

Assignment 11

Sairam V C Rebbapragada

Abstract—This document uses the concepts of matrix inverse, trace and determinant in solving a problem.

Download Python code from

https://github.com/Sairam13001/AI5106/blob/main/Assignment_11/assignment_11.py

Download latex-tikz codes from

https://github.com/Sairam13001/AI5106/blob/main/Assignment_11/assignment_11.tex

1 PROBLEM

Let $P_A(x)$ denote the characteristic polynomial of a matrix \mathbf{A} , then for which of the following matrices $P_A(x) - P_{A^{-1}}(x)$ a constant?

- 1) $\begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$
- 2) $\begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix}$
- 3) $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$
- 4) $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

2 EXPLANATION

The characteristic polynomial of a matrix \mathbf{A} is defined as

$$P_A(x) = \det(xI - A) \quad (2.0.1)$$

3 SOLUTION

Let matrix \mathbf{A} be

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (3.0.1)$$

$$\Rightarrow P_A(x) = \det(xI - A) \quad (3.0.2)$$

$$= \det \begin{pmatrix} x-a & -b \\ -c & x-d \end{pmatrix} \quad (3.0.3)$$

$$= x^2 - (a+d)x + (ad-bc) \quad (3.0.4)$$

$$= x^2 - \text{tr}(A)x + \det(A) \quad (3.0.5)$$

We know that for a 2×2 matrix, the inverse is calculated as:

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad (3.0.6)$$

$$\Rightarrow P_{A^{-1}}(x) = \det(xI - A^{-1}) \quad (3.0.7)$$

$$= x^2 - \text{tr}(A^{-1})x + \det(A^{-1}) \quad (3.0.8)$$

So, $P_A(x) - P_{A^{-1}}(x)$ results in

$$(\text{tr}(A^{-1}) - \text{tr}(A))x + \det(A) - \det(A^{-1}) \quad (3.0.9)$$

It can be observed that $P_A(x) - P_{A^{-1}}(x)$ will be constant when

$$\text{tr}(A^{-1}) - \text{tr}(A) = 0 \quad (3.0.10)$$

$$\Rightarrow \text{tr}(A^{-1}) = \text{tr}(A) \quad (3.0.11)$$

$$\Rightarrow \frac{a+d}{\det(A)} = a+d \quad (3.0.12)$$

(3.0.12) holds when either $a+d=0$ or $\det(A)=1$. It is easy to observe that the correct answer among the given options is $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ as its determinant is 1.

$$\text{Let } P_A(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 \quad (3.0.13)$$

From Cayley hamilton theorem:

$$A^n + a_{n-1}A^{n-1} + \dots + a_1A + a_0 = 0 \quad (3.0.14)$$

Multiplying with A^{-n} , we get:

$$a_0(A^{-1})^n + \dots + a_{n-1}A^{-1} + I = 0 \quad (3.0.15)$$

Dividing with a_0 (as $a_0 \neq 0$), we get:

$$A^{-n} + \frac{1}{a_0}(a_1A^{n-1} + \dots + 1) = 0 \quad (3.0.16)$$

From (3.0.16) we can write $P_{A^{-1}}(x)$ as:

$$x^n + \frac{1}{a_0}(a_1x^{n-1} + \dots + a_{n-1}x + 1)$$

So, for the inverse matrix of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we get:

$$P_{A^{-1}}(x) = x^2 - \left(\frac{a+d}{ad-bc} \right)x + \frac{1}{ad-bc} \quad (3.0.17)$$

From (3.0.17), we can calculate the eigen values for the required matrix as :

$$x^2 - 6x + 1 = 0 \quad (3.0.18)$$

$$\implies x = 3 + \sqrt{8} \text{ or } 3 - \sqrt{8} \quad (3.0.19)$$