Dartmouth College
Mathematics 25

Assignment 6
due Wednesday, November 4

1. Show that $\phi(n)$ is even for all integers $n \geq 3$.
2. Show that for each $k \geq 1$, there are at most a finite number of positive integers n with $\phi(n) = k$.
3. Show that if n is a positive integer with $n \not\equiv 0 \pmod 4$ then there is a positive integer $m \neq n$ with $\phi(m) = \phi(n)$.
4. Prove that for all $n \geq 1$, $\phi(n^2) = n\phi(n)$.
5. Let $\sigma(n)$ denote the sum of the positive divisors of n. We have shown σ is multiplicative, but not completely multiplicative. Verify, that for a prime p, and $k \geq 1$,
$$\sigma(p^{k+1}) = \sigma(p)\sigma(p^k) - p\sigma(p^{k-1}).$$
6. Verify the following (formal) identity
$$\sum_{k=0}^{\infty} \sigma(p^k)x^k = \frac{1}{1 - \sigma(p)x + px^2}.$$
Hint: Multiply the power series $\sum_{k=0}^{\infty} \sigma(p^k)x^k$ by $1 - \sigma(p)x + px^2$ and show that the resulting power series is the constant function 1. Note "formal" means you need not be concerned with convergence of the infinite series; it does have a domain of (absolute) convergence.
7. Extending the work in the previous problem show that
$$\frac{1}{1 - \sigma(p)x + px^2} = \left(\frac{1}{1 - px}\right)\left(\frac{1}{1 - x}\right) = \sum_{k=0}^{\infty}(px)^k \sum_{k=0}^{\infty} x^k.$$
What you have effectively shown is that the L-function of a special kind of modular form called an Eisenstein series is the product of the Riemann zeta function and a shift of itself. Modular forms played in integral role in Wiles' proof of Fermat's last theorem. Now if that doesn't sound hot, I don't know what does.