

Problem Statement:

Let $f(x) = \sin^3 x + \lambda \sin^2 x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Find the intervals in which λ should lie in order that $f(x)$ has exactly one minimum and exactly one maximum.

Solution:

Given function is ,

$$f(x) = \sin^3 x + \lambda \sin^2 x \quad (1)$$

Theoretical proof:

$$\text{Let } y=f(x)=\sin^3 x + \lambda \sin^2 x, -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Let $\sin x = t$

$$\frac{dy}{dt} = 3t^2 + 2t\lambda = t(3t + 2\lambda) \quad (2)$$

for exactly one minima and exactly one maxima $\frac{dy}{dx}$ must have two distinct roots $\in (-1,1)$

$$t = 0 \text{ and } t = \frac{-2\lambda}{3} \in (-1,1)$$

$$-1 < \frac{-2\lambda}{3} < 1 \quad (3)$$

$$\frac{-3}{2} < \lambda < \frac{3}{2} \quad (4)$$

$$\lambda \in \left(\frac{-3}{2}, \frac{3}{2}\right) \quad (5)$$

Objective function:

$$\min_x f(x) = \sin^3 x + \lambda \sin^2 x, -\frac{\pi}{2} < x < \frac{\pi}{2} \quad (6)$$

$$\max_x f(x) = \sin^3 x + \lambda \sin^2 x, -\frac{\pi}{2} < x < \frac{\pi}{2} \quad (7)$$

constraints:

$$x \in \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\} \quad (8)$$

Calculation of Minima using gradient descent algorithm:

Minima of the above (1), can be calculated from the following expression, Differentiating (10) yields,

$$x_{n+1} = x_n - \alpha \nabla h(x_n) \quad (9)$$

$$f(x) = \sin^3 x + \lambda \sin^2 x \quad (10)$$

$$\nabla f(x) = \sin x \cos x (3 \sin x + 2\lambda) \quad (11)$$

Taking $x_0 = -\frac{\pi}{2}$, $\alpha = 0.0001$ and precision = 0.000000001, values obtained using python are:

$$\text{Minima} = -2.5 \quad (12)$$

$$\text{Minima Point} = -1.5708 \quad (13)$$

Calculation of Maxima using gradient ascent algorithm:

Maxima of the above (1), can be calculated from the following expression, Differentiating (15) yields,

$$x_{n+1} = x_n - \alpha \nabla h(x_n) \quad (14)$$

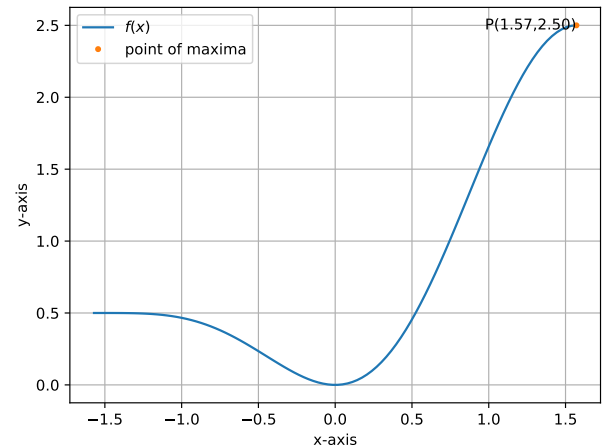
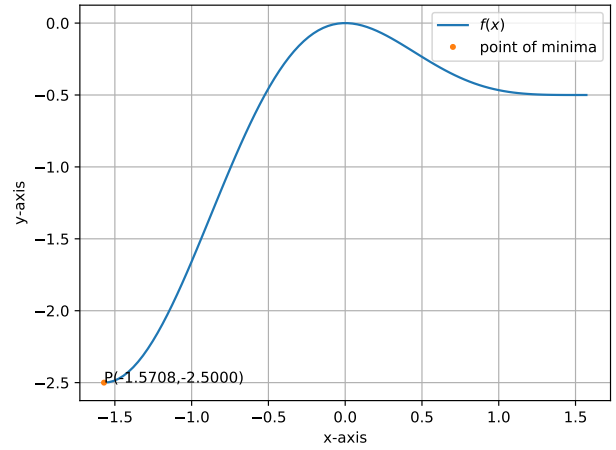
$$f(x) = \sin^3 x + \lambda \sin^2 x \quad (15)$$

$$\nabla f(x) = \sin x \cos x (3 \sin x + 2\lambda) \quad (16)$$

Taking $x_0 = \frac{\pi}{2}$, $\alpha = 0.0001$ and precision = 0.000000001, values obtained using python are:

$$\text{Maxima} = 2.5 \quad (17)$$

$$\text{Maxima Point} = 1.5707 \quad (18)$$

Plots :**Conclusion:**

1. At first, the given function has been differentiated to find $h'(x)$.
2. Later, the given function $h(x)$ is solved by gradient descent algorithm to find minima and the point at which $h(x)$ is minimum. Download the code to execute the above problem statement.

<https://github.com/Sairaghavendra36/Fwc-2022>