Assignment 11

Sairam V C Rebbapragada

***Abstract*—This document uses the concepts of matrix inverse, trace and determinant in solving a problem.**

Download Python code from

https://github.com/Sairam13001/AI5106/blob/main/Assignment_11/assignment_11.py

Download latex-tikz codes from

https://github.com/Sairam13001/AI5106/blob/main/Assignment_11/assignment_11.tex

1 Problem

Let $P_A(x)$ denote the characteristic polynomial of a matrix $\mathbf{A}$, then for which of the following matrices $P_A(x) - P_{A^{-1}}(x)$ a constant?

1) $\begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$
2) $\begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix}$
3) $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$
4) $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

2 Explanation

The characteristic polynomial of a matrix $\mathbf{A}$ is defined as

$$P_A(x) = det(xI - A) \tag{2.0.1}$$

3 Solution

Let matrix A be

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.0.1}$$

$$\implies P_A(x) = det(xI - A) \tag{3.0.2}$$

$$= det\begin{pmatrix} x-a & -b \\ -c & x-d \end{pmatrix} \tag{3.0.3}$$

$$= x^2 - (a+d)x + (ad - bc) \tag{3.0.4}$$

$$= x^2 - tr(A)x + det(A) \tag{3.0.5}$$

We know that for a 2×2 matrix, the inverse is calculated as:

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{3.0.6}$$

$$\implies P_{A^{-1}}(x) = det(xI - A^{-1}) \tag{3.0.7}$$

$$= x^2 - tr(A^{-1})x + det(A^{-1}) \tag{3.0.8}$$

So, $P_A(x) - P_{A^{-1}}(x)$ results in

$$\left(tr(A^{-1}) - tr(A)\right)x + det(A) - det(A^{-1}) \tag{3.0.9}$$

It can be observed that $P_A(x) - P_{A^{-1}}(x)$ will be constant when

$$tr(A^{-1}) - tr(A) = 0 \tag{3.0.10}$$

$$\implies tr(A^{-1}) = tr(A) \tag{3.0.11}$$

$$\implies \frac{a+d}{det(A)} = a + d \tag{3.0.12}$$

(3.0.12) holds when either $a + d = 0$ or $det(A) = 1$. It is easy to observe that the correct answer among the given options is $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ as its determinant is 1.