HOMEWORK 9

SOLUTIONS (SKETCHES)

4.2.23 **Use Menger's Theorem $\kappa(x, y) = \lambda(x, y)$ when $xy \notin E(G)$) to prove the Konig-Egervary Theorem ($\alpha'(G) = \beta(G)$ when G is bipartite).**

Proof. Let G be a X, Y bipartite graph. Construct H with $V(H) = V(G) \cup \{a, b\}, E(H) = E(G) \cup T$, where $T = \{ax, by : \forall x \in X, \forall y \in Y\}$

$\lambda(a, b) = \alpha'(G)$. Choose any maximum matching which we denote $\{x_i, y_i\}$. Then $a \to x_j \to y_j \to b$ is internally disjoint from $a \to x_k \to y_k \to b$ for all $i \neq j$ in our matching. Thus $\lambda_{G'}(a, b) \geq \alpha'(G)$. If $\lambda_{G'}(x, y) > \alpha'(G)$, we can see that this would provide a matching that would be greater than maximum.

In order to disconnect a, b in G', we need to delete vertices so that no edges from X to Y exist, or else that would be a path. Thus a disconnecting set of a, b is a vertex cover of G. Therefore $\kappa_{G'}(a, b) \geq \beta(G)$.

We have $\alpha'(G) = \lambda_{G'}(a, b) = \kappa_{G'}(a, b) \geq \beta(G)$. We know that $\alpha'(G) \leq \beta(G)$ because distinct vertices must be used to cover the edges of a matching.

Thus we have $\alpha' = \beta$ when G bipartite.

□

4.3.2 **In the network below, find a maximum flow from s to g. Prove that your answer is optimal by using the dual problem, and explain why this proves optimality.**

Proof. Maximum flow is 17. The minimum cut is to delete the 2×2 set of vertices of the bottom right (including t). Note that the edge with capacity 6 is directed from T to S, so we do not cut it. Then the capacity of a minimum cut is $3 + 4 + 5 + 2 + 3 = 17$. Any flow of value of 17 therefore must be maximum by 4.3.8.

□

4.3.5 **Use network flows to prove Menger's Theorem for internally disjoint paths in digraphs: $\kappa(x, y) = \lambda(x, y)$ when xy is not an edge.**

Proof. We know that $\kappa(x, y) \geq \lambda(x, y)$ for all digraphs.

Construct G' from G such that every vertex $v \in V(G) - \{x, y\}$ is replaced two vertices v^- and v^+. Connect these with vertex v^-v^+ with capacity 1. Adjacency is as before, all edges entering v enter v^- and edges leaving v leave v^+. Now for each edge u^+v^-, assign an arbitrarily large capacity. x is the source, y is the sink.

Now solve the min-max flow/cut problem. Suppose m is the maximum $val(f)$, f feasible and the minimum $|[S,T]|$. A maximum flow corresponds to m internally disjoint x,y paths (as in remark 4.3.13). Therefore, $\lambda(x,y) \geq m$.

Let $[S,T]$ be a minimum cut of capacity m. This means that $\kappa(x,y) \leq m$ because $[S,T]$ disconnects x,y. Because we assigned arbitrarily large values to the edges, we are only cutting those v^-v^+ edges that we've constructed. This corresponds directly to κ.

We have $\kappa(x,y) \leq m \leq \lambda(x,y)$, which means $\kappa = \lambda$, proving Menger's Theorem for digraphs.

□

4.3.13 **Several companies send representatives to a conference; the ith company sends m_i representatives. The organizers of the conference conduct simultaneous networking groups; the jth group can accommodate up to n_j participants. The organizers want to schedule all the participants into groups, but the participants from the same company must be in different groups. The groups need not all be filled.**

(a) Show how to use network flows to test whether the constraints can be satisfied.

Answer. Create a network with 4 "parts". From "left to right" we have source s, then k vertices $x_1, \dots, x_p$, one for each company. From s to x_i, draw an edge of capacity m_i. The 3rd part has l vertices $y_1, \dots, y_q$, one for each group. For each x_i, y_j, draw an edge of capacity 1 from x_i to y_j. Finally, the 4th part is the sink t. Draw edges y_j to t with capacity n_j.

We can assign everyone to a group if and only if $val(f) = m_1 + \cdots + m_p = M$ for the k companies. □

(b) Let p be the number of companies, and let q be the number of groups, indexed so that $m_1 \geq \cdots \geq m_p$ and $n_1 \leq \cdots \leq n_q$. Prove that there exists an assignment of participants to groups that satisfies all the constraints if and only if, for all $0 \leq k \leq p$ and $0 \leq l \leq q$, it holds that $k(q-l) + \sum_{j=1}^{l} n_j \geq \sum_{i=1}^{k} m_i$.

Proof. We can find a flow of value M if and only if there is a cut of capacity less than M. Consider a cut $[S,T]$. Let k be the number of vertices in S from the "companies" and l be the number of vertices of S from the "groups". Then the capacity of the cut is $\sum_{x_i \in T} m_i + \sum_{y_j \in S} n_j + k(q-l)$.

The first part is those edges that get disconnected from a, the second part is the sum of the capacities that get disconnected from the sink. The last part is the sum of the capacities that get deleted between the companies and the groups.

We want this greater than or equal to $M = \sum m_i$, which is the inequality in the book.

□

5.1.1 **Compute the clique number, the independence number, and the chromatic number of the graph below. Does either bound in Prop. 5.1.7 prove optimality for some proper coloring? Is the graph color-critical?**

Proof. $\omega(G) = 3$.

$\alpha(G) = 2$.

$\chi(G) = 4$.

5.1.7 says $\chi(G) \geq n(G)/\alpha(G) = 7/2 = 3.5$, thus we have optimality.

Further, this graph is color critical. By exhaustion from deleting one edge at a time. $\square$

5.1.19 **Find the error in the false argument below for Brook's Theorem (5.1.22)**

"We use induction on $n(G)$; the statement holds when $n(G) = 1$. For the induction step, suppose that G is not a complete graph or an odd cycle. Since $\kappa(g) \leq \delta(G)$, the graph G has a separating set S of size at most $\Delta(G)$. Let $G_1, \dots, G_m$ be the components of $G - S$, and let $H_i = G[V(G_i) \cup S]$. By the induction hypothesis, each H_i is $\Delta(G)-$colorable. Permute the names of the colors used on these subgraphs to agree on S. This yields a proper $\Delta(G)$-coloring of G."

Proof. It may be impossible to permute the colors of S to agree. Consider C_5 with some stuff attached to two adjacent vertices u, v. Let x, y be adjacent to u, v in C_5, and let them be the separating set. It's possible to construct a graph such that: In the P_3 component, x, y get the same color, but not in the other component.

$\square$

5.1.31 **Prove that a graph G is m-colorable if and only if $\alpha(G \square K_m) \geq n(G)$.**

Proof. G is m-colorable. $\chi(G \square K_m) = \max\{\chi(G), \chi(K_m)\} = m$.

$\chi(G \square K_m) \geq n(G \square K_m)/\alpha(G \square K_m)$ (Proposition 5.1.7).

Then $\alpha(G \square K_m) \geq n(G \square K_m)/\chi(G \square K_m) = n(G)m/m = n(G)$.

Suppose that $\alpha(G \square K_m) \geq n(G)$. Then $\alpha(G \square K_m) = n(G)$, because we can only take one vertex from each of the $n(G)$ copies of K_m. Let A be a maximum independent set of size $n(G)$. There are m disjoint copies of these (just rotate the K_m coordinate of each vertex in A, up to m times). Therefore we can color $G \square K_m$ with m colors, which means $\chi(G) \leq m$.

$\square$

5.1.38 **Prove that $\chi(G) = \omega(G)$ when $\bar{G}$ is bipartite.**

Proof. Let G be a graph such that $\bar{G}$ is bipartite. Suppose for now that $\bar{G}$ has no isolated vertices. Then construct some minimum edge cover in $\bar{G}$. We know this has size equal to $\alpha(\bar{G}) = \omega(G)$ (Corollary 3.1.24). Take this edge cover as a subgraph H. Every component of this subgraph is a star (otherwise P_3 is a subgraph of a component, violating "minimum edge cover"). For those components that are P_2, just label one of the vertices as the center of the star. Assign each of the leaves in H a color (every single leaf has a distinct color, even leaves of different stars). Assign the center of each star the color of one of its leaves. Then we have exactly as many colors as there are edges in H, which is $\beta'(\bar{G})$.

We claim that this produces a a vertex coloring of G. Suppose not. Clearly we cannot have the leaves in a tree be the same color. If they were, then they are not adjacent in G, which means that leaf, leaf, center produces C_3 in $\bar{G}$, a contradiction. For some vertex in one component of H, suppose that we could color it k for some color k in another component. Then they must be adjacent in $\bar{G}$. If they were both leaves, then we could form a smaller edge cover by taking just the edge connecting the two and dropping the two edges connecting them to their respective centers. If one or both is a center, then changing their color to another leaf does not reduce the number of colors used.

Therefore we've shown $\chi(G) = \omega(G) = \alpha(\bar{G}) = \beta'(\bar{G})$. Note that if $\bar{G}$ has isolated vertices, those vertices are adjacent to every vertex in G, thus increase χ by 1 each, and they also increase $\alpha(\bar{G}) = \omega(G)$ by 1 each, holding the equality. (We ditch the β' argument).

$\square$