

Assignment 8

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Abstract—This document uses the concepts of singular value decomposition in solving a problem.

3 SOLUTION

Download Python code from

https://github.com/Sairam13001/AI5106/blob/main/Assignment_8/assignment_8.py

Download latex-tikz codes from

https://github.com/Sairam13001/AI5106/blob/main/Assignment_8/assignment_8.tex

1 PROBLEM

Write the equation of line through $\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ and perpendicular to the plane

$$2x - y + 2z - 5 = 0 \quad (1.0.1)$$

Determine the co-ordinates of the point in which the plane is met by this line.

2 EXPLANATION

The general equation of a plane is given by

$$px + qy + rz = c \quad (2.0.1)$$

and can be expressed as

$$\mathbf{n}^T \mathbf{x} = c \quad (2.0.2)$$

where

$$\mathbf{n} = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \quad (2.0.3)$$

The equation of a line passing through the point \mathbf{a} and having direction vector \mathbf{b} is given by

$$\mathbf{x} = \mathbf{a} + \lambda \mathbf{b} \quad (2.0.4)$$

Rewriting given equation of plane in (2.0.2) form, we get

$$\begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5 \quad (3.0.1)$$

where : $\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $c = 5$

We need to represent equation of plane in parametric form,

$$\mathbf{x} = \mathbf{p} + \lambda_1 \mathbf{q} + \lambda_2 \mathbf{r} \quad (3.0.2)$$

Here \mathbf{p} is any point on plane and \mathbf{q}, \mathbf{r} are two vectors parallel to plane and hence \perp to \mathbf{n} . Find two vectors that are \perp to \mathbf{n}

$$\Rightarrow \begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \quad (3.0.3)$$

Put $a = 1$ and $b = 2$ in (3.0.3), $\Rightarrow c = 0$

Put $a = 0$ and $b = 2$ in (3.0.3), $\Rightarrow c = 1$

Hence $\mathbf{q} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

Let us find point \mathbf{p} on the plane. Put $x = 1, y = -1$

in (3.0.1), we get $\mathbf{p} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Since given plane is perpendicular to given line, if

we take $\mathbf{P} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ it will give us the shortest distance

to the plane.

Let \mathbf{Q} be the point on plane with shortest distance to \mathbf{P} . \mathbf{Q} can be expressed in (3.0.2) form as

$$\mathbf{Q} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \quad (3.0.4)$$

Equate \mathbf{P} and \mathbf{Q} , and compute λ_1, λ_2 using pseudoinverse from SVD

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} \quad (3.0.5)$$

$$\lambda_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix} \quad (3.0.6)$$

$$\begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix} \quad (3.0.7)$$

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (3.0.8)$$

$$\Rightarrow \mathbf{x} = \mathbf{M}^+ \mathbf{b} \quad (3.0.9)$$

where $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 1 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix}$

Diagonalize $\mathbf{M}\mathbf{M}^T$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 8 & 2 \\ 0 & 2 & 1 \end{pmatrix} \quad (3.0.10)$$

We get three eigen values as 9, 1 and 0. Normalizing the eigen vectors, \mathbf{U} is calculated as :

$$\mathbf{U} = \begin{pmatrix} \frac{1}{3\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{2}{3} \\ \frac{4}{3\sqrt{2}} & 0 & \frac{-1}{3} \\ \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{2}{3} \end{pmatrix} \quad (3.0.11)$$

Diagonalize $\mathbf{M}^T\mathbf{M}$

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \quad (3.0.12)$$

We get two eigen values as 9, 1. Normalizing the eigen vectors, \mathbf{V} is calculated as :

$$\mathbf{V} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad (3.0.13)$$

Taking square root of eigen values, We get $\mathbf{\Sigma}$ as :

$$\mathbf{\Sigma} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (3.0.14)$$

Thus, we performed Singular Value Decomposition for \mathbf{M} . It is easy to check that $\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \mathbf{M}$.

For calculating the pseudoinverse, we know

$$\mathbf{M}^+ = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^T \quad (3.0.15)$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \quad (3.0.16)$$

$$= \begin{pmatrix} \frac{5}{9} & \frac{2}{9} & \frac{-4}{9} \\ \frac{-4}{9} & \frac{2}{9} & \frac{5}{9} \end{pmatrix} \quad (3.0.17)$$

Substitute (3.0.17) in (3.0.9),

$$\mathbf{x} = \begin{pmatrix} \frac{5}{9} & \frac{2}{9} & \frac{-4}{9} \\ \frac{-4}{9} & \frac{2}{9} & \frac{5}{9} \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{28}{9} \\ \frac{-8}{9} \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \quad (3.0.18)$$

Substituting λ_1, λ_2 in (3.0.4)

$$\mathbf{Q} = \frac{1}{9} \begin{pmatrix} 37 \\ 31 \\ 1 \end{pmatrix} \quad (3.0.19)$$

Hence the equation of required line using (2.0.4) is:

$$\mathbf{x} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \quad (3.0.20)$$

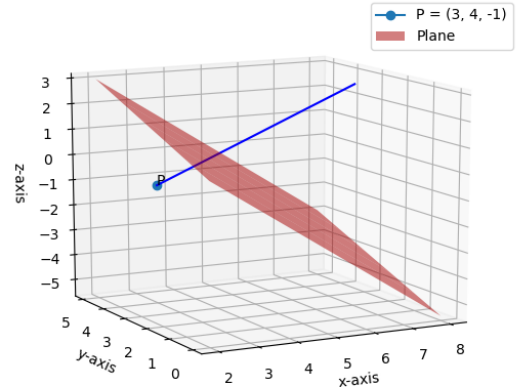


Fig. 0: Line passing through $(3 \ 4 \ -1)$ and perpendicular to the plane $2x - y + 2z = 5$.