Math 63 Winter 2009

Real Analysis

Final Exam

Monday, March 9

Your name (please print): ____

Instructions: This is an open book, open notes take home exam. **Use of calculators is not permitted**. **You must justify all of your answers to receive credit,** unless instructed otherwise in a given problem.

The exam is due by 3 PM on Sunday March 15. Please return it to my office 304 Kemeny Hall. If the office is closed please slide your exam under the door and write the time you have finished working on the exam. The building should be open during the final exam week, but if it is closed then please blitz me and I will meet you by the entrance and collect the exam.

The exam consists of 10 problems. Your total exam score is the sum of your scores for the 10 problems. Please do all your work in this exam booklet.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Grader's use only

1. ________ /15

2. ________ /15

3. ________ /15

4. ________ /15

5. ________ /15

6. ________ /15

7. ________ /15

8. ________ /15

9. ________ /15

10. ________ /15

Total: ________ /150

(1) For $n \in \mathbb{N}$ put $f_n(x) = \dfrac{\sin(2nx)}{2^n}$ be a function $f_n : [0,\pi] \to \mathbb{R}$. Prove that the function series $\displaystyle\sum_{n=1}^{\infty} f_n$ converges pointwise to some function $f : [0,\pi] \to \mathbb{R}$. Find $\displaystyle\int_0^{\pi} f dx$. Justify all your steps.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as follows

$$f(x) = \begin{cases} 1 - x & \text{if } x \in \mathbb{Q} \\ 3(x-1) & \text{if } x \notin \mathbb{Q} \end{cases}$$

Is the function continuous at $x = 1$? Prove your answer.

(3) Let $f : [-1, 1] \to \mathbb{R}$ be a function that has a removable discontinuity at 0. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous strictly increasing function and let $g = \phi \circ f : [-1, 1] \to \mathbb{R}$ be the composition function. Prove that g has a removable discontinuity at 0. Hint: the problem admits a relatively easy solution.

(4) Let $\alpha : [0,1] \to \mathbb{R}$ be a function $\alpha(x) = e^{x^{2009}}$ and let $f : [0,1] \to \mathbb{R}$ be a function such that $f(x) \geq 0$ for all x and $f^4 \in \mathcal{R}(\alpha)$. Prove that $f \in \mathcal{R}(\alpha)$ or provide an example when this is not true. Give the details.

(5) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin(|x|^3)$ is differentiable on the whole real line. Attention be careful.

(6) Is it possible to have a 3-times differentiable function $f : [-3, 3] \to \mathbb{R}$ that has a maximum at $x = -1$, a minimum at $x = 0$, a maximum at $x = 1$ and such that $f'''(x) \geq \pi$ for all $x \in [-3, 3]$. Prove your answer.

(7) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map such that the map $g : \mathbb{Q} \to \mathbb{R}$ defined by $g(x) = f(x)$ for all $x \in \mathbb{Q}$ is uniformly continuous. Prove that f is uniformly continuous or provide an example when this is not true.

(8) Prove or disprove the following statement. The set S of all real number that can be presented as finite products of square roots of not necessarily distinct positive integers is countable. For example the number $\sqrt{2}\sqrt{5}\sqrt{4}\sqrt{5} \in S$.

(9) Let $\sum_{n=1}^{\infty} a_n z^n$ be series such that for all n we have $a_n = \dfrac{i}{5^n}$ for some $i \in \mathbb{Z}$ and infinitely many coefficients a_n are nonzero. Prove that the radius of convergence of the series is at most 5.

(10) Let (X, d_X) be a metric space and let $f : X \to \mathbb{Z}$ be a continuous surjective map. Prove that X is not connected. Here the distance function $d_{\mathbb{Z}}$ on $\mathbb{Z}$ is defined by $d_{\mathbb{Z}}(x, y) = |x - y|$, for all $x, y \in \mathbb{Z}$.