Math 113: Functional Analysis

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Defn. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of (X, ρ). We say that $d > 0$ is a Lebesgue number for the cover if given any d-ball $B_d(x_0)$ with $x_0 \in X$, there exists $a_0 \in A$ such that $B_d(x_0) \subseteq U_{a_0}$.

Ex $X = \mathbb{R}$. $U_1 = (-\infty, 1)$, $U_2 = (0, 2)$, and $U_3 = (1, \infty)$. Here $d = 1/2$ is a LN for $\{U_1, U_2, U_3\}$. This is clear if $x_0 \in (1/2, 3/2)$.

Ex. (Hwk.) Given $x \in (0, 1)$, $\exists \delta_x > 0$ such that:

$$y \in B_{\delta_x}(x) = \{y \in (0,1) : |y - x| \leq \delta_x\}$$
$$\implies |1/x - 1/y| < 1$$

Then,

$$(0,1) = \bigcup_{x \in (0,1)} B_{\delta_x}(x)$$

has no Lebesgue number.

Lemma (Lebesgue Covering Lemma) Every open cover of a compact metric space has a Lebesgue number.
Proof. Pictures. Apr 9.

Thm. Suppose (X, ρ) is compact, and $F : (X, \rho) \to (Y, \sigma)$ is continuous. Then F is uniformly continuous.

Proof. Let $\epsilon > 0$. We need to find $\delta > 0$ such that $\forall x, y \in X$:

$$\rho(x, y) < \delta \implies \sigma(F(x), F(y)) < \epsilon$$

Since F is continous, $\forall z \in X$, $\exists \delta_z > 0$ such that:

$$\rho(x, z) < \delta_z \implies \sigma(F(x), F(z)) < \epsilon/2$$

That is,

$$F(B_{\delta_z}(z)) \subseteq B_{\epsilon/2}(F(z))$$

Let $\delta > 0$ be a Lebesgue number for the cover $\{B_{\delta_z}(z)\}_{z\in X}$.
Now supposed $\rho(x, y) < \delta$. Then $\exists z \in X$ such that:

$$B_\delta(x) \subseteq B_{\delta_z}(z)$$

and

$$\sigma(F(x), F(y)) \leq \sigma(F(x), F(y)) + \sigma(F(z), F(y)) < \epsilon/2 + \epsilon/2 = \epsilon$$

Defn. Let (X, ρ) be a metric space and $C(X)$ the $\mathbb{C}$-vector space of continous functions on X. We say $\mathcal{J} \subset C$ is equicontinous at $x \in X$ if $\forall \epsilon > 0, \exists \delta > 0$ such that:

$$\forall F \in \mathcal{J}, F(B_\delta(x)) \subseteq B_\epsilon(F(x))$$

We say $\mathcal{J}$ is equicontinous on X if $\forall x \in X$, $\mathcal{J}$ is equicontinous at x.

Ex. Let $X = [0, 1] \subset \mathbb{R}$. Let $F_n(x) = x^n \forall n \geq 1$. Let:

$$\mathcal{J} = \{F_n : n \in \mathbb{N}\}$$

Let $x_n = \frac{1}{2}^{1/n}$. Then, x_n arrow up to 1. Then,

$$|F_n(x_n) - F_n(1)| = |1/2 - 1| = 1/2$$

Thus, $\mathcal{J}$ is not equicontinous at 1.

Ex. (Hwk) Show that $\mathcal{J}$ is equicontinous on $[0, 1)$.

Defn. Let (F_n) be a sequence of ($\mathbb{C}$-valued) functions on X. Then, (F_n) is uniformly bounded if $\exists M > 0$ such that $\forall n \geq 1, \forall x \in X$:

$$|F_n(x)| < M$$

We say that (F_n) is pointwise bounded if $\forall x \in X$, $\exists M_x > 0$ such that:

$$|F_n(x)| < M_x$$

Defn. A metric space (top. space) is seperable if there is a countable dense subset $D \subset X$.

Ex. Since $\mathbb{Q}^n \subset \mathbb{R}^n$ is dense, $(\mathbb{R}^n, ||\cdot||_p)$ is separable.

Lemma. (Arzelà-Ascoli) Let (X, ρ) is a seperable metric space and that (F_n) is pointwise bounded and equicontinous in $C(X)$. Then, there is subsequence (F_{n_k}) such that:

$$\lim_{x\to\infty} F_{n_k}(x)$$

exists $\forall x \in X$.

Yap. Given a sequence (x_n), we get subsequence by finding $n_k \in \mathbb{N}$ such that $n_{k+1} > n_k$ and $(x_{n_k})_{k=1}^{\infty} \to x$ is a sequence.
A subsubsequence is determined by finding $n_{k_1} < n_{k_2} < \cdots$ and then we write:

$$(x_{n_{k_j}})_{j=1}^{\infty}$$

A subsequence is determined by an infinite subset $S_1 = \{n_1 \leq n_2 \leq \cdots\} \subset \mathbb{N}$. A subsubsequence is determined by an infinite subset $S_2 \subset S_1$,

$$S_2 = \{n_{k_1} < n_{k_2} < \cdots\} \subset S_1$$

Now, we write:

$$\lim_{n \in S_1} x_n = a \text{ instead of } \lim_{k \to \infty} x_{n_k}$$

Note that $\lim_{n \in S_1} x_n = a$ if:

$$\forall \epsilon > 0 \, \exists N : n \geq N, n \in S' \implies |x_n - a| < \epsilon$$

Rmk. Suppose $S_1 \subset \mathbb{N}$ determines a subsequence as above. Suppose $S' \subset \mathbb{N}$ is infinite and:

$$\{n \in S' : n \notin S_1\}$$

is finite. The $\lim_{n \in S_1} x_n = a$ then $\lim_{n \in S'} x_n = a$ as well.
Proof of the AA Lemma: Pictures. Apr 9 and 10.

Rmk. If X is compact, then $C(x) = C_b(X)$ is a complete metric space with respect to the uniform norm $|| \cdot ||_{\infty}$.

Thm. (Arzelà-Ascoli) Let (X, ρ) be a compact metric space and $(F_n) \subset C(X)$ be a sequence of functions that are point-wise bounded and equicontinous. Then (F_n) has a subsequence converging uniformly to some function $F \in C(X)$. Proof. Pictures.

Lemma. Suppose X is compact and that $\mathcal{J} \subset C(X)$ is equicontinous on X. Then, $\mathcal{J}$ is uniformly equicontinous on X, in that for all $\epsilon > 0 \exists \delta > 0$ such that for all $x, y \in X$ and all $F \in \mathcal{J}$,

$$\rho(x, y) < \delta \implies |F(x) - F(y)| < \epsilon$$

Rewriting,

$$F(B_\delta(x)) \subseteq B_\epsilon(F(x))$$

Proof left as homework.

Corr. Let X be a compact metric space. Let $\mathcal{J} \subset C(X)$ be a closed subset such that $\mathcal{J}$ is equicontinous and pointwise bounded. Then $\mathcal{J}$ is compact and uniformly bounded.

Thm. Suppose X is a compact metric space. Then $\mathcal{J} \subset C(X)$ is compact if and only if $\mathcal{J}$ is closed, uniformly bounded, and equicontinous on X. Proof. Pictures.

Defn. A topological space is called a Baire space if the countable intersection of dense open sets is dense.

Rmk. If ρ and σ are equivalent metrics on X then (X, ρ) is a Baire space if and only if (X, σ) is a Baire space.

Defn. If $S \subset X$ then the interior of S is:

$$Int(S) = \bigcup \{U \subset S : U \text{ is open in } X\}$$

Rmk. One can write $Int_X(S)$ as interior of S in X. Where you take the interior matters.

Lemma. A space X is a Baire space if and only if given a countable $\{F_n\}_{n=1}^{\infty}$ of closed sets, such that:

$$\bigcup_{n=1}^{\infty} F_n$$

has non-empty interior, then at least one F_n has (non-empty) interior.

Hwk. Write $O_n = F_n^C = X \setminus F_n$. O_n dense $\iff$ $Int(F_n) = \emptyset$.

Thm. Barise Category Thm. Every complete metric space is a Baire space.

Rmk. Note that $(0, 1)$ is homeomorphic to $\mathbb{R}$. Hence, $(0, 1)$ is a Baire space.

Hwk. More generally, every non-empty open subset V of a complete metric space admits an equivalent complete metric (Hwk). Hence V is a Baire space. Thus, if:

$$V = \bigcup_{n=1}^{\infty} V \cap F_n$$

of each F_n closed in X, thus $V \cap F_n$ is closed in V. Hence, at least one $V \cap F_n$ has non-empty interior in V. But V is open, so it has interior in X.

Thm. Suppose that X is a Baire space and that $(F_n) \subset C(X)$ such that $F_n \to F$ pointwise. Then,

$$A = \{x \in X : F \text{ is cts at } x\}$$

is dense in X.
Proof. Pictures.

Rmk. If $X = \mathbb{R}$, then we can assume:

$$A = \{x \in X : F \text{ is cts at } x\}$$

is uncountable.

Defn. A normed vector space $(V, ||\cdot||)$ is called a Banach space if V is complete in in the induced metric. Recall that $\mathbb{F}$ is always $\mathbb{C}$ or $\mathbb{R}$.

Yap. Normed vector spaces are special.

1. $B_r(x) = x + B_r(0)$. The topology is homogenous.
2. $|||v|| - ||w||| \leq ||v - w|| \implies v \to ||v||$ is continous.
3. $\overline{B_r(v)} = \{x \in V : ||x - v|| \leq r\}$.
4. $\epsilon B_r(0) = B_{\epsilon r}(0)$.
5. $\epsilon \overline{B_r(0)} = \overline{B_{\epsilon r}(0)}$.

Thm. Supposed that X and Y are normed vector spaces and $T : X \to Y$ is linear. Then, the following are equivalent:

1. T is continuous.
2. T is continous at a single point.
3. $\exists \alpha \geq 0$ such that $||T(x)|| \leq \alpha ||x||$ for all $v \in X$.

Proof.
(1) $\implies$ (2) is trivial.
(2) $\implies$ (3). Since T is continous at x_0, $\exists \delta > 0$ such that:

$$T(\overline{B_\delta(x_0)}) \subset B_1(T(x_0))$$

LHS $= T(B_\delta(x_0)) + T(x_0)$.
RHS $= B_1(0) + T(x)$.
This implies, $T(\overline{B_\delta(0)}) \subset B_1(0)$.
Now if $z \neq 0$,

$$||T(z)|| = ||\frac{||z||}{\delta} T(\delta \cdot \frac{z}{||z||})|| \leq \frac{||z||}{\delta}$$

Let $\alpha = \frac{1}{\delta}$.
(3) $\implies$ (1). Erased.

Rmk. Suppose that $||\cdot||_1$ and $||\cdot||_2$ (two different norms) induce equivalent metrics on V. Then, take the id : $(V, ||\cdot||_1) \to (V, ||\cdot||_2)$ is continous (same topology). Hence, $\exists c \geq 0$ such that:

$$||x_2|| \leq c \cdot ||x_1||$$

Clearly, that means $c > 0$. By symmetry, there exists a $d > 0$, such that:

$$||x_1|| \leq d \cdot ||x_2||$$

Thus, the metrics and the norms are strongly equivalent.

Defn. If X and Y are normed vector spaces, then $\mathcal{L}(X,Y)$ is the vector space of continous linear maps $T : X \to Y$. Define $||T|| = \sup_{||x||\leq 1} ||T(x)||$. If $X = Y$, then we write $\mathcal{L}(X)$ instead of $\mathcal{L}(X,X)$.

Lem. With $||T||$ as above, $\mathcal{L}(X,Y)$ is a normed vector space with:

$$||T(x)|| \leq ||T||||x|| \forall x \in X$$

If $S \in \mathcal{L}(Y,Z)$, then we write ST in place of $S \circ T$ and:

$$||ST|| \leq ||S||||T||$$

Defn. An algebra over $\mathbb{F}$ is a vector space A over $\mathbb{F}$ with a ring structure, with $\lambda(xy) = (\lambda x)y = x(\lambda y)$ for all $\lambda \in \mathbb{F}$ and $x, y \in A$.

Ex. $M_n(\mathbb{F})$, $\mathbb{R}[x]$, $\mathbb{C}[x]$, $C(X)$.

Defn. If $||\cdot||$ is a norm on an algebra A, then we call $(A, ||\cdot||)$ a normed algebra if $\forall x, y \in A$:

$$||xy|| \leq ||x||||y||$$

We call $(A, ||\cdot||)$ a Banach algebra if $(A, ||\cdot||)$ is a normed algebra and A is complete with respect to $||\cdot||$.

Prop. If X and Y are normed vector spaces, and Y is a Banach space, then $\mathcal{L}(X,Y)$ is a Banach space. If X is a Banach space, then $\mathcal{L}(X)$ is a Banach algebra.
Proof. Suppose that (T_n) is a Cauchy sequence in $\mathcal{L} :(X,Y)$. Then, for each $z \in X$, $T_n(X)$ is a Cauchy sequence in Y. Hebcem $\exists T(x) \in Y$ such that:

$$T_n(X) \to T(X)$$

It is not hard to see that $T : X \to Y$ is linear.
Since (T_n) is Cauchy in norm, its bounded i.e. $\exists m > 0$ such that:

$$||T_n|| \leq M \forall n \geq 1$$

Now if $||x|| \leq 1$, then:

$$||T(x)|| = \lim_{n\to\infty} ||T_n(x)|| \leq \text{limsup}_n \, ||T_n||||x|| \leq M||x||$$

This implies $T \in \mathcal{L}(X,Y)$.
Let $\epsilon > 0$. Let N be such that $m, n \geq N$. Thus,

$$||T_n - T_m|| \leq \frac{\epsilon}{2}$$

Now if $||x|| \leq 1$, and if $n \geq N$,

$$\begin{aligned} ||(T - T_n)(x)|| &= ||T(x) - T_n(x)|| \\ &= \lim_{m \to \infty} ||T_m(x) - T_n(x)|| \\ &\leq \text{limsup}_m \, ||T_m - T_n|| \cdot ||x|| \\ &\leq \frac{\epsilon}{2} < \epsilon \end{aligned}$$

Yap. Let $Y \subset X$ be a subsoace of a normed vector space X. Then we can form the quotient vector space $X/Y = \{x + Y : x \in X\}$ with $q : X \to X/Y$ the quotient map.
If $x \in X$, then:

$$\inf\{||x - y|| : y \in Y\}$$

depends only on $q(x)$. We call:

$$||q(x)|| := \inf\{||x - y|| : y \in Y\}$$

the quotient norm on X/Y.

Rmk. As Y is a subspace,

$$\begin{aligned} ||q(x)|| &= \inf\{||x + y|| : y \in Y\} \\ &= \inf\{||x + \alpha y|| : y \in Y\} \, \forall \alpha \in \mathbb{F} \setminus \{0\} \end{aligned}$$

Thm. If Y be a subspace of $(X, || \cdot ||)$. Then,

$$||q(x)|| = \inf\{||x - y|| : y \in Y\}$$

is a seminorm on X/Y which is a norm exactly when Y is closed. If X is a Banach space, and Y is closed in X, then X/Y is a Banach space.
Proof. Note that for $\alpha \neq 0$:

$$||\alpha q(x)|| = ||q(\alpha x)|| = \inf\{||\alpha x + \alpha y|| : y \in Y\} = |\alpha| \inf\{||x + y|| : y \in Y\} = |\alpha| ||q(x)||$$

Next, fix $x_1, x_2 \in X$. Then given $\epsilon > 0$, $\exists y_1, y_2 \in Y$ such that:

$$\begin{aligned} ||q(x_1)|| + ||q(x_2)|| + \epsilon &\geq ||x_1 - y_1|| + ||x_2 - y_2|| \\ &\geq ||x_1 - y_1 + x_2 - y_2|| \\ &\geq ||q(x_1 + x_2)|| \\ &= ||q(x_1)|| + ||q(x_2)|| \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $||q(x_1) + q(x_2)|| \leq ||q(x_1)|| + ||q(x_2)||$. Thus, it is a semi-norm. Sketch for closed. If quotient norm is 0, there exists a sequence y_n converging to x. If

Y closed, $x \in Y$ and $g(x) = 0$. Conversely, if quotient norm is a norm and $y_n \to x$. Then, $||q(x)|| = 0$. Then, $q(x) = 0$ and $x \in X$.
Now suppose X is a Banach space and Y is closed in X. Let $(q(z_0))$ be Cauchy in X/Y. Then we can pass to a subsequence and assume:

$$||q(z_{n+1}) - q(z_0)|| \le 1/2^n$$

Let $x_1 = z_1$. Since:

$$||q(z_2) - q(x_1)|| < 1/2$$

Thus,

$$\inf\{||z_2 - x_1 - y|| : y \in Y\} < 1/2$$

Hence, we can find x_2 such that $q(x_1) = q(z_2)$ and

$$||x_2 - x_1|| < 1/2$$

Continuing, we get (x_n) such that $q(x_n) = q(z_n)$ and:

$$||x_{n+1} - x_n|| < 1/2^n$$

(Hwk) (x_n) is Cauchy, thus, $x_n \to x$ in X.
But $||q(z)|| \le ||z||$. Hence, q is continous. Thus,

$$q(z_n) = q(x_n) \to q(x)$$

Rmk. Note that $q : X \to X/Y$ is a bounded linear map of norm at most 1.

Thm. Suppose that Y is a closed subspace of $(X, || \cdot ||)$. Then X is a Banach space if and only if both Y and X/Y are Banach spaces.
Proof. Pictures. 4/16

Thm. Every finite-dimensinal subspace Y of a normed vector space X is a Banach space and hence closed in X. If $\dim(Y) = n$, then every linear isomorphism $\Phi : \mathbb{F}^n \to Y$ is a homeomorphism.
Proof. Pictures. 4/16.

Corr. If Y is finite-dimensional vector space, then all norms $|| \cdot ||$ are strongly equivalent.
Proof. $\mathrm{id} : (Y, || \cdot ||_1) \to (Y, || \cdot ||_2)$ is a homeomorphism and continous linear maps are bounded.

Recall. Recall the topological definition of continous functions and open maps. If $F : X \to Y$ is a bijection, then $F^{-1} : Y \to X$ is continous if and only if F is open.

Thm. (The Big Three) (Open Mapping Theorem) Suppose that X and Y are Banach spaces and that $T \in \mathcal{L}(X, Y)$ is a surjection. Then T is an open map.

Lem. It will suffice to find $r > 0$ such that:

$$B_r(0) \subseteq T(B_1(0))$$

Proof. By homogeneity, $T(B_\delta(0))$ is a neighborhood of 0_Y for all $\delta > 0$. By linearity, $T(B_\delta(x))$ is a neighborhood of $T(x)$ for all $x \in X$ and $\delta > 0$. Thus, if V is open in X and $x \in V$, then $\exists \delta > 0$ such that:

$$B_\delta(x) \subseteq V$$

Then, $T(B_\delta(x))$ is a neighborhood of $T(x)$ in $T(Y)$. Thus, T is an open map. Thus, $T(V)$ is open.

Lem. It will suffice to find $r > 0$ such that:

$$B_r(0) \subseteq \overline{T(B_1(0))}$$

Proof. Assume $r > 0$ is such that the property holds. Let $y \in B_r(0)$. Then,

$$\exists y_1 \in T(B_1(0)) : ||y - y_1|| < r/2$$

Then, $y - y_1 \in B_{r/2}(0)$.
Hence, $\exists y_2 \in 1/2T(B_1(0)) = T(B_{1/2}(0))$ such that:

$$||y - y_1 - y_2|| < r/2^2$$

Continue to get a sequence (y_n) such that:

$$y_n \in 2^{-n+1}T(B_1(0)) = T(B_{2^{-n+1}}(0))$$

and:

$$||y - \sum_{i=1}^{n} y_i|| < 2^{-n}r$$

By construction, $\exists x_n \in X$ such that:

$$T(x_n) = y_n \qquad ||x_n|| < 2^{-n+1}$$

Since X is a Banach space, $x = \sum_{n=1}^{\infty} x_n$ converges and since T is continous (hence bounded), and also:

$$||x|| < \sum_{n=1}^{\infty} 2^{-n+1} = 2$$

and

$$T(x) = y$$

Since y was arbitrary, we have shown that:

$$B_r(0) \subset T(B_2(0))$$

This implies:

$$B_{r/2}(0) \subset T(B_1(0))$$

This suffices by lemma 1.

Lem. 3 If suffices to see that:

$$\overline{T(B_n(0))}$$

has interior for some $n \geq 1$.
Proof. By homogeneity, we can assume that $\overline{T(B_1(0))}$ has interior.
Thus, $\exists \epsilon > 0$ such that:

$$B_\epsilon(y) \subset \overline{T(B_1(0))}$$

Let $z \in B_\epsilon(0)$. Write

$$z = z/2 + y - (y - z/2) \in B_{\epsilon/2}(y) - B_{\epsilon/2}(y) \subseteq \overline{T(B_{1/2}(0))} - \overline{T(B_{1/2}(0))}$$

Rest in pictures.

Proof. Of the Open Mapping Theorem. Pictures. 4/18.

Ex. Let $1 \leq p < \infty$. Then, let:

$$l_0^p = \text{span}\{e_n : n \geq 1\} = \{x \in l^p : x(n) = 0 \text{ for all but finitely many } n\}$$

Notte l_0^p is dense in l^p.
Define: $T_0 : l_0^p \to l_0^p$ by:

$$T_0(e_n) = 1/n e_n$$

You can check that $||T|| = 1$. Also T_0 is a bijection.

$$T_0^{-1}(e_n) = n e_n$$

but $T_0^{-1} \notin \mathcal{L}(l_0^p)$

Thm. Suppose that X and Y are Banach spaces and $T \in \mathcal{L}(X, Y)$ is a bijection. Then $T^{-1} \in \mathcal{L}(Y, X)$.
Proof. T^{-1} is linear by general nonsense and T is open by the Open Mapping Theorem. Hence T^{-1} is continous hence bounded.

Yap. If X and Y are Banach spaces, then I can give $X \times Y$ a norm by:

$$||(x, y|| = max\{||x||, ||y||\})$$

This makes $X \times Y$ a Banach spaces.