Math 31 Lesson Plan

Day 26: Quotient Groups and Homomorphisms

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Supplies needed:

- Colored chalk
- Midterms
- Envelope for surveys

Goals for Students:

Students will:

- Understand how to apply the theorems from Section 13
- Feel more comfortable with the proofs of these theorems

[**Lecture Notes: Write everything in blue, and every equation, on the board. [Square brackets] indicate anticipated student responses. *Italics* are instructions to myself.**]

Return midterms

Today I want to try something new: Instead of having me work through the confusing proofs from Section 13 at the board, and then having you work through examples in your groups, I'm going to do the Examples of how to apply these Theorems at the board. Then hopefully we'll all understand what they mean. After that, I'll have you get into groups and Work through proofs from Section 13. I'll be wandering around the room to help you figure out the parts that you're still stuck on, and if everyone gets stuck at the same place, then we'll talk it through at the board.

The most confusing theorems were Theorem 13.3 and Theorem 13.5, but since these rely on the Fundamental Theorem of Homomorphisms, let's put that on the board first. What does this theorem say? [Let G, K be groups. If $\phi : G \to K$ is an epimorphism, then $G/\ker\phi \cong K$.]

Are there any questions about the Fundamental Theorem?

Theorem 13.3 Let $\phi : G \to K$ be an onto homomorphism. Then we have a one-to-one correspondence between subgroups of K, and subgroups of G that contain $\ker\phi$. Moreover, if $H \leq G$ contains $\ker\phi$, then $H \lhd G$ iff $\phi(H) \lhd K$.

We talked about this theorem on Monday, and we saw that

- If H is any subgroup of G, then $\phi(H) \leq K$
- If we remove the hypothesis that H contains $\ker\phi$ we don't get a 1-1 correspondence anymore.

We saw this in the case of the quotient projection $\rho : D_4 \to D_4/\langle 180\rangle$.

Questions about the example from Monday?

Let's do another Example: Let $G = \mathbb{Z}_{12}, K = \mathbb{Z}_4, \phi : G \to K$ be the identity projection. What are the subgroups of G? Of K? What's the kernel of ϕ? *Think-pair-share; Have students come write answers on the board when finished.* The subgroups of G are

$$\{e\}, \langle 1 \rangle = \mathbb{Z}_{12}, \langle 2 \rangle, \langle 3 \rangle, \langle 4 \rangle, \langle 6 \rangle.$$

Draw these in a subgroup lattice!

The subgroups of K are

$$\{e\}, \langle 1 \rangle = \mathbb{Z}_4, \langle 2 \rangle.$$

The kernel of ϕ is $\langle 4 \rangle$.

Which subgroups of G contain $\ker \phi$? Which subgroups of K do they correspond to? *Think-pair-share* Observe that $\phi(\langle 2 \rangle) = \phi(\langle 6 \rangle)$, but this doesn't contradict the statement about a 1-1 correspondence because $\langle 6 \rangle$ doesn't contain $\ker \phi = \langle 4 \rangle$.

Which subgroups of K are normal? [All of them,] so the last statement of the theorem isn't particularly interesting, in this case.

More questions about the statement of Theorem 13.3?

Who felt like they understood the proof of this Theorem after reading the section? *show of hands* Please divide into groups; and please make sure that each group has one of these people in it. Look back over the proof of Theorem 13.3, discuss it with the people in your group, try to make sure everyone understands it. I'll be coming around to help as well. 1:00

The other point that a lot of people had questions about was Theorem 13.5. Who can tell me what this theorem says?

Suppose $H \lhd K \lhd G$ and $H \lhd G$. Then $K/H \lhd G/H$, and

$$\frac{G/H}{K/H} \cong G/K.$$

Again, let's look at an Example: Let $G = D_4, H = \langle 180 \rangle, K = \langle 90 \rangle$. First let's check that this setup actually satisfies the hypotheses of the Theorem. Why is $H \lhd K$? [K is abelian] Why is $H \lhd G$? [It's the center] Why is $K \lhd G$? [Index 2]

What are the groups G/H? K/H? G/K? What are the elements of a quotient group in general? [Cosets]

Grab a partner; work on figuring out what the elements of these groups are. When you've got it, please come write your answers at the board.

$$G/H = \{H; H \circ 90; H \circ V; H \circ D\}; K/H = \{H; H \circ 90\}; G/K = \{K; K \circ V\}$$

Those groups look a little ugly, no? Can we think of any way to simplify the picture? Are these groups isomorphic to anything familiar? [Since K/H and G/K both have only two elements, they must be isomorphic to $\mathbb{Z}_2$.]

Last week, if you recall, I said that $D_4/\langle 180 \rangle \cong V_4$. This is true, but there's actually another way to think of V_4; as $\mathbb{Z}_2 \times \mathbb{Z}_2$, and I'd rather use that representation. (Your book proves this on page 91, just below the proof of Theorem 10.5.)

Claim: $G/H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

I'd like to use the Fundamental Theorem to prove this. In other words, what do we have to do? *Think-pair-share* I'm going to define a homomorphism $\phi : D_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ and show that the kernel is $\langle 180 \rangle$. Then the Fundamental Theorem will tell us that

$$D_4/\langle 180 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2,$$

which is what we wanted to prove.

Define $\phi : D_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ by $\phi(90) = (1, 0)$ and $\phi(H) = \phi(V) = (0, 1)$. If i want ϕ to be a homomorphism, this forces the definition of the other elements of the group: What are they?

$$\begin{aligned}
\phi(0) &= (0, 0) \\
\phi(180) &= \phi(90) * \phi(90) = (1, 0) + (1, 0) = (0, 0) \\
\phi(270) &= \phi(90) * \phi(180) = (1, 0) \\
\phi(D) &= \phi(V) * \phi(90) = (0, 1) + (1, 0) = (1, 1) \\
\phi(OD) &= \phi(90) * \phi(V) = (1, 0) + (0, 1) = (1, 1)
\end{aligned}$$

Have we checked taht ϕ is a homomorphism? [no] We haven't finished checking that ϕ is a homomorphism; we still have to check that, for example, $\phi(D \circ OD) = \phi(D) * \phi(OD)$. But we're nearly out of time so I'm going to leave that to you.

Assuming that ϕ is a homomorphism, can we apply the Fundamental Theorem? [yes] Why? [we can see easily that ϕ is onto and that $\ker \phi = \{0, 180\}$. Therefore, by the Fundamental Theorem, we know that $D_4/\langle 180 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ as claimed. $\square$]

So now, let's rewrite those groups we had up on the board earlier:

$$\frac{G/H}{K/H} \to \frac{\mathbb{Z}_2 \times \mathbb{Z}_2}{\mathbb{Z}_2}; \quad G/K \to \mathbb{Z}_2.$$

Thinking about it this way, do you believe that (in this example at least)

$$\frac{G/H}{K/H} \cong G/K?$$

The idea of taking a quotient group is that you identify/squish/glue together the elements in the normal subgroup. With that heuristic, it's (hopefully) easy to believe that $\mathbb{Z}_2 \times \mathbb{Z}_2/\mathbb{Z}_2 \cong \mathbb{Z}_2$; we've simply collapsed one of the factors in the direct product.

if time More questions about the statement of Theorem 13.5, or this application?

Who felt like they understood the proof of this Theorem after reading the section? *show of hands* Please divide into groups; and please make sure that each group has one of these people in it. Look back over the proof of Theorem 13.5, discuss it with the people in your group, try to make sure everyone understands it. I'll be coming around to help as well.