Assignment 5

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***Abstract*—This document uses the concepts of circles, tangents and distances in proving a statement.**

Download Python code from

https://github.com/Sairam13001/AI5006/blob/master/Assignment_4/assignment_5.py

Download latex-tikz codes from

https://github.com/Sairam13001/AI5006/blob/master/Assignment_4/assignment_5.tex

1 Problem

Prove that the line

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = 30 \tag{1.0.1}$$

touches the circle

$$\mathbf{x}^T \mathbf{x} - \begin{pmatrix} 10 & 2 \end{pmatrix} \mathbf{x} + 13 = 0 \tag{1.0.2}$$

and find the coordinates of the point of contact.

2 Explanation

Line joining the circle center to the tangent's point of contact with the circle is perpendicular to the tangent.

3 Solution

The equation of circle with center $\mathbf{c}$ can be expressed as

$$\mathbf{x}^T \mathbf{x} - 2\mathbf{c}^T \mathbf{x} + f = 0 \tag{3.0.1}$$

Comparing (1.0.2) with (3.0.1)

$$\mathbf{c} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}, f = 13 \tag{3.0.2}$$

$$r = \sqrt{||\mathbf{c}||^2 - f} = \sqrt{13} \tag{3.0.3}$$

Perpendicular distance from the point $\begin{pmatrix} c1 & c2 \end{pmatrix}$ to the line $\begin{pmatrix} a1 & a2 \end{pmatrix} \mathbf{x} = b$ is given by:

$$\frac{|a1c1 + a2c2 - b|}{\sqrt{a1^2 + b1^2}} \tag{3.0.4}$$

Calculating perpendicular distance from given circle center to the given line, we get:

$$\frac{|5 * 3 + 1 * 2 - 30|}{\sqrt{3^2 + 2^2}} = \frac{|-13|}{\sqrt{13}} = \sqrt{13} \tag{3.0.5}$$

As we can see, from (3.0.3) and (3.0.5), perpendicular distance is equal to radius, So it is proved that the given line is a tangent to the given circle.

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{3.0.6}$$

The general equation of a second degree can be expressed as (3.0.6). The point of contact q, of a line with a normal vector $\mathbf{n}$ to this conic is given by:

$$\mathbf{q} = \mathbf{V}^{-1} (\kappa \mathbf{n} - \mathbf{u}) \tag{3.0.7}$$

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \tag{3.0.8}$$

We know that, for a circle,

$$\mathbf{V} = \mathbf{I} \tag{3.0.9}$$

and from the properties of an Identity matrix,

$$\mathbf{I}^{-1} = \mathbf{I} \tag{3.0.10}$$

$$\mathbf{I}\mathbf{X} = \mathbf{X}. \tag{3.0.11}$$

Solving for the point of contact using the above equations we get,

$$\kappa = \pm \sqrt{\frac{\begin{pmatrix} -5 \\ -1 \end{pmatrix} \begin{pmatrix} -5 & -1 \end{pmatrix} - 13}{\begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 3 & 2 \end{pmatrix}}} \tag{3.0.12}$$

$$= \pm \sqrt{\frac{26 - 13}{13}} \tag{3.0.13}$$

$$= \pm \sqrt{1} \tag{3.0.14}$$

$$q = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} -5 \\ -1 \end{pmatrix} \tag{3.0.15}$$

$$= \begin{pmatrix} 8 \\ 3 \end{pmatrix} \tag{3.0.16}$$