Math 74: Algebraic Topology

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Problem 1. (1.3.18) For a path-connected, locally path-connected, and semilocally simply connected space X, call a path-connected covering $p\colon E \to X$ *abelian* if it is normal and has abelian deck transformation group. Show that X has an abelian covering space that is a covering space of every other abelian covering space of X and that such a 'universal' abelian covering space is unique up to equivalence. Describe this covering space explicitly for $X = S^1 \vee S^1$ and $S^1 \vee S^1 \vee S^1$.

Solution. Since X is path-connected, locally path-connected, and semilocally simply connected, we note that it it has a universal cover $\tilde{\rho} : \tilde{B} \to X$. Let $H \subsetneq G := \pi_1(X, x_0)$ be the commutator (i.e. generated by elements $[g, h]$ for $g, h \in G$). By the existence of covers theorem, there exists a covering space $\rho : (E, e_0) \to (X, x_0)$ such that $\rho_*(\pi_1(E, e_0)) = H$. We claim that (E, ρ) is the unique universal abelian covering space of X.

Note that since H is the commutator subgroup, it is normal. To see this, let $[a, b] \in H$ be a generator, and $g \in G$. Then,

$$\begin{aligned}
g[a,b]g^{-1} &= ga^{-1}b^{-1}abg^{-1} \\
&= ga^{-1}(g^{-1}g)b^{-1}(g^{-1}g)a(g^{-1}g)bg^{-1} \\
&= (ga^{-1}g^{-1})(gb^{-1}g^{-1})(gag^{-1})(gbg^{-1}) \\
&= (gag^{-1})^{-1}(gbg^{-1})^{-1}(gag^{-1})(gbg^{-1}) \\
&= [gag^{-1}, gbg^{-1}] \in H
\end{aligned}$$

Thus, H is normal in G and (E, ρ) is a normal covering space.

Moreover, by the normal covering theorem, we know that the deck transformation group is equal to G/H. However, since H is the commutator, we have that G/H is abelian (by definition of the commutator). Thus, (E, ρ) is an abelian covering space.

Moreover, if (E', ρ') was another normal cover corresponding to $H' \subsetneq G$ with abelian deck transformation group G/H', then since G/H' is abelian, we must have that $H \subseteq H'$ (the commutator must be quotiented out for the result to be abelian). Thus, we have that:

$$\rho_*(\pi_1(E, e_0)) \subseteq \rho'_*(\pi_1(E', e'_0))$$

Since E is path-connected and locally path-connected (as it is a cover of locally path-connected X), we can apply the general lifting theorem to get a map $f : (E, e_0) \to (E', e'_0)$ such that:

$$\rho' \circ f = \rho$$

By functoriality, we have that:

$$\rho'_* \circ f_* = \rho_*$$

Since ρ_* is injective, we have that f_* is injective. Thus,

$$f_*(\pi_1(E, e_0)) \subseteq \pi_1(E', e'_0)$$

is a subgroup. Thus, by the Galois correspondence, we have that $f : (E, e_0) \to (E', e'_0)$ is a covering map.

Uniqueness follows directly from the universal property. If A, B are two universal abelian covers, then by the universal property, there exists unique covering maps $f : A \to B$ and $g : B \to A$ that commute with the covering maps of A and B. However, then $g \circ f$ is a covering map from A to itself. By the uniqueness of lifts, we must have $g \circ f = \mathrm{id}_A$. Similarly, we have that $f \circ g = \mathrm{id}_B$. Thus, f and g are homeomorphisms, and the universal abelian cover is unique up to equivalence.

For $X = S^1 \vee S^1$, we have that $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$. The commutator subgroup H is generated by the element $[a, b]$. Thus, we want a cover of $S^1 \vee S^1$ corresponding to H. Let $E = \mathbb{R}^2$, where we each integer interval on the x-axis corresponds to a and on the y-axis corresponds to b.

Problem 2. (1.3.20) Construct non-normal covering spaces of a Klein bottle by a Klein bottle and by a torus.

Solution.

Problem 3. (1.3.29) Let Y be path-connected, locally path-connected, and simply connected. Let G_1 and G_2 be two subgroups of $\mathrm{Homeo}(Y)$ defining covering space actions on Y. Show that the orbit spaces Y/G_1 and Y/G_2 are homeomorphic if and only if G_1 and G_2 are conjugate subgroups of $\mathrm{Homeo}(Y)$.

Solution.

Problem 4. (2.1.10) Show that if A is a retract of X, then the map $H_n(A) \to H_n(X)$ induced by the inclusion of A in X is injective for all n.

Solution.

Problem 5. (2.1.11) Show that chain homotopy is an equivalence relation on the set of chain maps between two chain complexes.

Solution.