Math 121: Hodge Theory

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Problem 1. Let X be a differentiable manifold. Prove that $H^k_{\mathrm{dR}}(X, \mathbb{C}) \simeq H^k_{\mathrm{dR}}(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$.

Solution. We note that the de Rham cohomology with complex coefficients is defined as:

$$H^k_{dR}(X, \mathbb{C}) = \frac{\ker(d_{\mathbb{C}} : \Omega^k(X) \otimes_{\mathbb{R}} \mathbb{C} \to \Omega^{k+1}(X) \otimes_{\mathbb{R}} \mathbb{C})}{\mathrm{im}(d_{\mathbb{C}} : \Omega^{k-1}(X) \otimes_{\mathbb{R}} \mathbb{C} \to \Omega^k(X) \otimes_{\mathbb{R}} \mathbb{C})}$$

where $d_{\mathbb{C}}$ is the complexified map. From the last problem set (Problem 3.4), we know that this is equivalent to:

$$H^k_{dR}(X, \mathbb{C}) = \frac{\ker(d : \Omega^k(X) \to \Omega^{k+1}(X)) \otimes_{\mathbb{R}} \mathbb{C}}{\mathrm{im}(d : \Omega^{k-1}(X) \to \Omega^k(X)) \otimes_{\mathbb{R}} \mathbb{C}}$$

However, as tensoring with a vector space is exact, we note that for any real-vector spaces $A, B \subseteq A$:

$$\begin{aligned} &0 \to B \to A \to A/B \to 0 \\ \implies &0 \to B \otimes_{\mathbb{R}} \mathbb{C} \to A \otimes_{\mathbb{R}} \mathbb{C} \to (A/B) \otimes_{\mathbb{R}} \mathbb{C} \to 0 \end{aligned}$$

Thus, we have:

$$(A/B) \otimes_{\mathbb{R}} \mathbb{C} \equiv (A \otimes_{\mathbb{R}} \mathbb{C})/(B \otimes_{\mathbb{R}} \mathbb{C})$$

Applying this to the cohomology groups, we get:

$$\begin{aligned} H^k_{dR}(X, \mathbb{C}) &= \frac{\ker(d : \Omega^k(X) \to \Omega^{k+1}(X))}{\mathrm{im}(d : \Omega^{k-1}(X) \to \Omega^k(X))} \otimes_{\mathbb{R}} \mathbb{C} \\ &= H^k_{dR}(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \end{aligned}$$

Problem 2. This exercise is taken from HW1 as, unfortunately, the hint for question 1 was missing. As the techniques and the result are important, I put it back.
Let U be an open subset of $\mathbb{C}$ and $D \subset \Omega$ be a closed disk.

1. Let $f : U \to \mathbb{C}$ be a C^1 function. Show that for all $z \in D$, we have:

$$f(z) = \frac{1}{2i\pi} \int_{\partial D} \frac{f(\xi)}{\xi - z} d\xi + \frac{1}{2i\pi} \int_D \frac{\partial f}{\partial \bar{z}}(\xi) \frac{d\xi \wedge d\bar{\xi}}{\xi - z}.$$

   Hint: You can apply the Stokes formula to $\frac{f(\xi)}{\xi - z} d\xi$ *on* $D \setminus B(z, \epsilon)$ *and let* $\epsilon \to 0$.

2. Let g be a C^1 function on $\mathbb{C}$ with compact support and let:

$$f(z) = \frac{1}{2i\pi} \int_{\mathbb{C}} \frac{g(\xi)}{\xi - z} d\xi \wedge d\bar{\xi}.$$

   Show that f is a C^1 function and $\frac{\partial f}{\partial \bar{z}} = g$.

   Hint: you can differentiate under the integral sign after the change of variable $\xi' = \xi - z$, *then change back and conclude using the formula from the first question.*

3. Show that for any function g on U which is C^1, there exists f which is C^1 on U such that $\partial f / \partial \bar{z} = g$ on D.

4. In the last question, show that if g is C^∞, then f can be chosen C^∞. Show also that if g depends smoothly (or holomorphically) on other parameters, then so does f.

Solution.

Problem 3. **Holomorphic $\bar{\partial}$-Dolbeault Lemma.** Let U be an open subset of $\mathbb{C}^n$ and D an open polydisk with closure contained in U. Let $0 \leq p \leq n$, $1 \leq q \leq n$. The goal of this exercise is to prove that any (p,q)-form $\bar{\partial}$-closed on U has a restriction to D which is $\bar{\partial}$-exact.

1. Prove that we can reduce to the case where $p = 0$. *Hint: show that each form $\alpha \in \mathcal{A}^{p,q}(U)$ can be written as $\alpha = \sum_{|I|=p} \alpha_I \wedge dz^I$ with $\alpha_I \in \mathcal{A}^{0,q}(U)$ uniquely determined by α.*

2. Let $\alpha \in \Omega^{0,q}(U)$. Show that there exists $1 \leq k \leq n$ such that $\alpha = dz^k \wedge \gamma + \delta$ and γ, δ are forms in the subalgebra generated by dz^i, $1 \leq i \leq k-1$.

3. Prove the result by induction on k. *Hint: you can consider a form $\mu \in \mathcal{A}^{0,q-1}$ obtained from γ by replacing each coefficient $f \in C^\infty(D)$ by a function $g \in C^\infty$ such that $\partial g / \partial z^k = f$ on D. Show that if $\bar{\partial}\alpha = 0$, then we can choose μ such that $\bar{\partial}\mu = dz^k \wedge \gamma + \nu$ where ν can be expressed only in terms of $dz^1, \ldots, dz^{k-1}$ and $C^\infty(U)$.*

Solution.

Problem 4. Dolbeault cohomology of the open disk. Let D be an open disk in $\mathbb{C}$ or $D = \mathbb{C}$.

1. Let $g \in C^\infty(D)$. Show that there exists $f \in C^\infty(D)$ such that $\partial f/\partial \bar{z} = g$.

   Hint: choose a sequence of disks $D_n \subset D$ such that $D_n \subset D_{n+1}$ and $\bigcup_n D_n = D$. Construct $f_n \in C^\infty(D)$ such that $\partial f_n/\partial \bar{z} = g$ on D_n and such that $|f_{n+1} - f_n| \leq 2^{-n}$ on D_{n-1}. Show that f_n converges to a function f that solves the problem.

2. Compute the Dolbeault cohomology groups of D.

Solution.

Problem 5. Let $\mathbb{P}^3(\mathbb{C})$ denote the complex projective 3-space with homogeneous coordinates x_0, x_1, x_2, x_3. Consider the complex submanifold

$$X := \{x \in \mathbb{P}^3(\mathbb{C}) \mid x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\}.$$

Let M be the underlying C^∞ manifold of X and let I denote the corresponding complex structure. Show that (M, I) and $(M, -I)$ are isomorphic as complex manifolds. How can you generalize this example?

Solution. Define the map $\phi : (M, I) \to (M, I)$ by:

$$\phi([x_0 : \cdots : x_3]) \to [\overline{x_0} : \cdots : \overline{x_3}]$$

Which we see as picking a representative in $\mathbb{C}^4$, complex conjugating, and then quotienting back into $\mathbb{P}^3(C)$. We claim that this is an isomorphism between (M, I) and $(M, -I)$. We need to show that ϕ is well-defined, a diffeomorphism, and holomorphic with respect to the complex structure on the image.

To show that ϕ is well-defined, we need to show that $\phi(X) \subset X$ and that is it well-defined with respect to the choince of representative in $\mathbb{C}^4$. First, notice that $(\overline{z})^4 = \overline{z^4}$. Thus, by conjugating the equation, we have:

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0 \implies \overline{x_0}^4 + \overline{x_1}^4 + \overline{x_2}^4 + \overline{x_3}^4 = 0$$

Thus, $\phi(X) \subset X$. Moreover, for any $\lambda \in \mathbb{C}$,

$$\begin{aligned}\phi([\lambda x_0 : \cdots : \lambda x_3]) &= [\overline{\lambda x_0} : \cdots : \overline{\lambda x_3}] \\ &= [\overline{\lambda}\overline{x_0} : \cdots : \overline{\lambda}\overline{x_3}] \\ &= [\overline{x_0} : \cdots : \overline{x_3}]\end{aligned}$$

Thus, ϕ is independent of the choice of representative. Thus, ϕ is well-defined.

Next, we know that complex conjugation is smooth and bijective, as it is a linear map on $\mathbb{R}^8 \cong \mathbb{C}^4$. Moreover, it is self-inverse, thus is a diffeomorphism. Since it is well-defined with respect to quotienting, we conclude ϕ is a diffeomorphism.

Next, we claim that taking the complex conjugate is anti-holomorphic. Let $\psi(z) = \overline{z}$, be the complex conjugation map, where $z = x + iy$. Then, we have:

$$\frac{\partial \overline{z}}{\partial x} = 1 \qquad \frac{\partial \overline{z}}{\partial y} = -i$$

Thus, we have:

$$\frac{\partial \overline{z}}{\partial z} = \frac{1}{2}(1 - i(-i)) = 0$$
$$\frac{\partial \overline{z}}{\partial \overline{z}} = \frac{1}{2}(1 + i(-i)) = 1$$

Thus, ψ is anti-holomorphic. Since (anti-)holomorphicity for higher dimensions is defined coordinate-wise, we note that complex conjugation is anti-holomorphic. Thus, complex conjugation is holomorphic with respect to the flipped complex structure on the target space. Thus, ϕ is holomorphic with respect to the complex structure on the target space (using the same atlas).

In general, every part of this argument applies to any complex submanifold of $\mathbb{P}^n(\mathbb{C})$, so long as the submanifold is defined by a polynomial equation with real coefficients, as this guarantees that the complex conjugate of a solution is also a solution. The rest of the argument stays the same in those cases.