Yangians

The Yangian is sort of a "flat" deformation of the enveloping algebra of $\mathfrak{g}[z]$ as a Hopf algebra. In this way, it is a degeneration of the affine quantum group. So there are a couple of definitions of the Yangian. The first one is going to look like something that is directly defined out polynomials on the general linear Lie algebra. The second will come from the analog that affine Lie algebras are constructed from loop generators, and the Yangian is generated by the degenerate analog of the loop generators in the quantum group. Neither of these are the original definition, but both seem to dominate the literature.

1 RTT

Consider the general linear Lie algebra $\mathfrak{gl}_n$ with standard basis E_{ij} with $i, j = 1, \ldots, n$ and relations

$$[E_{i,j}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj}.$$

Let $E = (E_{ij})$ be the $n \times n$ matrix with i-jth entry E_{ij}. This satisfies

$$[(E^{r+1})_{ij}, (E^s)_{kl}] - [(E^r)_{ij}, (E^{s+1})_{kl}] = (E^r)_{kj}(E^s)_{il} - (E^s)_{kj}(E^r)_{il}, \tag{1}$$

where $r, s \geq 0$ and $E^0 = \mathrm{id}$. The following definition is the RTT definition, and is motivated by the relation in (1). Later we'll see that this definition leads naturally to a focus on R-matrices.

Definition 1.1 ([Mo, Defn 1.1.1])**.** The *Yangian* for $\mathfrak{gl}_n$ os a unital associative algebra over $\mathbb{C}$ with countably many generators $t_{ij}^{(r)}$ where $r = 1, 2, \ldots$ and $i, j = 1, \ldots, n$, and the defining relations

$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(s)} t_{il}^{(r)} - t_{kj}^{(r)} t_{il}^{(s)}, \tag{2}$$

where $r, s = 0, 1, \ldots$ and $t_{ij}^{(0)} = \delta_{ij}$. This algebra is denoted by $\mathcal{Y}\mathfrak{gl}_n$

Given a linear order on the set of generators, any element can be written uniquely as a linear combinations of ordered monomials in these generators ([Mo, Thm 1.4.1]). Introducing the formal generating series

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \cdots \in \mathcal{Y}\mathfrak{gl}_n[[u^{-1}]], \tag{3}$$

we can write (2) in the form

$$(u - v)[t_{ij}(u), t_{kl}(v)] = t_{kj}(u) t_{il}(v) - t_{kj}(v) t_{il}(u) \tag{4}$$

the indeterminants u and v are considered to be commuting with each other and the elements of the Yangian.

Proposition 1.1 ([Mo, Proposition 1.1.2]). *The system of relations in* (2)*is equivalent to the system*

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] = \sum_{a=1}^{\min(r,s)} \left(t_{kj}^{(a-1)} t_{il}^{(r+s-1)} - t_{kj}^{(r+s-1)} t_{il}^{(a-1)} \right). \tag{5}$$

Proposition 1.2 ([Mo, Proposition 1.1.3]). *The assignment*

$$\pi_n : t_{ij}(u) \mapsto \delta_{ij} + E_{ij} u^{-1} \tag{6}$$

defines a surjective homomorphism from the Yangian onto the universal enveloping algebra $\mathcal{U}\mathfrak{gl}_n$. Moreover, the assignment

$$E_{ij} \mapsto t_{ij}^{(1)} \tag{7}$$

defines an embedding $\mathcal{U}\mathfrak{gl}_n \hookrightarrow \mathcal{Y}\mathfrak{gl}_n$.

Let $T(u) = (t_{ij}(u))$ be the $n \times n$ matrix with i-$j$the entry $t_{ij}(u)$, i.e.

$$T(u) = \sum_{1 \leq i,j \leq n} e_{ij} \otimes t_{ij}(u)$$

where e_{ij} is the standard matrix unit. We really like to think about $T(u)$ as an operator on $\mathbb{C}^n$ with coefficients in $\mathcal{Y}\mathfrak{gl}_n[[u^{-1}]]$. Let

$$V = \mathrm{End}(\mathbb{C}^n).$$

We extend $T(u)$ to an operator on $(V)^{\otimes m} \otimes \mathcal{Y}\mathfrak{gl}_n$, letting $T_a(u)$ be the operator $T(u)$ corresponding to ath factor, i.e.

$$T_a(u) = \sum_{1 \leq i,j \leq n} \mathrm{id}^{\otimes(a-1)} \otimes e_{ij} \otimes \mathrm{id}^{\otimes m-a} \otimes t_{ij}(u).$$

Consider the permutation operator

$$P = \sum_{1 \leq i,j \leq n} e_{ij} \otimes e_{ji}.$$

The rational function

$$R(u) = 1 - Pu^{-1}, \qquad \text{which satisfies} \quad R(u)R(-u) = 1 - u^{-2},$$

is called the *Yang R-matrix*, where $1 = \mathrm{id} \otimes \mathrm{id}$. Denote by t the operator which switches e_{ij} with e_{ji}, and let $R^t = 1 - tPu^{-1}$. Since $(tP)^2 = ntP$, we have

$$R^t(u)^{-1} = 1 + tP(u-n)^{-1}.$$

Proposition 1.3 (Yang–Baxter equation, [Mo, Proposition 1.2.1]). *In the algebra $V^{\otimes 3}(u, v)$ we have the identity*

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u).$$

Then a third characterization of the relations in (2) is

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v).$$

Theorem 1.4 ([Mo, Thm 1.5.1])**.** *The Yangian $\mathcal{Y}\mathfrak{gl}_n$ is a Hopf algebra with*

$$\begin{aligned} \textit{comultiplication} \quad & \Delta : t_{ij}(u) \mapsto \sum_{k=1}^{n} t_{ik}(u) \otimes t_{kj}(u), \\ \textit{antipode} \quad & T(u) \mapsto T^{-1}(u), \textit{ and} \\ \textit{counit} \quad & T(u) \mapsto 1. \end{aligned}$$

On the generators, this looks like

$$\begin{aligned} \textit{comultiplication} \quad & \Delta : t_{ij}^{(r)} \mapsto \sum_{\substack{1\le k\le n \\ 0\le s\le r}} t_{ik}^{(s)} \otimes t_{kj}^{(r-s)}, \\ \textit{antipode} \quad & t_{ij}^{(r)} \mapsto ???, \textit{ and} \\ \textit{counit} \quad & t_{ij}^{(r)} \mapsto 0 \textit{ for } r \geq 1. \end{aligned}$$

1.1 Gradings

There are two natural ascending filtrations on $\mathcal{Y}\mathfrak{gl}_n$:

(1) $\deg(t_{ij}^{(r)}) = r$: The corresponding graded algebra $\mathrm{gr}\mathcal{Y}\mathfrak{gl}_n$ is commutative.

(2) $\deg'(t_{ij}^{(r)}) = r - 1$: The corresponding graded algebra $\mathrm{gr}'\mathcal{Y}\mathfrak{gl}_n$ is isomorphic to the enveloping algebra of $\mathfrak{gl}_n[z]$. The map here is

$$E_{ij} z^{r-1} \mapsto \bar{t}_{ij}^{(r)}, \qquad \text{for } r \geq 1.$$

References

[Mo] A. Molev, *Yangians and Classical Lie Algebras*, American Mathematical Society, Mathematical Surveys and Monographs, Volume 143 (1961).