Assignment 11

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***Abstract*—This document uses the concepts of matrix inverse, trace and determinant in solving a problem.**

Download Python code from

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https://github.com/Sairam13001/AI5106/blob/main/
    Assignment_11/assignment_11.py
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Download latex-tikz codes from

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https://github.com/Sairam13001/AI5106/blob/main/
    Assignment_11/assignment_11.tex
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1 Problem

Let $P_A(x)$ denote the characteristic polynomial of a matrix $\mathbf{A}$, then for which of the following matrices $P_A(x) - P_{A^{-1}}(x)$ a constant?

1) $\begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$
2) $\begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix}$
3) $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$
4) $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

2 Explanation

The characteristic polynomial of a matrix $\mathbf{A}$ is defined as

$$P_A(x) = det(xI - A) \tag{2.0.1}$$

3 Solution

Let matrix A be

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.0.1}$$

$$\implies P_A(x) = det(xI - A) \tag{3.0.2}$$
$$= det\begin{pmatrix} x-a & -b \\ -c & x-d \end{pmatrix} \tag{3.0.3}$$
$$= x^2 - (a+d)x + (ad - bc) \tag{3.0.4}$$
$$= x^2 - tr(A)x + det(A) \tag{3.0.5}$$

We know that for a 2×2 matrix, the inverse is calculated as:

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{3.0.6}$$
$$\implies P_{A^{-1}}(x) = det(xI - A^{-1}) \tag{3.0.7}$$
$$= x^2 - tr(A^{-1})x + det(A^{-1}) \tag{3.0.8}$$

So, $P_A(x) - P_{A^{-1}}(x)$ results in

$$\left(tr(A^{-1}) - tr(A)\right)x + det(A) - det(A^{-1}) \tag{3.0.9}$$

It can be observed that $P_A(x) - P_{A^{-1}}(x)$ will be constant when

$$tr(A^{-1}) - tr(A) = 0 \tag{3.0.10}$$
$$\implies tr(A^{-1}) = tr(A) \tag{3.0.11}$$
$$\implies \frac{a+d}{det(A)} = a + d \tag{3.0.12}$$

(3.0.12) holds when either $a + d = 0$ or $det(A) = 1$. It is easy to observe that the correct answer among the given options is $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ as its determinant is 1.

Eigen values for the inverse matrix can be calculated using the definition of eigen vector as:

Inverse of the matrix is:

$$\begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix} \tag{3.0.13}$$

Eigen vector definition:

$$det(xI - A^{-1}) = 0 \tag{3.0.14}$$
$$\implies det\begin{pmatrix} x-3 & 2 \\ 4 & x-3 \end{pmatrix} = 0 \tag{3.0.15}$$
$$\implies (x-3)^2 - 8 = 0 \tag{3.0.16}$$
$$\implies (x-3) = \pm\sqrt{8} \tag{3.0.17}$$

$\therefore$ The eigen values of Inverse matrix are

$$x = 3 + \sqrt{8} \text{ or } 3 - \sqrt{8} \tag{3.0.18}$$