# Lecture Notes

## Math 3 / Fall 2012

## Week 4

# Derivative rules

| | |
|---|---|
| **Sum/Difference** | $(f \pm g)' = f' \pm g'$ |
| **Product** | $(f \cdot g)' = f' \cdot g + f \cdot g'$ |
| **Quotient** | $(f/g)' = (f' \cdot g - f \cdot g')/g^2$ |
| **Chain** | $(f \circ g)' = g' \cdot (f' \circ g)$ |
| **Inverse** | $(f^{-1})' = 1/(f' \circ f^{-1})$ |

# Inverse rule

## Theorem

*Suppose $f^{-1}$ is the inverse function of $f$. If $f$ is differentiable at $f^{-1}(x)$ and $f'(f^{-1}(x)) \neq 0$, then $f^{-1}$ is differentiable at $x$ and*

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Because $x = f(f^{-1}(x))$, the Chain Rule gives that:

$$1 = (f \circ f^{-1})'(x) = f'(f^{-1}(x))(f^{-1})'(x).$$

Therefore $(f^{-1})'(x) = 1/f'(f^{-1}(x))$.

# Reciprocal rule

## Theorem

*If $f$ is differentiable at $x$ and $f(x) \neq 0$, then the reciprocal $\frac{1}{f}$ is also differentiable at $x$ and*

$$\left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{(f(x))^2}.$$

Because $\frac{1}{f} = g \circ f$ where $g(x) = \frac{1}{x}$, we can use the Chain Rule to compute the derivative:

$$\left(\frac{1}{f}\right)'(x) = f'(x)g'(f(x)) = f'(x)\left(\frac{-1}{(f(x))^2}\right) = -\frac{f'(x)}{(f(x))^2}$$

# Quotient rule

## Theorem

*If $f$ and $g$ are both differentiable at $x$ and $g(x) \neq 0$, then the quotient $f/g$ is also differentiable at $x$ and*

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Because:

$$\begin{aligned}\left(\frac{f}{g}\right)'(x) &= \left(f \cdot \frac{1}{g}\right)'(x) = f'(x)\left(\frac{1}{g}\right)(x) + f(x)\left(\frac{1}{g}\right)'(x) \\ &= f'(x)\frac{1}{g(x)} + f(x)\frac{-g'(x)}{(g(x))^2} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}\end{aligned}$$

# Derivative formulas

$$\frac{d}{dx}\left[x^p\right] = px^{p-1}$$

$$\frac{d}{dx}\left[e^x\right] = e^x \qquad \frac{d}{dx}\left[b^x\right] = \ln(b)b^x$$

$$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x} \qquad \frac{d}{dx}\left[\log_b(x)\right] = \frac{1}{\ln(b)x}$$

$$\frac{d}{dx}\left[\sin(x)\right] = \cos(x) \qquad \frac{d}{dx}\left[\cos(x)\right] = -\sin(x)$$

$$\frac{d}{dx}\left[\tan(x)\right] = \sec^2(x) \qquad \frac{d}{dx}\left[\cot(x)\right] = -\csc^2(x)$$

$$\frac{d}{dx}\left[\sec(x)\right] = \sec(x)\tan(x) \qquad \frac{d}{dx}\left[\csc(x)\right] = -\csc(x)\cot(x)$$

# Power functions

## Theorem

*For <u>any</u> exponent $p$,* $\frac{d}{dx}\left[x^p\right] = px^{p-1}$.

Recall that $x^p = e^{p \ln x}$ for $x > 0$
This is $f \circ g$ where $f(x) = e^x$ and $g(x) = p \ln x$
We know that $f'(x) = e^x$ and $g'(x) = p/x$
The Chain Rule then says that:

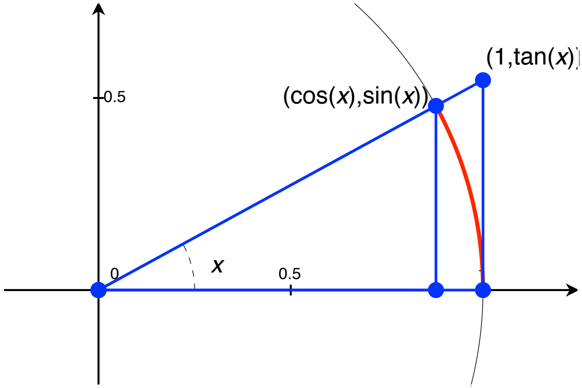
$$(f \circ g)'(x) = g'(x) f'(g(x)) = \frac{p}{x} e^{p \ln x} = p\frac{x^p}{x^1} = px^{p-1}$$

# Two fundamental limits

## Theorem

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \qquad \textit{and} \qquad \lim_{x\to 0} \frac{\cos(x) - 1}{x} = 0$$

Because:



# Two fundamental limits

$$
\begin{array}{rcccl}
\text{small } \triangle & \leq & \text{circular arc} & \leq & \text{large } \triangle \\
\frac{1}{2}\sin(x)\cos(x) & \leq & \left(\dfrac{x}{2\pi}\right)\cdot\pi & \leq & \frac{1}{2}\tan(x) \\
\cos(x) & \leq & \dfrac{x}{\sin(x)} & \leq & \dfrac{1}{\cos(x)} \\
\dfrac{1}{\cos(x)} & \geq & \dfrac{\sin(x)}{t} & \geq & \cos(x) \\
\downarrow & & \downarrow & & \downarrow \\
1 & & 1 & & 1
\end{array}
$$

# Two fundamental limits

$$
\begin{aligned}
\lim_{x\to 0}\frac{\cos(x)-1}{x} &= \lim_{x\to 0}\frac{\cos(x)-1}{x}\cdot\frac{\cos(x)+1}{\cos(x)+1}\\
&= \lim_{x\to 0}\frac{\cos^2(x)-1}{x}\cdot\frac{1}{\cos(x)+1}\\
&= \lim_{x\to 0}\frac{-\sin^2(x)}{x}\cdot\frac{1}{\cos(x)+1}\\
&= \lim_{x\to 0}-\sin(x)\cdot\frac{\sin(x)}{x}\cdot\frac{1}{\cos(x)+1}\\
&= -0\cdot 1\cdot\frac{1}{2} = 0
\end{aligned}
$$

# Derivatives of $\sin(x)$ and $\cos(x)$

**Theorem**

$$\frac{d}{dx}[\sin(x)] = \cos(x) \quad \textit{and} \quad \frac{d}{dx}[\cos(x)] = -\sin(x)$$

Because:

$$\begin{aligned}
\frac{d}{dx}[\sin(x)] &= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{(\sin(x)\cos(h) + \cos(x)\sin(h)) - \sin(x)}{h} \\
&= \lim_{h\to 0} \sin(x) \cdot \frac{\cos(h) - 1}{h} + \cos(x) \cdot \frac{\sin(h)}{h} \\
&= \sin(x) \cdot 0 + \cos(x) \cdot 1 = \cos(x)
\end{aligned}$$

$$\boxed{\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)}$$

# Derivatives of $\sin(x)$ and $\cos(x)$

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$$\frac{d}{dx}[\sin(x)] = \cos(x) \quad \textit{and} \quad \frac{d}{dx}[\cos(x)] = -\sin(x)$$

Because:

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\frac{d}{dx}[\cos(x)] &= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{(\cos(x)\cos(h) - \sin(x)\sin(h)) - \cos(x)}{h} \\
&= \lim_{h\to 0} \cos(x) \cdot \frac{\cos(h) - 1}{h} - \sin(x) \cdot \frac{\sin(h)}{h} \\
&= \cos(x) \cdot 0 - \sin(x) \cdot 1 = -\sin(x)
\end{aligned}$$

$$\boxed{\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)}$$

# Derivatives of $\tan(x)$ and $\sec(x)$

## Theorem

$$\frac{d}{dx}[\tan(x)] = \sec^2(x) \quad \textit{and} \quad \frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$$

Because:

$$\begin{aligned}
\frac{d}{dx}[\tan(x)] &= \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] \\
&= \frac{\frac{d}{dx}[\sin(x)]\cos(x) - \sin(x)\frac{d}{dx}[\cos(x)]}{\cos^2(x)} \\
&= \frac{\cos(x)\cos(x) + \sin(x)\sin(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} = \sec^2(x)
\end{aligned}$$

# Derivatives of $\tan(x)$ and $\sec(x)$

## Theorem

$$\frac{d}{dx}\left[\tan(x)\right] = \sec^2(x) \quad \textit{and} \quad \frac{d}{dx}\left[\sec(x)\right] = \sec(x)\tan(x)$$

Because:

$$\begin{aligned}
\frac{d}{dx}\left[\sec(x)\right] &= \frac{d}{dx}\left[\frac{1}{\cos(x)}\right] \\
&= \frac{-\frac{d}{dx}\left[\cos(x)\right]}{\cos^2(x)} \\
&= \frac{\sin(x)}{\cos^2(x)} \\
&= \frac{1}{\cos(x)} \cdot \frac{\sin(x)}{\cos(x)} = \sec(x)\tan(x)
\end{aligned}$$

# Derivative formulas

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# Derivative rules

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| **Sum/Difference** | $(f \pm g)' = f' \pm g'$ |
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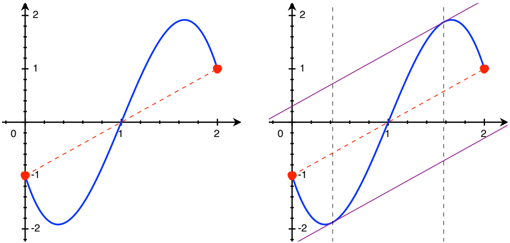
| **Inverse** | $(f^{-1})' = 1/(f' \circ f^{-1})$ |

# Mean value theorem

## Theorem

*Suppose that the function $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$. There is some $x$ in $(a, b)$ such that*

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$



# Monotonicity

Suppose $f$ is defined on some interval $I$ (of any kind)

- We say that $f$ is **increasing on** $I$ when

$$a < b \Rightarrow f(a) < f(b) \qquad \text{for all } a, b \text{ in } I$$

- We say that $f$ is **decreasing on** $I$ when

$$a < b \Rightarrow f(a) > f(b) \qquad \text{for all } a, b \text{ in } I$$

- We say that $f$ is **nondecreasing on** $I$ when

$$a < b \Rightarrow f(a) \leq f(b) \qquad \text{for all } a, b \text{ in } I$$

- We say that $f$ is **nonincreasing on** $I$ when

$$a < b \Rightarrow f(a) \geq f(b) \qquad \text{for all } a, b \text{ in } I$$

# Monotonicity and derivatives

Suppose the function $f$ is continuous on the interval $I$, and differentiable at every interior point of $I$.

- If $f'(x) > 0$ at every interior point of $I$

  then $f$ is increasing on $I$.
- If $f'(x) < 0$ at every interior point of $I$

  then $f$ is decreasing on $I$.
- If $f'(x) \geq 0$ at every interior point of $I$

  then $f$ is nondecreasing on $I$.
- If $f'(x) \leq 0$ at every interior point of $I$

  then $f$ is nonincreasing on $I$.

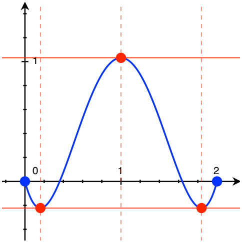
# Extreme value theorem

## Theorem

*Suppose $f$ is continuous on the closed interval $[a, b]$. Then there are numbers $x_{\min}$ and $x_{\max}$ in $[a, b]$ such that*

$$f(x_{\min}) \leq f(x) \leq f(x_{\max})$$

*for all numbers $x$ in $[a, b]$.*



# Extrema and derivatives

A **local maximum**/**local minimum** of $f$ is a value $x$ such that $x$ is the maximum/minimum of $f$ on some small interval around $x$.

**Theorem**

*Suppose $f$ has a local extremum at $x$.*

*If $f'(x)$ exists then $f'(x) = 0$.*

To find local extrema of a function $f$

look for places where $f'$ is zero or undefined

# Extrema and derivatives

- If $f$ changes from increasing to decreasing at $x$

  then $f$ has a local maximum at $x$
- If $f$ changes from decreasing to increasing at $x$

  then $f$ has a local minimum at $x$

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- If $f'$ changes from positive to negative at $x$

  then $f$ has a local maximum at $x$
- If $f'$ changes from negative to positive at $x$

  then $f$ has a local minimum at $x$

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- If $f'(x) = 0$ and $f''$ is negative around $x$

  then $f$ has a local maximum at $x$
- If $f'(x) = 0$ and $f''$ is positive around $x$

  then $f$ has a local minimum at $x$

# Mean value theorem: explanation

### Theorem

*Suppose that the function $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$. There is some $x$ in $(a, b)$ such that*

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

How would we prove the mean value theorem?
We first need to look at the special case where $f(a) = f(b)$...

# Mean value theorem: a special case

**Theorem**

*Suppose that the function $f$ is continuous on $[a, b]$, differentiable on $(a, b)$, and $f(a) = f(b)$. There is some $x$ in $(a, b)$ such that $f'(x) = 0$.*

The EVT says that there are some numbers $x_{\text{min}}$ and $x_{\text{max}}$ in $[a, b]$ where $f'$ is either undefined or 0.

- If $x_{\text{min}}$ is in $(a, b)$ then we must have $f'(x_{\text{min}}) = 0$.
- If $x_{\text{max}}$ is in $(a, b)$ then we must have $f'(x_{\text{max}}) = 0$.
- Otherwise, $f(a) = f(b) = f(x_{\text{min}}) = f(x_{\text{max}})$. So $f$ is constant on $[a, b]$ and then $f'(x) = 0$ for all $x$ in $(a, b)$.

# Mean value theorem: general case

In the general case, look at the function

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

This function $g$ is continuous on $[a, b]$, differentiable on $(a, b)$, and

$$g(b) = f(b) - \frac{f(b) - f(a)}{b - a}(b - a) = f(b) - (f(b) - f(a)) = g(a).$$

The special case of the mean value theorem applies to $g$ and there must be some $x$ in $(a, b)$ such that $g'(x) = 0$. Then

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} = 0$$

which means that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$