Workshop 8
Rank

Instructions:

Get into groups and work on the following exercises. Each group is expected to turn in *one neatly written copy* of their solutions at the end of the class period.

Exercise 1. Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix.

a. Show that $\operatorname{Nul} B \subset \operatorname{Nul} AB$. Conclude that $\dim \operatorname{Nul} B \leq \dim \operatorname{Nul} AB$.

b. Show that $\operatorname{Col} AB \subset \operatorname{Col} A$. Conclude that $\operatorname{rank} AB \leq \operatorname{rank} A$.

c. Use parts (a) and (b) together with the rank theorem to show that

$$\operatorname{rank} AB \leq \min \{\operatorname{rank} A, \operatorname{rank} B\}.$$

That is, show that $\operatorname{rank} AB \leq \operatorname{rank} A$ *and* $\operatorname{rank} AB \leq \operatorname{rank} B$. [*Hint:* Write down the conclusion of the rank theorem for each of A, B, and AB and compare.]

Exercise 2. Use the results of Problem 2 to show that if A and B are both $n \times n$ then

$$\dim \operatorname{Nul} AB \geq \max \{\dim \operatorname{Nul} A, \dim \operatorname{Nul} B\}.$$

That is, show that $\dim \operatorname{Nul} AB \geq \dim \operatorname{Nul} A$ *and* $\dim \operatorname{Nul} AB \geq \dim \operatorname{Nul} B$. [*Hint:* See the hint for the previous problem.]

Exercise 3. Let A be an $m \times n$ matrix and let A^T be its transpose, which is an $n \times m$ matrix.

a. Use the rank theorem to show that

$$\dim \text{ Col } A + \dim \text{ Nul } A^T = m.$$

b. Use part (a) to show that $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$ if and only if $A^T\mathbf{y} = \mathbf{0}$ has only the trivial solution.

c. If A is square (i.e. $m = n$) use part (a) to show that A is invertible if and only if A^T is invertible.

Exercise 4.* If A is an $m \times n$ matrix and $\operatorname{rank} A = 1$, show that there are vectors $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbb{R}^n$ so that $A = \mathbf{u}\mathbf{v}^T$. [*Hint:* Show that all of the columns of A are multiples of one another.]