Lecture Notes

Math 3 / Fall 2012

Week 9

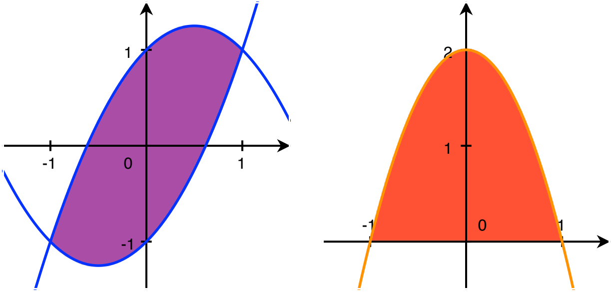
Cavalieri's principle

Suppose two regions in a plane are included between two parallel lines in that plane. If every line parallel to these two lines intersects both regions in line segments of equal length, then the two regions have equal areas.

Area between two functions

If $f(x) \geq g(x)$ for all x in $[a, b]$ then the area enclosed by the graphs $y = f(x)$ and $y = g(x)$ and the vertical lines $x = a$, $x = b$ is

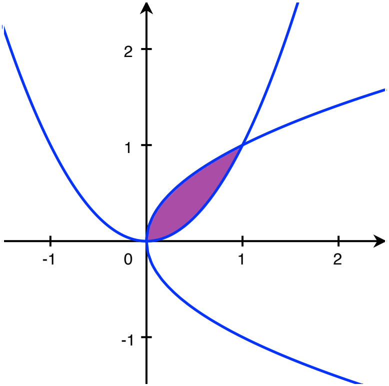
$$\int_a^b (f(x) - g(x))dx.$$



The right bounds

Example

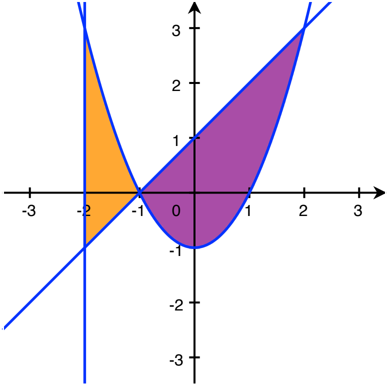
Find the area in the first quadrant enclosed by the parabolas $y = x^2$ and $y^2 = x$.



Divide and conquer

Example

Find the area bounded by $y = x^2 - 1$ and the lines $y = x + 1$ and $x = -2$.



Example

Find the area of the region bounded by $y = x^3 - 6x^2 - 16x$ and $y = 8x + 2x^2 - x^3$.

$$(x^3 - 6x^2 - 16x) - (8x + 2x^2 - x^3) = 2x^3 - 8x^2 - 24x = 2x(x+2)(x-6)$$

Intersections at $(-2, 0), (0, 0), (6, -96)$.

- From $x = -2$ to $x = 0$:

$$\left| \int_{-2}^{0} (2x^3 - 8x^2 - 24x) dx \right| = \left| 2\frac{(-2)^4}{4} - 8\frac{(-2)^3}{3} - 24\frac{(-2)^2}{2} \right| = \frac{56}{3}$$

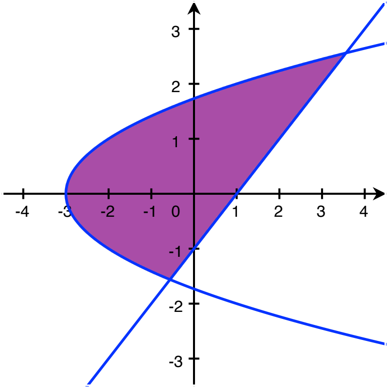
- From $x = 0$ to $x = 6$:

$$\left| \int_{0}^{6} (2x^3 - 8x^2 - 24x) dx \right| = \left| 2\frac{6^4}{4} - 8\frac{6^3}{3} - 24\frac{6^2}{2} \right| = 360$$

Change of perspective

Example

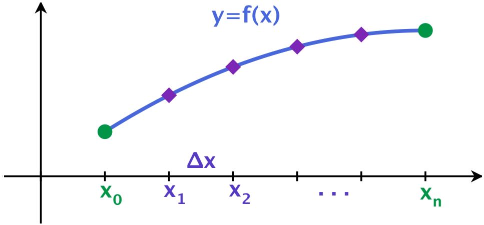
Find the area enclosed by the curves $x - y^2 = -3$ and $x - y = 1$.



Arc length

We can approximate the **arc length** of the graph $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$ via the sum:

$$\sqrt{(\Delta x_1)^2 + (\Delta y_1)^2} + \sqrt{(\Delta x_2)^2 + (\Delta y_2)^2} + \cdots + \sqrt{(\Delta x_n)^2 + (\Delta y_n)^2}$$



Integral formula for arc length

Suppose f is differentiable on (a, b). By the mean value theorem, $\Delta y_i = f'(c_i)\Delta x_i$ for some c_i in (x_{i-1}, x_i). So

$$\begin{aligned}\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} &= \sqrt{(\Delta x_i)^2 + (f'(c_i))^2(\Delta x_i)^2} \\ &= \sqrt{1 + (f'(c_i))^2}\,\Delta x_i\end{aligned}$$

As $n \to \infty$,

$$\sum_{i=1}^{n} \sqrt{1 + (f'(c_i))^2}\,\Delta x_i \to \int_a^b \sqrt{1 + (f'(x))^2}\,dx$$

Warning!!!

Arc length integrals are notoriously difficult!

1. $f(x) = x^2$ from $x = -3$ to 2
2. $f(x) = x^2 + 5$ from $x = -3$ to 2
3. $f(x) = -x^2 + \pi$ from $x = -3$ to 2
4. $f(x) = \sin(x)$ from $x = 0$ to $\frac{\pi}{2}$
5. $f(x) = e^x$ from $x = 0$ to 1
6. $f(x) = \sqrt{1 - x^2}$ from $x = -1$ to 1

Use the trapezoid rule or Simpson's rule to get approximate values...

Sneaky squares...

Evaluate the arc length of $f(x) = \dfrac{x^2 - 2\ln x}{4}$ from $x = 1$ to $x = 3$

$f'(x) = \frac{1}{2}(x - 1/x)$ and so

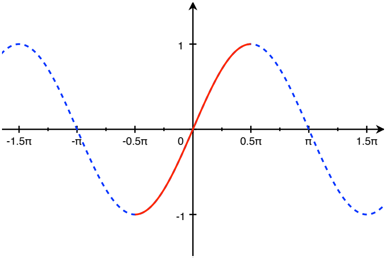
$$
\begin{aligned}
1 + (f'(x))^2 &= \frac{1}{4}\left(4 + \left(x - \frac{1}{x}\right)^2\right) \\
&= \frac{1}{4}\left(x^2 + 2 + \frac{1}{x^2}\right) = \frac{1}{4}\left(x + \frac{1}{x}\right)^2
\end{aligned}
$$

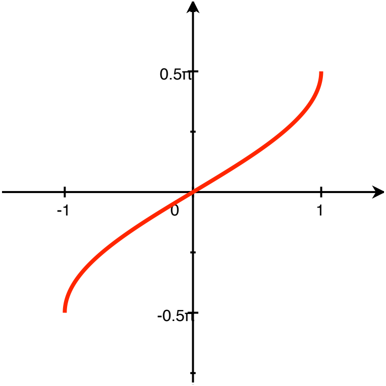
So

$$\int_1^3 \sqrt{1 + (f'(x))^2}\, dx = \frac{1}{2}\int_1^3 \left(x + \frac{1}{x}\right) dx = 2 + \frac{\ln 3}{2}$$

Arcsine

The function arcsin(x) or asin(x) or $\sin^{-1}(x)$ has domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$.





Arcsine: derivative

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$$

Because if

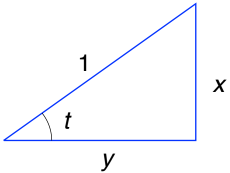
$$t = \arcsin(x)$$

then $\sin(t) = x$
and $\cos(t) = y = \sqrt{1-x^2}$.
So

$$\cos(t)\frac{dt}{dx} = 1$$

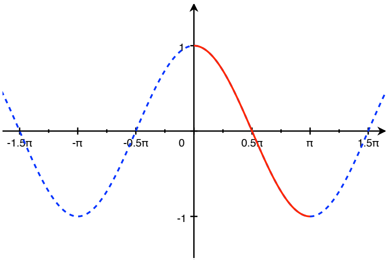
or

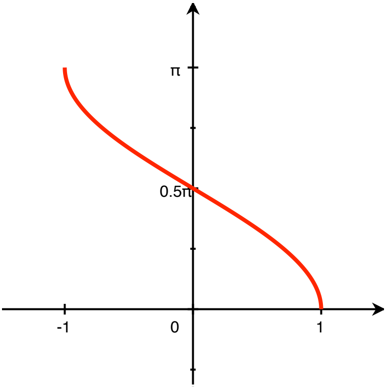
$$\frac{dt}{dx} = \frac{1}{y} = \frac{1}{\sqrt{1-x^2}}.$$



Arccosine

The function $\arccos(x)$ or $\operatorname{acos}(x)$ or $\cos^{-1}(x)$ has domain $[-1, 1]$ and range $[0, \pi]$.





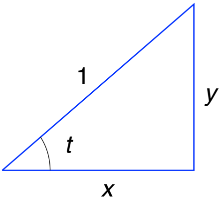
Arccosine: derivative

$$\frac{d}{dx}\left[\arccos x\right] = \frac{-1}{\sqrt{1-x^2}}$$

Because if

$$t = \arccos(x)$$

then $\cos(t) = x$
and $\sin(t) = y = \sqrt{1-x^2}$.



So

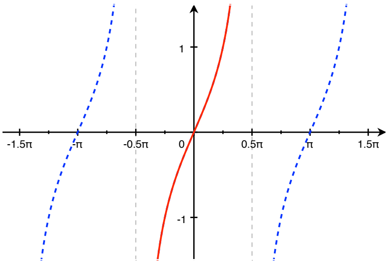
$$-\sin(t)\frac{dt}{dx} = 1$$

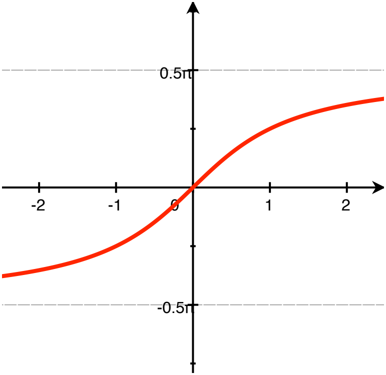
or

$$\frac{dt}{dx} = \frac{-1}{y} = \frac{1}{\sqrt{1-x^2}}.$$

Arctangent

The function arctan(x) or atan(x) or $\tan^{-1}(x)$ has domain $(-\infty, \infty)$ and range $(-\pi/2, \pi/2)$.





Arctangent: derivative

$$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$$

Because if

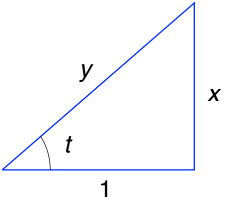
$$t = \arctan(x)$$

then $\tan(t) = x$
and $\sec(t) = y = \sqrt{1+x^2}$.
So

$$\sec^2(t)\frac{dt}{dx} = 1$$

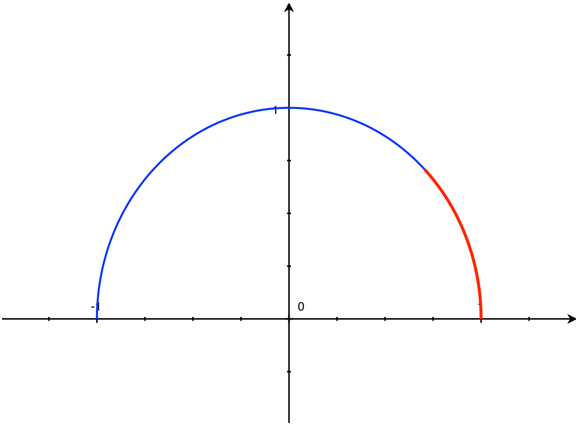
or

$$\frac{dt}{dx} = \frac{1}{y^2} = \frac{1}{1+x^2}.$$



Circular arc length

What is the length of the arc of the unit circle from $(1, 0)$ to $(a, \sqrt{1 - a^2})$?



Circular arc length

What is the length of the arc of the unit circle from $(1, 0)$ to $(a, \sqrt{1 - a^2})$?

Since

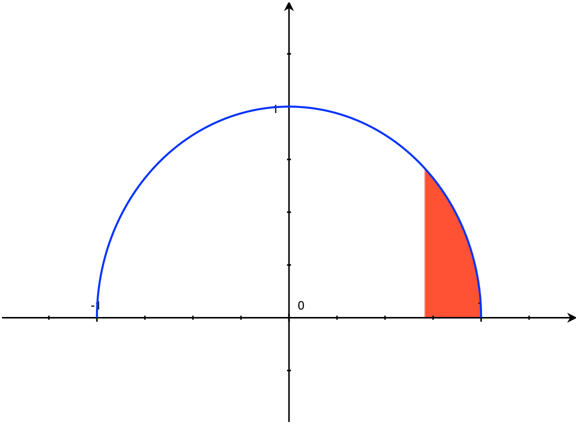
$$\frac{d}{dx}\left[\sqrt{1 - x^2}\right] = \frac{-x}{\sqrt{1 - x^2}}$$

the arc length is:

$$\begin{aligned}
\int_a^1 \sqrt{1 + \frac{x^2}{1 - x^2}}\, dx &= \int_a^1 \frac{1}{\sqrt{1 - x^2}}\, dx \\
&= \arcsin(1) - \arcsin(a) = \frac{\pi}{2} - \arcsin(a) \\
&= -\arccos(1) + \arccos(a) = \arccos(a)
\end{aligned}$$

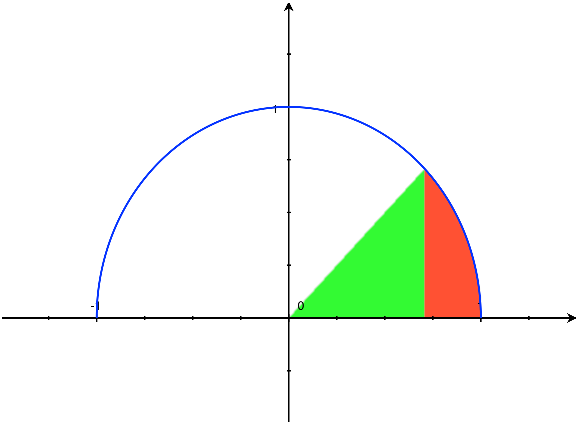
Circular area

$$\int_a^1 \sqrt{1 - x^2}\, dx$$



Circular area

$$\int_a^1 \sqrt{1-x^2}\,dx = \frac{1}{2}\left(\arccos(a) - a\sqrt{1-a^2}\right)$$



A difficult integral

$$\int \frac{2x+1}{x^2-6x+10}\,dx = \ln(x^2-6x+10) + 7\arctan(x-3) + C$$

Note that $x^2-6x+10 = (x-3)^2+1$. With $u = x-3$:

$$\int \frac{2x+1}{x^2-6x+10}\,dx = \int \frac{2u+7}{u^2+1}\,du = \int \frac{2u}{u^2+1}\,du + \int \frac{7}{u^2+1}\,du$$

With $w = u^2+1$:

$$\int \frac{2u}{u^2+1}\,du = \int \frac{1}{w}\,dw = \ln|w| + C = \ln(u^2+1) + C.$$

Also: $7\displaystyle\int \frac{1}{1+u^2}\,du = 7\arctan(u) + C.$