Assignment 11

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***Abstract*—This document uses the concepts of matrix inverse, trace and determinant in solving a problem.**

Download Python code from

https://github.com/Sairam13001/AI5106/blob/main/Assignment_11/assignment_11.py

Download latex-tikz codes from

https://github.com/Sairam13001/AI5106/blob/main/Assignment_11/assignment_11.tex

1 Problem

Let $P_A(x)$ denote the characteristic polynomial of a matrix $\mathbf{A}$, then for which of the following matrices $P_A(x) - P_{A^{-1}}(x)$ a constant?

1) $\begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$
2) $\begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix}$
3) $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$
4) $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$

2 Explanation

The characteristic polynomial of a matrix $\mathbf{A}$ is defined as

$$P_A(x) = det(xI - A) \tag{2.0.1}$$

3 Solution

Let matrix A be

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.0.1}$$

$$\implies P_A(x) = det(xI - A) \tag{3.0.2}$$

$$= det\begin{pmatrix} x-a & -b \\ -c & x-d \end{pmatrix} \tag{3.0.3}$$

$$= x^2 - (a+d)x + (ad-bc) \tag{3.0.4}$$

From Cayley Hamilton theorem, we can write:

$$A^2 - (a+d)A + (ad-bc) = 0 \tag{3.0.5}$$

Multiplying both sides with A^{-2} :

$$(ad-bc)A^{-2} - (a+d)A^{-1} + I = 0 \tag{3.0.6}$$

Dividing with $(ad-bc)$ on both sides:

$$(A^{-1})^{-2} - \left(\frac{a+d}{ad-bc}\right)A^{-1} + \left(\frac{1}{ad-bc}\right)I = 0$$

From above equation, we can write $P_{A^{-1}}(x)$ as:

$$x^2 - \left(\frac{a+d}{ad-bc}\right)x + \left(\frac{1}{ad-bc}\right) \tag{3.0.7}$$

So, $P_A(x) - P_{A^{-1}}(x)$ becomes:

$$\left(\frac{a+d}{ad-bc} - (a+d)\right)x + \left((ad-bc) - \frac{1}{ad-bc}\right)$$

Hence it can be observed that $P_A(x) - P_{A^{-1}}(x)$ becomes a constant when either $a+d=0$ or $ad-bc=1$.

From the given options it is easy to see that option 3 is the correct answer as its determinant $(ad-bc) = 1$.

From (3.0.7), eigenvalues of A^{-1} can be calculated as

$$x^2 - 6x + 1 = 0 \tag{3.0.8}$$

$$\implies x = 3 + \sqrt{8} \text{ or } 3 - \sqrt{8} \tag{3.0.9}$$