Workshop Problems 4

Problem 1. Let A, B and C be matrices. Assuming that the sizes of the matrices are "compatible" in each case, use the definition of matrix multiplication to prove the following:

a. $A(B + C) = AB + AC$

b. $(A + B)C = AC + BC$

c. $r(AB) = (rA)B = A(rB)$ for any scalar r.

Problem 2.

a. Let A be a 2×2 matrix. What can you say about A if $AB = BA$ for all 2×2 matrices B? Try taking B to be any one of the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

b. What can you say if A is a 3×3 matrix and $AB = BA$ for all 3×3 matrices B?

c. Can you guess what happens in general? That is, what do you think is true of A if it is an $n \times n$ matrix with the property that $AB = BA$ for all $n \times n$ matrices B?

Problem 3. Recall that if A is an $m \times n$ matrix then its transpose, A^T, is the $n \times m$ matrix whose columns are the rows of A. In terms of matrix entries we have

$$(A^T)_{ij} = A_{ji}.$$

Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix.

a. Show that the (i, j)-entry of $B^T A^T$ is

$$\sum_{k=1}^{n} a_{jk} b_{ki} = a_{j1} b_{1i} + a_{j2} b_{2i} + \cdots + a_{jn} b_{ni}.$$

b. Conclude from part (a) that $(AB)^T = B^T A^T$.

Problem 4. In this problem and the next we will work out a different "formula" for matrix multiplication that uses the rows of the matrices involved instead of the columns. Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. We can view $\mathbf{x}$ as an $n \times 1$ matrix. Then $\mathbf{x}^T$ is a $1 \times n$ matrix, sometimes called a *row vector*. Let B be an $n \times p$ matrix. Write B in terms of its rows:

$$B = \begin{pmatrix} \beta_1^T \\ \beta_2^T \\ \vdots \\ \beta_n^T \end{pmatrix}.$$

Here each β_i is a vector in $\mathbb{R}^p$ (Be sure you believe that this makes sense as a way to write B!). Since $\mathbf{x}^T$ is $1 \times n$ and B is $n \times p$, we can form the product $\mathbf{x}^T B$. Show that

$$\mathbf{x}^T B = x_1 \beta_1^T + x_2 \beta_1^T + \cdots + x_n \beta_n^T.$$

That is, if we multiply a matrix B by a row vector $\mathbf{x}^T$ on the left, then the result is a linear combination of the *rows* of B using the entries of $\mathbf{x}$ as weights. (Hint: Compute $B^T \mathbf{x}$ using the usual rule for matrix multiplication, and then apply the previous problem.)

Problem 5. Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix. Write A in terms of its rows:

$$A = \begin{pmatrix} \alpha_1^T \\ \alpha_2^T \\ \vdots \\ \alpha_m^T \end{pmatrix}$$

where each α_i is a vector in $\mathbb{R}^n$ (again, convince yourself that this makes sense). Show that

$$AB = \begin{pmatrix} \alpha_1^T B \\ \alpha_2^T B \\ \vdots \\ \alpha_m^T B \end{pmatrix}.$$

That is, the rows of AB are the rows of A multiplied by the matrix B. (Hint: As in the previous problem, start with the product of the transposes $B^T A^T$.)

The point of the previous two problems is that we now have two ways of thinking of matrix multiplication, and both give the same result! The original definition tells us that the j^{th} column of AB is a linear combinations of the columns of A using the entries of the j^{th} column of B as weights. Our new interpretation shows us that the i^{th} *row* of AB is a linear combination of the *rows* of B using the entries of the i^{th} *row* of A as weights. Both of these interpretations can be extremely useful.