Math 113: Functional Analysis

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Defn. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of (X, ρ). We say that $d > 0$ is a Lebesgue number for the cover if given any d-ball $B_d(x_0)$ with $x_0 \in X$, there exists $a_0 \in A$ such that $B_d(x_0) \subseteq U_{a_0}$.

Ex $X = \mathbb{R}$. $U_1 = (-\infty, 1)$, $U_2 = (0, 2)$, and $U_3 = (1, \infty)$. Here $d = 1/2$ is a LN for $\{U_1, U_2, U_3\}$. This is clear if $x_0 \in (1/2, 3/2)$.

Ex. (Hwk.) Given $x \in (0,1)$, $\exists \delta_x > 0$ such that:

$$y \in B_{\delta_x}(x) = \{y \in (0,1) : |y - x| \leq \delta_x\}$$
$$\implies |1/x - 1/y| < 1$$

Then,

$$(0,1) = \bigcup_{x \in (0,1)} B_{\delta_x}(x)$$

has no Lebesgue number.

Lemma (Lebesgue Covering Lemma) Every open cover of a compact metric space has a Lebesgue number.
Proof. Pictures. Apr 9.

Thm. Suppose (X, ρ) is compact, and $F : (X, \rho) \to (Y, \sigma)$ is continuous. Then F is uniformly continuous.

Proof. Let $\epsilon > 0$. We need to find $\delta > 0$ such that $\forall x, y \in X$:

$$\rho(x, y) < \delta \implies \sigma(F(x), F(y)) < \epsilon$$

Since F is continous, $\forall z \in X$, $\exists \delta_z > 0$ such that:

$$\rho(x, z) < \delta_z \implies \sigma(F(x), F(z)) < \epsilon/2$$

That is,

$$F(B_{\delta_z}(z)) \subseteq B_{\epsilon/2}(F(z))$$

Let $\delta > 0$ be a Lebesgue number for the cover $\{B_{\delta_z}(z)\}_{z \in X}$.
Now supposed $\rho(x, y) < \delta$. Then $\exists z \in X$ such that:

$$B_\delta(x) \subseteq B_{\delta_z}(z)$$

and

$$\sigma(F(x), F(y)) \leq \sigma(F(x), F(y)) + \sigma(F(z), F(y)) < \epsilon/2 + \epsilon/2 = \epsilon$$

Defn. Let (X, ρ) be a metric space and $C(X)$ the $\mathbb{C}$-vector space of continous functions on X. We say $\mathcal{J} \subset C$ is equicontinous at $x \in X$ if $\forall \epsilon > 0, \exists \delta > 0$ such that:

$$\forall F \in \mathcal{J}, F(B_\delta(x)) \subseteq B_\epsilon(F(x))$$

We say $\mathcal{J}$ is equicontinous on X if $\forall x \in X$, $\mathcal{J}$ is equicontinous at x.

Ex. Let $X = [0, 1] \subset \mathbb{R}$. Let $F_n(x) = x^n \forall n \geq 1$. Let:

$$\mathcal{J} = \{F_n : n \in \mathbb{N}\}$$

Let $x_n = \frac{1}{2}^{1/n}$. Then, x_n arrow up to 1. Then,

$$|F_n(x_n) - F_n(1)| = |1/2 - 1| = 1/2$$

Thus, $\mathcal{J}$ is not equicontinous at 1.

Ex. (Hwk) Show that $\mathcal{J}$ is equicontinous on $[0, 1)$.

Defn. Let (F_n) be a sequence of ($\mathbb{C}$-valued) functions on X. Then, (F_n) is uniformly bounded if $\exists M > 0$ such that $\forall n \geq 1, \forall x \in X$:

$$|F_n(x)| < M$$

We say that (F_n) is pointwise bounded if $\forall x \in X$, $\exists M_x > 0$ such that:

$$|F_n(x)| < M_x$$

Defn. A metric space (top. space) is seperable if there is a countable dense subset $D \subset X$.

Ex. Since $\mathbb{Q}^n \subset \mathbb{R}^n$ is dense, $(\mathbb{R}^n, ||\cdot||_p)$ is separable.

Lemma. (Arzelà-Ascoli) Let (X, ρ) is a seperable metric space and that (F_n) is pointwise bounded and equicontinous in $C(X)$. Then, there is subsequence (F_{n_k}) such that:

$$\lim_{x \to \infty} F_{n_k}(x)$$

exists $\forall x \in X$.

Yap. Given a sequence (x_n), we get subsequence by finding $n_k \in \mathbb{N}$ such that $n_{k+1} > n_k$ and $(x_{n_k})_{k=1}^{\infty} \to x$ is a sequence.
A subsubsequence is determined by finding $n_{k_1} < n_{k_2} < \cdots$ and then we write:

$$(x_{n_{k_j}})_{j=1}^{\infty}$$

A subsequence is determined by an infinite subset $S_1 = \{n_1 \leq n_2 \leq \cdots\} \subset \mathbb{N}$. A subsubsequence is determined by an infinite subset $S_2 \subset S_1$,

$$S_2 = \{n_{k_1} < n_{k_2} < \cdots\} \subset S_1$$

Now, we write:

$$\lim_{n \in S_1} x_n = a \text{ instead of } \lim_{k \to \infty} x_{n_k}$$

Note that $\lim_{n \in S_1} x_n = a$ if:

$$\forall \epsilon > 0 \, \exists N : n \geq N, n \in S' \implies |x_n - a| < \epsilon$$

Rmk. Suppose $S_1 \subset \mathbb{N}$ determines a subsequence as above. Suppose $S' \subset \mathbb{N}$ is infinite and:

$$\{n \in S' : n \notin S_1\}$$

is finite. The $\lim_{n \in S_1} x_n = a$ then $\lim_{n \in S'} x_n = a$ as well.
Proof of the AA Lemma: Pictures. Apr 9 and 10.

Rmk. If X is compact, then $C(x) = C_b(X)$ is a complete metric space with respect to the uniform norm $|| \cdot ||_{\infty}$.

Thm. (Arzelà-Ascoli) Let (X, ρ) be a compact metric space and $(F_n) \subset C(X)$ be a sequence of functions that are point-wise bounded and equicontinous. Then (F_n) has a subsequence converging uniformly to some function $F \in C(X)$. Proof. Pictures.

Lemma. Suppose X is compact and that $\mathcal{J} \subset C(X)$ is equicontinous on X. Then, $\mathcal{J}$ is uniformly equicontinous on X, in that for all $\epsilon > 0 \exists \delta > 0$ such that for all $x, y \in X$ and all $F \in \mathcal{J}$,

$$\rho(x, y) < \delta \implies |F(x) - F(y)| < \epsilon$$

Rewriting,

$$F(B_\delta(x)) \subseteq B_\epsilon(F(x))$$

Proof left as homework.

Corr. Let X be a compact metric space. Let $\mathcal{J} \subset C(X)$ be a closed subset such that $\mathcal{J}$ is equicontinous and pointwise bounded. Then $\mathcal{J}$ is compact and uniformly bounded.

Thm. Suppose X is a compact metric space. Then $\mathcal{J} \subset C(X)$ is compact if and only if $\mathcal{J}$ is closed, uniformly bounded, and equicontinous on X. Proof. Pictures.

Defn. A topological space is called a Baire space if the countable intersection of dense open sets is dense.

Rmk. If ρ and σ are equivalent metrics on X then (X, ρ) is a Baire space if and only if (X, σ) is a Baire space.

Defn. If $S \subset X$ then the interior of S is:

$$Int(S) = \bigcup \{U \subset S : U \text{ is open in } X\}$$

Rmk. One can write $Int_X(S)$ as interior of S in X. Where you take the interior matters.

Lemma. A space X is a Baire space if and only if given a countable $\{F_n\}_{n=1}^{\infty}$ of closed sets, such that:

$$\bigcup_{n=1}^{\infty} F_n$$

has non-empty interior, then at least one F_n has (non-empty) interior.

Hwk. Write $O_n = F_n^C = X \setminus F_n$. O_n dense $\iff$ $Int(F_n) = \emptyset$.

Thm. Barise Category Thm. Every complete metric space is a Baire space.

Rmk. Note that $(0, 1)$ is homeomorphic to $\mathbb{R}$. Hence, $(0, 1)$ is a Baire space.

Hwk. More generally, every non-empty open subset V of a complete metric space admits an equivalent complete metric (Hwk). Hence V is a Baire space. Thus, if:

$$V = \bigcup_{n=1}^{\infty} V \cap F_n$$

of each F_n closed in X, thus $V \cap F_n$ is closed in V. Hence, at least one $V \cap F_n$ has non-empty interior in V. But V is open, so it has interior in X.

Thm. Suppose that X is a Baire space and that $(F_n) \subset C(X)$ such that $F_n \to F$ pointwise. Then,

$$A = \{x \in X : F \text{ is cts at } x\}$$

is dense in X.
Proof. Pictures.

Rmk. If $X = \mathbb{R}$, then we can assume:

$$A = \{x \in X : F \text{ is cts at } x\}$$

is uncountable.

Defn. A normed vector space $(V, ||\cdot||)$ is called a Banach space if V is complete in in the induced metric. Recall that $\mathbb{F}$ is always $\mathbb{C}$ or $\mathbb{R}$.

Yap. Normed vector spaces are special.

1. $B_r(x) = x + B_r(0)$. The topology is homogenous.
2. $|||v|| - ||w||| \leq ||v - w|| \implies v \to ||v||$ is continous.
3. $\overline{B_r(v)} = \{x \in V : ||x - v|| \leq r\}$.
4. $\epsilon B_r(0) = B_{\epsilon r}(0)$.
5. $\epsilon \overline{B_r(0)} = \overline{B_{\epsilon r}(0)}$.

Thm. Supposed that X and Y are normed vector spaces and $T : X \to Y$ is linear. Then, the following are equivalent:

1. T is continuous.
2. T is continous at a single point.
3. $\exists \alpha \geq 0$ such that $||T(x)|| \leq \alpha ||x||$ for all $v \in X$.

Proof.
(1) $\implies$ (2) is trivial.
(2) $\implies$ (3). Since T is continous at x_0, $\exists \delta > 0$ such that:

$$T(\overline{B_\delta(x_0)}) \subset B_1(T(x_0))$$

LHS $= T(B_\delta(x_0)) + T(x_0)$.
RHS $= B_1(0) + T(x)$.
This implies, $T(\overline{B_\delta(0)}) \subset B_1(0)$.
Now if $z \neq 0$,

$$||T(z)|| = ||\frac{||z||}{\delta} T(\delta \cdot \frac{z}{||z||})|| \leq \frac{||z||}{\delta}$$

Let $\alpha = \frac{1}{\delta}$.
(3) $\implies$ (1). Erased.

Rmk. Suppose that $||\cdot||_1$ and $||\cdot||_2$ (two different norms) induce equivalent metrics on V. Then, take the id : $(V, ||\cdot||_1) \to (V, ||\cdot||_2)$ is continous (same topology). Hence, $\exists c \geq 0$ such that:

$$||x_2|| \leq c \cdot ||x_1||$$

Clearly, that means $c > 0$. By symmetry, there exists a $d > 0$, such that:

$$||x_1|| \leq d \cdot ||x_2||$$

Thus, the metrics and the norms are strongly equivalent.

Defn. If X and Y are normed vector spaces, then $\mathcal{L}(X,Y)$ is the vector space of continous linear maps $T : X \to Y$. Define $||T|| = \sup_{||x|| \leq 1} ||T(x)||$. If $X = Y$, then we write $\mathcal{L}(X)$ instead of $\mathcal{L}(X,X)$.

Lem. With $||T||$ as above, $\mathcal{L}(X,Y)$ is a normed vector space with:

$$||T(x)|| \leq ||T||||x|| \forall x \in X$$

If $S \in \mathcal{L}(Y,Z)$, then we write ST in place of $S \circ T$ and:

$$||ST|| \leq ||S||||T||$$

Defn. An algebra over $\mathbb{F}$ is a vector space A over $\mathbb{F}$ with a ring structure, with $\lambda(xy) = (\lambda x)y = x(\lambda y)$ for all $\lambda \in \mathbb{F}$ and $x, y \in A$.

Ex. $M_n(\mathbb{F})$, $\mathbb{R}[x]$, $\mathbb{C}[x]$, $C(X)$.

Defn. If $|| \cdot ||$ is a norm on an algebra A, then we call $(A, || \cdot ||)$ a normed algebra if $\forall x, y \in A$:

$$||xy|| \leq ||x||||y||$$

We call $(A, || \cdot ||)$ a Banach algebra if $(A, || \cdot ||)$ is a normed algebra and A is complete with respect to $|| \cdot ||$.

Prop. If X and Y are normed vector spaces, and Y is a Banach space, then $\mathcal{L}(X,Y)$ is a Banach space. If X is a Banach space, then $\mathcal{L}(X)$ is a Banach algebra.
Proof. Suppose that (T_n) is a Cauchy sequence in $\mathcal{L} :(X,Y)$. Then, for each $z \in X$, $T_n(X)$ is a Cauchy sequence in Y. Hebcem $\exists T(x) \in Y$ such that:

$$T_n(X) \to T(X)$$

It is not hard to see that $T : X \to Y$ is linear.
Since (T_n) is Cauchy in norm, its bounded i.e. $\exists m > 0$ such that:

$$||T_n|| \leq M \forall n \geq 1$$

Now if $||x|| \leq 1$, then:

$$||T(x)|| = \lim_{n \to \infty} ||T_n(x)|| \leq \text{limsup}_n \, ||T_n||||x|| \leq M||x||$$

This implies $T \in \mathcal{L}(X,Y)$.
Let $\epsilon > 0$. Let N be such that $m, n \geq N$. Thus,

$$||T_n - T_m|| \leq \frac{\epsilon}{2}$$

Now if $||x|| \leq 1$, and if $n \geq N$,

$$\begin{aligned} ||(T - T_n)(x)|| &= ||T(x) - T_n(x)|| \\ &= \lim_{m \to \infty} ||T_m(x) - T_n(x)|| \\ &\leq \text{limsup}_m \, ||T_m - T_n|| \cdot ||x|| \\ &\leq \frac{\epsilon}{2} < \epsilon \end{aligned}$$

Yap. Let $Y \subset X$ be a subsoace of a normed vector space X. Then we can form the quotient vector space $X/Y = \{x + Y : x \in X\}$ with $q : X \to X/Y$ the quotient map.
If $x \in X$, then:

$$\inf\{||x - y|| : y \in Y\}$$

depends only on $q(x)$. We call:

$$||q(x)|| := \inf\{||x - y|| : y \in Y\}$$

the quotient norm on X/Y.

Rmk. As Y is a subspace,

$$\begin{aligned} ||q(x)|| &= \inf\{||x + y|| : y \in Y\} \\ &= \inf\{||x + \alpha y|| : y \in Y\} \, \forall \alpha \in \mathbb{F} \setminus \{0\} \end{aligned}$$

Thm. If Y be a subspace of $(X, ||\cdot||)$. Then,

$$||q(x)|| = \inf\{||x - y|| : y \in Y\}$$

is a seminorm on X/Y which is a norm exactly when Y is closed. If X is a Banach space, and Y is closed in X, then X/Y is a Banach space.
Proof. Note that for $\alpha \neq 0$:

$$||\alpha q(x)|| = ||q(\alpha x)|| = \inf\{||\alpha x + \alpha y|| : y \in Y\} = |\alpha| \inf\{||x + y|| : y \in Y\} = |\alpha| ||q(x)||$$

Next, fix $x_1, x_2 \in X$. Then given $\epsilon > 0$, $\exists y_1, y_2 \in Y$ such that:

$$\begin{aligned} ||q(x_1)|| + ||q(x_2)|| + \epsilon &\geq ||x_1 - y_1|| + ||x_2 - y_2|| \\ &\geq ||x_1 - y_1 + x_2 - y_2|| \\ &\geq ||q(x_1 + x_2)|| \\ &= ||q(x_1)|| + ||q(x_2)|| \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $||q(x_1) + q(x_2)|| \leq ||q(x_1)|| + ||q(x_2)||$. Thus, it is a semi-norm.
Sketch for closed. If quotient norm is 0, there exists a sequence y_n converging to x. If

Y closed, $x \in Y$ and $g(x) = 0$. Conversely, if quotient norm is a norm and $y_n \to x$. Then, $||q(x)|| = 0$. Then, $q(x) = 0$ and $x \in X$.
Now suppose X is a Banach space and Y is closed in X. Let $(q(z_0))$ be Cauchy in X/Y. Then we can pass to a subsequence and assume:

$$||q(z_{n+1}) - q(z_0)|| \leq 1/2^n$$

Let $x_1 = z_1$. Since:

$$||q(z_2) - q(x_1)|| < 1/2$$

Thus,

$$\inf\{||z_2 - x_1 - y|| : y \in Y\} < 1/2$$

Hence, we can find x_2 such that $q(x_1) = q(z_2)$ and

$$||x_2 - x_1|| < 1/2$$

Continuing, we get (x_n) such that $q(x_n) = q(z_n)$ and:

$$||x_{n+1} - x_n|| < 1/2^n$$

(Hwk) (x_n) is Cauchy, thus, $x_n \to x$ in X.
But $||q(z)|| \leq ||z||$. Hence, q is continous. Thus,

$$q(z_n) = q(x_n) \to q(x)$$

Rmk. Note that $q : X \to X/Y$ is a bounded linear map of norm at most 1.

Thm. Suppose that Y is a closed subspace of $(X, || \cdot ||)$. Then X is a Banach space if and only if both Y and X/Y are Banach spaces.
Proof. Pictures. 4/16

Thm. Every finite-dimensinal subspace Y of a normed vector space X is a Banach space and hence closed in X. If $\dim(Y) = n$, then every linear isomorphism $\Phi : \mathbb{F}^n \to Y$ is a homeomorphism.
Proof. Pictures. 4/16.

Corr. If Y is finite-dimensional vector space, then all norms $|| \cdot ||$ are strongly equivalent.
Proof. $\mathrm{id} : (Y, || \cdot ||_1) \to (Y, || \cdot ||_2)$ is a homeomorphism and continous linear maps are bounded.

Recall. Recall the topological definition of continous functions and open maps. If $F : X \to Y$ is a bijection, then $F^{-1} : Y \to X$ is continous if and only if F is open.

Thm. (The Big Three) (Open Mapping Theorem) Suppose that X and Y are Banach spaces and that $T \in \mathcal{L}(X, Y)$ is a surjection. Then T is an open map.

Lem. It will suffice to find $r > 0$ such that:

$$B_r(0) \subseteq T(B_1(0))$$

Proof. By homogeneity, $T(B_\delta(0))$ is a neighborhood of 0_Y for all $\delta > 0$. By linearity, $T(B_\delta(x))$ is a neighborhood of $T(x)$ for all $x \in X$ and $\delta > 0$. Thus, if V is open in X and $x \in V$, then $\exists \delta > 0$ such that:

$$B_\delta(x) \subseteq V$$

Then, $T(B_\delta(x))$ is a neighborhood of $T(x)$ in $T(Y)$. Thus, T is an open map. Thus, $T(V)$ is open.

Lem. It will suffice to find $r > 0$ such that:

$$B_r(0) \subseteq \overline{T(B_1(0))}$$

Proof. Assume $r > 0$ is such that the property holds. Let $y \in B_r(0)$. Then,

$$\exists y_1 \in T(B_1(0)) : ||y - y_1|| < r/2$$

Then, $y - y_1 \in B_{r/2}(0)$.
Hence, $\exists y_2 \in 1/2T(B_1(0)) = T(B_{1/2}(0))$ such that:

$$||y - y_1 - y_2|| < r/2^2$$

Continue to get a sequence (y_n) such that:

$$y_n \in 2^{-n+1}T(B_1(0)) = T(B_{2^{-n+1}}(0))$$

and:

$$||y - \sum_{i=1}^{n} y_i|| < 2^{-n}r$$

By construction, $\exists x_n \in X$ such that:

$$T(x_n) = y_n \qquad ||x_n|| < 2^{-n+1}$$

Since X is a Banach space, $x = \sum_{n=1}^{\infty} x_n$ converges and since T is continous (hence bounded), and also:

$$||x|| < \sum_{n=1}^{\infty} 2^{-n+1} = 2$$

and

$$T(x) = y$$

Since y was arbitrary, we have shown that:

$$B_r(0) \subset T(B_2(0))$$

This implies:

$$B_{r/2}(0) \subset T(B_1(0))$$

This suffices by lemma 1.

Lem. 3 If suffices to see that:

$$\overline{T(B_n(0))}$$

has interior for some $n \geq 1$.
Proof. By homogeneity, we can assume that $\overline{T(B_1(0))}$ has interior.
Thus, $\exists \epsilon > 0$ such that:

$$B_\epsilon(y) \subset \overline{T(B_1(0))}$$

Let $z \in B_\epsilon(0)$. Write

$$z = z/2 + y - (y - z/2) \in B_{\epsilon/2}(y) - B_{\epsilon/2}(y) \subseteq \overline{T(B_{1/2}(0))} - \overline{T(B_{1/2}(0))}$$

Rest in pictures.

Proof. Of the Open Mapping Theorem. Pictures. 4/18.

Ex. Let $1 \leq p < \infty$. Then, let:

$$l_0^p = \text{span}\{e_n : n \geq 1\} = \{x \in l^p : x(n) = 0 \text{ for all but finitely many } n\}$$

Notte l_0^p is dense in l^p.
Define: $T_0 : l_0^p \to l_0^p$ by:

$$T_0(e_n) = 1/ne_n$$

You can check that $||T|| = 1$. Also T_0 is a bijection.

$$T_0^{-1}(e_n) = ne_n$$

but $T_0^{-1} \notin \mathcal{L}(l_0^p)$

Thm. Suppose that X and Y are Banach spaces and $T \in \mathcal{L}(X, Y)$ is a bijection. Then $T^{-1} \in \mathcal{L}(Y, X)$.
Proof. T^{-1} is linear by general nonsense and T is open by the Open Mapping Theorem. Hence T^{-1} is continous hence bounded.

Yap. If X and Y are Banach spaces, then I can give $X \times Y$ a norm by:

$$||(x, y|| = max\{||x||, ||y||\})$$

This makes $X \times Y$ a Banach spaces.

Thm. The map:

$$\Phi : l^q \to (l^p)^\vee$$

given by $\Phi(y) = \phi_y$ is an isometric isomorphism. That is, Φ is a isomorphism such that:

$$||\Phi(y)|| = ||y||$$

Rmk. If $z \in \mathbb{C}$, then:

$$\text{sgn}(z) = \frac{z}{|z|} \text{ if } z \neq 0 \text{ else } 0$$

Then, $z = |z| \,\text{sgn}(z)$ and $|z| = \text{sgn}(z)z$.

Lemma. If $y \in l^q$, then $||\phi_y|| = ||y||_q$. Proof. Pictures. 4/28.

Lemma. If $\phi \in (l^p)^\vee$, then $\exists y \in l^q$ such that:

$$\phi = \phi_y$$

Proof. Pictures. 4/28. Let $y_n = \phi(e_n)$. Then, if $x \in l^p$, then:

$$x = \sum_{n=1}^{\infty} x_n e_n$$

Note,

$$||x = \sum_{n=1}^{N} x_n e_n||_p^p = \sum_{n=N+1}^{\infty} |x_n|^p$$

converges in the Banach space l^p.
Hence,

$$\phi(x) = \sum_{n=1}^{\infty} x_n y_n$$

Then, if we can show $y \in l^q$, then $\phi = \phi_y$, and we are done.
Let $y^N \in l^q$ be given by:

$$y^N(n) = \begin{cases} y_n \text{ if } 1 \leq n \leq N \\ 0 \text{ otherwise} \end{cases}$$

Then, $\phi_{y^N} \in (l_p)^\vee$ and $||\phi_{y^N}|| \leq ||y^N||_q$.
If $x \in l^p$, then,

$$\phi_{y^N}(x) = \sum_{n=1}^{N} x_n y_n \to \sum_{n=1}^{\infty} x_y y_n = \phi(x)$$

Hence, by the Principle of Uniform Boundedness, $\exists M > 0$ such that:

$$||\phi_{y^N}|| \leq M \qquad \forall N \geq 1$$

Thus, if $q < \infty$, then,

$$\sum_{n=1}^{N} |y_n|^q \leq M^q < \infty$$

Rmk. If X is a normed vector space, then we know that X^* is a Banach space. Then, we can form the bidual $(X^*)^* = X^{**}$. We get a natural map ι

$$\iota : X \to X^{**}$$

$\iota(x)(\phi) = \phi(x)$ is evaluation at x.
Now,

$$||\iota(x)|| = \sup_{||\phi|| \leq 1} |\iota(x)(\phi)| = \sup_{||\phi|| \leq 1} |\phi(x)|(*)$$

Then, $(*) \leq ||x||$ since $||\phi|| \leq 1$.
Then, by Hahn-Banach, $\exists \phi \in X^*$ such that $||\phi|| = 1$ and $\phi(x) = ||x||$. Thus,

$$||\iota(x)|| = ||x||$$

We can identify X at $\iota(X)$ in X^{**}.
If X is a Banach space, then, $\iota(x)$ is complete, hence closed in X^{**}.
Otherwise, $\overline{\iota(X)}$ is a Banach space containing X as a dense subspace. Then, $\overline{\iota(X)}$ is the completion of X as a Banach space.

Defn. If X is a Banach space then X is reflexive if $\iota(X) = X^{**}$ is onto.

Rmk. If X is reflexive, then X^{**} are isometrically isomorphic. The converse can fail. It can be that they are isometrically isomorphic but ι is not the map. [Robert James 1951].

Ex. Let $1 < p < \infty$. Let $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$(l^p)^* = \{\phi_y^p : y \in l^q\}$$

and also:

$$(l^q)^* = \{\phi_x^q : x \in l^p\}$$

If $x \in l^p$, then, $\iota(x) \in (l^p)^{**}$, and:

$$\begin{aligned}\iota(x)(\phi_y^p) &= \phi_y^p(x) \\ &= \phi_x^q(y)\end{aligned}$$

Thus, $\iota : l^p \to (l^p)^(* *)$ is surjective and l^p is reflexive for $1 < p < \infty$.

Prop. l^p is reflexive for $1 < p < \infty$.

Rmk. Let X be a normed vector space and $D = \{d_n\}_{n=1}^{\infty} \subset X$ be a countable subset of X. Then the rational space of D, $\mathrm{span}_{\mathbb{Q}}(D)$ is the span of D viewing X as a rational vector space. Thus,

$$\mathrm{span}_{\mathbb{Q}}(D) = \bigcup_{n=1}^{\infty}\{\sum_{k=1}^{r} r_k d_k : r_k \in \mathbb{Q}\, d_k \in D\}$$

Then, $\mathrm{span}_{\mathbb{Q}}(D)$ is countable. Then if $\mathrm{span}(D)$ is dense, then since $\mathrm{span}_{\mathbb{Q}}(D)$ is dense in $\mathrm{span}(D)$. Thus, X is separable.
Same works over $\mathbb{C}$ replacing $\mathbb{Q}$ by $\mathbb{Q} + i\mathbb{Q}$.

Corr. l^p is seperatble for $1 \leq p < \infty$. Let $D = \{e_n\}_{n=1}^{\infty}$. But l^{∞} is not separable.

Hwk. If X^* is separable, then X is.

Thm. If $1 \leq p \leq \infty$, and if q is the conjugate exponent, then:

$$\Phi : l^q \to (l^p)^*$$

is an isometric isomorphism when $\Phi(y) = \phi_y^p$ and

$$\phi_y^p = \sum_{n=1}^{\infty} x_n y_n$$

Rmk. If $(l^p)^* = \{\phi_y^p : y \in l^q\}$ and if $q \leq \infty$, then:

$$(l^q)^* = \{\phi_x^q : x \in l^p\}$$

Defn. Recall: If X is a Banach space, then X is reflexive if $\iota(X) = X^{**}$ is surjective.

Prop. If $1 < p < \infty$, then l^p is reflexive.
Proof. COnsider $\iota : l^p \to (l^p)^{**}$. If $\phi_y^p \in (l^p)^*$, then:

$$\iota(x)(\phi_y^p) = \phi_y^p(x) = \phi_x^q(y)$$

Let $\psi \in (l^p)^{**}$. Then,

$$\psi(\phi_y^p) = \psi(\Phi(y))$$

Now $y \to \psi(\Phi(y))$ is a in $(l^q)^*$. Hence, $\exists x \in l^p$ such that:

$$\psi(\Phi(y)) = \phi_x^q(y)$$

Thus, $\iota(x) = \psi$. Thus, ι is onto.

Rmk. If X is a normed vector space over $\mathbb{F}$ and $D \subset X$ is countable and $\overline{\text{span}(D)} = X$, then X is separable.

Corr. If $1 \leq p < \infty$, then l^p is separable.
Proof. $D = \{e_n : n \geq 1\}$.

Ex. l^∞ is not separable.
Proof. Let $A \subseteq \mathbb{N}$. Let:

$$x_A(n) = \mathbf{1}_{n \in A}$$

If $A \neq B$, then $||x_A - x_B||_\infty = 1$. But $\{x_A : A \subseteq \mathbb{N}\}$ is uncountable. You cannot find a countable dense subset anymore (cant fit all of them into countably many 1/4 balls).

Hwk. $(l^\infty)^*$ is not separable. Thus, l^1 is not reflexive.

Defn. Suppose $T \in \mathcal{L}(X, Y)$. Then we define:

$$T^* : Y^* \to X^*$$

by:

$$T^*(\phi)(x) = \phi(T(x))$$

Prop. If X and Y are normed vector spaces, and $T \in \mathcal{L}(X, Y)$, then $T^* \in \mathcal{L}(Y^*, X^*)$ and $||T^*|| = ||T||$.

Proof. First, check that T^* is linear (easy).

Next, show the norm equality. Look at:

$$\begin{aligned} ||T^*(\phi)|| &= \sup_{||x|| \leq 1} |T^*(\phi)(x)| \\ &= \sup_{||x|| \leq 1} |\phi(T(x))| \\ &\leq ||\phi|| ||T|| ||x|| \\ &\leq ||\phi|| ||T|| \end{aligned}$$

Thus,

$$||T^*|| \leq ||T||$$

Fix $\epsilon > 0$. Then, $\exists x_0 \in X$ such that $||x_0|| = 1$ and:

$$||T(x_0)|| > ||T|| - \epsilon$$

But $\exists \phi \in Y^*$ such that $||\phi|| = 1$ and:

$$\phi(T(x_0)) = ||T(x_0)||$$

Then,

$$\begin{aligned} ||T^*|| &\geq ||T^*(\phi)|| \\ &\geq |T^*(\phi)(x_0)| \\ &= |\phi(T(x_0))| \\ &= ||T(x_0)|| \qquad > ||T|| - \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have:

$$||T^*|| \geq ||T||$$

Thus, $||T^*|| = ||T||$.

Thm. Let X and Y be Banach spaces and suppose that:

$$T : X \to Y \qquad S : Y^* \to X^*$$

are functions (not linear or bounded) such that $\forall \phi \in Y^*$ and $x \in X$:

$$S(\phi)(x) = \phi(T(x))$$

Then, $T \in \mathcal{L}(X,Y)$ and $S \in \mathcal{L}(Y^*, X^*)$, with $S = T^*$.
Proof. Supposed that $x, y \in X$ and $\alpha \in \mathbb{F}$. Then, if $\phi \in Y^*$, we have:

$$\begin{aligned}
\phi(T(\alpha x + y)) &= S(\phi)(\alpha x + y) \\
&= \alpha(S(\phi)(x)) + S(\phi)(y) \\
&= \alpha\phi(T(x)) + \phi(T(y)) \\
&= \phi(\alpha T(x) + T(y))
\end{aligned}$$

Since $\phi \in Y^*$ is arbitrary, we have $T(\alpha x + y) = \alpha T(x) + T(y)$. (something about separate points).
To see that T is bounded, use CGT. Suppose that $x_n \to x$ in X and $T(x_n) \to y$ in Y. But $\forall \phi \in Y^*$, we have:

$$\begin{aligned}
\phi(y) &= \lim \phi(T(x_n)) \\
&= \lim S(\phi)(x_n) \\
&= S(\phi)(x) \\
&= \phi(T(x))
\end{aligned}$$

Thus, $y = T(x)$. Thus, T is bounded.
But $T^*(\phi) = S(\phi)$.

Yap. Let (X, τ) be a topological space. Then $\beta \subset \tau$ is a basis for τ if given any $U \in \tau$ and $x \in U$, then $\exists V \in \beta$ such that $x \in V \subset U$.
We say S is a neighborhood of $x \in X$ if $\exists U \in \tau$ such that $x \in U \subset S$.
Wrote $\mathcal{N}(x)$ for the set of all neighborhoods of x. We say that $\alpha \subseteq \mathcal{N}(x)$ is a neighborhood basis at x if $U \in \mathcal{N}(x)$, there exists $V \in \alpha$ such that:

$$x \in V \subseteq U$$

Ex. (1) In a metric space, the collection of all open balls is a basis for the metric topology.
(2) In $\mathbb{R}^n$, every point has a neighborhood basis consisting of compact sets. Such spaces are called locally compact.

Lemma. Low Hanging Fruit: Let (X, τ) be a topological space and $\alpha(x)$ has a neighborhood basis at $x \in X$ consisting of open sets. Then:

$$\beta = \bigcup_{x \in X} \alpha(x)$$

is a basis for τ.

Lemma. Low Hanging Fruit: $\beta \in \tau$ is a bssis for τ if and only if given $U \in \tau$,

$$U = \bigcup_{V \in \beta \subset U} V$$

Defn. Let (X, τ) be a topological space. Then,

1. (X, τ) is seperable if it ihas a countable dense subset.
2. (X, τ) is 2nd countable if it has a countable basis.
3. (X, τ) 1st countable if every point has a countable neighborhood basis.

Rmk.
1. Every 2nd countable space is seperable. The converse holds in metric spaces. Hwk 8.
2. Metric spaces are 1st countable.

Ex.
1. Let X be a set. Then $\tau = \mathcal{P}(X)$ is the discrete topology. This is the metric topology coming from the the discrete metric.
2. $\tau = \{X, \phi\}$.

Lemma. Let $S \subset \mathcal{P}(X)$. Then, there is a smallest topology $\tau(S)$ that contains S.
Proof. Let $\tau(S) = \bigcap \{\tau' : \tau' \text{ is a top and } S \in \tau'\}$.

Prop. Let $\beta \subset \mathcal{P}(X)$ be a coer of X. Then β is a basis for $\tau(B)$ if and only if $U, V \in \beta$ and $x \in U \cap V$, then $\exists W \in \beta$ such that $x \in W \subset U \cap V$.
Proof. Hwk.

One lecture on weak topology omitted.

Defn. An ordered set $(X, \leq)$ is directed if given $x, y \in X$, $\exists z \in X$ such that $x \leq z$ and $y \leq z$.

Ex.
- $X = \mathbb{N}$.
- Let (Y, τ) be a topological space and $y_0 \in Y$. Then $X = \mathcal{N}(y_0)$ be the collection of neighborhoods of y_0 induced by reverse inclusion. Then, $U \leq V \iff V \subset U$. Then $\mathcal{N}(y_0)$ is directed. If $U, V \in \mathcal{N}(y_0)$, then so is $U \cap V$.
- We can do the same thing for open neighborhoods.

Defn.
- A net x is a set X is a function $x : \ ambda \to X$ where Λ is a directed set. As with seqeucnes, we usually write X_λ for $x(\lambda)$ and $(x_\lambda)_{\lambda \in \Lambda}$ in place of $x : \Lambda \to X$, or just x.
- If X is a topological space, then we say that a net $(x_\lambda)_{\lambda \in \Lambda}$ converges to $x_0 \in X$ if (x_λ) is eventually in every neighborhood of x_0. That is, if $U \in \mathcal{N}(x_0)$, then $\exists \lambda_0 \in \Lambda$ such that:
$$\lambda \geq \lambda_0 \implies x_\lambda \in U$$
- We say that $x_0 \in X$ is an accmulation point of a net x_λ if (x_λ) is frequently in every neighborhood of x_0. That is if U is a neighborhood of x_0 and $\lambda_0 \in \Lambda$, then $\exists \lambda \geq \lambda_0$ such that $x_\lambda \in U$.

Prop. Suppose X is a topological space and $E \subset X$. Then, $x_0 \in \overline{E}$ if and only if there exists a net $(x_\lambda) \subset E$ with $x_\lambda \to x_0$.
Proof. Suppose we have $x_\lambda \subset E$ with $x_\lambda \to x_0$. If $x_0 \notin \overline{E}$, then $\exists U \in \mathcal{O}(x_0)$ such that $U \cap E = \emptyset$. But then we would eventually have (x_λ) in U. Contradiction.

Now suppose that $x_0 \in \overline{E}$. Let $\Lambda = \mathcal{O}(x_0)$.
If $W \in \Lambda$, then $W \cap E \neq \emptyset$. Then, we let $x_W \in W \cap E$. Then, $(x_W)_{W \in \Lambda}$ converges to x_0.
If $U_0 \in \mathcal{O}(x_0)$, then $U \geq U_0$, $x_U \in U \subset U_0$.

Ex. $S = \{\sqrt{n} e_n \in l^2 : n \geq 1\}$, then $0 \in \overline{S}^w$ (weak topology).

Defn. A subset C of a vector space V over $\mathcal{F}$ is convex if $x, y \in C$ and $t \in [0,1]$, then $tx + (1-t)y \in C$.

Ex. In a normed vector space X, every open ball $B_r(x_0)$ is convex if $x_0 \in X$ and $r > 0$. To see this, supposed $x, y \in B_r(x_0)$, and $t \in [0,1]$. Then,
$$\begin{aligned} ||tx + (1-t)y - x_0|| &\leq ||t(x - x_0)|| + ||(1-t)(y - x_0)|| \\ &< tr + (1-t)r \\ &= r \end{aligned}$$

Lemma. Let C be an open convex neighborhood of 0 in a normed vector space X. Then,

$$m(x) = \inf\{s > 0, s^{-1}x \in C\}$$

Then m is a Minkowski functional on X such that:

$$C = \{x \in X : m(x) < 1\}$$

Proof. If $x \in X$, then $\frac{1}{n}x \to 0$. So $\frac{1}{n}x$ is eventually in C. So $m(x) < \infty$ for all $x \in X$.

Note that if $t \geq 0$, then $m(tx) = tm(x)$. (This is one of the axioms for a Minkowski functional).
Also let $s^{-1}x$, $t^{-1}y \in C$. Then:

$$(s+t)^{-1}(x+y) = \frac{s}{s+t}s^{-1}x + \frac{t}{s+t}t^{-1}y \in C$$

Thus, $m(x+y) \leq s+t$.
Since $s^{-1}x, t^{-1}y \in C$ is arbitrary, we have:

$$m(x+y) \leq m(x) + m(y)$$

This is the second axiom for a Minkowski functional.
Thus, $m : X \to \mathbb{R}$ is a Minkowski functional. Now if $x \in C$, then $(1+1/n)x \to x$. Thus, for some n,

$$m(x) = frac11 + 1/n < 1$$

If $m(x)K1$, then $\exists s < 1$ such that $s^{-1}x \in C$.

But $0 \in C$ and $x = (1-s)0 + s(s^{-1}x) \in C$.

Thm. (Hahn-Banach Seperation Theorem). Let A and B be disjoint non-empty convex subsets of some normed vector space X. If A is also open, then there exists $\phi \in X^*$ and $t \in \mathbb{R}$ such that:

$$\Re(\phi(x)) < t \leq \Re(\phi(y))$$

for all $x \in A$ and $y \in B$.

Proof. We start with $\mathbb{F} = \mathbb{R}$. Then, we can use the Basic Extension Lemma. Fix $x_0 \in A$ and $y_0 \in B$. Let $z_0 = x_0 - y_0$. Let:

$$C = A - B + z_0$$

We can check that C is convex.
Since $C = \bigcup_{y \in B} A - y + z_0$. Thus, C is union of open sets, hence open neighborhood of 0.

Let $m : X \to \mathbb{R}$ be the corresponding Minkowski functional.
We claim $z_0 \notin C$. If $x - y + z_0 = z_0$, then $x = y \in A \cap B - \emptyset$.
Thus, $m(z_0) \geq 1$.
Define:

$$\phi_0 : \mathbb{R}z_0 \to \mathbb{R}$$

by $\phi_0(\alpha z_0) = \alpha$.
Since $m(x) \geq 0$, then for all x if $\alpha \leq 0$, then:

$$\phi_0(\alpha z_0) \leq m(\alpha z_0)$$

If $\alpha \geq 0$, then:

$$\phi_0(\alpha z_0) = \alpha \leq \alpha m(z_0) = m(\alpha z_0)$$

Thus, $\phi_0(z) \leq m(z)$ for all $z \in \mathbb{R}z_0$.
Thus, the Basic Extension Lemma gives us $\phi : X \to \mathbb{R}$ such that, $\phi(x) \leq m(x)$.
Thus, $\phi(x) < 1$ if $x \in C$.
Also, $-\phi(x) = \phi(-x) < 1$ if $x \in -C$.
Thus, $\forall \epsilon > 0$,

$$|\phi(x)| < \epsilon \text{ if } x \in \epsilon C \cap -\epsilon C$$

Since $\epsilon C \cap -\epsilon C$ is a neighborhood of 0 for all $\epsilon < 0$, this means ϕ is continous at 0.
Thus, $\phi \in X^*$. If $x \in A$ and $y \in B$, then:

$$x - y + z_0 \in C$$

Then,

$$m(x - y + z_0) < 1$$

Thus,

$$\phi(x - y + z_0) < 1$$

Since $\phi(z_0) = \phi_0(z_0) - 1$, we have:

$$\phi(x) < \phi(y)$$

Since ϕ is linear and since A and B are convex, $\phi(A)$ and $\phi(B)$ are intervals. Since A is open and ϕ is linear, $\phi(A)$ must be open.
Then, we can let t be the right-hand endpoint of $\phi(A)$.

Now if $\mathbb{F} = \mathbb{C}$, then we can treat X as a real space and produce a real linear functional $\psi : X \to \mathbb{R}$ such that:

$$\psi(A) < t \leq \phi(B)$$

Now let $\phi(x) = \psi(x) - i\phi(ix)$. ϕ is continous since ψ is and continuity implies boundedness.