Math 121: Hodge Theory

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Problem 1. Let $U \subset \mathbb{C}^n$ be a domain (i.e., a non-empty connected subset) and let $f : U \to \mathbb{C}$ be a holomorphic function.

1. Prove that f satisfies analytic continuation: if it vanishes on an open subset of U, then it vanishes everywhere.
2. Prove that f satisfies the maximum principle: if $|f|$ admits a local maximum in U, then f is constant.
3. Let M be a compact complex manifold. Prove that any holomorphic function on M is constant.

Solution.

1. Let $V \subseteq U$ be open such that $f|_V \equiv 0$. Let G be the set of points in $z \in U$ such that f vanishes in a neighborhood of z. We want to show that $G = U$. We do this by showing that G is a non-empty clopen set.

   By definition, G is a union of open sets, and thus, G is open.

   Let $p \in V$ be arbitrary. Since U is open, and f is holomorphic, there exists an open ball $B_r(p) \subseteq U$ of radius r around p such that f has a convergent power series expansion around p:
   $$f(z) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha (z-p)^\alpha$$
   where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index, $(z-p)^\alpha = (z_1-p_1)^\alpha_1 \cdots (z_n-p_n)^\alpha_n$ and $c_\alpha \in \mathbb{C}$. Moreover, since V is open, $B_r(p) \cap V \subseteq U$ is an open neighborhood of p such that f vanishes on it. Thus, we have that $c_\alpha = 0$ for all $\alpha \in \mathbb{N}^n$. Since $p \in V$ was arbitrary, $V \subseteq G$. Thus, G is non-empty.

To show that G is closed, let $(w_n) \in G$ be a sequence that converges to $w \in U$. Let B be a small open ball around w and consider the power series expansion of f around w in B. We have:

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha (z-w)^\alpha$$

If all $c_\alpha = 0$, we are done. Otherwise, there exists, some minimum α' (with dictionary order) such that $c_\alpha \neq 0$. Then, $(z-w)^{\alpha'}$ is a factor of every other non-zero term (since we picked the dictionary order). Thus, we can write:

$$f(z) = c_{\alpha'}(z-w)^{\alpha'}(1 + g(z-w))$$

where $g \to 0$ as $z \to w$. Moroever, we know that eventually $(w_k) \in B$. Since $w_k \neq w$, we have that $(z-w)^{\alpha'} \neq 0$ and $(1 + g(z-w)) \neq 0$. Thus, $f(w_k) \neq 0$, which is a contradiction. Thus, no such α' exists and $f(z) = 0$ on all of B. Thus, $w \in G$ and G is closed.

Since G is a non-empty open and closed subset of U, and U is connected, we have that $G = U$. Thus, f vanishes everywhere.

2. Note that we showed the result for $U \subset \mathbb{C}$ in class (using the Cauchy formula in polar form). Assume $|f|$ has a local maximum at $(a_1, \cdots, a_n) \in U$. Then, there exists a poly-disk $D = D_1 \times \cdots \times D_n$ such that for $p \in D$, we have:

$$|f(p)| \leq |f(a_1, \cdots, a_n)|$$

We need to show that for any $(z_1, \cdots, z_n) \in D$, we have $f(z_1, \cdots, z_n) = f(a_1, \cdots, a_n)$.

Consider the function $f_1 : \mathbb{C} \to \mathbb{C}$ defined as:

$$f_1(z) = f(z, a_2, \cdots, a_n)$$

Then, $f_1(z)$ attains a maximum on D_1 at a_1. Thus, by the result in the single-variable case, f_1 is constant on D_1. Thus, we have:

$$f(z_1, a_2, \cdots, a_n) = f(a_1, \cdots, a_n)$$

and $f(z_1, a_2, \cdots, a_n)$ is a local minimum. Thus, we can repeat this process n times to get:

$$f(z_1, z_2, \cdots, z_n) = f(a_1, a_2, \cdots, a_n)$$

Thus, f is constant on an open subset D. Then, we can define $g : U \to \mathbb{C}$ by letting it be:

$$g(z) = f(z) - f(a_1, \cdots, a_n)$$

Then this function is vanishes on D, thus, by the previous part, vanishes on U. Thus, f is constant on U.

3. Let $f : M \to \mathbb{C}$ be a holomorphic function. Since f is holomorphic, it is continous. Moreover, $|f| : M \to \mathbb{R}$ is also continous. Since M is compact, $|f|$ attains a maximum at some point $p \in M$.
   Let $\{(U_i, \phi_i)\}_{i=1}^n$ be a cover of charts (we can pick finite as M is compact). Then $p \in U_i$ for some i. Then, we have that $f \circ \phi_i^{-1} : \phi_i(U_i) \to \mathbb{C}$ attains a maximum at $\phi_i(p)$. Thus, by the previous part, $f \circ \phi_i^{-1}$ is constant on $\phi_i(U_i)$. Thus, f is constant on U_i (as ϕ^{-1} is bijective).
   Then, using part 1), and assuming that M is connected, we can prove that f is constant on all of M, propagating via open intersections of chart domains.

Problem 2.

1. Show that the assignment $L \mapsto (L_{\mathbb{R}}, \mathrm{mult}(i))$ gives an equivalence between the category of complex vector spaces and the category of pairs (V, J), where V is a real vector space, $J : V \to V$ is an $\mathbb{R}$-linear operator satisfying $J^2 = -\mathrm{Id}_{L_{\mathbb{R}}}$, and a morphism $(V, J) \to (W, K)$ is defined as an $\mathbb{R}$-linear map $f : V \to W$ that intertwines J and K, i.e., such that $K \circ f = f \circ J$.

2. Let (V, J) be a pair as above:

   (a) Let $\iota : v \mapsto v$ be the $\mathbb{R}$-linear automorphism of $V \otimes_{\mathbb{R}} \mathbb{C}$ induced from conjugation on the second factor. Show that V, seen inside $V \otimes_{\mathbb{R}} \mathbb{C}$ via $v \mapsto v \otimes 1$, is isomorphic to the fixed locus of ι.

   (b) By diagonalizing J over $V \otimes_{\mathbb{R}} \mathbb{C}$, show that we have a decomposition $V_{\mathbb{C}} = W \oplus \bar{W}$ where J acts on W by multiplication by i and $\bar{W} = \iota(W)$.

   (c) Show that the projection map $V \to W$ is an $\mathbb{R}$-linear isomorphism that intertwines J and multiplication by i on W.

   (d) Each element $v \in V_{\mathbb{C}}$ can be written as $v = v_1 + v_2$ along the above decomposition. Show that $v \in V$ if and only if $v_2 = \bar{v}_1$.

Problem 3. Let E, F be real vector spaces and let G be a complex vector space, all finite-dimensional.

1. Show that $E \otimes_{\mathbb{R}} G$ admits a natural structure of a complex vector space. Construct a basis in terms of bases of E and G. Show that $E \otimes_{\mathbb{R}} G = (E \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} G$.

2. Show that $\mathrm{Hom}_{\mathbb{R}}(E, G) = \mathrm{Hom}_{\mathbb{C}}(E \otimes \mathbb{C}, G)$.

3. Show that

   $$(E \oplus F) \otimes_{\mathbb{R}} \mathbb{C} = (E \otimes_{\mathbb{R}} \mathbb{C}) \oplus (F \otimes_{\mathbb{R}} \mathbb{C}), \quad (E \otimes_{\mathbb{R}} F) \otimes_{\mathbb{R}} \mathbb{C} = (E \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (F \otimes_{\mathbb{R}} \mathbb{C})$$

   and

   $$\Lambda^n E \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^n (E \otimes_{\mathbb{R}} \mathbb{C}).$$

4. Let $f : E \to F$ be a linear map and let $f_{\mathbb{C}} = f \otimes_{\mathbb{R}} \mathbb{C} : E \otimes_{\mathbb{R}} \mathbb{C} \to F \otimes_{\mathbb{R}} \mathbb{C}$ be the induced map. Show that

   $$\ker(f_{\mathbb{C}}) = \ker(f) \otimes_{\mathbb{R}} \mathbb{C}, \quad \mathrm{Im}(f_{\mathbb{C}}) = \mathrm{Im}(f) \otimes_{\mathbb{R}} \mathbb{C}.$$

Solution.

1. Let $e \in E$ and $g \in G$. Then, we define the scalar multiplication by $z \in \mathbb{C}$ by using the complex structure of G:

   $$z \cdot (e \otimes g) = e \otimes zg$$

   and extend linearly. For $z \in \mathbb{R}$, this definition respects the tensor product over $\mathbb{R}$ as:

   $$z \cdot (e \otimes g) = e \otimes zg = ze \otimes g$$

   If $\{e_i\}_i$ is a basis of E and $\{g_j\}_j$ is a basis of G, then the set $\{e_i \otimes_{\mathbb{R}} g_j\}_{i,j}$ is a $\mathbb{R}$-linear basis of $E \otimes_{\mathbb{R}} G$ (before we introduce the complex structure). With respect to the complex structure, note that this set is $\mathbb{C}$-linearly independent as if:

   $$\sum_{i,j} z_{ij}(e_i \otimes g_j) = 0$$

   then,

   $$\sum_{i,j} e_i \otimes z_{ij} g_j = 0$$

   which contradicts the $\mathbb{R}$-linear independence of $\{e_i \otimes_{\mathbb{R}} g_j\}_{i,j}$. Moreover, the set clearly $\mathbb{C}$-spans $E \otimes_{\mathbb{R}} G$ as it $\mathbb{R}$-spans it. Thus, the set is a $\mathbb{C}$-linear basis of $E \otimes_{\mathbb{R}} G$.

Next, we define the homomorphisms $\Phi : E \otimes_{\mathbb{R}} G \to (E \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} G$ and $\Psi : (E \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} G \to E \otimes_{\mathbb{R}} G$ as follows:

$$\begin{aligned}\Phi(e \otimes g) &= (e \otimes_{\mathbb{R}} 1) \otimes_{\mathbb{C}} g \\ \Psi((e \otimes_{\mathbb{R}} z) \otimes_{\mathbb{C}} g) &= e \otimes_{\mathbb{R}} zg\end{aligned}$$

and extending linearly. We show that these are inverses as follows:

$$\begin{aligned}\Psi \circ \Phi((e \otimes_{\mathbb{R}} g)) &= \Psi((e \otimes_{\mathbb{R}} 1) \otimes_{\mathbb{C}} g) \\ &= e \otimes_{\mathbb{R}} g \\ \Phi \circ \Psi((e \otimes_{\mathbb{R}} z) \otimes_{\mathbb{C}} g) &= \Phi(e \otimes_{\mathbb{R}} zg) \\ &= (e \otimes_{\mathbb{R}} 1) \otimes_{\mathbb{C}} zg \\ &= z \cdot (e \otimes_{\mathbb{R}} 1) \otimes_{\mathbb{C}} g \\ &= (e \otimes_{\mathbb{R}} z) \otimes_{\mathbb{C}} g\end{aligned}$$

where we use the same natural complex structure on $E \otimes_{\mathbb{R}} \mathbb{C}$. Thus, Φ and Ψ are inverses. Thus,

$$E \otimes_{\mathbb{R}} G \cong (E \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} G$$

2. We show this by constructing the linear maps $\Phi : \mathrm{Hom}_{\mathbb{R}}(E, G) \to \mathrm{Hom}_{\mathbb{C}}(E \otimes_{\mathbb{R}} \mathbb{C}, G)$ and $\Psi : \mathrm{Hom}_{\mathbb{C}}(E \otimes_{\mathbb{R}} \mathbb{C}, G) \to \mathrm{Hom}_{\mathbb{R}}(E, G)$ as follows, for $f \in \mathrm{Hom}_{\mathbb{R}}(E, G)$ and $g \in \mathrm{Hom}_{\mathbb{C}}(E \otimes_{\mathbb{R}} \mathbb{C}, G)$:

$$\begin{aligned}\Phi(f)(e \otimes_{\mathbb{R}} z) &= zf(e) \in G \\ \Psi(g)(e) &= g(e \otimes_{\mathbb{R}} 1) \in G\end{aligned}$$

and extending linearly. We show that $\Phi(g)$ is $\mathbb{C}$-linear and $\Psi(g)$ is $\mathbb{R}$-linear.

$$\begin{aligned}\Phi(f)(z \cdot e_1 \otimes_{\mathbb{R}} z_1 + e_2 \otimes_{\mathbb{R}} z_2) &= \Phi(f)(e_1 \otimes_{\mathbb{R}} zz_1 + e_2 \otimes_{\mathbb{R}} z_2) \\ &= zz_1 f(e_1) + z_2 f(e_2) \\ &= z \cdot \Phi(f)(e_1 \otimes_{\mathbb{R}} z_1) + \Phi(f)(e_2 \otimes_{\mathbb{R}} z_2)\end{aligned}$$

Thus, $\Phi(f)$ is $\mathbb{C}$-linear.

$$\begin{aligned}\Psi(g)(ze_1 + e_2) &= g((ze_1 + e_2) \otimes_{\mathbb{R}} 1) \\ &= g((e_1 \otimes_{\mathbb{R}} z) + (e_2 \otimes_{\mathbb{R}} 1)) \\ &= \Phi(g)(ze_1) + \Phi(g)(e_2)\end{aligned}$$

Thus, $\Psi(g)$ is $\mathbb{R}$-linear. Finally, we wanna show that Φ and Ψ are inverses, for $e \in E$ and $z \in \mathbb{C}$.

$$\begin{aligned}\Psi(\Phi(f))(e) &= \Phi(f)(e \otimes_{\mathbb{R}} 1) \\ &= f(e) \\ \Phi(\Psi(g))(e \otimes_{\mathbb{R}} z) &= z\Psi(g)(e) \\ &= zg(e \otimes_{\mathbb{R}} 1) \\ &= g(e \otimes_{\mathbb{R}} z)\end{aligned}$$

Thus, Φ and Ψ are inverses. Thus, we have the isomorphism:

$$\mathrm{Hom}_{\mathbb{R}}(E,G) \cong \mathrm{Hom}_{\mathbb{C}}(E \otimes_{\mathbb{R}} \mathbb{C}, G)$$

3. We do this by constructing explicit maps.

   - Let $e \in E$, $f \in F$ and $z \in \mathbb{C}$. Define the two maps:

$$\begin{aligned}
\Phi : (E \oplus F) \otimes_{\mathbb{R}} \mathbb{C} &\to (E \otimes_{\mathbb{R}} \mathbb{C}) \oplus (F \otimes_{\mathbb{R}} \mathbb{C}) \\
(e,f) \otimes_{\mathbb{R}} z &\mapsto (e \otimes_{\mathbb{R}} z, f \otimes_{\mathbb{R}} z) \\
\Psi : (E \otimes_{\mathbb{R}} \mathbb{C}) \oplus (F \otimes_{\mathbb{R}} \mathbb{C}) &\to (E \oplus F) \otimes_{\mathbb{R}} \mathbb{C} \\
(e \otimes z_1, f \otimes z_2) &\mapsto (e,0) \otimes_{\mathbb{R}} z_1 + (0,f) \otimes_{\mathbb{R}} z_2
\end{aligned}$$

     and extend linearly. We show that these are inverses as follows:

$$\begin{aligned}
\Phi \circ \Psi(e \otimes_{\mathbb{R}} z_1, f \otimes_{\mathbb{R}} z_2) &= \Phi((e,0) \otimes_{\mathbb{R}} z_1 + (0,f) \otimes_{\mathbb{R}} z_2) \\
&= \Phi((e,0) \otimes_{\mathbb{R}} z_1) + \Phi((0,f) \otimes_{\mathbb{R}} z_2) \\
&= (e \otimes_{\mathbb{R}} z_1, 0) + (0, f \otimes_{\mathbb{R}} z_2) \\
&= (e \otimes_{\mathbb{R}} z_1, f \otimes_{\mathbb{R}} z_2) \\
\Psi \circ \Phi((e,f) \otimes_{\mathbb{R}} z) &= \Psi(e \otimes_{\mathbb{R}} z, f \otimes_{\mathbb{R}} z) \\
&= (e,0) \otimes_{\mathbb{R}} z + (0,f) \otimes_{\mathbb{R}} z \\
&= (e,f) \otimes_{\mathbb{R}} z
\end{aligned}$$

     Thus, Φ and Ψ are inverses. Thus, we have the isomorphism:

$$(E \oplus F) \otimes_{\mathbb{R}} \mathbb{C} \cong (E \otimes_{\mathbb{R}} \mathbb{C}) \oplus (F \otimes_{\mathbb{R}} \mathbb{C})$$

   - Let $e \in E$, $f \in F$ and $z \in \mathbb{C}$. Note that a simple tensor in $(E \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (F \otimes_{\mathbb{R}} \mathbb{C})$ is $(e \otimes_{\mathbb{R}} z) \otimes_{\mathbb{C}} (f \otimes_{\mathbb{R}} 1)$ as we are tensoring over $\mathbb{C}$. Define the two maps:

$$\begin{aligned}
\Phi : (E \otimes_{\mathbb{R}} F) \otimes_{\mathbb{R}} \mathbb{C} &\to (E \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (F \otimes_{\mathbb{R}} \mathbb{C}) \\
e \otimes f \otimes_{\mathbb{R}} z &\mapsto (e \otimes_{\mathbb{R}} z) \otimes_{\mathbb{C}} (f \otimes_{\mathbb{R}} 1) \\
\Psi : (E \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (F \otimes_{\mathbb{R}} \mathbb{C}) &\to (E \otimes F) \otimes_{\mathbb{R}} \mathbb{C} \\
(e \otimes z) \otimes (f \otimes 1) &\mapsto e \otimes_{\mathbb{R}} f \otimes_{\mathbb{R}} z
\end{aligned}$$

     and extend linearly. We show that these are inverses as follows:

$$\begin{aligned}
\Phi \circ \Psi((e \otimes_{\mathbb{R}} z) \otimes_{\mathbb{C}} (f \otimes_{\mathbb{R}} 1)) &= \Phi(e \otimes_{\mathbb{R}} f \otimes_{\mathbb{R}} z) \\
&= (e \otimes_{\mathbb{R}} z) \otimes_{\mathbb{C}} (f \otimes_{\mathbb{R}} 1) \\
\Psi \circ \Phi(e \otimes_{\mathbb{R}} f \otimes_{\mathbb{R}} z) &= \Psi(e \otimes_{\mathbb{R}} z) \otimes_{\mathbb{C}} (f \otimes_{\mathbb{R}} 1) \\
&= e \otimes_{\mathbb{R}} f \otimes_{\mathbb{R}} z
\end{aligned}$$

     Thus, Φ and Ψ are inverses. Thus, we have the isomorphism:

$$(E \otimes_{\mathbb{R}} F) \otimes_{\mathbb{R}} \mathbb{C} \cong (E \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (F \otimes_{\mathbb{R}} \mathbb{C})$$

- Let $e_1, \cdots, e_n \in E$ and $z_1, \cdots, z_n \in \mathbb{C}$. Assuming that $\bigwedge^n (E \otimes_{\mathbb{R}} \mathbb{C})$ is wedging over $\mathbb{C}$. Define the two maps:

$$\begin{aligned}
\Phi : \bigwedge^n E \otimes_{\mathbb{R}} \mathbb{C} &\to \bigwedge^n (E \otimes_{\mathbb{R}} \mathbb{C}) \\
e_1 \wedge \cdots \wedge e_n \otimes_{\mathbb{R}} z_1 &\mapsto (e_1 \otimes_{\mathbb{R}} z_1) \wedge (e_2 \otimes_{\mathbb{R}} 1) \wedge \cdots \wedge (e_n \otimes_{\mathbb{R}} 1) \\
\Psi : \bigwedge^n (E \otimes_{\mathbb{R}} \mathbb{C}) &\to \bigwedge^n E \otimes_{\mathbb{R}} \mathbb{C} \\
(e_1 \otimes_{\mathbb{R}} z_1) \wedge \cdots \wedge (e_n \otimes_{\mathbb{R}} z_n) &\mapsto e_1 \wedge \cdots \wedge e_n \otimes_{\mathbb{R}} z_1 \cdots z_n
\end{aligned}$$

  and extend linearly. We show that these are inverses as follows:

$$\begin{aligned}
\Phi \circ \Psi((e_1 \otimes_{\mathbb{R}} z_1) \wedge \cdots \wedge (e_n \otimes_{\mathbb{R}} z_n)) &= \Phi(e_1 \wedge \cdots \wedge e_n \otimes_{\mathbb{R}} z_1 \cdots z_n) \\
&= (e_1 \otimes_{\mathbb{R}} z_1 \cdots z_n) \wedge (e_2 \otimes_{\mathbb{R}} 1) \wedge \cdots \wedge (e_n \otimes_{\mathbb{R}} 1) \\
&= (e_1 \otimes_{\mathbb{R}} z_1) \wedge \cdots \wedge (e_n \otimes_{\mathbb{R}} z_n) \\
\Psi \circ \Phi(e_1 \wedge \cdots \wedge e_n \otimes_{\mathbb{R}} z_1) &= \Psi((e_1 \otimes_{\mathbb{R}} z_1) \wedge (e_2 \otimes_{\mathbb{R}} 1) \wedge \cdots \wedge (e_n \otimes_{\mathbb{R}} 1)) \\
&= e_1 \wedge \cdots \wedge e_n \otimes_{\mathbb{R}} z_1
\end{aligned}$$

  Thus, Φ and Ψ are inverses. Thus, we have the isomorphism:

$$\bigwedge^n E \otimes_{\mathbb{R}} \mathbb{C} \cong \bigwedge^n (E \otimes_{\mathbb{R}} \mathbb{C})$$

4. Let $e \in \ker(f)$. Then, for any $e \otimes_{\mathbb{R}} z \in \ker(f) \otimes_{\mathbb{R}} \mathbb{C}$, we have:

$$f_{\mathbb{C}}(e \otimes_{\mathbb{R}} z) = f(e) \otimes_{\mathbb{R}} z = 0 \otimes_{\mathbb{R}} z = 0$$

   Thus, $e \otimes_{\mathbb{R}} z \in \ker(f_{\mathbb{C}})$ for any $z \in \mathbb{C}$. Thus, $\ker(f) \otimes \mathbb{C} \subseteq \ker(f_{\mathbb{C}})$.
   Similarly, let $e \otimes_{\mathbb{R}} z \in \ker(f_{\mathbb{C}})$. If $z = 0$, then, $e \otimes_{\mathbb{R}} z = 0 \otimes 0$ and $0 \in \ker(f)$. Thus, assume $z \neq 0$. Then, $f_{\mathbb{C}}(e \otimes_{\mathbb{R}} z) = 0$ implies that $f(e) \otimes_{\mathbb{R}} z = 0$. Since $z \neq 0$, we have $f(e) = 0$. Thus, $e \in \ker(f)$. Thus, $\ker(f_{\mathbb{C}}) \subseteq \ker(f) \otimes_{\mathbb{R}} \mathbb{C}$. Hence, we conclude $\ker(f_{\mathbb{C}}) = \ker(f) \otimes_{\mathbb{R}} \mathbb{C}$.

   Let $p \in \text{img}(f)$. Then, there exists $e \in E$ such that $f(e) = p$. Then, we have $f_{\mathbb{C}}(e \otimes_{\mathbb{R}} z) = f(e) \otimes_{\mathbb{R}} z = p \otimes_{\mathbb{R}} z$. Thus, $\text{img}(f) \otimes_{\mathbb{R}} \mathbb{C} \subseteq \text{img}(f_{\mathbb{C}})$.
   Conversely, let $p \otimes_{\mathbb{R}} z \in \text{img}(f_{\mathbb{C}})$. Then, there exists $e \otimes_{\mathbb{R}} z' \in E \otimes_{\mathbb{R}} \mathbb{C}$ such that $f_{\mathbb{C}}(e \otimes_{\mathbb{R}} z') = p \otimes_{\mathbb{R}} z$. Then, by the definition of $f_{\mathbb{C}}$, we have $z = z'$ and $p = f(e)$. Thus, $p \in \text{img}(f)$. Thus, $\text{img}(f_{\mathbb{C}}) \subseteq \text{img}(f) \otimes_{\mathbb{R}} \mathbb{C}$. Hence, we conclude $\text{img}(f_{\mathbb{C}}) = \text{img}(f) \otimes_{\mathbb{R}} \mathbb{C}$.

Problem 4. Let $U \subset \mathbb{C}$ be an open subset and let $D \subset \Omega$ be a closed disk.

1. Let $f : U \to \mathbb{C}$ be a $\mathcal{C}^1$ function. Show that for all $z \in D$, we have:

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\xi)}{\xi - z}\, d\xi + \frac{1}{2\pi i} \int_D \frac{\partial f}{\partial \bar{z}}(\xi) \frac{d\xi \wedge d\bar{\xi}}{\xi - z}.$$

   Hint: Apply Stokes' theorem to $\frac{f(\xi)}{\xi - z} d\xi$ on $D \setminus B(z, \varepsilon)$ and let $\varepsilon \to 0$.

2. Let g be a $\mathcal{C}^1$ function on $\mathbb{C}$ with compact support and define

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g(\xi)}{\xi - z} d\xi \wedge d\bar{\xi}.$$

   Show that f is $\mathcal{C}^1$ and $\frac{\partial f}{\partial \bar{z}} = g$. *Hint:* Differentiate under the integral sign after substituting $\xi' = \xi - z$.

3. Show that for any function g on U which is $\mathcal{C}^1$, there exists a function f on U, also $\mathcal{C}^1$, such that $\frac{\partial f}{\partial \bar{z}} = g$ on D.

4. In the previous question, show that if g is $\mathcal{C}^\infty$, then f can also be chosen to be $\mathcal{C}^\infty$.

Problem 5. Let E and F be two holomorphic vector bundles on a complex manifold X. Given an open cover $\{U_\alpha\}$ of X that trivializes E, the vector bundle E is described on overlaps $U_\alpha \cap U_\beta$ by holomorphic transition functions:

$$\rho_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{C}).$$

1. Prove the cocycle condition: $\rho_{\alpha\gamma} = \rho_{\beta\gamma} \circ \rho_{\alpha\beta}$.

2. Let E' be the quotient of $\bigsqcup_\alpha U_\alpha \times \mathbb{C}^n$ by the equivalence relation on $U_\alpha \cap U_\beta \times \mathbb{C}^n$ given by
$$(x, v) \sim (x, \rho_{\alpha\beta}(x)(v)).$$
Prove that E' is a holomorphic vector bundle and that it is isomorphic to E as vector bundles over X, i.e., there exists a biholomorphism $f : E \to E'$ commuting with projection to X.

3. Conversely, assume that E and F are isomorphic as holomorphic vector bundles. How are their transition functions related?

4. Using the transition maps of E and F, construct the following vector bundles by writing down explicitly their transition functions: $E \otimes_{\mathbb{C}} F$, $E \oplus F$, $\Lambda^n E$.

Solution.

1. Let $\{U_\alpha\}$ be an open cover of X that trivializes E, with respective maps $\tau_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^n$. Then, note that $\rho_{\alpha\beta} : \tau_\alpha(\pi^{-1}(U_\alpha \cap U_\beta)) \to U_\beta \times \mathbb{C}^n$ is given by:
$$\rho_{\alpha\beta} = \tau_\beta \circ \tau_\alpha^{-1}$$
with others defined similarly. Let $x \in U_\alpha \cap U_\beta \cap U_\gamma$ and $E_x = \pi^{-1}(a)$ be the fiber over x in E. Then, we calculate:
$$\begin{aligned}\rho_{\beta\gamma} \circ \rho_{\alpha\beta}(E_x) &= \tau_\gamma \circ \tau_\beta^{-1} \circ \tau_\beta \circ \tau_\alpha^{-1}(E_x) \\ &= \tau_\gamma \circ \tau_\alpha^{-1}(E_x) \\ &= \rho_{\alpha\gamma}(E_x)\end{aligned}$$
Thus, the cocycle condition holds on every such fiber. Thus, it holds in general. We can also realize this by picking a basis for the image of E_x in each trivialization, and noticing that ρ are just change of basis matrices, which satisfy the cocycle condition.

2. We first check that E' is a holomorphic vector bundle. We write an element of E' as $(x, [v])$, for $x \in U_\alpha$, where $[v]$ is the equivalence class of $\{\rho_{\alpha\beta}(v) : x \in U_\beta\}$.

   Defining $\pi'(x, [v]) = x$, we note that the projection π' is clearly well-defined and can be made holomorphic by inheriting the holomorphic structure from the base space.

Furthermore, we define the trivialization maps $\tau'_\beta : \pi'^{-1}(U_\beta) \to U_\beta \times \mathbb{C}^n$ as follows:

$$\tau'_\beta(x, [v]) = (x, v)$$

where v is the representative of $[v]$ such that $(x, v) \in U_\beta \times \mathbb{C}^n$. Then, τ'^{-1}_β is given by taking the equivalence class:

$$\tau'^{-1}_\beta(x, v) = (x, [v])$$

Thus, the transition maps $\rho'_{\alpha\beta}$ are given by, for $x \in U_\alpha \cap U_\beta$:

$$\begin{aligned}
\rho'_{\alpha\beta}(x, v) &= \tau'_\beta \circ \tau'^{-1}_\alpha(x, v) \\
&= (x, [v])) \\
&= (x, [\rho_{\alpha\beta}(v)]) \\
&= (x, \rho_{\alpha\beta}(v))
\end{aligned}$$

Thus, the transition functions $\rho'_{\alpha\beta}$ are holomorphic as $\rho_{\alpha\beta}$ are holomorphic. Thus, E' is a holomorphic vector bundle.

Next, to show that E' is isomorphic to E as a vector bundle, we need to find a biholomorphism $f : E \to E'$ that respects the fibers. We can define f as follows: for $x \in U_\alpha$, and $\epsilon \in \pi^{-1}(x) = E_x$,

$$f(\epsilon) = \tau'^{-1}_\alpha \circ \tau_\alpha(\epsilon)$$

To see that f is well-defined, let $x \in U_\alpha \cap U_\beta$. Then, we have:

$$\begin{aligned}
f(\epsilon) &= \tau'^{-1}_\alpha \circ \tau_\alpha(\epsilon) \\
&= \tau'^{-1}_\alpha(x, v) \\
&= (x, [v]) \\
&= (x, [\rho_{\alpha\beta}(v)]) \\
&= \tau'^{-1}_\beta(x, \rho_{\alpha\beta}(v)) \\
&= \tau'^{-1}_\beta \circ \tau_\beta(\epsilon) \\
&= f(\epsilon)
\end{aligned}$$

Since it is a composition of holomorphic maps, f is holomorphic. Next, we define the inverse $f^{-1} : E' \to E$ as follows: for $x \in U_\alpha$ and $(x, [v]) \in E'$, with v being the representative that came from $U_\alpha \times \mathbb{C}^n$, we have:

$$f^{-1}(x, [v]) = \tau^{-1}_\alpha \circ \tau'_\alpha(x, [v])$$

To see that this is well-defined, let $x \in U_\alpha \cap U_\beta$. Then, we have:

$$\begin{aligned}
f^{-1}(x,[v]) &= \tau_\alpha^{-1} \circ \tau'_\alpha(x,[v]) \\
&= \tau_\alpha^{-1}(x,v) \\
&= \tau_\beta^{-1}(x,\rho_{\alpha\beta}(v)) \\
&= \tau_\beta^{-1} \circ \tau'_\beta(x,[\rho_{\alpha\beta}(v)]) \\
&= \tau_\beta^{-1} \circ \tau'_\beta(x,[v]) \\
&= f^{-1}(x,[v])
\end{aligned}$$

Since f^{-1} is also a composition of holomorphic maps, f^{-1} is holomorphic. Moreover, it is easy to see that f^{-1} and f are inverses. Thus, f is a biholomorphism.

Finally, we can see that f and f^{-1} respect the fibers as they are compositons of fiber-preserving maps. Thus, they commute with the projection maps. Moreover, we note that on a particular fiber, E_x, with respect to the trivializations, f is a vector space isomorphism. That is, $\tau'_\alpha \circ f \circ \tau_\alpha^{-1}$ is an isomorphism on $\{x\} \times \mathbb{C}^n$ as it is linear and has an inverse.

3. Since E and F are isomorphic as holomorphic vector bundles, there exists a biholomorphism $f : E \to F$ that respects the fibers, and is a vector space isomorphism when restricted to each fiber.

   By refining the covers over which the trivializations for E and F are defined, choose a cover $\{U_\alpha\}$ such that both E and F are trivialized over it. Let $x \in U_\alpha \cap U_\beta$. Let $\tau_{\alpha,E}$ and $\tau_{\alpha,F}$ be trivializations and ρ_E and ρ_F be the transition functions from $\tau_\alpha(\pi^{-1}(U_\alpha)) \to U_\beta \times \mathbb{C}^n$ (sloppily identifying the two projections π). Let $E_{x,\alpha}$, $E_{x,\beta}$ be the trivilized fibers over x and similarly for f. Then, f is such that:

$$\begin{matrix}
\tau_{\alpha,F} \circ f \circ \tau_{\alpha,E}^{-1} & \tau_{\beta,F} \circ f \circ \tau_{\alpha,E}^{-1} \\
\tau_{\alpha,F} \circ f \circ \tau_{\beta,E}^{-1} & \tau_{\beta,F} \circ f \circ \tau_{\beta,E}^{-1}
\end{matrix}$$

   are all isomorphisms. Composition by ρ_E and ρ_F and their inverses permutes them. Write $f_\alpha := \tau_{\alpha,F} \circ f \circ \tau_{\alpha,E}^{-1}$. Thus, we have the commutative diagram connecting the two transition functions (where every arrow is invertible):

$$\begin{tikzcd}
E_{x,\alpha} \arrow[r, "\rho_E"] \arrow[d, "f_\alpha"'] & E_{x,\beta} \arrow[d, "f_\beta"] \\
F_{x,\alpha} \arrow[r, "\rho_F"'] & F_{x,\beta}
\end{tikzcd}$$

4. We can pick a cover $\{U_\alpha\}$ that trivializes both E and F. Then, pick $x \in U_\alpha \cap U_\beta$ and ρ_E and ρ_F be the transiton functions from the U_α trivialization to the U_β trivialization. Then,

(a) $\rho_E \otimes_{\mathbb{C}} \rho_F$ is a transition function for $E \otimes_{\mathbb{C}} F$.

(b) $\rho_E \oplus \rho_F$ be a transition function for $E \oplus F$.

(c) $\bigwedge^n \rho_E$ be a transition function for $\bigwedge^n E$.

These are all holomorphic, as they are holomorphic on each component.