Math 113: Functional Analysis

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May 26, 2025

Problem 54. Suppose that X is a reflexive Banach space. Show that the unit ball $B = \{x \in X : ||x|| \leq 1\}$ is weakly compact. (Hint provided).

Solution.

Problem 59. Let E be a nonempty subset of a Hilbert space H. Let Y be a subspace spanned by E. Then $E^{\perp\perp}$ is the closure of Y in H.

Solution.

Problem 60. Let $X = l^2$. Show that the sequence $\{e_n\}$ of standard basis vectors converges weakly to 0.

Solution.

Problem 61. Let H be a hilbert space. If $x, y \in H$, define $\Theta_{x,y} : H \to H$ by:

$$\Theta_{x,y}(z) = (z \mid y)x$$

Compute the norm of $\Theta_{x,y}$ and its adjoint $\Theta_{x,y}^*$.

Solution.

Problem 64. Let $P \in \mathcal{L}(H)$ be the orthogonal projection onto a nonzero subspace W. Show that $P = P^* = P^2$ and that $||P|| = 1$. Conversely, show that if $P \in \mathcal{L}(H)$ and $P = P^* = P^2$, then P is an orthogonal projection onto its range.

Solution.

Problem 65.(Dini's Theorem) Suppose that X is a compact metric space and that $C(X)$ is the Banach space of real-valued functions on X. Show that if $(f_n) \subset C(X)$ is such that there is a $f \in C(X)$ such that $f_n(X) \nearrow f(x)$ for all $x \in X$, then $f_n \to f$ in $C(X)$. Equivalently, show that $f_n \to f$ uniformly on X. (Hint provided).

Solution.

Problem 66. A linear map $V : H \to H$ is called an isometry if $||V(x)|| = ||x||$ for all $x \in H$. Show that the following are equivalent:

1. V is an isometry.
2. $(V(x) \mid V(y)) = (x \mid y)$ for all $x, y \in H$.
3. $V^*V = I$.

Solution.

Problem 67. A surjective isometry $U : H \to H$ is called a unitary. Show that the following are equivalent for $U \in \mathcal{H}$.

1. U is a unitary.

2. U is invertible with $U^{-1} = U^*$.

3. If $\{e_n\}$ an orthonormal basis for H, then $\{U(e_n)\}$ is an orthonormal basis for H.

(Remark: (c) implies (a) is not true unless U is both linear and bounded.)

Solution.