Assignment 9

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***Abstract*—This document uses the concepts of elementary row operations of a matrix in solving a problem.**

Download Python code from

https://github.com/Sairam13001/AI5106/blob/main/Assignment_9/assignment_9.py

Download latex-tikz codes from

https://github.com/Sairam13001/AI5106/blob/main/Assignment_9/assignment_9.tex

1 Problem

Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1.0.1}$$

Prove, using elementary row operations that A is invertible if and only if $(ad - bc) \neq 0$

2 Explanation

If $\mathbf{A}$ is an invertible matrix, then some sequence of elementary row operations will transform $\mathbf{A}$ into the identity matrix, $\mathbf{I}$.

3 Solution

The goal is to effect the transformation $(\mathbf{A}|\mathbf{I}) \to \left(\mathbf{I}|\mathbf{A}^{-1}\right)$. Augmenting $\mathbf{A}$ with the 2×2 identity matrix, we get:

$$\begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix} \tag{3.0.1}$$

Now, if $a = 0$, switch the rows. If c is also 0, then the process of reducing $\mathbf{A}$ to $\mathbf{I}$ cannot even begin. So, one necessary condition for $\mathbf{A}$ to be invertible is that the entries a and c are not both 0.

1) Assume that $a \neq 0$, Then :

$$\begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix} \xleftrightarrow{R_1 = R_1/a} \begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{pmatrix} \tag{3.0.2}$$

$$\xleftrightarrow{R_2 = R_2 - cR_1} \begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & \frac{-c}{a} & 1 \end{pmatrix}$$

Next, assuming that $ad - bc \neq 0$, we get:

$$\xleftrightarrow{R_1 = R_1 - \frac{b}{ad-bc} R_2} \begin{pmatrix} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & \frac{ad-bc}{a} & \frac{-c}{a} & 1 \end{pmatrix}$$

$$\xleftrightarrow{R_2 = R_2 \frac{a}{ad-bc}} \begin{pmatrix} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

Therefore, if $ad - bc \neq 0$, then the matrix is invertible and it's inverse is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{3.0.3}$$

2) In (3.0.2), we have assumed that $a \neq 0$. Now consider $a = 0$, then, as we have seen before, it is mandatory that $c \neq 0$:

$$\begin{pmatrix} 0 & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix} \xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} c & d & 0 & 1 \\ 0 & b & 1 & 0 \end{pmatrix} \tag{3.0.4}$$

$$\xleftrightarrow{R_1 = R_1/c} \begin{pmatrix} 1 & \frac{d}{c} & 0 & \frac{1}{c} \\ 0 & b & 1 & 0 \end{pmatrix}$$

$$\xleftrightarrow{R_1 = R_1 - R_2 \times \frac{d}{bc}} \begin{pmatrix} 1 & 0 & -\frac{d}{bc} & \frac{1}{c} \\ 0 & b & 1 & 0 \end{pmatrix}$$

$$\xleftrightarrow{R_2 = R_2/b} \begin{pmatrix} 1 & 0 & -\frac{d}{bc} & \frac{1}{c} \\ 0 & 1 & \frac{1}{b} & 0 \end{pmatrix}$$

Therefore, When we consider $a = 0$ the matrix is invertible if $bc \neq 0$, which is included in the condition $ad - bc \neq 0$.

3) Similarly, consider $c = 0$, then, as we have seen before, it is mandatory that $a \neq 0$:

$$\begin{pmatrix} a & b & 1 & 0 \\ 0 & d & 0 & 1 \end{pmatrix} \xleftrightarrow{R_1 = R_1/a} \begin{pmatrix} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d & 0 & 1 \end{pmatrix} \tag{3.0.5}$$

$$\xleftrightarrow{R_1 = R_1 - R_2 \times \frac{b}{ad}} \begin{pmatrix} 1 & 0 & \frac{1}{a} & -\frac{b}{ad} \\ 0 & d & 0 & 1 \end{pmatrix}$$

$$\xleftrightarrow{R_2 = R_2/d} \begin{pmatrix} 1 & 0 & \frac{1}{a} & -\frac{b}{ad} \\ 0 & 1 & 0 & \frac{1}{d} \end{pmatrix}$$

Therefore, When we consider $c = 0$, the matrix is invertible if $ad \neq 0$, which is included in the condition $ad - bc \neq 0$.

Hence, it is proved from above three cases that the given matrix is invertible iff $ad - bc \neq 0$.