Math 74: Algebraic Topology

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Problem 1.(2.3.1)
If $T_n(X, A)$ denotes the torsion subgroup of $H_n(X, A)$, show that the functors $(X, A) \mapsto T_n(X, A)$ with the obvious induced homomorphisms $T_n(X, A) \to T_n(Y, B)$ and boundary maps $T_n(X, A) \to T_{n-1}(A)$ do not satisfy a homology theory even if excluding the dimension axiom. Do the same for the 'mod-torsion' functor $MT_n(X, A) = H_n(X, A)/T_n(X, A)$.

Solution. Let $X = \mathbb{RP}^2$ and A be a circle in X. The long exact sequence in homology gives us:

$$\cdots \to H_2(X, A) \to H_1(A) \to H_1(X) \to H_1(X, A) \to H_0(A) \to H_0(X) \to \cdots$$

Then, note that we have $H_1(X) = \mathbb{Z}/2\mathbb{Z}$, $H_1(A) = \mathbb{Z}$ and $H_0(X) = H_0(A) = \mathbb{Z}$.

$$\cdots \to H_2(X, A) \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to H_1(X, A) \to \mathbb{Z} \to \mathbb{Z} \to \cdots$$

The generator of $H_1(A)$ maps to a boundary in $H_1(X)$, thus, the first map is 0. Thus, the second map is injective. Moroever, the last map is induced by the inclusion of A into X, both of which are path-connected, thus the last $H_0(A) \to H_0(X)$ is an isomorphism. Thus, the image of $H_1(X, A) \to H_0(A)$ is trivial, i.e. the map is 0. Thus, $H_1(X, A) = 0$. Overall, we have:

$$\to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \to \cdots$$

Applying the torsion functors, we get:

$$T_1(A) = 0 \to T_1(X) = \mathbb{Z}/2\mathbb{Z} \to T_1(X, A) = 0 \to \cdots$$

which is not exact. Thus, the torsion functor does not satisfy the exactness axiom.

Problem 2.(2.3.5, with $G = \mathbb{Z}$) Regarding a cochain $\varphi \in C^1(X)$ as a function on paths in X to $\mathbb{Z}$, show that if φ is a cocycle, then

1. $\varphi(f \cdot g) = \varphi(f) + \varphi(g)$,
2. φ takes the value 0 on constant paths,
3. $\varphi(f) = \varphi(g)$ if $f \simeq_p g$, and
4. φ is a coboundary if and only if $\varphi(f)$ depends only on the endpoints of f for all paths f in X.

Solution.

Problem 3. Verify the remark in Hatcher after exercise 2.3.5: If X is path-connected, the previous problem together with the universal coefficient theorem induces an isomorphism $H^1(X) \cong \mathrm{Hom}(\pi_1(X), \mathbb{Z})$.

Solution.