Borsuk-Ulam Theorem via the Liftasaurus

Theorem 1 *No continuous map*

$$f : S^2 \to S^1$$

satisfies that $f(-x) = -f(x)$ *for every* $x \in S^2$.

Proof: We will prove this by contradiction. To do so, let us suppose we have a continuous map

$$f : S^2 \to S^1,$$

that preserves anti-podal points, or, in other words, satisfies that $f(-x) = -f(x)$ for every $x \in S^2$. Recall that the if we identify anti-podal points, then we form nice covers (the general case is exercise 1 from the exam). Let

$$p : S^2 \to C^2$$

be the covering map from S^2 to th cross surface C^2 that identifies anti-podal points; and let

$$q : S^1 \to S^1$$

be covering map from the circle to itself that identifies anti-podal points (given by the z^2 map from exercise 6 section 54 of Munkres). We have the following commutative diagram.

Sub-lemma 1 *Assuming there exist an* f *such that* $f(-x) = -f(x)$ *for every* $x \in S^2$*, there exist a continuous map* $\tilde{f}$ *such that the following diagram commutes:*

$$\begin{array}{ccc} S^2 & \xrightarrow{f} & S^1 \\ \downarrow^{p} & & \downarrow^{q} \\ C^2 & \xrightarrow{\tilde{f}} & S^1 \end{array}$$

In other words, $\tilde{f}p = qf$.

Proof:(Lemma 1) We will be using lemma 22.2 of Munkres. Perhaps it is best to simply state who will play the roll of all the object in Munkres' lemma. The X from Munkres' lemma will be our S^2, Munkres' p is (fortunately) our quotient map p, Munkres' Y is our C^2, Munkres' g is our continuous qf, and Munkres' Z is our S^1 in the lower right hand corner. Notice that

$$g(x) = qf(x) = q(-f(x)) = qf(-x) = g(-x),$$

hence g is constant on $p^{-1}(x) = \{x, -x\}$ as needed to utilize lemma 22.2 and assert that Munkres' f, hence our needed $\tilde{f}$, exist, is continuous and satisfies our sought after $\tilde{f}p = g = qf$ condition. **q.e.d**

Since S^2 is path connected, we may choose a path λ that connects a pair of anti-podal points, $\{x, -x\} \subset S^2$. Following λ's journey through the diagram in lemma 1, we have the following immediate consequence of our sublemma.

Lemma 1

$$[\tilde{f}(p(\lambda))] = [q(f(\lambda))]$$

We will finish our proof off by contradicting this equality. Our contradiction will be that, as elements of $\pi_1(S^1, q(f(x)))$, that $[q(f(\lambda))]$ and $[\tilde{f}(p(\lambda))]$ are distinct. This will be an immediate consequence of the following pair of lemmas.

Lemma 2

$$[\tilde{f}(p(\lambda))] = 0$$

Proof:(Lemma 2) Notice $\lambda(0) = -\lambda(1)$, hence $p(\lambda)$ is loop at $p(x)$ in C^2. Recall that $\pi_1(C^2) = \frac{Z}{2Z}$ and $\pi_1(S^1) = Z$. Furthermore, notice that every homomorphism of $\frac{Z}{2Z}$ into the integers is the zero, since the nontrivial element of $\frac{Z}{2Z}$ must go to an element $m \in Z$ with the property that $2m = 0$, hence $m = 0$. Now $\tilde{f}_\star$ is such an homomorphism hence $\tilde{f}_\star = 0$. In particular,

$$0 = \tilde{f}_\star[p(\lambda)] = [\tilde{f}(p(\lambda))],$$

as needed. **q.e.d**

Lemma 3

$$[q(f(\lambda))] \neq 0$$

Proof:(Lemma 3) Notice, from the path lifting lemma that $f(\lambda)$ is the unique lift of $q(f(\lambda))$ starting at $f(x)$. Furthermore, notice since $f($ preserves anti podal pairs that $f(\lambda)$ has distinct end points, namely

$$f(\lambda)(0) = f(x) \neq -f(x) = f(\lambda)(1).$$

If $[q(f(\lambda))]$ were indeed equal to 0 in $\pi_1(S^1, q(f(x)))$, then there would be a homotopy rel $\{0,1\}$ between $q(f(\lambda))$ constant path $q(f(x))$. By the homotopy lifting lemma, this homotopy would lift to a homotopy rel $\{0,1\}$ between $f(\lambda)$ and the constant path $f(x)$. Such a homotopy immediately contradicts the fact that $f(\lambda)$ has distinct end points. **q.e.d**

q.e.d (theorem 1)

This same argument tells us some information about maps from S^1 to itself. View the circle as $R/\mathbf{Z}$ via the usual action of the integers, $\phi(m)(x) = m + x$. Furthermore, from our key theorem this action gives us a canonical isomorphism between $\pi_1(S^1, x)$ and $\mathbf{Z}$. In particular, any continuous map

$$f : S^1 \to S^1$$

induces a mapping

$$f_\star : \pi_1(S^1, x) \to \pi_1(S^1, f(x))$$

and hence a homomorphism

$$f_\star : \mathbf{Z} \to \mathbf{Z}.$$

Any homomorphism ψ from $\mathbf{Z}$ to itself is given by $\psi(m) = deg(f)m$ for some integer $deg(f)$. Our next result is about this integer. Namely in the second problem of the final you will be asked to prove the following theorem.

Theorem 2 *If*

$$f : S^1 \to S^1$$

satisfies that $f(-x) = -f(x)$ *for every* $x \in S^1$, *then* $deg(f)$ *is odd.*