Worksheet #27

(1) Solve the following wave equation

$$u_{tt} = c^2 u_{xx} \qquad \text{for } 0 < x < 1, \ t > 0$$
$$u(0,t) = u(1,t) = 0 \qquad \text{for } t > 0$$
$$u(x,0) = \sin(5\pi x) + 2\sin(7\pi x) \quad \text{for } 0 < x < 1$$
$$u_t(x,0) = 0 \qquad \text{for } 0 < x < 1$$

Guess $u(x,t)$ is separable.

$\Rightarrow \ u(x,t) = X(x) T(t)$ & Plug into PDE

$$T'' X = c^2 X'' T \ \Rightarrow \ \frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda \qquad \text{constant}$$

$\Rightarrow$ ① $X'' + \lambda X = 0$

② $T'' + c^2 \lambda X = 0$

① $\Rightarrow \quad X(x) = C_1 \cos(\sqrt{\lambda} \, x) + C_2 \sin(\sqrt{\lambda} \, x)$

BC Tell us

$\quad X(0) = X(1) = 0$

$\quad X(0) = C_1 = 0$

$\quad X(1) = C_2 \sin(\sqrt{\lambda}) = 0 \qquad \Rightarrow \sqrt{\lambda} = n\pi \quad n = 1, 2, \dots$

$\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow \lambda = (n\pi)^2$

$\Rightarrow X(x) = C_2 \sin(n\pi x)$

② $\Rightarrow T(t) = k_1 \cos(n\pi c t) + k_2 \sin(n\pi c t)$

$\quad u_t(x,0) = X(x) T'(0) = 0 \qquad \Rightarrow \ T'(0) = 0$

now $\quad T'(t) = -k_1(c n\pi) \sin(n\pi c t) + k_2 (n\pi c) \cos(n\pi c t)$

$\qquad T'(0) = k_2(n\pi c) = 0 \quad \Rightarrow k_2 = 0.$

$\Rightarrow \quad u_n(x,t) = k_n \sin(n\pi x) \cos(n\pi c t)$

$$\rightarrow U(x,t) = \sum_{n=1}^{\infty} k_n \sin(n\pi x) \cos(n\pi c t)$$

$$U(x,0) = \sum_{n=1}^{\infty} k_n \sin(n\pi x) = f(x) = \sin(5\pi x) + 2\sin(7\pi x)$$

$$\rightarrow k_n = \frac{2}{1} \int_0^1 [\sin(5\pi x) + 2\sin(7\pi x)] \sin(n\pi x) \, dx$$

$$= \begin{cases} 1 & n = 5 \\ 2 & n = 7 \\ 0 & \text{otherwise} \end{cases} \quad \text{by orthogonality of sin}$$

$$\rightarrow U(x,t) = \sin(5\pi x) \cos(5\pi c t) + 2\sin(7\pi x)\cos(7\pi c t)$$