56) Suppose $M \lhd G$ and $N \lhd G$. Prove that $NM \lhd G$.

* Although this does not necessarily solve the problem, we can show that <u>$MN = NM$</u>.

$\forall\, nm \in NM$, $nm = (mm^{-1})(nm) = m(m^{-1}nm) \in MN$, since $N \lhd G$. so $NM \subseteq MN$.

$\forall\, mn \in MN$, $mn = mn(m^{-1}m) = (mnm^{-1})m \in NM$, since $N \lhd G$. so $MN \subseteq NM$.

$\therefore\ MN = NM$.

First, we show that NM is a subgroup.

Obviously $e_G \in NM$.

<u>Existence of Inverses</u>: $(nm)^{-1} = m^{-1}n^{-1}$ $\Rightarrow$ If you showed that $MN = NM$ you're done, but I'll assume you haven't.

$$= m^{-1}n^{-1}(mm^{-1})$$
$$= (m^{-1}n^{-1}m)m^{-1} \in NM \quad \text{because } N \lhd G$$

<u>Closure</u>

$$(n_1m_1)(n_2m_2) = n_1m_1n_2(m_1^{-1}m_1)m_2$$
$$= \left(n_1(m_1n_2m_1^{-1})\right)(m_1m_2) \in NM, \text{ again because } N \lhd G.$$

$\therefore\ NM \leq G$.

<u>Notice</u> that the above proof ~~that~~ of $NM = MN \leq G$ requires <u>only N</u> (or M) to be normal in G.

For $NM \lhd G$, we need <u>both</u> to be normal, however.

let $g \in G$; then

$$g(nm)g^{-1} = gn(g^{-1}g)mg^{-1}$$
$$= (gng^{-1})(gmg^{-1}) \in NM, \text{ since } \underline{\text{both}}\ N \text{ and } M \text{ are normal in } G.$$

$\therefore\ NM \lhd G$.