Math 121: Hodge Theory

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Problem 1. Let X be a differentiable manifold. Prove that $H^k_{\mathrm{dR}}(X, \mathbb{C}) \simeq H^k_{\mathrm{dR}}(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$.

Solution.

Problem 2. This exercise is taken from HW1 as, unfortunately, the hint for question 1 was missing. As the techniques and the result are important, I put it back.
Let U be an open subset of $\mathbb{C}$ and $D \subset \Omega$ be a closed disk.

1. Let $f : U \to \mathbb{C}$ be a C^1 function. Show that for all $z \in D$, we have:
$$f(z) = \frac{1}{2i\pi} \int_{\partial D} \frac{f(\xi)}{\xi - z} d\xi + \frac{1}{2i\pi} \int_D \frac{\partial f}{\partial \bar{z}}(\xi) \frac{d\xi \wedge d\bar{\xi}}{\xi - z}.$$
Hint: You can apply the Stokes formula to $\frac{f(\xi)}{\xi - z} d\xi$ *on* $D \setminus B(z, \epsilon)$ *and let* $\epsilon \to 0$.

2. Let g be a C^1 function on $\mathbb{C}$ with compact support and let:
$$f(z) = \frac{1}{2i\pi} \int_{\mathbb{C}} \frac{g(\xi)}{\xi - z} d\xi \wedge d\bar{\xi}.$$
Show that f is a C^1 function and $\frac{\partial f}{\partial \bar{z}} = g$.
Hint: you can differentiate under the integral sign after the change of variable $\xi' = \xi - z$*, then change back and conclude using the formula from the first question.*

3. Show that for any function g on U which is C^1, there exists f which is C^1 on U such that $\partial f / \partial \bar{z} = g$ on D.

4. In the last question, show that if g is C^∞, then f can be chosen C^∞. Show also that if g depends smoothly (or holomorphically) on other parameters, then so does f.

Solution.

Problem 3. Holomorphic $\bar{\partial}$-Dolbeault Lemma. Let U be an open subset of $\mathbb{C}^n$ and D an open polydisk with closure contained in U. Let $0 \le p \le n$, $1 \le q \le n$. The goal of this exercise is to prove that any (p,q)-form $\bar{\partial}$-closed on U has a restriction to D which is $\bar{\partial}$-exact.

1. Prove that we can reduce to the case where $p = 0$. *Hint: show that each form $\alpha \in \mathcal{A}^{p,q}(U)$ can be written as $\alpha = \sum_{|I|=p} \alpha_I \wedge dz^I$ with $\alpha_I \in \mathcal{A}^{0,q}(U)$ uniquely determined by α.*

2. Let $\alpha \in \Omega^{0,q}(U)$. Show that there exists $1 \le k \le n$ such that $\alpha = dz^k \wedge \gamma + \delta$ and γ, δ are forms in the subalgebra generated by dz^i, $1 \le i \le k-1$.

3. Prove the result by induction on k. *Hint: you can consider a form $\mu \in \mathcal{A}^{0,q-1}$ obtained from γ by replacing each coefficient $f \in C^\infty(D)$ by a function $g \in C^\infty$ such that $\partial g/\partial z^k = f$ on D. Show that if $\bar{\partial}\alpha = 0$, then we can choose μ such that $\bar{\partial}\mu = dz^k \wedge \gamma + \nu$ where ν can be expressed only in terms of $dz^1, \dots, dz^{k-1}$ and $C^\infty(U)$.*

Solution.

Problem 4. **Dolbeault cohomology of the open disk.** Let D be an open disk in $\mathbb{C}$ or $D = \mathbb{C}$.

1. Let $g \in C^\infty(D)$. Show that there exists $f \in C^\infty(D)$ such that $\partial f / \partial \bar{z} = g$.

   Hint: choose a sequence of disks $D_n \subset D$ such that $D_n \subset D_{n+1}$ and $\bigcup_n D_n = D$. Construct $f_n \in C^\infty(D)$ such that $\partial f_n / \partial \bar{z} = g$ on D_n and such that $|f_{n+1} - f_n| \leq 2^{-n}$ on D_{n-1}. Show that f_n converges to a function f that solves the problem.

2. Compute the Dolbeault cohomology groups of D.

Solution.

Problem 5. Let $\mathbb{P}^3(\mathbb{C})$ denote the complex projective 3-space with homogeneous coordinates x_0, x_1, x_2, x_3. Consider the complex submanifold

$$X := \{x \in \mathbb{P}^3(\mathbb{C}) \mid x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\}.$$

Let M be the underlying C^∞ manifold of X and let I denote the corresponding complex structure. Show that (M, I) and $(M, -I)$ are isomorphic as complex manifolds. How can you generalize this example?

Solution.