HW3

12.2

#16 $$\sum_{n=1}^{\infty} \frac{10^n}{(-9)^{n-1}} = \sum_{n=1}^{\infty} 10 \left(\frac{10}{-9}\right)^{n-1}$$

Hence the series is geometric with $a = 10$ & common ratio $(r) = \frac{10}{-9}$

Since $|r| = \frac{10}{9} > 1$, series is divergent.

#20 $$\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}} = \sum_{n=1}^{\infty} e \left(\frac{e}{3}\right)^{n-1}$$

Geometric series with $a = e$ & $r = e/3 < 1$

Since $|r| < 1$, the series is convergent & it converges to

$$\frac{a}{1-r} = \frac{e}{1-e/3} = \frac{3e}{3-e}.$$

#22 $$\sum_{n=1}^{\infty} \frac{n+1}{2n-3}$$

Let's compute $\lim_{n\to\infty} a_n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{2n-3} = \lim_{n\to\infty} \frac{1+1/n}{2-3/n}$$

$$= \tfrac{1}{2} \neq 0$$

Hence by the Test for divergence, $\sum_{n=1}^{\infty} \frac{n+1}{2n-3}$ is divergent.

\# 38 $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$ (Express S_n as a telescoping sum)

$$a_n = \ln \frac{n}{n+1}$$

$$= (\ln n) - \ln(n+1) \quad \text{[log rule]}$$

$S_n =$ sum of the 1^{st} n terms

$$= (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + \cdots + (\ln n - \ln(n+1))$$

$$= \ln 1 - \ln(n+1)$$

$$= -\ln(n+1)$$

Now $\lim_{n\to\infty} S_n = \lim_{n\to\infty} -\ln(n+1) = -\infty \Rightarrow \sum_{n=1}^{\infty} \ln\frac{n}{n+1}$ is divergent.

#12 $1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\frac{1}{4\sqrt{4}}+\cdots$

$$=\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}=\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$$

This is a p-series with $p=3/2>1$.

Hence by p series test, this series is cgt.

#22 $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$

Let $f(x)=\frac{1}{x(\ln x)^2}$

f is continuous, positive on $[2,\infty)$

Also f is decreasing on $[2,\infty)$ because $x(\ln x)^2$ is increasing on $[2,\infty)$.

Hence we can apply the Integral test.

$$\int_2^{\infty}\frac{1}{x(\ln x)^2}\,dx=\lim_{t\to\infty}\int_2^{t}\frac{1}{x(\ln x)^2}\,dx \quad \text{—(*)}$$

First compute $\int_2^{t}\frac{1}{x(\ln x)^2}\,dx$. for $t\geq 2$

Let $u=\ln x \Rightarrow du=\frac{1}{x}dx$

$x=2,\ u=\ln 2$ & if $x=t,\ u=\ln t$

$$\int_2^t \frac{1}{x(\ln x)^2}\,dx$$

$$= \int_{\ln 2}^{\ln t} u^{-2}\,du$$

$$= \left. -u^{-1} \right]_{\ln 2}^{\ln t}$$

$$= -\left[\frac{1}{\ln t} - \frac{1}{\ln 2}\right]$$

$$= \frac{1}{\ln 2} - \frac{1}{\ln t}$$

From ⊛ (P.3),

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{t \to \infty}\left(\frac{1}{\ln 2} - \frac{1}{\ln t}\right)$$

$$= \frac{1}{\ln 2}$$

Hence by Integral test, the series

$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ is convergent.

#4 $\sum_{n=2}^{\infty} \frac{n^3}{n^4-1}$

$$\frac{n^3}{n^4-1} > \frac{n^3}{n^4} = \frac{1}{n} \quad \text{for all } n \geq 2$$

~~Hence~~ We know that $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges,

hence by comparison test, $\sum_{n=2}^{\infty} \frac{n^3}{n^4-1}$ is

divergent.

OR you can use limit comparison test with $\sum \frac{1}{n}$.

#20 $\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$

Let $a_n = \frac{n+4^n}{n+6^n}$

& $b_n = \frac{4^n}{6^n}$

Let's try limit comparison test.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n+4^n)\,6^n}{(n+6^n)\,4^n}$$

$$= \lim_{n\to\infty} \frac{6^n n + 24^n}{4^n n + 24^n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{(24)^n}\left(6^n n + 24^n\right)}{\frac{1}{(24)^n}\left(4^n n + 24^n\right)}$$

$$= \lim_{n\to\infty} \frac{n/4^n + 1}{n/6^n + 1}$$

$$= \frac{0+1}{0+1} = 1$$

$$\left[\lim_{n\to\infty} \frac{n}{4^n} = \lim_{x\to\infty} \frac{x}{4^x} \right.$$

$$= \lim_{x\to\infty} \frac{1}{4^x \ln 4}$$

(L'Hospital's rule)

$$\left. = 0 \right]$$

Hence $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$

We know that $\sum_{n=1}^{\infty} \frac{4^n}{6^n}$ converges

(geometric series with $|r| = \frac{4}{6} < 1$)

Hence by limit comparison test, $\sum_{n=1}^{\infty} a_n$ is cgt

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$$\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$$

$$a_n = \frac{n+5}{\sqrt[3]{n^7+n^2}} \,. \quad \text{Let } b_n = \frac{n}{\sqrt[3]{n^7}} = \frac{n}{n^{7/3}}$$

$$= \frac{1}{n^{4/3}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n+5)}{\sqrt[3]{n^7+n^2}} (n^{4/3})$$

$$= \lim_{n\to\infty} \frac{n^{7/3}+5n^{4/3}}{(n^7+n^2)^{1/3}}$$

$$= \lim_{n\to\infty} \frac{1+5\frac{n^{4/3}}{n^{7/3}}}{\left(1+\frac{n^2}{n^7}\right)^{1/3}}$$

$$= \lim_{n\to\infty} \frac{1+5/n}{\left(1+\frac{1}{n^5}\right)^{1/3}} = 1 > 0$$

Hence by limit comparison test,

$\sum_{n=1}^{\infty} a_n$ is cgt because $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$

is cgt (P- series with $p = 4/3 > 1$)

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$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$$

$e^{1/n} > 1$ for all $n \geq 1$

Hence $\frac{e^{1/n}}{n} > \frac{1}{n}$ for all $n \geq 1$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (P-series with $P=1$)
or Harmonic series

& hence by Comparison test $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$ is

dgt.

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$$\sum_{n=2}^{\infty} \frac{1}{n^P \ln n}.$$

(i) If $P=1$ the series is $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

~~Consi~~ Let $f(x) = \frac{1}{x \ln x}$ is continuous, positive & decreasing on $[2,\infty)$

& $\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{t \to \infty} \int_2^{t} \frac{1}{x \ln x}\, dx$

$$= \lim_{t \to \infty} \int_{\ln 2}^{\ln t} \frac{1}{u} \, du$$

$$= \lim_{t \to \infty} \left[\ln|\ln t| - \ln|\ln(2)| \right]$$

$$= \infty$$

Hence by Integral test $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is dgt.

(ii) <u>If $p<1$</u>, $\longrightarrow \boxed{n^p < n}$ for $n \geq 2$

Hence $\frac{1}{n^p(\ln n)} > \frac{1}{n(\ln n)}$

From $p=1$ case, $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$ is dgt.

& hence by comparison test,

$\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ is dgt.

(iii) <u>If $p>1$</u>

Note that $\ln n > 1$ for all $n > e$

Hence $\ln n > 1$ for all $n \geq 3$

$\Rightarrow$ ~~$\frac{1}{n^p \ln n}$~~

$\Rightarrow \quad n^p \ln n > n^p$ for all $n \geq 3$

$\Rightarrow \quad \frac{1}{n^p \ln n} < \frac{1}{n^p}$ for all $n \geq 3$

We know that $\sum_{n=3}^{\infty} \frac{1}{n^p}$ is cgt

(p- series with $\underline{p > 1}$ (assumption)) & hence

by Comparison test $\sum_{n=3}^{\infty} \frac{1}{n^p \ln n}$ is cgt.

$$\sum_{n=2}^{\infty} \frac{1}{n^p \ln n} = \frac{1}{2^p \ln 2} + \sum_{n=3}^{\infty} \frac{1}{n^p \ln n}$$

(series in the question) finite # cgt

Hence $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ is cgt.

Combining (i), (ii), (iii), the series <u>converges</u> if $\underline{p > 1}$ & diverges if $p \leq 1$.