Math 74: Algebraic Topology

Sair Shaikh

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Problem 1. (0.10) A space is called contractible if the identity map is nullhomotopic. Show that X is contractible if and only if for every space Y, every map $f : X \to Y$ is nullhomotopic. Similarly, show that X is contractible if and only if for every Y, every map $f : Y \to X$ is nullhomotopic.

Solution. ($\Longleftarrow$) Assume for every Y every map $f : X \to Y$ is nullhomotopic. Then, in particular, picking $Y = X$ and $f = \mathrm{id}_X$, we have that id_X is nullhomotopic. Thus, X is contractible.
($\Longrightarrow$) Assume that X is contractible and Y be any space. Thus, there exists a homotopy $H : X \times I \to X$ from id_X to a constant map $\lambda_c : X \to X$ for some $c \in X$. Then, for any map $f : X \to Y$, we claim that $H' : X \times I \to Y$ defined by $H'(x,t) = f(H(x,t))$ is a homotopy from f to the constant map $f(c)$. Similarly, for any map g from Y to X, we claim that $H'' : Y \times I \to X$ defined by $H''(y,t) = H(g(y),t)$ is a homotopy from g to the constant map $g(c)$. Note that H' and H'' are continous as they are compositions of continous maps. Moreover, we check, for all $x \in X$ and $y \in Y$,

$$\begin{aligned}
H'(x,0) &= f(H(x,0)) = f(x) \\
H'(x,1) &= f(H(x,1)) = f(\lambda_c(x)) = f(c) \\
H''(y,0) &= H(g(y),0) = g(y) \\
H''(y,1) &= H(g(y),1) = \lambda_c(g(y)) = c
\end{aligned}$$

Thus, H' is a homotopy from f to the constant map valued at $f(c)$ and H'' is a homotopy from g to the constant map valued at c.

Problem 2. Let $A \subset X$ and suppose $r : X \to A$ is a continuous map such that $r(a) = a$ for all $a \in A$ (i.e., r is a retraction of X onto A). If $a_0 \in A$, show that

$$r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$$

is surjective. (Hint: Consider also the inclusion map of A into X.)

Solution. Let $\iota : A \to X$ be the inclusion map. Then, note that for all $a \in A$:

$$r \circ \iota(a) = a$$

Thus, for any $f : I \to A$, we have $r \circ \iota \circ f = f$. Moreover, we have $\iota_* : \pi_1(A, a_0) \to \pi_1(X, a_0)$ with $\iota_*([f]) = [\iota \circ f]$. Let $p \in \pi_1(A, a_0)$ be arbitrary. Then, we claim that $\iota_*([p]) \in \pi_1(X, a_0)$ maps to $[p]$ under r_*. We compute:

$$\begin{aligned} r_*(\iota_*([p])) &= r_*([\iota \circ p]) \\ &= [r \circ \iota \circ p] \\ &= [p] \end{aligned}$$

Thus, as $[p]$ was arbitrary, we have shown that r_* is surjective.

Problem 3. (1.1.3) If X is a path-connected space, show that $\pi_1(X)$ is abelian if and only if every change-of-basepoint isomorphism ($\hat{\alpha}$ in class or β_h in Hatcher) depends only on the endpoints of the path.

Solution. ($\implies$) Assume $\pi_1(X)$ is abelian. Let $\alpha_1, \alpha_2 : I \to X$ be two paths from x_0 to x_1. Then, $\hat{\alpha_i} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ are given by $\hat{\alpha_i}([f]) = [\overline{\alpha_i} \cdot f \cdot \alpha_i]$. Note that $\pi_1(X, x_1)$ is abelian as $\pi_1(X)$ is. We show, for any $[f] \in \pi_1(X, x_0)$ that:

$$\begin{aligned}
\hat{\alpha_1}([f]) &= [\overline{\alpha_1} \cdot f \cdot \alpha_1] \\
&= [\overline{\alpha_1} \cdot f \cdot \alpha_2 \cdot \overline{\alpha_2} \cdot \alpha_1] \\
&= [\overline{\alpha_1} \cdot f \cdot \alpha_2] \cdot [\overline{\alpha_2} \cdot \alpha_1] \\
&= [\overline{\alpha_2} \cdot \alpha_1] \cdot [\overline{\alpha_1} \cdot f \cdot \alpha_2] \\
&= [\overline{\alpha_2} \cdot \alpha_1 \cdot \overline{\alpha_1} \cdot f \cdot \alpha_2] \\
&= [\overline{\alpha_2} \cdot f \cdot \alpha_2] \\
&= \hat{\alpha_2}([f])
\end{aligned}$$

Thus, $\hat{\alpha_1}$ depends only on the endpoints of the path.

($\impliedby$) Let $x_0 \in X$ and $[f], [g] \in \pi_1(X, x_0)$ be arbitary. Assume $\hat{f} = \hat{g}$, since f and g have the same endpoints (they are loops based at x_0). It suffices to show that $[f] = [\overline{g} \cdot f \cdot g]$. We compute:

$$\begin{aligned}
[f] &= [\overline{f} \cdot f \cdot f] \\
&= \hat{f}([f]) \\
&= \hat{g}([f]) \\
&= [\overline{g} \cdot f \cdot g]
\end{aligned}$$

Thus, $\pi_1(X, x_0)$ is abelian. Since x_0 was arbitrary, we have shown that $\pi_1(X)$ is abelian.

Problem 4. (1.1.6) Note that a loop based at x_0 can be regarded as a continuous map of pointed spaces

$$(S^1, (1,0)) \to (X, x_0).$$

Let $[S^1, X]$ be the set of homotopy classes of maps from S^1 to X without conditions on basepoints. Then, there is a map $\Phi : \pi_1(X, x_0) \to [S^1, X]$ that forgets the base points.

(a) Show that $\Phi([f]) = \Phi([g])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$.

(b) Deduce that if X is path connected, then $[S^1, X]$ is in bijection with conjugacy classes of $\pi_1(X, x_0)$.

Solution.

Problem 5. Suppose that $p : E \to B$ is a covering map where B is connected. Show that if $p^{-1}(b_0)$ has k elements for some $b_0 \in B$, then $p^{-1}(b)$ has k elements for every $b \in B$.

Solution.

Problem 6. Let $q : X \to Y$ and $r : Y \to Z$ be covering maps such that r has finite degree. Show that $p = r \circ q$ is a covering map.

Solution.