

Problem 1. Prove the pasting lemma: Suppose $X = A \cup B$ is a topological space with A, B closed in X . If $f: X \rightarrow Y$ is a map such that the restrictions $f|_A$ and $f|_B$ are continuous, then f is continuous.

Solution. This follows from the definition of continuity. Let $U \subseteq Y$ be open. Then $f^{-1}(U) = (f|_A)^{-1}(U) \cup (f|_B)^{-1}(U)$. Since $f|_A$ and $f|_B$ are continuous, $(f|_A)^{-1}(U)$ and $(f|_B)^{-1}(U)$ are open in A and B , respectively. Because A and B are closed in X , these sets are also open in X . Hence, $f^{-1}(U)$ is open in X , and f is continuous.

Problem 2. In a connected space X , a point $x \in X$ is called a *cut point* if $X \setminus \{x\}$ is disconnected.

1. Suppose that $f : X \rightarrow Y$ is a homeomorphism of connected spaces. Show that $x \in X$ is a cut point if and only if $f(x) \in Y$ is a cut point.
2. Show that none of the spaces $(0, 1)$, $(0, 1]$, $[0, 1]$, and $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ are homeomorphic to each other.
3. Show that \mathbb{R} is not homeomorphic to \mathbb{R}^n for any $n \geq 2$.
4. The bouquet B_n of n circles is the space obtained by gluing n disjoint copies of S^1 at a single point in each circle. Show that B_n and B_m are not homeomorphic for $n \neq m$.

Solution. HELLO

Problem 3. Define \mathbb{RP}^n to be the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ by $(x_1, \dots, x_{n+1}) \simeq (ax_1, \dots, ax_{n+1})$ for all nonzero scalars a . Let $[x_1 : \dots : x_{n+1}] \in \mathbb{RP}^n$ denote the image of $(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \setminus \{0\}$ under the quotient map. Show that $i: \mathbb{R}^n \rightarrow \mathbb{RP}^n$ given by

$$i(x_1, \dots, x_n) = [1 : x_1 : \dots : x_n]$$

is a topological embedding (i.e., a homeomorphism onto its image) and the complement of $i(\mathbb{R}^n)$ is homeomorphic to \mathbb{RP}^{n-1} .

Solution.

Problem 4. Suppose that \mathcal{C} is a category, A, B, C are objects of \mathcal{C} , and $f \in \text{hom}(A, B)$ and $g \in \text{hom}(B, C)$ are isomorphisms.

1. Show that f has a unique inverse in $\text{hom}(B, A)$.
2. Show that id_A is an isomorphism.
3. Show that the inverse of f is an isomorphism.
4. Show that $g \circ f$ is an isomorphism.
5. Show that $\text{Aut}(A)$ and $\text{Aut}(B)$ are isomorphic groups.
6. Show that if $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor, then $F(f) \in \text{hom}_{\mathcal{D}}(F(A), F(B))$ is an isomorphism.

Solution.

Problem 5. For a space X , let $\pi_0(X)$ be the space of path components of X . Recall that the image of a path-connected space is path-connected. Thus, for a continuous map $f : X \rightarrow Y$, there is an induced map $\pi_0(f) : \pi_0(X) \rightarrow \pi_0(Y)$ taking a path component A to the path component containing $f(A)$. Show that this makes π_0 a functor from the category of topological spaces to the category of sets.

Solution.

Problem 6. (Munkres, *Topology*, 51.2) Given spaces X and Y , let $[X, Y]$ denote the set of homotopy classes of maps of X into Y .

1. Let $I = [0, 1]$. Show that for any X , the set $[X, I]$ has a single element.
2. Show that if Y is path connected, then $[I, Y]$ has a single element.

Solution.

Problem 7. (Munkres, *Topology*, 51.3) A space X is said to be *contractible* if the identity map $i_X : X \rightarrow X$ is nullhomotopic.

1. Show that I and \mathbb{R} are contractible.
2. Show that a contractible space is path connected.
3. Show that if Y is contractible, then for any X , the set $[X, Y]$ has a single element.
4. Show that if X is contractible and Y is path connected then $[X, Y]$ has a single element.