Math 74: Algebraic Topology

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> **Problem 1**. (0.10) A space is called contractible if the identity map is nullhomotopic. Show that X is contractible if and only if for every space Y, every map $f : X \to Y$ is nullhomotopic. Similarly, show that X is contractible if and only if for every Y, every map $f : Y \to X$ is nullhomotopic.

Solution. ($\Longleftarrow$) Assume for every Y every map $f : X \to Y$ is nullhomotopic. Then, in particular, picking $Y = X$ and $f = \mathrm{id}_X$, we have that id_X is nullhomotopic. Thus, X is contractible.

($\Longrightarrow$) Assume that X is contractible and Y be any space. Thus, there exists a homotopy $H : X \times I \to X$ from id_X to a constant map $\lambda_c : X \to X$ for some $c \in X$. Then, for any map $f : X \to Y$, we claim that $H' : X \times I \to Y$ defined by $H'(x, t) = f(H(x, t))$ is a homotopy from f to the constant map $f(c)$. Similarly, for any map g from Y to X, we claim that $H'' : Y \times I \to X$ defined by $H''(y, t) = H(g(y), t)$ is a homotopy from g to the constant map $g(c)$. Note that H' and H'' are continous as they are compositions of continous maps. Moreover, we check, for all $x \in X$ and $y \in Y$,

$$\begin{aligned}
H'(x, 0) &= f(H(x, 0)) = f(x) \\
H'(x, 1) &= f(H(x, 1)) = f(\lambda_c(x)) = f(c) \\
H''(y, 0) &= H(g(y), 0) = g(y) \\
H''(y, 1) &= H(g(y), 1) = \lambda_c(g(y)) = c
\end{aligned}$$

Thus, H' is a homotopy from f to the constant map valued at $f(c)$ and H'' is a homotopy from g to the constant map valued at c.

Problem 2. Let $A \subset X$ and suppose $r : X \to A$ is a continuous map such that $r(a) = a$ for all $a \in A$ (i.e., r is a retraction of X onto A). If $a_0 \in A$, show that

$$r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$$

is surjective. (Hint: Consider also the inclusion map of A into X.)

Solution. Let $\iota : A \to X$ be the inclusion map. Then, note that for all $a \in A$:

$$r \circ \iota(a) = a$$

Thus, for any $f : I \to A$, we have $r \circ \iota \circ f = f$. Moreover, we have $\iota_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ with $\iota_*([f]) = [\iota \circ f]$. Let $p \in \pi_1(A, a_0)$ be arbitrary. Then, we claim that $\iota_*([p]) \in \pi_1(X, a_0)$ maps to $[p]$ under r_*. We compute:

$$\begin{aligned} r_*(\iota_*([p])) &= r_*([\iota \circ p]) \\ &= [r \circ \iota \circ p] \\ &= [p] \end{aligned}$$

Thus, as $[p]$ was arbitrary, we have shown that r_* is surjective.

Problem 3. (1.1.3) If X is a path-connected space, show that $\pi_1(X)$ is abelian if and only if every change-of-basepoint isomorphism ($\hat{\alpha}$ in class or β_h in Hatcher) depends only on the endpoints of the path.

Solution. ($\implies$) Assume $\pi_1(X)$ is abelian. Let $\alpha_1, \alpha_2 : I \to X$ be two paths from x_0 to x_1. Then, $\hat{\alpha_i} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ are given by $\hat{\alpha_i}([f]) = [\overline{\alpha_i} \cdot f \cdot \alpha_i]$. Note that $\pi_1(X, x_1)$ is abelian as $\pi_1(X)$ is. We show, for any $[f] \in \pi_1(X, x_0)$ that:

$$\begin{aligned}
\hat{\alpha_1}([f]) &= [\overline{\alpha_1} \cdot f \cdot \alpha_1] \\
&= [\overline{\alpha_1} \cdot f \cdot \alpha_2 \cdot \overline{\alpha_2} \cdot \alpha_1] \\
&= [\overline{\alpha_1} \cdot f \cdot \alpha_2] \cdot [\overline{\alpha_2} \cdot \alpha_1] \\
&= [\overline{\alpha_2} \cdot \alpha_1] \cdot [\overline{\alpha_1} \cdot f \cdot \alpha_2] \\
&= [\overline{\alpha_2} \cdot \alpha_1 \cdot \overline{\alpha_1} \cdot f \cdot \alpha_2] \\
&= [\overline{\alpha_2} \cdot f \cdot \alpha_2] \\
&= \hat{\alpha_2}([f])
\end{aligned}$$

Thus, $\hat{\alpha_1}$ depends only on the endpoints of the path.

($\impliedby$) Let $x_0 \in X$ and $[f], [g] \in \pi_1(X, x_0)$ be arbitary. Assume $\hat{f} = \hat{g}$, since f and g have the same endpoints (they are loops based at x_0). It suffices to show that $[f] = [\overline{g} \cdot f \cdot g]$. We compute:

$$\begin{aligned}
[f] &= [\overline{f} \cdot f \cdot f] \\
&= \hat{f}([f]) \\
&= \hat{g}([f]) \\
&= [\overline{g} \cdot f \cdot g]
\end{aligned}$$

Thus, $\pi_1(X, x_0)$ is abelian. Since x_0 was arbitrary, we have shown that $\pi_1(X)$ is abelian.

Problem 4. (1.1.6) Note that a loop based at x_0 can be regarded as a continuous map of pointed spaces

$$(S^1, (1,0)) \to (X, x_0).$$

Let $[S^1, X]$ be the set of homotopy classes of maps from S^1 to X without conditions on basepoints. Then, there is a map $\Phi : \pi_1(X, x_0) \to [S^1, X]$ that forgets the base points.

(a) Show that $\Phi([f]) = \Phi([g])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$.

(b) Deduce that if X is path connected, then $[S^1, X]$ is in bijection with conjugacy classes of $\pi_1(X, x_0)$.

Solution.

(a) Let $f, g : S^1 \to X$ be two loops based at x_0.
($\implies$) Assume $\Phi([f]) = \Phi([g])$. Then there exists a homotopy $H : S^1 \times I \to X$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ (we can pick the representative for the conjugacy classes as there exists homotopies between any two representatives that we can compose).

We define $\alpha_t : I \to X$ be the path that x_0 takes under the homotopy H in the interval $[0, t]$ (appropriately rescaled). We then define the homotopy $H' : S^1 \times I :\to X$ as:

$$H'(x, t) = \alpha_t \cdot H(x, t) \cdot \overline{\alpha_t}$$

Clearly, $\alpha_t = \overline{\alpha_t} = H(0, t)$ by definition. H' is also continuous as it is a concatenation of continuous paths. Moreover, we check:

$$\begin{aligned} H'(x, 0) &= \alpha_0 \cdot H(x, 0) \cdot \overline{\alpha_0} \\ &= f(x) \\ H'(x, 1) &= \alpha_1 \cdot H(x, 1) \cdot \overline{\alpha_1} \\ &= \alpha_1 \cdot g(x) \cdot \overline{\alpha_1} \\ H'((1, 0), t) &= \alpha_t \cdot H((1, 0), t) \cdot \overline{\alpha_t} \\ &= \alpha_t \cdot \overline{\alpha_t} \\ &= x_0 \end{aligned}$$

Thus, H' is a homotopy from f to $\alpha_1 \cdot g \cdot \overline{\alpha_1}$. Note that α_1 is the path of x_0 under the homotopy H. Thus, $\alpha_1(0) = H((0, 1), 0) = f(0) = x_0$ and $\alpha_1(1) = H((0, 1), 1) = g(0) = x_0$. Thus, α_1 is a loop based at x_0. We have shown that $[f] = [\alpha_1 \cdot g \cdot \overline{\alpha_1}]$, thus $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$.

($\Longleftarrow$) Assume $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Then, there exists a path $\alpha : I \to X$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_0$ such that $[f] = [\overline{\alpha} \cdot g \cdot \alpha]$, via a path homotopy H. Let β_t be the part of the loop α on the interval $[1-t, 1]$ (rescaled appropriately). Note that $\beta_1 = \alpha_1$. We define the homotopy $H' : S^1 \times I \to X$ as:

$$H'(x,t) = \beta_t \cdot H(x,t) \cdot \overline{\beta_t}$$

H' is a well-defined continous map as its a concatenation of paths (which are compatible by definition). Finally, we check:

$$\begin{aligned} H'(x,0) &= \beta_0 \cdot H(x,0) \cdot \overline{\beta_0} \\ &= f(x) \\ H'(x,1) &= \beta_1 \cdot H(x,1) \cdot \overline{\beta_1} \\ &= \beta_1 \cdot \overline{\alpha_1} \cdot g(x) \cdot \alpha_1 \cdot \overline{\beta_1} \\ &= g(x) \end{aligned}$$

Thus, H' is a homotopy from f to g. Thus, $\Phi([f]) = \Phi([g])$.

(b) In the previous part, we already showed that Φ provides an injection from the conjugacy classes of $\pi_1(X, x_0)$ to $[S^1, X]$. We need to show that Φ provides a surjection from the conjugacy classes of $\pi_1(X, x_0)$ to $[S^1, X]$. Since every element in a conjugacy class has the same image, it is sufficient to show that Φ is surjective.

Let $f : S^1 \to X$ be a loop. By path-connectedness, we can pick a path $\alpha : I \to X$ such that $\alpha(0) = x_0$ and $\alpha(1) = f((1,0))$. Then,

$$\alpha \cdot f \cdot \overline{\alpha}$$

is a loop based at x_0. We claim that $\Phi([\alpha \cdot f \cdot \overline{\alpha}]) = [f]$. Similar to before, define α_t to be the part of α on the interval $[1-t, 1]$. Define the homotopy $H : S^1 \times I \to X$ as follows:

$$H(x,t) = \alpha_t \cdot f(x) \cdot \overline{\alpha_t}$$

Similar to before, H is a well-defined continous map. We check:

$$\begin{aligned} H(x,0) &= \alpha_0 \cdot f(x) \cdot \overline{\alpha_0} \\ &= f(x) \\ H(x,1) &= \alpha_1 \cdot f(x) \cdot \overline{\alpha_1} \\ &= \alpha \cdot f(x) \cdot \overline{\alpha} \end{aligned}$$

Thus, $\Phi([\alpha \cdot f \cdot \overline{\alpha}]) = [f]$. Thus, Φ is surjective. Therefore, there is a bijection between the conjugacy classes of $\pi_1(X, x_0)$ and $[S^1, X]$.

Problem 5. Suppose that $p : E \to B$ is a covering map where B is connected. Show that if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$, then $p^{-1}(b)$ has k elements for every $b \in B$.

Solution. Let $b \in B$ be arbitrary. Since p is a covering map, there exists open U such that $b \in U$ and $p^{-1}(U) = \bigcup_{\alpha \in A} U_\alpha$ where U_α are disjoint open subsets in E such that $p|_{U_\alpha} : U_\alpha \to U$ is a homeomorphism, where A is some space. Since each $p|_{U_\alpha}$ is a homeomorphism onto U, we must have $p^{-1}(b) \cap U_\alpha$ be a singleton. Thus, $|p^{-1}(b)| \leq |A|$. Since the U_α are disjoint, each U_α must contain a distinct element, thus, $|p^{-1}(b)| = |A|$. Similarly, $\forall b' \in U$, the same arguments hold, thus, $|p^{-1}(b')| = |A|$. Thus, the size of the pre-image is constant for any two points within the same evenly covered open.

Let $G_i = \{b \in B : p^{-1}(b) = i\}$ for $i \in \mathbb{N} \cup \{\infty\}$. For every $b \in G_i$, we know that there exists an evenly covered open U_b such that every point in U_b has i pre-images. Thus, $U_b \in G_i$. Thus, G_i is open for all i. Moreover, as $B \setminus G_i = \bigcup_{j \neq i} G_j$, is a union of opens, hence open. Thus, G_i is also closed for all i.

Since B is connected, the only non-empty clopen set is B. We know that G_k is non-empty as $b_0 \in G_k$. Since G_k is clopen, $G_k = B$. Thus, $\forall b \in B$, $p^{-1}(b) = k$.

Problem 6. Let $q : X \to Y$ and $r : Y \to Z$ be covering maps such that r has finite degree. Show that $p = r \circ q$ is a covering map.

Solution. Let $z \in Z$ be some point. We need to show there exists a $U \subseteq Z$ with $z \in U$ such that U is evenly covered by p. Since r is a covering map of finite degree, call it d, there exists an evenly covered $U \subset Z$ such that:

$$r^{-1}(U) = \bigcup_{i=1}^{d} V_i$$

where V_i are disjoint open sets in Y such that $r|_{V_i} : V_i \to U$ is a homeomorphism.

We know that each V_i has a unique point y_i such that $r(y_i) = z$. Since q is a covering map, there exists an evenly covered open $W_i \subseteq Y$ such that $y_i \in W_i$. Let $U' = \bigcap_{i=1}^{d} r(V_i \cap W_i)$. Then, $z \in U'$ as $y_i \in V_i \cap W_i$ for all i. We will show that U' is evenly covered by p.

As $U' \subseteq U$ it is evenly covered by r by restricting all homomorphisms to the pre-image of U' in each $V_i \cap W_i$. Thus, we have:

$$r^{-1}(U') = \bigcup_{i=i}^{d} B_i$$

where $B_i \subset V_i \cap W_i$ are disjoint open sets in Y such that $r|_{B_i} : B_i \to U'$ is a homeomorphism.

Since $B_i \subseteq W_i$, and W_i is evenly covered by q, B_i is also evenly covered by restricting all the homeomorphisms onto W_i to the pre-image of B_i. Thus, we have:

$$q^{-1}(B_i) = \bigcup_{\alpha \in A_i} C_\alpha$$

where C_α are disjoint open sets in X such that $q|_{C_\alpha} : C_\alpha \to B_i$ is a homeomorphism. Thus, each C_α is homeomorphic to U' under $p = r \circ q$.

Moreover, since the B_i are disjoint, for C_α for $\alpha \in A_i$ is disjoint from C_β if $\beta \notin A_i$ as their images are disjoint. W_α already disjoint of W_β for $\beta \in A_i$ by construction.

Let $A = \bigcup_{i=1}^{d} A_i$. Then, we have:

$$p^{-1}(U') = \bigcup_{\alpha \in A} C_\alpha$$

where C_α are disjoint open sets in X such that $p|_{C_\alpha} : C_\alpha \to U'$ is a homeomorphism for all $\alpha \in A$, where U' is an open set containing z. Since z was arbitrary, this shows that p is a covering map.