Math 113: Functional Analysis

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Problem 42. Let $\mathfrak{c}$ be the subspace of l^∞ of sequences $x = (x_n)$ such that $\lim_n x_n$ exists and let $\mathfrak{c}_0$ be the subspace of $\mathfrak{c}$ for which the limit is 0.

- If $y \in l^1$, then let ϕ_y be the functional on $\mathfrak{c}_0$ given by:

$$\phi_y(x) = \sum_{n=1}^{\infty} x_n y_n$$

  Show that $y \to \phi_y$ is an isometric isomorphism of l^1 onto $\mathfrak{c}_0^*$.

- Describe the dual of $\mathfrak{c}$.

- Is either $\mathfrak{c}_0$ or $\mathfrak{c}$ reflexive?

Solution.

1. Let $\Phi : l^1 \to \mathfrak{c}_0^*$ be the map given by $\Phi(y) = \phi_y$. Since the map is given to us, we assume it is well-defined (it is also immediate via Holder's inequality with $p = 1$ and $q = \infty$). We need to show that Φ is an isometric isomorphism, thus we need to show it is linear, isometric and surjective.

First, to show that Φ is linear, let $y, z \in l^1$ and $\alpha \in \mathbb{F}$. Then, we have for all $x \in \mathfrak{c}_0$:

$$\begin{aligned}
\Phi(\alpha y + y)(x) &= \sum_{n=1}^{\infty} x_n(\alpha y + z)_n \\
&= \sum_{n=1}^{\infty} x_n(\alpha y_n + z_n) \\
&= \alpha \sum_{n=1}^{\infty} x_n y_n + \sum_{n=1}^{\infty} x_n z_n \\
&= \alpha\Phi(y)(x) + \Phi(z)(x) \\
&= (\alpha\Phi(y) + \Phi(z))(x)
\end{aligned}$$

Thus, Φ is linear. Next, we show that Φ is isometric. Let $y \in l^1$ be arbitrary. Then, for any $x \in \mathfrak{c}_0$, we have:

$$\begin{aligned}
|\Phi(y)(x)| &= \left| \sum_{n=1}^{\infty} x_n y_n \right| \\
&\leq \sum_{n=1}^{\infty} |x_n||y_n| \\
&\leq \sum_{n=1}^{\infty} ||x||_{\infty} |y_n| \\
&= ||x||_{\infty} \sum_{n=1}^{\infty} |y_n| \\
&= ||y||_1 ||x||_{\infty}
\end{aligned}$$

Thus, $||\Phi(y)|| \leq ||y||_1$. To show the other direction, recall the definition of the operator norm:

$$||\Phi(y)|| = \sup_{||x||_{\infty} \leq 1} |\Phi(y)(x)|$$

for $x \in \mathfrak{c}_0$. Thus, to show that $||\Phi(y)|| \geq ||y||_1$, for $\epsilon > 0$, we need to find an $x \in \mathfrak{c}_0$ such that $||x||_{\infty} \leq 1$ and $|\Phi(y)(x)| \geq ||y||_1 - \epsilon$.

Note that we have:

$$\sum_{n=1}^{\infty} |y_n| = \lim_{N \to \infty} \sum_{n=1}^{N} |y_n| = ||y||_1$$

Thus, there exists an N such that:

$$||y||_1 - \sum_{n=1}^{N} |y_n| < \epsilon$$

Next, define $x \in l^\infty$ as follows:

$$x_n = \begin{cases} \operatorname{sgn}(y_n) & \text{if } n \leq N \\ 0 & \text{otherwise} \end{cases}$$

where sgn returns the sign of a real number, and 0 for 0.

Clearly, as $x_n \in \{0, 1, -1\}$, for all $n \in \mathbb{N}$, we have that $||x||_\infty \leq 1$. Moreover, as x_n is zero after a finite number of terms, we have that $\lim_n x_n = 0$. Thus, $x \in \mathfrak{c}_0$. Finally, we calculate:

$$\begin{aligned}
|\Phi(y)(x)| &= \left| \sum_{n=1}^{\infty} x_n y_n \right| \\
&= \left| \sum_{n=1}^{N} \operatorname{sgn}(y_n) y_n + \sum_{n=N+1}^{\infty} (0) y_n \right| \\
&= \left| \sum_{n=1}^{N} |y_n| \right| \\
&= \sum_{n=1}^{N} |y_n| \\
&\geq ||y||_1 - \epsilon
\end{aligned}$$

Thus, we have that $||\Phi(y)|| \geq ||y||_1$. Thus, we conclude that $||\Phi(y)|| = ||y||_1$ and Φ is isometric.

Since Φ is isometric map of normed vector spaces, it is also injective (i.e. the only things mapped to 0 must have norm 0, which is just 0). Thus, we only need to show that Φ is surjective.

Let $\phi \in \mathfrak{c}^*$ be arbitrary. Define the sequence y as:

$$y_n = \phi(e_n)$$

where e_n is the sequence is 1 at the nth index and 0 elsewhere (note $e_n \in \mathfrak{c}_0$ as all but one entry are 0). We claim that $y \in l^1$. To see this, note that:

$$\begin{aligned}
\sum_{n=1}^{N} |y_n| &= \sum_{n=1}^{N} |\phi(e_n)| \\
&= \sum_{n=1}^{N} \phi(e_n) \operatorname{sgn}(\phi(e_n)) \\
&= \phi(x^{(N)})
\end{aligned}$$

where:

$$x^{(N)} = \begin{cases} \operatorname{sgn}(\phi(e_n)) & \text{if } n \leq N \\ 0 & \text{otherwise} \end{cases}$$

Note that for all N, $x^{(N)} \in \mathfrak{c}_0$ as it is 0 after a finite number of terms. Moreover, $\left|\left|x^{(N)}\right|\right|_\infty = 1$. Thus, as ϕ is bounded on $\mathfrak{c}_0$, we have that:

$$\begin{aligned} \sum_{n=1}^{N} |y_n| &= \phi(x^{(N)}) \\ &\leq ||\phi|| \cdot \left|\left|x^{(N)}\right|\right|_\infty \\ &= ||\phi|| \end{aligned}$$

Then taking the limit as $N \to \infty$, we have that:

$$\begin{aligned} ||y||_1 &= \lim_{N\to\infty} \sum_{n=1}^{N} |y_n| \\ &\leq \lim_{N\to\infty} ||\phi|| \\ &= ||\phi|| \end{aligned}$$

Thus, $y \in l^1$. Finally, we calculate $\Phi(y)$. For all $x \in \mathfrak{c}_0$, we have:

$$\begin{aligned} \Phi(y)(x) &= \sum_{n=1}^{\infty} x_n y_n \\ &= \sum_{n=1}^{\infty} x_n \phi(e_n) \\ &= \sum_{n=1}^{\infty} \phi(x_n e_n) \\ &= \phi\left(\sum_{n=1}^{\infty} x_n e_n\right) \\ &= \phi(x) \end{aligned}$$

Thus, we have shown that $\Phi(y) = \phi$. Thus, Φ is surjective.

Thus, we have shown that Φ is an isometric isomorphism of l^1 onto $\mathfrak{c}_0^*$.

2. We claim that the dual of $\mathfrak{c}$ is also l^1. Note that if $x \in \mathfrak{c}$ and $b = \lim_n x_n$, if we write $\mathbf{1}$ for the constant sequence of 1s, we have that:

$$\lim_n (x - b\mathbf{1})_n = \lim_n (x_n - b) = 0$$

Thus, $x - b\mathbf{1} \in \mathfrak{c}_0$. Thus, for each $x \in \mathfrak{c}$, there exists an $x' \in \mathfrak{c}_0$, such that $x_m = (\lim_n x_n) + x'_m$ for all $m \in \mathbb{N}$. Thus, every bounded linear functional on $\mathfrak{c}$ can be written, for $x \in \mathfrak{c}$, as:

$$\phi(x) = (\lim_n x_n)\phi(\mathbf{1}) + \phi(x')$$

by linearity. Thus, each map ϕ is characterized by its values on $\mathfrak{c}_0$ and $\phi(\mathbf{1})$ (which may or may not be independent of each other).

Next, note that $S(l^1) \cong l^1$, where S is the shift by one operator defined in Problem 39. Clearly, $S(l^1) \subset l^1$ as removing the first element keeps a finite l^1 norm finite. Moreover, every sequence in l^1 can be obtained by shifting a sequence in l^1 (say, take the original sequence and append a 0 in the first position). Thus, we can write:

$$l^1 \cong \mathbb{F} \oplus l^1$$

by the map $x \to (x_1, S(x))$. If we define the norm on $\mathbb{F} \oplus l^1$ as:

$$||(\alpha, x)|| = |\alpha| + ||x||_1$$

this mapping is clearly isometric (it agrees with the l^1 norm on the domain by definition). In this formulation, we can construct an map from $\Phi : \mathbb{F} \otimes l^1$ to $\mathfrak{c}$, for all $(\alpha, y) \in \mathbb{F} \otimes l^1$ by:

$$\Phi(\alpha, y)(x) = \alpha \lim_n x_n + \sum_{n=1}^{\infty} x_n y_n$$

This is clearly surjective, by our characterization of all bounded linear functionals on $\mathfrak{c}$ earlier. It is also easily seen to be linear. We show it is isometric as:

$$\begin{aligned} |\Phi(\alpha, y)(x)| &= \left| \alpha \lim_n x_n + \sum_{n=1}^{\infty} x_n y_n \right| \\ &\leq |\alpha| |\lim_n x_n| + ||x||_\infty \left| \sum_{n=1}^{\infty} y_n \right| \\ &\leq (|\alpha| + ||y||_1) \cdot ||x||_\infty \end{aligned}$$

noting that $|\lim_n x_n| \leq ||x||_\infty$. Thus, we have:

$$||\Phi(\alpha, y)|| \leq |\alpha| + ||y||_1$$

To show that this bound is achievable, we can reuse the same proof as before.

Let $(\alpha, y) \in \mathbb{F} \oplus l^1$ be arbitrary. Note that we have:

$$\sum_{n=1}^{\infty} |y_n| = \lim_{N \to \infty} \sum_{n=1}^{N} |y_n| = ||y||_1$$

Thus, there exists an N such that:

$$\sum_{n=N+1}^{\infty} |y_n| = ||y||_1 - \sum_{n=1}^{N} |y_n| < \epsilon$$

Next, define $x \in l^\infty$ as follows:

$$x_n = \begin{cases} \operatorname{sgn}(y_n) & \text{if } n \leq N \\ \operatorname{sgn}(\alpha) & \text{otherwise} \end{cases}$$

where sgn returns the sign of a real number, and 0 for 0.

Clearly, as $x_n \in \{0, 1, -1\}$, for all $n \in \mathbb{N}$, we have that $||x||_\infty \leq 1$. Moreover, as x_n is constant after a finite number of terms, we have that $\lim_n x_n = \operatorname{sgn}(\alpha)$. Thus, $x \in \mathfrak{c}$. Finally, we calculate:

$$\begin{aligned} |\Phi(\alpha, y)(x)| &= \left| \alpha \lim_n x_n + \sum_{n=1}^{\infty} x_n y_n \right| \\ &= \left| \alpha \operatorname{sgn}(\alpha) + \sum_{n=1}^{N} \operatorname{sgn}(y_n) y_n + \operatorname{sgn}(\alpha) \sum_{n=N+1}^{\infty} y_n \right| \end{aligned}$$

Next, note that:

$$\operatorname{sgn}(\alpha) \sum_{n=N+1}^{\infty} y_n \leq \sum_{n=N+1}^{\infty} |y_n| < \epsilon$$

Thus, we have:

$$\begin{aligned} |\Phi(\alpha, y)(x)| &\geq \left| \alpha \operatorname{sgn}(\alpha) + \sum_{n=1}^{N} \operatorname{sgn}(y_n) y_n \right| - \epsilon \\ &= |\alpha| + \sum_{n=1}^{N} |y_n| - \epsilon \\ &= ||(\alpha, y)|| - \epsilon \end{aligned}$$

Thus, we have that Φ is isometric. Thus, we have shown isometric isomorphisms:

$$l^1 \cong \mathbb{F} \oplus l^1 \cong \mathfrak{c}^*$$

Since the composition of isometric isomorphisms is an isometric isomorphism, we have that $\mathfrak{c}^* \cong l^1$.

3. We claim that neither $\mathfrak{c}$ nor $\mathfrak{c}_0$ are reflexive. Note that we showed that the dual of both of these spaces is isometrically isomorphic to l^1. However, we know that the dual of l^1 is isometrically isomorphic to l^∞. Thus, we have that the double dual of these spaces is isometrically isomorphic to l^∞. However, these subspaces of l^∞ are not isometrically isomorphic to l^∞ as l^∞ is not separable, but these subspaces are separable as their dual, l^1 is separable (Problem 41).

Problem 43. Show that X is reflexive if and only if X^* is.

Solution. Assume X is reflexive. Then X is isometrically isomorphic to X^{**} via the map:

$$\begin{aligned}\iota : X &\to X^{**}\\ \iota(x)(f) &= f(x)\end{aligned}$$

for $x \in X$ and $f \in X^*$. To show that X^* is reflexive, we need to show that the map given by:

$$\begin{aligned}\lambda : X^* &\to X^{***}\\ \lambda(f)(y) &= y(f)\end{aligned}$$

for $f \in X^*$ and $y \in X^{**}$ is onto.

Thus, let $F \in X^{***}$. Since X is reflexive, every element of X^{**} can be written as $\iota(x)$ for a unique x in X. Then, we can define a well-defined functional $f \in X^*$ by:

$$f(x) = F(\iota(x))$$

i.e. $f = F \circ \iota$. This is linear and bounded as its a composition of linear bounded functionals. Then, note that for any $\iota(x) \in X^{**}$.

$$\begin{aligned}\lambda(f)(\iota(x)) &= \iota(x)(f)\\ &= f(x)\\ &= F(\iota(x))\end{aligned}$$

Thus, $F = \lambda(l)$. Since F was arbitrary, we have shown that λ is onto and thus X^* is reflexive.

Now, assume that X^* is reflexive. Then, by the previous argument, we have that X^{**} is reflexive. Let ι and λ be as before. Assume for the sake of contradiction that ι is not onto.

Since X is Banach and ι is an isometric injection, we have that $\iota(X) \subset X^{**}$ is a closed proper subspace. Then by the 2nd corollary to the Hahn-Banach theorem, we have a non-zero functional $F \in X^{***}$ such that $F(\iota(x)) = 0$ for all $x \in X$. Since λ is onto, there exists a $f \in X^*$ such that $\lambda(f) = F$. Then, following the unpacking of the definitions, we get, for all $x \in X$:

$$\begin{aligned}0 &= F(\iota(x))\\ &= \lambda(f)(\iota(x))\\ &= \iota(x)(f)\\ &= f(x)\end{aligned}$$

Thus, $f \equiv 0$, which is a contradiction. Thus, X is reflexive.

> **Problem 44**. Let $\beta \subset \mathcal{P}(X)$ be a cover of X. Show that β is a basis for $\tau(\beta)$ if and only if given U and V in β and $x \in U \cap V$ there is a $W \in \beta$ such that $x \in W \subset U \cap V$.

Solution. Assume β is a basis for $\tau(\beta)$. Then, if U and V are in β, they are in $\tau(\beta)$. Since $\tau(\beta)$ is a topology, $U \cap V$ is in $\tau(\beta)$. As β is a basis for $\tau(\beta)$, for any $x \in U \cap V$, there exists a $W \in \beta$ such that $x \in W \subset U \cap V$. Thus, the condition holds.

Now, assume that the condition holds. Define τ' to be the collection of arbitrary unions of elements of β. We claim that $\tau' = \tau(\beta)$ and β is a basis for τ'. Clearly, as $\beta \subset \tau(\beta)$, $\tau(\beta)$ must contain arbitrary unions of elements of β as it is a topology. Thus, we have that $\tau' \subset \tau(\beta)$.

To show the other direction, we need to show that τ' is a topology that contains β, since then $\tau(\beta) \subset \tau'$ by definition. Thus, we check the axioms:

1. Since β is a cover, $X = \bigcup_{U\in\beta} U$. Thus, X is in τ'. $\emptyset$ is the empty union of elements of β, so it is in τ' as well.

2. τ' is closed under arbitrary unions by definition, as arbitary unions of arbitrary unions of elements of β are still arbitrary unions of elements of β.

3. Let U and V be in τ'. Then, by definition, $U = \bigcup_{i\in I} U_i$ and $V = \bigcup_{j\in J} V_j$ for some index sects I, J and with $U_i, V_j \in \beta$. Then, we have:

$$\begin{aligned} U \cap V &= \left(\bigcup_{i\in I} U_i\right) \cap \left(\bigcup_{j\in J} V_j\right) \\ &= \bigcup_{i\in I}\bigcup_{j\in J}(U_i \cap V_j) \end{aligned}$$

   i.e. something is in $U \cap V$ iff it is in some U_i and some V_j, if and only if it is in some $U_i \cap V_j$. However, by the given condition, for any $x \in U_i \cap V_j$, there exists a $W_{x,i,j} \in \beta$ such that $x \in W_{x,i,j} \subset U_i \cap V_j$. Then, clearly,

$$U_i \cap V_j = \bigcup_{x\in U_i\cap V_j} W_{x,i,j}$$

   as each $W_{x,i,j}$ is contained in $U_i \cap V_j$ and conversely each x is contained in some $W_{x,i,j}$. Thus, we finally write:

$$\begin{aligned} U \cap V &= \bigcup_{i\in I}\bigcup_{j\in J}(U_i \cap V_j) \\ &= \bigcup_{i\in I}\bigcup_{j\in J}\left(\bigcup_{x\in U_i\cap V_j} W_{x,i,j}\right) \end{aligned}$$

   Thus, τ' is closed under finite intersections.

Thus, τ' is a topology and $\tau' = \tau(\beta)$.

Finally, we need to show that β is a basis for τ'. Let $U \in \tau'$ and $x \in U$. Then, by the given property, there exists a $V \in \beta$ such that $x \in V \subset U \cap U = U$. Thus, β is a basis for $\tau' = \tau(\beta)$.

Problem 45. If X is a finite dimensional normed space, show that the weak topology is the same as the norm topology. (Hint. Use the dual basis.)

Solution. Let τ_W be the weak topology and τ_N be the norm topology. We already know that $\tau_W \subseteq \tau_N$ as every weakly open set is norm open. Thus, we need to show that if X is finite dimensional, then $\tau_N \subseteq \tau_W$.

Since X is finite dimensional, all norms on X are equivalent, thus generate the same topology. Let $\{e_1, \cdots, e_n\}$ be a basis for X. Let $||x||_\infty = \max_i |x_i|$ be the l^∞ norm on X, where $x = \sum_{i=1}^n x_i e_i$. We will show that open balls in this topology are weakly open.

For all $\epsilon > 0$ and $x_0 \in X$, let:

$$B_\epsilon(x_0) = \{x \in X : ||x - x_0||_\infty < \epsilon\}$$

be an open ball. Let $\{f_1, \cdots, f_n\} \subset X^*$ be the dual basis. These are bounded functionals as all functionals in a finite dimensional space are bounded (Problem 33). Then, we have the following subbasis elements for the weak topology, for $1 \leq i \leq n$:

$$\begin{aligned} U(f_i, x_0, \epsilon) &= \{x \in X : |f_i(x_0) - f_i(x)| < \epsilon\} \\ &= \{x \in X : |(x_0)_i - x_i| < \epsilon\} \end{aligned}$$

Then, the intersection of these for $i = 1, \cdots, n$ gives us:

$$\begin{aligned} \bigcap_{i=1}^n U(f_i, x_0, \epsilon) &= \bigcap_{i=1}^n \{x \in X : |(x_0)_i - x_i| < \epsilon\} \\ &= \{x \in X : |(x_0)_i - x_i| < \epsilon, i = 1, \cdots, n\} \\ &= \{x \in X : ||x - x_0||_\infty < \epsilon\} \\ &= B_\epsilon(x_0) \end{aligned}$$

Thus, we have shown that $B_\epsilon(x_0)$ is weakly open for all $x_0 \in X$ and $\epsilon > 0$. Since these open balls form a basis for the norm topology, every norm open set is weakly open. Thus, we have shown that $\tau_N \subseteq \tau_W$.

Thus, we have shown that $\tau_W = \tau_N$ when X is finite dimensional.

Problem 46. Show that if X is an infinite dimensional normed space, then every nonempty weakly open set is unbounded. (Hint provided.)

Solution. Let X be an infinite dimensional normed space. Let U be a non-empty weakly open set. We will show that U is unbounded.

Since U is non-empty, there exists a point $x_0 \in U$. Since U is a neighborhood of x_0, there exists an element of the neighborhood basis of the weak topology contained in U. Thus, there exists $\phi_1, \cdots, \phi_n \in X^*$ and $\epsilon > 0$ such that:

$$B := U(\{\phi_1, \cdots, \phi_n\}, x, \epsilon) = \{x \in X : |\phi_i(x) - \phi_i(x_0)| < \epsilon, 1 \leq i \leq n\} \subset U$$

Define $\phi : X \to \mathbb{F}^n$ by $\phi(x) = (\phi_1(x), \cdots, \phi_n(x))$. Consider $\ker(\phi)$. We note that:

$$x \in \ker(\phi) \iff x \in \ker(\phi_i), \forall 1 \leq i \leq n$$

Thus,

$$\ker(\phi) = \bigcap_{i=1}^{n} \ker(\phi_i)$$

By Problem 33, since each ϕ_i is bounded, it has a closed kernel. An intersection of closed sets is closed, thus ϕ has a closed kernel. Then, finally, by Problem 33, ϕ is bounded.

Moreover, note that by rank-nullity, as X is infinite dimensional and $\dim(\text{img}(\phi)) \leq n$, we have that $\ker(\phi)$ is infinite dimensional, hence a non-empty linear subspace.

Next, consider $\phi^{-1}(\phi(x_0))$. From undergraduate linear algebra, we know that this is a coset of $\ker(\phi)$, i.e.:

$$\phi^{-1}(\phi(x_0)) = x_0 + \ker(\phi)$$

Moreover, we claim that $\phi^{-1}(\phi(x_0)) \subset B$. To see this, let $z \in \ker(\phi)$ be arbitrary. Then, $z \in \ker(\phi_i)$ for all i. Thus, we have:

$$|\phi_i(x_0 + z) - \phi_i(x_0)| = |\phi_i(x_0 - x_0) + \phi_1(z)| = 0 < \epsilon$$

for all i. Thus, $x_0 + z \in B$. Thus, $x_0 + \ker(\phi) \subset B$.

Now, since $\ker(\phi)$ is a non-empty linear subspace, it contains λz for some $z \in \ker(\phi)$ and all $\lambda \in \mathbb{F}$. Then, we calculate the metric distance between x_0 and $x_0 + \lambda z \in x_0 + \ker(\phi)$:

$$||x_0 - (x_0 + \lambda z)|| = ||\lambda z|| = |\lambda| \cdot ||z||$$

Since $||z||$ is a constant, we can choose $|\lambda| > N$ for any $N \in \mathbb{R}$. Thus, $x_0 + \ker(\phi)$ is unbounded. Thus, B is unbounded. Thus, U is unbounded.

Problem 48. Let $f : (X, \tau) \to (Y, \sigma)$ be a function between topological spaces. Show that f is continous if and only if f takes convergent nets to convergent nets. That is, f is continous if and only if given $x_\lambda \to x$ in X, we have $f(x_\lambda) \to f(x)$ in Y.

Solution. Assume that f is continous. Let $x_\lambda \to x$ be a convergent net in X. Then, $f(x_\lambda)$ is a net in Y. We need to show that $f(x_\lambda) \to f(x)$. Thus, we need to show that $f(x_\lambda)$ is eventually in every neighborhood of $f(x)$.

Let $V \in \mathcal{N}(f(x))$ be a neighborhood of $f(x)$. Then, there exists an open $U \in \sigma$ such that $f(x) \in U \subset V$. Since f is continous, $f^{-1}(U)$ is open in τ. Moreover, as $x \in f^{-1}(U)$ as $f(x) \in U$. Thus, $f^{-1}(U)$ is a neighborhood of x. Since $x_\lambda \to x$, there exists an index λ_0 such that for all $\lambda \geq \lambda_0$, $x_\lambda \in f^{-1}(U)$. Thus, for all $\lambda \geq \lambda_0$, we have that $f(x_\lambda) \in U$. Since $U \subset V$, we have that $f(x_\lambda) \in V$. Thus, $f(x_\lambda)$ is eventually in V. Since V was arbitrary, $f(x_\lambda) \to f(x)$.

For the other direction, we prove the contrapositive. Assume that f is not continous. Then, there exists an open set $V \in \sigma$ such that $f^{-1}(V)$ is not open in τ. Then, there exists a point $x \in f^{-1}(V)$ such that there are no open neighborhoods containing x that are contained in $f^{-1}(V)$ (using a definition of open from point set).

Let $\Lambda = \mathcal{N}(x)$ be the neighborhoods of x ordered by reverse inclusion. We construct a net (x_λ) in X by picking a point in each neighborhood of x that is not in $f^{-1}(V)$. We claim that this net converges to x. To see this, let $U \in \mathcal{N}(x)$ be a neighborhood of x. Then for any $V \geq U \in \Lambda$, since we have $V \subseteq U$, we have that $x_V \in U$. Thus, x_λ is eventually in U. Since U was an arbitrary neighborhood of x, we have that $x_\lambda \to x$.

However, $x_\lambda \notin f^{-1}(V)$ for all $\lambda \in \Lambda$. Thus, $f(x_\lambda) \notin V$ for all $\lambda \in \Lambda$. Thus, $f(x_\lambda)$ is not eventually in V, a neighborhood of $f(x)$. Thus, $f(x_\lambda)$ does not converge to $f(x)$.

By the contrapositive, we have shown that if f takes convergent nets to convergent nets, then f is continous and we are done.

Problem 49. Let X be a normed vector space. Show that a net (x_λ) converges to x weakly if and only if $\phi(x_\lambda) \to \phi(x)$ for all $\phi \in X^*$. Does a weakly convergent net (x_λ) have to be bounded?

Solution. Let $x_\lambda \subset X$ be a net.

First assume that $x_\lambda \to x$ in the weak topology. By definition, we have that each $\phi \in X^*$ is continous with respect to the weak topology. Thus, by the previous problem, $\forall \phi \in X^*$, we have:

$$x_\lambda \to x \implies \phi(x_\lambda) \to \phi(x)$$

Next, assume $\phi(x_\lambda) \to \phi(x)$ for all $\phi \in X^*$. We need to show that $x_\lambda \to x$ in the weak topology. Thus, let $V \in \mathcal{N}(x)$ be a neighborhood of x. We need to show that x_λ is eventually in V.

Let β be the neighborhood basis of x with respect to the weak topology that we defined in class. Since V is a neighborhood of x, we know there exists a $U \in \beta$ such that $U \subset V$. Then, U is of the form:

$$U = U(\{\phi_1, \cdots, \phi_n\}, x_0, \epsilon) = \{x' \in X : |\phi_i(x') - \phi_i(x)| < \epsilon, \forall 1 \leq i \leq n\}$$

for some $\phi_1, \cdots, \phi_n \in X^*$ and $\epsilon > 0$. We will show that x_λ is eventually in $U \subset V$.

Note that $\phi_i(x_\lambda) \to \phi_i(x)$ for all $1 \leq i \leq n$. Take $B_\epsilon(\phi_i(x))$ to be the open ball of radius ϵ around $\phi_i(x)$. Then, there exists an index λ_i such that for all $\lambda \geq \lambda_i$, we have:

$$\phi_i(x_\lambda) \in B_\epsilon(\phi_i(x)) \iff |\phi_i(x_\lambda) - \phi_i(x)| < \epsilon$$

Since the λ_i are a part of a directed set, any two of them have an element dominating them. By (very simple) induction, any finite collection of them has an element dominating them. Thus, let λ_0 be the element such that:

$$\lambda_0 \geq \lambda_i \qquad \forall 1 \leq i \leq n$$

Then, for all $\lambda \geq \lambda_0$, we have that for all $1 \leq i \leq n$:

$$|\phi_i(x_\lambda) - \phi_i(x)| < \epsilon$$

Thus, for all $\lambda \geq \lambda_0$, we have that:

$$x_\lambda \in U$$

Thus, x_λ is eventually in U, thus eventually in V. Since V was an arbitrary neighborhood of x, we have that $x_\lambda \to x$ in the weak topology.

TODO

Problem 51. Let (x_λ) be a net in the compact space X. Show that (x_λ) has an accmulation point. (Hint provided.)

Solution. Let (x_λ) be a net in compact space X. We need to show that there exists a point $x \in X$ such that for any neighborhood U of x and index λ_0, we have that there exists $\lambda \geq \lambda_0$ such that $x_\lambda \in U$.

Let $F_{\lambda'} = \overline{\{x_\lambda : \lambda \geq \lambda'\}}$. Then, the family $\{F_\lambda\}_\lambda$ is a family of closed sets. Moreover, let $\lambda_1, \cdots, \lambda_n$ be a finite collection of indices. Since the indices are from a directed set, there exists an index dominating any pair of them. Thus, by (very simple) induction, we can find an index λ'' dominating all of them (for any finite collection), i.e. $\lambda \geq \lambda''$ implies $\lambda \geq \lambda_i$ for all $1 \leq i \leq n$. Thus, by definition, we have that:

$$x_{\lambda''} \in \bigcap_{i=1}^{n} F_{\lambda_i}$$

Thus, the family $\{F_\lambda\}_\lambda$ is a family of closed sets with the finite intersection property. As X is compact, we have (Problem 13) that:

$$\bigcap_{\lambda} F_\lambda \neq \emptyset$$

Thus, there exists a point $x \in \bigcap_\lambda F_\lambda$. We claim that x is an accumulation point of x_λ.

Let U be a neighborhood of x and λ_0 be an index. Then,

$$x \in F_{\lambda_0} = \overline{\{x_\lambda : \lambda \geq \lambda_0\}}$$

Thus, by the definition of closure, we have that every open neighborhood of x intersects $\{x_\lambda : \lambda \geq \lambda_0\}$. Thus, there exists a point in $\{x_\lambda : \lambda \geq \lambda_0\} \cap U$. Thus, there exists a $\lambda \geq \lambda_0$ such that $x_\lambda \in U$.

As U and λ_0 were arbitrary, x is an accumulation point of (x_λ).

Not every weakly convergent net is bounded. Let $\Lambda = \mathbb{R}_{\leq 0}$ be the negative real numbers with their usual order. Then we have $x_\lambda \subseteq \mathbb{R}$ a net, defined by:

$$x_\lambda = -\lambda$$

Then, $x_\lambda \to 0$ in the norm topology, thus x_λ is weakly convergent to 0 (every weakly open set is norm open). However, for any $N \in \mathbb{R}$, if we pick $\lambda < N$, we have that $x_\lambda = -\lambda > N$. Thus, x_λ is unbounded.

Problem 52. Let (x_n) be a sequence in a metric space X. Show that x is an accumulation point of (x_n) if and only if (x_n) has a subsequence converging x.

Solution. Assume x is an accumulation point of (x_n). Let $B_{1/n}(x)$ be the open ball of radius $1/n$ around x. Since x is an accmulation point and $B_1(x)$ is a neighborhood of x, we have that there exists $n_1 > 1$ such that $x_{n_1} \in B_1(x)$. Then, since $B_{1/2}(x)$ is a neighborhood of x, there exists $n_2 > n_1$ such that $x_{n_2} \in B_{1/2}(x) \subset B_1(x)$. Continuing in this manner, we can find a sequence of indices $n_1 < n_2 < \cdots < n_k < \cdots$ such that $x_{n_k} \in B_{1/k}(x)$. We claim that this subsequence converges to x.

For any $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that $1/N < \epsilon$. Then, for all $k \geq N$, we have that $x_{n_k} \in B_{1/k}$. However, as $1/k < 1/N < \epsilon$, we have that $x_{n_k} \in B_{1/N}(x) \subseteq B_\epsilon(x)$. Thus, $x_{n_k} \to x$.

Next, assume that (x_n) has a subsequence (x_{n_k}) converging to x. We need to show that x is an accumulation point of (x_n).

Let U be a neighborhood of x and $N \in \mathbb{N}$ be some index. We need to show that there exists an $m > N$ such that $x_m \in U$.

Since every neighborhood of x contains an open neighborhood, we assume, without loss of generality, that U is open. Since x_{n_k} is a subsequence of x, we have that $n_k \geq k$. Moreover, since x_{n_k} converges to x, there exists an index N_1 such that for all $k \geq N_1$, we have that $x_{n_k} \in U$.

Let $N_0 > \max(N, N_1)$ be an index. Then, $n_{N_0} \geq N_0 > N$ and as $N_0 > N_1$, we have that $x_{N_0} \in U$. Thus, x is an accumulation point of (x_n).