Euler Equations

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A second order homogeneous linear ODE of the form

$$t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0, \tag{1}$$

with α, β some constants, is called an *Euler equation*. In this note we will show how a simple change of variables (analogous to a substitution in an integral) can be used to transform an Euler equation into a second order homogeneous linear ODE with constant coefficients. Since we know how to find the general solution to the latter type of equation, we can therefore always find the general solution to any Euler equation.

We begin with the change of variable

$$x = \ln t.$$

Since the set of real numbers x corresponds in a one-to-one fashion with the set of positive numbers t, we can view any function y of t as a function of x, and vice-versa. For example, the function of t

$$y = 12t - \sin t$$

becomes the function of x

$$y = 12e^x - \sin(e^x)$$

since $t = e^x$.

Our immediate task is to determine how changing the independent variable from t to x changes the given differential equation. The chain rule tells us how the derivative of y with respect to the "new" variable x is related to the derivative of y with respect to the "old" variable t. Indeed:

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}t. \tag{2}$$

Here we have used the fact that $t = e^x$ so that $dt/dx = e^x = t$. Likewise, we can use the chain rule again (together with the product rule) to get an analogous expression for the second derivative of y with respect to x. Using the above we have

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\frac{dy}{dx} \\
&= \frac{d}{dx}\left(\frac{dy}{dt}t\right) \\
&= \left(\frac{d}{dx}\frac{dy}{dt}\right)t + \frac{dy}{dt}\frac{dt}{dx} \\
&= \frac{d^2y}{dt^2}\frac{dt}{dx}t + \frac{dy}{dt}\frac{dt}{dx} \\
&= \frac{d^2y}{dt^2}t^2 + \frac{dy}{dt}t
\end{aligned}$$

where we have again used the fact that $dt/dx = t$. Combining this with (2) gives the relations

$$\frac{dy}{dt}t = \frac{dy}{dx} \tag{3}$$

$$\frac{dy}{dt}t^2 = \frac{d^2y}{dx^2} - \frac{dy}{dx}. \tag{4}$$

We now substitute (3) and (4) into the Euler equation (1) to see what it becomes as a result of our change of variables. We get

$$\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) + \alpha\frac{dy}{dx} + \beta y = 0.$$

Putting the dy/dx terms together, we are led to the following.

Conclusion. Under the change of variables $x = \ln t$, the Euler equation

$$t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0$$

is transformed into the constant-coefficient equation

$$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0. \tag{5}$$

The moral of the story is that we now have a procedure for finding the general solution to Euler equations:

1. Perform the change of variable $x = \ln t$, yielding the constant coefficient equation (5).
2. Find the general solution to (5) using the techniques we already know. This solution will be in terms of the independent variable x.
3. Eliminate x by substituting $x = \ln t$, yielding the general solution to (1) in terms of the original independent variable t.

We demonstrate this procedure with a few examples.

Example 1. Find the general solution to the Euler equation

$$t^2y'' - 3ty' + 7y = 0, \qquad t > 0. \tag{6}$$

Solution. According to the work above, the variable change $x = \ln t$ transforms the given differential equation into

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 7y = 0 \tag{7}$$

(we have $\alpha = -3, \beta = 7$). The characteristic equation of this DE is

$$r^2 - 4r + 7 = 0$$

which has the roots

$$r = \frac{4 \pm \sqrt{16 - 28}}{2} = 2 \pm \sqrt{3}i.$$

Therefore, the general solution to (7) is

$$y = c_1e^{2x}\cos\sqrt{3}x + c_2e^{2x}\sin\sqrt{3}x.$$

To get the solution to (6) we simply remember the relationship $x = \ln t$:

$$y = c_1t^2\cos(\sqrt{3}\ln t) + c_2t^2\sin(\sqrt{3}\ln t).$$

Example 2. Find the general solution to the equation

$$4t^2y'' + 8ty' - 3y = 0, \qquad t > 0. \tag{8}$$

Solution. Once we divide (8) by 4, we recognize this as an Euler equation with $\alpha = 2$ and $\beta = -3/4$. Consequently, we know that the variable substitution $x = \ln t$ transforms (8) into the equivalent equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{3}{4}y = 0. \tag{9}$$

The characteristic equation of the latter is

$$r^2 + r - \frac{3}{4} = \left(r + \frac{3}{2}\right)\left(r - \frac{1}{2}\right) = 0$$

which has the roots $r_1 = 1/2$ and $r_2 = -3/2$. Therefore, the general solution to (9) is

$$y = c_1 e^{x/2} + c_2 e^{-3x/2}.$$

Switching back to the variable t through our substitution $x = \ln t$ we are led to the general solution of (8):

$$y = c_1 t^{1/2} + c_2 t^{-3/2}.$$

Example 3. Find the general solution to the equation

$$t^2 y'' - 2y = 0. \tag{10}$$

Solution. Notice that we considered this example in class: we found the general solution after an "oracle" had given us two fundamental solutions. Now we can find these fundamental solutions ourselves!

The equation (10) is an Euler equation with $\alpha = 0$ and $\beta = -2$. The usual change of variables $x = \ln t$ therefore turns it into

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0 \tag{11}$$

which has characteristic equation

$$r^2 - r - 2 = (r-2)(r+1) = 0$$

so that the general solution is

$$y = c_1 e^{2x} + c_2 e^{-x}.$$

Changing back to t via $x = \ln t$ we find that the general solution to (11) is

$$y = c_1 t^2 + c_2 t^{-1}.$$

This is exactly what we showed in class!