Math 74: Algebraic Topology

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Problem 1.

Def. 1. *A* topological group *G is a group with a topology such that the maps $G \times G \to G$ given $(g, h) \mapsto g * h$ where $*$ is the group operation and $G \to G$ given by $g \mapsto g^{-1}$ are continuous.*

Let G be a topological group with identity element x_0.

1. Let C be the connected component of G containing the identity element x_0. Show that C is a normal subgroup and every other connected component is homemorphic to C. (*Hint:* Show that if $g \in G$, then gC is the component of G containing g.)

2. Let $\Omega(G, x_0)$ be the set of all loops in G based at x_0. Define an operation $f \times g$ for $f, g \in \Omega(G, x_0)$ by
$$(f \times g)(s) = f(s) * g(s).$$
Show that the operation $\times$ makes $\Omega(G, x_0)$ into a group.

3. Show that $\times$ induces a group operation on $\pi_1(G, x_0)$.

4. Show that $\times$ agrees with the usual concatenation group operation $\cdot$ on $\pi_1(G, x_0)$. (*Hint:* Compute $(f \cdot e_{x_0}) * (e_{x_0} \cdot g)$.)

5. Show that $\pi_1(G, x_0)$ is an abelian group.

6. Let A be a (possibly empty) finite set of points in $\mathbb{R}^2$. For what values of $|A|$ can $\mathbb{R}^2 \setminus A$ be given the structure of a topological group?

Problem 2.(50 points)

Def. 2. *The* real Grassmannian $\mathrm{Gr}(k, n)$ *is the space of k-dimensional subspaces of $\mathbb{R}^n$. More precisely, let $M^*(k, n)$ be the set of $k \times n$ matrices of rank k with the subspace topology in the space of all $k \times n$ matrices $M(k, n) \cong \mathbb{R}^{kn}$. Then, we define* $\mathrm{Gr}(k, n) = M^*(k, n)/\sim$ *where $A \sim B$ if and only if A and B have the same row space.*

Note that $\mathrm{Gr}(1, n) = \mathbb{RP}^{n-1}$.

1. Show that $\mathrm{Gr}(k, n)$ is homeomorphic to $\mathrm{Gr}(n-k, n)$ for all $0 \leq k \leq n$.
2. Show that $\mathrm{Gr}(2, 4)$ is homeomorphic to $S^2 \times S^2/\sim$ where $(x, y) \sim (-x, -y)$ for all $(x, y) \in S^2 \times S^2$.
3. Describe a CW complex structure on $\mathrm{Gr}(2, 4)$. (*Hint:* You may wish to use the previous part.)
4. Compute $\pi_1(\mathrm{Gr}(2, 4))$.
5. Compute the homology of $\mathrm{Gr}(2, 4)$.

Problem 3.(50 points)

Def. 3. *Let $\mathcal{C}, \mathcal{D}$ be (locally small) categories and let $F, G : \mathcal{C} \to \mathcal{D}$ be functors.*

- *A* natural transformation *η from F to G is the data of a morphism $\eta_X \in \hom_{\mathcal{D}}(F(X), G(X))$ for every object X in $\mathcal{C}$ such that for every $f \in \hom_{\mathcal{C}}(X, Y)$, $\eta_Y \circ F(f) = G(f) \circ \eta_X$.*
- *A* natural isomorphism *η from F to G is a natural transformation such that η_X is an isomorphism for every object X of $\mathcal{C}$. If such an η exists, the functors F and G are said to be* naturally isomorphic.
- *The functor $F : \mathcal{C} \to \mathcal{D}$ is an* equivalence of categories *if there is a functor $F' : \mathcal{D} \to \mathcal{C}$ such that $F' \circ F$ is naturally isomorphic to the identity functor on $\mathcal{C}$ and $F \circ F'$ is naturally isomorphic to the identity functor on $\mathcal{D}$. If such a functor exists, $\mathcal{C}$ and $\mathcal{D}$ are said to be equivalent.*

For a topological space X, let $\Pi(X)$ be the fundamental groupoid of X.

1. Show that if $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories and X and Y are objects of $\mathcal{C}$ such that $F(X)$ and $F(Y)$ are isomorphic, then X and Y are isomorphic. Deduce that the forgetful functor $F : \text{Top} \to \text{Sets}$ and the fundamental group functor $\pi_1 : \text{Top} \to \text{Groups}$ are not equivalences of categories.
2. Show that if X is path connected, the inclusion $\pi_1(X, x) = \hom_{\Pi(X)}(x, x) \to \Pi(X)$ is an equivalence of categories for every $x \in X$.
3. Show that any continuous function $f : X \to Y$ induces a functor $\Pi(f) : \Pi(X) \to \Pi(Y)$.
4. Show that if $f : X \to Y$ and $g : X \to Y$ are continuous and homotopic, then the functors $\Pi(f)$ and $\Pi(g)$ are naturally isomorphic.
5. Deduce that if X and Y are homotopy equivalent, then $\Pi(X)$ and $\Pi(Y)$ are equivalent. Use this to reprove the fact from class that if $f : X \to Y$ is a homotopy equivalence, then $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is an isomorphism.
6. Give an example of homotopic continuous functions f and g such that $\Pi(f)$ and $\Pi(g)$ are not equal.

Problem 4.(50 points)

Def. 4. *Given an open cover* $\mathcal{U} = \{U_i\}_{i\in I}$ *of a topological space* X*, we define the Čech complex as*

$$\check{C}^k(X,\mathcal{U}) = \prod_{(i_0,\ldots,i_k)\in I^{k+1}} \mathcal{F}(U_{i_0}\cap\ldots\cap U_{i_k})$$

where $\mathcal{F}(Y) = \{f\colon Y\to\mathbb{Z} : f \text{ is continuous}\}$ *where* $\mathbb{Z}$ *has the discrete topology. For* $f\in\check{C}^k(X,\mathcal{U})$*, denote by* $f_{i_0,\ldots,i_k}$ *the component in* $\mathcal{F}(U_{i_0}\cap\ldots\cap U_{i_k})$*. Then, define*

$$d\colon \check{C}^k(X,\mathcal{U})\to\check{C}^{k+1}(X,\mathcal{U})$$

by

$$(df)_{i_0,\ldots,i_{k+1}} = \sum_{j=0}^{k+1}(-1)^j f_{i_0,\ldots,\hat{i_j},\ldots i_k}|_{U_{i_0}\cap\ldots\cap U_{i_k}}$$

1. Verify that the Čech complex is a cochain complex for any $X,\mathcal{U}$.
2. Denoting the cohomology groups of the Čech complex by $\check{H}^k(X,\mathcal{U})$, describe $\check{H}^0(X,\mathcal{U})$ for any $X,\mathcal{U}$.
3. Find two covers of S^1 whose Čech cohomology groups do not coincide.
4. A cover $\mathcal{V} = \{V_j\}_{j\in J}$ is a *refinement* of $\mathcal{U} = \{U_i\}_{i\in I}$ if for every $j\in J$, there is an $i\in I$ such that $V_j\subseteq U_i$. Show that if $\mathcal{V}$ is a refinement of $\mathcal{U}$, then there is a chain map $\check{C}^\bullet(X,\mathcal{U})\to\check{C}^\bullet(X,\mathcal{V})$ given by restriction.

We say that $\mathcal{U}$ is a *good cover* if every non-empty intersection $U_{i_0}\cap\ldots\cap U_{i_k}$ is contractible. Here is a theorem that can be proved using a similar argument that we deployed for barycentric subdivision in singular homology once one proves that the Čech cohomology of a contractible space with respect to any cover agrees with the Čech cohomology of point, e.g., by proving homotopy invariance.

Theorem 0.1. *If* $\mathcal{U}$ *is a good cover of* X *and the* $\mathcal{V}$ *is a refinement of* $\mathcal{U}$*, the chain map from the previous part induces an isomorphism on Čech cohomology. Moreover,* $\check{H}^k(X,\mathcal{U})\cong H^k(X)$ *for all* k*.*

5. Find good covers and explicitly compute the cohomology of a sphere S^n and an orientable genus g surface M_g using the previous theorem.
   (*Remark/challenge:* These computations give you explicit generators. There is a natural way to phrase the cup product in Čech cohomology – can you compute it in these examples? To be clear, I'm not grading you on this remark, but just leaving something to think about.)