Math 123 Homework Assignment #2

Due Friday, April 22

Part I:

1. Suppose that A is a C^*-algebra.

   (a) Suppose that $e \in A$ satisfies $xe = x$ for all $x \in A$. Show that $e = e^*$ and that $\|e\| = 1$. Conclude that e is a unit for A.

   (b) Show that for any $x \in A$, $\|x\| = \sup_{\|y\| \le 1} \|xy\|$. (Do *not* assume that A has an approximate identity.)

   ANS: In part (b), just take $y = \|x\|^{-1}x^*$.

2. Suppose that A is a Banach algebra with an involution $x \mapsto x^*$ that satisfies $\|x\|^2 \le \|x^*x\|$. Then show that A is a Banach $*$-algebra (i.e., $\|x^*\| = \|x\|$). In fact, show that A is a C^*-algebra.

   ANS: Since A is a Banach algebra, $\|x\|^2 \le \|x^*x\| \le \|x^*\|\,\|x\|$, which implies that $\|x\| \le \|x^*\|$. Replacing x by x^*, we get $\|x^*\| \le \|x^{**}\| = \|x\|$. Thus, A is a Banach $*$-algebra, and the C^*-norm equality follows from the first calculation and that fact that in any Banach $*$-algebra, $\|x^*x\| \le \|x\|^2$.

3. Let I be a set and suppose that for each $i \in I$, A_i is a C^*-algebra. Let $\bigoplus_{i\in I} A_i$ be the subset of the direct product $\prod_{i\in I} A_i$ consisting of those $a \in \prod_{i\in I} A_i$ such that $\|a\| := \sup_{i\in I} \|a_i\| < \infty$. Show that $\left(\bigoplus_{i\in I} A_i, \|\cdot\|\right)$ is a C^*-algebra with respect to the usual pointwise operations:

$$\begin{aligned}(a + \lambda b)(i) &:= a(i) + \lambda b(i)\\ (ab)(i) &:= a(i)b(i)\\ a^*(i) &:= a(i)^*.\end{aligned}$$

We call $\bigoplus_{i\in I} A_i$ the *direct sum* of the $\{ A_i \}_{i\in I}$.

ANS: The real issue is to see that the direct sum is complete. So suppose that $\{ a_n \}$ is Cauchy in $\bigoplus_{i\in I} A_i$. Then, clearly, each $\{ a_n(i) \}$ is Cauchy in A_i, and hence there is $a(i) \in A_i$ such that $a_n(i) \to a(i)$. If $\epsilon > 0$, choose N so that $n, m \ge N$ imply that $\|a_n - a_m\| < \epsilon/3$. I claim that if $n \ge N$, then $\|a_n - a\| < \epsilon$. This will do the trick.

But for each $i \in N$, there is a $N(i)$ such that $n \ge N(i)$ implies that $\|a_n(i) - a(i)\| < \epsilon/3$. Then if $n \ge N$, we have

$$\|a_n(i) - a(i)\| \le \|a_n(i) - a_{N(i)}(i)\| + \|a_{N(i)}(i) - a(i)\| < \frac{2\epsilon}{3}.$$

But then $n \geq N$ implies that

$$\sup_{i \in I} \|a_n(i) - a(i)\| \leq \frac{2\epsilon}{3} < \epsilon$$

as required.

4. Let A^1 be the vector space direct sum $A \oplus \mathbf{C}$ with the $*$-algebra structure given by

$$(a, \lambda)(b, \mu) := (ab + \lambda b + \mu a, \lambda\mu)$$
$$(a, \lambda)^* := (a^*, \bar{\lambda}).$$

Show that there is a norm on A^1 making it into a C^*-algebra such that the natural embedding of A into A^1 is isometric. (Hint: If $1 \in A$, then show that $(a, \lambda) \mapsto (a + \lambda 1_A, \lambda)$ is a $*$-isomorphism of A^1 onto the C^*-algebra direct sum of A and $\mathbf{C}$. If $1 \notin A$, then for each $a \in A$, let L_a be the linear operator on A defined by left-multiplication by a: $L_a(x) = ax$. Then show that the collection B of operators on A of the form $\lambda I + L_a$ is a C^*-algebra with respect to the operator norm, and that $a \mapsto L_a$ is an isometric $*$-isomorphism.)

ANS: If $1 \in A$, then it is easy to provide an inverse to the given map.

The interesting bit is when A is non-unital to begin with. Since A is complete, $B(A)$ is a Banach algebra with respect to the operator norm. The set $B = \{\, \lambda I + L_x : \lambda \in \mathbf{C}, x \in A \,\}$ is clearly a subalgebra which admits an involution: $(\lambda I + L_x)^* = \overline{\lambda} I + L_{x^*}$. Notice that we have

$$\|L_x\| = \sup_{\|y\|=1} \|xy\| = \|x\|$$

(problem 1(b) above). Since $L_{\lambda x} = \lambda L_x$, $L_{(x+y)} = L_x + L_y$, $L_{xy} = L_x \circ L_y$, and $L_{x^*} = L_x^*$, the map $x \mapsto L_x$ is an isometric $*$-isomorphism of A onto $B_0 = \{\, L_x \in B(A) : x \in A \,\}$. It follows that B_0 is complete and therefore closed in $B(A)$. Therefore, since $I \notin B_0$ (because $e \notin A$) and since the invertible elements in $B(A)$ are open, there is a $\delta > 0$ such that $\|I - L_x\| \geq \delta$ for all $x \in A$. So to see that B is also closed, suppose that $\lambda_n I + L_{x_n} \to L$ in $B(A)$. Passing to a subsequence and relabeling, we may assume that $\lambda_n \neq 0$ for all n. (If infinitely many λ_n are zero, then $L \in B_0$.) Thus, $|\lambda_n| \|I + \lambda_n^{-1} L_{x_n}\| \to \|L\|$. Since $\|I + \lambda_n^{-1} L_{x_n}\| \geq \|\delta\|$, it follows that $\{\, \lambda_n \,\}$ must be bounded, and hence must have a convergent subsequence. Therefore $L \in B$, and B is a Banach algebra.

Finally,

$$\begin{aligned}
\|\lambda I + L_x\|^2 &= \sup_{\|y\|=1} \|\lambda y + xy\|^2 = \sup_{\|y\|=1} \|(\lambda y + xy)^*((\lambda y + xy)\| \\
&= \sup_{\|y\|=1} \|y^*(\overline{\lambda} I + L_{x^*})\big((\lambda I + L_x)(y)\big)\| \leq \sup_{\|y\|=1} \|(\lambda I + L_x)^*\big((\lambda I + L_x)(y)\big)\| \\
&= \|(\lambda I + L_x)^*(\lambda I + L_x)\|.
\end{aligned}$$

It now follows from problem 2 that B is a C*-algebra. It is immediate that $(x, \lambda) \mapsto \lambda I + L_x$ is an (algebraic) isomorphism of A^1 onto B (note that you need to use the fact that A in non-unital to see that this map is injective). Of course, $\|(x, \lambda)\| := \|\lambda I + L_x\|_B$ is the required norm on A^1.

5. In this question, ideal always means ‘closed two-sided ideal.’

(a) Suppose that I and J are ideals in a C^*-algebra A. Show that IJ — defined to be the closed linear span of products from I and J — equals $I \bigcap J$.

(b) Suppose that J is an ideal in a C^*-algebra A, and that I is an ideal in J. Show that I is an ideal in A.

ANS: Clearly $IJ \subseteq I \bigcap J$. Suppose $a \in I \bigcap J$, and that $\{ e_\alpha \}_{\alpha \in A}$ is an approximate identity for J. Then ae_α converges to a in J. On the other hand, for each α, $ae_\alpha \in IJ$. Thus, $a \in IJ$. This proves part (a).

For part (b), consider $a \in A$ and $b \in I$. Again let $\{ e_\alpha \}_{\alpha \in A}$ be an approximate identity for J. Then $ab = \lim_\alpha a(e_\alpha b) = \lim_\alpha (ae_\alpha)b$, and the latter is in I, since I is closed and $ae_\alpha \in J$ for all α. This suffices as everything in sight is $*$-closed, so I must be a two-sided ideal in A.

6. Suppose that a and b are elements in a C^*-algebra A and that $0 \le a \le b$. Show that $\|a\| \le \|b\|$. What happens if we drop the assumption that $0 \le a$? (Hint: use Lemma Z.)

Part II:

7. Suppose that A is a unital C^*-algebra and that $f : \mathbf{R} \to \mathbf{C}$ is continuous. Show that the map $x \mapsto f(x)$ is a continuous map from $A_{\text{s.a.}} = \{ x \in A : x = x^* \}$ to A.

ANS: Suppose that $f : \mathbf{R} \to \mathbf{C}$ is continuous and that $x_n \to x$ in $A_{\text{s.a.}}$. We need to see that $f(x_n) \to f(x)$ in A. Since we may write $f = f_1 + if_2$ with f_i real-valued and since $f(x_n) = f_1(x_n) + if_2(x_n)$, we may as well assume that f itself is real-valued. Furthermore, since addition and multiplication are norm-continuous in A, we certainly have $p(x_n) \to p(x)$ for any polynomial; this is proved in the same was as one proves that any polynomial is continuous in calculus. Clearly there is a constant $M \in \mathbf{R}^+$ so that $\|x_n\| \le M$ for all n. Thus $\rho(x_n) \le M$ and $\sigma(x_n) \subseteq [-M, M]$ for all n. Similarly, $\sigma(x) \subseteq [-M, M]$ as well. By the Weierstrass approximation theorem, given $\epsilon > 0$, there is a polynomial p such that $|f(t) - p(t)| < \epsilon/3$ for all $t \in [-M, M]$. Thus for each n,

$$\|f(x_n) - p(x_n)\| = \sup_{t \in \sigma(x_n)} |f(t) - p(t)| < \epsilon/3. \tag{†}$$

(Notice that $f(x_n)$ is the image of $f|_{\sigma(x_n)}$ by the isometric $*$-isomorphism of $C\big(\sigma(x_n)\big)$ onto the abelian C*-subalgebra of A generated by e and x_n. Then (†) follows because $f(x_n) - p(x_n)$ is the image of $(f - p)|_{\sigma(x_n)}$ which has norm less than $\epsilon/3$ in $C\big(\sigma(x_n)\big)$ since $\sigma(x_n) \subseteq [-M, M]$.) Of course, (†) holds with x_n replaced by x as well. Now choose N so that $n \ge N$ implies that $\|p(x_n) - p(x)\| < \epsilon/3$. Therefore for all $n \ge N$,

$$\|f(x_n) - f(x)\| \le \|f(x_n) - p(x_n)\| + \|p(x_n) - p(x)\| + \|p(x) - f(x)\| < \epsilon.$$

The conclusion follows.

8. Prove Corollary AC: Show that every separable C^*-algebra contains a sequence which is an approximate identity. (Recall that we showed in the proof of Theorem AB that if $x \in A_{\text{s.a.}}$, and if $x \in \{ x_1, \dots, x_n \} = \lambda$, then $\|x - xe_\lambda\|^2 < 1/4n$.)

ANS: Let $\{ e_\lambda \}_{\lambda \in \Lambda}$ be the net constructed in the proof of the Theorem. If $D = \{ x_k \}_{k=1}^\infty$ is dense in $A_{\text{s.a.}}$, then define $e_n = e_{\lambda_n}$ where $\lambda_n = \{ x_1, \dots, x_n \}$. Since properties (1)–(3) are clear, we only need to show that $xe_n \to x$ for all $x \in A$. (This will suffice by taking adjoints.) As we saw in the proof of the Theorem, $\|xe_n - x\|^2 = \|x^*x - x^*xe_n\|$, so we may as well assume that $x \in A_{\text{s.a.}}$. But then if $x \in \{ z_1, \dots, z_n \} = \lambda$, we have $\|x - xe_\lambda\|^2 \le 1/4n$.

So fix $x \in A_{\text{s.a.}}$ and $\epsilon > 0$. Choose $y \in D$ such that $\|x - y\| < \epsilon/3$. Finally, choose N so that $y \in \{ x_1, \dots, x_N \} = \lambda_N$, and such that $1/4N < \epsilon/3$. Then, since $\|e_n\| \le 1$, $n \ge N$ implies that

$$\|x - xe_n\| \le \|x - y\| + \|y - ye_n\| + \|ye_n - xe_n\| < \epsilon.$$

This suffices.

9. Suppose that $\pi : A \to B(\mathcal{H})$ is a representation. Prove that the following are equivalent.

(a) π has no non-trivial closed invariant subspaces; that is, π is irreducible.

(b) The commutant $\pi(A)' := \{ T \in B(\mathcal{H}) : T\pi(a) = \pi(a)T \text{ for all } a \in A \}$ consists solely of scalar multiples of the identity; that is $\pi(A)' = \mathbf{C}I$.

(c) No non-trivial projection in $B(\mathcal{H})$ commutes with every operator in $\pi(A)$.

(d) Every vector in $\mathcal{H}$ is cyclic for π.

(Suggestions. Observe that $\pi(A)'$ is a C^*-algebra. If $A \in \pi(A)'_{\text{s.a.}}$ and $A \ne \alpha I$ for some $\alpha \in \mathbf{C}$, then use the Spectral Theorem to produce nonzero operators $B_1, B_2 \in \pi(A)'$ with $B_1B_2 = B_2B_1 = 0$. Observe that the closure of the range of B_1 is a non-trivial invariant subspace for π.)

ANS: $(a) \implies (b)$: Since $\pi(A)'$ is a (norm) closed selfadjoint subalgebra of $B(\mathcal{H})$, it is a C*-algebra (a von-Neumann algebra in fact). Therefore, $\pi(A)'$ is spanned by its self-adjoint elements. Thus, if $\pi(A)'$ does not consist of solely scalar operators, then there is a $T \in \pi(A)'_{\text{s.a.}}$ with $\sigma(T)$ not a single point. Thus Urysohn's Lemma implies that there are real-valued functions $f_1, f_2 \in C(\sigma(T))$ of norm one which satisfy $f_1f_2 = 0$. Let $B_i = f_i(T)$ for $i = 1, 2$. Note that each $B_i \in \pi(A)'_{\text{s.a.}}$ and $B_1B_2 = B_2B_1 = 0$. Let $V = [B_1\mathcal{H}]$. Since $\|B_1\| = 1$, $V \ne \{ 0 \}$. Since $\pi(x)B_1\xi = B_1\pi(x)\xi$ for all $x \in A$ and $\xi \in \mathcal{H}$, it follows that V is a non-zero closed invariant subspace for π. But since $\|B_2\| = 1$, there is an $\eta \in \mathcal{H}$ such that $B_2\eta \ne 0$. Yet $\langle B_1\xi, B_2\eta \rangle = \langle \xi, B_1B_2\eta \rangle = 0$ for all $\xi \in \mathcal{H}$. Thus $B_2\eta \perp V$, and V is a non-trivial invariant subspace.

$(c) \implies (d)$: If $\xi \in \mathcal{H}$ is non-zero, then $V = [\pi(A)\xi]$ is a non-zero, closed invariant subspace for π. Thus it will suffice to prove that the projection P onto any invariant subspace V is in $\pi(A)'$. But if V is invariant, then so is $V^\perp$. Thus for any $x \in A$ and any $\xi \in \mathcal{H}$, we have $\pi(x)P\xi \in V$ and $\pi(x)(I - P)\xi \in V^\perp$. Thus for all $\xi, \eta \in \mathcal{H}$, $\langle P\pi(x)\xi, \eta \rangle = \langle P\pi(x)P\xi, \eta \rangle + \langle P\pi(x)(I - P)\xi, \eta \rangle = \langle \pi(x)P\xi, \eta \rangle$. This suffices.

The implications $(b) \implies (c)$ and $(d) \implies (a)$ are immediate.

Part III:

10. As in footnote 1 of problem #8 on the first assignment, use the maximum modulus theorem to view the disk algebra, $A(D)$, as a Banach subalgebra of $C(\mathbf{T})$.[1] Let $f \in A(D)$ be the identity function: $f(z) = z$ for all $z \in \mathbf{T}$. Show that $\sigma_{C(\mathbf{T})}(f) = \mathbf{T}$, while $\sigma_{A(D)}(f) = \overline{D}$. This shows that, unlike the case of C^*-algebras where we have "spectral permanence," we can have $\sigma_A(b)$ a proper subset of $\sigma_B(b)$ when B is a unital subalgebra of A.

> **ANS**: The spectrum of any element of $C(X)$ is simply its range, so we immediately have $\sigma_{C(\mathbf{T})}(f) = \mathbf{T}$. But $\lambda - f$ is invertible in $A(D)$ only when $(\lambda - f)^{-1}$ has an analytic extension to D, but if $\lambda \in D$, then this is impossible since
> $$\int_{|z|=1} \frac{1}{\lambda - z}\, dz = 2\pi i \quad \text{if } \lambda \in D.$$
> On the other hand, if $|\lambda| > 1$, then $\lambda - f$ is clearly in $G\big(A(D)\big)$. Therefore $\sigma_{A(D)}(f) = \overline{D}$ as claimed.

11. Suppose that U is an bounded operator on a complex Hilbert space $\mathcal{H}$. Show that the following are equivalent.

(a) U is isometric on $\ker(U)^\perp$.

(b) $UU^*U = U$.

(c) UU^* is a projection[2].

(d) U^*U is a projection.

An operator in $B(\mathcal{H})$ satisfying (a), and hence (a)–(d), is called a *partial isometry* on $\mathcal{H}$. The reason for this terminology ought to be clear from part (a).

Conclude that if U is a partial isometry, then UU^* is the projection on the (necessarily closed) range of U, that U^*U is the projection on the $\ker(U)^\perp$, and that U^* is also a partial isometry.

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[1]Although it is not relevant to the problem, we can put an involution on $C(\mathbf{T})$, $f^*(z) = \overline{f(\bar{z})}$, making $A(D)$ a Banach $*$-subalgebra of $C(T)$. You can then check that neither $C(\mathbf{T})$ nor $A(D)$ is a C^*-algebra with respect to this involution.

[2]A a bounded operator P on a complex Hilbert space $\mathcal{H}$ is called a *projection* if $P = P^* = P^2$. The term *orthogonal projection* or *self-adjoint projection* is, perhaps, more accurate. Note that $\mathcal{M} = P(\mathcal{H})$ is a *closed* subspace of $\mathcal{H}$ and that P is the usual projection with respect to the direct sum decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. However, since we are only interested in these sorts of projections, we will settle for the undecorated term "projection."

(Hint: Replacing U by U^*, we see that (b)$\Longleftrightarrow$(c) implies (b)$\Longleftrightarrow$(c)$\Longleftrightarrow$(d). Then use (b)–(d) to prove (a). To prove (c)$\Longrightarrow$(b), consider $(UU^*U - U)(UU^*U - U)^*$.)

ANS: That (b) implies (c) is easy. To see that (c) implies (b), note that $(UU^*U-U)(UU^*U-U)^* = (UU^*)^3 - 2(UU^*)^2 + UU^*$, which is zero. But in a C^*-algebra, $x^*x = 0$ implies that $x = 0$. Therefore $UU^*U - U = 0$.

Now replacing U by U^* gives us the fact that (b), (c), and (d) are equivalent.

But if U^*U is a projection, then the range of U^*U is exactly $\ker(U^*U)^\perp$. I claim $\ker(T^*T) = \ker(T)$ for any bounded operator. Obviously, $\ker(T) \subseteq \ker(T^*T)$. On the other hand, if $T^*T(x) = 0$, then $\langle T^*Tx, x\rangle = 0 = \langle Tx, Tx\rangle = |Tx|^2$. This proves the claim.

It follows from the previous paragraph that if $x \in \ker(U)^\perp$, then $U^*Ux = x$. But then $|Ux|^2 = \langle Ux, Ux\rangle = \langle U^*Ux, x\rangle = \langle x, x\rangle = |x|^2$. Thus, (d) implies (a).

Finally, if (a) holds, then the polarization identity implies that $\langle Ux, Uy\rangle = \langle x, y\rangle$ for all $x, y \in \ker(U)^\perp$. Now suppose $x \in \ker(U)^\perp$. On the one hand, $z \in \ker(U)^\perp$ implies that $\langle U^*Ux, z\rangle = \langle Ux, Uz\rangle = \langle x, z\rangle$. While on the other hand, $z \in \ker(U)$ implies that $\langle U^*Ux, z\rangle = \langle Ux, Uz\rangle = 0 = \langle x, z\rangle$. We have shown that $\langle U^*Ux, y\rangle = \langle x, y\rangle$ for all $y \in \mathcal{H}$ and $x \in \ker(U)^\perp$; therefore the restriction of U^*U to $\ker(U)^\perp$ is the identity. But U^*U is certainly zero on $\ker(U)$. In other words, U^*U is the projection onto $\ker(U)^\perp$, and (a) implies (d).

Of course we just proved above that if U is partial isometry, then U^*U is the projection onto $\ker(U)^\perp$. I'm glad everyone (eventually anyway) realized this is what I meant. Sorry if you wasted time here. Of course, taking adjoints in part (b) shows that U^* is a partial isometry, so $UU^* = U^{**}U^*$ is the projection onto $\ker(U^*)^\perp$. It is standard nonsense that, for any bounded operator T, $\ker(T^*) = T(\mathcal{H})^\perp$ (see, for example, *Analysis Now*, 3.2.5). Thus, UU^* is the projection onto $\ker(U^*)^\perp$, which is the closure of the range of U. However, the range of U is the isometric image of the closed, hence complete, subspace $\ker(U)^\perp$. Thus the range of U is complete, and therefore, closed. Thus, UU^* is the projection onto the range of U as claimed.