

Assignment 11

Sairam V C Rebbapragada

Abstract—This document uses the concepts of matrix inverse, trace and determinant in solving a problem.

Download Python code from

https://github.com/Sairam13001/AI5106/blob/main/Assignment_9/assignment_11.py

Download latex-tikz codes from

https://github.com/Sairam13001/AI5106/blob/main/Assignment_9/assignment_11.tex

1 PROBLEM

Let $P_A(x)$ denote the characteristic polynomial of a matrix \mathbf{A} , then for which of the following matrices $P_A(x) - P_{A^{-1}}(x)$ a constant?

- 1) $\begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$
- 2) $\begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix}$
- 3) $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$
- 4) $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

2 EXPLANATION

The characteristic polynomial of a matrix \mathbf{A} is defined as

$$P_A(x) = \det(xI - A) \quad (2.0.1)$$

3 SOLUTION

Let matrix \mathbf{A} be

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (3.0.1)$$

$$\Rightarrow P_A(x) = \det(xI - A) \quad (3.0.2)$$

$$= \det \begin{pmatrix} x-a & -b \\ -c & x-d \end{pmatrix} \quad (3.0.3)$$

$$= x^2 - (a+d)x + (ad-bc) \quad (3.0.4)$$

$$= x^2 - \text{tr}(A)x + \det(A) \quad (3.0.5)$$

We know that for a 2×2 matrix, the inverse is calculated as:

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad (3.0.6)$$

$$\Rightarrow P_{A^{-1}}(x) = \det(xI - A^{-1}) \quad (3.0.7)$$

$$= x^2 - \text{tr}(A^{-1})x + \det(A^{-1}) \quad (3.0.8)$$

So, $P_A(x) - P_{A^{-1}}(x)$ results in

$$(\text{tr}(A^{-1}) - \text{tr}(A))x + \det(A) - \det(A^{-1}) \quad (3.0.9)$$

It can be observed that $P_A(x) - P_{A^{-1}}(x)$ will be constant when

$$\text{tr}(A^{-1}) - \text{tr}(A) = 0 \quad (3.0.10)$$

$$\Rightarrow \text{tr}(A^{-1}) = \text{tr}(A) \quad (3.0.11)$$

$$\Rightarrow \frac{a+d}{\det(A)} = a+d \quad (3.0.12)$$

(3.0.12) holds when either $a+d=0$ or $\det(A)=1$. It is easy to observe that the correct answer among the given options is $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$ as its determinant is 1.