Counting Subrings of $\mathbb{Z}^n$

In this talk we will discuss joint work with Ramin Takloo-Bighash in which we investigate subring growth of $\mathbb{Z}^n$ for small n using p-adic integration. Much recent work by du Sautoy, Grunewald, Segal, and many others, has been devoted to studying the number of subgroups of an infinite group G which have index k. We call this number $a_G(k)$, and consider the zeta function of G, $\zeta_G(s) = \sum_{k\geq 1} a_G(k)k^{-s}$. The analytic properties of this zeta function can tell us much about G. This function often has a nice Euler product expansion, with local factors that can be expressed as p-adic integrals.

We will show how to evaluate some simple p-adic integrals and compute $\zeta_{\mathbb{Z}^n}(s)$ for all n. We will then turn to the question of counting subrings and the analogous zeta function which can be attacked with similar techniques. Let $a_n(k)$ be the number of subrings of $\mathbb{Z}^n$ of index k and $N_n(B)$ be the number of subrings of $\mathbb{Z}^n$ of index at most B. We will give an idea of how to show that $\sum_{k=0}^{\infty} a_3(k)k^{-s} = \frac{\zeta(s)^3\zeta(3s-1)}{\zeta(2s)^2}$ and that for each $n \leq 5$, there is a positive real number C_n such that $N_n(B) \sim C_n B (\log B)^{\binom{n}{2}-1}$.

This talk will assume little background and should be accessible to first year graduate students (even non-number theorists).