Math 121: Hodge Theory

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Problem 1.

(a) Check the equivalence between the two definitions of the Hodge structure of weight k given in class.

(b) Check the a morphism Hodge structures is strict for the Hodge filtration.

(c) Show that the kernel, cokernel, and image of a morphism of Hodge structures are Hodge structures.

(d) Let $\phi : X \to Y$ a surjective holomorphic map of complex compact manifolds such that X is kählerian. Show that ϕ^* is injective.

Solution.

(a) First assume that we have filtration of $V_{\mathbb{C}}$:

$$0 = F^{k+1} \subset F^k V_{\mathbb{C}} \subset F^{n-1} V_{\mathbb{C}} \subset \cdots \subset F^0 V_{\mathbb{C}} = V_{\mathbb{C}}$$

that satisfies the condition:

$$F^p V_{\mathbb{C}} \oplus \overline{F^{k-p+1} V_{\mathbb{C}}} = V_{\mathbb{C}}$$

Then, we define:

$$H^{p,q} = F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}}$$

Clearly, with this definition, we have:

$$\begin{aligned}
\overline{H^{p,q}} &= \overline{F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}}} \\
&= \overline{F^p V_{\mathbb{C}}} \cap F^q V_{\mathbb{C}} \\
&= H^{q,p}
\end{aligned}$$

We will show that:

$$F^iV = \bigoplus_{p\geq i} H^{p,k-p}$$

Notice that:

$$\begin{aligned}
F^iV_{\mathbb{C}} &= F^iV_{\mathbb{C}} \cap V_{\mathbb{C}} \\
&= F^iV_{\mathbb{C}} \cap (F^{i+1}V_{\mathbb{C}} \oplus \overline{F^{k-(i+1)+1}V_{\mathbb{C}}}) \\
&= F^iV_{\mathbb{C}} \cap (F^{i+1}V_{\mathbb{C}} \oplus \overline{F^{k-i}V_{\mathbb{C}}}) \\
&= (F^iV_{\mathbb{C}} \cap F^{i+1}V_{\mathbb{C}}) \oplus (F^iV_{\mathbb{C}} \cap \overline{F^{k-i}V_{\mathbb{C}}}) \\
&= F^{i+1}V_{\mathbb{C}} \ \oplus (F^iV_{\mathbb{C}} \cap \overline{F^{k-i}V_{\mathbb{C}}}) \\
&= F^{i+1}V_{\mathbb{C}} \oplus H_{i,k-1} \\
&= \bigoplus_{p\geq i} H^{p,k-p}
\end{aligned}$$

Where the third equality uses the fact that $A \cap (B \oplus C) = A \cap B \oplus A \cap C$ if $B \subseteq A$ as if $a = b + c \in A \cap (B \oplus C)$ $(a \in A, b \in B, c \in C)$, then since $b \in B \subset A$, we have $a - b = c \in A$. Thus, $b \in A \cap B$ and $c \in A \cap C$ and $A \cap (B \oplus C) \subseteq A \cap B \oplus A \cap C$. The other inclusion is clear.

Thus, noting the result for $i = 0$, we have recovered the first definition of a Hodge structure of weight k.

Now, assume that we have a decomposition of $V_{\mathbb{C}}$ into the direct sum of the $H^{p,q}$:

$$V_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$$

with $H^{p,q} = \overline{H^{q,p}}$. Then, we define the filtration similarly:

$$F^pV_{\mathbb{C}} = \bigoplus_{i\geq p} H^{i,k-i}$$

Then, we have that:

$$\begin{aligned}
F^pV_{\mathbb{C}} \oplus \overline{F^{k-p+1}V_{\mathbb{C}}} &= \bigoplus_{i\geq p} H^{i,k-i} \oplus \overline{\bigoplus_{j\geq k-p+1} H^{j,k-j}} \\
&= \bigoplus_{i\geq p} H^{i,k-i} \oplus \bigoplus_{j\geq k-p+1} H^{k-j,j} \\
&= \bigoplus_{i\geq p} H^{i,k-i} \oplus \bigoplus_{j'\leq p-1} H^{j',k-j'} \\
&= \bigoplus_{i\geq 0} H^{i,k-i} \\
&= V_{\mathbb{C}}
\end{aligned}$$

Thus, we have recovered the second definition of a Hodge structure of weight k.

(b) Recall the defnition of a morphism of Hodge structures. Let $V_{\mathbb{Z}}$ and $V'_{\mathbb{Z}}$ be two hodge structures of weight k and $k+2r$. Then a morphism of Hodge structures of type (r,r) is a group morphism ϕ whose $\mathbb{C}$-linear extension (which we will also call ϕ for this part) satisfies:

$$\phi(V^{p,q}) \subseteq V'^{p+r,q+r} \iff \phi(F^pV) \subset F^{p+r}V'$$

We need to show that this is strict for the Hodge filtration, i.e.

$$\text{img}(\phi) \cap F^{p+r}V' = \phi(F^pV)$$

Let $\alpha \in \text{img}(\phi) \cap F^{p+r}V'$. Then, there exists $\beta \in V$ such that $\phi(\beta) = \alpha$. Then, using the decomposition of V, we can write:

$$\begin{aligned}
\alpha &= \phi(\beta) \\
&= \phi\left(\sum_{i+j=k} \beta^{i,j}\right) \\
&= \sum_{i+j=k} \phi(\beta^{i,j})
\end{aligned}$$

Then, each $\phi(\beta^{i,j})$ is of type $(i+r, j+r)$. Then, if $i < p$, note that $\phi(\beta^{i,j}) \in V'^{i+r,j+r} \cap F^{p+r}$. But, as noted from the previous problem, we have:

$$F^{p+r}V' = \bigoplus_{i \geq p+r} H^{i,k-i}$$

Thus, we have that $\phi(\beta^{i,j}) = 0$ for $i < p$. Thus, we can write:

$$\alpha = \sum_{i \geq p} \phi(\beta^{i,k-i}) \in F^{p+r}V'$$

noting the decomposition for $F^{p+r}V'$ from the previous problem and that $\phi(\beta^{i,j}) \in V'^{i+r,j+r}$ (as noted before). Thus, we have that:

$$\text{img}(\phi) \cap F^{p+r}V' \subseteq \phi(F^pV)$$

The other direction is implied in the definition of a Hodge structure morphism.

(c) Let $\phi : V_{\mathbb{Z}} \to V'_{\mathbb{Z}}$ be a morphism of Hodge structures of type (r,r), where V is of weight k and V' is of weight $k+2r$. We first prove that $\text{img}(\phi)$ and $\ker(\phi)$ have natural Hodge structures.

For $\operatorname{img}(\phi)$, first note that:

$$\operatorname{img}(\phi_{\mathbb{C}}) = \operatorname{img}(\phi) \otimes_{\mathbb{Z}} \mathbb{C} \qquad \ker(\phi_{\mathbb{C}}) = \ker(\phi) \otimes_{\mathbb{Z}} \mathbb{C}$$

Thus, we can define the filtration on $\operatorname{img}(\phi_{\mathbb{C}})$ by:

$$F^p \operatorname{img}(\phi_{\mathbb{C}}) := \operatorname{img}(\phi) \cap F^p V'_{\mathbb{C}}$$

It is easy to see that these define a decreasing filtration on $\operatorname{img}(\phi)$ and $\ker(\phi)$ by the filtration structures on the codomain and domain. Then, note for $\operatorname{img}(\phi)$ that we have:

$$\begin{aligned}
F^p \operatorname{img}(\phi_{\mathbb{C}}) \oplus \overline{F^{k+2r-p+1} \operatorname{img}(\phi_{\mathbb{C}})} &= \operatorname{img}(\phi) \cap F^p V'_{\mathbb{C}} \oplus \overline{\operatorname{img}(\phi) \cap F^{k+2r-p+1} V'_{\mathbb{C}}} \\
&= \phi(F^{p-r} V) \oplus \overline{\phi(F^{k+r-p+1} V_{\mathbb{C}})} \\
&= \phi(F^{p-r} V \oplus \overline{F^{k+r-p+1} V_{\mathbb{C}}}) \\
&= \phi(V_{\mathbb{C}}) \\
&= \operatorname{img}(\phi_{\mathbb{C}})
\end{aligned}$$

Thus, $\operatorname{img}(\phi)$ has an Hodge structure of weight $k+2r$ and for $\ker(\phi)$, we note that:

$$\ker(\phi_{\mathbb{C}}) = \bigoplus_{p+q=k} \ker(\phi_{\mathbb{C}}) \cap V^{p,q}$$

as the $V^{p,q}$ are disjoint. Let $K^{p,q} = \ker(\phi) \cap V^{p,q}$. Then, we have that:

$$\overline{\ker(\phi_{\mathbb{C}}) \cap V^{p,q}} = \overline{\ker(\phi_{\mathbb{C}})} \cap V^{q,p}$$

But $\ker(\phi_{\mathbb{C}})$ is closed under complex conjugation as $\phi_{\mathbb{C}}$ is $\mathbb{C}$-linear. Thus, we have shown that:

$$\ker(\phi_{\mathbb{C}}) = \bigoplus_{p+q=k} K^{p,q} \qquad K^{p,q} = \overline{K^{q,p}}$$

Thus, we have that $\ker(\phi_{\mathbb{C}})$ is a Hodge structure of weight k.

Finally, for the cokernel, note that:

$$\begin{aligned}
\operatorname{coker}(\phi_{\mathbb{C}}) &= V'_{\mathbb{C}} / \operatorname{img}(\phi_{\mathbb{C}}) \\
&= (V' \otimes_{\mathbb{Z}} \mathbb{C}) / (\operatorname{img}(\phi) \otimes_{\mathbb{Z}} \mathbb{C}) \\
&= (V'_{\mathbb{C}} / \operatorname{img}(\phi_{\mathbb{C}})) \otimes_{\mathbb{Z}} \mathbb{C} \\
&= \operatorname{coker}(\phi) \otimes_{\mathbb{Z}} \mathbb{C}
\end{aligned}$$

Let

$$\operatorname{coker}(\phi_{\mathbb{C}})^{p,q} = V'^{p,q} / \operatorname{img}(\phi)^{p,q}$$

where $\mathrm{img}(\phi)^{p,q}$ is the (p,q)-part of $\mathrm{img}(\phi_{\mathbb{C}})$. Then, we have:

$$\begin{aligned}\mathrm{img}(\phi_{\mathbb{C}})^{p,q} &= (\mathrm{img}(\phi_{\mathbb{C}}) \cap F^p V'_{\mathbb{C}}) \cap \overline{(\mathrm{img}(\phi_{\mathbb{C}}) \cap F^q V'_{\mathbb{C}})} \\ &= \mathrm{img}(\phi_{\mathbb{C}}) \cap V'^{p,q} \subseteq V'^{p,q}\end{aligned}$$

as the image is closed under complex conjugation. Thus, using these compatible decompositions, we note:

$$\begin{aligned}\bigoplus_{p+q=k+2r} \mathrm{coker}(\phi_{\mathbb{C}})^{p,q} &= \bigoplus_{p+q=k+2r} V'^{p,q} / \mathrm{img}(\phi_{\mathbb{C}})^{p,q} \\ &= \left(\bigoplus_{p+q=k+2r} V'^{p,q} \right) / \left(\bigoplus_{p+q=k+2r} \mathrm{img}(\phi_{\mathbb{C}})^{p,q} \right) \\ &= V'{}_{\mathbb{C}} / \mathrm{img}(\phi_{\mathbb{C}}) \\ &= \mathrm{coker}(\phi_{\mathbb{C}})\end{aligned}$$

Moreover, we have that:

$$\begin{aligned}\overline{\mathrm{coker}(\phi_{\mathbb{C}})^{p,q}} &= \overline{V'^{p,q} / \mathrm{img}(\phi_{\mathbb{C}})^{p,q}} \\ &= V'^{q,p} / \mathrm{img}(\phi_{\mathbb{C}})^{q,p} \\ &= \mathrm{coker}(\phi_{\mathbb{C}})^{q,p}\end{aligned}$$

as the quotient map is $\mathbb{C}$-linear, thus respects the complex structure. Thus, we have that $\mathrm{coker}(\phi_{\mathbb{C}})$ is a Hodge structure of weight $k+2r$.

(d) (I took several hints from the proof in Voisin, and the proof gradually looked more and more like hers).

Note that the pullback is:

$$\phi^* : H^k(Y, \mathbb{Z}) \to H^k(X, \mathbb{Z})$$

It suffices to show that ϕ^* is injective with $\mathbb{R}$ coefficients. Let $\dim_{\mathbb{C}}(X) = n$ and $\dim_{\mathbb{C}}(Y) = m$. Let $\alpha \in H^{2m}(Y, \mathbb{R})$. Then, we have $\phi^*\alpha \in H^{2m}(X, \mathbb{R})$. Let $r = n - m > 0$ (as ϕ surjective). Then, we have that $\omega^r \wedge \phi^*\alpha \in H^{2n}(X)$ is a top-degree form on X, where ω is the Kähler form. Voisin claims that this map is always non-negative, and positive on at least an open set. Thus, the integral:

$$\int_X \omega^r \wedge \phi^*\alpha > 0$$

Thus, $\phi^*\alpha \geq 0$. We conclude that ϕ^* is injective on $H^k(Y, \mathbb{R})$ using Poincare duality to get a top-form on Y, and noticing that the pullback distributes over the cup product (and is, in particular, a ring homomorphism).

Problem 2.1. Let $H_{\mathbb{R}}$ be a $\mathbb{R}$-vector space, and $H_{\mathbb{C}} := H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

(a) Show that a decomposition

$$H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q} \qquad H^{p,q} = \overline{H^{q,p}}$$

determines a continous action $\rho : \mathbb{C}^{\times} \to \mathrm{GL}(H_{\mathbb{C}})$ of $\mathbb{C}^{\times}$ on $H^{\mathbb{C}}$ given by:

$$z \cdot \alpha^{p,q} = z^p \overline{z^q} \alpha^{p,q}$$

for $\alpha^{p,q} \in H^{p,q}$. Show that this action satisfies:

$$\rho(\overline{z}) = \overline{\rho(z)}$$

where the conjugacy on $GL(H_{\mathbb{C}})$ is defined by:

$$\overline{g}(u) = \overline{g(\overline{u})}$$

Show that one also has $\rho(t) = t^k \,\mathrm{id}$ for $t \in \mathbb{R}^{\times}$.
Conversely, let $\rho : \mathbb{C}^{\times} \to GL(H_{\mathbb{C}})$ be a continous action of $\mathbb{C}^*$ on $H_{\mathbb{C}}$ satisfying $\rho(t) = t^k \,\mathrm{id}$ for $t \in \mathbb{R}^*$ and $\rho(\overline{z}) = \overline{\rho(z)}$. Show that there exists a decomposition of $H_{\mathbb{C}}$ into the direct sum of the $H^{p,q}$ such that $\rho(z)$ acts as above.

(b) Applying the diagonalization theorem for the actions of torsion abelian groups to the torsion points of $\mathbb{C}^*$, show that there exists a decompositon into a directt sum:

$$H = \bigoplus_{\chi} H_{\chi}$$

where χ belongs to the set of characters of $\mathbb{C}^*$ and $\mathbb{C}^*$ acts by $z \to \chi(z)\,\mathrm{id}$ on H_{χ}.

(c) Show that only the characters $\chi_{p,q} : z \to z^p \overline{z}^q$ with $p + q = k$ appear in the decomposition of $H_{\mathbb{C}}$.

(d) Let $H^{p,q} := H_{\chi_{p,q}}$. Show that $H^{p,q} = \overline{H^{q,p}}$.

(e) Let $V_{\mathbb{Z}}$ be a Hodge structure of weight $2k$ and $\mathbb{C}^* \to GL(V_{\mathbb{R}})$ the corresponding group morphism defined in the previous question. Show that the group of Hodge classes is equal to the group of stable (i.e. fixed up to a scaler) vectors of $V_{\mathbb{Z}}$ under the action of $\mathbb{C}^*$.

Solution.

Problem 3.(The Hodge Decomposition for Curves) Let X be a compact connected complex curve. We have the differential:

$$d : \mathcal{O} \to \Omega_X$$

between the sheaf of homolormphic functions and the sheaf of holomorphic differentials.

(a) Show that d is surjective with kernel equal to the constant sheaf $\mathbb{C}$. Hence, we have an exact sequence:

$$0 \to \mathbb{C} \to \mathcal{O} \to \to \Omega_X \to 0$$

(b) Deduce from Serre duality that $H^1(X, \Omega_X) \cong \mathbb{C}$. Deduce from Poincare duality that $H^2(X, \mathbb{C}) = \mathbb{C}$.

(c) Show that (6.15) induces a short exact sequence:

$$0 \to H^0(X, \Omega_X) \to H^1(X, \mathbb{C}) \to H^1(X, \mathcal{O}_X) \to 0$$

(d) Show that the map which to a holomorphic form α associates the class of $\overline{\alpha}$ in $H^1(X, \mathcal{O})$ is injective.

(e) Deduce from Serre duality that it is also surjective and that we have the decomposiiton:

$$H^1(X, \mathbb{C}) = H^0(X, \Omega_X) \oplus \overline{H^0(X, \Omega_X)}$$

with

$$\overline{H^0(X, \Omega_X)} \cong H^1(X, \mathcal{O})$$

Solution.