Math 74: Algebraic Topology

Sair Shaikh

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Problem 1.

Definition 0.1. *A* topological group G *is a group with a topology such that the maps* $G \times G \to G$ *given* $(g, h) \mapsto g * h$ *where* $*$ *is the group operation and* $G \to G$ *given by* $g \mapsto g^{-1}$ *are continuous.*

Let G be a topological group with identity element x_0.

1. Let C be the connected component of G containing the identity element x_0. Show that C is a normal subgroup and every other connected component is homemorphic to C. (*Hint:* Show that if $g \in G$, then gC is the component of G containing g.)

2. Let $\Omega(G, x_0)$ be the set of all loops in G based at x_0. Define an operation $f \times g$ for $f, g \in \Omega(G, x_0)$ by
$$(f \times g)(s) = f(s) * g(s).$$
Show that the operation $\times$ makes $\Omega(G, x_0)$ into a group.

3. Show that $\times$ induces a group operation on $\pi_1(G, x_0)$.

4. Show that $\times$ agrees with the usual concatenation group operation $\cdot$ on $\pi_1(G, x_0)$. (*Hint:* Compute $(f \cdot e_{x_0}) * (e_{x_0} \cdot g)$.)

5. Show that $\pi_1(G, x_0)$ is an abelian group.

6. Let A be a (possibly empty) finite set of points in $\mathbb{R}^2$. For what values of $|A|$ can $\mathbb{R}^2 \setminus A$ be given the structure of a topological group?

Solution.

1. First, we show that C is a subgroup. Since C is connected, so is $C \times C \subset G \times G$. As $* : C \times C \to G$ is the restriction of a continuous map, it is continuous. Thus,

$*(C \times C) = \{g*h : g, h \in C\}$ is connected. Moreover, as $x_0 \in C$, $x_0 * x_0 = x_0 \in *(C \times C)$. Thus, as C is the largest connected subset containing x_0, we have that $*(C \times C) \subseteq C$. Thus, C is closed under multiplication. Similarly, as taking an inverse is continuous, we have that $C^{-1} = \{g^{-1} : g \in C\}$ is also connected and contains $x_0 = x_0^{-1}$. Thus, $C^{-1} \subseteq C$. Thus C is closed under taking inverses, and hence C is a subgroup.

Let $L_h : G \to G$ be the left multiplication map, given by $L_h(g) = h * g$. Since multiplication $* : G \times G \to G$ is continuous, and L_h is just its restriction, L_h is continuous for all $h \in G$. Moreover, we claim that for all $h \in G$, $L_{h^{-1}}$ is a continuous inverse of L_h. Indeed, we can check, for all $g \in G$:

$$\begin{aligned} L_{h^{-1}}(L_h(g)) &= L_{h^{-1}}(h * g) \\ &= h^{-1} * h * g \\ &= g \\ L_h(L_{h^{-1}}(g)) &= L_h(h^{-1} * g) \\ &= h * h^{-1} * g \\ &= g \end{aligned}$$

using the associativity of the group operation. Thus, L_h is a homeomorphism for all $h \in G$. The same argument shows that R_h, the right multiplication map, is also a homeomorphism for all $h \in G$.

Define $\phi_g : G \to G$ by letting $\phi_g(h) = L_g(R_{g^{-1}}(h)) = g * h * g^{-1}$ for all $g \in G$. Note that ϕ_g is a homeomorphism, since it is the composition of two homeomorphisms. It is easy to see that its inverse is $\phi_{g^{-1}}$. Then, note that $x_0 \in \phi_g(C)$ as:

$$\phi_g(x_0) = g * x_0 * g^{-1} = g * g^{-1} = x_0 \in \phi_g(C)$$

Similarly, $x_0 \in \phi_g^{-1}(C)$. Since $\phi_g(C)$ is connected, this implies $\phi_g(C) \subset C$. Similarly, since $\phi_g^{-1}(C)$ is connected, we have $\phi_g^{-1}(C) \subseteq C$. Thus, we have:

$$C = \phi_g(\phi_{g^{-1}}(C)) \subseteq \phi_g(C) \subseteq C$$

Thus, we must have equality everywhere, i.e. $\phi_g(C) = C$. Thus, C is normal.

Next, let $g \in G$ and C_g be the connected component of g. We also have:

$$L_g(C) = \{g * h : h \in C\} = gC$$

As L_g is a homeomorphism, gC is homeomorphic to C, hence connected. Moreover, as $x_0 \in C$, $g = g * x_0 \in gC$. Thus, $gC \subseteq C_g$.
Next, note that $L_{g^{-1}}(C_g) = g^{-1}C_g$ is a connected subset of G containing $x_0 = g^{-1} * g$

as $g \in C_g$. Thus, $g^{-1}C_g \subseteq C$.
Thus, we have:

$$C = g^{-1}gC \subseteq g^{-1}(C_g) \subseteq C$$

Thus, we must equality everywhere. Thus, $C = g^{-1}C_g$. Thus, applying L_g, we get:

$$gC = C_g$$

Thus, gC is the connected component of g in G. We know that $gC \cong C$ as L_g is a homeomorphism and $gC = L_g(C)$.

2. We need to check that $\Omega(G, x_0)$ is closed under $\times$, has an identity, inverses, and that $\times$ is associative.

   - Let $f, g \in \Omega(G, x_0)$. Then, $f \times g$ is a map $I \to G$, thus a path. It is a continuous map as its a composition of continuous maps $*$ after $(f, g) : I \to G \times G$. Moreover,

     $$f \times g(0) = f(0) * g(0) = x_0 * x_0 = x_0$$

     and

     $$f \times g(1) = f(1) * g(1) = x_0 * x_0 = x_0$$

     Thus, $f \times g$ is a loop based at x_0. Thus, $f \times g \in \Omega(G, x_0)$.
   - The identity element is the constant loop $e_{x_0} : I \to G$ given by $e_{x_0}(t) = x_0$ for all $t \in I$. Note that for any $f \in \Omega(G, x_0)$, we have:

     $$\begin{aligned}(f \times e_{x_0})(t) &= f(t) * e_{x_0}(t) \\ &= f(t) * x_0 \\ &= f(t)\end{aligned}$$

     and similarly,

     $$\begin{aligned}(e_{x_0} \times f)(t) &= e_{x_0}(t) * f(t) \\ &= x_0 * f(t) \\ &= f(t)\end{aligned}$$

     Thus, e_{x_0} is the identity element in $\Omega(G, x_0)$.
   - For $f \in \Omega(G, x_0)$, we define the inverse $f^{-1} \in \Omega(G, x_0)$ by $f^{-1}(t) = f(t)^{-1}$ for all $t \in I$, where the latter inverse is in G. Note that since $x_0^{-1} = x_0$, this still is a loop based at x_0. It is also continuous as the inverse map is continuous on G. Then, we have:

     $$\begin{aligned}(f \times f^{-1})(t) &= f(t) * f^{-1}(t) \\ &= f(t) * f(t)^{-1} \\ &= e_{x_0}\end{aligned}$$

and similarly,

$$\begin{aligned}(f^{-1} \times f)(t) &= f^{-1}(t) * f(t) \\ &= f(t)^{-1} * f(t) \\ &= e_{x_0}\end{aligned}$$

Thus, f^{-1} is the inverse of f in $\Omega(G, x_0)$.

- The associativity of $\times$ follows from the associativity of the group operation $*$ as follows. Let $f, g, h \in \Omega(G, x_0)$. Then, we have for all $t \in I$:

$$\begin{aligned}((f \times g) \times h)(t) &= (f \times g)(t) * h(t) \\ &= f(t) * g(t) * h(t) \\ &= f(t) * (g \times h)(t) \\ &= (f \times (g \times h))(t)\end{aligned}$$

  Thus, $\times$ is associative.

We have shown that $\Omega(G, x_0)$ is a group under the operation $\times$.

3. We define the map $[\cdot] : \Omega(G, x_0) \to \pi_1(G, x_0)$ by sending $f \in \Omega(G, x_0)$ to the homotopy class $[f] \in \pi_1(G, x_0)$. This is a quotient map by the homotopy equivalence relation. We define the map $\times$ on $\pi_1(G, x_0)$ by $[f] \times [g] = [f \times g]$. We need to check that this is well-defined.
Let $f \cong f'$ and $g \cong g'$ be two pairs of homotopic loops in $\Omega(G, x_0)$. We need to show that $[f \times g] = [f' \times g']$. Let F and G be the homotopies from f to f' and g to g', respectively. We define the map $H : I \times I \to G$ by $H(t, s) = F(t, s) * G(t, s)$. Since F and G are continuous, $(F, G) : I \times I \to G \times G$ is also continuous. Thus, as $*$ is continuous, H is a composition of continuous, thus continuous. We check that H is a homotopy from $f \times g$ to $f' \times g'$.

- Setting $s = 0$, we have:

$$H(t, 0) = F(t, 0) * G(t, 0) = f(t) * g(t)$$

- Setting $s = 1$, we have:

$$H(t, 1) = F(t, 1) * G(t, 1) = f'(t) * g'(t)$$

- Setting $t = 0$, we have:

$$H(0, s) = F(0, s) * G(0, s) = x_0 * x_0 = x_0$$

- Setting $t = 1$, we have:

$$H(1, s) = F(1, s) * G(1, s) = x_0 * x_0 = x_0$$

Thus, H is a homotopy from $f \times g$ to $f' \times g'$. Thus, we have that $[f \times g] = [f' \times g']$. Thus, $\times$ is well-defined on $\pi_1(G, x_0)$.

The group axioms then follow from the group structure on $\Omega(G, x_0)$ ($[\cdot]$ commutes with $\times$).

4. For $[f], [g] \in \pi_1(G, x_0)$, we want to show that:
$$[f] \times [g] = [f] \cdot [g]$$
Note that $[f \cdot e_{x_0}] = [f]$ and $[e_{x_0} \cdot g] = [g]$ in $\pi_1(G, x_0)$. Thus, we have:
$$[f \times g] = [(f \cdot e_{x_0}) \times (e_{x_0} \cdot g)]$$
We evaluate this. For $t \in I$, we have:
$$\begin{aligned}(f \cdot e_{x_0}) \times (e_{x_0} \cdot g)(t) &= (f \cdot e_{x_0})(t) * (e_{x_0} \cdot g)(t) \\ &= \left\{\begin{matrix} f(2t) & t \le \frac{1}{2} \\ x_0 & t \ge \frac{1}{2} \end{matrix}\right\} * \left\{\begin{matrix} x_0 & t \le \frac{1}{2} \\ g(2t-1) & t \ge \frac{1}{2} \end{matrix}\right\} \\ &= \left\{\begin{matrix} f(2t) & t \le \frac{1}{2} \\ g(2t-1) & t \ge \frac{1}{2} \end{matrix}\right\} \\ &= f \cdot g\end{aligned}$$
Thus, we have:
$$[f \times g] = [(f \cdot e_{x_0}) \times (e_{x_0} \cdot g)] = [f \cdot g]$$
Thus,
$$[f] \times [g] = [f] \cdot [g]$$
and the two operations agree on $\pi_1(G, x_0)$.

5. Let $[f], [g] \in \pi_1(G, x_0)$ be arbitrary. We compute:
$$\begin{aligned}[f] \times [g] &= [(e_{x_0} \cdot f)] \times [(g \cdot e_{x_0})] \\ &= [(e_{x_0} \cdot f) \times (g \cdot e_{x_0})] \\ &= \left[t \to \left\{\begin{matrix} x_0 & t \le \frac{1}{2} \\ f(2t-1) & t \ge \frac{1}{2} \end{matrix}\right\} * \left\{\begin{matrix} g(2t) & t \le \frac{1}{2} \\ x_0 & t \ge \frac{1}{2} \end{matrix}\right\} \right] \\ &= \left[t \to \left\{\begin{matrix} g(2t) & t \le \frac{1}{2} \\ f(2t-1) & t \ge \frac{1}{2} \end{matrix}\right\} \right] \\ &= [g \cdot f] \\ &= [g] \times [f]\end{aligned}$$
As $[f], [g]$ were arbitrary, we have that $\pi_1(G, x_0)$ is abelian under both operations $\times$ and $\cdot$.

6. For $|A| = 0$, we have $\mathbb{R}^2 \setminus A = \mathbb{R}^2$, which has a trivial π_1. This is a topological group under the usual vector addition.

   For $|A| \geq 1$, we know that $\mathbb{R}^2 \setminus A$ retracts onto the wedge of $|A|$ circles, thus has π_1 isomorphic to the free group on $|A|$ generators. For $|A| > 1$, this is not abelian, thus $\mathbb{R}^2 \setminus A$ cannot be a topological group. For $A = 1$, one gets a space homeomorphic to $\mathbb{R}^2 \setminus \{0\}$, which is homeomorphic to $\mathbb{C}^\times$ on which one can place the group operation of complex multiplication to get a topological group.

Problem 3. (50 points)

Definition 0.2. *Let $\mathcal{C}, \mathcal{D}$ be (locally small) categories and let $F, G : \mathcal{C} \to \mathcal{D}$ be functors.*

- *A* natural transformation *η from F to G is the data of a morphism $\eta_X \in \hom_{\mathcal{D}}(F(X), G(X))$ for every object X in $\mathcal{C}$ such that for every $f \in \hom_{\mathcal{C}}(X, Y)$, $\eta_Y \circ F(f) = G(f) \circ \eta_X$.*
- *A* natural isomorphism *η from F to G is a natural transformation such that η_X is an isomorphism for every object X of $\mathcal{C}$. If such an η exists, the functors F and G are said to be* naturally isomorphic.
- *The functor $F : \mathcal{C} \to \mathcal{D}$ is an* equivalence of categories *if there is a functor $F' : \mathcal{D} \to \mathcal{C}$ such that $F' \circ F$ is naturally isomorphic to the identity functor on $\mathcal{C}$ and $F \circ F'$ is naturally isomorphic to the identity functor on $\mathcal{D}$. If such a functor exists, $\mathcal{C}$ and $\mathcal{D}$ are said to be equivalent.*

For a topological space X, let $\Pi(X)$ be the fundamental groupoid of X.

1. Show that if $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories and X and Y are objects of $\mathcal{C}$ such that $F(X)$ and $F(Y)$ are isomorphic, then X and Y are isomorphic. Deduce that the forgetful functor $F : \text{Top} \to \text{Sets}$ and the fundamental group functor $\pi_1 : \text{Top} \to \text{Groups}$ are not equivalences of categories.
2. Show that if X is path connected, the inclusion $\pi_1(X, x) = \hom_{\Pi(X)}(x, x) \to \Pi(X)$ is an equivalence of categories for every $x \in X$.
3. Show that any continuous function $f : X \to Y$ induces a functor $\Pi(f) : \Pi(X) \to \Pi(Y)$.
4. Show that if $f : X \to Y$ and $g : X \to Y$ are continuous and homotopic, then the functors $\Pi(f)$ and $\Pi(g)$ are naturally isomorphic.
5. Deduce that if X and Y are homotopy equivalent, then $\Pi(X)$ and $\Pi(Y)$ are equivalent. Use this to reprove the fact from class that if $f : X \to Y$ is a homotopy equivalence, then $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is an isomorphism.
6. Give an example of homotopic continuous functions f and g such that $\Pi(f)$ and $\Pi(g)$ are not equal.

Solution.

1. Assume $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories and there exists an isomorphism $f \in \hom_{\mathcal{D}}(F(X), F(Y))$. Since F is an equivalence of categories, there exists functor

$F' : \mathcal{D} \to \mathcal{C}$ such that $F' \circ F$ is naturally isomorphic to $\mathrm{id}_{\mathcal{C}}$. Since F' is a functor, $F'(f) \in \hom_{\mathcal{C}}(F'F(X), F'F(Y))$ is an isomorphism. Moreover, we have isomorphisms $\eta_X \in \hom_{\mathcal{C}}(F'F(X), X)$ and $\eta_Y \in \hom_{\mathcal{C}}(F'F(Y), Y)$. Thus, we have an isomorphism:

$$\eta_Y \circ F'(f) \circ \eta_X^{-1} : X \to Y$$

We assumed Homework 1 Problem 4 in this proof.

Take $X = \mathbb{R}$ with the indiscrete topology (just $\emptyset$ and X are open) and $Y = \mathbb{R}$ with the discrete topology. The only continuous maps $X \to Y$ are constant (any non-empty preimage is all of X), which are not invertible. Thus, X and Y are not isomorphic. However, the forgetful functor maps them to the underlying sets $\mathbb{R}$ and $\mathbb{R}$, which are isomorphic as sets using the identity map.

Let $X = S^2$ and $Y = \mathbb{R}^2$ with their usual topologies. We know that these are not isomorphic since their H_2 groups are not isomorphic. However, applying π_1 functor, as both of these spaces are simply connected, we have $\pi_1(X) = \pi_1(Y) = \{0\}$.

Thus, in both cases, the property we just we do not have an equivalence of categories.

2. Let $x \in X$ be arbitrary. We define a functor $F' : \Pi(X) \to \pi_1(X, x)$ as follows. For $y \in \Pi(X)$, we define $F'(y) = x$, i.e. F' is the constant map with value x on objects. For $[\gamma] \in \hom_{\Pi(X)}(y, z)$, we define $F'([\gamma]) = [\alpha_y \cdot \gamma \cdot \alpha_z^{-1}]$, where α_y and α_z are some choice of paths from x to y and z, respectively with α_x being the identity on x. We check that this is a functor:

   - Let $[\gamma] \in \hom_{\Pi(X)}(y, z)$ and $[\delta] \in \hom_{\Pi(X)}(z, w)$. Then, we have:

   $$\begin{aligned} F'([\gamma]) \cdot F'([\delta]) &= [\alpha_y \cdot \gamma \cdot \alpha_z^{-1}] \cdot [\alpha_z \cdot \delta \cdot \alpha_w^{-1}] \\ &= [\alpha_y \cdot \gamma \cdot \delta \cdot \alpha_w^{-1}] \\ &= F'([\gamma \cdot \delta]) \end{aligned}$$

     Thus, F' preserves composition.

   - Let $[\mathrm{id}_y] \in \hom_{\Pi(X)}(y, y)$. Then, we have:

   $$F'([\mathrm{id}_y]) = [\alpha_y \cdot \mathrm{id}_y \cdot \alpha_y^{-1}] = [\alpha_y \cdot \alpha_y^{-1}] = [\mathrm{id}_x]$$

     Thus, F' preserves identities.

   Thus, F' is a functor. Now, we check that $F' \circ F \cong \mathrm{id}_{\pi_1(X,x)}$ and $F \circ F' \cong \mathrm{id}_{\Pi(X)}$.

First, we check $F' \circ F \cong \mathrm{id}_{\pi_1(X,x)}$. For $x \in \pi_1(X,x)$ (the only object), note that $F' \circ F(x) = F'(x) = x$. Thus, let $\eta_x = [\mathrm{id}_x] \in \hom_{\pi_1(X,x)}(F'F(x), x)$. This is clearly an isomorphism. Moreover, it clearly satisfies the naturality condition, since for any other morphism $[\gamma] \in \hom_{\pi_1(X,x)}(x,x)$ (noting x is the only object), we have:

$$\begin{aligned}\eta_x \circ F'F([\gamma]) &= F'F([\gamma]) \cdot \eta_x \\ &= F'([\gamma]) \cdot [\mathrm{id}_x] \\ &= [\mathrm{id}_x \cdot \gamma \cdot \mathrm{id}_x] \\ &= [\mathrm{id}_x] \cdot [\gamma] \\ &= \mathrm{id}_{\pi_1(X,x)}([\gamma]) \circ \eta_x\end{aligned}$$

Thus, $F' \circ F \cong \mathrm{id}_{\pi_1(X,x)}$.

Finally, we check $F \circ F' \cong \mathrm{id}_{\Pi(X)}$. For any object $y \in \Pi(X)$, note that:

$$F \circ F'(y) = F(x) = x$$

Thus, we let $\eta_y = [\alpha_y] \in \hom_{\Pi(X)}(FF'(y), y)$. This is clearly an isomorphism (everything in a groupoid is). Moreover, for any morphism $[\gamma] \in \hom_{\Pi(X)}(y,z)$, we have:

$$\begin{aligned}\eta_z \circ FF'([\gamma]) &= FF'([\gamma]) \cdot \eta_z \\ &= F([\alpha_y \cdot \gamma \cdot \alpha_z^{-1}]) \cdot [\alpha_z] \\ &= [\alpha_y \cdot \gamma \cdot \alpha_z^{-1}] \cdot [\alpha_z] \\ &= [\alpha_y \cdot \gamma] \\ &= [\gamma] \circ [\alpha_y] \\ &= \mathrm{id}_{\Pi(X)}([\gamma]) \circ \eta_y\end{aligned}$$

Thus, $F \circ F' \cong \mathrm{id}_{\Pi(X)}$. Thus, we have show than F is an equivalence of categories.

3. Define the functor $\Pi(f)$ by sending $x \in X$ to $\Pi(f)(x) = f(x)$ and $\gamma \in \hom_{\Pi(X)}(x,y)$ to $f_*([\gamma]) := [f \circ \gamma] \in \hom_{\Pi(Y)}(f(x), f(y))$ (the composition makes sense if you see γ as a map $I \to X$). Then, we check that this is a functor:

   - Let $\gamma \in \hom_{\Pi(X)}(x,y)$ and $\delta \in \hom_{\Pi(X)}(y,z)$. Then, we have:

     $$f_*([\gamma]) \cdot f_*([\delta]) = [(f \circ \gamma) \cdot (f \circ \delta)] = [f \circ (\gamma \cdot \delta)] = f_*([\gamma \cdot \delta])$$

     where the 2nd equality follows from noting that $f \circ (\gamma \cdot \delta)$ is doing $f \cdot \gamma$ first and then $f \cdot \delta$. Thus, f_* preserves composition.

   - Let id_x be the constant path on $x \in X$ in $\Pi(X)$. Then, we have:

     $$f_*([\mathrm{id}_x]) = [f \circ \mathrm{id}_x] = [\mathrm{id}_{f(x)}]$$

     Thus, f_* preserves identities.

Thus, $\Pi(f)$ is a functor.

4. Let $H : X \times I \to Y$ be a homotopy from f to g. For each $x \in X$, we let $\eta_x \in \hom_{\Pi(Y)}(f(x), g(x))$ be $\eta_x : I \to Y$ given by $\eta_x(t) = H(x,t)$. Since $\Pi(Y)$ is a groupoid, η_x is an isomorphism. Next, let $[\gamma] \in \hom_{\Pi(X)}(y,z)$ be a path in X. Then, we construct a path-homotopy that $H' : I \times I \to Y$ from $\mathrm{id}_{f(y)} \cdot \eta_y \cdot g_*([\gamma])$ to $f_*([\gamma]) \cdot \eta_z \cdot \mathrm{id}_{g(z)}$. We define H' by:

$$H'(t,s) = \begin{cases} H(\gamma(3ts), 0) & t \leq \frac{1}{3} \\ H(\gamma(s), 3t-1) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ H(\gamma(s + (3t-2)(1-s)), 1) & t \geq \frac{2}{3} \end{cases}$$

First, we check that H' is continuous. By the pasting lemma, we only need to check it is well-defined on the boundaries.

- For $t = \frac{1}{3}$:
$$H(\gamma(3ts), 0) = H(\gamma(s), 0) = H(\gamma(s), 3t-1)$$
- For $t = \frac{2}{3}$:
$$H(\gamma(s + (3t-2)(1-s)), 1) = H(\gamma(s), 1) = H(\gamma(s), 3t-1)$$

Finally, we check that H' is a path-homotopy as described:

- Noting that $\gamma(0) = y$, we have
$$\begin{aligned} H'(t,0) &= \begin{cases} H(y, 0) & t \leq \frac{1}{3} \\ H(y, 3t-1) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ H(\gamma(3t-2), 1) & t \geq \frac{2}{3} \end{cases} \\ &= \mathrm{id}_{f(y)} \cdot \eta_y \cdot g_*([\gamma]) \end{aligned}$$
- Noting that $\gamma(1) = z$, we have:
$$\begin{aligned} H'(t,1) &= \begin{cases} H(\gamma(3t), 0) & t \leq \frac{1}{3} \\ H(z, 3t-1) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ H(z, 1) & t \geq \frac{2}{3} \end{cases} \\ &= f_*([\gamma]) \cdot \eta_z \cdot \mathrm{id}_{g(z)} \end{aligned}$$
- $H'(0,s) = H(\gamma(0), 0) = f(\gamma(0)) = f(y)$
- $H'(1,s) = H(\gamma(s + (1-s)), 1) = g(\gamma(1)) = g(z)$

Thus, H' is a path-homotopy. Therefore, we have:

$$[f_*([\gamma]) \cdot \eta_z \cdot \mathrm{id}_{g(z)}] = [\mathrm{id}_{f(y)} \cdot \eta_y \cdot g_*([\gamma])]$$

Since the constant paths $\mathrm{id}_{\_}$ are the identity morphisms for their respective objects, we conclude:

$$[f_*([\gamma]) \cdot \eta_z] = [\eta_y \cdot g_*([\gamma])]$$

Thus, we conclude that $\Pi(f)$ and $\Pi(g)$ are naturally isomorphic.

5. If X and Y are homotopy equivalent, there exist maps $f : X \to Y$ to $g : Y \to X$ such that $f \circ g \cong \mathrm{id}_Y$ (homotopic). Thus, we have that the functor $\Pi(f \circ g)$ is naturally isomorphic to $\Pi(\mathrm{id}_Y)$. Next, we claim that $\Pi(f \circ g) = \Pi(f) \circ \Pi(g)$ and $\Pi(\mathrm{id}_Y) = \mathrm{id}_{\Pi(Y)}$. To see this, note:

   - On objects $y \in \Pi(Y)$, we have:

     $$\Pi(f) \circ \Pi(g)(y) = \Pi(f)(g(y)) = f \circ g(y) = \Pi(f \circ g)(y)$$

     and:

     $$\Pi(\mathrm{id}_Y)(y) = \mathrm{id}_Y(y) = y$$

   - On morphisms $[\gamma] \in \hom_{\Pi(Y)}(y, z)$, we have:

     $$\Pi(f) \circ \Pi(g)([\gamma]) = \Pi(f)([g \circ \gamma]) = [f \circ g \circ \gamma] = \Pi(f \circ g)([\gamma])$$

     and:

     $$\Pi(\mathrm{id}_Y)([\gamma]) = [\mathrm{id}_Y \circ \gamma] = [\gamma]$$

   Thus, we have that $\Pi(f) \circ \Pi(g)$ is naturally isomorphic to $\mathrm{id}_{\Pi(Y)}$. By the same argument, we have that $\Pi(g) \circ \Pi(f)$ is naturally isomorphic to $\mathrm{id}_{\Pi(X)}$. Thus, we have that $\Pi(X)$ and $\Pi(Y)$ are equivalent as categories.

   We can conclude the next part by showing that equivalence of categories is an equivalence relation (assuming functor composition is associative, not difficult. reflexivity and symmetry are immediate from the definition). However, we will do this more directly.

   Note that $\Pi(f)$ and $\Pi(g)$ induce maps $f_* : \mathrm{Aut}_{\Pi(X)}(x) = \pi_1(X, x) \to \mathrm{Aut}_{\Pi(Y)}(f(x)) = \pi_1(Y, f(x))$ and $g_* : \mathrm{Aut}_{\Pi(Y)}(f(x)) = \pi_1(Y, f(x)) \to \mathrm{Aut}_{\Pi(X)}(gf(x)) = \pi_1(X, gf(x))$. These are group homomorphisms as $\Pi(f)$ and $\Pi(g)$ are functors, thus preserve compositions and the identity.

Next, as we have a natural isomorphism from $\Pi(g \circ f)$ to $\mathrm{id}_{\Pi(X)}$, we have an isomorphism $\eta_x \in \hom_{\Pi(X)}(gf(x), x)$ such that for any $[\gamma] \in \mathrm{Aut}_{\Pi(X)}(x)$, we have:

$$\eta_x \circ g_* f_*([\gamma]) = [\gamma] \circ \eta_x$$

Thus, we have that:

$$g_* f_*([\gamma]) = \eta_x^{-1} \circ [\gamma] \circ \eta_x$$

We show that this is an isomorphism. First, assume $g_* f_*([\gamma]) = [\mathrm{id}_{gf(x)}]$. Then, we have:

$$\mathrm{id}_{gf(x)} = \eta_x^{-1} \circ [\gamma] \circ \eta_x$$

This yields:

$$[\gamma] = \eta_x \circ \mathrm{id}_{gf(x)} \circ \eta_x^{-1} = \eta_x \circ \eta_x^{-1} = \mathrm{id}_x$$

Thus, $g_* f_*$ is injective. Next, let $[\delta] \in \mathrm{Aut}_{\Pi(X)}(gf(x))$ be arbitrary. Then, we have:

$$\begin{aligned} g_* f_*(\eta_x \circ [\delta] \circ \eta_x^{-1}) &= \eta_x^{-1} \circ \eta_x \circ [\delta] \circ \eta_x^{-1} \circ \eta_x \\ &= [\delta] \end{aligned}$$

Thus, $g_* f_*$ is surjective. Thus, $g_* f_*$ is an isomorphism.

Using the same argument with the natural isomorphism from $\Pi(f \circ g)$ to $\mathrm{id}_{\Pi(Y)}$, we note that $f_* g_*$ is also an isomorphism.

Finally, since $g_* f_*$ is an isomorphism, we have that f_* is injective. Since $f_* g_*$ is also an isomorphism, we have that f_* is surjective as well. Thus, as f_* is a group homomorphism, we conclude that f_* is an isomorphism.

6. Consider $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$. Since $\mathbb{R}^n$ is contractible, id is homotopic to a constant map, call it λ. Then, notice that $\Pi(\mathrm{id}) = \mathrm{id}_{\Pi(\mathbb{R}^n)}$ which maps $\Pi(\mathbb{R}^n)$ to itself (which has one object for each point in $\mathbb{R}^n$). But $\Pi(\lambda)$ maps every object $x \in \Pi(\mathbb{R}^n)$ to the same object $\lambda(x) \in \Pi(\mathbb{R}^n)$. Thus, these maps are not equal.