Math 121: Hodge Theory

Sair Shaikh

May 27, 2025

Problem 1.

(a) Check the equivalence between the two definitions of the Hodge structure of weight k given in class.

(b) Check the a morphism Hodge structures is strict for the Hodge filtration.

(c) Show that the kernel, cokernel, and image of a morphism of Hodge structures are Hodge structures.

(d) Let $\phi : X \to Y$ a surjective holomorphic map of complex compact manifolds such that X is kählerian. Show that ϕ^* is injective.

Solution. (a) Assume first that we have a Hodge structure of weight k V such that:

$$V_{\mathbb{C}} = \bigoplus V^{p,q} \qquad\qquad V^{p,q} = \overline{V}^{q,p}$$

Define the Hodge filtration by:

$$F^p V_{\mathbb{C}} = \bigoplus_{i \geq p} V^{i,k-i}$$

Then note that if $a > b$, we have that:

$$\begin{aligned} F^a V_{\mathbb{C}} &= \bigoplus_{i \geq a} V^{i,k-i} \\ &\subseteq \bigoplus_{i \geq b} V^{i,k-i} \\ &= F^b V_{\mathbb{C}} \end{aligned}$$

Moreover, we have that:

$$\begin{aligned}
F^pV_{\mathbb{C}} \cap \overline{F^{k-p+1}V_{\mathbb{C}}} &= \bigoplus_{i\geq p} V^{i,k-i} \cap \bigoplus_{j\geq k-p+1} \overline{V^{j,k-j}} \\
&= \bigoplus_{i\geq p} V^{i,k-i} \cap \bigoplus_{j\geq k-p+1} V^{k-j,j} \\
&= \bigoplus_{i\geq p} V^{i,k-i} \cap \bigoplus_{j'\leq p-1} V^{j',k-j'}
\end{aligned}$$

Problem 3.(The Hodge Decomposition for Curves) Let X be a compact connected complex curve. We have the differential:

$$d : \mathcal{O} \to \Omega_X$$

between the sheaf of homolormphic functions and the sheaf of holomorphic differentials.

(a) Show that d is surjective with kernel equal to the constant sheaf $\mathbb{C}$. Hence, we have an exact sequence:

$$0 \to \mathbb{C} \to \mathcal{O} \to\to \Omega_X \to 0$$

(b) Deduce from Serre duality that $H^1(X, \Omega_X) \cong \mathbb{C}$. Deduce from Poincare duality that $H^2(X, \mathbb{C}) = \mathbb{C}$.

(c) Show that (6.15) induces a short exact sequence:

$$0 \to H^0(X, \Omega_X) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X) \to 0$$

(d) Show that the map which to a holomorphic form α associates the class of $\overline{\alpha}$ in $H^1(X, \mathcal{O})$ is injective.

(e) Deduce from Serre duality that it is also surjective and that we have the decomposiiton:

$$H^1(X, \mathbb{C}) = H^0(X, \Omega_X) \oplus \overline{H^0(X, \Omega_X)}$$

with

$$\overline{H^0(X, \Omega_X)} \cong H^1(X, \mathcal{O})$$

Solution.

Problem 4.

Solution.