

# Math 113: Functional Analysis

Sair Shaikh

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Defn. Let  $\{U_\alpha\}_{\alpha \in A}$  be an open cover of  $(X, \rho)$ . We say that  $d > 0$  is a Lebesgue number for the cover if given any  $d$ -ball  $B_d(x_0)$  with  $x_0 \in X$ , there exists  $a_0 \in A$  such that  $B_d(x_0) \subseteq U_{a_0}$ .

Ex  $X = \mathbb{R}$ .  $U_1 = (-\infty, 1)$ ,  $U_2 = (0, 2)$ , and  $U_3 = (1, \infty)$ . Here  $d = 1/2$  is a LN for  $\{U_1, U_2, U_3\}$ . This is clear if  $x_0 \in (1/2, 3/2)$ .

Ex. (Hwk.) Given  $x \in (0, 1)$ ,  $\exists \delta_x > 0$  such that:

$$y \in B_{\delta_x}(x) = \{y \in (0, 1) : |y - x| \leq \delta_x\} \\ \implies |1/x - 1/y| < 1$$

Then,

$$(0, 1) = \bigcup_{x \in (0, 1)} B_{\delta_x}(x)$$

has no Lebesgue number.

Lemma (Lebesgue Covering Lemma) Every open cover of a compact metric space has a Lebesgue number.

Proof. Pictures. Apr 9.

Thm. Suppose  $(X, \rho)$  is compact, and  $F : (X, \rho) \rightarrow (Y, \sigma)$  is continuous. Then  $F$  is uniformly continuous.

Proof. Let  $\epsilon > 0$ . We need to find  $\delta > 0$  such that  $\forall x, y \in X$ :

$$\rho(x, y) < \delta \implies \sigma(F(x), F(y)) < \epsilon$$

Since  $F$  is continuous,  $\forall z \in X$ ,  $\exists \delta_z > 0$  such that:

$$\rho(x, z) < \delta_z \implies \sigma(F(x), F(z)) < \epsilon/2$$

That is,

$$F(B_{\delta_z}(z)) \subseteq B_{\epsilon/2}(F(z))$$

Let  $\delta > 0$  be a Lebesgue number for the cover  $\{B_{\delta_z}(z)\}_{z \in X}$ .  
Now supposed  $\rho(x, y) < \delta$ . Then  $\exists z \in X$  such that:

$$B_\delta(x) \subseteq B_{\delta_z}(z)$$

and

$$\sigma(F(x), F(y)) \leq \sigma(F(x), F(y)) + \sigma(F(z), F(y)) < \epsilon/2 + \epsilon/2 = \epsilon$$

Defn. Let  $(X, \rho)$  be a metric space and  $C(X)$  the  $\mathbb{C}$ -vector space of continuous functions on  $X$ . We say  $\mathcal{J} \subset C$  is equicontinuous at  $x \in X$  if  $\forall \epsilon > 0, \exists \delta > 0$  such that:

$$\forall F \in \mathcal{J}, F(B_\delta(x)) \subseteq B_\epsilon(F(x))$$

We say  $\mathcal{J}$  is equicontinuous on  $X$  if  $\forall x \in X$ ,  $\mathcal{J}$  is equicontinuous at  $x$ .

Ex. Let  $X = [0, 1] \subset \mathbb{R}$ . Let  $F_n(x) = x^n \forall n \geq 1$ . Let:

$$\mathcal{J} = \{F_n : n \in \mathbb{N}\}$$

Let  $x_n = \frac{1}{2}^{1/n}$ . Then,  $x_n$  arrow up to 1. Then,

$$|F_n(x_n) - F_n(1)| = |1/2 - 1| = 1/2$$

Thus,  $\mathcal{J}$  is not equicontinuous at 1.

Ex. (Hwk) Show that  $\mathcal{J}$  is equicontinuous on  $[0, 1)$ .

Defn. Let  $(F_n)$  be a sequence of ( $\mathbb{C}$ -valued) functions on  $X$ . Then,  $(F_n)$  is uniformly bounded if  $\exists M > 0$  such that  $\forall n \geq 1, \forall x \in X$ :

$$|F_n(x)| < M$$

We say that  $(F_n)$  is pointwise bounded if  $\forall x \in X, \exists M_x > 0$  such that:

$$|F_n(x)| < M_x$$

Defn. A metric space (top. space) is separable if there is a countable dense subset  $D \subset X$ .

Ex. Since  $\mathbb{Q}^n \subset \mathbb{R}^n$  is dense,  $(\mathbb{R}^n, \|\cdot\|_p)$  is separable.

Lemma. (Arzelà-Ascoli) Let  $(X, \rho)$  is a separable metric space and that  $(F_n)$  is pointwise bounded and equicontinuous in  $C(X)$ . Then, there is subsequence  $(F_{n_k})$  such that:

$$\lim_{x \rightarrow \infty} F_{n_k}(x)$$

exists  $\forall x \in X$ .

Yap. Given a sequence  $(x_n)$ , we get subsequence by finding  $n_k \in \mathbb{N}$  such that  $n_{k+1} > n_k$  and  $(x_{n_k})_{k=1}^\infty \rightarrow x$  is a sequence.

A subsubsequence is determined by finding  $n_{k_1} < n_{k_2} < \dots$  and then we write:

$$(x_{n_{k_j}})_{j=1}^\infty$$

A subsequence is determined by an infinite subset  $S_1 = \{n_1 \leq n_2 \leq \dots\} \subset \mathbb{N}$ . A subsubsequence is determined by an infinite subset  $S_2 \subset S_1$ ,

$$S_2 = \{n_{k_1} < n_{k_2} < \dots\} \subset S_1$$

Now, we write:

$$\lim_{n \in S_1} x_n = a \text{ instead of } \lim_{k \rightarrow \infty} x_{n_k}$$

Note that  $\lim_{n \in S_1} x_n = a$  if:

$$\forall \epsilon > 0 \exists N : n \geq N, n \in S' \implies |x_n - a| < \epsilon$$

Rmk. Suppose  $S_1 \subset \mathbb{N}$  determines a subsequence as above. Suppose  $S' \subset \mathbb{N}$  is infinite and:

$$\{n \in S' : n \notin S_1\}$$

is finite. The  $\lim_{n \in S_1} x_n = a$  then  $\lim_{n \in S'} x_n = a$  as well.

Proof of the AA Lemma: Pictures. Apr 9 and 10.

Rmk. If  $X$  is compact, then  $C(x) = C_b(X)$  is a complete metric space with respect to the uniform norm  $\|\cdot\|_\infty$ .

Thm. (Arzelà-Ascoli) Let  $(X, \rho)$  be a compact metric space and  $(F_n) \subset C(X)$  be a sequence of functions that are point-wise bounded and equicontinuous. Then  $(F_n)$  has a subsequence converging uniformly to some function  $F \in C(X)$ . Proof. Pictures.

Lemma. Suppose  $X$  is compact and that  $\mathcal{J} \subset C(X)$  is equicontinuous on  $X$ . Then,  $\mathcal{J}$  is uniformly equicontinuous on  $X$ , in that for all  $\epsilon > 0 \exists \delta > 0$  such that for all  $x, y \in X$  and all  $F \in \mathcal{J}$ ,

$$\rho(x, y) < \delta \implies |F(x) - F(y)| < \epsilon$$

Rewriting,

$$F(B_\delta(x)) \subseteq B_\epsilon(F(x))$$

Proof left as homework.

Corr. Let  $X$  be a compact metric space. Let  $\mathcal{J} \subset C(X)$  be a closed subset such that  $\mathcal{J}$  is equicontinuous and pointwise bounded. Then  $\mathcal{J}$  is compact and uniformly bounded.

Thm. Suppose  $X$  is a compact metric space. Then  $\mathcal{J} \subset C(X)$  is compact if and only if  $\mathcal{J}$  is closed, uniformly bounded, and equicontinuous on  $X$ . Proof. Pictures.

Defn. A topological space is called a Baire space if the countable intersection of dense open sets is dense.

Rmk. If  $\rho$  and  $\sigma$  are equivalent metrics on  $X$  then  $(X, \rho)$  is a Baire space if and only if  $(X, \sigma)$  is a Baire space.

Defn. If  $S \subset X$  then the interior of  $S$  is:

$$Int(S) = \bigcup \{U \subset S : U \text{ is open in } X\}$$

Rmk. One can write  $Int_X(S)$  as interior of  $S$  in  $X$ . Where you take the interior matters.

Lemma. A space  $X$  is a Baire space if and only if given a countable  $\{F_n\}_{n=1}^\infty$  of closed sets, such that:

$$\bigcup_{n=1}^\infty F_n$$

has non-empty interior, then at least one  $F_n$  has (non-empty) interior.

Hwk. Write  $O_n = F_n^C = X \setminus F_n$ .  $O_n$  dense  $\iff Int(F_n) = \emptyset$ .

Thm. Baire Category Thm. Every complete metric space is a Baire space.

Rmk. Note that  $(0, 1)$  is homeomorphic to  $\mathbb{R}$ . Hence,  $(0, 1)$  is a Baire space.

Hwk. More generally, every non-empty open subset  $V$  of a complete metric space admits an equivalent complete metric (Hwk). Hence  $V$  is a Baire space. Thus, if:

$$V = \bigcup_{n=1}^\infty V \cap F_n$$

of each  $F_n$  closed in  $X$ , thus  $V \cap F_n$  is closed in  $V$ . Hence, at least one  $V \cap F_n$  has non-empty interior in  $V$ . But  $V$  is open, so it has interior in  $X$ .

Thm. Suppose that  $X$  is a Baire space and that  $(F_n) \subset C(X)$  such that  $F_n \rightarrow F$  pointwise. Then,

$$A = \{x \in X : F \text{ is cts at } x\}$$

is dense in  $X$ .

Proof. Pictures.

Rmk. If  $X = \mathbb{R}$ , then we can assume:

$$A = \{x \in X : F \text{ is cts at } x\}$$

is uncountable.

Defn. A normed vector space  $(V, \|\cdot\|)$  is called a Banach space if  $V$  is complete in the induced metric. Recall that  $\mathbb{F}$  is always  $\mathbb{C}$  or  $\mathbb{R}$ .

Yap. Normed vector spaces are special.

1.  $B_r(x) = x + B_r(0)$ . The topology is homogenous.
2.  $|||v|| - ||w||| \leq ||v - w|| \implies v \rightarrow ||v||$  is continuous.
3.  $\overline{B_r(v)} = \{x \in V : ||x - v|| \leq r\}$ .
4.  $\epsilon B_r(0) = B_{\epsilon r}(0)$ .
5.  $\overline{\epsilon B_r(0)} = \overline{B_{\epsilon r}(0)}$ .

Thm. Supposed that  $X$  and  $Y$  are normed vector spaces and  $T : X \rightarrow Y$  is linear. Then, the following are equivalent:

1.  $T$  is continuous.
2.  $T$  is continuous at a single point.
3.  $\exists \alpha \geq 0$  such that  $||T(x)|| \leq \alpha ||x||$  for all  $v \in X$ .

Proof.

(1)  $\implies$  (2) is trivial.

(2)  $\implies$  (3). Since  $T$  is continuous at  $x_0$ ,  $\exists \delta > 0$  such that:

$$T(\overline{B_\delta(x_0)}) \subset B_1(T(x_0))$$

$$\text{LHS} = T(B_\delta(x_0)) + T(x_0).$$

$$\text{RHS} = B_1(0) + T(x_0).$$

This implies,  $T(\overline{B_\delta(0)}) \subset B_1(0)$ .

Now if  $z \neq 0$ ,

$$||T(z)|| = \left| \left| \frac{||z||}{\delta} T\left(\delta \cdot \frac{z}{||z||}\right) \right| \right| \leq \frac{||z||}{\delta}$$

Let  $\alpha = \frac{1}{\delta}$ .

(3)  $\implies$  (1). Erased.

Rmk. Suppose that  $||\cdot||_1$  and  $||\cdot||_2$  (two different norms) induce equivalent metrics on  $V$ . Then, take the id :  $(V, ||\cdot||_1) \rightarrow (V, ||\cdot||_2)$  is continuous (same topology). Hence,  $\exists c \geq 0$  such that:

$$||x_2|| \leq c \cdot ||x_1||$$

Clearly, that means  $c > 0$ . By symmetry, there exists a  $d > 0$ , such that:

$$||x_1|| \leq d \cdot ||x_2||$$

Thus, the metrics and the norms are strongly equivalent.

Defn. If  $X$  and  $Y$  are normed vector spaces, then  $\mathcal{L}(X, Y)$  is the vector space of continuous linear maps  $T : X \rightarrow Y$ . Define  $\|T\| = \sup_{\|x\| \leq 1} \|T(x)\|$ . If  $X = Y$ , then we write  $\mathcal{L}(X)$  instead of  $\mathcal{L}(X, X)$ .

Lem. With  $\|T\|$  as above,  $\mathcal{L}(X, Y)$  is a normed vector space with:

$$\|T(x)\| \leq \|T\| \|x\| \forall x \in X$$

If  $S \in \mathcal{L}(Y, Z)$ , then we write  $ST$  in place of  $S \circ T$  and:

$$\|ST\| \leq \|S\| \|T\|$$

Defn. An algebra over  $\mathbb{F}$  is a vector space  $A$  over  $\mathbb{F}$  with a ring structure, with  $\lambda(xy) = (\lambda x)y = x(\lambda y)$  for all  $\lambda \in \mathbb{F}$  and  $x, y \in A$ .

Ex.  $M_n(\mathbb{F}), \mathbb{R}[x], \mathbb{C}[x], C(X)$ .

Defn. If  $\|\cdot\|$  is a norm on an algebra  $A$ , then we call  $(A, \|\cdot\|)$  a normed algebra if  $\forall x, y \in A$ :

$$\|xy\| \leq \|x\| \|y\|$$

We call  $(A, \|\cdot\|)$  a Banach algebra if  $(A, \|\cdot\|)$  is a normed algebra and  $A$  is complete with respect to  $\|\cdot\|$ .

Prop. If  $X$  and  $Y$  are normed vector spaces, and  $Y$  is a Banach space, then  $\mathcal{L}(X, Y)$  is a Banach space. If  $X$  is a Banach space, then  $\mathcal{L}(X)$  is a Banach algebra.

Proof. Suppose that  $(T_n)$  is a Cauchy sequence in  $\mathcal{L}(X, Y)$ . Then, for each  $z \in X$ ,  $T_n(z)$  is a Cauchy sequence in  $Y$ . Hence  $\exists T(z) \in Y$  such that:

$$T_n(z) \rightarrow T(z)$$

It is not hard to see that  $T : X \rightarrow Y$  is linear.

Since  $(T_n)$  is Cauchy in norm, it's bounded i.e.  $\exists M > 0$  such that:

$$\|T_n\| \leq M \forall n \geq 1$$

Now if  $\|x\| \leq 1$ , then:

$$\|T(x)\| = \lim_{n \rightarrow \infty} \|T_n(x)\| \leq \limsup_n \|T_n\| \|x\| \leq M \|x\|$$

This implies  $T \in \mathcal{L}(X, Y)$ .

Let  $\epsilon > 0$ . Let  $N$  be such that  $m, n \geq N$ . Thus,

$$\|T_n - T_m\| \leq \frac{\epsilon}{2}$$

Now if  $\|x\| \leq 1$ , and if  $n \geq N$ ,

$$\begin{aligned} \|(T - T_n)(x)\| &= \|T(x) - T_n(x)\| \\ &= \lim_{m \rightarrow \infty} \|T_m(x) - T_n(x)\| \\ &\leq \limsup_m \|T_m - T_n\| \cdot \|x\| \\ &\leq \frac{\epsilon}{2} < \epsilon \end{aligned}$$

Yap. Let  $Y \subset X$  be a subspace of a normed vector space  $X$ . Then we can form the quotient vector space  $X/Y = \{x + Y : x \in X\}$  with  $q : X \rightarrow X/Y$  the quotient map.

If  $x \in X$ , then:

$$\inf\{\|x - y\| : y \in Y\}$$

depends only on  $q(x)$ . We call:

$$\|q(x)\| := \inf\{\|x - y\| : y \in Y\}$$

the quotient norm on  $X/Y$ .

Rmk. As  $Y$  is a subspace,

$$\begin{aligned} \|q(x)\| &= \inf\{\|x + y\| : y \in Y\} \\ &= \inf\{\|x + \alpha y\| : y \in Y\} \forall \alpha \in \mathbb{F} \setminus \{0\} \end{aligned}$$

Thm. If  $Y$  be a subspace of  $(X, \|\cdot\|)$ . Then,

$$\|q(x)\| = \inf\{\|x - y\| : y \in Y\}$$

is a seminorm on  $X/Y$  which is a norm exactly when  $Y$  is closed. If  $X$  is a Banach space, and  $Y$  is closed in  $X$ , then  $X/Y$  is a Banach space.

Proof. Note that for  $\alpha \neq 0$ :

$$\|\alpha q(x)\| = \|q(\alpha x)\| = \inf\{\|\alpha x + \alpha y\| : y \in Y\} = |\alpha| \inf\{\|x + y\| : y \in Y\} = |\alpha| \|q(x)\|$$

Next, fix  $x_1, x_2 \in X$ . Then given  $\epsilon > 0$ ,  $\exists y_1, y_2 \in Y$  such that:

$$\begin{aligned} \|q(x_1)\| + \|q(x_2)\| + \epsilon &\geq \|x_1 - y_1\| + \|x_2 - y_2\| \\ &\geq \|x_1 - y_1 + x_2 - y_2\| \\ &\geq \|q(x_1 + x_2)\| \\ &= \|q(x_1)\| + \|q(x_2)\| \end{aligned}$$

Since  $\epsilon > 0$  is arbitrary,  $\|q(x_1) + q(x_2)\| \leq \|q(x_1)\| + \|q(x_2)\|$ . Thus, it is a semi-norm. Sketch for closed. If quotient norm is 0, there exists a sequence  $y_n$  converging to  $x$ . If

$Y$  closed,  $x \in Y$  and  $g(x) = 0$ . Conversely, if quotient norm is a norm and  $y_n \rightarrow x$ . Then,  $\|q(x)\| = 0$ . Then,  $q(x) = 0$  and  $x \in X$ .

Now suppose  $X$  is a Banach space and  $Y$  is closed in  $X$ . Let  $(q(z_0))$  be Cauchy in  $X/Y$ . Then we can pass to a subsequence and assume:

$$\|q(z_{n+1}) - q(z_0)\| \leq 1/2^n$$

Let  $x_1 = z_1$ . Since:

$$\|q(z_2) - q(x_1)\| < 1/2$$

Thus,

$$\inf\{\|z_2 - x_1 - y\| : y \in Y\} < 1/2$$

Hence, we can find  $x_2$  such that  $q(x_1) = q(z_2)$  and

$$\|x_2 - x_1\| < 1/2$$

Continuing, we get  $(x_n)$  such that  $q(x_n) = q(z_n)$  and:

$$\|x_{n+1} - x_n\| < 1/2^n$$

(Hwk)  $(x_n)$  is Cauchy, thus,  $x_n \rightarrow x$  in  $X$ .

But  $\|q(z)\| \leq \|z\|$ . Hence,  $q$  is continuous. Thus,

$$q(z_n) = q(x_n) \rightarrow q(x)$$

Rmk. Note that  $q : X \rightarrow X/Y$  is a bounded linear map of norm at most 1.

Thm. Suppose that  $Y$  is a closed subspace of  $(X, \|\cdot\|)$ . Then  $X$  is a Banach space if and only if both  $Y$  and  $X/Y$  are Banach spaces.

Proof. Pictures. 4/16

Thm. Every finite-dimensional subspace  $Y$  of a normed vector space  $X$  is a Banach space and hence closed in  $X$ . If  $\dim(Y) = n$ , then every linear isomorphism  $\Phi : \mathbb{F}^n \rightarrow Y$  is a homeomorphism.

Proof. Pictures. 4/16.

Corr. If  $Y$  is finite-dimensional vector space, then all norms  $\|\cdot\|$  are strongly equivalent.

Proof.  $\text{id} : (Y, \|\cdot\|_1) \rightarrow (Y, \|\cdot\|_2)$  is a homeomorphism and continuous linear maps are bounded.

Recall. Recall the topological definition of continuous functions and open maps. If  $F : X \rightarrow Y$  is a bijection, then  $F^{-1} : Y \rightarrow X$  is continuous if and only if  $F$  is open.

Thm. (The Big Three) (Open Mapping Theorem) Suppose that  $X$  and  $Y$  are Banach spaces and that  $T \in \mathcal{L}(X, Y)$  is a surjection. Then  $T$  is an open map.



Lem. It will suffice to find  $r > 0$  such that:

$$B_r(0) \subseteq T(B_1(0))$$

Proof. By homogeneity,  $T(B_\delta(0))$  is a neighborhood of  $0_Y$  for all  $\delta > 0$ . By linearity,  $T(B_\delta(x))$  is a neighborhood of  $T(x)$  for all  $x \in X$  and  $\delta > 0$ . Thus, if  $V$  is open in  $X$  and  $x \in V$ , then  $\exists \delta > 0$  such that:

$$B_\delta(x) \subseteq V$$

Then,  $T(B_\delta(x))$  is a neighborhood of  $T(x)$  in  $T(Y)$ . Thus,  $T$  is an open map. Thus,  $T(V)$  is open.

Lem. It will suffice to find  $r > 0$  such that:

$$B_r(0) \subseteq \overline{T(B_1(0))}$$

Proof. Assume  $r > 0$  is such that the property holds. Let  $y \in B_r(0)$ . Then,

$$\exists y_1 \in T(B_1(0)) : \|y - y_1\| < r/2$$

Then,  $y - y_1 \in B_{r/2}(0)$ .

Hence,  $\exists y_2 \in 1/2T(B_1(0)) = T(B_{1/2}(0))$  such that:

$$\|y - y_1 - y_2\| < r/2^2$$

Continue to get a sequence  $(y_n)$  such that:

$$y_n \in 2^{-n+1}T(B_1(0)) = T(B_{2^{-n+1}}(0))$$

and:

$$\|y - \sum_{i=1}^n y_i\| < 2^{-n}r$$

By construction,  $\exists x_n \in X$  such that:

$$T(x_n) = y_n \quad \|x_n\| < 2^{-n+1}$$

Since  $X$  is a Banach space,  $x = \sum_{n=1}^{\infty} x_n$  converges and since  $T$  is continuous (hence bounded), and also:

$$\|x\| < \sum_{n=1}^{\infty} 2^{-n+1} = 2$$

and

$$T(x) = y$$

Since  $y$  was arbitrary, we have shown that:

$$B_r(0) \subset T(B_2(0))$$

This implies:

$$B_{r/2}(0) \subset T(B_1(0))$$

This suffices by lemma 1.

Lem. 3 It suffices to see that:

$$\overline{T(B_n(0))}$$

has interior for some  $n \geq 1$ .

Proof. By homogeneity, we can assume that  $\overline{T(B_1(0))}$  has interior.

Thus,  $\exists \epsilon > 0$  such that:

$$B_\epsilon(y) \subset \overline{T(B_1(0))}$$

Let  $z \in B_\epsilon(0)$ . Write

$$z = z/2 + y - (y - z/2) \in B_{\epsilon/2}(y) - B_{\epsilon/2}(y) \subseteq \overline{T(B_{1/2}(0))} - \overline{T(B_{1/2}(0))}$$

Rest in pictures.

Proof. Of the Open Mapping Theorem. Pictures. 4/18.

Ex. Let  $1 \leq p < \infty$ . Then, let:

$$l_0^p = \text{span}\{e_n : n \geq 1\} = \{x \in l^p : x(n) = 0 \text{ for all but finitely many } n\}$$

Notte  $l_0^p$  is dense in  $l^p$ .

Define:  $T_0 : l_0^p \rightarrow l_0^p$  by:

$$T_0(e_n) = 1/ne_n$$

You can check that  $\|T\| = 1$ . Also  $T_0$  is a bijection.

$$T_0^{-1}(e_n) = ne_n$$

but  $T_0^{-1} \notin \mathcal{L}(l_0^p)$

Thm. Suppose that  $X$  and  $Y$  are Banach spaces and  $T \in \mathcal{L}(X, Y)$  is a bijection. Then  $T^{-1} \in \mathcal{L}(Y, X)$ .

Proof.  $T^{-1}$  is linear by general nonsense and  $T$  is open by the Open Mapping Theorem. Hence  $T^{-1}$  is continuous hence bounded.

Yap. If  $X$  and  $Y$  are Banach spaces, then I can give  $X \times Y$  a norm by:

$$\|(x, y)\| = \max\{\|x\|, \|y\|\}$$

This makes  $X \times Y$  a Banach space.

[Missed 3 classes due to Covid. Did Open Mapping Theorem, Closed Graph Theorem, Principle of Uniform Boundedness, defined norm-dual, and did Hahn-Banach with 3 important corollaries.]

Thm. The map:

$$\Phi : l^q \rightarrow (l^p)^\vee$$

given by  $\Phi(y) = \phi_y$  is an isometric isomorphism. That is,  $\Phi$  is a isomorphism such that:

$$\|\Phi(y)\| = \|y\|$$

Rmk. If  $z \in \mathbb{C}$ , then:

$$\text{sgn}(z) = \frac{z}{|z|} \text{ if } z \neq 0 \text{ else } 0$$

Then,  $z = |z| \text{sgn}(z)$  and  $|z| = \text{sgn}(z)z$ .

Lemma. If  $y \in l^q$ , then  $\|\phi_y\| = \|y\|_q$ . Proof. Pictures. 4/28.

Lemma. If  $\phi \in (l^p)^\vee$ , then  $\exists y \in l^q$  such that:

$$\phi = \phi_y$$

Proof. Pictures. 4/28. Let  $y_n = \phi(e_n)$ . Then, if  $x \in l^p$ , then:

$$x = \sum_{n=1}^{\infty} x_n e_n$$

Note,

$$\|x = \sum_{n=1}^N x_n e_n\|_p^p = \sum_{n=N+1}^{\infty} |x_n|^p$$

converges in the Banach space  $l^p$ .

Hence,

$$\phi(x) = \sum_{n=1}^{\infty} x_n y_n$$

Then, if we can show  $y \in l^q$ , then  $\phi = \phi_y$ , and we are done.

Let  $y^N \in l^q$  be given by:

$$y^N(n) = \begin{cases} y_n & \text{if } 1 \leq n \leq N \\ 0 & \text{otherwise} \end{cases}$$

Then,  $\phi_{y^N} \in (l^p)^\vee$  and  $\|\phi_{y^N}\| \leq \|y^N\|_q$ .

If  $x \in l^p$ , then,

$$\phi_{y^N}(x) = \sum_{n=1}^N x_n y_n \rightarrow \sum_{n=1}^{\infty} x_n y_n = \phi(x)$$

Hence, by the Principle of Uniform Boundedness,  $\exists M > 0$  such that:

$$\|\phi_{y^N}\| \leq M \quad \forall N \geq 1$$

Thus, if  $q < \infty$ , then,

$$\sum_{n=1}^N |y_n|^q \leq M^q < \infty$$

Rmk. If  $X$  is a normed vector space, then we know that  $X^*$  is a Banach space. Then, we can form the bidual  $(X^*)^* = X^{**}$ . We get a natural map  $\iota$

$$\iota : X \rightarrow X^{**}$$

$\iota(x)(\phi) = \phi(x)$  is evaluation at  $x$ .

Now,

$$\|\iota(x)\| = \sup_{\|\phi\| \leq 1} |\iota(x)(\phi)| = \sup_{\|\phi\| \leq 1} |\phi(x)| (*)$$

Then,  $(*) \leq \|x\|$  since  $\|\phi\| \leq 1$ .

Then, by Hahn-Banach,  $\exists \phi \in X^*$  such that  $\|\phi\| = 1$  and  $\phi(x) = \|x\|$ . Thus,

$$\|\iota(x)\| = \|x\|$$

We can identify  $X$  at  $\iota(X)$  in  $X^{**}$ .

If  $X$  is a Banach space, then,  $\iota(x)$  is complete, hence closed in  $X^{**}$ .

Otherwise,  $\overline{\iota(X)}$  is a Banach space containing  $X$  as a dense subspace. Then,  $\overline{\iota(X)}$  is the completion of  $X$  as a Banach space.

Defn. If  $X$  is a Banach space then  $X$  is reflexive if  $\iota(X) = X^{**}$  is onto.

Rmk. If  $X$  is reflexive, then  $X^{**}$  are isometrically isomorphic. The converse can fail. It can be that they are isometrically isomorphic but  $\iota$  is not the map. [Robert James 1951].

Ex. Let  $1 < p < \infty$ . Let  $\frac{1}{p} + \frac{1}{q} = 1$ . Then,

$$(l^p)^* = \{\phi_y^p : y \in l^q\}$$

and also:

$$(l^q)^* = \{\phi_x^q : x \in l^p\}$$

If  $x \in l^p$ , then,  $\iota(x) \in (l^p)^{**}$ , and:

$$\begin{aligned} \iota(x)(\phi_y^p) &= \phi_y^p(x) \\ &= \phi_x^q(y) \end{aligned}$$

Thus,  $\iota : l^p \rightarrow (l^p)^{**}$  is surjective and  $l^p$  is reflexive for  $1 < p < \infty$ .

Prop.  $l^p$  is reflexive for  $1 < p < \infty$ .

Rmk. Let  $X$  be a normed vector space and  $D = \{d_n\}_{n=1}^\infty \subset X$  be a countable subset of  $X$ . Then the rational space of  $D$ ,  $\text{span}_{\mathbb{Q}}(D)$  is the span of  $D$  viewing  $X$  as a rational vector space. Thus,

$$\text{span}_{\mathbb{Q}}(D) = \bigcup_{n=1}^{\infty} \left\{ \sum_{k=1}^r r_k d_k : r_k \in \mathbb{Q} d_k \in D \right\}$$

Then,  $\text{span}_{\mathbb{Q}}(D)$  is countable. Then if  $\text{span}(D)$  is dense, then since  $\text{span}_{\mathbb{Q}}(D)$  is dense in  $\text{span}(D)$ . Thus,  $X$  is separable.

Same works over  $\mathbb{C}$  replacing  $\mathbb{Q}$  by  $\mathbb{Q} + i\mathbb{Q}$ .

Corr.  $l^p$  is separable for  $1 \leq p < \infty$ . Let  $D = \{e_n\}_{n=1}^\infty$ . But  $l^\infty$  is not separable.

Hwk. If  $X^*$  is separable, then  $X$  is.

Thm. If  $1 \leq p \leq \infty$ , and if  $q$  is the conjugate exponent, then:

$$\Phi : l^q \rightarrow (l^p)^*$$

is an isometric isomorphism when  $\Phi(y) = \phi_y^p$  and

$$\phi_y^p = \sum_{n=1}^{\infty} x_n y_n$$

Rmk. If  $(l^p)^* = \{\phi_y^p : y \in l^q\}$  and if  $q \leq \infty$ , then:

$$(l^q)^* = \{\phi_x^q : x \in l^p\}$$

Defn. Recall: If  $X$  is a Banach space, then  $X$  is reflexive if  $\iota(X) = X^{**}$  is surjective.

Prop. If  $1 < p < \infty$ , then  $l^p$  is reflexive.

Proof. Consider  $\iota : l^p \rightarrow (l^p)^{**}$ . If  $\phi_y^p \in (l^p)^*$ , then:

$$\iota(x)(\phi_y^p) = \phi_y^p(x) = \phi_x^q(y)$$

Let  $\psi \in (l^p)^{**}$ . Then,

$$\psi(\phi_y^p) = \psi(\Phi(y))$$

Now  $y \rightarrow \psi(\Phi(y))$  is a in  $(l^q)^*$ . Hence,  $\exists x \in l^p$  such that:

$$\psi(\Phi(y)) = \phi_x^q(y)$$

Thus,  $\iota(x) = \psi$ . Thus,  $\iota$  is onto.

Rmk. If  $X$  is a normed vector space over  $\mathbb{F}$  and  $D \subset X$  is countable and  $\overline{\text{span}(D)} = X$ , then  $X$  is separable.

Corr. If  $1 \leq p < \infty$ , then  $l^p$  is separable.

Proof.  $D = \{e_n : n \geq 1\}$ .

Ex.  $l^\infty$  is not separable.

Proof. Let  $A \subseteq \mathbb{N}$ . Let:

$$x_A(n) = \mathbf{1}_{n \in A}$$

If  $A \neq B$ , then  $\|x_A - x_B\|_\infty = 1$ . But  $\{x_A : A \subseteq \mathbb{N}\}$  is uncountable. You cannot find a countable dense subset anymore (cant fit all of them into countably many  $1/4$  balls).

Hwk.  $(l^\infty)^*$  is not separable. Thus,  $l^1$  is not reflexive.

Defn. Suppose  $T \in \mathcal{L}(X, Y)$ . Then we define:

$$T^* : Y^* \rightarrow X^*$$

by:

$$T^*(\phi)(x) = \phi(T(x))$$

Prop. If  $X$  and  $Y$  are normed vector spaces, and  $T \in \mathcal{L}(X, Y)$ , then  $T^* \in \mathcal{L}(Y^*, X^*)$  and  $\|T^*\| = \|T\|$ .

Proof. First, check that  $T^*$  is linear (easy).

Next, show the norm equality. Look at:

$$\begin{aligned} \|T^*(\phi)\| &= \sup_{\|x\| \leq 1} |T^*(\phi)(x)| \\ &= \sup_{\|x\| \leq 1} |\phi(T(x))| \\ &\leq \|\phi\| \|T\| \\ &\leq \|\phi\| \|T\| \end{aligned}$$

Thus,

$$\|T^*\| \leq \|T\|$$

Fix  $\epsilon > 0$ . Then,  $\exists x_0 \in X$  such that  $\|x_0\| = 1$  and:

$$\|T(x_0)\| > \|T\| - \epsilon$$

But  $\exists \phi \in Y^*$  such that  $\|\phi\| = 1$  and:

$$\phi(T(x_0)) = \|T(x_0)\|$$

Then,

$$\begin{aligned} \|T^*\| &\geq \|T^*(\phi)\| \\ &\geq |T^*(\phi)(x_0)| \\ &= |\phi(T(x_0))| \\ &= \|T(x_0)\| > \|T\| - \epsilon \end{aligned}$$

Since  $\epsilon > 0$  is arbitrary, we have:

$$\|T^*\| \geq \|T\|$$

Thus,  $\|T^*\| = \|T\|$ .

Thm. Let  $X$  and  $Y$  be Banach spaces and suppose that:

$$T : X \rightarrow Y \quad S : Y^* \rightarrow X^*$$



are functions (not linear or bounded) such that  $\forall \phi \in Y^*$  and  $x \in X$ :

$$S(\phi)(x) = \phi(T(x))$$

Then,  $T \in \mathcal{L}(X, Y)$  and  $S \in \mathcal{L}(Y^*, X^*)$ , with  $S = T^*$ .

Proof. Supposed that  $x, y \in X$  and  $\alpha \in \mathbb{F}$ . Then, if  $\phi \in Y^*$ , we have:

$$\begin{aligned} \phi(T(\alpha x + y)) &= S(\phi)(\alpha x + y) \\ &= \alpha(S(\phi)(x)) + S(\phi)(y) \\ &= \alpha\phi(T(x)) + \phi(T(y)) \\ &= \phi(\alpha T(x) + T(y)) \end{aligned}$$

Since  $\phi \in Y^*$  is arbitrary, we have  $T(\alpha x + y) = \alpha T(x) + T(y)$ . (something about separate points).

To see that  $T$  is bounded, use CGT. Suppose that  $x_n \rightarrow x$  in  $X$  and  $T(x_n) \rightarrow y$  in  $Y$ . But  $\forall \phi \in Y^*$ , we have:

$$\begin{aligned} \phi(y) &= \lim \phi(T(x_n)) \\ &= \lim S(\phi)(x_n) \\ &= S(\phi)(x) \\ &= \phi(T(x)) \end{aligned}$$

Thus,  $y = T(x)$ . Thus,  $T$  is bounded.

But  $T^*(\phi) = S(\phi)$ .

Yap. Let  $(X, \tau)$  be a topological space. Then  $\beta \subset \tau$  is a basis for  $\tau$  if given any  $U \in \tau$  and  $x \in U$ , then  $\exists V \in \beta$  such that  $x \in V \subset U$ .

We say  $S$  is a neighborhood of  $x \in X$  if  $\exists U \in \tau$  such that  $x \in U \subset S$ .

Wrote  $\mathcal{N}(x)$  for the set of all neighborhoods of  $x$ . We say that  $\alpha \subseteq \mathcal{N}(x)$  is a neighborhood basis at  $x$  if  $U \in \mathcal{N}(x)$ , there exists  $V \in \alpha$  such that:

$$x \in V \subseteq U$$

Ex. (1) In a metric space, the collection of all open balls is a basis for the metric topology.  
 (2) In  $\mathbb{R}^n$ , every point has a neighborhood basis consisting of compact sets. Such spaces are called locally compact.

Lemma. Low Hanging Fruit: Let  $(X, \tau)$  be a topological space and  $\alpha(x)$  has a neighborhood basis at  $x \in X$  consisting of open sets. Then:

$$\beta = \bigcup_{x \in X} \alpha(x)$$

is a basis for  $\tau$ .

Lemma. Low Hanging Fruit:  $\beta \in \tau$  is a bssis for  $\tau$  if and only if given  $U \in \tau$ ,

$$U = \bigcup_{V \in \beta \subset U} V$$

Defn. Let  $(X, \tau)$  be a topological space. Then,

1.  $(X, \tau)$  is seperable if it ihas a countable dense subset.
2.  $(X, \tau)$  is 2nd countable if it has a countable basis.
3.  $(X, \tau)$  1st countable if every point has a countable neighborhood basis.

Rmk. 1. Every 2nd countable space is seperable. The converse holds in metric spaces.  
Hwk 8.

2. Metric spaces are 1st countable.

- Ex. 1. Let  $X$  be a set. Then  $\tau = \mathcal{P}(X)$  is the discrete topology. This is the metric topology coming from the the discrete metric.
2.  $\tau = \{X, \emptyset\}$ .

Lemma. Let  $S \subset \mathcal{P}(X)$ . Then, there is a smallest topology  $\tau(S)$  that contains  $S$ .

Proof. Let  $\tau(S) = \bigcap \{\tau' : \tau' \text{ is a top and } S \in \tau'\}$ .

Prop. Let  $\beta \subset \mathcal{P}(X)$  be a coer of  $X$ . Then  $\beta$  is a basis for  $\tau(\beta)$  if and only if  $U, V \in \beta$  and  $x \in U \cap V$ , then  $\exists W \in \beta$  such that  $x \in W \subset U \cap V$ .

Proof. Hwk.

[One lecture on omitted. Defined weak topology. ]

Defn. An ordered set  $(X, \leq)$  is directed if given  $x, y \in X$ ,  $\exists z \in X$  such that  $x \leq z$  and  $y \leq z$ .

Ex. –  $X = \mathbb{N}$ .

- Let  $(Y, \tau)$  be a topological space and  $y_0 \in Y$ . Then  $X = \mathcal{N}(y_0)$  be the collection of neighborhoods of  $y_0$  induced by reverse inclusion. Then,  $U \leq V \iff V \subset U$ . Then  $\mathcal{N}(y_0)$  is directed. If  $U, V \in \mathcal{N}(y_0)$ , then so is  $U \cap V$ .
- We can do the same thing for open neighborhoods.

Defn. – A net  $x$  is a set  $X$  is a function  $x : \Lambda \rightarrow X$  where  $\Lambda$  is a directed set. As with sequences, we usually write  $x_\lambda$  for  $x(\lambda)$  and  $(x_\lambda)_{\lambda \in \Lambda}$  in place of  $x : \Lambda \rightarrow X$ , or just  $x$ .

- If  $X$  is a topological space, then we say that a net  $(x_\lambda)_{\lambda \in \Lambda}$  converges to  $x_0 \in X$  if  $(x_\lambda)$  is eventually in every neighborhood of  $x_0$ . That is, if  $U \in \mathcal{N}(x_0)$ , then  $\exists \lambda_0 \in \Lambda$  such that:

$$\lambda \geq \lambda_0 \implies x_\lambda \in U$$

- We say that  $x_0 \in X$  is an accumulation point of a net  $x_\lambda$  if  $(x_\lambda)$  is frequently in every neighborhood of  $x_0$ . That is if  $U$  is a neighborhood of  $x_0$  and  $\lambda_0 \in \Lambda$ , then  $\exists \lambda \geq \lambda_0$  such that  $x_\lambda \in U$ .

Prop. Suppose  $X$  is a topological space and  $E \subset X$ . Then,  $x_0 \in \overline{E}$  if and only if there exists a net  $(x_\lambda) \subset E$  with  $x_\lambda \rightarrow x_0$ .

Proof. Suppose we have  $x_\lambda \subset E$  with  $x_\lambda \rightarrow x_0$ . If  $x_0 \notin \overline{E}$ , then  $\exists U \in \mathcal{O}(x_0)$  such that  $U \cap E = \emptyset$ . But then we would eventually have  $(x_\lambda)$  in  $U$ . Contradiction.

Now suppose that  $x_0 \in \overline{E}$ . Let  $\Lambda = \mathcal{O}(x_0)$ .

If  $W \in \Lambda$ , then  $W \cap E \neq \emptyset$ . Then, we let  $x_W \in W \cap E$ . Then,  $(x_W)_{W \in \Lambda}$  converges to  $x_0$ .

If  $U_0 \in \mathcal{O}(x_0)$ , then  $U \geq U_0$ ,  $x_U \in U \subset U_0$ .

Ex.  $S = \{\sqrt{n}e_n \in l^2 : n \geq 1\}$ , then  $0 \in \overline{S}^w$  (weak topology).

Defn. A subset  $C$  of a vector space  $V$  over  $\mathcal{F}$  is convex if  $x, y \in C$  and  $t \in [0, 1]$ , then  $tx + (1 - t)y \in C$ .

Ex. In a normed vector space  $X$ , every open ball  $B_r(x_0)$  is convex if  $x_0 \in X$  and  $r > 0$ . To see this, supposed  $x, y \in B_r(x_0)$ , and  $t \in [0, 1]$ . Then,

$$\begin{aligned} \|tx + (1 - t)y - x_0\| &\leq \|t(x - x_0)\| + \|(1 - t)(y - x_0)\| \\ &< tr + (1 - t)r \\ &= r \end{aligned}$$

Lemma. Let  $C$  be an open convex neighborhood of 0 in a normed vector space  $X$ . Then,

$$m(x) = \inf\{s > 0, s^{-1}x \in C\}$$

Then  $m$  is a Minkowski functional on  $X$  such that:

$$C = \{x \in X : m(x) < 1\}$$

Proof. If  $x \in X$ , then  $\frac{1}{n}x \rightarrow 0$ . So  $\frac{1}{n}x$  is eventually in  $C$ . So  $m(x) < \infty$  for all  $x \in X$ .

Note that if  $t \geq 0$ , then  $m(tx) = tm(x)$ . (This is one of the axioms for a Minkowski functional).

Also let  $s^{-1}x, t^{-1}y \in C$ . Then:

$$(s+t)^{-1}(x+y) = \frac{s}{s+t}s^{-1}x + \frac{t}{s+t}t^{-1}y \in C$$

Thus,  $m(x+y) \leq s+t$ .

Since  $s^{-1}x, t^{-1}y \in C$  is arbitrary, we have:

$$m(x+y) \leq m(x) + m(y)$$

This is the second axiom for a Minkowski functional.

Thus,  $m : X \rightarrow \mathbb{R}$  is a Minkowski functional. Now if  $x \in C$ , then  $(1 + 1/n)x \rightarrow x$ .

Thus, for some  $n$ ,

$$m(x) = \frac{1}{1 + 1/n} < 1$$

If  $m(x) < 1$ , then  $\exists s < 1$  such that  $s^{-1}x \in C$ .

But  $0 \in C$  and  $x = (1-s)0 + s(s^{-1}x) \in C$ .

Thm. (Hahn-Banach Separation Theorem). Let  $A$  and  $B$  be disjoint non-empty convex subsets of some normed vector space  $X$ . If  $A$  is also open, then there exists  $\phi \in X^*$  and  $t \in \mathbb{R}$  such that:

$$\Re(\phi(x)) < t \leq \Re(\phi(y))$$

for all  $x \in A$  and  $y \in B$ .

Proof. We start with  $\mathbb{F} = \mathbb{R}$ . Then, we can use the Basic Extension Lemma. Fix  $x_0 \in A$  and  $y_0 \in B$ . Let  $z_0 = x_0 - y_0$ . Let:

$$C = A - B + z_0$$

We can check that  $C$  is convex.

Since  $C = \bigcup_{y \in B} A - y + z_0$ . Thus,  $C$  is union of open sets, hence open neighborhood of 0.

Let  $m : X \rightarrow \mathbb{R}$  be the corresponding Minkowski functional.  
We claim  $z_0 \notin C$ . If  $x - y + z_0 = z_0$ , then  $x = y \in A \cap B = \emptyset$ .  
Thus,  $m(z_0) \geq 1$ .  
Define:

$$\phi_0 : \mathbb{R}z_0 \rightarrow \mathbb{R}$$

by  $\phi_0(\alpha z_0) = \alpha$ .  
Since  $m(x) \geq 0$ , then for all  $x$  if  $\alpha \leq 0$ , then:

$$\phi_0(\alpha z_0) \leq m(\alpha z_0)$$

If  $\alpha \geq 0$ , then:

$$\phi_0(\alpha z_0) = \alpha \leq \alpha m(z_0) = m(\alpha z_0)$$

Thus,  $\phi_0(z) \leq m(z)$  for all  $z \in \mathbb{R}z_0$ .

Thus, the Basic Extension Lemma gives us  $\phi : X \rightarrow \mathbb{R}$  such that,  $\phi(x) \leq m(x)$ .

Thus,  $\phi(x) < 1$  if  $x \in C$ .

Also,  $-\phi(x) = \phi(-x) < 1$  if  $x \in -C$ .

Thus,  $\forall \epsilon > 0$ ,

$$|\phi(x)| < \epsilon \text{ if } x \in \epsilon C \cap -\epsilon C$$

Since  $\epsilon C \cap -\epsilon C$  is a neighborhood of 0 for all  $\epsilon > 0$ , this means  $\phi$  is continuous at 0.

Thus,  $\phi \in X^*$ . If  $x \in A$  and  $y \in B$ , then:

$$x - y + z_0 \in C$$

Then,

$$m(x - y + z_0) < 1$$

Thus,

$$\phi(x - y + z_0) < 1$$

Since  $\phi(z_0) = \phi_0(z_0) = 1$ , we have:

$$\phi(x) < \phi(y)$$

Since  $\phi$  is linear and since  $A$  and  $B$  are convex,  $\phi(A)$  and  $\phi(B)$  are intervals. Since  $A$  is open and  $\phi$  is linear,  $\phi(A)$  must be open.

Then, we can let  $t$  be the right-hand endpoint of  $\phi(A)$ .

Now if  $\mathbb{F} = \mathbb{C}$ , then we can treat  $X$  as a real space and produce a real linear functional  $\psi : X \rightarrow \mathbb{R}$  such that:

$$\psi(A) < t \leq \psi(B)$$

Now let  $\phi(x) = \psi(x) - i\psi(ix)$ .  $\phi$  is continuous since  $\psi$  is and continuity implies boundedness.

[Skipped a class of notes. Took pictures. 5/9 ]

Ex. The sequence  $(\sqrt{n}e_n) \subset l^2$  has no weakly convergent sequence.

Ex. For each  $\alpha \in l^\infty$ , let:

$$D_\alpha = \{z \in \mathbb{C} : |z| \leq \|\alpha\|_\infty\}$$

Then if  $\alpha = (\alpha_n)$  then  $\alpha_n \in D_\alpha$  for all  $n \geq 1$ .

We now know that:

$$Z = \prod_{\alpha \in l^\infty} D_\alpha$$

is compact in the product topology. Let  $(z_n) \subset Z$  be the sequence given by:

$$z_n(\alpha) = \alpha_n$$

But we claim  $(z_n)$  has no convergent subsequence.

Proof. Suppose  $(z_{n_k})$  converges to  $z_0 \in Z$ . Then it converges pointwise. Thus for all  $\alpha \in l^\infty$ , we have:

$$z_{n_k}(\alpha) \rightarrow z_0(\alpha)$$

Let  $\alpha_0 \in l^\infty$  be given by:

$$\alpha_0(n) = \begin{cases} (-1)^k & \text{if } n = n_k \\ 0 & \text{otherwise} \end{cases}$$

But now:

$$z_{n_k}(\alpha_0) = (-1)^k$$

But

$$\lim_{k \rightarrow \infty} z_{n_k}(\alpha_0)$$

does not exist.

Defn. Let  $x : \Lambda \rightarrow X$  be a net in a set  $x$ . Then a net  $y : M \rightarrow X$  is a subnet of  $x$  if  $\exists h : M \rightarrow \Lambda$  such that  $y = x \circ h$  and for all  $\lambda_0 \in \Lambda$ , there exists  $m(\lambda_0)$  such that:

$$m \geq m(\lambda_0) \implies h(m) \geq \lambda_0$$

Rmk. If  $h : M \rightarrow \Lambda$  is monotone in that  $m_1 \geq m_2$  then  $h(m_1) \geq h(m_2)$ , then given  $\lambda_0 \in \Lambda$  it suffices to find  $m(\lambda_0)$  such that  $h(m(\lambda_0)) \geq \lambda_0$ .

Yap. We usually write:

$$(y_m)_{m \in M} \text{ as } (x_{h(m)})_{m \in M} \text{ or } (x_{\lambda_m})_{m \in M}$$

Example: If you have a subsequence  $(x_n)_{n \in \mathbb{N}}$  and  $h : \mathbb{N} \rightarrow \mathbb{N}$  strictly increasing, we can write:

$$x_k = h(k) \quad n_{k+1} > n_k$$



Lemma. Let  $(x_\lambda)_{\lambda \in \Lambda}$  be a net in a topological space. Then  $x_0$  is an accumulation point of  $(x_\lambda)$  if and only if there is a subnet  $(x_{h(m)})_{m \in M}$  converging to  $x_0$ .

Proof. Suppose that  $(x_{h(m)})_{m \in M}$  converges to  $x_0$ .

Let  $U \in \mathcal{O}(x_0)$  and  $\lambda_0 \in \Lambda$ .

Since the subnet converges to  $x_0$ , there is a  $m_0 \in M$  such that  $m \geq m_0$  implies  $x_{h(m)} \in U$ .

But there exists  $m_1 = m(\lambda_0)$  such that  $m \geq m_1$  implies  $h(m) \geq \lambda_0$ .

Let  $m_2$  be such that  $m_2 \geq m_1$  and  $M_2 \geq m_0$ . Then,

$$h(m_2) \geq \lambda_0 \text{ and } x_{h(m_2)} \in U$$

Thus,  $x_0$  is an accumulation point of  $(x_\lambda)$ .

Now suppose that  $x_0$  is an accumulation point of  $(x_\lambda)$ . Let:

$$\mathcal{M} \{ (U, \lambda) \in \mathcal{O}(x_0) \times \Lambda : x_\lambda \in U \}$$

and  $(U_1, \lambda_1) \leq (U_2, \lambda_2)$  if  $U_2 \subset U_1$  and  $\lambda_1 \geq \lambda_2$ .

Claim:  $\mathcal{M}$  is a directed set. To see this, let  $(U_1, \lambda_1), (U_2, \lambda_2) \in \mathcal{M}$ . Let  $\lambda_3 \geq \lambda_1$  and  $\lambda_3 \geq \lambda_2$ . Let  $\lambda_4 \geq \lambda_3$  such that  $x_{\lambda_4} \in U_1 \cap U_2$ . Then,  $(U_1 \cap U_2, \lambda_4) \geq (U_k, \lambda_k)$  for  $k = 1, 2$ .

Now define  $h : M \rightarrow \Lambda$  by  $h(U, \lambda) = \lambda$ . Clearly,

$$(x_{(U, \lambda)})_{(U, \lambda) \in \mathcal{M}} \rightarrow x_0$$

Corr. A topological space is compact if and only if every net has a convergent subnet.

Not. If  $X$  is a normed vector space, and  $Y$  a collection of linear functional on  $X$ . Then we write  $\sigma(X, Y)$  for the initial topology on  $X$  induced by  $\mathcal{F} = Y$  (smallest topology that makes all of these functional continuous, likely want separation so you get Hausdorff etc).

Ex.  $\sigma(X, X^*)$  is the weak topology on  $X$ .

$\sigma(X^*, X^{**})$  is the weak topology on  $X^*$ .

Yap. But now we want to consider:

$$\sigma(X^*, X) := \sigma(X^*, \iota(X))$$

Defn. If  $X$  is a normed vector space, then:

$$\sigma(X^*, X)$$

is called the weak-\* topology on  $X^*$ .

Lemma. Let  $X$  be a normed vector space. Then a sub-basis for the weak-\* topology on  $X^*$  consists of all sets of the form:

$$\mathcal{U}(x, \phi_0, \epsilon) = \{\phi \in X^* : |\phi(x) - \phi_0(x)| \leq \epsilon\}$$

where  $x \in X, \phi_0 \in X^*$  and  $\epsilon > 0$ .

Then we get a neighborhood basis for  $\phi_0 \in X^*$  by taking:

$$U(\{x_1, \dots, x_n\}, \phi_0, \epsilon) = \{\phi \in X^* : |\phi(x_k) - \phi_0(x_k)| < \epsilon \forall k\}$$

Lemma. A net  $(\phi_\lambda)$  in  $X^*$  converges in weak-\* to  $\phi_0$  if  $\phi_\lambda(x) \rightarrow \phi_0(x)$  for all  $x \in X$ .

Thm. (Alaoglu's Theorem.) If  $X$  is a normed vector space and:

$$B^* = \{\phi \in X^* : \|\phi\| \leq 1\}$$

is the closed unit ball, then  $B^*$  is compact in the weak-\* topology.

Proof. Let  $D_r = \{z \in \mathcal{F} : |z| \leq r\}$ . Let:

$$Z = \prod_{x \in X} D_{\|x\|}$$

Then we know that  $Z$  is compact in the product topology by Tychonoff's theorem.

Define:

$$j : B^* \rightarrow Z$$

by  $j(\phi)(x) = \phi(x)$ . Note that  $j$  is injective.

If  $\phi_\lambda \rightarrow \phi_0$  in the weak-\* topology, then  $\phi_\lambda(x) \rightarrow \phi_0(x)$  for all  $x \in X$ . Thus,  $j(\phi_\lambda)(x) \rightarrow j(\phi_0)(x)$  for all  $x \in X$ . Thus,  $j(\phi_\lambda) \rightarrow j(\phi_0)$  in  $Z$ .

Now, claim that  $j$  has closed range in  $Z$ . Suppose  $j(\phi_\lambda) \rightarrow z_0$  in  $Z$ . Then,

$$j(\phi_\lambda)(x) \rightarrow z_0(x) \quad \forall x \in X$$

That is:

$$\phi_\lambda(x) \rightarrow z_0(x)$$

Then,

$$\begin{aligned} z_0(x+y) &= \lim_{\lambda} \phi_\lambda(x+y) \\ &= \lim_{\lambda} \phi_\lambda(x) + \phi_\lambda(y) \\ &= z_0(x) + z_0(y) \end{aligned}$$

We also clearly have:

$$|z_0(x)| \leq \|x\|$$

Thus,  $z_0 \in j(B^*)$ . Thus,  $j(B^*)$  is compact.

If  $j(\phi_\lambda) \rightarrow j(\phi)$  in  $Z$ , then:

$$j(\phi_\lambda)(x) \rightarrow j(\phi)(x) \quad \forall x \in X$$

Then,  $\phi_\lambda(x) \rightarrow \phi(x)$  for all  $x \in X$ . Thus,  $\phi_\lambda \rightarrow \phi$  in the weak-\* topology (thus inverse is continuous). Thus,  $j$  is a homeomorphism onto  $j(B^*)$  and then  $B^*$  is compact.

Defn. We defined the weak-\* topology on  $X^*$ ,  $\sigma(X^*, X)$ , to be the initial topology on  $X^*$  induced by  $\mathcal{J}$  a family of linear functional on  $X$ .

Defn. A subbases for  $\sigma(X^*, X)$  is given by the sets:

$$\mathcal{U}(x, \phi_0, \epsilon) = \{\phi \in X^* : |\phi(x) - \phi_0(x)| < \epsilon\}$$

for all  $x \in X$ ,  $\phi_0 \in X^*$  and  $\epsilon > 0$ .

$$\mathcal{U}(x, \phi_0, \epsilon) = \omega(\iota(x), \phi_0, \epsilon) = \{\phi \in X^* : |\iota(x)(\phi) - \iota(x)(\phi_0)| < \epsilon\}$$

Defn. If  $V$  is a vecto space over  $\mathcal{F}$ , then we call  $(\cdot, \cdot) : V \times V \rightarrow \mathcal{F}$  a sesquilinear form on  $V$  if for all  $x, y, z \in V$  and  $\alpha \in \mathcal{F}$ , we have:

$$- (x + \alpha y, z) = (x, z) + \alpha(y, z)$$

$$- (x, \alpha y + z) = \bar{\alpha}(x, y) + (x, z)$$

If we also have:

$$(x, y) = \overline{(y, x)}$$

then we say that  $(\cdot, \cdot)$  is a self-adjoint and if:

$$(x, x) \geq 0$$

then we say  $(\cdot, \cdot)$  is positive.

Lem. (Polarization Identity). If  $\mathbb{F} = \mathbb{C}$  and  $(\cdot, \cdot)$  is a sesquilinear form on  $V$ , then:

$$(x, y) = \frac{1}{4} \sum_{k=0}^3 i^k (x + i^k y, x + i^k y)$$

Proof. Check that:

$$4(x, y) = (x + y, x + y) + i(x + iy, x + iy) - (x - y, x - y) - i(x - iy, x - iy) \quad (1)$$

$$(2)$$

Corr. If  $\mathbb{F} = \mathbb{C}$ , then  $(\cdot, \cdot)$  is self-adjoint if and only if  $(x, x) \in \mathbb{R}$  for all  $x \in V$ . Hence on a complex vector space, a positive sesquilinear form is always self-adjoint.

Defn. A positive self-adjoint sesquilinear form on  $V$  is called a pre-inner product. If  $(x, x) = 0$  if and only if  $x = 0$ , then we call it an inner product.

Defn. If  $(\cdot, \cdot)$  is a pre-inner product on  $V$ , then we define:

$$||v|| = (v, v)^{frac{1}{2}}$$

for all  $v \in V$ .

Rmk. The polarization identity then becomes for  $\mathbb{C}$ :

$$(x, y) = \frac{1}{4} \sum_{k=0}^3 i^k \|x + i^k y\|^2$$

for  $\mathbb{R}$ ,

$$(x, y) = \frac{1}{3} (\|x + y\|^2 - \|x - y\|^2)$$

Lemm. (Cauchy-Schwarz Inequality). If  $(\cdot, \cdot)$  is a pre-inner product on  $V$ , then  $\forall x, y \in V$ :

$$|(x, y)| \leq \|x\| \|y\|$$

Proof. For all  $\alpha \in \mathcal{F}$ ,

$$\begin{aligned} 0 &\leq \|\alpha x + y\|^2 = (\alpha x + y, \alpha x + y) \\ &= |\alpha|^2 \|x\|^2 + \alpha(x, y) + \overline{\alpha}(x, y) + \|y\|^2 \\ &= |\alpha|^2 \|x\|^2 + 2\Re(\alpha(x, y)) + \|y\|^2 \end{aligned}$$

Let  $\tau \in \mathcal{F}$  be such that  $\tau(x, y) = |(x, y)|$ . If  $\alpha = t\tau$  with  $t \in \mathbb{R}$ , then:

$$0 \leq t^2 \|x\|^2 + 2t|(x, y)| + \|y\|^2$$

Then, by the quadratic formula,

$$\begin{aligned} 4|(x, y)|^2 - 4\|x\|^2 \|y\|^2 &\leq 0 \\ \implies |(x, y)| &\leq \|x\| \|y\| \end{aligned}$$

Corr. Let  $(\cdot, \cdot)$  be a pre-inner product on  $V$ . Then:

$$\|x\| = (x, x)^{\frac{1}{2}}$$

is a semi-norm on  $V$  which is a norm exactly when  $(\cdot, \cdot)$  is an inner product.

Proof. The homogeneity is clear. Use Cauchy Schwarz to see:

$$\|x + y\|^2 \leq (\|x\| + \|y\|)^2$$

Defn. A vector space  $H$  with an inner product is called a Hilbert space if the normed vector space  $(H, \|\cdot\|)$  is complete with respect to  $\|x\| = (x, x)^{\frac{1}{2}}$ .

Ex. (1) If  $H = \mathbb{F}^n$  and

$$(x, y) = \sum_{k=1}^n x_k \overline{y_k}$$

Here,

$$\|x\| = \|x\|_2$$

(2) Let  $H = l^2$ . Now,

$$(x, y) = \sum_{n=1}^{\infty} x_n \overline{y_n}$$

Again,

$$||x|| = ||x||_2$$

(3) Let  $H = L^2(x, \mathcal{M}, \mu)$  and:

$$(f, g) = \int_X (f(x) \overline{g(x)}) d\mu(x)$$

Thm. (Parallelogram Law). If  $V$  is a pre-inner product space, then for all  $x, y \in H$ ,

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2$$

Thm. (Jordan von Neumann Thm.) Let  $X$  be a normed vector space where norm satisfies the parallelogram law. Then  $X$  admits an inner product such that  $||x|| = (x, x)^{\frac{1}{2}}$ .

Prop. If  $H$  is an inner product space and if  $x_n \rightarrow x$  and  $y_n \rightarrow y$  in  $H$ . Then,

$$(x_n, y_n) \rightarrow (x, y)$$

Proof. Since  $H$  is a normed vector space, the norm is continuous. Thus,

$$||x_n|| \rightarrow ||x|| \quad ||y_n|| \rightarrow ||y||$$

Now, using Cauchy-Schwarz,

$$\begin{aligned} |(x_n, y_n) - (x, y)| &\leq |(x_n - x, y_n)| + |(x, y_n - y)| \\ &\leq ||x_n - x|| \cdot ||y_n|| + ||x|| \cdot ||y_n - y|| \\ &\rightarrow 0 \end{aligned}$$

Defn. Let  $H$  be an inner product space. We say  $x \in H$  is orthogonal to  $g \in H$  or just  $x \perp y$  if  $(x, y) = 0$ . We say  $E \subset H$  is orthogonal to  $F \subset H$  if  $\forall x \in E, y \in F, x \perp y$ . We would write  $E \perp F$ .

Thm. (Pythagorean Theorem). Suppose  $H$  is an inner product space and that  $x_1, \dots, x_n \in H$  are pairwise orthogonal. Then,

$$||\sum_{k=1}^n x_k||^2 = \sum_{k=1}^n ||x_k||^2$$

Proof. Simple.

Lemm. Let  $C$  be a nonempty closed, convex subset of a Hilbert space  $H$ . Then  $\forall y \in H$ ,  $\exists! x \in C$  such that:

$$d(y, C) = \|y - x\|$$

Proof. We can replace  $C$  by  $C - \{y\}$  and assume  $y = 0$ .

Let  $\alpha = \inf\{\|x\| : x \in C\} = d(0, C)$ .

Let  $(x_n) \subset C$  be a sequence such that  $\|x_n\| \rightarrow \alpha$ .

Then,

$$\begin{aligned} 2(\|x_n\|^2 + \|x_m\|^2) &= \|x_n + x_m\|^2 + \|x_n - x_m\|^2 \\ &= 4 \left\| \frac{x_n + x_m}{2} \right\|^2 + \|x_n - x_m\|^2 \\ &\geq 4\alpha^2 + \|x_n - x_m\|^2 \end{aligned}$$

But LHS converges to  $4\alpha^2$ . Thus,  $(x_n)$  is Cauchy. Then,

$$x_n \rightarrow x \quad \|x\| = \alpha$$

Suppose  $y \in C$  and  $\|y\| = \alpha$ . Then,

$$\begin{aligned} 4\alpha^2 &= 2(\|x\|^2 + \|y\|^2) \\ &= \|x + y\|^2 + \|x - y\|^2 \\ &\geq 4\alpha^2 + \|x - y\|^2 \end{aligned}$$

Thus,  $\|x - y\|^2 = 0$ , thus  $x = y$ .

Missed a lecture. 5/16.

Thm. (Bessel's Inequality).  $H$  is an inner product space  $\{e_n : n \in \mathbb{N}\}$  be an orthonormal subset. Then  $\forall x \in H$ :

$$\|x\|^2 \geq \sum_{n=1}^{\infty} |(x | e_n)|^2$$

Corr. Let  $\{e_n : n \in \mathbb{N}\}$  be an orthonormal subset of a Hilbert space  $H$  and let:

$$\mathcal{E} = \overline{\text{span}\{e_n : n \in \mathbb{N}\}}$$

Then, for all  $h \in H$ ,

$$\sum_{n=1}^{\infty} (h | e_n) e_n$$

converges in  $H$  and:

$$P_{\mathcal{E}}(h) = \sum_{n=1}^{\infty} (h | e_n) e_n$$

Proof. Pictures.

Defn. An orthonormal subset  $\{e_{\alpha}\}_{\alpha \in A}$  is a basis for  $H$  if  $\text{span}\{e_{\alpha} : \alpha \in A\}$  is dense.

Prop. Every Hilbert space  $H$  has an orthonormal basis.

Proof. Let  $\Lambda$  be a collection of orthonormal subsets index by inclusion. Then  $\Lambda$  is inductively ordered. Then Zorn's lemma gives us a maximal element in  $\Lambda$ :

$$\{e_{\alpha} : \alpha \in A\}$$

If  $w = \text{span}\{e_{\alpha}\} \subset H$  is proper, then  $\exists e_0 \in w^{\perp}$  with  $\|e_0\| = 1$ . This is a contradiction to maximality.

Corr. If  $H$  is a separable  $\infty$ -dimensional Hilbert space, then every orthonormal basis for  $H$  is countably infinite.

Proof. Let  $\{e_{\alpha} : \alpha \in A\}$  be an orthonormal subset of  $H$ . Then the balls:

$$B_{1/\sqrt{2}}(e_n)$$

are pairwise disjoint. Hence there exists at most countably many  $e_n$  in  $H$ .

Thm. Let  $\{e_n; n \in \mathbb{N}\}$  be an orthonormal subset of a Hilbert space. Then the following are equivalent:

- This is a basis (i.e. it has a dense linear span).
- $(h | e_n) = 0$  for all  $n \in \mathbb{N}$  implies  $h = 0$ .
- $\forall h \in H, h = \sum_{n=1}^{\infty} (h | e_n) e_n$ .
- For all  $h, k \in H, (h | k) = \sum_{n=1}^{\infty} (h | e_n) (e_n | k)$ .



– For all  $h \in H$ ,  $\|h\|^2 = \sum_{n=1}^{\infty} |(h|e_n)|^2$ .

Proof. (1)  $\implies$  (2). Let  $w_0 = \text{span}\{e_n\}$ . Then,

$$w_0^{\perp\perp} = \overline{w_0} = H$$

Hence,

$$w_0^{\perp} = \overline{w_0}^{\perp} = \{0\}$$

(2)  $\implies$  (1). If  $w_0^{\perp} = \{0\}$ , then:

$$\overline{w_0} = w_0^{\perp\perp} = H$$

(1)  $\implies$  (3). Apply the  $P_{\mathcal{E}}$  corollary with  $\mathcal{E} = H$ .

(3)  $\implies$  (4).  $\phi_k(\cdot) = (\cdot | k)$  is continuous.

(4)  $\implies$  (5). Let  $k = h$ .

(5)  $\implies$  (1). Suppose  $w_0 = \overline{\text{span}\{e_n\}} \neq H$ . Then,  $\exists h \in H \setminus \{0\}$  such that  $(h|e_n) = 0$  for all  $n \in \mathbb{N}$ .

Missed one line.

Thm. Let  $H$  be a separable  $\infty$ -dimensional Hilbert space. Then, there is an isometric linear isomorphism.

$$U : H \rightarrow l^2$$

such that:

$$(U(h) | U(k))_{l^2} = (h | k)_H$$

Proof. Let  $\{e_n : n \in \mathbb{N}\}$  be an orthonormal basis for  $H$ . Let  $H_0 = \text{span}\{e_n : n \in \mathbb{N}\}$ . Define:

$$U_0 : H_0 \rightarrow l^2$$

by  $U_0(\sum_{k=1}^n \alpha_k e_k) = (\alpha_1, \dots, \alpha_n, 0, 0, \dots)$ .

Since  $\{e_n\}$  is a Hamel basis for  $H_0$ ,  $U_0$  is well-defined and isometric by Parseval. Since  $H_0$  is dense in  $H$  and  $U_0(H_0) = c_{0,0}$  (seqs that vanish after some point) is dense in  $l^2$ . Thus,  $U_0$  extends to an isometry of  $H$  into  $l^2$  with dense range. Since  $U$  is an isometry,  $U(H)$  is complete. Hence its closed and  $U$  is a bijection.

We want to show that this preserves the inner product. Let:

$$[x, y] = (U(x) | U(y))_{l^2}$$

is an inner product on  $H$ . Then,

$$\begin{aligned} (Ux | Uy) &= \frac{1}{4} \sum_{k=0}^3 i^k \|U(x + i^k y)\|^2 \\ &= \frac{1}{4} \sum_{k=0}^3 i^k \|x + i^k y\|^2 \\ &= (x | y) \end{aligned}$$

If  $\mathbb{F} = \mathbb{R}$ , use:

$$(Ux \mid Uy) = \frac{1}{4}(\|U(x) + U(y)\|^2 + \|U(x) - U(y)\|^2)$$

to get the same.

Ex. Let  $H = \mathbb{F}^n$ . If  $A \in M_n(\mathbb{F})$ , we set:

$$L_A \in \mathcal{L}(\mathbb{F})$$

by  $L_A(h) = Ah$ . where:

$$(Ah)_k = \sum_{j=1}^n a_{kj} h_j$$

where:

$$a_{ij} = (T(e_j) \mid e_i)$$

Now if  $A = M_n(\mathbb{F})$ , let  $A^* = (b_{ij})$ , where

$$b_{ij} = \overline{a_{ji}}$$

then, we have:

$$\begin{aligned} (L_A(h) \mid k) &= (Ah \mid k) \\ &= (h \mid A^*k) \\ &= (h \mid L_{A^*}(k)) \end{aligned}$$

Gien  $T \in \mathcal{L}(\mathbb{F}^n)$ . Then,  $\exists! T^* \in \mathcal{L}(\mathbb{F}^n)$  such that:

$$(T(h) \mid k) = (h \mid T^*(k))$$

Thmn. Let  $H$  be a Hilbert space and  $T \in \mathcal{L}(H)$ . Then  $\exists! T^* \in \mathcal{L}(H)$  such that for all  $h, k \in H$ :

$$(T(h) \mid k) = (h \mid T^*(k))$$

Furthermore,

- $\|T^*\| = \|T\|$
- $(T + \lambda S)^* = T^* + \bar{\lambda} S^*$
- $T^{**} = T$
- $(TS)^* = S^* T^*$
- $\|T^* T\| = \|T\|^2$

Proof. Clearly, if  $T^*$  exists, it is unique. For existence, fix  $k \in H$ . Then,

$$h \mapsto (T(h) \mid k)$$

is linear and

$$|(T(h) \mid k)| \leq \|T\| \cdot \|k\| \cdot \|h\|$$

Thus, this is bounded in norm by  $\|T\| \cdot \|k\|$ . Thus this is a functional in  $H^*$ . Thus,  $\exists! z(k)$  such that  $\|z(k)\| \leq \|T\| \|k\|$  and:

$$(T(h) \mid k) = (h \mid z(k)) \quad \forall h \in H$$

Now, check that:

$$z(k + \lambda k') = z(k) + \lambda z(k')$$

and we can define:

$$T^*(k) = z(k) \quad \|z(k)\| \leq \|T\| \|k\|$$

for all  $k \in H$ . Thus,  $\|T^*\| \leq \|T\|$ .

Since  $T^* \in \mathcal{L}(H)$ , hence we have  $T^{**} \in \mathcal{L}(H)$  and:

$$(T^*(h) \mid k) = (h \mid T^{**}(k))$$

But:

$$(T^*(h) \mid k) = (h \mid T(k))$$

since this holds for all  $h$  and  $k$ , we have that:  $T = T^{**}$ .

Thus,  $\|T\| = \|T^{**}\| \leq \|T^*\|$ . Thus,  $\|T^*\| = \|T\|$ .

Next, note:

$$(h \mid (TS)^*(k)) = (TS(h) \mid k) = (h \mid S^*T^*(k))$$

Thus,  $T^*S^* = (TS)^*$ .

Finally,

$$\begin{aligned} \|T(h)\|^2 &= (T(h) \mid T(h)) \\ &= (T^*T(h) \mid h) \\ &\leq \|T^*T\| \|h\|^2 \end{aligned}$$

Thus,  $\|T\|^2 \leq \|T^*T\| \leq \|T^*\| \|T\| = \|T\|^2$ .

Ex.  $H = \mathbb{R}^2$ . Let  $T \in \mathcal{L}(\mathbb{R}^2)$  be given by:

$$T(x, y) = (-y, x)$$

Note that  $(T(h) \mid h) = 0$  for all  $h \in \mathbb{R}^2$ .

Prop. Let  $H$  be a complex Hilbert space. If  $T \in \mathcal{L}(H)$  and:

$$(T(h) \mid h) = 0 \quad \forall h \in H$$

then  $T = 0$ .

Proof. Let  $[x, y] = (T(x) \mid y)$ . Then  $[\cdot, \cdot]$  is a sesquilinear form on  $H$ .

Thus, we have the polarization identity:

$$\begin{aligned} (T(x) \mid y) &= [x, y] \\ &= \frac{1}{4} \sum_{k=0}^3 i^k (T(x + i^k y) \mid x + i^k y) \\ &= 0 \end{aligned}$$

Thus,  $T = 0$ .