Problem 1. Let $U \subset \mathbb{C}^n$ be a domain (i.e., a non-empty connected subset) and let $f : U \to \mathbb{C}$ be a holomorphic function.

1. Prove that f satisfies analytic continuation: if it vanishes on an open subset of U, then it vanishes everywhere.

2. Prove that f satisfies the maximum principle: if $|f|$ admits a local maximum in U, then f is constant.

3. Let M be a compact complex manifold. Prove that any holomorphic function on M is constant.

Problem 2.

1. Show that the assignment $L \mapsto (L_{\mathbb{R}}, \mathrm{mult}(i))$ gives an equivalence between the category of complex vector spaces and the category of pairs (V, J), where V is a real vector space, $J : V \to V$ is an $\mathbb{R}$-linear operator satisfying $J^2 = -\mathrm{Id}_{L_{\mathbb{R}}}$, and a morphism $(V, J) \to (W, K)$ is defined as an $\mathbb{R}$-linear map $f : V \to W$ that intertwines J and K, i.e., such that $K \circ f = f \circ J$.

2. Let (V, J) be a pair as above:

   (a) Let $\iota : v \mapsto v$ be the $\mathbb{R}$-linear automorphism of $V \otimes_{\mathbb{R}} \mathbb{C}$ induced from conjugation on the second factor. Show that V, seen inside $V \otimes_{\mathbb{R}} \mathbb{C}$ via $v \mapsto v \otimes 1$, is isomorphic to the fixed locus of ι.

   (b) By diagonalizing J over $V \otimes_{\mathbb{R}} \mathbb{C}$, show that we have a decomposition $V_{\mathbb{C}} = W \oplus \bar{W}$ where J acts on W by multiplication by i and $\bar{W} = \iota(W)$.

   (c) Show that the projection map $V \to W$ is an $\mathbb{R}$-linear isomorphism that intertwines J and multiplication by i on W.

   (d) Each element $v \in V_{\mathbb{C}}$ can be written as $v = v_1 + v_2$ along the above decomposition. Show that $v \in V$ if and only if $v_2 = \bar{v}_1$.

Problem 3. Let E, F be real vector spaces and let G be a complex vector space, all finite-dimensional.

1. Show that $E \otimes_\mathbb{R} G$ admits a natural structure of a complex vector space. Construct a basis in terms of bases of E and G. Show that $E \otimes_\mathbb{R} G = (E \otimes_\mathbb{R} \mathbb{C}) \otimes_\mathbb{C} G$.

2. Show that $\mathrm{Hom}_\mathbb{R}(E, G) = \mathrm{Hom}_\mathbb{C}(E \otimes \mathbb{C}, G)$.

3. Show that

$$(E \oplus F) \otimes_\mathbb{R} \mathbb{C} = (E \otimes_\mathbb{R} \mathbb{C}) \oplus (F \otimes_\mathbb{R} \mathbb{C}), \quad (E \otimes_\mathbb{R} F) \otimes_\mathbb{R} \mathbb{C} = (E \otimes_\mathbb{R} \mathbb{C}) \otimes_\mathbb{C} (F \otimes_\mathbb{R} \mathbb{C})$$

and

$$\Lambda^n E \otimes_\mathbb{R} \mathbb{C} = \Lambda^n (E \otimes_\mathbb{R} \mathbb{C}).$$

4. Let $f : E \to F$ be a linear map and let $f_\mathbb{C} = f \otimes_\mathbb{R} \mathbb{C} : E \otimes_\mathbb{R} \mathbb{C} \to F \otimes_\mathbb{R} \mathbb{C}$ be the induced map. Show that

$$\ker(f_\mathbb{C}) = \ker(f) \otimes_\mathbb{R} \mathbb{C}, \quad \mathrm{Im}(f_\mathbb{C}) = \mathrm{Im}(f) \otimes_\mathbb{R} \mathbb{C}.$$

Solution.

1. Let $e \in E$ and $g \in G$. Then, we define the scalar multiplication by $z \in \mathbb{C}$ by using the complex structure of G:

$$z \cdot (e \otimes g) = e \otimes zg$$

and extend linearly. For $z \in \mathbb{R}$, this definition respects the tensor product over $\mathbb{R}$ as:

$$z \cdot (e \otimes g) = e \otimes zg = ze \otimes g$$

If $\{e_i\}_i$ is a basis of E and $\{g_j\}_j$ is a basis of G, then the set $\{e_i \otimes_\mathbb{R} g_j\}_{i,j}$ is a $\mathbb{R}$-linear basis of $E \otimes_\mathbb{R} G$ (before we introduce the complex structure). With respect to the complex structure, note that this set is $\mathbb{C}$-linearly independent as if:

$$\sum_{i,j} z_{ij} (e_i \otimes g_j) = 0$$

then,

$$\sum_{i,j} e_i \otimes z_{ij} g_j = 0$$

which contradicts the $\mathbb{R}$-linear independence of $\{e_i \otimes_\mathbb{R} g_j\}_{i,j}$. Moreover, the set clearly $\mathbb{C}$-spans $E \otimes_\mathbb{R} G$ as it $\mathbb{R}$-spans it. Thus, the set is a $\mathbb{C}$-linear basis of $E \otimes_\mathbb{R} G$.

Next, we define the homomorphisms $\Phi : E \otimes_{\mathbb{R}} G \to (E \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} G$ and $\Psi : (E \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} G \to E \otimes_{\mathbb{R}} G$ as follows:

$$\Phi(e \otimes g) = (e \otimes_{\mathbb{R}} 1) \otimes_{\mathbb{C}} g$$
$$\Psi((e \otimes_{\mathbb{R}} z) \otimes_{\mathbb{C}} g) = e \otimes_{\mathbb{R}} zg$$

and extending linearly. We show that these are inverses as follows:

$$\Psi \circ \Phi((e \otimes_{\mathbb{R}} g)) = \Psi((e \otimes_{\mathbb{R}} 1) \otimes_{\mathbb{C}} g)$$
$$= e \otimes_{\mathbb{R}} g$$
$$\Phi \circ \Psi((e \otimes_{\mathbb{R}} z) \otimes_{\mathbb{C}} g) = \Phi(e \otimes_{\mathbb{R}} zg)$$
$$= (e \otimes_{\mathbb{R}} 1) \otimes_{\mathbb{C}} zg$$
$$= z \cdot (e \otimes_{\mathbb{R}} 1) \otimes_{\mathbb{C}} g$$
$$= (e \otimes_{\mathbb{R}} z) \otimes_{\mathbb{C}} g$$

where we use the same natural complex structure on $E \otimes_{\mathbb{R}} \mathbb{C}$. Thus, Φ and Ψ are inverses. Thus,

$$E \otimes_{\mathbb{R}} G \cong (E \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} G$$

2. We show this by constructing the linear maps $\Phi : \mathrm{Hom}_{\mathbb{R}}(E, G) \to \mathrm{Hom}_{\mathbb{C}}(E \otimes_{\mathbb{R}} \mathbb{C}, G)$ and $\Psi : \mathrm{Hom}_{\mathbb{C}}(E \otimes_{\mathbb{R}} \mathbb{C}, G) \to \mathrm{Hom}_{\mathbb{R}}(E, G)$ as follows, for $f \in \mathrm{Hom}_{\mathbb{R}}(E, G)$ and $g \in \mathrm{Hom}_{\mathbb{C}}(E \otimes_{\mathbb{R}} \mathbb{C}, G)$:

$$\Phi(f)(e \otimes_{\mathbb{R}} z) = zf(e) \in G$$
$$\Psi(g)(e) = g(e \otimes_{\mathbb{R}} 1) \in G$$

and extending linearly. We show that $\Phi(g)$ is $\mathbb{C}$-linear and $\Psi(g)$ is $\mathbb{R}$-linear.

$$\Phi(f)(z \cdot e_1 \otimes_{\mathbb{R}} z_1 + e_2 \otimes_{\mathbb{R}} z_2) = \Phi(f)(e_1 \otimes_{\mathbb{R}} zz_1 + e_2 \otimes_{\mathbb{R}} z_2)$$
$$= zz_1 f(e_1) + z_2 f(e_2)$$
$$= z \cdot \Phi(f)(e_1 \otimes_{\mathbb{R}} z_1) + \Phi(f)(e_2 \otimes_{\mathbb{R}} z_2)$$

Thus, $\Phi(f)$ is $\mathbb{C}$-linear.

$$\Psi(g)(ze_1 + e_2) = g((ze_1 + e_2) \otimes_{\mathbb{R}} 1)$$
$$= g((e_1 \otimes_{\mathbb{R}} z) + (e_2 \otimes_{\mathbb{R}} 1))$$
$$= \Phi(g)(ze_1) + \Phi(g)(e_2)$$

Thus, $\Psi(g)$ is $\mathbb{R}$-linear. Finally, we wanna show that Φ and Ψ are inverses, for $e \in E$ and $z \in \mathbb{C}$.

$$\Psi(\Phi(f))(e) = \Phi(f)(e \otimes_{\mathbb{R}} 1)$$
$$= f(e)$$
$$\Phi(\Psi(g))(e \otimes_{\mathbb{R}} z) = z\Psi(g)(e)$$
$$= zg(e \otimes_{\mathbb{R}} 1)$$
$$= g(e \otimes_{\mathbb{R}} z)$$

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Thus, Φ and Ψ are inverses. Thus, we have the isomorphism:

$$\mathrm{Hom}_{\mathbb{R}}(E, G) \cong \mathrm{Hom}_{\mathbb{C}}(E \otimes_{\mathbb{R}} \mathbb{C}, G)$$

3.

4.

Problem 4. Let $U \subset \mathbb{C}$ be an open subset and let $D \subset \Omega$ be a closed disk.

1. Let $f : U \to \mathbb{C}$ be a $\mathcal{C}^1$ function. Show that for all $z \in D$, we have:
    $$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\xi)}{\xi - z} d\xi + \frac{1}{2\pi i} \int_D \frac{\partial f}{\partial \bar{z}}(\xi) \frac{d\xi \wedge d\bar{\xi}}{\xi - z}.$$

    Hint: Apply Stokes' theorem to $\frac{f(\xi)}{\xi - z} d\xi$ on $D \setminus B(z, \varepsilon)$ and let $\varepsilon \to 0$.

2. Let g be a $\mathcal{C}^1$ function on $\mathbb{C}$ with compact support and define
    $$f(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g(\xi)}{\xi - z} d\xi \wedge d\bar{\xi}.$$

    Show that f is $\mathcal{C}^1$ and $\frac{\partial f}{\partial \bar{z}} = g$. *Hint:* Differentiate under the integral sign after substituting $\xi' = \xi - z$.

3. Show that for any function g on U which is $\mathcal{C}^1$, there exists a function f on U, also $\mathcal{C}^1$, such that $\frac{\partial f}{\partial \bar{z}} = g$ on D.

4. In the previous question, show that if g is $\mathcal{C}^\infty$, then f can also be chosen to be $\mathcal{C}^\infty$.

Problem 5. Let E and F be two holomorphic vector bundles on a complex manifold X. Given an open cover $\{U_\alpha\}$ of X that trivializes E, the vector bundle E is described on overlaps $U_\alpha \cap U_\beta$ by holomorphic transition functions:

$$\rho_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{C}).$$

1. Prove the cocycle condition: $\rho_{\alpha\gamma} = \rho_{\beta\gamma} \circ \rho_{\alpha\beta}$.

2. Let E' be the quotient of $\bigsqcup_\alpha U_\alpha \times \mathbb{C}^n$ by the equivalence relation on $U_\alpha \cap U_\beta \times \mathbb{C}^n$ given by
$$(x, v) \sim (x, \rho_{\alpha\beta}(x)(v)).$$

   Prove that E' is a holomorphic vector bundle and that it is isomorphic to E as vector bundles over X, i.e., there exists a biholomorphism $f : E \to E'$ commuting with projection to X.

3. Conversely, assume that E and F are isomorphic as holomorphic vector bundles. How are their transition functions related?

4. Using the transition maps of E and F, construct the following vector bundles by writing down explicitly their transition functions: $E \otimes_\mathbb{C} F$, $E \oplus F$, $\Lambda^n E$.

Solution.

1. Let $\{U_\alpha\}$ be an open cover of X that trivializes E, with respective maps $\tau_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^n$. Then, note that $\rho_{\alpha\beta} : \tau_\alpha(\pi^{-1}(U_\alpha \cap U_\beta)) \to U_\beta \times \mathbb{C}^n$ is given by:
$$\rho_{\alpha\beta} = \tau_\beta \circ \tau_\alpha^{-1}$$
with others defined similarly. Let $x \in U_\alpha \cap U_\beta \cap U_\gamma$ and $E_x = \pi^{-1}(a)$ be the fiber over x in E. Then, we calculate:
$$\begin{aligned}
\rho_{\beta\gamma} \circ \rho_{\alpha\beta}(E_x) &= \tau_\gamma \circ \tau_\beta^{-1} \circ \tau_\beta \circ \tau_\alpha^{-1}(E_x) \\
&= \tau_\gamma \circ \tau_\alpha^{-1}(E_x) \\
&= \rho_{\alpha\gamma}(E_x)
\end{aligned}$$

   Thus, the cocycle condition holds on every such fiber. Thus, it holds in general. We can also realize this by picking a basis for the image of E_x in each trivialization, and noticing that ρ are just change of basis matrices, which satisfy the cocycle condition.

2. We first check that E' is a holomorphic vector bundle. We write an element of E' as $(x, [v])$, for $x \in U_\alpha$, where $[v]$ is the equivalence class of $\{\rho_{\alpha\beta}(v) : x \in U_\beta\}$.

   Defining $\pi'(x, [v]) = x$, we note that the projection π' is clearly well-defined and can be made holomorphic by inheriting the holomorphic structure from the base space.

Furthermore, we define the trivialization maps $\tau'_\beta : \pi'^{-1}(U_\beta) \to U_\beta \times \mathbb{C}^n$ as follows:

$$\tau'_\beta(x, [v]) = (x, v)$$

where v is the representative of $[v]$ such that $(x, v) \in U_\beta \times \mathbb{C}^n$. Then, τ'^{-1}_β is given by taking the equivalence class:

$$\tau'^{-1}_\beta(x, v) = (x, [v])$$

Thus, the transition maps $\rho'_{\alpha\beta}$ are given by, for $x \in U_\alpha \cap U_\beta$:

$$
\begin{aligned}
\rho'_{\alpha\beta}(x, v) &= \tau'_\beta \circ \tau'^{-1}_\alpha(x, v) \\
&= (x, [v])) \\
&= (x, [\rho_{\alpha\beta}(v)]) \\
&= (x, \rho_{\alpha\beta}(v))
\end{aligned}
$$

Thus, the transition functions $\rho'_{\alpha\beta}$ are holomorphic as $\rho_{\alpha\beta}$ are holomorphic. Thus, E' is a holomorphic vector bundle.

Next, to show that E' is isomorphic to E as a vector bundle, we need to find a biholomorphism $f : E \to E'$ that respects the fibers. We can define f as follows: for $x \in U_\alpha$, and $\epsilon \in \pi^{-1}(x) = E_x$,

$$f(\epsilon) = \tau'^{-1}_\alpha \circ \tau_\alpha(\epsilon)$$

To see that f is well-defined, let $x \in U_\alpha \cap U_\beta$. Then, we have:

$$
\begin{aligned}
f(\epsilon) &= \tau'^{-1}_\alpha \circ \tau_\alpha(\epsilon) \\
&= \tau'^{-1}_\alpha(x, v) \\
&= (x, [v]) \\
&= (x, [\rho_{\alpha\beta}(v)]) \\
&= \tau'^{-1}_\beta(x, \rho_{\alpha\beta}(v)) \\
&= \tau'^{-1}_\beta \circ \tau_\beta(\epsilon) \\
&= f(\epsilon)
\end{aligned}
$$

Since it is a composition of holomorphic maps, f is holomorphic. Next, we define the inverse $f^{-1} : E' \to E$ as follows: for $x \in U_\alpha$ and $(x, [v]) \in E'$, with v being the representative that came from $U_\alpha \times \mathbb{C}^n$, we have:

$$f^{-1}(x, [v]) = \tau^{-1}_\alpha \circ \tau'_\alpha(x, [v])$$

To see that this is well-defined, let $x \in U_\alpha \cap U_\beta$. Then, we have:

$$\begin{aligned}
f^{-1}(x, [v]) &= \tau_\alpha^{-1} \circ \tau_\alpha'(x, [v]) \\
&= \tau_\alpha^{-1}(x, v) \\
&= \tau_\beta^{-1}(x, \rho_{\alpha\beta}(v)) \\
&= \tau_\beta^{-1} \circ \tau_\beta'(x, [\rho_{\alpha\beta}(v)]) \\
&= \tau_\beta^{-1} \circ \tau_\beta'(x, [v]) \\
&= f^{-1}(x, [v])
\end{aligned}$$

Since f^{-1} is also a composition of holomorphic maps, f^{-1} is holomorphic. Moreover, it is easy to see that f^{-1} and f are inverses. Thus, f is a biholomorphism.

Finally, we can see that f and f^{-1} respect the fibers as they are compositons of fiber-preserving maps. Thus, they commute with the projection maps. Moreover, we note that on a particular fiber, E_x, with respect to the trivializations, f is a vector space isomorphism. That is, $\tau_\alpha' \circ f \circ \tau_\alpha^{-1}$ is an isomorphism on $\{x\} \times \mathbb{C}^n$ as it is linear and has an inverse.

3. Since E and F are isomorphic as holomorphic vector bundles, there exists a biholomorphism $f : E \to F$ that respects the fibers, and is a vector space isomorphism when restricted to each fiber.

By refining the covers over which the trivializations for E and F are defined, choose a cover $\{U_\alpha\}$ such that both E and F are trivialized over it. Let $x \in U_\alpha \cap U_\beta$. Let $\tau_{\alpha,E}$ and $\tau_{\alpha,F}$ be trivializations and ρ_E and ρ_F be the transition functions from $\tau_\alpha(\pi^{-1}(U_\alpha)) \to U_\beta \times \mathbb{C}^n$ (sloppily identifying the two projections π). Let $E_{x,\alpha}$, $E_{x,\beta}$ be the trivilized fibers over x and similarly for f. Then, f is such that:

$$\begin{array}{cc}
\tau_{\alpha,F} \circ f \circ \tau_{\alpha,E}^{-1} & \tau_{\beta,F} \circ f \circ \tau_{\alpha,E}^{-1} \\
\tau_{\alpha,F} \circ f \circ \tau_{\beta,E}^{-1} & \tau_{\beta,F} \circ f \circ \tau_{\beta,E}^{-1}
\end{array}$$

are all isomorphisms. Composition by ρ_E and ρ_F and their inverses permutes them. Write $f_\alpha := \tau_{\alpha,F} \circ f \circ \tau_{\alpha,E}^{-1}$. Thus, we have the commutative diagram connecting the two transition functions (where every arrow is invertible):

$$\begin{array}{ccc}
E_{x,\alpha} & \xrightarrow{\rho_E} & E_{x,\beta} \\
\downarrow{f_\alpha} & & \downarrow{f_\beta} \\
F_{x,\alpha} & \xrightarrow[\rho_F]{} & F_{x,\beta}
\end{array}$$

4. We can pick a cover $\{U_\alpha\}$ that trivializes both E and F. Then, pick $x \in U_\alpha \cap U_\beta$ and ρ_E and ρ_F be the transiton functions from the U_α trivialization to the U_β trivialization. Then,

(a) $\rho_E \otimes_{\mathbb{C}} \rho_F$ is a transition function for $E \otimes_{\mathbb{C}} F$.

(b) $\rho_E \oplus \rho_F$ be a transition function for $E \oplus F$.

(c) $\bigwedge^n \rho_E$ be a transition function for $\bigwedge^n E$.

These are all holomorphic, as they are holomorphic on each component.

Questions:

1. In 5b), is it okay to just take a representative that comes from a particular $U_\alpha \times \mathbb{C}^n$ pre-quotienting? What's a better way to write this?