Math 121: Hodge Theory

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1 Background and Definitions

Definition 1.1. *Let $E/\mathbb{C}$ be an elliptic curve. Then, the elliptic curve E is given by a Weierstrass equation*

$$E : y^2 = x^3 + g_1 x + g_2$$

together with a special point $O \in E$ at infinity.

Definition 1.2. *Let $E/\mathbb{C}$ be an elliptic curve. E has a abelian group structure with the identity element O.*

Definition 1.3. *Let $E_1/\mathbb{C}$ and $E_2/\mathbb{C}$ be elliptic curves. An isogeny $\phi : E_1 \to E_2$ is a non-zero morphism of varieties with $\phi(O_1) = O_2$. As the map is not constant, it is surjective. An isogeny from E_1 to itself is called an endomorphism. The set of endomorphisms $\mathrm{End}(E_1)$ is a ring with the addition and multiplication given by the pointwise addition and composition, respectively.*

Definition 1.4. *An elliptic function (relative to a lattice Λ) is a meromorphic function $f(z)$ on $\mathbb{C}$ that satisfies:*

$$f(z + \omega) = f(z)$$

for all $z \in \mathbb{C}$ and $\omega \in \Lambda$.

Definition 1.5. *Let $\Lambda \subset \mathbb{C}$ be a lattice. The Weierstrauss $\wp$-function relative to Λ is defined by the series:*

$$\wp(z; \Lambda) = \frac{1}{z^2} + \sum_{\substack{w \in \Lambda \\ w \neq 0}} \left(\frac{1}{(z - w)^2} - \frac{1}{\omega^2} \right)$$

The Eisenstein series $G_{2k}(\Lambda)$ of weight $2k$ is the series:

$$G_{2k}(\Lambda) = \sum_{\substack{w \in \Lambda \\ w \neq 0}} w^{-2k}$$

Proposition 1.6. *Let $\Lambda \subset \mathbb{C}$ be a lattice. Then,*

$$\mathbb{C}(\Lambda) = \mathbb{C}(\wp(z), \wp'(z))$$

i.e. every elliptic function is a rational combination of $\wp(z)$ and $\wp'(z)$.

Proposition 1.7. *The Laurent series for $\wp(z)$ around $z = 0$ is given by:*

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1) G_{2k+2} z^{2k}$$

Proof. For $|z| < |w|$, we have:

$$\begin{aligned} \frac{1}{(w-z)^2} - \frac{1}{w^2} &= \frac{1}{w^2}\left(\frac{1}{(1-z/w)^2} - 1\right) \\ &= \sum_{k=1}^{\infty} (k+1) z^k / w^{2+k} \end{aligned}$$

Substituting this gives the desired result. □

Theorem 1.8. *For all $z \in \mathbb{C} \setminus \Lambda$, we have the relation:*

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

Let $g_2 = 60G_4(\Lambda)$ and $g_3 = 140G_6(\Lambda)$. Moreover,

$$f(x) = 4x^3 - g_2 x - g_3$$

has distinct roots, so its discriminant:

$$\Delta(\Lambda) = g_2^3 - 27g_3^3$$

is nonzero.

Proof. Write down the Laurent expansions:

$$\begin{aligned} \wp'(z)^2 &= 4z^{-6} - 24G_4 z^{-2} - 80G_6 + \cdots \\ \wp(z)^3 &= z^{-6} + 9G_4 z_1^{-1} 5G_6 + \cdots \\ \wp(z) &= z^{-2} + 3G_4 z^2 + \cdots \end{aligned}$$

Then, compare:

$$f(z) = wp'(z)^2 - 4\wp(z)^3 + 60G_4\wp(z) + 140G_6$$

Note that this is holomorphic at $z = 0$ and satisfies $f(0) = 0$. Since it is an elliptic function relative to Λ, from [AEC VI.3.1b], it is holomorphic away from Λ. Thus, it is constant, thus 0. □

Proposition 1.9. *Let $E/\mathbb{C}$ be the Elliptic curve:*

$$E : y^2 = 4x^3 - g_2 x - g_3$$

Then the map:

$$\phi : \mathbb{C}/\Lambda \to E(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C}), \qquad z \mapsto [\wp(z), \wp'(z), 1]$$

is a complex analytic isomorphism of complex Lie groups, i.e. it is an isomorphism of Riemann surfaces that is also a group homomorphism.

Theorem 1.10. *Let $\phi_\alpha : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ be $\phi_\alpha(z) = \alpha z \pmod{\Lambda_2}$. This map is holomorphic. Moreover,*

1. *The association:*

$$\{\alpha \in \mathbb{C} : \alpha\Lambda_1 \subset \Lambda_2\} \to \left\{ \substack{\textit{holomorphic maps} \\ \phi:\mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2 \\ \textit{with } \phi(0)=0} \right\}$$
$$\alpha \mapsto \phi_\alpha$$

   is a bijection.

2. *Let E_1 and E_2 be elliptic curves corresponding to lattices Λ_1 and Λ_2, respectively. Then the natural inclusion:*

$$\{\ \textit{isogenies } \phi : E_1 \to E_2\} \to \left\{ \substack{\textit{holomorphic maps} \\ \phi:\mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2 \\ \textit{with } \phi(0)=0} \right\}$$

   is a bijection.

Proof.

1. Assume $\phi_\alpha = \phi_\beta$. Then, $\alpha z \equiv \beta z \pmod{\Lambda_2}$ for all $z \in \mathbb{C}$. Thus, $z \mapsto (\alpha - \beta)z \equiv 0$. As Λ_2 is a discrete group, the map must be constant, hence $\alpha = \beta$. Thus, the map is injective.
   Next, let $\phi : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$ be a holomorphic map with $\phi(0) = 0$. Then, as $\mathbb{C}$ is simply connected, we lift to $\phi : \mathbb{C} \to \mathbb{C}$ satisfying:

$$f(z + w) \equiv f(z) \pmod{\Lambda_2}$$

   for all $w \in \Lambda_1$. As Λ_2 is discrete, we have $f(z + w) - f(z)$ is constant. Moreover, we have:

$$f'(z + w) = f'(z)$$

   so $f'(z)$ is a holomorphic elliptic function. It follows from [AEC VI.2.1] that $f'(z)$ is constant, so $f(z) = \alpha z + \gamma$. But $\gamma = 0$ as $f(0) = 0$. Then, we note $f(\Lambda_1) \subset \Lambda_2$ implies $\alpha\Lambda_1 \subseteq \Lambda_2$. Hence, $\phi = \phi_\alpha$.

2. Note that an isogeneny is given locally by everywhere defined rational functions, thus the map induced between the corresponding complex tori is holomorphic. Thus, the association is well-defined and injective.
   To show surjectivity, consider a map ϕ_α, where $\alpha \in \mathbb{C}^*$ with $\alpha\Lambda_1 \subseteq \Lambda_2$. The induced map on Weierstrass $\wp$-functions satisfies:
$$\wp(\alpha(z+w), \Lambda_2) = \wp(\alpha z + \alpha w, \Lambda_2) = \wp(\alpha z, \Lambda_2)$$
   and similarly for $\wp'$. Thus, $\wp(\alpha z, \Lambda_2)$ and $\wp'(\alpha z, \Lambda_2)$ are in $\mathbb{C}(\Lambda_1)$. The result follows from [AEC VI.3.2]. Thus, the maps is surjective.

□

Theorem 1.11. *(Uniformization Theorem) Let $A, B \in \mathbb{C}$ satisfy $4A^3 + 27B^2 \neq 0$. Then, there is a unique lattice $\Lambda \subset \mathbb{C}$ satisfying:*
$$g_2(\Lambda) = A \qquad g_3(\Lambda) = B$$

Theorem 1.12. *The following categories are equivalent:*

1. *Elliptic curves over $\mathbb{C}$ with isogenies.*
2. *Elliptic curves over $\mathbb{C}$ with complex analytic maps taking O to O.*
3. *Lattices $\Lambda \subset \mathbb{C}$ up to homothety with maps:*
$$\mathrm{Hom}(\Lambda_1, \Lambda_2) = \{\alpha \in \mathbb{C} : \alpha\Lambda_1 \subset \Lambda_2\}$$

Proof. This is precisely the content of the last few theorems. □

Definition 1.13. *Let K ba number field. An order R of K is a subring of K that is a finitely generated $\mathbb{Z}$-module and satisfies $R \otimes \mathbb{Q} = K$.*

Theorem 1.14. *Let $E/\mathbb{C}$ be an elliptic curve with associated lattice $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$. Then, one of the following is true:*

1. $\mathrm{End}(E) = \mathbb{Z}$.
2. *The field $\mathbb{Q}(\tau)$ is an imaginary quadratic extension of $\mathbb{Q}$ and $\mathrm{End}(E)$ is isomorphic to an order in $\mathbb{Q}(\tau)$.*

Proof. Note that we have:
$$\mathrm{End}(E) \cong R := \{\alpha : \alpha\Lambda \subset \Lambda\}$$
Thus, there are integers a, b, c, d such that:
$$\alpha = a + b\tau \qquad \alpha\tau = c + d\tau$$

Eliminating τ, we get:

$$\alpha^2 - (a+d)\alpha + (ad - bc) = 0$$

Thus, R is an integral extension of $\mathbb{Z}$.
If $R \neq \mathbb{Z}$, pick $\alpha \in R \setminus \mathbb{Z}$. Then, $b \neq 0$, so eliminating α gives:

$$b\tau^2 - (a-d)\tau - c = 0$$

Thus, $\mathbb{Q}(\tau)$ is an imginary quadratic extension of $\mathbb{Q}$ as $\tau \notin \mathbb{R}$. Then, since R is integral over $\mathbb{Z}$, R is an order in $\mathbb{Q}(\tau)$. □

Definition 1.15. *An elliptic curve $E/\mathbb{C}$ is said to have complex multiplication (CM) if* $\mathrm{End}(E)$ *is an order R of an imaginary quadratic field. (Corollary III 9.4, also VI)*

2 CM Elliptic Curvers are defined over $\overline{\mathbb{Q}}$

Proposition 2.1. *Two elliptic curves E and E' are isomorphic over $\overline{\mathbb{Q}}$ if and only if $j(E) = j(E')$.*

Proof. This is just computation. Refer to [AEC III 1.4] □

Definition 2.2. *We denote by* $\mathrm{Ell}(R)$ *the set of elliptic curves E with* $\mathrm{End}(E) \cong R$ *up to isomorphism.*

Let α is a non-zero fractional ideal of K ($\mathbb{Z}$ module of rank 2 not contained in $\mathbb{R}$), then α is a lattice in $\mathbb{C}$. Then, we have:

$$\begin{aligned}\mathrm{End}(E_\alpha) &\cong \{z \in \mathbb{C} : z\alpha \subset \alpha\}\\ &= \{z \in K : z\alpha \subset \alpha\}\\ &= R_K \qquad (\text{ } \alpha \text{ is a fractional ideal})\end{aligned}$$

Since we care about lattices up to homothety, we define:

Definition 2.3. *Let:*

$$CL(R_K) = \frac{\{\textit{non-zero fractional ideals of } R_K\}}{\{\textit{non-zero principal ideals of } R_K\}}$$

Then, there is a map:

$$\begin{aligned}\mathrm{CL}(R_K) &\to \mathrm{Ell}(R_K)\\ \alpha &\mapsto E_\alpha\end{aligned}$$

Moreover, define:

$$\alpha\Lambda = \left\{\sum_{i=1}^{n} \alpha_i\lambda_i : \alpha_i \in \alpha, \lambda_i \in \Lambda\right\}$$

Proposition 2.4. *There is a well-defined simply transitive action $[\alpha] * \Lambda = E_{\alpha^{-1}\Lambda}$ of $CL(R_K)$ on* $\mathrm{Ell}(R_K)$. *In particular,*

$$\# CL(R_K) = \# \,\mathrm{Ell}(R_K)$$

Proof. OUCH. □

Lemma 2.5. *Let $E/\mathbb{C} \in \mathrm{Ell}(R_K)$. Then $j(E) \in \overline{\mathbb{Q}}$.*

Proof. First, note that if $\phi : E \to E$ is an endomorphism of E, then $\phi^\sigma : E^\sigma \to E^\sigma$ is an endomorphism of E^σ. Thus,

$$\mathrm{End}(E^\sigma) \cong \mathrm{End}(E)$$

Next, let $\sigma \in \text{Aut}(\mathbb{C})$. Then E^σ is obtained by letting σ act on the coefficients of a Weierstrauss equation for E, and $j(E)$ is a rational combination of those coefficients, so it is clear that:

$$j(E^\sigma) = j(E)^\sigma$$

The previous theorem implies that $\text{End}(E^\sigma) \cong R_K$. Then, (1.2$b$) implies that E^σ is in one of finitely many $\mathbb{C}$-isomorphism classes of elliptic curves. Thus, $j(E^\sigma)$ can take on finitely many values as σ ranges over $\text{Aut}(\mathbb{C})$. Therefore, $[\mathbb{Q}(j(E)) : \mathbb{Q}]$ is finite, and $j(E) \in \overline{\mathbb{Q}}$. (Why?) □

Theorem 2.6.

$$\text{Ell}(R_K) \cong \text{Ell}_{\overline{\mathbb{Q}}}(R_K) := \frac{\{E/\overline{\mathbb{Q}} : \text{End}(E) \cong R_K\}}{\textit{isomorphism over } \overline{\mathbb{Q}}}$$

Proof. Fixing an embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$, there is a natural map:

$$\epsilon : \text{Ell}_{\mathbb{Q}}(R_K) \to \text{Ell}_{\mathbb{C}}(R_K)$$

We need to show that ϵ is a bijection.

To show surjection, note the following:

- $j(E) \in \overline{\mathbb{Q}}$ from (b).
- There is an elliptic curve $E'/\mathbb{Q}(j(E))$ with $j(E') = j(E)$. [AEC III.1.4c]
- E' is isomorphic to E over $\mathbb{C}$ from [AEC III.1.4b].

Thus, $\epsilon(E') = E$.

Next let $E', E \in \text{Ell}_{\mathbb{Q}}(R_K)$ be such that $\epsilon(E') = \epsilon(E)$. Then, $j(E') = j(E)$. Thus, E_1 and E_2 are isomorphic over $\overline{\mathbb{Q}}$ by [AEC III.1.4b]. □

3 CM for Abelian Varieties

Definition 3.1. *A morphism between abelian varieties $f : A \to B$ is a rational map that respects the group structure. If f is birrational, it is also biregular, hence an isomorphism.*

Definition 3.2. *For an abelian variety A/k, $\mathrm{End}(A)$ is a free $\mathbb{Z}$-module of finite rank and every element is defined over a separably algebraic extension of k. Let $\mathrm{End}_{\mathbb{Q}}(A) = \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then, $\mathrm{End}_{\mathbb{Q}}(A)$ is a $\mathbb{Q}$ algebra and $\mathrm{End}(A)$ is an order in $\mathrm{End}_{\mathbb{Q}}(A)$.*

Definition 3.3. *Let R be an algebra over $\mathbb{Q}$ with an identity element 1. An abelian variety of type R is a pair (A, ι) where ι is an isomorphism of R into $\mathrm{End}_{\mathbb{Q}}(A)$ such that $\iota(1) = 1_A$.*

The analytic representation of $\mathrm{End}^0(A)$ is given by:

$$\mathrm{End}^0(A) \cong \{M \in \mathcal{M}_g(\mathbb{C}) : M\mathbb{Q}\Lambda \subset \mathbb{Q}\Lambda\}$$

as it is a g-dimensional complex representation of $\mathrm{End}^0(A)$. Since $\mathbb{R}\Lambda = \mathbb{C}^n$, any $\mathbb{C}$-linear endomorphism that is identity on $\mathbb{Q}\Lambda$ is identity on the whole of $\mathbb{C}^n$. Hence, $\mathbb{Q}\Lambda$ is a faithful $\mathrm{End}^0(A)$-module. Thus,

$$[End^0(A) : \mathbb{Q}]_{red} \leq \dim_{\mathbb{Q}} \mathbb{Q}\Lambda = 2 \dim A$$

Definition 3.4. *An abelian variety $A/\mathbb{C}$ has complex multiplication if:*

$$[End^0(A) : \mathbb{Q}]_{red} = 2 \dim A$$

Theorem 3.5. *(Shimura, 12, Prop. 26) Let $(F; \{\phi\})$ be a CM-type and (A, ι) ab abelian variety of type $(F; \{\phi\})$. Then, there exists an abelian variety of type $(F; \{\phi_i\})$ isomorphic to (A, ι), defined over an algebraic number field of finite degree.*

Proof. (Sketch)

□

Lets fill out the proofs in reverse. Writing notes here.

1. The isomorphism theorem directly requires [AEC III.1.4bc] and $j(E) \in \overline{\mathbb{Q}}$.
2. $j(E) \in \overline{\mathbb{Q}}$ has one ??? and requires [AEC III.1.4b] and (1.2b).
3. (1.2b) requires defining class field, understanding fractional/principle ideals, and the action of $\mathrm{CL}(R_K)$ on $\mathrm{Ell}(R_K)$. This is a somewhat longer proof. Relies on [AEC VI.4.1.1], [AEC VI.5.5].