Assignment 12

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***Abstract*—This document uses the concepts of matrix trace, eigen values, minimal polynomial etc in solving a problem.**

Download latex-tikz codes from

https://github.com/Sairam13001/AI5106/blob/main/Assignment_12/assignment_12.tex

1 Problem

Let $M_n(\mathbb{R})$ be the ring of $n \times n$ matrices over $\mathbb{R}$. Which of the following are true for every $n \geq 2$?

1) there exists matrices A, B $\in M_n(\mathbb{R})$ such that $AB - BA = I_n$, where I_n denotes the identity $n \times n$ matrix.
2) if A, B $\in M_n(\mathbb{R})$ and $AB = BA$, then A is diagonalizable over $\mathbb{R}$ if and only if B is diagonalizable over $\mathbb{R}$.
3) if A, B $\in M_n(\mathbb{R})$, then AB and BA have same minimal polynomial.
4) if A, B $\in M_n(\mathbb{R})$, then AB and BA have same eigenvalues in $\mathbb{R}$.

2 Explanation

If A, B are two matrices of same size and k is any scalar, then we know that

$$trace(A + B) = trace(A) + trace(B) \tag{2.0.1}$$

$$trace(kA) = k \times trace(A) \tag{2.0.2}$$

$$trace(AB) = trace(BA) \tag{2.0.3}$$

3 Solution

1) Consider option 1 is true. It means that there exists matrices A, B such that $AB - BA = I_n$.

$$\implies tr(AB - BA) = tr(I_n) \tag{3.0.1}$$

$$\implies tr(AB) - tr(BA) = n \tag{3.0.2}$$

$$\implies 0 = n \tag{3.0.3}$$

But it is given that $n \geq 2$. This contradicts our assumption. Hence, option 1 is FALSE.

2) Let A be a zero matrix of order n and B be any non-diagonalizable matrix of order n. We know that a zero matrix is always diagonalizable.

As, A = $0_{n\times n}$

$$AB = 0 = BA \tag{3.0.4}$$

We can observe here that when $AB = BA$, matrix A is diagonalizable even though matrix B is not diagonalizable. Hence, option 2 is FALSE.

3) The minimal polynomial m(x), of a matrix A, is the polynomial P of minimum degree such that P(A) = 0.

$$\text{Let } A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.0.5}$$

$$\implies AB = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.0.6}$$

We can observe here that

$$m_{BA}(x) = x \text{ while} \tag{3.0.7}$$

$$m_{AB}(x) = x^2 \tag{3.0.8}$$

As minimal polynomial of AB and BA are not same, option 3 is also FALSE.

4) Let λ be an eigen value of AB, it implies that there is some $x \neq 0$ such that,

$$ABx = \lambda x \tag{3.0.9}$$

Let y = Bx. Then $y \neq 0$. Otherwise we would get from (3.0.9) that $\lambda = 0$ or $x = 0$. Now we have,

$$BAy = BABx = B(ABx)$$

$$= B(\lambda x) = \lambda Bx = \lambda y$$

It follows that λ is an eigen value of BA. Hence, option 4 is TRUE.