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Matrix Assignment - Circle

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CONTENTS

I. PROBLEM

Find the coordinates of the point at which the circle $x^2+y^2-4x-2y+4=0$ and $x^2+y^2-12x-8y+36=0$ touch each other. Also find equations common tangents touching the circles in the distinct points.

II. SOLUTION

The equation of a circles is given as,

$$\mathbf{x}^\top \mathbf{V_1}\mathbf{x}+2\mathbf{u_1}^\top\mathbf{x}+f_1=0 \tag{1}$$
$$\mathbf{x}^\top \mathbf{V_2}\mathbf{x}+2\mathbf{u_2}^\top\mathbf{x}+f_2=0 \tag{2}$$

By comparing the given circle equations with the equation of circles we get,

$$\mathbf{V_1}=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix};\mathbf{u_1}^\top=\begin{pmatrix}-2 & -1\end{pmatrix};f_1=4$$
$$\mathbf{V_2}=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix};\mathbf{u_2}^\top=\begin{pmatrix}-6 & -4\end{pmatrix};f_2=36$$

To find the radius of the circles,

$$r_1=\sqrt{\mathbf{u_1}^\top u-f_1}$$
$$r_2=\sqrt{\mathbf{u_2}^\top u-f_2}$$

$r_1=1$ and $r_2=4$ are the radius of the circles. The Center of Circles are,

$$\mathbf{C_1}=-\mathbf{u_1}=\begin{pmatrix}2\\1\end{pmatrix} \text{ and } \mathbf{C_2}=-\mathbf{u_2}=\begin{pmatrix}6\\4\end{pmatrix}$$

The locus of the intersection is

$$\begin{vmatrix}\mathbf{V_1}+\mu\mathbf{V_2} & \mathbf{u_1}+\mu\mathbf{u_2}\\ \mathbf{u_1}+\mu\mathbf{u_2}^\top & f_1+\mu f_2\end{vmatrix}=0$$

$$\begin{vmatrix}1+\mu & 0 & -2-6\mu\\ 0 & 1+\mu & -1-4\mu\\ -1-6\mu & -1-4\mu & 4+36\mu\end{vmatrix}=0$$

$(1+\mu)[(1+\mu)(4+36\mu)-(-1-4\mu)(-1-4\mu)]+(-2-6\mu)[-(1+\mu)(-2-6\mu)]=0$

$((1+\mu)[4+40\mu+36\mu^2-1-8\mu-16\mu^2]+(-2-6\mu)[2+8\mu+6\mu^2]=0$

$(3+32\mu+20\mu^2+3\mu+32\mu^2+20\mu^3)+(-4-28\mu-60\mu^2-36\mu^3)=0$

$16\mu^3+8\mu^2-7\mu+1=0$

$(\mu+1)(4\mu-1)(4\mu-1)=0$

$$\mu = -1,\tfrac{1}{4},\tfrac{1}{4}$$

The intersection of conics is given by

$$\mathbf{x}^\top(\mathbf{V_1}+\mu\mathbf{V_2})\mathbf{x}+2(\mathbf{u_1}+\mu\mathbf{u_2})^\top\mathbf{x}+(f_1+\mu f_2)=0 \tag{3}$$

For μ = -1: upon checking for $|\mathbf{V_1}+\mu\mathbf{V_2}|<0$. Thus,the parameters for the pair of straight lines can be expressed as

$$\mathbf{V_1}+\mu\mathbf{V_2} = 0,\ \mathbf{u_1}+\mu\mathbf{u_2} = \begin{pmatrix}4\\3\end{pmatrix},\ f_1+\mu f_2 = -32$$

Thus, the desired pair of straight line obtained is

$$\begin{pmatrix}8 & 6\end{pmatrix}x=32 \tag{4}$$

from (4)

$$\mathbf{n} = \begin{pmatrix}8\\6\end{pmatrix}$$

Finding the point of contact from line to the circles,

$$\boxed{\mathbf{q}_i=\mathbf{V_1}^{-1}(k_i\mathbf{n}-\mathbf{u_1})} \tag{5}$$

Here,

$$k_i=\pm\sqrt{\frac{f_0}{\mathbf{n_i^T}\mathbf{V_1}^{-1}\mathbf{n_i}}} \tag{6}$$

$$f_0=\mathbf{u_1^T}\mathbf{V_1}^{-1}\mathbf{u_1}-f_1 \tag{7}$$

On solving the above (4),(6),(7) we get f_0 = 1 and k_i = $\frac{1}{10}$ and substituting the f_0 ,k_i,$\mathbf{n}$ in (5) we get

$$\text{point of contact : } \begin{pmatrix} 2.8 \\ 1.6 \end{pmatrix}$$

For μ = $\frac{1}{4}$: upon checking for $|\mathbf{V_1} + \mu\mathbf{V_2}| <$ 0.Thus, the parameters for the pair of straight lines can be expressed as

$\mathbf{V_1} + \mu\mathbf{V_2}$ = $\begin{pmatrix} \frac{5}{4} & 0 \\ 0 & \frac{5}{4} \end{pmatrix}$, $\mathbf{u_1} + \mu\mathbf{u_2}$ = $\begin{pmatrix} -\frac{7}{2} \\ -2 \end{pmatrix}$,

$f_1 + \mu f_2$ = 13 , $\mathbf{c} = -\mathbf{V^{-1}u}$

$\mathbf{P} = \mathbf{I}$ and $f\left(\lambda\right)$ = $\left(|\lambda I - \mathbf{V}|\right)$

Thus, the desired pair of straight lines are

$$\begin{pmatrix} \sqrt{\lambda_1} & \pm\sqrt{\lambda_2} \end{pmatrix} \mathbf{P}^\top(\mathbf{x} - \mathbf{c}) = 0$$

$$\begin{pmatrix} \sqrt{5} & \sqrt{5} \end{pmatrix} x = 9.8 \tag{8}$$

$$\begin{pmatrix} \sqrt{5} & -\sqrt{5} \end{pmatrix} x = 2.8 \tag{9}$$

These two pair of straight are perpendicular to each other as they satisfies the condition

$$\mathbf{n_1^\top n_2} = \mathbf{n_1.n_2}$$

(8) and (9) are in the form of $\mathbf{n}^\top\mathbf{x} = c$ by solving them we get,

$$\begin{pmatrix} \mathbf{n_1}^\top \\ \mathbf{n_2}^\top \end{pmatrix} \mathbf{x} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} \mathbf{n_1}^\top \\ \mathbf{n_2}^\top \end{pmatrix}^{-1} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

Intersection point $\mathbf{x}$ = $\begin{pmatrix} 2.817 \\ 1.565 \end{pmatrix}$

To find the point of intersection of direct common tangent..,

$$\mathbf{h} = \left(\frac{r_1\mathbf{C_2} - r_2\mathbf{C_1}}{r_1 - r_2}\right)$$

By solving we get,

$$\mathbf{h} = \begin{pmatrix} \frac{2}{3} \\ \\ 0 \end{pmatrix}$$

To find the point of contact of first circle we have,

$$\boxed{\mathbf{q}_i = \mathbf{V^{-1}}(k_i\mathbf{n_i} - \mathbf{u})}....(i = 3, 4) \tag{10}$$

Here,

$$k_i = \pm\sqrt{\frac{f_0}{\mathbf{n_i^T V_1^{-1} n_i}}} \tag{11}$$

$$f_0 = \mathbf{u_1^T V_1^{-1} u_1} - f_1 \tag{12}$$

$$\mathbf{n_3} = \mathbf{P_1}\begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \tag{13}$$

$$\mathbf{n_4} = \mathbf{P_1}\begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} \tag{14}$$

$$\mathbf{P_1} = (\mathbf{V_1h} + \mathbf{u_1})(\mathbf{V_1h} + \mathbf{u_1})^T - \mathbf{V_1}(\mathbf{h^TV_1h} + \mathbf{2u_1^Th} + f_1) \tag{15}$$

On solving the above (11),(12),(13),(14),(15) and substituting in (10) we get two point of contacts for first circle

$$\mathbf{q_1} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \ \mathbf{q_2} = \begin{pmatrix} 1.04 \\ 1.28 \end{pmatrix}$$

finding the equation of tangent for the point of contact $\mathbf{q_1}$ is,

$$(\mathbf{V_1q_1} + \mathbf{u_1})^\top + \mathbf{u_1}^\top\mathbf{q_1} + f_1 = 0 \tag{16}$$

upon substituting and expanding the (16) we get,

$$\begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} = 0$$

finding the equation of tangent for the point of contact $\mathbf{q_2}$ is,

$$(\mathbf{V_1q_2} + \mathbf{u_1})^\top + \mathbf{u_1}^\top\mathbf{q_2} + f_1 = 0 \tag{17}$$

upon substituting and expanding the (17) we get,

$$\begin{pmatrix} -0.96 & 0.28 \end{pmatrix} \mathbf{x} = \text{-0.64}$$

To find the point of contact of second circle we have,

$$\boxed{\mathbf{q}_i = \mathbf{V_2^{-1}}(k_i\mathbf{n_i} - \mathbf{u})}....(i = 5, 6) \tag{18}$$

Here,

$$k_i = \pm\sqrt{\frac{f_0}{\mathbf{n_i^T V_2^{-1} n_i}}} \tag{19}$$

$$f_0 = \mathbf{u_2^T V_2^{-1} u_2} - f_2 \tag{20}$$

$$\mathbf{n_5} = \mathbf{P_2}\begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} \tag{21}$$

$$\mathbf{n_6} = \mathbf{P_2}\begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} \tag{22}$$

$$\mathbf{P_2} = (\mathbf{V_2h} + \mathbf{u_2})(\mathbf{V_2h} + \mathbf{u_2})^T - \mathbf{V_2}(\mathbf{h^TV_2h} + \mathbf{2u_2^Th} + f_2) \tag{23}$$

On solving the above (19),(20),(21),(22),(23) and substituting in (18) we get two point of contacts for second circle

$$\mathbf{q_3} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \mathbf{q_4} = \begin{pmatrix} 2.16 \\ 5.12 \end{pmatrix}$$

finding the equation of tangent for the point of contact $\mathbf{q_3}$ is,

$$(\mathbf{V_2 q_3} + \mathbf{u_2})^\top + \mathbf{u_2}^\top \mathbf{q_3} + f_2 = 0 \tag{24}$$

upon substituting and expanding the (24) we get,

$$\begin{pmatrix} 0 & -1 \end{pmatrix} \mathbf{x} = 0$$

finding the equation of tangent for the point of contact $\mathbf{q_4}$ is,

$$(\mathbf{V_2 q_4} + \mathbf{u_2})^\top + \mathbf{u_2}^\top \mathbf{q_4} + f_2 = 0 \tag{25}$$

upon substituting and expanding the (25) we get,

$$\begin{pmatrix} -3.84 & 1.12 \end{pmatrix} \mathbf{x} = \text{-2.56}$$

III. Figure

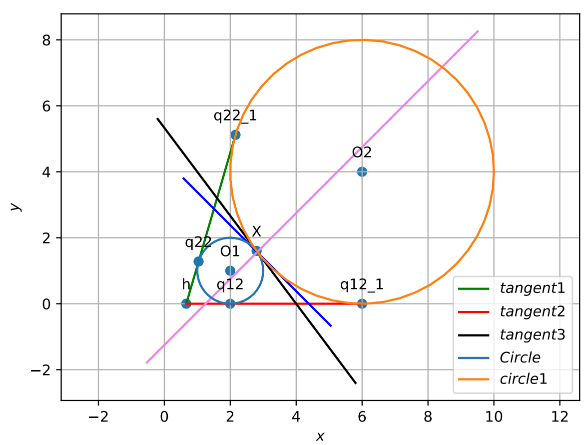

Fig. 1. Intersection of conics and labeling the common tangents touching the circles in the distinct points.