Assignment 6

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***Abstract*—This document uses vector algebra to trace a given curve.**

Download Python code from

https://github.com/Sairam13001/AI5106/tree/main/Assignment_6/assignment_6_fig.py

Download latex-tikz codes from

https://github.com/Sairam13001/AI5106/tree/main/Assignment_6/assignment_6.tex

1 Problem

Trace the curve

$$14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0 \tag{1.0.1}$$

2 Explanation

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \tag{2.0.4}$$

3 Solution

Comparing (1.0.1) with (2.0.1), we get

$$\mathbf{V} = \begin{pmatrix} 14 & -2 \\ -2 & 11 \end{pmatrix} \tag{3.0.1}$$

$$\mathbf{u}^T = \begin{pmatrix} -22 & -29 \end{pmatrix} \tag{3.0.2}$$

If $|\mathbf{V}| > 0$, then (2.0.2) is an ellipse.

$$|V| = \begin{vmatrix} 14 & -2 \\ -2 & 11 \end{vmatrix} = 150 > 0 \tag{3.0.3}$$

(2.0.2) can be expressed as

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \qquad |V| \neq 0 \tag{3.0.4}$$

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \qquad |V| = 0 \tag{3.0.5}$$

with center as

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad |V| \neq 0 \tag{3.0.6}$$

Calculating the center for given curve we get,

$$\mathbf{c} = -\frac{1}{|14 \times 11 - (-2 \times -2)|}\begin{pmatrix} 11 & 2 \\ 2 & 14 \end{pmatrix}\begin{pmatrix} -22 \\ -29 \end{pmatrix} \tag{3.0.7}$$

$$= \frac{1}{150}\begin{pmatrix} 300 \\ 450 \end{pmatrix} \tag{3.0.8}$$

$$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{3.0.9}$$

For

$$|\mathbf{V}| > 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 > 0 \tag{3.0.10}$$

(3.0.4) becomes

$$\lambda_1 y_1^2 + \lambda_2 y_1^2 = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{3.0.11}$$

which is the equation of an ellipse with major and minor axes parameters

$$\sqrt{\frac{\lambda_1}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}}, \sqrt{\frac{\lambda_2}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}} \tag{3.0.12}$$

The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 14 & 2 \\ 2 & \lambda - 11 \end{vmatrix} = 0 \tag{3.0.13}$$

$$\implies \lambda^2 - 25\lambda + 150 = 0 \tag{3.0.14}$$

The eigenvalues are the roots of (3.0.14) given by

$$\lambda_1 = 15, \lambda_2 = 10 \tag{3.0.15}$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{3.0.16}$$

$$\implies (\lambda\mathbf{I} - \mathbf{V})\,\mathbf{p} = 0 \tag{3.0.17}$$

where λ is the eigenvalue. For $\lambda_1 = 15$,

$$(\lambda_1 \mathbf{I} - \mathbf{V}) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \quad (3.0.18)$$

$$\implies \mathbf{p}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad (3.0.19)$$

such that $\|\mathbf{p}_1\| = 1$. Similarly, the eigenvector corresponding to λ_2 can be obtained as

$$\mathbf{p}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (3.0.20)$$

It is easy to verify that

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \mathbf{P}\mathbf{D}\mathbf{P}^T \quad \because \mathbf{P}^{-1} = \mathbf{P}^T \quad (3.0.21)$$

$$\text{or, } \mathbf{D} = \mathbf{P}^T\mathbf{V}\mathbf{P} \quad (3.0.22)$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \quad (3.0.23)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 15 & 0 \\ 0 & 10 \end{pmatrix} \quad (3.0.24)$$

Calculating the ellipse parameters using (3.0.12), we get

$$\begin{aligned} \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} &= \\ &= \begin{pmatrix} -22 & -29 \end{pmatrix} \frac{1}{150} \begin{pmatrix} 11 & 2 \\ 2 & 14 \end{pmatrix} \begin{pmatrix} -22 \\ -29 \end{pmatrix} \\ &= \frac{1}{150} \begin{pmatrix} 300 & 450 \end{pmatrix} \begin{pmatrix} 22 \\ 29 \end{pmatrix} \\ &= 131 \end{aligned}$$

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 131 - 71 = 60 \quad (3.0.25)$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{60}{15}} = 2 \quad (3.0.26)$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} = \sqrt{\frac{60}{10}} = \sqrt{6} \quad (3.0.27)$$

Thus, the given curve is found to be an ellipse from (3.0.3) with center at $\begin{pmatrix} 2 & 3 \end{pmatrix}$ and the major and minor axes lengths are calculated as $\sqrt{6}$, 2. An ellipse with these parameters along with one having center as origin are plotted as shown.

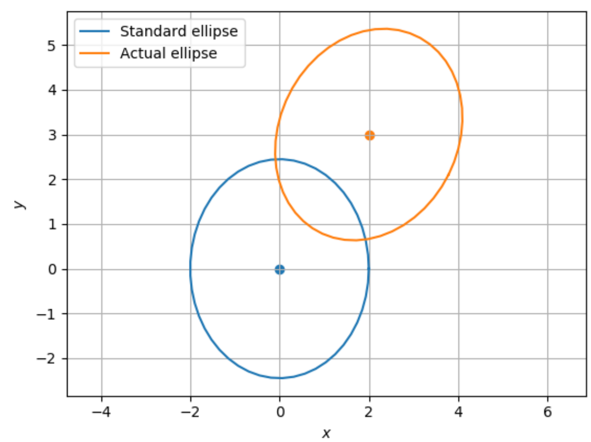

Fig. 0: Ellipse with center (2 3) and having the axes lengths as $\sqrt{6}$ and 2 along with an ellipse with center as origin