Math 113: Functional Analysis

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Problem 30. Suppose that X and Y are normed vector spaces.

1. Show that $\mathcal{L}(X,Y)$ is a normed vector space with respect to the the operator norm defined in lecture such that:

$$||T(x)|| \leq ||T||||x||$$

2. Show that if $S \in \mathcal{L}(Y,Z)$. Then,

$$||ST|| \leq ||S||||T||$$

3. Show that:

$$||T|| = \inf\{a \geq 0 : ||T(x)|| \leq a||x|| \quad \forall x \in X\}$$

Solution.

1. First, note that $\mathcal{L}(X,Y)$ is a vector space, through pointwise addition and scalar multiplication defined in Y, i.e. for $T, S \in \mathcal{L}(X,Y)$ and $\alpha \in \mathbb{F}$ we let:

$$(\alpha T + S)(x) := \alpha T(x) + S(x)$$

for all $x \in X$. Thus, we only need to show that the operator norm is a norm and satisfies the given property. Recall the definition of the operator norm for $T \in \mathcal{L}(X,Y)$:

$$||T|| = \sup_{||x|| \leq 1} ||T(x)||$$

Non-Neg. Let $T \in \mathcal{L}(X,Y)$ be arbitrary. Then, for all $x \in X$ with $||x|| \leq 1$, $||T(x)|| \geq 0$ by the non-negativity of the norm on Y. Thus, $||T|| \geq 0$.

Homogeneity. Let $T \in \mathcal{L}(X,Y)$ and $\alpha \in \mathbb{F}$. Then, for every $x \in X$, with $||x|| \leq 1$, we have that:

$$\begin{aligned} ||(\alpha T)(x)|| &= ||\alpha T(x)|| \\ &= |\alpha| \cdot ||T(x)|| \\ &\leq |\alpha| \cdot ||T|| \end{aligned}$$

using the homogeneity of the norm on Y. Thus,

$$||\alpha T|| \leq |\alpha| \cdot ||T(x)||$$

Similarly, we also have:

$$\begin{aligned} |\alpha| \cdot ||T(x)|| &= ||\alpha T(x)|| \\ &= ||(\alpha T)(x)|| \\ &\leq ||\alpha T|| \end{aligned}$$

Thus, we have:

$$|\alpha| \cdot ||T|| \leq ||\alpha T||$$

Thus, we have shown that:

$$||\alpha T|| = |\alpha| \cdot ||T||$$

△ ineq. Let $T, S \in \mathcal{L}(X,Y)$. For every $x \in X$, with $||x|| \leq 1$, we have:

$$\begin{aligned} ||(T+S)(x)|| &= ||T(x) + S(x)|| \\ &\leq ||T(x)|| + ||S(x)|| \\ &\leq ||T|| + ||S|| \end{aligned}$$

using the triangle inequality for the norm in Y. Thus,

$$||T+S|| \leq ||T|| + ||S||$$

Pos. Def. To prove this, we first will show that $||T(x)|| \leq ||T||||x||$ for all $x \in X$. For $x \in X$, by homogeneity (and non-negativity) of the norm on X:

$$\left|\left|\frac{1}{||x||}x\right|\right| = \frac{1}{||x||} \cdot ||x|| = 1$$

Thus, we note that:

$$\left|\left|T\left(\frac{1}{||x||}x\right)\right|\right| \leq ||T||$$

By the linearity of T and the homogeneity of the norm in Y, this implies:

$$\begin{aligned} \frac{1}{||x||}||T(x)|| &= \left|\left|\frac{1}{||x||}T(x)\right|\right| \\ &= \left|\left|T\left(\frac{1}{||x||}x\right)\right|\right| \\ &\leq ||T|| \end{aligned}$$

Thus,

$$||T(x)|| \leq ||T|| \cdot ||x||$$

Now, let $T \in \mathcal{L}(X, Y)$ be such that $||T|| = 0$. Then, for all $x \in X$, we have that:

$$\begin{aligned} ||T(x)|| &\leq ||T|| \cdot ||x|| \\ &= 0 \end{aligned}$$

However, by the non-negativity of the norm in Y, we must have that $||T(x)|| = 0$ for all $x \in X$. Then, by positive definiteness of the norm in Y, we have that $T(x) = 0$ for all $x \in X$. Thus, T is the zero map.
Conversely, if T is the zero map, then for all $x \in X$ with $||x|| \leq 1$, we have that:

$$||T|| = ||0 \cdot T|| = 0 \cdot ||T|| = 0$$

by homogeneity. Thus, the norm is positive definite.

Therefore, we have shown that the operator norm is a norm on $\mathcal{L}(X, Y)$ and satisfies for all $T \in \mathcal{L}(X, Y)$ and $x \in X$:

$$||T(x)|| \leq ||T|| \cdot ||x||$$

2. For any $x \in X$, with $||x|| = 1$, by applying the property from part 1 twice, we have that:

$$\begin{aligned} ||ST(x)|| &= ||S(T(x))|| \\ &\leq ||S|| \cdot ||T(x)|| \\ &\leq ||S|| \cdot ||T|| \cdot ||x|| \\ &= ||S|| \cdot ||T|| \end{aligned}$$

Thus,

$$||ST|| \leq ||S|| \cdot ||T||$$

3. Let $\alpha(T)$ be the defined infimum.

Since we have that $||T(x)|| \leq ||T|| \cdot ||x||$ for all $x \in X$, $||T||$ is in the set we are taking the infimum over. Thus, $\alpha(T) \leq ||T||$.

Moreover, by the definition of $\alpha(T)$, we have that for all $x \in X$ with $||x|| \leq 1$,

$$||T(x)|| \leq \alpha(T)||x|| = \alpha(T)$$

Thus, $\alpha(T)$ is an upperbound on $||T(x)||$ with $||x|| \leq 1$. Therefore, by the definition of the supremum, we have that:

$$||T|| \leq \alpha(T)$$

Thus, we have shown that:

$$||T|| = \alpha(T) = \inf\{a \geq 0 : ||T(x)|| \leq a||x|| \quad \forall x \in X\}$$

Problem 31. Suppose that X and Y are Banach spaces with $T \in \mathcal{L}(X, Y)$. Suppose that E is a closed proper subspace of X such that $E \subset \ker(T)$. Show that there is a unique operator $\overline{T} \in \mathcal{L}(X/E, Y)$ such that $\overline{T}(q(x)) = T(x)$ for all $x \in X$ where $q : X \to X/E$ is the quotient map. Moreover, $||\overline{T}|| = ||T||$.

Solution. We claim the map $\overline{T} : X/E \to Y$ given by:

$$\overline{T}([x]) = T(x)$$

satisfies the desired properties. Clearly, $\overline{T}(q(x)) = T(x)$ for all $x \in X$ by definition. We need to show that $\overline{T}$ is well-defined, linear, and satisfies $||\overline{T}|| = ||T||$ (hence is bounded/continous).

Let $x, y \in X$ be such that $[x] = [y]$. Then, note that $x - y \in E \subset \ker(T)$. Thus, $T(x) - T(y) = T(x - y) = 0$. Thus, $T(x) = T(y)$. Then, by the definition of $\overline{T}$, we have $\overline{T}([x]) = \overline{T}([y])$. Thus, $\overline{T}$ is well-defined.

Next, let $[x], [y] \in X/E$ and $\alpha \in \mathbb{F}$ be arbitrary. Then, noting the linearity of q and T, we have that:

$$\begin{aligned}
\overline{T}(\alpha[x] + [y]) &= \overline{T}([\alpha x + y]) \\
&= T(\alpha x + y) \\
&= \alpha T(x) + T(y) \\
&= \alpha \overline{T}([x]) + \overline{T}([y])
\end{aligned}$$

Thus, $\overline{T}$ is linear.

Finally, we need to show that $||\overline{T}|| = ||T||$. First, we show that $||q|| = 1$. We already showed that $||q|| \leq 1$. Next, let $x \in X$ such that $||x|| \leq 1$. Then, if $x \in E$, $q(x) = 0$. Thus, we consider $x \in X \setminus E$. Since q is bounded, for all $e \in E$:

$$\begin{aligned}
||q(x)||_{X/E} &= ||q(x + e)||_{X/E} \\
&\leq ||q|| \cdot ||x + e||_X
\end{aligned}$$

Thus,

$$||q(x)||_{X/E} \leq ||q|| \cdot \inf_{e \in E} ||x + e||_X = ||q|| \cdot ||q(x)||_{X/E}$$

Dividing through by $||q(x)|| > 0$ (as $x \notin E$ closed), we have:

$$||q|| \geq 1$$

Thus, $||q|| = 1$. Next, note that for all $[x] \in X/E$ and for all $e \in E \subset \ker(T)$,

$$\begin{aligned}
||\overline{T}([x])|| &= ||T(x)|| \\
&= ||T(x + e)|| \\
&\leq ||T|| \cdot ||x + e||_X
\end{aligned}$$

Thus, we have that:

$$||\overline{T}([x])|| \leq ||T|| \inf_{e \in E} ||x + e||_X = ||T|| \cdot ||[x]||_{X/E}$$

Thus, by the definiton of the operator norm in Part 3 of Problem 30, we note that since $||T||$ is a bound for $\overline{T}$, and $||\overline{T}||$ is the infimum of these bounds, $||\overline{T}|| \leq ||T||$. This also shows $\overline{T}$ is bounded, thus, $\overline{T} \in \mathcal{L}(X/E, Y)$. Finally, from Problem 30 Part 2, we also have:

$$||T|| \leq ||\overline{T}||||q|| = ||\overline{T}||$$

Thus, $||\overline{T}|| = ||T||$.

Problem 33. Let E and X be Banach spaces with E finite dimensional.

1. Show that every linear map $S : E \to X$ is bounded.
2. Show that a linear map $T : X \to E$ is bounded if and only if $\ker(T)$ is closed.

Solution.

1. Let $e_1, \ldots, e_n$ be a basis for E. Then $S(e_1), \ldots, S(e_n)$ span the image of S. Let $B = \max\{||S(e_i)|| : 1 \leq i \leq n\}$. Then, let $x = \sum_{i=1}^n a_i e_i \in E$ be arbitrary (where $a_i \in \mathbb{F}$). Then, using the triangle inequality,

$$\begin{aligned} ||S(x)|| &= \left|\left|S\left(\sum_{i=1}^n a_i e_i\right)\right|\right| \\ &= ||\sum_{i=1}^n a_i S(e_i)|| \\ &\leq \sum_{i=1}^n |a_i| \cdot ||S(e_i)|| \\ &\leq B \sum_{i=1}^n |a_i| \\ &= B \cdot ||x||_\infty \end{aligned}$$

where we are using the identification of E with $\mathbb{F}^n$ to define the $||\cdot||_\infty$ (i.e. its defined with respect to our picked basis). However, as E is finite dimensional, we use the fact that all norms are equivalent to obtain a constant $C > 0$ such that:

$$||x||_\infty < C||x||$$

Thus, we have that:

$$||S(x)|| \leq BC||x||$$

Thus, S is bounded.

2. Assume that T is bounded. Since X and E are Banach spaces (E is Banach since all finite dimensional spaces are Banach), we have that T is continuous. Thus, the preimage of a closed set is closed. Since E is a normed vector space, it is a metric space, and hence Hausdorff. Thus the singleton $\{0\}$ is closed in E. Thus, $T^1(\{0\}) = \ker(T)$ is closed in X.

Next, assume that $\ker(T)$ is closed. Then, the quotient space $X/\ker(T)$ is a Banach space with norm given by the quotient norm. Then using results from Problem 31, we have the map $\overline{T} : X/\ker(T) \to E$ given by:

$$\overline{T}([x]) = T(x)$$

is well-defined and linear. Moreover, if $\overline{T}$ is bounded, then so is T as $||\overline{T}|| = ||T||$. Thus, we need to show that $\overline{T}$ is bounded.

However, by the first isomorphism theorem for vector spaces, $X/\ker(T)$ is isomorphic to $\text{img}(T) \subset E$, which is finite dimensional. Thus, $\overline{T}$ is a map from a finite dimensional Banach space, and is thus bounded by part 1. Thus, since $||\overline{T}|| = ||T||$, we have that T is bounded.

Problem 34. Supposed that E and M are closed subspaces of a Banach space X. If E is finite dimensional, show that $E + M = \{x + y : x \in E, y \in M\}$ is closed.

Solution. Let $q : X \to X/M$ be the quotient map. Since M is closed, X/M is a Banach space. Since $\dim(q(E)) \leq \dim(E) < \infty$ (rank-nullity applied to $q|_E$) so $q(E)$ is a finite dimensional subspace of X/M, thus Banach, thus closed. Then, since q is continous, the pre-image of closed sets is closed. However, note that $q^{-1}(q(E)) = E + M$. Thus, $E + M$ is closed in X.

Problem 35. Suppose that X and Y are Banach spaces for $T \in \mathcal{L}(X, Y)$. Show that T is injective with closed range if and only if:

$$\inf\{||T(x)|| : ||x|| = 1\} > 0$$

Solution. Let $m := \inf\{||T(x)|| : ||x|| = 1\}$.

First, assume that T is injective with closed range. Then, $\mathrm{img}(T)$ is Banach as it is closed. Moreover, as T is injective, it is a bijection onto its image. Thus, T is a surjective continuous linear map between Banach spaces, and is thus open (Open Mapping Theorem). Since T is a continous open bijection, it is a homeomorphism and has a continous inverse $T^{-1} : \mathrm{img}(T) \to X$. Thus, T^{-1} is bounded. Hence, there exists a constant $C > 0$ such that $\forall x \in X$:

$$||x|| \leq C||T(x)||$$

Thus, $\forall x \in X$ with $||x|| = 1$, we rearrange to get:

$$||T(x)|| \geq \frac{1}{C}$$

Thus, $m \geq \frac{1}{C} > 0$.

Next, assume that $m > 0$. Then for all $x \in X$ with $||x|| = 1$, we have that:

$$||T(x)|| \geq m$$

Then, for all $x \in X$, since $\frac{x}{||x||}$ has norm 1, we have that (by homogeneity of the norm on Y and linearity of T):

$$\begin{aligned} \frac{1}{||x||} ||T(x)|| &= \left|\left| T\left(\frac{x}{||x||}\right) \right|\right| \\ &\geq m \end{aligned}$$

Thus,

$$||T(x)|| \geq m||x||$$

Now, if $T(x) = 0$, then $||T(x)|| = 0$. Then, since $m > 0$, the inequality above implies that $||x|| = 0$. Then, by positive definiteness of the norm, we have $x = 0$. Thus, T is injective.

As T is injective, it is a bijection onto its image. Thus, we can define $T^{-1} : \mathrm{img}(T) \to X$ (as a linear map). Then, for all $y \in \mathrm{img}(T)$, there exists $x \in X$ such that $T(x) = y$. Then, we have:

$$\begin{aligned} ||T^{-1}(y)|| = ||T^{-1}(T(x))|| &= ||x|| \\ &\leq \frac{1}{m}||T(x)|| = \frac{1}{m}||y|| \end{aligned}$$

Thus, T^{-1} is bounded.

Now, let $(y_n) \subset \text{img}(T)$ be a Cauchy sequence. Then, since T^{-1} is bounded, $(T^{-1}(y_n))$ is also a Cauchy sequence. To see this, let $\epsilon > 0$. Then, since (y_n) is Cauchy, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, we have that:

$$||y_m - y_n|| < m\epsilon$$

Then, we note that:

$$\begin{aligned} ||T^{-1}(y_m) - T^{-1}(y_n)|| &= ||T^{-1}(y_m - y_n)|| \\ &\leq \frac{1}{m}||y_m - y_n|| \\ &< \frac{1}{m} \cdot m\epsilon \\ &= \epsilon \end{aligned}$$

Thus, $(T^{-1}(y_n))$ is Cauchy. Since X is complete, $(T^{-1}(y_n)) \to x \in X$. Let $y = T(x) \in \text{img}(T)$. Then, since T is continous, we have that:

$$(y_n) \to y$$

Thus, $\text{img}(T)$ is complete (hence Banach). Thus, $\text{img}(T)$ is a closed subspace of Y. Thus, T is injective with closed range.

Problem 38. Let X be a normed vector space. A Banach space $\tilde{X}$ is called a completion of X is there is an isometric isomorphism $\iota : X \to \tilde{X}$ onto a dense subspace of $\tilde{X}$. Show that any two completions $(\tilde{X}_1, \iota_1)$ and $(\tilde{X}_2, \iota_2)$ are isometrically isomorphic by an isomorphism:

$$\Phi : \tilde{X}_1 \to \tilde{X}_2$$

such that $\Phi(\iota_1(x)) = \iota_2(x)$ for all $x \in X$.

Solution.

Lemma 1. *If $\phi : X \to Y$ is an isometric isomorphism of normed vector spaces, then $\phi^{-1} : Y \to X$ is also an isometric isomorphism.*

Proof. Since ϕ is an isomorphism, we know that ϕ^{-1} is a well-defined linear isomorphism. Thus, we only need to show that ϕ^{-1} is an isometry.

Let $y \in Y$ be arbitrary. Then, since ϕ is an isometry, we have that:

$$\begin{aligned} ||\phi^{-1}(y)||_X &= ||\phi(\phi^{-1}(y))||_Y \\ &= ||y||_Y \end{aligned}$$

Thus, we have shown that ϕ^{-1} is an isometry. □

Lemma 2. *Let X and Y be Banach spaces and D a dense subspace of X. If $T_0 \in \mathcal{L}(D, Y)$, then there exists a unique bounded linear operator $T \in \mathcal{L}(X, Y)$ such that $T(x) = T_0(x)$ for all $x \in D$.*

Proof. (This was optional question 32. Please move past this proof if we are allowed to use these without proof.)
Since D is a dense subspace of X, for all $x \in X$, there exists a Cauchy sequence $(x_n) \subset D$ such that $(x_n) \to x$ (we can construct this since D meets every open set around x, and we can pick points in $B_{1/n}(x) \cap D$, which is obviously Cauchy and converges to x). Then, $(T_0(x_n))$ is a Cauchy sequence in Y (since T is bounded). Thus, $T_0(x_n) \to y \in Y$. We can then define $T : X \to Y$ by:

$$T(x) = y$$

where y is the limit of $(T_0(x_n))$.

We check that this is well-defined. Let (x_n) and (x'_n) be two Cauchy sequences in D converging to $x \in X$. Then,

$$\begin{aligned} \lim_{n\to\infty} ||T(x_n) - T(x'_n)|| &= \lim_{n\to\infty} ||T(x_n - x'_n)|| \\ &\leq ||T|| \cdot \lim_{n\to\infty} ||x_n - x'_n|| \end{aligned}$$

Now, let $\epsilon > 0$. Then, since (x_n) is Cauchy, there exists $N_1 \in \mathbb{N}$ such that for all $m, n \geq N_1$, we have that:

$$||x_n - x_m|| < \frac{\epsilon}{2}$$

Taking the limit as $m \to \infty$, we have that:

$$||x_n - x|| < \frac{\epsilon}{2}$$

Similarly, there exists $N_2 \in \mathbb{N}$ such that $\forall n > N_2$:

$$||x'_n - x|| < \frac{\epsilon}{2}$$

Thus, letting $N > \max\{N_1, N_2\}$, we have that for all $n \geq N$:

$$\begin{aligned} ||(x_n - x'_n) - 0|| &\leq ||x_n - x|| + ||x - x'_n|| \\ &= ||x_n - x|| + ||x'_n - x|| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Thus, $\lim_{n\to\infty} ||x_n - x'_n|| = 0$. Thus, we have that:

$$\lim_{n\to\infty} ||T(x_n) - T(x'_n)|| \leq ||T|| \cdot \lim_{n\to\infty} ||x_n - x'_n|| = 0$$

Since $||\cdot||$ is non-negative, by the Squeeze Theorem, we have that $\lim_{n\to\infty} ||T(x_n) - T(x'_n)|| = 0$. Thus, the sequences converge to the same limit. Thus, T is well-defined.

Moreover, if $x_n \to x \in D$, then $T(x) = \lim_{n\to\infty} T_0(x_n) = T_0(x)$.

Next, note that T is linear as a linear combination of two Cauchy sequences converges to the the same linear combination of their limits (easy via triangle inequality and picking $\epsilon/2$ and $\epsilon/2\alpha$). Thus, if $(T_0(x_n)) \to y$ and $(T_0(x'_n)) \to y'$, then $(T_0(x_n + \alpha x'_n)) = (T_0(x_n) + \alpha T_0(x'_n)) \to y + \alpha y'$. Thus, $T(x + \alpha x') = T(x) + \alpha T(x')$.

Next, we need to show that T is bounded. If $(x_n) \subset D$ is Cauchy and converges to $x \in X$, then we have, by continuity of norm:

$$\begin{aligned} ||T(x)|| &= \left|\left| \lim_{n\to\infty} T_0(x_n) \right|\right| \\ &= \lim_{n\to\infty} ||T_0(x_n)|| \\ &\leq \lim_{n\to\infty} ||T_0|| \cdot ||x_n|| \\ &\leq ||T_0|| \cdot \lim_{n\to\infty} ||x_n|| \\ &\leq ||T_0|| \cdot ||x|| \end{aligned}$$

Thus, $T \in \mathcal{L}(X, Y)$.

Finally, to show that T is unique, let $T_1, T_2 \in \mathcal{L}(X, Y)$ with $T_1 = T_2$ on D. Then, for all $x \in X$, if $(x_n) \subset D \to x$, then, by continuity, we have that:

$$T_1(x) = \lim_{n\to\infty} T_1(x_n) = \lim_{n\to\infty} T_2(x_n) = T_2(x)$$

Thus, $T_1 = T_2$ on X. Hence, T is unique. $\square$

Now, for the main proof: Since ι_1 is an isometric isomorphism onto $\iota_1(X)$, then, $\iota_1^{-1} : \iota_1(X) \to X$ is also an isometric isomorphism. Thus, we can define the map $\Phi_0 : \iota_1(X) \to \tilde{X}_2$ given by:

$$\Phi_0(x) = \iota_2 \circ \iota_1^{-1}(x)$$

Since ι_2 is an isometric isomorphism, and a composition of isometric isomorphisms is an isometric isomorphism, we have that Φ_0 is an isometric isomorphism.

Since $\iota_1(X)$ is a dense subspace of $\tilde{X}_1$, and $\tilde{X}_2$ is Banach, by Lemma 2, we can extend Φ_0 uniquely to a bounded linear operator $\Phi : \tilde{X}_1 \to \tilde{X}_2$ such that $\Phi(x) = \Phi_0(x)$ for all $x \in \iota_1(X)$.

We need to show that Φ is an isometric isomorphism.

First, we show that Φ is an isometry. Let $x \in \tilde{X}_1$ be arbitrary. Then, there exists a Cauchy sequence $(x_n) \subset \iota_1(X)$ such that $(x_n) \to x$. Then, by the isometry of Φ_0 and continuity of the norm:

$$\begin{aligned}
||\Phi(x)|| &= \left|\left|\lim_{n\to\infty} \Phi(x_n)\right|\right| \\
&= \left|\left|\lim_{n\to\infty} \Phi_0(x_n)\right|\right| \\
&= \lim_{n\to\infty} ||\Phi_0(x_n)|| \\
&= \lim_{n\to\infty} ||x_n|| \\
&= \left|\left|\lim_{n\to\infty} x_n\right|\right| \\
&= ||x||
\end{aligned}$$

Next, we show that Φ is injective. Let $x \in \tilde{X}_1$ be such that $\Phi(x) = 0$. Then, take a Cauchy sequence $(x_n) \subset \iota_1(X)$ such that $(x_n) \to x$. Then, we have by continuity of the norm and isometry of Φ, we have:

$$||x|| = ||\Phi(x)|| = 0$$

Thus, by positive definiteness of the norm, we have that $x = 0$. Thus, Φ is injective.

Finally, we show that Φ is surjective. Let $y \in \tilde{X}_2$ be arbitrary. Then, since $\text{img}(\Phi) = \text{img}(\iota_2)$ is dense in $\tilde{X}_2$, there exists a Cauchy sequence $(y_n) \to y$ such that $(y_n) \subset \text{img}(\Phi_0)$. Then, since Φ_0 is an isometric isomorphism, it has an inverse Φ_0^{-1} that is an isometric isomorphism. Then, we have a Cauchy sequence $(\Phi_0^{-1}(y_n)) \subset \iota_1(X) = \text{img}(\Phi_0^{-1})$ converging to x. Since Φ is continous, we have that:

$$\begin{aligned}\Phi(x) &= \lim_{n\to\infty} \Phi(\Phi_0^{-1}(y_n)) \\ &= \lim_{n\to\infty} y_n = y\end{aligned}$$

as $\Phi = \Phi_0$ on $\iota_1(X)$. Thus, Φ is surjective.

Thus, we have shown that Φ is an isometric isomorphism.

Problem 39. Lets find a use for a genuine Minkowski functional. In this problem, we'll let $l^\infty_\mathbb{R}$ be the real Banach space of bounded sequences in $\mathbb{R}$. Define m on $l^\infty_\mathbb{R}$:

$$m(x) = \text{limsup}_n\, x_n$$

We clearly have $m(tx) = tm(x)$ if $t \geq 0$ and it is not hard to check that $m(x+y) \leq m(x) + m(y)$ for all $x, y \in l^\infty_\mathbb{R}$. We want to show that there are Banach limits or what I prefer to call a generalized limit on $l^\infty_\mathbb{R}$. This is we want to show that there is a functional $L \in {l^\infty_\mathbb{R}}^*$ such that:

$$L(S(x)) = L(x)$$

where $S \in \mathcal{L}(l^\infty_\mathbb{R})$ is given by $S(x)_n = x_{n+1}$ and such that $\liminf_n x_n \leq L(x) \leq \text{limsup}_n\, x_n$. (Hint provided).

Solution.

Lemma 3. *If x_n is a sequence in $\mathbb{R}$, and $A_n = \frac{1}{n}\sum_{i=1}^n x_i$ converge, then $\lim_n A_n \leq \text{limsup}_n\, x_n$ and $\lim_n A_n \geq \liminf_n x_n$.*

Proof. Let $k \in \mathbb{N}$ be arbitrary. Then, $\forall n \geq k$ note that:

$$\begin{aligned} A_n &= \frac{1}{n}\sum_{i=1}^{k} x_k + \frac{1}{n}\sum_{i=k+1}^{n} x_i \\ &\leq \frac{1}{n}\sum_{i=1}^{k} x_k + \sup_{i\geq k} x_i \end{aligned}$$

Taking the limsup with respect to n on both sides, we have that:

$$\begin{aligned} \text{limsup}_n\, A_n &\leq \text{limsup}_n \left(\frac{1}{n}\sum_{i=1}^{k} x_i + \sup_{i\geq k} x_i\right) \\ &= 0 + \sup_{i\geq k} x_i \end{aligned}$$

Since this is true for arbitrary k, we have that:

$$\text{limsup}_n\, A_n \leq \lim_{k\to\infty} \sup_{i\geq k} x_i = \text{limsup}_k\, x_k$$

However, since A_n converges, we have that $\text{limsup}_n\, A_n = \lim_n A_n$. Thus, we have shown that:

$$\lim_n A_n \leq \text{limsup}_n\, x_n$$

The proof for $\liminf$ follows similarly by flipping the inequalities. $\square$

Define $m_n(x) = \frac{1}{n}(x_1 + \ldots + x_n)$. Then define $Y = \{x \in l^\infty_\mathbb{R} : \lim_n m_n(x) \text{ exists } \}$. Then, Y is a subspace as for any $x, y \in Y$ and $\alpha \in \mathbb{R}$, if $m_n(x) \to a$ and $m_n(y) \to b$, then:

$$\lim_{n\to\infty} m_n(x + \alpha y) = \lim_{n\to\infty} \frac{1}{n}\left(\sum_{i=1}^n x_i + \alpha y_i\right) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n x_i + \alpha \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n y_i = a + \alpha b$$

Thus, $x + \alpha y \in Y$ and Y is a subspace (this also shows m_n is linear).

Define $L_0 : Y \to \mathbb{R}$ by:

$$L_0(x) = \lim_{n\to\infty} m_n(x)$$

Then, for $y \in Y$, we use the lemma to show that:

$$L_0(x) \le m(x)$$

where $m(x) = \text{limsup}_n\, x_n$.

Thus, by the Basic Extension Lemma, we can extend L_0 to a linear functional $L : l^\infty_\mathbb{R} \to \mathbb{R}$ such that $L(x) = L_0(x)$ for all $x \in Y$ and $L(x) \le m(x)$ for all $x \in l^\infty_\mathbb{R}$. Moreover, for all $x \in l^\infty_\mathbb{R}$, we have that:

$$\begin{aligned} L(x) &= -L(-x) \\ &\ge -m(-x) \\ &= \text{limsup}_n(-x_n) \\ &= -\liminf_n x_n \end{aligned}$$

Thus,

$$\liminf_n x_n \le L(x) \le \liminf_n x_n$$

Next, we show that $x - S(x) \in Y$. To see this, we unpack the definition to note that:

$$(x - S(x))_n = x_n - x_{n+1}$$

Thus, we get that:

$$\begin{aligned} m_n(x - S(x)) &= \frac{1}{n}((x_1 - x_2) + (x_2 - x_3) + \cdots + (x_n - x_{n+1})) \\ &= \frac{1}{n}(x_1 - x_{n+1}) \end{aligned}$$

Taking the limit as $n \to \infty$, we have that:

$$\lim_{n\to\infty} m_n(x - S(x)) = 0$$

Thus, $x - S(x) \in Y$.

Thus, we have that:

$$L(x - S(x)) = 0 \implies L(x) = L(S(x))$$

by linearity of L. Thus, $L(x)$ is a generalized limit of x. Moreover, if x converges, then $\text{limsup}_n\, x_n = \liminf_n x_n$, thus, by the squeeze theorem, $L(x)$ equals this limit.

Problem 40. Prove the following Lemma from lecture. Let X be a complex vector space. Every real linear functional of X is the real part of a complex linear functional on X. In fact, if $\phi = \Re(\psi)$ then $\psi(x) = \phi(x) - i\phi(ix)$.

Solution. Let ϕ be a real linear functional on X. We prove existence and uniqueness separately.

Existence: Let $\psi : X \to \mathbb{C}$ be given by:

$$\psi(x) = \phi(x) - i\phi(x)$$

Clearly, $\phi(x) = \Re(\psi)$. We need to show that ψ is complex linear.

Let $x, y \in X$ and $\alpha \in \mathbb{C}$ be arbitrary. Then, we have that:

$$\begin{aligned}\psi(\alpha x + y) &= \phi(\alpha x + y) - i\phi(\alpha x + y)\\ &= \alpha\phi(x) - i\alpha\phi(x) + \phi(y) - i\phi(y)\\ &= \alpha(\phi(x) - i\phi(x)) + (\phi(y) - i\phi(y))\\ &= \alpha\psi(x) + \psi(y)\end{aligned}$$

Thus, ψ is complex linear.

Uniqueness: Let ψ be a linear functional such that $\phi = \Re(\psi)$. Then, we need to show that $\psi(x) = \phi(x) - i\phi(ix)$.
Let $g(x) = \Im(\psi)$. Then, since ψ is complex linear, we have that:

$$\begin{aligned}\phi(ix) + ig(ix) &= \psi(ix)\\ &= i\psi(x)\\ &= i\phi(x) - g(x)\end{aligned}$$

Then, comparing real parts, we have that:

$$g(x) = -\phi(ix)$$

Thus,

$$\psi(x) = \phi(x) - i\phi(ix)$$

Thus, every real linear function ϕ is the real part of a unique complex linear functional $\psi = \phi(x) - i\phi(ix)$ on X.

Problem 41. Suppose that X is a normed vector space such that X^* is seperable. Show that X is seperable. (Hint provided).

Solution. Since X^* is separable, there exists a countable dense subset $\{f_n\}_{n\in\mathbb{N}} \subset X^*$. Then, for each $n \in \mathbb{N}$, note that:

$$||f_n|| = \inf\{a : |f_n(x)| \leq a||x|| \quad \forall x \in X\}$$

Noting that norms are non-negative and homogeneous, and f_n is linear, we divide through by $||x||$ to get:

$$||f_n|| = \inf\{a : |f_n(x)| \leq a \quad \forall x \in X, ||x|| = 1\}$$

Thus, for $\frac{1}{2}||f_n|| > 0$, we can find x_n, with $||x_n|| = 1$ such that:

$$|f_n(x'_n)| \geq ||f_n|| - \frac{1}{2}||f_n|| = \frac{1}{2}||f_n||$$

Define $S = \text{span}\{x_n : n \in \mathbb{N}\}$. Clearly, S is a subspace. We claim that S is dense in X.

For contradiction, assume that S is not dense in X. Then, there exists a (non-empty) open set that S does not meet, thus $\overline{S} \neq X$ (we use the definition of the closure of A to contain all points in X such that any open set around that point meets A). Thus, $\overline{S}$ is a proper closed subspace of X.

Then, by a corollary of the Hahn-Banach Theorem, we can find a functional $f \in X^*$ such that $f(x) = 0$ for all $x \in \overline{S}$ and $||f|| = 1$. We show that this contradicts the denseness of $\{f_n\}_{n\in\mathbb{N}}$.

For all $n \in \mathbb{N}$, we have the following cases:

- Assume $||f_n|| \geq \frac{1}{2}$. Then, for x_n, with $||x_n|| = 1$, we have:

$$|f_n(x_n) - f(x_n)| = |f_n(x_n)| \geq \frac{1}{2}||f_n|| \geq \frac{1}{4}$$

  Thus, $||f_n - f|| \geq 1/4$ is bounded away from 0.

- Assume $||f_n|| < \frac{1}{2}$. Then,

$$\begin{aligned} ||f|| &= ||f - f_n + f_n|| \\ &\leq ||f - f_n|| + ||f_n|| \end{aligned}$$

  Thus,

$$||f - f_n|| \geq ||f|| - ||f_n|| \geq 1 - \frac{1}{2} = \frac{1}{2}$$

  Thus, $||f_n - f|| \geq 1/2$ is bounded away from 0.

Thus, there exists an open neighborhood of f (or radius $< \frac{1}{4}$) that does not meet $\{f_n\}_{n\in\mathbb{N}}$, which is a contradiction since this set is dense in X^*.

Thus, $\operatorname{span}\{x_n : n \in \mathbb{N}\}$ is dense in X. Moreover, since $\operatorname{span}_{\mathbb{Q}}\{x_n : n \in \mathbb{N}\}$ is a countable dense subset of $\operatorname{span}\{x_n : n \in \mathbb{N}\}$, X is separable (we made a remark about this in class).