Math 74: Algebraic Topology

Sair Shaikh

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Problem 1.(2.3.1)
If $T_n(X, A)$ denotes the torsion subgroup of $H_n(X, A)$, show that the functors $(X, A) \mapsto T_n(X, A)$ with the obvious induced homomorphisms $T_n(X, A) \to T_n(Y, B)$ and boundary maps $T_n(X, A) \to T_{n-1}(A)$ do not satisfy a homology theory even if excluding the dimension axiom. Do the same for the 'mod-torsion' functor $MT_n(X, A) = H_n(X, A)/T_n(X, A)$.

Solution. Let $X = \mathbb{RP}^2$ and A be a circle in X. The long exact sequence in homology gives us:

$$\cdots \to H_2(X, A) \to H_1(A) \to H_1(X) \to H_1(X, A) \to H_0(A) \to H_0(X) \to \cdots$$

Then, note that we have $H_1(X) = \mathbb{Z}/2\mathbb{Z}$, $H_1(A) = \mathbb{Z}$ and $H_0(X) = H_0(A) = \mathbb{Z}$.

$$\cdots \to H_2(X, A) \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to H_1(X, A) \to \mathbb{Z} \to \mathbb{Z} \to \cdots$$

The generator of $H_1(A)$ maps to a boundary in $H_1(X)$, thus, the first map is 0. Thus, the second map is injective. Moroever, the last map is induced by the inclusion of A into X, both of which are path-connected, thus the last $H_0(A) \to H_0(X)$ is an isomorphism. Thus, the image of $H_1(X, A) \to H_0(A)$ is trivial, i.e. the map is 0. Thus, $H_1(X, A) = 0$. Overall, we have:

$$\to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \to \cdots$$

Applying the torsion functors, we get:

$$T_1(A) = 0 \to T_1(X) = \mathbb{Z}/2\mathbb{Z} \to T_1(X, A) = 0 \to \cdots$$

which is not exact. Thus, the torsion functor does not satisfy the exactness axiom.

Problem 2.(2.3.5, with $G = \mathbb{Z}$) Regarding a cochain $\varphi \in C^1(X)$ as a function on paths in X to $\mathbb{Z}$, show that if φ is a cocycle, then

1. $\varphi(f \cdot g) = \varphi(f) + \varphi(g)$,
2. φ takes the value 0 on constant paths,
3. $\varphi(f) = \varphi(g)$ if $f \simeq_p g$, and
4. φ is a coboundary if and only if $\varphi(f)$ depends only on the endpoints of f for all paths f in X.

Solution.

1. Recall that a cochain $\varphi \in C^1(X)$ is a cocycle if $\delta\varphi = 0$. However, we have, by definition, for $\sigma : \Delta^2 \to X$ that:
$$\delta(\varphi)(\sigma) = \varphi(\delta\sigma) = 0$$
Thus, φ is 0 on all boundaries.

   We construct $\sigma : \Delta^2 \to X$ with sides f, g and $f \cdot g$ as follows:

   INSERT PICTURE HERE.

   Then, we have:
$$\varphi(\delta\sigma) = \varphi(g) - \varphi(f \cdot g) + \varphi(f) = 0$$
Thus, we have $\varphi(f \cdot g) = \varphi(f) + \varphi(g)$.

2. Let id_e be the constant path at point $e \in X$. The constant path is the boundary of the 2-simplex $\sigma : \Delta^2 \to X$ with all vertices mapped to e (thus all edges constant paths id_e). Then, we have:
$$\varphi(\delta\sigma) = \varphi(\mathrm{id}_e) - \varphi(\mathrm{id}_e) + \varphi(\mathrm{id}_e) = 0$$
Thus, $\varphi(\mathrm{id}_e) = 0$.

3. If $f \simeq_p g$ are paths from x_0 to x_1, then we can construct two 2-simplices as in the diagram:

   INSERT PICTURE HERE.

Then, we have:

$$\varphi(\delta(\sigma_1 - \sigma_2)) = \varphi(-\,\mathrm{id}_{x_1}) + \varphi(f) - \varphi(\psi) - \varphi(g) - \varphi(\mathrm{id}_{x_0}) + \varphi(\psi) = 0$$

Then, using the previous part, we have:

$$\varphi(f) = \varphi(g)$$

Note that these three parts together imply that $\varphi(f^{-1}) = -\varphi(f)$, since $f \cdot f^{-1} \simeq_p \mathrm{id}_{x_0}$, and then $\varphi(f^{-1}) + \varphi(f) = \varphi(\mathrm{id}_{x_0}) = 0$, where x_0 is the start point of f.

4. If φ is a coboundary there exists a 0-cochain $\psi \in C^0(X)$ such that $\varphi = \delta\psi$. Thus, for $f : \Delta^1 \to X$ with endpoints x_0 and x_1, we have:

$$\begin{aligned}
\varphi(f) &= \delta\psi(f) \\
&= \psi(\delta f) \\
&= \psi(x_1) - \psi(x_0)
\end{aligned}$$

Thus, $\varphi(f)$ depends only on the endpoints of f.

Conversely, assume $\varphi(f)$ depends only on the endpoints of f. Let X' be a path-connected component. Pick a basepoint $x \in X'$. Then, for any $x' \in X'$, we can construct a path $f_{x'} : \Delta^1 \to X'$ from x to x'. Then, we define $\psi : X' \to \mathbb{Z}$ by:

$$\psi(x') := \varphi(f_{x'})$$

since $\varphi(f)$ depends only on the endpoints, this is well-defined. Similarly, we do this for all path-connected components of X. Then, if f is a path from x_0 to x_1 in path-connected component with basepoint x, we construct f_1 and f_2, paths from x to x_1 and x_2, respectively. Then, we have:

$$\begin{aligned}
\varphi(f) &= \varphi(f_1^{-1} \cdot f_2) \\
&= -\varphi(f_1) + \varphi(f_2) \\
&= -\psi(x_1) + \psi(x_2) \\
&= \psi(\delta f) \\
&= \delta\psi(f)
\end{aligned}$$

Thus, φ is a coboundary.

Problem 3. Verify the remark in Hatcher after exercise 2.3.5: If X is path-connected, the previous problem together with the universal coefficient theorem induces an isomorphism $H^1(X) \cong \text{Hom}(\pi_1(X), \mathbb{Z})$.

Solution. We calculate $Ext^1(H_0(X), \mathbb{Z})$. This represents the isomorphism classes of extensions:

$$0 \to \mathbb{Z} \to A \to H_0(X) \to 0$$

Since X is path-connected, we have $H_0(X) \cong \mathbb{Z}$. Thus, there are no extensions and $Ext^1(H_0(X), \mathbb{Z}) = 0$. Then, note the universal coefficient theorem gives us the exact sequence:

$$0 \to H^1(X) \to \text{Hom}(H_1(X), \mathbb{Z}) \to 0$$

Thus, we have $H^1(X) \cong \text{Hom}(H_1(X), \mathbb{Z})$.

Define $\Phi : Z^1(X) \to \text{Hom}(\pi_1(X), \mathbb{Z})$ as follows:

$$\Phi(\varphi)([\gamma]) = \varphi(\gamma)$$

where $\varphi \in Z^1(X)$ is a cocycle and $[\gamma] \in \pi_1(X)$. This is well-defined by part (3), and a homomorphism by part (1) and (2) of the previous problem. We claim that Φ is surjective.

Let $\rho : \pi_1(X) \to \mathbb{Z}$ be a homomorphism. We define a cocycle $\varphi \in Z^1(X)$ for path f from x_1 to x_2 as follows. Let $\alpha_{x_1}, \alpha_{v_2}$ be paths from x_0 (the basepoint of $\pi_1(X)$) to x_1 and x_2, respectively. Then, we define:

$$\varphi(f) = \rho([\alpha_{x_1} \cdot f \cdot \alpha_{x_2}^{-1}])$$

One can verify this is a cocycle. Let $\sigma : \Delta^2 \to X$ be a 2-simplex with sides $[v_1, v_2]$, $[v_2, v_3]$ and $[v_3, v_1]$. Then,

$$\begin{aligned}
\varphi(\delta\sigma) &= \rho([\alpha_{v_2} \cdot [v_2, v_3] \cdot \alpha_{v_3}^{-1}]) - \rho([\alpha_{v_1} \cdot [v_1, v_3] \cdot \alpha_{v_3}^{-1}]) + \rho([\alpha_{v_1} \cdot [v_1, v_2] \cdot \alpha_{v_2}^{-1}]) \\
&= \rho([\alpha_{v_2} \cdot [v_2, v_3] \cdot \alpha_{v_3}^{-1} \cdot \alpha_{v_3} \cdot [v_3, v_1] \cdot \alpha_{v_1}^{-1} \cdot \alpha_{v_1} \cdot [v_1, v_2] \cdot \alpha_{v_2}^{-1}]) \\
&= \rho([\alpha_{v_2} \cdot [v_2, v_3] \cdot [v_3, v_1] \cdot [v_1, v_2] \cdot \alpha_{v_2}^{-1}]) \\
&= \rho([\alpha_{v_2} \cdot \text{id}_{v_2} \cdot \alpha_{v_2}^{-1}]) \\
&= \rho(\text{id}_{v_2}) = 0
\end{aligned}$$

Thus, φ is a cocycle and Φ is surjective.

Next, we investigate $\ker(\Phi)$. Assume $\Phi(\varphi) = 0$. Then $\varphi(\gamma) = 0$ for all $\gamma \in \pi_1(X)$. Then, φ trivially depends only on the endpoints of γ, thus is a coboundary by part (4) of the previous problem. Thus, $\ker(\Phi) \subseteq B^1(X)$. Moreover, if φ is a coboundary, it depends only on the endpoints of any path. Thus, as paths in $\pi_1(X)$ are loops, φ is constant on $\pi_1(X)$. Thus, $\varphi(\gamma) = \varphi(\text{id}) = 0$ for all $\gamma \in \pi_1(X)$. Thus, $B^1(X) \subseteq \ker(\Phi)$. Thus, $\ker(\Phi) = B^1(X)$.

Then, by the first isomorphism theorem, we have:

$$H^1(X) = Z^1(X)/B^1(X) \cong \text{Hom}(\pi_1(X), \mathbb{Z})$$