4/28/2011
Barnett

SOLUTIONS

Math 46, Applied Math (Spring 2011): Midterm 1

2 hours, 50 points total, 6 questions worth varying number of points. Good luck!

1. [9 points] A fluid of density ρ rests in a gravitational field strength g (units of acceleration). Surface waves at a frequency f (units of inverse time) may propagate in the fluid, and have wavelength λ. Let us first assume ρ, g, λ and f (and only these variables) are related by a physical law.

[4] (a) How many (independent) dimensionless quantities are there? Give them. [Hint: a dimensions matrix will help].

$$\begin{array}{c} \\ M \\ L \\ T \end{array} \begin{array}{cccc} \rho & g & \lambda & f \end{array}$$
$$\begin{array}{c} M \\ L \\ T \end{array} \begin{bmatrix} 1 & & & \\ -3 & 1 & 1 & \\ & -2 & & -1 \end{bmatrix}$$

full rank, $r=3$

$p = m - r = 4-3 = 1$

$\pi_1 = \frac{g}{\lambda f^2}$ or any power of this.

(Note: ρ cannot be involved in this law!)

[2] (b) Write the most specific formula you can for how the frequency f must depend on the other three parameters.

Pi Thm tells you $\pi_1 = C$, for some const C

$$\Rightarrow \frac{g}{\lambda f^2} = C \quad \Rightarrow \quad f = c\sqrt{\frac{g}{\lambda}}$$

[3] (c) If now a fifth parameter, the surface tension s (units mass per time squared), is also involved in the physical law, use the Buckingham Pi Theorem to deduce the most specific formula for how f must depend on the other four parameters. [Hint: in your answer, f must only appear once.]

Extra column of dim. matrix: $\begin{array}{c} \\ M \\ L \\ T \end{array} \begin{array}{c} s \\ \begin{bmatrix} 1 \\ \\ -2 \end{bmatrix} \end{array}$

look for a π_2 that doesn't involve f (since want it to appear only once). $p = 5-3 = 2$.

$\pi_2 = \frac{sf^2}{\rho\lambda^3}$ or $\frac{s}{\rho g \lambda^2}$

bad since involves f good.

Pi Thm says $\pi_1 = h(\pi_2)$ (some func.) ie $\frac{g}{\lambda f^2} = h\left(\frac{s}{\rho g \lambda^2}\right)$ since h unknown anyway.

ie $f = \sqrt{\frac{g}{\lambda}}\, h\left(\frac{s}{\rho g \lambda^2}\right)$

2. [11 points] A mass released from rest on an aging spring is described by the model

$$my'' = -ke^{-at}y, \qquad y(0) = L, \qquad y'(0) = 0,$$

where the dynamical variable $y(t)$ is the displacement of the mass vs time.

[2] (a) What are the possible timescales ? [Hint: a dimensions matrix will help]

$$\begin{array}{c c} & \begin{array}{cccc} m & k & a & L \end{array} \\ \begin{array}{c} M \\ L \\ T \end{array} & \left[\begin{array}{cccc} 1 & 1 & & \\ & & & 1 \\ & -2 & -1 & \end{array}\right] \end{array}$$

$t_c = a^{-1}$ aging time

$t_c = \sqrt{\frac{m}{k}}$ initial period of oscillation.

(that's it; others are not linearly indep of these).

[5] (b) Choosing an appropriate timescale to give a nonsingular problem in the limit of small aging rate a, and a lengthscale, non-dimensionalize the problem, and give the resulting *small* parameter ε:

$y_c = L$ the only possible lengthscale.

Since a small, choose t_c not to involve this (ie., choose the longer t_c).

$\Rightarrow t_c = \sqrt{\frac{m}{k}}$

rescale $m\frac{y_c}{t_c^2}\bar{y}'' = -ke^{-at_c\bar{t}}y_c\bar{y}$, sub y_c, t_c

$$\cancel{m}\cancel{L}\cdot\frac{k}{\cancel{m}}\bar{y}'' = -\cancel{k}e^{-a\sqrt{\frac{m}{k}}\bar{t}}\cancel{L}\bar{y}$$

cancel stuff: $\bar{y}'' = -e^{-\varepsilon\bar{t}}\bar{y}$ $\leftarrow$ ODE

IC.

ICs : $y_c\bar{y}(0) = L$ ie $L\bar{y}(0) = L$ ie $\bar{y}(0) = 1$

[You may also choose $t_c = \frac{ma}{k}$, in which case $\bar{y}'' = -\varepsilon^2 e^{-\varepsilon\bar{t}}\bar{y}$, but this is less useful since $t_c \ll \sqrt{\frac{m}{k}}$ and the unperturbed is $\bar{y}'' = 0$ which doesn't even oscillate !]

[Hint: don't forget the ICs]

Answer for $\varepsilon =$ $a\sqrt{\frac{m}{k}}$

(ratio of timescales)

note: not ours from b! Actually it's $t_c = a^{-1}$, the other choice.

[4] (c) One choice of timescale results in the following non-dimensionalized IVP,

$$y'' = -\frac{1}{\varepsilon^2} e^{-t} y, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Find the WKB approximation to the solution to this IVP (give your answer in terms of ε and elementary functions only):

rearrange to std form for WKB (note: indep. var is t not the usual x).

$$\varepsilon^2 y'' + e^{-t} y = 0$$

oscillatory case

$k^2(t) = e^{-t}$ so $k(t) = e^{-t/2}$.

$$y_{WKB}(t) = \frac{c_1}{e^{-t/4}} \cos\left(\frac{1}{\varepsilon}\int_0^t k(s)\,ds\right) + \frac{c_2}{e^{-t/4}} \sin\left(\frac{1}{\varepsilon}\int_0^t k(s)\,ds\right)$$

$$\int_0^t e^{-s/2}\,ds = -2e^{-s/2}\Big|_0^t = 2\left(1 - e^{-t/2}\right)$$

@t=0

$$\text{ICs: } y'_{WKB}(0) = \frac{1}{4} c_1 e^{t/4} \cos(0) - c_1 e^{t/4} \underbrace{\sin(0)}_{0} \frac{k(0)}{\varepsilon} + \frac{c_2 e^{t/4}}{4} \cancel{\sin(0)} + c_2 e^0 \frac{k(0)}{\varepsilon} \cos(0)$$

sorry about this! Harder than planned.

$$= \frac{c_1}{4} + \frac{c_2}{\varepsilon} \quad \text{so} \quad c_2 = -\frac{\varepsilon}{4} c_1 = -\varepsilon/4.$$

$$y_{WKB}(0) = c_1 = 1.$$

Ans: $y_{WKB}(t) = e^{t/4}\left[\cos\left(\frac{2(1-e^{-t/2})}{\varepsilon}\right) - \frac{\varepsilon}{4}\sin\left(\frac{2(1-e^{-t/2})}{\varepsilon}\right)\right]$

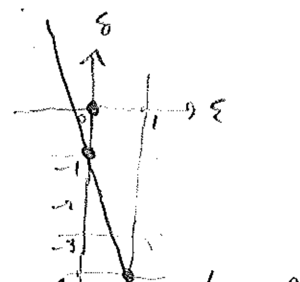
[BONUS: until roughly what time t do you expect this to be accurate?]

accurate until $k(t) \sim \varepsilon$ ie $e^{-t/2} \sim \varepsilon$ ie $t \approx 2 \ln \frac{1}{\varepsilon}$

3. [5 points] Find the leading order perturbation approximation of all roots of $\varepsilon x^4 - x + 1 = 0$, $\quad \varepsilon \ll 1$.

Regular root(s) set $\varepsilon = 0$: $\quad -x + 1 = 0 \quad$ so $\quad x = 1$.

Rescale $x = \frac{y}{\delta}$ so $\underbrace{\frac{\varepsilon}{\delta^4} y^4 - \frac{1}{\delta} y}_{\text{balance via } \delta = \varepsilon^{1/3}} + 1 = 0$.



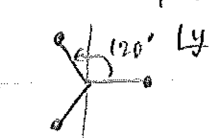
so $\quad \varepsilon^{-1/3} y^4 - \varepsilon^{-1/3} y + 1 = 0$

$\Rightarrow \quad y^4 - y + \cancel{\varepsilon^{1/3}} = 0$ leading order.

$$y(y^3 - 1) = 0$$

$y = 0$ (image of regular root), $\underbrace{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}}_{\text{cube roots of unity.}}$ or $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}$

roots (leading orders): $\quad x = 1 + \ldots, \quad \frac{1}{\varepsilon^{1/3}}, \quad \frac{e^{2\pi i/3}}{\varepsilon^{1/3}}, \quad \frac{e^{4\pi i/3}}{\varepsilon^{1/3}}$



(4 roots).

4. [7 points] Consider the following IVP, where ε is a small parameter,

$$y' = \frac{y}{1+\varepsilon y}, \qquad y(0) = 1.$$

[5] (a) Use a perturbation expansion to find a 2-term approximation: ie only up to y_1

binomial

$$y(1+\varepsilon y)^{-1} = y\left(1 - \varepsilon y + O(\varepsilon^2)\right).$$

Substitute reg. series $y = y_0 + \varepsilon y_1 + \cdots$

$$y_0' + \varepsilon y_1' + \cdots = (y_0 + \varepsilon y_1 + \cdots)\left(1 - \varepsilon(y_0 + \cdots) + \cdots\right)$$

ε^0: $y_0' = y_0$ gen. soln. $y_0(t) = Ae^t$ matches IC by $A=1$, $y_0 = e^t$.

ε^1: $y_1' = y_1 - y_0^2$ ie $y_1' - y_1 = -y_0^2 = -e^{2t}$ linear 1st order.

can use integrating factor e^{-t}: $y_1 = e^t\left(\int e^{-t}(-e^{2t})\,dt + c\right)$

$$= -e^{2t} + ce^t$$

ICs for $y_1(0) = 0$ (by pert. exp. of IC) so $c = +1$, $y_1 = e^t - e^{2t}$

put together:

$$y_a(t) = e^t + \varepsilon(e^t - e^{2t})$$

[2] (b) Find the residual function of the *unperturbed* solution. Is it uniformly convergent to zero as $\varepsilon \to 0$, on $t \in (0, \infty)$?

ie $y_0(t) = e^t$.

residual $r(t) = $ (LHS of ODE) $-$ (RHS of ODE) applied to y_0.

$$= y_0' - \frac{y_0}{1+\varepsilon y_0}$$

$$= e^t - \frac{e^t}{1+\varepsilon e^t} = \cancel{\frac{e^t + \varepsilon e^{2t} - e^t}{1+\varepsilon e^t}}$$

$$= \varepsilon \frac{e^{2t}}{1+\varepsilon e^t} = \varepsilon e^{2t} + O(\varepsilon^2) = O(\varepsilon)$$

No, since e^{2t} unbounded on $(0,\infty)$, not unif. convergent as $\varepsilon \to 0$.

5. [9 points] Use singular perturbation methods to find a uniform approximate solution to the boundary-value problem

never changes sign in $(0,1]$.

$$\varepsilon y'' - \frac{1}{1+2x}y' + y = 0, \qquad \varepsilon \ll 1, \qquad y(0)=1, \qquad y(1)=0$$

As always, remember to check and explain the location of any boundary layer(s).

[grading: given one minor slip (nonconceptual) for free here].

Outer soln. $-\frac{1}{1+2x}y_0' + y_0 = 0$ separates,

$$\int \frac{dy_0}{y_0} = \int (1+2x)\,dx$$

$\Rightarrow \ln|y_0| = x + x^2 + c$ or $y_0(x) = ce^{x+x^2}$

relative sign suggests BL @ $x=1$ (try a BL @ $x=0$ & you'll find exp. growth in ξ, unstable).

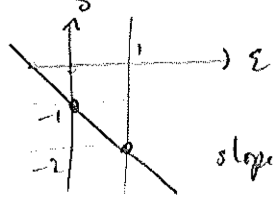
so BC @ $x=0$ $y_0(0)=1$ gives $c=1$, $y_0(x) = e^{x+x^2}$

Inner Layer $\xi = \frac{1-x}{\delta}$, $x = 1-\delta\xi$ so $\frac{\varepsilon}{\delta^2}Y'' + \frac{1}{1+2(1-\delta\xi)}\frac{Y'}{\delta} + Y = 0$. rescaled

$$\frac{1}{3+O(\delta)} = \frac{1}{3} + O(\delta)$$

2 (tricky) only this matters. small

balance 1st two terms



slope $= -1$. $\delta = \varepsilon$, the usual case.

$\Rightarrow \frac{1}{\varepsilon}Y_i'' + \frac{1}{3\varepsilon}Y_i' + O(1)Y_i' + Y = 0$ mult. by ε.

$Y_i'' + \frac{1}{3}Y_i' + \cancel{O(\varepsilon)} = 0$ drop to leading order.

Gen. soln. $Y_i(\xi) = Ae^{-\xi/3} + B$ match BC@ $x=1$ i.e. $\xi = 0$.

$Y_i(0) = 0$ so $A = -B$.

$\Rightarrow Y_i = A(1-e^{-\xi/3})$.

Match $\lim_{x\to 1} y_0(x) = c_m = \lim_{\xi\to\infty} Y_i(\xi)$

$e^{1+1} = e^2$ $\quad A$ $\quad$ so $A = e^2$.

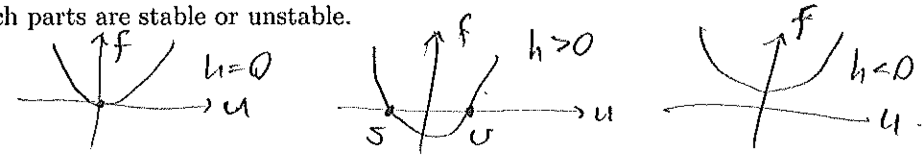
$$y_u(x) = y_0(x) + y_i(x) - c_m = e^{x+x^2} + e^2\left(\cancel{1} - e^{-\frac{1-x}{3\varepsilon}}\right) - \cancel{e^2}$$

$$= e^{x+x^2} - e^{2-\frac{1-x}{3\varepsilon}}$$

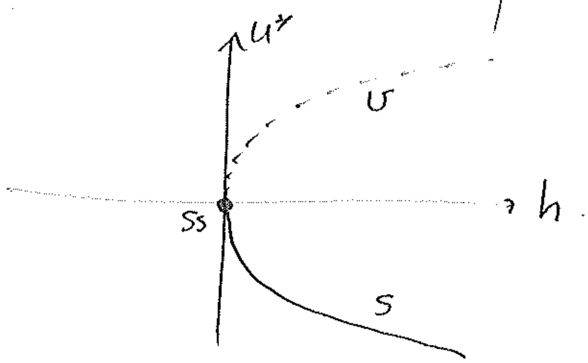
6. [9 points] Short answer questions.

[3] (a) Sketch a bifurcation diagram, with respect to the parameter h, for the autonomous ODE $u' = u^2 - h$. Label your axes, and which parts are stable or unstable.

$f(u,h) = u^2 - h.$



trying these by guessing h values & sketching.



(3) (b) Write a little-o relation stating that $\log \varepsilon$ blows up more weakly than any negative power of ε, as $\varepsilon \to 0^+$, then prove it.

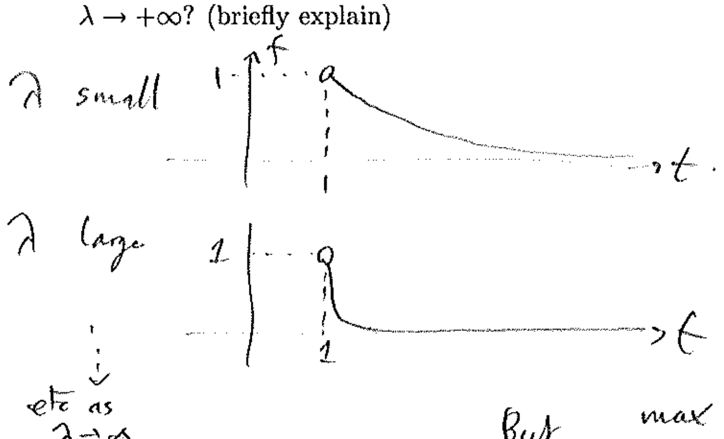
$$\log \varepsilon = o(\varepsilon^{-\alpha}) \qquad \forall \alpha > 0$$

If true, the following vanishes:

$$\lim_{\varepsilon \to 0^+} \frac{\log \varepsilon}{\varepsilon^{-\alpha}} \xrightarrow[\text{since both blow up}]{\text{l'Hopital}} \frac{\varepsilon^{-1}}{-\alpha \varepsilon^{-1-\alpha}} = \frac{1}{-\alpha \varepsilon^{-\alpha}} = -\frac{\varepsilon^{\alpha}}{\alpha} \to 0 \text{ as } \varepsilon \to 0.$$

limit $= 0$ so little-o holds.

(3) (c) Is $f(\lambda, t) = 1/t^\lambda$ pointwise, and/or uniformly, convergent to zero on the interval $t \in (1, \infty)$, as $\lambda \to +\infty$? (briefly explain)



for any fixed t,

$$\lim_{\lambda \to \infty} \frac{1}{t^\lambda} = 0 \text{ so}$$

pointwise convergent.

But $\max_{1 < t < \infty} \frac{1}{t^\lambda} = 1, \ \forall \lambda$ (strictly we should write sup for max since it's not achieved...)

which doesn't converge to 0.

$\Rightarrow$ not uniformly convergent.