Assignment 8

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***Abstract*—This document uses the concepts of singular value decomposition in solving a problem.**

Download Python code from

https://github.com/Sairam13001/AI5106/blob/main/Assignment_8/assignment_8.py

Download latex-tikz codes from

https://github.com/Sairam13001/AI5106/blob/main/Assignment_8/assignment_8.tex

1 Problem

Write the equation of line through $\begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ and perpendicular to the plane

$$2x - y + 2z - 5 = 0 \tag{1.0.1}$$

Determine the co-ordinates of the point in which the plane is met by this line.

2 Explanation

The general equation of a plane is given by

$$px + qy + rz = c \tag{2.0.1}$$

and can be expressed as

$$\mathbf{n}^T\mathbf{x} = c \tag{2.0.2}$$

where

$$\mathbf{n} = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \tag{2.0.3}$$

The equation of a line passing through the point a and having direction vector $\mathbf{b}$ is given by

$$\mathbf{x} = \mathbf{a} + \lambda\mathbf{b} \tag{2.0.4}$$

3 Solution

Rewriting given equation of plane in (2.0.2) form, we get

$$\begin{pmatrix} 2 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5 \tag{3.0.1}$$

where : $\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $c = 5$

We need to represent equation of plane in parametric form,

$$\mathbf{x} = \mathbf{p} + \lambda_1\mathbf{q} + \lambda_2\mathbf{r} \tag{3.0.2}$$

Here $\mathbf{p}$ is any point on plane and $\mathbf{q}, \mathbf{r}$ are two vectors parallel to plane and hence $\perp$ to $\mathbf{n}$. Find two vectors that are $\perp$ to $\mathbf{n}$

$$\implies \begin{pmatrix} 2 & -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{3.0.3}$$

Put $a = 1$ and $b = 2$ in (3.0.3), $\implies c = 0$
Put $a = 0$ and $b = 2$ in (3.0.3), $\implies c = 1$
Hence $\mathbf{q} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$
Let us find point $\mathbf{p}$ on the plane. Put $x = 1, y = -1$ in (3.0.1), we get $\mathbf{p} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$
Since given plane is perpendicular to given line, if we take $\mathbf{P} = \begin{pmatrix} 3 \\ 4 \\ -1 \end{pmatrix}$ it will give us the shortest distance to the plane.
Let $\mathbf{Q}$ be the point on plane with shortest distance to $\mathbf{P}$. $\mathbf{Q}$ can be expressed in (3.0.2) form as

$$\mathbf{Q} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda_1\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_2\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \tag{3.0.4}$$

Equate $\mathbf{P}$ and $\mathbf{Q}$, and compute λ_1, λ_2 using *pseudoinverse* from *SVD*

$$\begin{pmatrix}1\\-1\\1\end{pmatrix} + \lambda_1\begin{pmatrix}1\\2\\0\end{pmatrix} + \lambda_2\begin{pmatrix}0\\2\\1\end{pmatrix} = \begin{pmatrix}3\\4\\-1\end{pmatrix} \tag{3.0.5}$$

$$\lambda_1\begin{pmatrix}1\\2\\0\end{pmatrix} + \lambda_2\begin{pmatrix}0\\2\\1\end{pmatrix} = \begin{pmatrix}2\\5\\-2\end{pmatrix} \tag{3.0.6}$$

$$\begin{pmatrix}1 & 0\\2 & 2\\0 & 1\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}2\\5\\-2\end{pmatrix} \tag{3.0.7}$$

$$\mathbf{Mx} = \mathbf{b} \tag{3.0.8}$$

$$\implies \mathbf{x} = \mathbf{M}^{+}\mathbf{b} \tag{3.0.9}$$

where $\mathbf{M} = \begin{pmatrix}1 & 0\\2 & 2\\0 & 1\end{pmatrix}$, $\mathbf{x} = \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}2\\5\\-2\end{pmatrix}$

Diagonalize $\mathbf{MM}^T$

$$\mathbf{MM}^T = \begin{pmatrix}1 & 0\\2 & 2\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 0\\0 & 2 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 0\\2 & 8 & 2\\0 & 2 & 1\end{pmatrix} \tag{3.0.10}$$

We get three eigen values as 9, 1 and 0. Normalizing the eigen vectors, $\mathbf{U}$ is calculated as :

$$\mathbf{U} = \begin{pmatrix}\frac{1}{3\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{2}{3}\\ \frac{4}{3\sqrt{2}} & 0 & \frac{-1}{3}\\ \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{2}{3}\end{pmatrix} \tag{3.0.11}$$

Diagonalize $\mathbf{M}^T\mathbf{M}$

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix}1 & 2 & 0\\0 & 2 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\2 & 2\\0 & 1\end{pmatrix} = \begin{pmatrix}5 & 4\\4 & 5\end{pmatrix} \tag{3.0.12}$$

We get two eigen values as 9, 1. Normalizing the eigen vectors, $\mathbf{V}$ is calculated as :

$$\mathbf{V} = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix} \tag{3.0.13}$$

Taking square root of eigen values, We get $\mathbf{\Sigma}$ as :

$$\mathbf{\Sigma} = \begin{pmatrix}3 & 0\\0 & 1\\0 & 0\end{pmatrix} \tag{3.0.14}$$

Thus, we performed Singular Value Decomposition for $\mathbf{M}$. It is easy to check that $\mathbf{U\Sigma V}^T = \mathbf{M}$.

For calculating the psuedoinverse, we know

$$\mathbf{M}^{+} = \mathbf{V\Sigma}^{-1}\mathbf{U}^T \tag{3.0.15}$$

$$= \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{3} & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & \frac{1}{3\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ \frac{2}{3} & \frac{-1}{3} & \frac{2}{3}\end{pmatrix} \tag{3.0.16}$$

$$= \begin{pmatrix}\frac{5}{9} & \frac{2}{9} & \frac{-4}{9}\\ \frac{-4}{9} & \frac{2}{9} & \frac{5}{9}\end{pmatrix} \tag{3.0.17}$$

Substitute (3.0.17) in (3.0.9),

$$\mathbf{x} = \begin{pmatrix}\frac{5}{9} & \frac{2}{9} & \frac{-4}{9}\\ \frac{-4}{9} & \frac{2}{9} & \frac{5}{9}\end{pmatrix}\begin{pmatrix}2\\5\\-2\end{pmatrix} = \begin{pmatrix}\frac{28}{9}\\ \frac{-8}{9}\end{pmatrix} = \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} \tag{3.0.18}$$

Substituting λ_1, λ_2 in (3.0.4)

$$\mathbf{Q} = \frac{1}{9}\begin{pmatrix}37\\31\\1\end{pmatrix} \tag{3.0.19}$$

Hence the equation of required line using (2.0.4) is:

$$\mathbf{x} = \begin{pmatrix}3\\4\\-1\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\2\end{pmatrix} \tag{3.0.20}$$

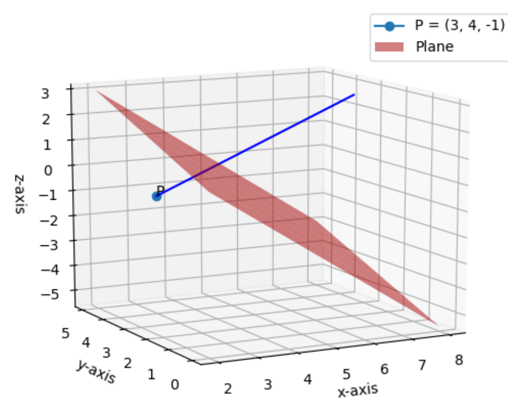

Fig. 0: Line passing through $\begin{pmatrix}3 & 4 & -1\end{pmatrix}$ and perpendicular to the plane $2x - y + 2z = 5$.

Verifying solution to (3.0.8) with *least squares* method

$$\mathbf{M}^T(\mathbf{b} - \mathbf{M}\mathbf{x}) = 0 \tag{3.0.21}$$
$$\implies \mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{3.0.22}$$

Substituting $\mathbf{M}, \mathbf{b}$ from (3.0.7) in (3.0.22)

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix} \tag{3.0.23}$$
$$\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \end{pmatrix} \tag{3.0.24}$$
$$\implies 5\lambda_1 + 4\lambda_2 = 12 \tag{3.0.25}$$
$$4\lambda_1 + 5\lambda_2 = 8 \tag{3.0.26}$$
$$\lambda_1 = \frac{28}{9} \tag{3.0.27}$$
$$\text{and } \lambda_2 = \frac{-8}{9} \tag{3.0.28}$$
$$\implies \mathbf{x} = \begin{pmatrix} \frac{28}{9} \\ \frac{-8}{9} \end{pmatrix} \tag{3.0.29}$$

Comparing (3.0.18) and (3.0.29) solution is verified.