

Assignment 5

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Abstract—This document uses the concepts of circles, tangents and distances in proving a statement.

Download Python code from

https://github.com/Sairam13001/AI5006/blob/master/Assignment_5/assignment_5.py

Download latex-tikz codes from

https://github.com/Sairam13001/AI5006/blob/master/Assignment_5/assignment_5.tex

1 PROBLEM

Prove that the line

$$(3 \ 2)\mathbf{x} = 30 \quad (1.0.1)$$

touches the circle

$$\mathbf{x}^T \mathbf{x} - (10 \ 2)\mathbf{x} + 13 = 0 \quad (1.0.2)$$

and find the coordinates of the point of contact.

2 EXPLANATION

The vector equation of a line can be expressed as

$$\mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad (2.0.1)$$

The general equation of a second degree can be expressed as :

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.2)$$

3 SOLUTION

Comparing (1.0.2) with (2.0.2)

$$\mathbf{u} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}, f = 13 \quad (3.0.1)$$

If \mathbf{n} is the normal vector of a line, equation of that line can be written as

$$\mathbf{n}^T \mathbf{x} = c \quad (3.0.2)$$

Comparing (1.0.1) with (3.0.2)

$$\mathbf{n} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad (3.0.3)$$

The point of contact \mathbf{q} , of a line with a normal vector \mathbf{n} to the conic in (2.0.2) is given by:

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa \mathbf{n} - \mathbf{u}) \quad (3.0.4)$$

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \quad (3.0.5)$$

We know that, for a circle,

$$\mathbf{V} = \mathbf{I} \quad (3.0.6)$$

and from the properties of an Identity matrix,

$$\mathbf{I}^{-1} = \mathbf{I} \quad (3.0.7)$$

$$\mathbf{I} \mathbf{x} = \mathbf{x} \quad (3.0.8)$$

Solving for the point of contact using the above equations we get,

$$\kappa = \pm \sqrt{\frac{(-5 \ -1) \begin{pmatrix} -5 \\ -1 \end{pmatrix} - 13}{(3 \ 2) \begin{pmatrix} 3 \\ 2 \end{pmatrix}}} \quad (3.0.9)$$

$$= \pm \sqrt{\frac{26 - 13}{13}} \quad (3.0.10)$$

$$= \pm \sqrt{1} \quad (3.0.11)$$

$$\mathbf{q} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} -5 \\ -1 \end{pmatrix} \quad (3.0.12)$$

$$= \begin{pmatrix} 8 \\ 3 \end{pmatrix} \quad (3.0.13)$$

If the line in (2.0.1) touches (2.0.2) at exactly one point \mathbf{q} , then

$$\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u}) = 0 \quad (3.0.14)$$

It can seen that for the given line

$$\mathbf{m} = \begin{pmatrix} 1 \\ -1.5 \end{pmatrix} \quad (3.0.15)$$

Solving (3.0.14) for given line and circle, we get

$$= \begin{pmatrix} 1 & -1.5 \end{pmatrix} \left(\begin{pmatrix} 8 \\ 3 \end{pmatrix} + \begin{pmatrix} -5 \\ -1 \end{pmatrix} \right) \quad (3.0.16)$$

$$= \begin{pmatrix} 1 & -1.5 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad (3.0.17)$$

$$= 0 \quad (3.0.18)$$

Hence, it is proved that the given line touches the given circle at only one point and so it is a tangent.

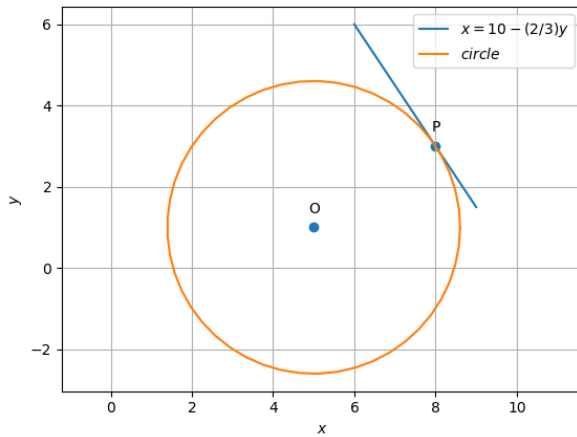


Fig. 0: Circle with center (5 1) and having the line $(3\ 2)x = 30$ as tangent with (8 3) as point of contact.