

Assignment 6

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Abstract—This document uses vector algebra to trace a given curve.

Download Python code from

https://github.com/Sairam13001/AI5106/tree/main/Assignment_6/assignment_6.py

Download latex-tikz codes from

https://github.com/Sairam13001/AI5106/tree/main/Assignment_6/assignment_6.tex

1 PROBLEM

Trace the curve

$$14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0 \quad (1.0.1)$$

2 EXPLANATION

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (2.0.1)$$

and can be expressed as

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.2)$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad (2.0.3)$$

$$\mathbf{u}^T = (d \quad e) \quad (2.0.4)$$

3 SOLUTION

Comparing (1.0.1) with (2.0.1), we get

$$\mathbf{V} = \begin{pmatrix} 14 & -2 \\ -2 & 11 \end{pmatrix} \quad (3.0.1)$$

$$\mathbf{u}^T = (-22 \quad -29) \quad (3.0.2)$$

If $|\mathbf{V}| > 0$, then (2.0.2) is an ellipse.

$$|\mathbf{V}| = \begin{vmatrix} 14 & -2 \\ -2 & 11 \end{vmatrix} = 150 > 0 \quad (3.0.3)$$

(2.0.2) can be expressed as

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad |\mathbf{V}| \neq 0 \quad (3.0.4)$$

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = -\eta \begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{y} \quad |\mathbf{V}| = 0 \quad (3.0.5)$$

with center as

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad |\mathbf{V}| \neq 0 \quad (3.0.6)$$

Calculating the center for given curve we get,

$$\mathbf{c} = -\frac{1}{|14 * 11 - (-2 * -2)|} \begin{pmatrix} 11 & 2 \\ 2 & 14 \end{pmatrix} \begin{pmatrix} -22 \\ -29 \end{pmatrix} \quad (3.0.7)$$

$$= \frac{1}{150} \begin{pmatrix} 300 \\ 450 \end{pmatrix} \quad (3.0.8)$$

$$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad (3.0.9)$$

For

$$|\mathbf{V}| > 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 > 0 \quad (3.0.10)$$

(3.0.4) becomes

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (3.0.11)$$

which is the equation of an ellipse with major and minor axes parameters

$$\sqrt{\frac{\lambda_1}{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}}, \sqrt{\frac{\lambda_2}{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}} \quad (3.0.12)$$

The characteristic equation of \mathbf{V} is obtained by evaluating the determinant

$$|\lambda \mathbf{I} - \mathbf{V}| = \begin{vmatrix} \lambda - 14 & 2 \\ 2 & \lambda - 11 \end{vmatrix} = 0 \quad (3.0.13)$$

$$\Rightarrow \lambda^2 - 25\lambda + 150 = 0 \quad (3.0.14)$$

The eigenvalues are the roots of (3.0.14) given by

$$\lambda_1 = 15, \lambda_2 = 10 \quad (3.0.15)$$

Calculating the major and minor axes lengths using (3.0.12), we get

$$\begin{aligned}\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} &= \\ &= (-22 - 29) \frac{1}{150} \begin{pmatrix} 11 & 2 \\ 2 & 14 \end{pmatrix} \begin{pmatrix} -22 \\ -29 \end{pmatrix} \\ &= \frac{1}{150} (300 \ 450) \begin{pmatrix} 22 \\ 29 \end{pmatrix} \\ &= 131\end{aligned}$$

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 131 - 71 = 60 \quad (3.0.16)$$

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{60}{15}} = 2 \quad (3.0.17)$$

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_2}} = \sqrt{\frac{60}{10}} = \sqrt{6} \quad (3.0.18)$$

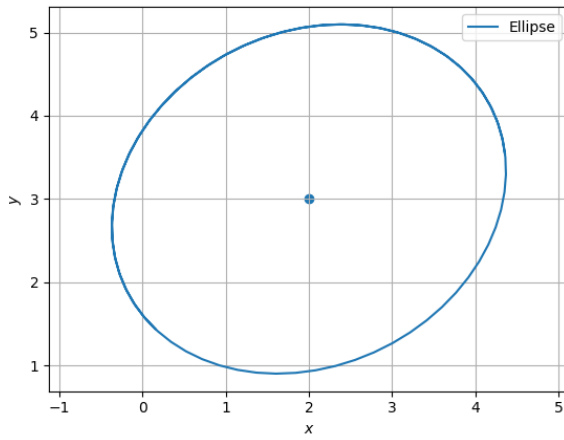


Fig. 0: Ellipse with center (2 3) and having the axes lengths as $\sqrt{6}$ and 2.