CS 40: Computational Complexity

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Problem 6. Prove that

$$\text{STRONGCON} := \{\langle G\rangle : G \text{ is a strongly connected directed graph }\}$$

is NL-complete.

Proof. To show that STRONGCON is NL-complete, we need to show that STRONGCON $\in$ NL and STRONGCON is NL-hard. We do each of these seperately.

First, we show that STRONGCON is in NL. It suffices to show that STRONGCON $\in$ coNL as NL = coNL. Thus, we consider the complement:

$$\overline{\text{STRONGCON}} = \{\langle G\rangle : \exists u, v : \text{ there is no path } u \to v\}$$

We need to show that this is in NL. We do this in the following way:

1. Guess u and v non-deterministically. Concretely we can do this by counting up from 1 to $|V|$ (we can figure out $|V|$ by parsing the input and maintaining a counter) in a binary counter and choosing whether to stop at each particular point. Thus, this can be done in $O(\log(|V|))$ space.

2. Simulate the NDTM associated to $\overline{\text{STCONN}}$ on $\langle G, u, v\rangle$. We have G on our input tape and u, v on the work-tape, so this requires only a slight modification and can be done in log-space. Note that since NL = coNL, $\overline{\text{STCONN}} \in$ NL thus this NDTM exists.

3. If the simulation ACCEPTS, then there is no path from u to v, thus, G is not strongly connected, and we ACCEPT. Otherwise, we REJECT. This takes no space.

Thus, we conclude that $\overline{\text{STRONGCON}} \in$ NL. Thus, STRONGCON $\in$ coNL = NL.

Finally, we need to show STRONGCON is NL-hard. Since STCONN is NL-hard, it suffices to show a log-space reduction from STCONN to STRONGCON.

Let $\langle G, s, t\rangle$ be some encoding of a graph and two vertices. We first create a graph G' with the property that:

$$\text{there is a path } s \to t \text{ in } G \iff G' \text{ is strongly connected}$$

Let G' have the same vertex set as G, but with the following edges added:

- Edge (u, s) for all $u \in V(G)$ if they do not exist
- Edge (t, v) for all $v \in V(G)$ if they do not exist

We show that G' satisfies the given property. First assume there is a path from $s \to t$ in G. Let $u, v \in V(G')$ be any vertices. We know there exists an edge (u, s), path $s \to t$, and edge (t, v). Thus, there is a path from $u \to v$. Since u, v were arbitrary, G' is strongly connected. Next, assume that G' is strongly connected. Then, there is at least one path from $s \to t$ in G'. Let p be (one of) the shortest path(s) from s to t. We claim that p only uses edges already present in G. Indeed, if any edge (u, s) appeared in p, the path p can be shortened by taking the section after this edge. By the minimality of p, no such edge exists in p. Similarly, we see that no edge (t, v) can exist in p. In particular, p avoids all the edges we appended to G to create G'. Thus, p only uses edges in G, hence there is a path from $s \to t$ in G.

Note that in terms of our languages, we can write this property as:

$$\langle G, s, t\rangle \in \text{STCONN} \iff \langle G'\rangle \in \text{STRONGCON}$$

We only need to show that we can construct G' from G in log-space. We only need to stream the list of vertices and the list of edges for G'. Streaming the list of vertices and the edges already contained in G requires no additional space, as we just read the input and regurgitate it. Thus, we only need to worry about outputting (u, s) and (t, u) for all $u \in V(G)$. But this can be done in logspace as it only requires one to maintain a counter to track which vertex we are on as we loop over the vertex set, which is of size $O(\log(|V|))$, which is at most logarithmic in the description of the entire graph (s and t can be read from the input tape each time).

Thus, we have shown a logspace reduction of STCONN to STRONGCON. Hence, STRONGCONN is NL-hard, therefore NL-complete.

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