CS6846 – Quantum Algorithms and Cryptography

Simon's and Bernstein-Vazirani Algorithms

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Simon's Algorithm

Given a function $f : \{0,1\}^n \to \{0,1\}^n$ such that:

- f is two-to-one
- $\forall x, y : f(x) = f(y) \iff y = x \oplus s$ for some fixed $s \neq 0^n$

find the value of s.

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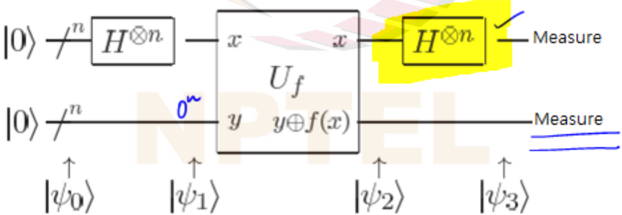


Simon's Algorithm

Given a function $f : \{0,1\}^n \to \{0,1\}^n$ such that (i) f is two-to-one, (ii) $\forall x, y : f(x) = f(y) \iff y = x \oplus s$ for some fixed $s \neq 0^n$, find the value of s.

Algorithm:

- Prepare a superposition $H^{\otimes n}(|0^n\rangle) = \frac{1}{2^{n/2}} \sum_x |x\rangle$.
- Apply the unitary function U_f to this, with ancillary 0^n qubits.
- Measure the last n registers in the computational basis. Discard.
- Apply the quantum fourier transform $H^{\otimes n}$ to first n registers.
- Measure this state. —— 1 output.
- Repeat above steps "many" times.

Simon's Algorithm: Analysis

1). First Hadamard gives

$$|\psi_1\rangle = \frac{1}{2^{n/2}} \sum_x |x\rangle \; |0\rangle^n.$$

$\hookrightarrow$ untouched.

$y = f(x_1) = f(x_2)$ iff $x_1 = x_2 \oplus s$.

2). Apply f.

$$|\psi_2\rangle = 2^{-n/2} \sum_x |x, f(x)\rangle.$$

Ignore normalizations.

3). Measure last n bits. Say I get y.

$$|\psi_3\rangle = \left(\frac{|x\rangle + |x \oplus s\rangle}{\sqrt{2}}\right)|y\rangle.$$

Simon's Algorithm: Analysis

4) Apply $H^{\otimes n}$ on $\dfrac{|x\rangle + |x \oplus s\rangle}{\sqrt{2}}$

$$= \sum_z 2^{-n/2 - \frac{1}{2}} (-1)^{\langle x; z \rangle} |z\rangle + \sum_z 2^{-n/2 - \frac{1}{2}} (-1)^{\langle x \oplus s; z \rangle} |z\rangle$$

$$= 2^{-n/2 - \frac{1}{2}} \underline{\underline{\sum_z}} (-1)^{\langle x; z \rangle} \left(1 + (-1)^{\langle s; z \rangle}\right) |z\rangle$$

$$= 2^{-n/2 - \frac{1}{2}} \sum_{z \perp s} (-1)^{\langle x; z \rangle} \underline{\underline{(1+1)}} |z\rangle$$

$$+ \cancel{\text{"} \sum_{z \not\perp s} (-1)^{\langle x; z \rangle} (1-1) |z\rangle}.$$

5) Measure to get $z \perp s$. Do "enough" times to get n Linearly Indpt z.

Simon's Algorithm: Analysis

How many times do I need to repeat ?

Claim: $O(n)$ repetitions give constant probability

$$\Pr(z_1 \dots z_{k+1} \text{ are linearly independent}) = 1 - \underbrace{\frac{2^k}{2^{n-1}}}$$

Prob. z_{k+1} is in span of $z_1 \dots z_k$.

Pr of selecting appropriate z_i in each iteration is :

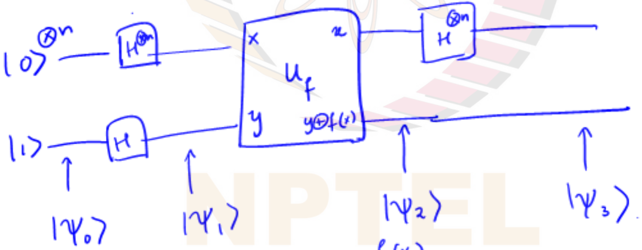
$$\prod_{i=1}^{n-1}\left(1 - \frac{1}{2^i}\right) \geq \frac{1}{4}.$$

Exercise

Given oracle access to $f : \{0,1\}^n \to \{0,1\}$ where $f(x) = \langle x, s\rangle \pmod 2$ for all $x \in \{0,1\}^n$. What is s?

Classically: n queries

Quantumly: 1 query.



$$|\psi_2\rangle = \sum_{x \in \{0,1\}^n} \frac{(-1)^{f(x)} |x\rangle |-\rangle}{2^{n/2}}$$

Scratch Pad

- Applying Hadamard on top n bits:

$$|\psi_3\rangle = \sum_x \sum_z \frac{(-1)^{f(x)+\langle x; z\rangle}}{2^n} |z\rangle\, |-\rangle$$

- Measure top n qubits.

  Consider amplitude on $z = s$.

$$\frac{\frac{1}{2^n} \sum_x (-1)^{\underbrace{\langle x; s+s\rangle}_{\longrightarrow \; ? \; 0}}}{}$$

$$= 1.$$

Scratch Pad

Interesting term. $(-1)^{\langle x; s\rangle + \langle x; z\rangle} = (-1)^{\langle x;\ \underline{\underline{s+z}}\rangle}$

What happens when $z \neq s$?

Amplitude $= 0$. Why?

Fix $s+z = 1{*}{*}\cdots{*}$

$x_0 = 0\,x'$, $x_1 = 1\,x'$

$\underbrace{(-1)^{\langle x_0;\ z+s\rangle}}$ vs $(-1)^{\langle x_1;\ z+s\rangle}$

$\downarrow$ $\downarrow$

Consider inner product

$0 \cdot 1 + \underbrace{\langle x';\ {*}\cdots{*}\rangle}_{\Delta}$ $1 \cdot 1 + \Delta$

$\langle x; s\rangle \bmod 2$.

Is this balanced?

$s = 1{*}{*}{*}{*}$.

$x_0 = 0\,x'$

$x_1 = 1\,x'$