Assignment 11

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***Abstract*—This document uses the concepts of matrix inverse, trace and determinant in solving a problem.**

Download Python code from

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https://github.com/Sairam13001/AI5106/blob/main/
    Assignment_11/assignment_11.py
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Download latex-tikz codes from

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https://github.com/Sairam13001/AI5106/blob/main/
    Assignment_11/assignment_11.tex
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1 Problem

Let $P_A(x)$ denote the characteristic polynomial of a matrix $\mathbf{A}$, then for which of the following matrices $P_A(x) - P_{A^{-1}}(x)$ a constant?

1) $\begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$
2) $\begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix}$
3) $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$
4) $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

2 Explanation

The characteristic polynomial of a matrix $\mathbf{A}$ is defined as

$$P_A(x) = det(xI - A) \tag{2.0.1}$$

3 Solution

Let matrix A be

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.0.1}$$

$$\implies P_A(x) = det(xI - A) \tag{3.0.2}$$

$$= det\begin{pmatrix} x-a & -b \\ -c & x-d \end{pmatrix} \tag{3.0.3}$$

$$= x^2 - (a+d)x + (ad - bc) \tag{3.0.4}$$

$$= x^2 - tr(A)x + det(A) \tag{3.0.5}$$

We know that for a 2×2 matrix, the inverse is calculated as:

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{3.0.6}$$

$$\implies P_{A^{-1}}(x) = det(xI - A^{-1}) \tag{3.0.7}$$

$$= x^2 - tr(A^{-1})x + det(A^{-1}) \tag{3.0.8}$$

So, $P_A(x) - P_{A^{-1}}(x)$ results in

$$\left(tr(A^{-1}) - tr(A)\right)x + det(A) - det(A^{-1}) \tag{3.0.9}$$

It can be observed that $P_A(x) - P_{A^{-1}}(x)$ will be constant when

$$tr(A^{-1}) - tr(A) = 0 \tag{3.0.10}$$

$$\implies tr(A^{-1}) = tr(A) \tag{3.0.11}$$

$$\implies \frac{a+d}{det(A)} = a + d \tag{3.0.12}$$

(3.0.12) holds when either $a + d = 0$ or $det(A) = 1$. It is easy to observe that the correct answer among the given options is $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ as its determinant is 1.

$$\text{Let } P_A(x) = x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 \tag{3.0.13}$$

From Cayley hamilton theorem:

$$A^n + a_{n-1}A^{n-1} + ... + a_1 A + a_0 = 0 \tag{3.0.14}$$

Multiplying with A^{-n}, we get:

$$a_0(A^{-1})^n + ... + a_{n-1}A^{-1} + I = 0 \tag{3.0.15}$$

Dividing with a_0 (as $a_0 \neq 0$), we get:

$$A^{-n} + \frac{1}{a_0}\left(a_1 A^{n-1} + ... + 1\right) = 0 \tag{3.0.16}$$

From (3.0.16) we can write:

$$P_{A^{-1}}(x) = x^n + \frac{1}{a_0}\left(a_1 x^{n-1} + ... + a_{n-1}x + 1\right)$$

So, for the inverse matrix of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we get:

$$P_{A^{-1}}(x) = x^2 - \left(\frac{a+d}{ad-bc}\right)x + \frac{1}{ad-bc} \tag{3.0.17}$$

From (3.0.17), we can calculate the eigen values for the required matrix as :

$$x^2 - 6x + 1 = 0 \tag{3.0.18}$$

$$\implies x = 3 + \sqrt{8} \text{ or } 3 - \sqrt{8} \tag{3.0.19}$$