb{Z}^3$.

Finally, for $k = 0$, we note that $H_0(X, A) = H_0(X, B) = 0$ as X, A and B are all path-connected (Problem 3).

Overall, we summarize the results:

$$H_k(X, A) = \begin{cases} \mathbb{Z}^4 & k = 1 \\ \mathbb{Z}^2 & k = 2 \\ 0 & k \geq 3, k = 0 \end{cases} \qquad H_k(X, B) = \begin{cases} \mathbb{Z}^3 & k = 1 \\ \mathbb{Z} & k = 2 \\ 0 & k \geq 3, k = 0 \end{cases}$$