Selected Solutions for Math 43

April 15, 2005

I thought that two of the problems in §2.5, namely #20 and #21, were especially interesting. Furthermore, proper solutions required a bit of care. (In particular, the author's hint on problem #20 didn't seem relevant to me.) The moral of these two problems, using $20 words, is that while it might not always be possible to find a global harmonic conjugate to given harmonic function in a domian D, we can always find local conjugates. In fact, we can find a conjugate in any disk that fits inside D.

§2.5 #20: Suppose that u is a harmonic function on the disk $D = \{\, z \in \mathbf{C} : |z - z_0| < d \,\}$. Show that u has a harmonic conjugate in D.

Solution: We are asked to find a function v such that $f(x+iy) := u(x,y)+iv(x,y)$ is analytic in D. For this, it suffices to show that v has continuous partial derivatives satisfying the Cauchy-Riemann equations throughout D with respect to u. Suppose that $z_0 = x_0 + iy_0$. If $(a,b) \in D$, then since D is a disk, notice that the line segment from (a,b) to $(a,0)$ lies wholly within D. Therefore we can define

$$v(a,b) := \int_{y_0}^{b} u_x(a,t)\,dt.$$

Now we easily see that the Fundamental Theorem of Calculus implies that

$$v_y(a,b) = u_x(a,b). \tag{1}$$

However, we can also calculate as follows:

$$\begin{aligned} v_x(a,b) &= \frac{\partial}{\partial x}\int_{y_0}^{b} u_x(a,t)\,dt \\ &= \int_{y_0}^{b} u_{xx}(a,t)\,dt \end{aligned}$$

which, since u is harmonic, is

$$= \int_{y_0}^{b} -u_{yy}(a,t)\,dt$$

which, by the Fundmental Theorem of Calculus, is

$$= -u_y(a,b).$$

Together with (1), this shows that u and v satisfy the Cauchy-Riemann equations. Therefore v is a Harmonic conjugate for u.

§2.5 #21: Let $u(x,y) = \ln|x+iy| = \frac{1}{2}\ln(x^2+y^2)$. We want to show that u is harmonic in $D := \mathbf{C} \setminus \{\,0\,\}$, but that u has no harmonic conjugate in (all of) D.

Solution: We start by showing that $u(x,y)$ and $v(x,y) = \operatorname{Arg}(x+iy)$ are Harmonic. At this point, we have to do this by brute force (because we haven't yet proved that $\operatorname{Log}(z)$ is analytic).[1] Straightforward computations show that

$$u_x(x,y) = \frac{x}{x^2+y^2} \qquad\qquad u_{xx} = \frac{y^2-x^2}{x^2+y^2}.$$

Then, without differentiation, we conclude that by symmetry that

$$u_y(x,y) = \frac{y}{x^2+y^2} \qquad\qquad u_{yy} = \frac{x^2-y^2}{x^2+y^2}.$$

In particular, $u_{xx} + u_{yy} = 0$ and u is Harmonic throughout D.

Now we *define*

$$w(x,y) := \operatorname{Arg}(x+iy). \tag{2}$$

We can use inverse trig functions to compute w, but we have to pay attention to which quadrant $x+iy$ is in and the definition of the range of the inverse trig functions to get the correct value for $w(x,y)$. But if $y>0$, then since $\cos^{-1}$ takes values in $[0,\pi]$, we have

$$w(x,y) = \cos^{-1}\Bigl(\frac{x}{\sqrt{x^2+y^2}}\Bigr).$$

An uninspiring computation reveals that

$$\frac{\partial}{\partial x}\cos^{-1}\Bigl(\frac{x}{\sqrt{x^2+y^2}}\Bigr) = -\frac{|y|}{x^2+y^2}.$$

[1] If you're paying attention, this will also give a different proof that $\operatorname{Log}(z)$ is analytic using the Cauchy-Riemann equations.

(Note the $|y|$ in the formula. This comes from the often overlooked fact that $\sqrt{y^2} = |y|$ in general — not y!) But if $y > 0$, then $|y| = y$, and

$$w_x(x,y) = -\frac{|y|}{x^2+y^2} = -\frac{y}{x^2+y^2} = -u_y(x,y).$$

But if $y < 0$, then since $\cos^{-1}$ takes values in $[0,\pi]$,

$$w(x,y) = -\cos^{-1}\Big(\frac{x}{\sqrt{x^2+y^2}}\Big),$$

and

$$w_x(x,y) = \frac{|y|}{x^2+y^2} = -\frac{y}{x^2+y^2} = -u_y(x,y).$$

Also if $x > 0$, then we can also write

$$w(x,y) = \tan^{-1}(\frac{y}{x}).$$

Once again, differentiation reveals that

$$w_x(x,y) = -u_y(x,y).$$

Let D^* be the domain from by the complex plane minus the nonpositive real axis. The above computations show that $w_x = -u_y$ throughout D^*. A similar set of computations shows that

$$w_y(x,y) = u_x(x,y)$$

throughout D^*. Since the partial derivatives of w are continuous throughout D^*, $w(x,y) = \operatorname{Arg}(x+iy)$ is a harmonic conjugate for u in D^*. But w has a jump discontinuity at each point on the negative real axis. If v were a harmonic conjugate for u in D, then v would also be a harmonic conjugate in D^* as well. As we proved in lecture, there would have to be a real constant a such that $v(x,y) = w(x,y) + a$ for all $(x,y) \in D^*$. But then v would have discontinuities on the negative real axis. This means v could not be a harmonic conjugate in D.