Math 74: Algebraic Topology

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> **Problem 1.** Let $p : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ be the quotient map from HW1. For $n \geq 2$, show that $p|_{S^n}$ is a degree two cover and deduce that $\pi_1(\mathbb{RP}^n) \simeq \mathbb{Z}/2\mathbb{Z}$. (The fundamental group is $\mathbb{Z}$ for $n = 1$ as $S^1 \cong \mathbb{RP}^1$.)

Solution. Recall the quotient map identifies points in $\mathbb{R}^{n+1} \setminus \{0\}$ as follows:

$$\forall \lambda \neq 0 \in \mathbb{R} : (x_0, \cdots, x_n) \sim \lambda(x_0, \cdots, x_n)$$

Let $(x_0, \cdots, x_n) \in S^n$ be a point on the sphere. Then, we additionally know that:

$$x_0^2 + x_1^2 + \cdots + x_n^2 = 1$$

The equivalence class of this point on the sphere contains all points such that $\lambda(x_0, \cdots, x_n)$ such that:

$$(\lambda x_0)^2 + \cdots + (\lambda x_n)^2 = \lambda^2(x_0^2 + \cdots + x_n^2) = 1$$

Thus, we have $\lambda = \pm 1$. Thus, restricting p to S^n, we get a quotient map that identifies antipodal points on the sphere, i.e. $x \sim -x$.
Thus, each $[p] \in \mathbb{RP}^n$ has exactly two pre-images in S^n, i.e. $p|_{S^n}$ is a two to one mapping.

Moreover, let $[x] \in \mathbb{RP}^n$ be a point. Then, $p|_{S^n}^{-1}([x]) = \{x, -x\}$. Let V and V' be two disjoint open sets in S^n around x and $-x$ (S^n is Hausdorff). Let $W = V \cap -V'$ and $W' = V' \cap -V$. Then, W and W' are still disjoint. Moreover, W and W' are such that they contain antipodal points (by definition).
Let $U = p(W)$. By the definition of p, we note that:

$$\rho|_{S^n}^{-1}(U) = W \bigsqcup W'$$

Since W and W' are open, U is open in $\mathbb{RP}^n$. Moreover, $p|_W(W) = U$ is a continous bijection of compact Hausdroff spaces, and is thus a homeomorphism. Similarly for W'. Thus, as $[x]$ was arbitrary, we have found an open set U around x whose pre-image is a disjoint union of two isomorphic copies of U. Thus, $p|_{S^n}$ is a degree 2 covering map.

Moreover, we note that for $n \geq 2$, S^n is simply connected. Thus, for any point $x \in \mathbb{RP}^n$, the map:

$$\pi_1(\mathbb{RP}^n, x) \to p|_{S^n}^{-1}(x)$$

is a bijection. Thus, $|\pi_1(\mathbb{RP}^n, x)| = 2$. Thus,

$$\pi_1(\mathbb{RP}^n, x) = \mathbb{Z}/2\mathbb{Z}$$

Problem 2. Let $g\colon S^1 \to S^1$ be the covering map $g(z) = z^n$ for $n \in \mathbb{Z}$. Under the isomorphism $\pi_1(S^1, (1,0)) \cong \mathbb{Z}$ that we proved, compute $g_*\colon \mathbb{Z} \to \mathbb{Z}$ and the map $\phi_g : \mathbb{Z} \to g^{-1}(1,0)$ defined by lifting loops so that they still start at $(1,0)$.

Solution.

Problem 3. Show that there are no retractions $r\colon X \to A$ in the following cases:

1. $X = \mathbb{R}^3$ and A is any subspace homeomorphic to S^1.
2. $X = S^1 \times D^2$ and A is its boundary torus $S^1 \times S^1$.
3. X is the Möbius band and A is its boundary circle.

Solution.

Problem 4. Use the intermediate value theorem to prove the 1-dimensional version of the Brouwer fixed point theorem: If $f\colon I \to I$ is continuous, there is a point $x \in I$ such that $f(x) = x$.

Solution.

Problem 5. Use the intermediate value theorem to prove the 1-dimensional version of the Borsuk-Ulam theorem: If $f : S^1 \to \mathbb{R}$ is continuous, there is a point $x \in S^1$ such that $f(x) = f(-x)$.

Solution.