Assignment 7

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***Abstract*—This document uses Gram-Schmidt process to perform QR decomposition of a matrix.**

Download Python code from

https://github.com/Sairam13001/AI5106/blob/main/Assignment_7/assignment_7.py

Download latex-tikz codes from

https://github.com/Sairam13001/AI5106/blob/main/Assignment_7/assignment_7.tex

1 Problem

$$14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0 \quad (1.0.1)$$

Express this conic section in the from

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \quad (1.0.2)$$

and perform QR decomposition for the matrix V.

2 Explanation

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2.0.1)$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \quad (2.0.2)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.0.3)$$

The QR decomposition (also called the QR factorization) of a matrix is a decomposition of the matrix into an orthogonal matrix and a triangular matrix. A QR decomposition of a real square matrix $\mathbf{V}$ is a decomposition of $\mathbf{V}$ as $\mathbf{V} = \mathbf{QR}$, where $\mathbf{Q}$ is an orthogonal matrix $\left(\mathbf{Q}^T\mathbf{Q} = \mathbf{I}\right)$ and $\mathbf{R}$ is an upper triangular matrix.

3 Solution

Comparing (1.0.1) with (2.0.1), we get

$$\mathbf{V} = \begin{pmatrix} 14 & -2 \\ -2 & 11 \end{pmatrix} \quad (3.0.1)$$

$$\text{If, } \mathbf{V} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \text{ Consider, } \mathbf{V} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix}$$

$$\text{Where, } \mathbf{a} = \begin{pmatrix} a1 \\ a2 \end{pmatrix} \text{and } \mathbf{b} = \begin{pmatrix} b1 \\ b2 \end{pmatrix} \quad (3.0.2)$$

$$\text{Then, } \mathbf{u_1} = \mathbf{a} \text{ , } \mathbf{e_1} = \frac{\mathbf{u_1}}{\|\mathbf{u_1}\|}$$

$$\mathbf{u_2} = \mathbf{b} - (\mathbf{b}\cdot\mathbf{e_1})\,\mathbf{e_1} \text{ , } \mathbf{e_2} = \frac{\mathbf{u_2}}{\|\mathbf{u_2}\|} \quad (3.0.3)$$

$$\text{and, } \mathbf{V} = \begin{pmatrix} \mathbf{e_1} & \mathbf{e_2} \end{pmatrix} \begin{pmatrix} \mathbf{a}\cdot\mathbf{e_1} & \mathbf{b}\cdot\mathbf{e_1} \\ 0 & \mathbf{b}\cdot\mathbf{e_2} \end{pmatrix} = \mathbf{QR} \quad (3.0.4)$$

Performing QR decomposition on $\mathbf{V}$ we get,

$$\mathbf{e_1} = \frac{1}{10\sqrt{2}} \begin{pmatrix} 14 \\ -2 \end{pmatrix} \quad (3.0.5)$$

$$\mathbf{e_2} = \frac{1}{\sqrt{50}} \begin{pmatrix} 1 \\ 7 \end{pmatrix} \quad (3.0.6)$$

$$\mathbf{Q} = \begin{pmatrix} \mathbf{e_1} & \mathbf{e_2} \end{pmatrix} \quad (3.0.7)$$

$$\mathbf{R} = \begin{pmatrix} 10\sqrt{2} & -5\sqrt{2} \\ 0 & \frac{75}{\sqrt{50}} \end{pmatrix} \quad (3.0.8)$$

It is easy to verify that $\mathbf{QR} = \mathbf{V}$ and $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$. Thus, $\mathbf{V}$ is decomposed into an orthogonal matrix and an upper triangular matrix.