1. (15) Let $E$ and $F$ be metric spaces and $f : E \to F$ a function. Complete the following definitions.

(a) We say that $f$ is continuous at $p_0 \in E$ if ...

(b) We say that $f$ is continuous on $E$ if ...

(c) We say that $f$ is uniformly continuous on $E$ if ...

2. (10) Let $E$ and $F$ be metric spaces and $\{f_n\}_{n=1}^{\infty}$ a sequence of functions $f_n : E \to F$. Complete the following definitions.

(a) We say that $\{f_n\}_{n=1}^{\infty}$ converges (pointwise) to $f : E \to F$ if ...

(b) We say that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to $f : E \to F$ if ...

3. (10) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions $f_n : E \to F$. Suppose that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to $f : E \to F$. Show that $f$ is continuous.

**ANS**: Given $p_0 \in E$ and $\epsilon > 0$ we need to find $\delta > 0$ such that $d_E(p, p_0) < \delta$ implies that $d_F(f(p), f(p_0)) < \epsilon$. Since $p_0$ is arbitrary, this will suffice.

To that end, choose $N \in \mathbf{N}$ such that $n \geq N$ implies

$$d_F(f_n(p), f(p) < \frac{\epsilon}{3} \quad \text{for all } p \in E.$$

Since $f_N$ is continuous, there is a $\delta > 0$ such that $d_E(p, p_0) < \delta$ implies $d_F(f_N(p), f_N(p_0)) < \epsilon/3$.

Now if $d_E(p, p_0) < \delta$, then

$$\begin{aligned} d_F(f(p), f(p_0)) &\leq d(f(p), f_N(p)) + d(f_N(p), f_N(p_0)) + d(f_N(p_0), f(p_0)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

This completes the proof.

4. (10) Give an example of a function $f : \mathbf{R} \to \mathbf{R}$ that is continuous, but not uniformly continuous. Justify any assertions.

**ANS**: The function $f(x) = x^2$ is a polynomial and hence continuous on all of $\mathbf{R}$. If it were uniformly continuous, there would be a $\delta > 0$ such that $|x - y| < \delta$ implies $|x^2 - y^2| < 1$. But there is a $n > \frac{1}{\delta}$. Let $x = n + \frac{\delta}{2}$ and $y = n$. Then $|x - y| = \frac{\delta}{2} < \delta$, but $|x^2 - y^2| = \delta n + \frac{\delta^2}{4} \geq \delta n > 1$. This is a contradiction. Hence $f$ is not uniformly continuous.

5. (10) Prove or provide a fully justified counterexample: If $\{ f_n \}_{n=1}^{\infty}$ is a sequence of continuous functions from $[0, 1] \to \mathbf{R}$ which converges pointwise to $f : [0, 1] \to \mathbf{R}$, then the convergence is uniform.

> **ANS**: Let $f_n = x^n$. On $[0, 1]$, $f_n \to f$ where $f(1) = 1$ and $f(x) = 0$ otherwise. In the convergence were uniform on $[0, 1]$, $f$ would have to be continuous, but it isn't.

## Math 63 — Exam II — Take Home Portion

Problems #1 through #5 are to be completed in class on Tuesday. The remaining problems are to be turned in at the beginning of class on Friday. Your solutions are to be fully justified and *neatly* written on *one side only* of $8\frac{1}{2}'' \times 11''$ paper with smooth edges and stapled in the upper left-hand corner. Please start each problem on a separate page.

On the take-home portion only, you may refer to your text and class notes. Otherwise, you are to work alone and neither receive assistance from nor provide assistance to anyone else.

COMMENTS: Problems 9 and 10 are a bit harder. Don't waste too much time on them.

6. (15) Let $E$, $F$ and $G$ be metric spaces. Suppose that $f : E \to F$ is uniformly continuous on $E$ and that $g : F \to G$ is uniformly continuous on $F$. Show that $g \circ f$ is uniformly continuous on $E$.

> **ANS**: Let $\epsilon > 0$. We need to show that there is a $\delta > 0$ such that $d_E(p, q) < \delta$ implies $d_G(g(f(p)), g(f(q))) < \epsilon$. But the uniform continuity of $g$ implies there is a $\delta_1 > 0$ such that $d_F(k, l) < \delta_1$ implies $d_G(g(k), g(l)) < \epsilon$. Then the uniform continuity of $f$ implies that there is a $\delta > 0$ such that $d_E(p, q) < \delta$ implies that $d_F(f(p), f(q)) < \delta_1$. Hence $d_E(p, q) < \delta$ also implies $d_G(g(f(p)), g(f(q))) < \epsilon$ as required.

7. (16) Recall that a function $f : E \to \mathbf{R}$ is bounded if there is a $M \in \mathbf{R}$ such that $|f(p)| \le M$ for all $p \in E$.

(a) Show that if $\{f_n\}_{n=1}^{\infty}$ is a sequence of bounded functions converging uniformly to $f : E \to \mathbf{R}$, then $f$ is bounded.

(b) Show that the conclusion in the previous part need not hold if uniform convergence is replaced by pointwise convergence.

**ANS**: (a) Since $f_n \to f$ uniformly, there is a $N \in \mathbf{N}$ such that $n \geq N$ implies $|f_n(p) - f(p)| le 1$ for all $p \in E$. Since $f_N$ is bounded, there is a $M_N$ such that $|f_n(p)| \leq M_N$. Thus on the one hand,

$$-1 < f(p) - f_N(p) < 1 \quad \text{for all } p \in E.$$

Hence

$$-1 + f_N(p) < f(p) < f_N(p) + 1 \quad \text{for all } p \in E.$$

Whence

$$-1 - M_N < f(p) < 1 + M_N \quad \text{for all } p \in E.$$

This shows that $|f(p)| \leq M_N + 1$ for all $p$ and that $f$ is bounded.

(b) Just let

$$f_n(x) = \begin{cases} x & \text{if } |x| \leq n \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Then each $f_n$ is bounded (by $n$), and $\{f_n\}$ converges pointwise to the identity function, $x \mapsto x$, on $\mathbf{R}$ which is certainly not bounded.

8. (15) Let $S := [a, b] \times [c, d] = \{ (x, y) \in \mathbf{R}^2 : a \leq x \leq b \text{ and } c \leq y \leq d \}$. Show that $S$ is connected. (Hint: first show that $\{k\} \times [c, d]$ and $[a, b] \times \{l\}$ are connected for any $k \in [a, b]$ or $l \in [c, d]$.)

**ANS**: The function $f : [a, b] \mapsto S$ given by $f(x) = (x, d)$ is certainly continuous.[1] Hence its image, $V_0 := [a, b] \times \{d\}$, is connected. Similarly, the sets $T_x := \{x\} \times [c, d]$ are continuous images of $[c, d]$, and hence connected. Note that

$$V_0 \cap T_x = \{(x, d)\} \neq \emptyset.$$

Since

$$S = V_0 \cup \bigcup_{x \in [a,b]} T_x,$$

it follows that $S$ is connected. (It is the union of a family of sets all of which have nonempty intersection with $V_0$.)

ALTERNATE APPROACH: I actually like this proof better, but for some reason I never assigned problem 29a in Chapter IV which says that arcwise connected spaces are connected. But it is pretty easy to see that $S$ is arcwise connected. Let $(r, s)$ and $(u, v)$ be elements of $S$. Define $f : [0, 1] \to \mathbf{R}^2$ by $f(t) = t \cdot (r, s) + (1 - t) \cdot (u, v) = (tr + (1 - t)u, ts, (1 - t)v)$. Then $f$ is certainly continuous — its component functions are polynomials — and it is not hard to see that $f(t) \in S$ for all $t \in [0, 1]$. Hence $S$ is arcwise connected. So if you believe that arcwise connected sets are connected, we're done.

[1] We proved $f : E \to \mathbf{R}^n$ is continuous if and only if its component functions are continuous. Here the components are polynomials.

9. (12) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions $f_n : E \to \mathbf{R}$ on a compact metric space $E$. Suppose that for each $p \in E$, $f_1(p) \geq f_2(p) \geq f_3(p) \geq \dots 0$ and that $f_n \to 0$ pointwise. Show that the convergence must be uniform.

**ANS**: This is known as Dini's Theorem! Let $\epsilon > 0$. We need to find $N \in \mathbf{N}$ such that $n \geq N$ implies $|f_n(p)| < \epsilon$ for all $p \in E$.

Fix $p \in E$. Then there exists $N(p) \in \mathbf{N}$ such that $|f_{N(p)}(p)| < \epsilon/2$. Since $f_{N(p)}$ is continuous, there is a $\delta(p) > 0$ such that $q \in D_{\delta(p)}(p)$ implies $|f_{N(p)}(q)| < \epsilon$. Since $0 \leq f_n(q) \leq f_{N(p)}(q)$ for any $n \geq N(p)$, it follows that

$$|f_n(q)| < \epsilon \quad \text{for any } q \in D_{\delta(p)}(p) \text{ and } n \geq N(p).$$

Since $E$ is compact, there are $p_1, \dots, p_k \in E$ such that

$$E = \bigcup_{j=1}^{k} D_{\delta(p_j)}(p_j).$$

Let $N = \max\{N(p_1), \dots, N(p_k)\}$. Then if $n \geq N$ and $q \in E$, we must have $q$ in some $D_{\delta(p_j)}(p_j)$, and hence

$$|f_n(q)| < \epsilon.$$

This is what we wanted to show.

Alternate Argument: Here is a slick approach due to Sahil. Fix $\epsilon > 0$. We need to find $N \in \mathbf{N}$ such that $n \geq N$ implies $f_n(p) < \epsilon$ for all $p \in E$.

$$U_n = \{\, p \in E : f_n(p) < \epsilon \,\}.$$

Since $f_n$ is continuous, $U_n$ is open in $E$ and since $f_{n+1}(p) \leq f_n(p)$ for all $p$, $U_{n+1} \subset U_n$. On the other hand, for every $p \in E$, $f_n(p) \to 0$. Hence $p \in U_n$ for sufficiently large $n$. That means

$$E = \bigcup_n U_n.$$

Since $E$ is compact, there is a $N$ such that

$$E = \bigcup_{n=1}^{N} U_n = U_N.$$

This just means that $f_N(p) \leq \epsilon$ for all $p$. But then $f_n(p) < \epsilon$ for all $p$ and all $n \geq N$ as required.

10. (12) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions $f_n : [a, b] \subset \mathbf{R} \to \mathbf{R}$. Suppose that each $f_n$ is an increasing function in that $a \leq x \leq y \leq b$ implies that $f_n(x) \leq f_n(y)$. Suppose that $f_n \to f$ pointwise on $[a, b]$. Prove that $f$ is increasing and that if $f$ is continuous, then the convergence is uniform.

**ANS**: First notice that if $x_n \geq 0$ and $x_n \to x$, then $x \geq 0$. (This is easy: if $x < 0$, then $x_n$ would eventually be in $D_{|x|/2}(x)$ which would contradict $x_n \geq 0$. It is also proved in the Proposition on page 50 of the text.) Thus if $x \leq y$ in $[a, b]$, we have $f_n(y) - f_n(x) \geq 0$ while $f_n(y) - f_n(x) \to f(y) - f(x)$. Hence $f(y) - f(x) \geq 0$ and $f$ is increasing.

Now suppose that $f$ is continuous. Let $\epsilon > 0$. We need to find $N \in \mathbf{N}$ such that $n \geq N$ implies

$$|f_n(x) - f(x)| < \epsilon \quad \text{for all } x \in [0,1]. \tag{1}$$

First, I claim that for $x_0 \in [a,b]$ there is a $\delta = \delta(x_0) > 0$ and $N = N(x_0) \in \mathbf{N}$ such that $x \in D_\delta(x_0)$[2] and $n \geq N$ implies that

$$|f_n(x) - f(x)| < \epsilon.$$

In fact, we can choose $\delta > 0$ such that $x \in D_{2\delta}(x_0)$ implies $|f(x) - f(x_0)| < \frac{\epsilon}{4}$. Thus if $x, y \in D_{2\delta}(x_0)$ we have

$$|f(x) - f(y)| < \frac{\epsilon}{2}. \tag{2}$$

Now choose $N$ such that $n \geq N$ implies

$$|f_n(x_0 + \delta) - f(x_0 + \delta)| < \frac{\epsilon}{2} \quad \text{and} \quad |f_n(x_0 - \delta) - f(x_0 - \delta)| < \frac{\epsilon}{2}.$$

(Here and in the sequel, if $x_0 + \delta > b$, then we replace $x_0 + b$ by $b$. Similarly, if $x_0 - \delta < a$, we replace $x_0 - \delta$ by $a$.)

Now assume $x \in D_\delta(x_0)$ and $n \geq N$. Since $f_n$ and $f$ are increasing,

$$\begin{aligned} f_n(x) - f(x) &\leq f_n(x_0 + \delta) - f(x_0 - \delta) \\ &\leq |f_n(x_0 + \delta) - f(x_0 + \delta)| + |f(x_0 + \delta) - f(x_0 - \delta)| \end{aligned}$$

which, in view of our choice of $N$ and (2), is

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

On the other hand,

$$\begin{aligned} f(x) - f_n(x) &\leq f(x_0 + \delta) - f_n(x_0 - \delta) \\ &\leq |f(x_0 + \delta) - f(x_0 - \delta)| + |f(x_0 - \delta) - f_n(x_0 - \delta)| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore $-\epsilon < f_n(x) - f(x) < \epsilon$, and

$$|f_n(x) - f(x)| < \epsilon \quad \text{for all } x \in D_\delta(x_0) \text{ and } n \geq N.$$

This establishes the claim.

Now

$$[a,b] = \bigcup_{x_0 \in [a,b]} D_{\delta(x_0)}(x_0),$$

[2]Here $D_r(x_0)$ means $\{\, x \in [a,b] : |x - x_0| < r \,\}$. This avoids awkward statements like "$|x - x_0| < r$ and $x \in [a,b]$".

and $[a,b]$ is compact. Hence there are $x_1, \ldots, x_m \in [a,b]$ such that

$$[a,b] = \bigcup_{k=1}^{m} D_{\delta(x_k)}(x_k).$$

Let $N = \max\{N(x_1), \ldots, N(x_m)\}$. Now if $n \geq N$ and $x \in [a,b]$, we must have $x \in D_{\delta(x_k)}(x_k)$ for some $k$ and $n \geq N(x_k)$. Hence

$$|f_n(x) - f(x)| < \epsilon.$$

This establishes (1) and completes the proof.

NAME : ______________________________

# Math 63

19 February 2013
Dana P Williams

Problems #1 through #5 are to be completed in class on Tuesday. The remaining problems are to be turned in at the beginning of class on Friday. Your solutions are to be fully justified and *neatly* written on *one side only* of $8\frac{1}{2}'' \times 11''$ paper with smooth edges and stapled in the upper left-hand corner. Please start each problem on a separate page.

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| Problem | Points | Score |
|---|---|---|
| 1 | 15 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 15 | |
| 7 | 16 | |
| 8 | 15 | |
| 9 | 12 | |
| 10 | 12 | |
| Total | 125 | |