Assignment 10

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***Abstract*—This document uses the concepts of representation of transformations by matrices in solving a problem.**

Download Python code from

https://github.com/Sairam13001/AI5106/blob/main/Assignment_9/assignment_10.py

Download latex-tikz codes from

https://github.com/Sairam13001/AI5106/blob/main/Assignment_9/assignment_10.tex

1 Problem

Let $\mathbf{T}$ be the linear operator on $\mathbb{R}^2$ defined by

$$T(x_1, x_2) = (-x_2, x_1) \tag{1.0.1}$$

Prove that if β is any ordered basis for $\mathbb{R}^2$ and $[\mathbf{T}]_\beta = \mathbf{A}$ then $\mathbf{A}_{12}\mathbf{A}_{21} \neq 0$.

2 Explanation

If the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, then $ad - bc \neq 0$.

3 Solution

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{3.0.1}$$

$$\mathbf{T}(\mathbf{x}) = \mathbf{T}\mathbf{x} \tag{3.0.2}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \tag{3.0.3}$$

The matrix of $\mathbf{T}$ in the standard ordered basis is

$$\mathbf{T} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.0.4}$$

Let $\{\alpha_1, \alpha_2\}$ be any basis. Write $\alpha_1 = (a, b)$ and $\alpha_2 = (c, d)$. Then

$$T\alpha_1 = (-b, a), T\alpha_2 = (-d, c) \tag{3.0.5}$$

We need to write $T\alpha_1$ and $T\alpha_2$ in terms of α_1, α_2. We can do this by row reducing the augmented matrix.

$$\begin{pmatrix} a & c & -b & -d \\ b & d & a & c \end{pmatrix} \tag{3.0.6}$$

Since $\{\alpha_1, \alpha_2\}$ is a basis, the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible. Thus, $ad - bc \neq 0$. Which means that both a and b cannot be zero at the same time. So, the matrix row-reduces to

$$\begin{pmatrix} 1 & 0 & \frac{ac+bd}{ad-bc} & \frac{c^2+d^2}{ad-bc} \\ 0 & 1 & \frac{a^2+b^2}{ad-bc} & \frac{ac+bd}{ad-bc} \end{pmatrix} \tag{3.0.7}$$

1) Assuming $a \neq 0$, we can show this as follows

$$\begin{pmatrix} a & c & -b & -d \\ b & d & a & c \end{pmatrix}$$
$$\xleftrightarrow{R_1 = R_1/a} \begin{pmatrix} 1 & \frac{c}{a} & -\frac{b}{a} & -\frac{d}{a} \\ b & d & a & c \end{pmatrix}$$
$$\xleftrightarrow{R_2 = R_2 - R_1 \times b} \begin{pmatrix} 1 & \frac{c}{a} & -\frac{b}{a} & -\frac{d}{a} \\ 0 & \frac{ad-bc}{a} & \frac{a^2+b^2}{a} & \frac{ac+bd}{a} \end{pmatrix}$$
$$\xleftrightarrow{R_2 = R_2 \frac{a}{ad-bc}} \begin{pmatrix} 1 & \frac{c}{a} & -\frac{b}{a} & -\frac{d}{a} \\ 0 & 1 & \frac{a^2+b^2}{ad-bc} & \frac{ac+bd}{ad-bc} \end{pmatrix}$$
$$\xleftrightarrow{R_1 = R_1 - R_2 \times \frac{c}{a}} \begin{pmatrix} 1 & 0 & -\frac{ac+bd}{ad-bc} & -\frac{c^2+d^2}{ad-bc} \\ 0 & 1 & \frac{a^2+b^2}{ad-bc} & \frac{ac+bd}{ad-bc} \end{pmatrix}$$

2) Assuming $b \neq 0$, We arrive at the same result

as follows

$$\begin{aligned}
&\begin{pmatrix} a & c & -b & -d \\ b & d & a & c \end{pmatrix} \\
&\xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} b & d & a & c \\ a & c & -b & -d \end{pmatrix} \\
&\xleftrightarrow{R_1 = R_1/b} \begin{pmatrix} 1 & \frac{d}{b} & \frac{a}{b} & \frac{c}{b} \\ a & c & -b & -d \end{pmatrix} \\
&\xleftrightarrow{R_2 = R_2 - R_1 \times a} \begin{pmatrix} 1 & \frac{d}{b} & \frac{a}{b} & \frac{c}{b} \\ 0 & \frac{bc-ad}{b} & -\frac{a^2+b^2}{b} & -\frac{ac+bd}{b} \end{pmatrix} \\
&\xleftrightarrow{R_2 = R_2 \frac{b}{bc-ad}} \begin{pmatrix} 1 & \frac{d}{b} & \frac{a}{b} & \frac{c}{b} \\ 0 & 1 & \frac{a^2+b^2}{ad-bc} & \frac{ac+bd}{ad-bc} \end{pmatrix} \\
&\xleftrightarrow{R_1 = R_1 - R_2 \times \frac{d}{b}} \begin{pmatrix} 1 & 0 & -\frac{ac+bd}{ad-bc} & -\frac{c^2+d^2}{ad-bc} \\ 0 & 1 & \frac{a^2+b^2}{ad-bc} & \frac{ac+bd}{ad-bc} \end{pmatrix}
\end{aligned}$$

Thus, from above two cases:

$$[\mathbf{T}]_\beta = \mathbf{A} = \begin{pmatrix} -\frac{ac+bd}{ad-bc} & -\frac{c^2+d^2}{ad-bc} \\ \frac{a^2+b^2}{ad-bc} & \frac{ac+bd}{ad-bc} \end{pmatrix}. \tag{3.0.8}$$

Now $ad - bc \neq 0$ implies that atleast one of a or b is non-zero and atleast one of c or d is non-zero, which follows that:

$$a^2 + b^2 > 0 \text{ and } c^2 + d^2 > 0 \tag{3.0.9}$$

$$\text{Thus, } \left(a^2 + b^2\right)\left(c^2 + d^2\right) \neq 0. \tag{3.0.10}$$

Hence, it is proved that $\mathbf{A}_{12}\mathbf{A}_{21} \neq 0$.