Matrix Assignment - Circle

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CONTENTS

I. PROBLEM

Find the coordinates of the point at which the circle $x^2 + y^2 - 4x - 2y + 4 = 0$ and $x^2 + y^2 - 12x - 8y + 36 = 0$ touch each other. Also find equations common tangents touching the circles in the distinct points.

II. SOLUTION

The equation of a circles is given as,

$$\mathbf{x}^\top \mathbf{V_1} \mathbf{x} + 2\mathbf{u_1}^\top \mathbf{x} + f_1 = 0$$

$$\mathbf{x}^\top \mathbf{V_2} \mathbf{x} + 2\mathbf{u_2}^\top \mathbf{x} + f_2 = 0$$

By comparing the given circle equations with the equation of circles we get,

$$\mathbf{V_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; \mathbf{u_1}^\top = \begin{pmatrix} -2 & -1 \end{pmatrix} ; f_1 = 4$$
$$\mathbf{V_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; \mathbf{u_2}^\top = \begin{pmatrix} -6 & -4 \end{pmatrix} ; f_2 = 36$$

To find the radius of the circles,

$$r_1 = \sqrt{\mathbf{u_1}^\top u - f_1}$$
$$r_2 = \sqrt{\mathbf{u_2}^\top u - f_2}$$

$r_1 = 1$ and $r_2 = 4$ are the radius of the circles.

The Center of Circles are

$$\mathbf{C_1} = -\mathbf{u_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } \mathbf{C_2} = -\mathbf{u_2} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

The locus of the intersection is

$$\begin{vmatrix} \mathbf{V_1} + \mu \mathbf{V_2} & \mathbf{u_1} + \mu \mathbf{u_2} \\ \mathbf{u_1} + \mu \mathbf{u_2}^\top & f_1 + \mu f_2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1+\mu & 0 & -2-6\mu \\ 0 & 1+\mu & -1-4\mu \\ -1-6\mu & -1-4\mu & 4+36\mu \end{vmatrix} = 0$$

$(1+\mu)[(1+\mu)(4+36\mu) - (-1-4\mu)(-1-4\mu)] + (-2-6\mu)[-(1+\mu)(-2-6\mu)] = 0$

$((1+\mu)[4+40\mu+36\mu^2 - 1 - 8\mu - 16\mu^2] + (-2-6\mu)[2+8\mu+6\mu^2] = 0$

$(3+32\mu+20\mu^2+3\mu+32\mu^2+20\mu^3) + (-4-28\mu-60\mu^2-36\mu^3) = 0$

$$16\mu^3 + 8\mu^2 - 7\mu + 1 = 0$$

$$(\mu+1)(4\mu-1)(4\mu-1) = 0$$

$$\mu = -1, \tfrac{1}{4}, \tfrac{1}{4}$$

The intersection of conics is given by

$$\mathbf{x}^\top (\mathbf{V_1} + \mu \mathbf{V_2}) \mathbf{x} + 2(\mathbf{u_1} + \mu \mathbf{u_2})^\top \mathbf{x} + (f_1 + \mu f_2) = 0 \tag{1}$$

For μ = -1:

upon checking for $|\mathbf{V_1} + \mu \mathbf{V_2}| < 0$. Thus,the parameters for the pair of straight lines can be expressed as

$$\mathbf{V_1} + \mu \mathbf{V_2} = 0,$$
$$\mathbf{u_1} + \mu \mathbf{u_2} = \begin{pmatrix} 4 \\ 3 \end{pmatrix},$$
$$f_1 + \mu f_2 = -32$$

Thus, the desired pair of straight line obtained is

$$\begin{pmatrix} 8 & 6 \end{pmatrix} x = 32 \tag{2}$$

from (2)

$$\mathbf{n} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$$

Finding the point of contact from line to the circles,

$$\boxed{\mathbf{q}_i = \mathbf{V_1^{-1}}(k_i\mathbf{n} - \mathbf{u_1})} \tag{3}$$

Here,

$$k_i = \pm\sqrt{\frac{f_0}{\mathbf{n_i^T V_1^{-1} n_i}}} \tag{4}$$

$$f_0 = \mathbf{u_1^T V_1^{-1} u_1} - f_1 \tag{5}$$

On solving the above (2),(4),(5) we get $f_0 = 1$ and $k_i = \frac{1}{10}$

on substituting the f_0 ,k_i,$\mathbf{n}$ in (3) we get

$$\text{point of contact : } \begin{pmatrix} 2.8 \\ 1.6 \end{pmatrix}$$

For $\mu = \frac{1}{4}$:

upon checking for $|\mathbf{V_1} + \mu\mathbf{V_2}| < 0$.Thus, the parameters for the pair of straight lines can be expressed as

$$\mathbf{V_1} + \mu\mathbf{V_2} = \begin{pmatrix} \frac{5}{4} & 0 \\ 0 & \frac{5}{4} \end{pmatrix}$$
$$\mathbf{u_1} + \mu\mathbf{u_2} = \begin{pmatrix} -\frac{7}{2} \\ -2 \end{pmatrix},$$
$$f_1 + \mu f_2 = 13$$
$$\mathbf{c} = -\mathbf{V^{-1}u}$$
$$\mathbf{P} = \mathbf{I}$$
$$f\left(\lambda\right) = \left(|\lambda I - \mathbf{V}|\right)$$

Thus, the desired pair of straight lines are

$$\begin{pmatrix} \sqrt{\lambda_1} & \pm\sqrt{\lambda_2} \end{pmatrix} \mathbf{P}^\top(\mathbf{x} - \mathbf{c}) = 0$$

$$\begin{pmatrix} \sqrt{5} & \sqrt{5} \end{pmatrix} x = 9.8 \tag{6}$$

$$\begin{pmatrix} \sqrt{5} & -\sqrt{5} \end{pmatrix} x = 2.8 \tag{7}$$

These two pair of straight are perpendicular to each other as they satisfies the condition

$$\mathbf{n_1^\top n_2} = \mathbf{n_1.n_2}$$

(6) and (7) are in the form of $\mathbf{n}^\top\mathbf{x} = c$ by solving them we get,

$$\begin{pmatrix} \mathbf{n_1}^\top \\ \mathbf{n_2}^\top \end{pmatrix} \mathbf{x} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} \mathbf{n_1}^\top \\ \mathbf{n_2}^\top \end{pmatrix}^{-1} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

$$\text{Intersection point } \mathbf{x} = \begin{pmatrix} 2.817 \\ 1.565 \end{pmatrix}$$

To find the point of intersection of direct common tangent..,

$$\mathbf{h} = \left(\frac{r_1\mathbf{C_2} - r_2\mathbf{C_1}}{r_1 - r_2}\right)$$

By solving we get,

$$\mathbf{h} = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}$$

To find the point of contact of first circle we have,

$$\boxed{\mathbf{q}_i = \mathbf{V^{-1}}(k_i\mathbf{n_i} - \mathbf{u})} ....(i = 3, 4) \tag{8}$$

Here,

$$k_i = \pm\sqrt{\frac{f_0}{\mathbf{n_i^T V_1^{-1} n_i}}} \tag{9}$$

$$f_0 = \mathbf{u_1^T V_1^{-1} u_1} - f_1 \tag{10}$$

$$\mathbf{n_3} = \mathbf{P_1}\begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \tag{11}$$

$$\mathbf{n_4} = \mathbf{P_1}\begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} \tag{12}$$

$$\mathbf{P_1} = (\mathbf{V_1h} + \mathbf{u_1})(\mathbf{V_1h} + \mathbf{u_1})^T - \mathbf{V_1}(\mathbf{h^T V_1 h} + \mathbf{2u_1^T h} + f_1) \tag{13}$$

On solving the above (9),(10),(11),(12),(13) and substituting in (8) we get two point of contacts for first circle

$$\mathbf{q_1} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \ \mathbf{q_2} = \begin{pmatrix} 1.04 \\ 1.28 \end{pmatrix}$$

finding the equation of tangent for the point of contact $\mathbf{q_1}$ is,

$$(\mathbf{V_1 q_1} + \mathbf{u_1})^\top + \mathbf{u_1}^\top\mathbf{q_1} + f_1 = 0 \tag{14}$$

upon substituting and expanding the (14) we get,

$$\begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} = 0$$

finding the equation of tangent for the point of contact $\mathbf{q_2}$ is,

$$\left(\mathbf{V_1 q_2} + \mathbf{u_1}\right)^\top + \mathbf{u_1}^\top \mathbf{q_2} + f_1 = 0 \tag{15}$$

upon substituting and expanding the (15) we get,

$$\begin{pmatrix} -0.96 & 0.28 \end{pmatrix} \mathbf{x} = -0.64$$

To find the point of contact of second circle we have,

$$\boxed{\mathbf{q}_i = \mathbf{V_2^{-1}}(k_i \mathbf{n_i} - \mathbf{u})} \ldots (i = 5, 6) \tag{16}$$

Here,

$$k_i = \pm\sqrt{\frac{f_0}{\mathbf{n_i^T V_2^{-1} n_i}}} \tag{17}$$

$$f_0 = \mathbf{u_2^T V_2^{-1} u_2} - f_2 \tag{18}$$

$$\mathbf{n_5} = \mathbf{P_2}\begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \tag{19}$$

$$\mathbf{n_6} = \mathbf{P_2}\begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} \tag{20}$$

$$\mathbf{P_2} = (\mathbf{V_2 h} + \mathbf{u_2})(\mathbf{V_2 h} + \mathbf{u_2})^T - \mathbf{V_2}(\mathbf{h^T V_2 h} + 2\mathbf{u_2^T} \tag{21}$$

On solving the above (17),(18),(19),(20),(21) we get two point of contacts for second circle

$$\mathbf{q_3} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \quad \mathbf{q_4} = \begin{pmatrix} 2.16 \\ 5.12 \end{pmatrix}$$

finding the equation of tangent for the point of contact $\mathbf{q_3}$ is,

$$\left(\mathbf{V_2 q_3} + \mathbf{u_2}\right)^\top + \mathbf{u_2}^\top \mathbf{q_3} + f_2 = 0 \tag{22}$$

upon substituting and expanding the (22) we get,

$$\begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} = 0$$

finding the equation of tangent for the point of contact $\mathbf{q_4}$ is,

$$\left(\mathbf{V_2 q_4} + \mathbf{u_2}\right)^\top + \mathbf{u_2}^\top \mathbf{q_4} + f_2 = 0 \tag{23}$$

upon substituting and expanding the (23) we get,

$$\begin{pmatrix} -3.84 & 1.12 \end{pmatrix} \mathbf{x} = -2.56$$

III. Figure

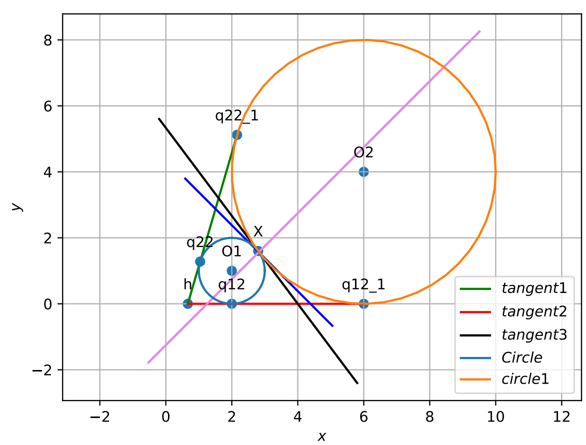

Fig. 1. Intersection of conics and labeling the common tangents touching the circles in the distinct points.