Math 31 Lesson Plan

Day 16: Review; Start Section 8

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Supplies needed:

- homework
- Colored chalk
- Quizzes!

Goals for students: Students will:

- improve their understanding of the applications and implications of important theorems from the textbook
- Solidify their understanding of order and subgroups

[**Lecture Notes: Write everything in blue, and every equation, on the board. [Square brackets] indicate anticipated student responses. *Italics* are instructions to myself.**]

Return homework

- Implications of theorems
- Worksheet;
- Section 8

We'll continue with Section 8 tomorrow, and so probably we won't finish Section 12 till Friday. So there will almost certainly be no new reading assignment for Friday. But we'll figure that out for sure tomorrow.

Before we get started, I want to go back to the direct products stuff we were talking about yesterday. What can you tell me about the groups $\langle 2\rangle$ & $\langle 10\rangle$ in $\mathbb{Z}_{12}$? What about $\langle 1\rangle$ and $\langle 2\rangle$ in $\mathbb{Z}_3$? [same] However, Does that mean $\langle(2,1)\rangle = \langle(10,2)\rangle \in \mathbb{Z}_{12} \times \mathbb{Z}_3$? *think-pair-share* [no, because $o(10) \in \mathbb{Z}_{12}$ and $o(1) \in \mathbb{Z}_3$ don't match up. We can see this by writing down the cyclic subgroups:

Are there any more questions about the example from yesterday? Or general questions about order, direct products, cyclic subgroups, etc?

I'm going to put some questions on the board, about implications of some of the big theorems from Sections 4 and 5. I'd like you to get into groups of 4 – and I want every group to have one person from the front row (and side, ie Ian and Tom, if needed). In your groups, please discuss these questions. Take until 1:25 to work on answering these questions, and/or work on the worksheet from yesterday. Then I'd like every group to pair up with another group and make sure you all agree about the statements on the board.

For all of these, if they're true, try to come up with a logical argument or a proof. If not, try to come up with a counterexample!

1. Why doesn't Theorem 5.3 show that any finite subgroup is cyclic? Give an example of a finite, non-cyclic subgroup.
2. In a finite cyclic group, will elements of the same order generate the same subgroup? What about in non-cyclic groups?
3. If $a, b, c \in G$ are elements of a group $(G, *)$, and $a * b = b * c$, is it true that $a = c$?
4. If a, b, c are as above, and $a * b = c * b$, must $a = c$? What about if $a * b = a * c$; must we have $b = c$?

1:30

Section 8: Symmetric Groups Today we're going to talk about disjoint cycle notation exclusively. We'll talk about transpositions, the alternating group and even/odd permutations tomorrow.

For now, here are two permutations in 2-line notation:

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix};\ k = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 5 & 2 & 3 & 1 \end{pmatrix}.$$

How would we write them in disjoint cycle notation?

$$h = (153)(264); k = (1426)(35).$$

To write a permutation p of a set X in disjoint cycle notation, we write down one element x of X, then $p(x)$, then $p(p(x))$, etc. We will eventually get back to x if X is finite. When that happens, we close the parentheses, and start over with some element of X that we haven't written down yet. This procedure will always result in a list of disjoint cycles, whose product is p.

Can we also write $h = (264)(153)$? What about $(642)(153)$? What about $(624)(153)$?

We can rewrite a product of disjoint cycles by reordering the cycles; we can also start the cycle at any of its elements. However, we must keep the order of the elements in each cycle the same!

How would we find $h \circ k$? To find $h \circ k(1)$, look where k takes 1; then look where h takes $k(1)$. This number is $h \circ k(1)$. To find the next element in the cycle notation of $h \circ k$, look where k takes $h \circ k(1)$; then look where h takes $k(h \circ k(1))$. This is the next element. For example,

$$h \circ k = (12465)(3).$$

When you're calculating the product of two permutations, apply each one once, from right to left!

Are there more questions about multiplying permutations, or disjoint cycle notation?

Is $D_4 = S_4$? Think about the generating relation $gf^ig = f^{-i}$ in the square.

Please get into your groups again. I'd like you to discuss How can we find the order of an element in S_n, based on its disjoint cycle notation? Then we'll put a nice tidy proof on the board; I realize I haven't been very good at modeling nice tidy coherent proofs on the board lately, so I'll try to fix that with this one.

Probably won't have time

Claim: *If $f \in S_n$ can be written as $f = f_1 \circ f_2 \circ \ldots \circ f_r$, where the f_i are disjoint cycles, then $o(f)$ is the lcm of the lengths of the f_i.*

Proof: Observe that the order of a cycle is the length ℓ of the cycle: If f_i^m is the identity, then we must have $f_i^m(x) = x$ for all $x \in \{1, 2, \ldots, n\}$. Since each successive application of the cycle f_i moves the element x one step to the right in the cycle, f_i^m will not act as the identity permutation for any $m < \ell$; but $f_i^\ell(x) = x$. Thus $\ell = o(f_i)$ as claimed.

Since each of the elements f_i affects distinct elements of $\{1, \ldots, n\}$,

$$f^m = f_1^m \circ f_2^m \circ \ldots \circ f_r^m.$$

(Alternatively, since the product of disjoint cycles is unaffected by the order of the cycles, we can group all the f_i's together.)

If f^m acts as the identity –that is, $f^m(x) = x$ for every $x \in \{1, 2, \ldots, n\}$, then we must have $f_i^m(x) = x$. Hence m must be a multiple of $o(f_i)$ for each i. Since the order of the cycle f_i is the length of the cycle, this tells us that the order of f – the smallest integer such that f^m is the identity permutation – must be the lcm of the lengths of the cycles, as claimed. $\square$