Math 118. Combinatorics.
Problem Set 2. Due on Friday, 2/4/11.

1. For fixed k, give the exponential generating function for the number of surjective maps from $[n]$ onto $[k]$.

2. Let $f(n)$ be the number of words of length n on the alphabet $\{a, b, c, d\}$ that contain a an odd number of times. Find and expression for $F(x) = \sum_{n\geq 0} f(n)x^n$ and also a formula for $f(n)$.

3. Given two sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$, their Hadamard product is the sequence $\{a_n b_n\}_{n\geq 0}$. Show that if $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ have rational generating functions, then so does their Hadamard product.

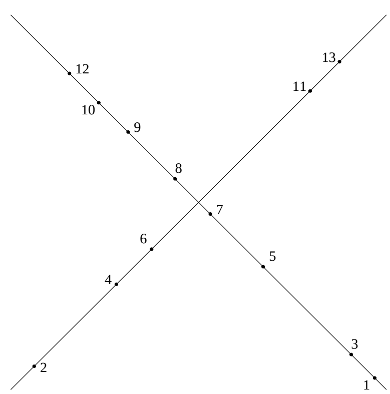
4. A set partition of $[n]$ is called *noncrossing* if it contains no two blocks B and B' such that $i, k \in B$ and $j, l \in B'$ for some $i < j < k < l$. Show that the number of noncrossing partitions of $[n]$ equals the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

5. (a) Find the bivariate generating functions $A(u, z) = \sum_{D\in\mathcal{D}} u^{\ell(D)} z^{|D|}$ and $B(u, z) = \sum_{D\in\mathcal{D}} u^{r(D)} z^{|D|}$, where $\mathcal{D}$ is the class of Dyck paths, and for $D \in \mathcal{D}$, $|D|$ is half of the number of steps, $\ell(D)$ is the number of up-steps before the first down-step, and $r(D)$ is the number of times that D returns to the x-axis (the starting point does not count as a return).

   (b) Give a bijection that explains why $A(u, z) = B(u, z)$.

6. Show that $e^x = \sum_{n\geq 0} \frac{x^n}{n!} \in \mathbb{C}[[x]]$ is not algebraic.

7. Consider two crossing lines in the plane with slopes 1 and -1, forming an X-shape. Place n points anywhere on these lines, with no two of them having the same x- or y-coordinate, and label them $1, 2, \ldots, n$ by increasing y-coordinate. Reading the labels of the points by increasing x-coordinate determines a permutation. For example, in the picture below we get 2 12 10 4 9 6 8 7 5 11 13 3 1.



Let r_n be the number of permutations of $[n]$ that can be obtained in this way (note that $r_3 = 6$ and $r_4 = 20$). Find an ordinary generating function or an expression for r_n.