Assignment 5

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Abstract—**This document uses the concepts of circles, tangents and distances in proving a statement.**

Download Python code from

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https://github.com/Sairam13001/AI5006/blob/
master/Assignment_5/assignment_5.py
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Download latex-tikz codes from

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https://github.com/Sairam13001/AI5006/blob/
master/Assignment_5/assignment_5.tex
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1 Problem

Prove that the line

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = 30 \tag{1.0.1}$$

touches the circle

$$\mathbf{x}^T \mathbf{x} - \begin{pmatrix} 10 & 2 \end{pmatrix} \mathbf{x} + 13 = 0 \tag{1.0.2}$$

and find the coordinates of the point of contact.

2 Explanation

The vector equation of a line can be expressed as

$$\mathbf{x} = \mathbf{q} + \mu \mathbf{m} \tag{2.0.1}$$

The general equation of a second degree can be expressed as :

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{2.0.2}$$

3 Solution

Comparing (1.0.2) with (2.0.2)

$$\mathbf{u} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}, f = 13 \tag{3.0.1}$$

If $\mathbf{n}$ is the normal vector of a line, equation of that line can be written as

$$\mathbf{n}^T \mathbf{x} = c \tag{3.0.2}$$

Comparing (1.0.1) with (3.0.2)

$$\mathbf{n} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{3.0.3}$$

The point of contact $\mathbf{q}$, of a line with a normal vector $\mathbf{n}$ to the conic in (2.0.2) is given by:

$$\mathbf{q} = \mathbf{V}^{-1} \left(\kappa \mathbf{n} - \mathbf{u} \right) \tag{3.0.4}$$

$$\kappa = \pm \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\mathbf{n}^T \mathbf{V}^{-1} \mathbf{n}}} \tag{3.0.5}$$

We know that, for a circle,

$$\mathbf{V} = \mathbf{I} \tag{3.0.6}$$

and from the properties of an Identity matrix,

$$\mathbf{I}^{-1} = \mathbf{I} \tag{3.0.7}$$

$$\mathbf{I}\mathbf{X} = \mathbf{X} \tag{3.0.8}$$

Solving for the point of contact using the above equations we get,

$$\kappa = \pm \sqrt{\frac{\begin{pmatrix} -5 & -1 \end{pmatrix} \begin{pmatrix} -5 \\ -1 \end{pmatrix} - 13}{\begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}}} \tag{3.0.9}$$

$$= \pm \sqrt{\frac{26 - 13}{13}} \tag{3.0.10}$$

$$= \pm \sqrt{1} \tag{3.0.11}$$

$$q = \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} -5 \\ -1 \end{pmatrix} \tag{3.0.12}$$

$$= \begin{pmatrix} 8 \\ 3 \end{pmatrix} \tag{3.0.13}$$

If the line in (2.0.1) touches (2.0.2) at exactly one point $\mathbf{q}$, then

$$\mathbf{m}^T \left(\mathbf{V}\mathbf{q} + \mathbf{u} \right) = 0 \tag{3.0.14}$$

It can seen that for the given line

$$\mathbf{m} = \begin{pmatrix} 1 \\ -1.5 \end{pmatrix} \tag{3.0.15}$$

Solving (3.0.14) for given line and circle, we get

$$= \begin{pmatrix} 1 & -1.5 \end{pmatrix} \left(\begin{pmatrix} 8 \\ 3 \end{pmatrix} + \begin{pmatrix} -5 \\ -1 \end{pmatrix} \right) \tag{3.0.16}$$

$$= \begin{pmatrix} 1 & -1.5 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{3.0.17}$$

$$= 0 \tag{3.0.18}$$

Hence, it is proved that the given line touches the given circle at only one point and so it is a tangent.