



Matrix Assignment - Circle

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CONTENTS

I Problem

II Solution

III Figure

I. PROBLEM

Find the coordinates of the point at which the circle $x^2 + y^2 - 4x - 2y + 4 = 0$ and $x^2 + y^2 - 12x - 8y + 36 = 0$ touch each other. Also find equations common tangents touching the circles in the distinct points.

II. SOLUTION

The equation of a circles is given as,

$$\mathbf{x}^\top \mathbf{V}_1 \mathbf{x} + 2\mathbf{u}_1^\top \mathbf{x} + f_1 = 0$$

$$\mathbf{x}^\top \mathbf{V}_2 \mathbf{x} + 2\mathbf{u}_2^\top \mathbf{x} + f_2 = 0$$

By comparing the given circle equations with the equation of circles we get,

$$\mathbf{V}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \mathbf{u}_1^\top = (-2 \quad -1); f_1 = 4$$

$$\mathbf{V}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \mathbf{u}_2^\top = (-6 \quad -4); f_2 = 36$$

To find the radius of the circles,

$$r_1 = \sqrt{\mathbf{u}_1^\top \mathbf{u}_1 - f_1}$$

$$r_2 = \sqrt{\mathbf{u}_2^\top \mathbf{u}_2 - f_2}$$

$r_1 = 1$ and $r_2 = 4$ are the radius of the circles.

The Center of Circles are

$$\mathbf{C}_1 = -\mathbf{u}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ and } \mathbf{C}_2 = -\mathbf{u}_2 = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

1 The locus of the intersection is

$$\begin{vmatrix} \mathbf{V}_1 + \mu \mathbf{V}_2 & \mathbf{u}_1 + \mu \mathbf{u}_2 \\ \mathbf{u}_1 + \mu \mathbf{u}_2^\top & f_1 + \mu f_2 \end{vmatrix} = 0$$

3

$$\begin{vmatrix} 1 + \mu & 0 & -2 - 6\mu \\ 0 & 1 + \mu & -1 - 4\mu \\ -1 - 6\mu & -1 - 4\mu & 4 + 36\mu \end{vmatrix} = 0$$

$$(1 + \mu)[(1 + \mu)(4 + 36\mu) - (-1 - 4\mu)(-1 - 4\mu)] + (-2 - 6\mu)[-(1 + \mu)(-2 - 6\mu)] = 0$$

$$((1 + \mu)[4 + 40\mu + 36\mu^2 - 1 - 8\mu - 16\mu^2] + (-2 - 6\mu)[2 + 8\mu + 6\mu^2]) = 0$$

$$(3 + 32\mu + 20\mu^2 + 3\mu + 32\mu^2 + 20\mu^3) + (-4 - 28\mu - 60\mu^2 - 36\mu^3) = 0$$

$$16\mu^3 + 8\mu^2 - 7\mu + 1 = 0$$

$$(\mu + 1)(4\mu - 1)(4\mu - 1) = 0$$

$$\mu = -1, \frac{1}{4}, \frac{1}{4}$$

The intersection of conics is given by

$$\mathbf{x}^\top (\mathbf{V}_1 + \mu \mathbf{V}_2) \mathbf{x} + 2(\mathbf{u}_1 + \mu \mathbf{u}_2)^\top \mathbf{x} + (f_1 + \mu f_2) = 0 \quad (1)$$

For $\mu = -1$:

upon checking for $|\mathbf{V}_1 + \mu \mathbf{V}_2| < 0$. Thus, the parameters for the pair of straight lines can be expressed as

$$\mathbf{V}_1 + \mu \mathbf{V}_2 = 0,$$

$$\mathbf{u}_1 + \mu \mathbf{u}_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix},$$

$$f_1 + \mu f_2 = -32$$

Thus, the desired pair of straight line obtained is

$$(8 \ 6) x = 32 \quad (2)$$

Finding the point of contact from line to the circles,

$$\boxed{\mathbf{q}_i = \mathbf{V}_1^{-1}(k_i \mathbf{n} - \mathbf{u}_1)} \quad (3)$$

Here,

$$k_i = \pm \sqrt{\frac{f_0}{\mathbf{n}_i^T \mathbf{V}_1^{-1} \mathbf{n}_i}} \quad (4)$$

$$f_0 = \mathbf{u}_1^T \mathbf{V}_1^{-1} \mathbf{u}_1 - f_1 \quad (5)$$

On solving the above equations we get $f_0 = 1$ and $k_i = \frac{1}{10}$

on substituting the f_0, k_i, \mathbf{n} in eq(3) we get

$$\text{point of contact : } \begin{pmatrix} 2.8 \\ 1.6 \end{pmatrix}$$

For $\mu = \frac{1}{4}$:

upon checking for $|\mathbf{V}_1 + \mu \mathbf{V}_2| < 0$. Thus, the parameters for the pair of straight lines can be expressed as

$$\mathbf{V}_1 + \mu \mathbf{V}_2 = \begin{pmatrix} \frac{5}{4} & 0 \\ 0 & \frac{5}{4} \end{pmatrix}$$

$$\mathbf{u}_1 + \mu \mathbf{u}_2 = \begin{pmatrix} -\frac{7}{2} \\ -2 \end{pmatrix},$$

$$f_1 + \mu f_2 = 13$$

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u}$$

$$\mathbf{P} = \mathbf{I}$$

$$f(\lambda) = (|\lambda \mathbf{I} - \mathbf{V}|)$$

Thus, the desired pair of straight lines are

$$(\sqrt{\lambda_1} \pm \sqrt{\lambda_2}) \mathbf{P}^T (\mathbf{x} - \mathbf{c}) = 0$$

$$(\sqrt{5} \quad \sqrt{5}) x = 9.8 \quad (6)$$

$$(\sqrt{5} \quad -\sqrt{5}) x = 2.8 \quad (7)$$

These two pair of straight are perpendicular to each other as they satisfies the condition

$$\mathbf{n}_1^T \mathbf{n}_2 = \mathbf{n}_1 \cdot \mathbf{n}_2$$

eq(3) and eq(4) are in the form of $\mathbf{n}^T \mathbf{x} = c$ by solving them we get,

$$\begin{pmatrix} \mathbf{n}_1^T \\ \mathbf{n}_2^T \end{pmatrix} \mathbf{x} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} \mathbf{n}_1^T \\ \mathbf{n}_2^T \end{pmatrix}^{-1} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

$$\text{Intersection point } \mathbf{x} = \begin{pmatrix} 2.817 \\ 1.565 \end{pmatrix}$$

To find the point of intersection of direct common tangent...

$$\mathbf{h} = \begin{pmatrix} r_1 \mathbf{C}_2 - r_2 \mathbf{C}_1 \\ r_1 - r_2 \end{pmatrix}$$

By solving we get,

$$\mathbf{h} = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}$$

To find the point of contact of first circle we have,

$$\boxed{\mathbf{q}_i = \mathbf{V}^{-1}(k_i \mathbf{n}_i - \mathbf{u})} \dots (i = 3, 4) \quad (8)$$

Here,

$$k_i = \pm \sqrt{\frac{f_0}{\mathbf{n}_i^T \mathbf{V}_1^{-1} \mathbf{n}_i}} \quad (9)$$

$$f_0 = \mathbf{u}_1^T \mathbf{V}_1^{-1} \mathbf{u}_1 - f_1 \quad (10)$$

$$\mathbf{n}_3 = \mathbf{P}_1 \begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \quad (11)$$

$$\mathbf{n}_4 = \mathbf{P}_1 \begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} \quad (12)$$

$$\mathbf{P}_1 = (\mathbf{V}_1 \mathbf{h} + \mathbf{u}_1)(\mathbf{V}_1 \mathbf{h} + \mathbf{u}_1)^T - \mathbf{V}_1 (\mathbf{h}^T \mathbf{V}_1 \mathbf{h} + 2\mathbf{u}_1^T \mathbf{h} + f_1) \quad (13)$$

On solving the above equations $k_i, f_0, \mathbf{n}_3, \mathbf{n}_4, \mathbf{P}_1$ we get two point of contacts for first circle

$$\mathbf{q}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \mathbf{q}_2 = \begin{pmatrix} 1.04 \\ 1.28 \end{pmatrix}$$

finding the equation of tangent for the point of contact \mathbf{q}_1 is,

$$(\mathbf{V}_1 \mathbf{q}_1 + \mathbf{u}_1)^T + \mathbf{u}_1^T \mathbf{q}_1 + f_1 = 0$$

upon substituting and expanding the equation we get,

$$(0 \quad -1) \mathbf{x} = 0$$

finding the equation of tangent for the point of contact \mathbf{q}_2 is,

$$(\mathbf{V}_1 \mathbf{q}_2 + \mathbf{u}_1)^\top + \mathbf{u}_1^\top \mathbf{q}_2 + f_1 = 0$$

upon substituting and expanding the equation we get,

$$(-0.96 \quad 0.28) \mathbf{x} = -0.64$$

To find the point of contact of second circle we have,

$$\boxed{\mathbf{q}_i = \mathbf{V}_2^{-1}(k_i \mathbf{n}_i - \mathbf{u})} \dots (i = 5, 6) \quad (14)$$

Here,

$$k_i = \pm \sqrt{\frac{f_0}{\mathbf{n}_i^\top \mathbf{V}_2^{-1} \mathbf{n}_i}} \quad (15)$$

$$f_0 = \mathbf{u}_2^\top \mathbf{V}_2^{-1} \mathbf{u}_2 - f_2 \quad (16)$$

$$\mathbf{n}_5 = \mathbf{P}_2 \left(\frac{\sqrt{|\lambda_1|}}{\sqrt{|\lambda_2|}} \right) \quad (17)$$

$$\mathbf{n}_6 = \mathbf{P}_2 \left(\frac{\sqrt{|\lambda_1|}}{-\sqrt{|\lambda_2|}} \right) \quad (18)$$

$$\mathbf{P}_2 = (\mathbf{V}_2 \mathbf{h} + \mathbf{u}_2)(\mathbf{V}_2 \mathbf{h} + \mathbf{u}_2)^\top - \mathbf{V}_2(\mathbf{h}^\top \mathbf{V}_2 \mathbf{h} + 2\mathbf{u}_2^\top \mathbf{h} + f_2) \quad (19)$$

On solving the above equations $k_i, f_0, \mathbf{n}_5, \mathbf{n}_6, \mathbf{P}_2$ we get two point of contacts for second circle

$$\mathbf{q}_3 = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \quad \mathbf{q}_4 = \begin{pmatrix} 2.16 \\ 5.12 \end{pmatrix}$$

finding the equation of tangent for the point of contact \mathbf{q}_3 is,

$$(\mathbf{V}_2 \mathbf{q}_3 + \mathbf{u}_2)^\top + \mathbf{u}_2^\top \mathbf{q}_3 + f_2 = 0$$

upon substituting and expanding the equation we get,

$$(0 \quad -1) \mathbf{x} = 0$$

finding the equation of tangent for the point of contact \mathbf{q}_4 is,

$$(\mathbf{V}_2 \mathbf{q}_4 + \mathbf{u}_2)^\top + \mathbf{u}_2^\top \mathbf{q}_4 + f_2 = 0$$

upon substituting and expanding the equation we get,

$$(-3.84 \quad 1.12) \mathbf{x} = -2.56$$

III. FIGURE

