Math 113: Functional Analysis

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Problem 54. Suppose that X is a reflexive Banach space. Show that the unit ball $B = \{x \in X : ||x|| \leq 1\}$ is weakly compact. (Hint provided).

Solution. Let X^* be the dual of X. By Alaoglu's theorem, we know that $B^{**} = \{f \in X^{**} : ||f|| \leq 1\}$ is compact in $\sigma(X^{**}, X^*)$. Thus, it suffices to show that B under the weak topology on X is homeomorphic to B^{**} in the weak-* topology on X^{**} under the natural map ι.

Since X is reflexive, we already know that $\iota : X \to X^{**}$ is a bijection. Recall the definition of an basic open set in the weak* topology on X^{**}, for $f_0 \in X^{**}$, $\phi_1, \cdots, \phi_k \in X^*$ and $\epsilon > 0$:

$$U = \{f \in X^{**} : |f(\phi_i) - f_0(\phi_i)| < \epsilon, 1 \leq i \leq k\}$$

Noting that ι is bijective, we may rewrite this as:

$$\begin{aligned} U &= \{\iota(x) \in X^{**} : |\iota(x)(\phi_i) - \iota(x_0)(\phi_i)| < \epsilon, 1 \leq i \leq k\} \\ &= \{\iota(x) \in X^{**} : |\phi_i(x) - \phi_i(x_0)| < \epsilon, 1 \leq i \leq k\} \end{aligned}$$

The preimage of U under ι then is precisely:

$$\iota^{-1}(U) = \{x \in X : |\phi_i(x) - \phi_i(x_0)| < \epsilon, 1 \leq i \leq k\}$$

which is a basic open set in the weak topology on X. Thus, ι is continous.

Similarly, let V be a basic open set in the weak topology on X, written, for some $x_0 \in X$, $\phi_1, \cdots \phi_k \in X^*$ and $\epsilon > 0$ as:

$$V = \{x \in X : |\phi_i(x) - \phi_i(x_0)| < \epsilon, 1 \leq i \leq k\}$$

Then, we have that:

$$\begin{aligned} \iota(V) &= \{\iota(x) \in X^{**} : |(\phi_i(x)) - \phi_i(x_0)| < \epsilon, 1 \leq i \leq k\} \\ &= \{\iota(x) \in X^{**} : |\iota(x)(\phi_i) - \iota(x_0)(\phi_i)| < \epsilon, 1 \leq i \leq k\} \end{aligned}$$

which is a basic open set in the weak* topology on X^{**}. Thus, ι^{-1} is also continous.

Thus, ι is a homeomorphism from X with the weak topology to X^{**} with the weak* topology. Finally, note that as ι is isometric,

$$\iota(B) = \{\iota(x) : ||\iota(x)|| \leq 1\} = B^{**}$$

Thus, as B^{**} is compact in the weak* topology, we have that B is weakly compact in the weak topology on X.

Problem 59. Let E be a nonempty subset of a Hilbert space H. Let Y be a subspace spanned by E. Then $E^{\perp\perp}$ is the closure of Y in H.

Solution. We show both inclusions separately.

Let $y \in Y$ be arbitrary. Then, we have that $y = \sum_{i=1}^{n} a_i e_i$ for some $e_i \in E$ and $a_i \in \mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). Then, for all $w \in E^{\perp}$, we have that:

$$\begin{aligned}(y \mid w) &= \left(\sum_{i=1}^{n} a_i e_i \mid w\right) \\ &= \sum_{i=1}^{n} a_i (e_i \mid w) \\ &= 0\end{aligned}$$

Thus, $y \in E^{\perp\perp}$. Therefore, $Y \subseteq E^{\perp\perp}$ (as y was arbitrary). As noted in class, since the inner-product is continous, $E^{\perp\perp}$ is closed. Thus, we have that $\overline{Y} \subseteq E^{\perp\perp}$.

Next, let $z \in E^{\perp\perp}$. Since $\overline{Y}$ is closed, we have that $H = \overline{Y} \oplus \overline{Y}^{\perp}$. Thus, we can write $z = y + y'$ where $y \in \overline{Y}$ and $y' \in \overline{Y}^{\perp}$. It suffices to show $y' = 0$ and thus, $z = y \in \overline{Y}$, implying $E^{\perp\perp} \subseteq \overline{Y}$ (as z was arbitrary).

Let $w \in \overline{Y}^{\perp}$. Then, as $E \subset \overline{Y}$, we have that:

$$(w \mid e) = 0 \qquad \forall e \in E$$

Thus, we have $w \in E^{\perp}$, thus, $\overline{Y}^{\perp} \subseteq E^{\perp}$. In particular, $y' \in E^{\perp}$. Moreover, $(y \mid y') = 0$ as $y \in \overline{Y}$ and $y' \in \overline{Y}^{\perp}$. Since $z \in E^{\perp\perp}$, we have:

$$\begin{aligned}0 &= (z \mid y') \\ &= (y \mid y') + (y' \mid y') \\ &= (y' \mid y')\end{aligned}$$

Thus, $y' = 0$ by positive definiteness of the inner product. Therefore, $z = y \in \overline{Y}$. Thus, we have that $E^{\perp\perp} \subseteq \overline{Y}$.

Therefore, we have that $E^{\perp\perp} = \overline{Y}$.

Problem 60. Let $X = l^2$. Show that the sequence $\{e_n\}$ of standard basis vectors converges weakly to 0.

Solution. Let $\phi \in (l^2)^*$ and $\epsilon > 0$. We need to show that there exists an N such that for all $n \geq N$, $e_n \in U := \{x \in l^2 : |\phi(x)| < \epsilon\}$. Since $(l^2) \cong (l^2)^*$ (isometric isomorphism) via the map $y \to \Phi_y$ defined by:

$$\Phi_y(x) = \sum_{n=1}^{\infty} x_n y_n$$

Thus, there exists a $y \in l^2$ such that $\phi = \Phi_y$. Then, we have:

$$\phi(e_n) = y_n$$

Then, since $y \in l^2$, we have that $\sum_{n=1}^{\infty} |y_n|^2 < \infty$. Thus, there exists an N, such that $\sum_{n=N+1}^{\infty} |y_n|^2 < \epsilon^2$. Thus, for all $n \geq N$, we have that:

$$|\phi(e_n)| = |y_n| < \epsilon$$

Thus, $e_n \in U$ for all $n \geq N$. Therefore, $\{e_n\}$ converges weakly to 0.

Problem 61. Let H be a hilbert space. If $x, y \in H$, define $\Theta_{x,y} : H \to H$ by:

$$\Theta_{x,y}(z) = (z \mid y)x$$

Compute the norm of $\Theta_{x,y}$ and its adjoint $\Theta_{x,y}^*$.

Solution. First, note that if $x = 0$ or $y = 0$, then $\Theta_{x,y}(z) = (z \mid y)x = 0$ for all $z \in H$. Thus, $||\Theta_{x,y}|| = 0$. Thus, assume $x \neq 0$ and $y \neq 0$. For $z \in H$ by homogeneity of the norm and Cauchy-Schwarz we have:

$$\begin{aligned} ||\Theta_{x,y}(z)|| &= ||(z \mid y)x|| \\ &= |(z \mid y)| \, ||x|| \\ &\leq ||z|| \, ||y|| \, ||x|| \end{aligned}$$

Thus, $||\Theta_{x,y}|| \leq ||y|| \, ||x||$. Let $z = \frac{y}{||y||}$. Then, $||z|| = 1$. Moreover, we have:

$$\begin{aligned} \left|\left|\Theta_{x,y}\left(\frac{y}{||y||}\right)\right|\right| &= \left|\left(\frac{y}{||y||} \mid y\right)\right| ||x|| \\ &= \frac{1}{||y||} \, ||y||^2 \, ||x|| \\ &= ||y|| \, ||x|| \end{aligned}$$

Thus, the bound is achieved. Therefore, $||\Theta_{x,y}|| = ||y|| \, ||x||$. This also captures the $x = 0$ or $y = 0$ case.

Next, let $z, w \in H$. Then, we have that:

$$\begin{aligned} (\Theta_{x,y} z \mid w) &= ((z \mid y)x \mid w) \\ &= (z \mid y)(x \mid w) \\ &= (z \mid y)\overline{(w \mid x)} \\ &= (z \mid (w \mid x)y) \end{aligned}$$

Thus, we can define $\Theta_{x,y}^*(w) = (w \mid x)y$. Notice that:

$$\Theta_{x,y}^* = \Theta_{y,x}$$

Thus, by the previous argument, we have that:

$$\left|\left|\Theta_{x,y}^*\right|\right| = ||\Theta_{y,x}|| = ||y|| \, ||x||$$

Problem 64. Let $P \in \mathcal{L}(H)$ be the orthogonal projection onto a nonzero subspace W. Show that $P = P^* = P^2$ and that $||P|| = 1$. Conversely, show that if $P \in \mathcal{L}(H)$ and $P = P^* = P^2$, then P is an orthogonal projection onto its range.

Solution. We assume W is closed as the definition of P requires it. Let P^* be the adjoint of P. Note that $x - Px \in W^\perp$ for all $x \in H$. Let $h, k \in H$. Note that:

$$\begin{aligned}(Ph \mid k) &= (Ph \mid Pk + (k - Pk)) \\ &= (Ph \mid Pk) + (Ph \mid k - Pk) \\ &= (Ph \mid Pk) + 0 \\ &= (Ph \mid Pk) + (h - Ph \mid Pk) \\ &= (Ph + (h - Ph) \mid Pk) \\ &= (h \mid Pk)\end{aligned}$$

Thus, P is self-adjoint. Moreover, note that:

$$\begin{aligned}(P^2 h \mid k) &= (Ph \mid Pk) \\ &= (Ph \mid Pk) + (Ph \mid k - Pk) \\ &= (Ph \mid k)\end{aligned}$$

for all $h, k \in H$. Thus, $P^2 = P$.

First, note that by the Pythagorean theorem, for all $h \in H$,

$$\begin{aligned}||Ph||^2 &\leq ||Ph||^2 + ||h - Ph||^2 \\ &= ||h||^2\end{aligned}$$

Thus, $||Ph|| \leq ||h||$. Therefore, $||P|| \leq 1$.

Next, pick $h \in W$ with $||h|| = 1$ (we can do this as W is a subspace and we can divide any non-zero vector by its norm to get norm 1). Then, $Ph = h$, thus $||Ph|| = ||h||$. Thus, $||P|| \geq 1$ (by the sup definition of the operator norm). Therefore, $||P|| = 1$.

Conversely, let $P \in \mathcal{L}(H)$ such that $P = P^* = P^2$. We need to show that P is an orthogonal projection onto $V := \mathrm{img}(P)$. Thus, we need to show that V is closed and $H = V \oplus V^\perp$.

Let $x \in H$ and $Py \in V$ be arbitrary. Then note:

$$\begin{aligned}(x - Px \mid Py) &= (x \mid Py) - (Px \mid Py) \\ &= (x \mid Py) - (x \mid P^* Py) \\ &= (x \mid Py) - (x \mid P^2 y) \\ &= (x \mid Py) - (x \mid Py) \\ &= 0\end{aligned}$$

Thus, as Py was arbitrary, $x - Px \in V^\perp$. Therefore, we have that $H = V + V^\perp$. Next, let $v \in V \cap V^\perp$. Then, we have $(v \cap v) = 0$ thus $v = 0$. Thus, $V \cap V^\perp = \{0\}$. Therefore, we have that $H = V \oplus V^\perp$.

Then, note that $V^\perp$ is closed as the inner product is continous (as we noted in class). Then, we note that $H = V^\perp \oplus V^{\perp\perp}$ by the orthgonal projections theorem we proved in class. Moreover, $V^{\perp\perp} = \overline{V}$ (Problem 59). Thus, we have:

$$H = V \oplus V^\perp = \overline{V} \oplus V^\perp$$

Then, let $x \in \overline{V} \setminus V$. Then, as $H = V \oplus V^\perp$, we must have $x \in V^\perp$. However, $\overline{V} \cap V^\perp = \{0\}$, thus $x = 0$. But $0 \in V$ as V is a subspace. Thus, we have a contradiction, and $\overline{V} \subseteq V$ is empty. Thus, $V = \overline{V}$, i.e. V is closed.

Thus, we have shown that P is an orthogonal projection onto its range $V = \mathrm{img}(P)$.

Problem 65.(Dini's Theorem) Suppose that X is a compact metric space and that $C(X)$ is the Banach space of real-valued functions on X. Show that if $(f_n) \subset C(X)$ is such that there is a $f \in C(X)$ such that $f_n(X) \nearrow f(x)$ for all $x \in X$, then $f_n \to f$ in $C(X)$. Equivalently, show that $f_n \to f$ uniformly on X. (Hint provided).

Solution. Let $(f_n) \subset C(X)$ be such that $f_n(x) \nearrow f(x)$ for all $x \in X$. We need to show that $f_n \to f$ uniformly on X.

Let $\epsilon > 0$ be arbitrary. Let $E_n = \{x \in X : |f(x) - f_n(x)| < \epsilon\}$. Then, as $f_n(x) \leq f_{n+1}(x)$, we have that $|f(x) - f_{n+1}(x)| \leq |f(x) - f_n(x)|$ for all $x \in X$ for all n. Thus, we have that $E_n \subseteq E_{n+1}$ for all n. Moreover, for each $x \in X$, as $f_n(x) \to f(x)$, there exists N_x such that for all $n \geq N_x$,

$$|f(x) - f_n(x)| < \epsilon$$

Thus, $x \in E_n$ for all $n \geq N_x$. Therefore, we have that $\{E_n\}$ is a cover for X.

Moreover, since f and f_n are continous, so is $f - f_n$ for all n. As $\{c \in \mathbb{R} : |c| < \epsilon\}$ is open, we have that E_n is the continous preimage of an open set, thus is open. Thus, $\{E_n\}$ is an open cover for X.

Since X is compact, there exista finite subcover $\{E_{n_1}, \ldots, E_{n_k}\}$ such that:

$$X = \bigcup_{i=1}^{k} E_{n_i}$$

Let $N = \max\{n_1, \ldots, n_k\}$. Then, as noted previously, $E_{n_i} \subseteq E_N$ for all i. Thus for any $x \in X$, there exists an i such that $x \in E_{n_i} \subseteq E_N$. Thus, $X = E_N$.

Moreover, we also note that the containments $E_N \subseteq E_{N+1} \subseteq \cdots$ imply that $E_m = X$ for all $m \geq N$. Thus, we have that for $m \geq N$, for all $x \in X$,:

$$x \in E_m \iff |f(x) - f_m(x)| < \epsilon$$

Thus, $(f_n) \to f$ uniformly on X.

Problem 66. A linear map $V : H \to H$ is called an isometry if $||V(x)|| = ||x||$ for all $x \in H$. Show that the following are equivalent:

1. V is an isometry.
2. $(V(x) \mid V(y)) = (x \mid y)$ for all $x, y \in H$.
3. $V^*V = I$.

Solution. We show (1) $\implies$ (2), (2) $\implies$ (3), and (3) $\implies$ (1).

((1) $\implies$ (2)). Let $x, y \in H$ be arbitrary. Then, by the polarization identity (for $\mathbb{F} = \mathbb{C}$), we have:

$$\begin{aligned}
(Vx \mid Vy) &= \frac{1}{4}\left(||Vx + Vy||^2 - ||Vx - Vy||^2 + i\,||Vx + iVy||^2 - i\,||Vx - iVy||^2\right) \\
&= \frac{1}{4}\left(||V(x+y)||^2 - ||V(x-y)||^2 + i\,||V(x+iy)||^2 - i\,||V(x-iy)||^2\right) \\
&= \frac{1}{4}\left(||x+y||^2 - ||x-y||^2 + i\,||x+iy||^2 - i\,||x-iy||^2\right) \\
&= (x \mid y)
\end{aligned}$$

and for $\mathbb{F} = \mathbb{R}$, we have:

$$\begin{aligned}
(Vx \mid Vy) &= \frac{1}{4}\left(||Vx + Vy||^2 - ||Vx - Vy||^2\right) \\
&= \frac{1}{4}\left(||V(x+y)||^2 - ||V(x-y)||^2\right) \\
&= \frac{1}{4}\left(||x+y||^2 - ||x-y||^2\right) \\
&= (x \mid y)
\end{aligned}$$

Thus, we have that $(Vx \mid Vy) = (x \mid y)$ for all $x, y \in H$.

((2) $\implies$ (3)). Let $y \in H$ be arbitrary. Then, for all $x \in H$, we have:

$$\begin{aligned}
(x \mid y) &= (Vx \mid Vy) \\
&= (x \mid V^*Vy)
\end{aligned}$$

Thus, $(x \mid (y - V^*Vy)) = 0$ for all $x \in H$. Letting $x = y - V^*Vy$, we have that $y - V^*Vy = 0$ by the positive definiteness of the inner product. Thus, $V^*Vy = y$. Since y was arbitrary, $V^*V = I$.

((3) $\implies$ (1)). Let $x \in H$ be arbitrary. Then, we have:

$$\begin{aligned}
||Vx||^2 &= (Vx \mid Vx) \\
&= (x \mid V^*Vx) \\
&= (x \mid Ix) \\
&= (x \mid x) \\
&= ||x||^2
\end{aligned}$$

Thus, $||Vx|| = ||x||$ for all $x \in H$. Therefore, V is an isometry.

Thus, we have shown that the three statements are equivalent.

Problem 67. A surjective isometry $U : H \to H$ is called a unitary. Show that the following are equivalent for $U \in \mathcal{H}$.

1. U is a unitary.
2. U is invertible with $U^{-1} = U^*$.
3. If $\{e_n\}$ an orthonormal basis for H, then $\{U(e_n)\}$ is an orthonormal basis for H.

(Remark: (c) implies (a) is not true unless U is both linear and bounded.)

Solution. We show (1) $\implies$ (2), (2) $\implies$ (3), and (3) $\implies$ (1).

((1) $\implies$ (2)). Let U be a unitary. Then, as U is an isometry, we have that:

$$||Ux|| = 0 \iff ||x|| = 0 \iff x = 0$$

Thus, U is injective. Since U is surjective, we have that U is a bijection. By a direct corollary of the Open Mapping theorem, we have that U is invertible (proved in class). Then, as $U^*U = I$ (Problem 66), we have that $U^* = U^{-1}$ by the uniqueness of the inverse.

((2) $\implies$ (3)). Let $\{e_n\}$ be an orthonormal basis for H. Let $h \in H$ be such that:

$$(h \mid U(e_n)) = 0$$

for all n. It suffices to show that this implies $h = 0$ by one of the alternative characterizations of an orthonormal basis (proved in class).

As U is inveritble, hence surjective, there exists $h' \in H$ such that $U(h') = h$. Moreover, as $U^* = U^{-1}$ implies $U^*U = I$, we use the results of Problem 66 to get:

$$(U(h') \mid U(e_n)) = (h' \mid e_n) = 0$$

for all n. Then, as $\{e_n\}$ is an orthonormal basis for H, we have that $h' = 0$. Then, $h = U(h') = 0$. Thus, $\{U(e_n)\}$ is an orthonormal basis.

((3) $\implies$ (1)). Assume that if $\{e_n\}$ is an orthonormal basis for H, then $\{U(e_n)\}$ is an orthonormal basis for H. Since $\{Ue_n\}$ is an orthonormal basis, we can write $x \in H$ as:

$$x = \sum_{n=1}^{\infty} (x \mid Ue_n) Ue_n$$

Then, as U is linear and continous, we can write:

$$\begin{aligned} x &= \sum_{n=1}^{\infty}(x \mid Ue_n)Ue_n \\ &= \lim_{N\to\infty}\sum_{n=1}^{N}(x \mid Ue_n)Ue_n \\ &= U\left(\lim_{N\to\infty}\sum_{n=1}^{N}(x \mid Ue_n)e_n\right) \\ &= U\left(\sum_{n=1}^{\infty}(x \mid Ue_n)e_n\right) \end{aligned}$$

Thus, U is surjective. Now, let $x, y \in H$ be arbitrary such that:

$$x = \sum_{n=1}^{\infty}(x \mid e_n)e_n \qquad y = \sum_{n=1}^{\infty}(y \mid e_n)e_n$$

Then, we have that:

$$Ux = \sum_{n=1}^{\infty}(x \mid e_n)Ue_n \qquad Uy = \sum_{n=1}^{\infty}(y \mid e_n)Ue_n$$

using the linearity and boundedness of U (as above). Thus, we have:

$$(Ux \mid Ue_n) = (x \mid e_n)$$

and similarly for Uy. Then, using alternative characterizations of an orthnormal basis, we have:

$$\begin{aligned} (Ux \mid Uy) &= \sum_{n=1}^{\infty}(Ux \mid Ue_n)(Ue_n \mid Uy) \\ &= \sum_{n=1}^{\infty}(x \mid e_n)(e_n \mid y) \\ &= (x \mid y) \end{aligned}$$

Thus, by Problem 66, we have that U is an isometry. Thus, U is a unitary.

Thus, we have shown that the three statements are equivalent.