

Math 113: Functional Analysis

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Defn. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of (X, ρ) . We say that $d > 0$ is a Lebesgue number for the cover if given any d -ball $B_d(x_0)$ with $x_0 \in X$, there exists $a_0 \in A$ such that $B_d(x_0) \subseteq U_{a_0}$.

Ex $X = \mathbb{R}$. $U_1 = (-\infty, 1)$, $U_2 = (0, 2)$, and $U_3 = (1, \infty)$. Here $d = 1/2$ is a LN for $\{U_1, U_2, U_3\}$. This is clear if $x_0 \in (1/2, 3/2)$.

Ex. (Hwk.) Given $x \in (0, 1)$, $\exists \delta_x > 0$ such that:

$$y \in B_{\delta_x}(x) = \{y \in (0, 1) : |y - x| \leq \delta_x\}$$

$$\implies |1/x - 1/y| < 1$$

Then,

$$(0, 1) = \bigcup_{x \in (0, 1)} B_{\delta_x}(x)$$

has no Lebesgue number.

Lemma (Lebesgue Covering Lemma) Every open cover of a compact metric space has a Lebesgue number.

Proof. Pictures. Apr 9.

Thm. Suppose (X, ρ) is compact, and $F : (X, \rho) \rightarrow (Y, \sigma)$ is continuous. Then F is uniformly continuous.

Defn. Let (X, ρ) be a metric space and $C(X)$ the \mathbb{C} -vector space of continuous functions on X . We say $\mathcal{J} \subset C$ is equicontinuous at $x \in X$ if $\forall \epsilon > 0, \exists \delta > 0$ such that:

$$\forall F \in \mathcal{J}, F(B_\delta(x)) \subseteq B_\epsilon(F(x))$$

We say \mathcal{J} is equicontinuous on X if $\forall x \in X$, \mathcal{J} is equicontinuous at x .

Ex. Let $X = [0, 1] \subset \mathbb{R}$. Let $F_n(x) = x^n \forall n \geq 1$. Let:

$$\mathcal{J} = \{F_n : n \in \mathbb{N}\}$$

Let $x_n = \frac{1}{2}^{1/n}$. Then, x_n arrow up to 1. Then,

$$|F_n(x_n) - F_n(1)| = |1/2 - 1| = 1/2$$

Thus, \mathcal{J} is not equicontinuous at 1.

Ex. (Hwk) Show that \mathcal{J} is equicontinuous on $[0, 1)$.

Defn. Let (F_n) be a sequence of (\mathbb{C} -valued) functions on X . Then, (F_n) is uniformly bounded if $\exists M > 0$ such that $\forall n \geq 1, \forall x \in X$:

$$|F_n(x)| < M$$

We say that (F_n) is pointwise bounded if $\forall x \in X, \exists M_x > 0$ such that:

$$|F_n(x)| < M_x$$

Defn. A metric space (top. space) is seperable if there is a countable dense subset $D \subset X$.

Ex. Since $\mathbb{Q}^n \subset \mathbb{R}^n$ is dense, $(\mathbb{R}^n, \|\cdot\|_p)$ is separable.

Lemma. (Arzelà-Ascoli) Let (X, ρ) is a seperable metric space and that (F_n) is pointwise bounded and equicontinuous in $C(X)$. Then, there is subsequence (F_{n_k}) such that:

$$\lim_{x \rightarrow \infty} F_{n_k}(x)$$

exists $\forall x \in X$.

Yap. Given a sequence (x_n) , we get subsequence by finding $n_k \in \mathbb{N}$ such that $n_{k+1} > n_k$ and $(x_{n_k})_{k=1}^{\infty} \rightarrow x$ is a sequence.

A subsubsequence is determined by finding $n_{k_1} < n_{k_2} < \dots$ and then we write:

$$(x_{n_{k_j}})_{j=1}^{\infty}$$

A subsequence is determined by an infinite subset $S_1 = \{n_1 \leq n_2 \leq \dots\} \subset \mathbb{N}$. A subsubsequence is determined by an infinite subset $S_2 \subset S_1$,

$$S_2 = \{n_{k_1} < n_{k_2} < \dots\} \subset S_1$$

Now, we write:

$$\lim_{n \in S_1} x_n = a \text{ instead of } \lim_{k \rightarrow \infty} x_{n_k}$$

Note that $\lim_{n \in S_1} x_n = a$ if:

$$\forall \epsilon > 0 \exists N : n \geq N, n \in S' \implies |x_n - a| < \epsilon$$

Rmk. Suppose $S_1 \subset \mathbb{N}$ determines a subsequence as above. Suppose $S' \subset \mathbb{N}$ is infinite and:

$$\{n \in S' : n \notin S_1\}$$

is finite. The $\lim_{n \in S_1} x_n = a$ then $\lim_{n \in S'} x_n = a$ as well.

Proof of the AA Lemma: Pictures. Apr 9 and 10.

Rmk. If X is compact, then $C(x) = C_b(X)$ is a complete metric space with respect to the uniform norm $\|\cdot\|_\infty$.

Thm. (Arzelà-Ascoli) Let (X, ρ) be a compact metric space and $(F_n) \subset C(X)$ be a sequence of functions that are point-wise bounded and equicontinuous. Then (F_n) has a subsequence converging uniformly to some function $F \in C(X)$. Proof. Pictures.

Lemma. Suppose X is compact and that $\mathcal{J} \subset C(X)$ is equicontinuous on X . Then, \mathcal{J} is uniformly equicontinuous on X , in that for all $\epsilon > 0 \exists \delta > 0$ such that for all $x, y \in X$ and all $F \in \mathcal{J}$,

$$\rho(x, y) < \delta \implies |F(x) - F(y)| < \epsilon$$

Rewriting,

$$F(B_\delta(x)) \subseteq B_\epsilon(F(x))$$

Proof left as homework.

Corr. Let X be a compact metric space. Let $\mathcal{J} \subset C(X)$ be a closed subset such that \mathcal{J} is equicontinuous and pointwise bounded. Then \mathcal{J} is compact and uniformly bounded.

Thm. Suppose X is a compact metric space. Then $\mathcal{J} \subset C(X)$ is compact if and only if \mathcal{J} is closed, uniformly bounded, and equicontinuous on X . Proof. Pictures.