1. (**10 Points**) Consider the matrix $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$. It has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 5$ with corresponding eigenvectors $\zeta_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\zeta_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$. Please find the solution to the initial value problem

$$\mathbf{x}'(t) = \mathbf{A}\,\mathbf{x}(t),\ \mathbf{x}(0) = \begin{pmatrix} 9 \\ 5 \end{pmatrix}.$$

Don't forget to show all of your work.

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{5t} \tag{1}$$

Substitute initial condition

$$\vec{x}(0) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{0} + c_2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{0} = \begin{pmatrix} 9 \\ 5 \end{pmatrix} \tag{2}$$

Therefore

$$\begin{pmatrix} c_1 + 4c_2 \\ -c_1 + 3c_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \end{pmatrix} \tag{3}$$

Thus,

$$c_2 = 2, c_1 = 1 \tag{4}$$

and

$$\vec{x}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + 2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} e^{5t} \tag{5}$$

2. (**10 Points**) Consider the matrix $\mathbf{B} = \begin{pmatrix} 2 & 4 \\ -1 & 2 \end{pmatrix}$. solve the differential equation

$$\mathbf{x}'(t) = \mathbf{B}\,\mathbf{x}(t),$$

Eigenvalues and eigenvectos of $\mathbf{B}$

$$det(\mathbf{B} - \lambda I) = det \begin{pmatrix} 2-\lambda & 4 \\ -1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 + 4 = 0 \tag{6}$$

$$(2-\lambda)^2 + 4 = \lambda^2 - 4\lambda + 8 = 0 \tag{7}$$

Therefore

$$\lambda = \frac{4 \pm \sqrt{16-32}}{2} = 2 \pm 2i \tag{8}$$

Eigenvector for $2+2i$

$$\begin{pmatrix} 2-2-2i & 4 \\ -1 & 2-2-2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{9}$$

$$\begin{pmatrix} -2i & 4 \\ -1 & -2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{10}$$

$$-x_1 - 2ix_2 = 0 \rightarrow x_1 = -2ix_2 \tag{11}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2ix_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -2i \\ 1 \end{pmatrix} \tag{12}$$

Eigenvector corresponding to $2+2i$ is

$$\begin{pmatrix} -2i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{13}$$

Eigenvector corresponding to $2-2i$ is

$$\begin{pmatrix} 2i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - i \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{14}$$

A fundamental set of solutions

$$u(t) = e^{2t}(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos 2t - \begin{pmatrix} -2 \\ 0 \end{pmatrix} \sin 2t) = e^{2t} \begin{pmatrix} 2\sin 2t \\ \cos 2t \end{pmatrix}$$

$$v(t) = e^{2t}(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin 2t + \begin{pmatrix} -2 \\ 0 \end{pmatrix} \cos 2t) = e^{2t} \begin{pmatrix} -2\cos 2t \\ \sin 2t \end{pmatrix}.$$

The general solution is

$$\vec{x}(t) = c_1 e^{2t} \begin{pmatrix} 2\sin 2t \\ \cos 2t \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} -2\cos 2t \\ \sin 2t \end{pmatrix}$$

3. (**15 Points**) Consider the initial value problem

$$(x+3)y'' - (x+5)y' = 0,\ y(2) = 1,\ y'(2) = -1.$$

By the existence and uniqueness theorem for second-order linear ODEs, there is a solution $\phi(x)$ to this initial value problem on the interval $-3 < x < +\infty$. Furthermore, since $x_0 = 2$ is an ordinary point, there is an interval I containing x_0 on which $\phi(x)$ is analytic. Let $\sum_{n=0}^{\infty} a_n(x-2)^n$ be the power series expansion of $\phi(x)$ centered at x_0 on I. Find the coefficients a_0, a_1, a_2, a_3 and a_4. Please show all of your work.

$$y = \sum_{n=0}^{\infty} a_n(x-2)^n$$
$$y' = \sum_{n=1}^{\infty} na_n(x-2)^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-2)^{n-2}$$

$$y(2) = a_0 = 1$$
$$y'(2) = a_1 = -1$$

Differentia equation can be rewritten as

$$(x+3)y'' - (x+5)y' = (x-2)y'' + 5y'' - (x-2)y' - 7y' = 0$$

Then

$$(x-2)y'' + 5y'' - (x-2)y' - 7y' \tag{15}$$

$$= (x-2)\sum_{n=2}^{\infty} n(n-1)a_n(x-2)^{n-2} + 5\sum_{n=2}^{\infty} n(n-1)a_n(x-2)^{n-2} \tag{16}$$

$$-(x-2)\sum_{n=1}^{\infty} na_n(x-2)^{n-1} - 7\sum_{n=1}^{\infty} na_n(x-2)^{n-1} \tag{17}$$

$$= \sum_{n=2}^{\infty} n(n-1)a_n(x-2)^{n-1} + 5\sum_{n=2}^{\infty} n(n-1)a_n(x-2)^{n-2} \tag{18}$$

$$-\sum_{n=1}^{\infty} na_n(x-2)^{n} - 7\sum_{n=1}^{\infty} na_n(x-2)^{n-1} \tag{19}$$

$$= \sum_{n=1}^{\infty}(n+1)na_{n+1}(x-2)^n + \sum_{n=0}^{\infty} 5(n+2)(n+1)a_{n+2}(x-2)^n \tag{20}$$

$$-\sum_{n=1}^{\infty} na_n(x-2)^n - \sum_{n=0}^{\infty} 7(n+1)a_{n+1}(x-2)^n \tag{21}$$

$$= \sum_{n=1}^{\infty}(n+1)na_{n+1}(x-2)^n + 10a_2 + \sum_{n=1}^{\infty} 5(n+2)(n+1)a_{n+2}(x-2)^n \tag{22}$$

$$-\sum_{n=1}^{\infty} na_n(x-2)^n - 7a_1 - \sum_{n=1}^{\infty} 7(n+1)a_{n+1}(x-2)^n \tag{23}$$

$$= 10a_2 - 7a_1 + \sum_{n=1}^{\infty}(n+1)na_{n+1}(x-2)^n + \sum_{n=1}^{\infty} 5(n+2)(n+1)a_{n+2}(x-2)^n \tag{24}$$

$$-\sum_{n=1}^{\infty} na_n(x-2)^n - \sum_{n=1}^{\infty} 7(n+1)a_{n+1}(x-2)^n \tag{25}$$

$$= 10a_2 - 7a_1 \tag{26}$$

$$+\sum_{n=1}^{\infty}[(n+1)na_{n+1} - 7(n+1)a_{n+1} + 5(n+2)(n+1)a_{n+2} - na_n](x-2)^n = 0 \tag{27}$$

Therefore

$$a_0 = 1$$

$$a_1 = -1$$

$$a_2 = \frac{7a_1}{10} = -\frac{7}{10}$$

$$a_{n+2} = \frac{(n+1)na_{n+1} - 7(n+1)a_{n+1} - na_n}{-5(n+2)(n+1)}, n = 1, 2, 3. \cdots .$$

$$n = 1 \to a_3 = \frac{2\cdot 1a_2 - 7\cdot 2a_2 - a_1}{-5\cdot 3\cdot 2} = \frac{12\frac{7}{10}+1}{-30} = -\frac{94}{300}$$

$$n = 2 \to a_4 = \frac{3\cdot 2a_3 - 7\cdot 3a_3 - 2a_2}{-5\cdot 4\cdot 3} = \frac{-15a_3 - 2a_2}{60} = \frac{-15\frac{-94}{300} - 2\frac{-7}{10}}{60} = \frac{\frac{94}{20}+\frac{28}{20}}{60} = \frac{\frac{122}{20}}{60} = \frac{122}{1200}$$

In case you need it, here's more space for this problem.

4. (**15 Points**) Find the general solution to the non-homogeneous equation

$$\mathbf{x}'(t) = \mathbf{C}\,\mathbf{x}(t) + \begin{pmatrix} t \\ t+2 \end{pmatrix},$$

where $\mathbf{C}$ is a 2×2-matrix with real entries such that

$$\mathbf{D} = \mathbf{T}^{-1}\mathbf{C}\mathbf{T} = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$$

for $\mathbf{T} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

$$\mathbf{T^{-1}} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\mathbf{x}'(t) = \mathbf{C}\,\mathbf{x}(t) + \begin{pmatrix} t \\ t+2 \end{pmatrix} \rightarrow \mathbf{x}'(t) = \mathbf{TDT^{-1}}\,\mathbf{x}(t) + \begin{pmatrix} t \\ t+2 \end{pmatrix}$$

$$\mathbf{T}^{-1}\mathbf{x}'(t) = \mathbf{T^{-1}TDT^{-1}}\mathbf{x}(t) + \mathbf{T^{-1}}\begin{pmatrix} t \\ t+2 \end{pmatrix}$$

Let

$$\mathbf{y} = \mathbf{T^{-1}}\mathbf{x}(t)$$

Then

$$\mathbf{y}'(t) = \mathbf{D}\mathbf{y}(t) + \mathbf{T^{-1}}\begin{pmatrix} t \\ t+2 \end{pmatrix}$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} t \\ t+2 \end{pmatrix}$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} t+1 \\ 1 \end{pmatrix}$$

Therefore

$$y_1' = -2y_1 + t + 1 \rightarrow y_1' + 2y_1 = t + 1 \tag{28}$$

$$y_2' = 3y_2 + 1 \rightarrow y_2' - 3y_2 = 1 \tag{29}$$

Using integrating factor y_1 and y_2 can be found as

$$y_1 = c_1 e^{-2t} + \frac{t}{2} + \frac{1}{4} \tag{30}$$

$$y_2 = c_2 e^{3t} - \frac{1}{3} \tag{31}$$

Therefore

$$\mathbf{x} = \mathbf{Ty} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 e^{-2t} + \frac{t}{2} + \frac{1}{4} \\ c_2 e^{3t} - \frac{1}{3} \end{pmatrix} = \begin{pmatrix} c_1 e^{-2t} + \frac{t}{2} + \frac{1}{4} - c_2 e^{3t} + \frac{1}{3} \\ c_1 e^{-2t} + \frac{t}{2} + \frac{1}{4} + c_2 e^{3t} - \frac{1}{3} \end{pmatrix} \tag{32}$$

Further simplification yields

$$\mathbf{x}(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-2t} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}e^{3t} + \begin{pmatrix} \frac{t}{2} + \frac{7}{12} \\ \frac{t}{2} - \frac{1}{12} \end{pmatrix}$$

In case you need it, here's more space for this problem.

5. (**10 Points**)

(a) (**5 Points**) Find the eigenvalues of

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & \text{-2} \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

$$\begin{aligned}
det(\mathbf{A} - \lambda I) &= det \begin{pmatrix} -\lambda & 0 & \text{-2} \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{pmatrix} && (33) \\
&= -\lambda \; det \begin{pmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix} - 2 \; det \begin{pmatrix} 1 & 2-\lambda \\ 1 & 0 \end{pmatrix} && (34) \\
&= -\lambda(2-\lambda)(3-\lambda) + 2(2-\lambda) = (2-\lambda)(2-\lambda(3-\lambda)) && (35) \\
&= (2-\lambda)(\lambda^2 - 3\lambda + 2) = (2-\lambda)(\lambda-2)(\lambda-1) = 0 && (36)
\end{aligned}$$

Therefore, eigenvalues are $\lambda = 2$, 1.

(b) (**5 Points**) The 3×3 matrix

$$\mathbf{B} = \begin{pmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$, and $\lambda_3 = 5$. Find an eigenvector of $\mathbf{B}$ corresponding to the eigenvalue $\lambda_2 = 3$.

$$(\mathbf{B} - 3I)\vec{x} = \vec{0} \tag{37}$$

$$\begin{pmatrix} 0 & -2 & 0 \\ -1 & 0 & -2 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{38}$$

Therefore

$$x_2 = 0$$

$$-x_1 - 2x_3 = 0 \rightarrow x_1 = -2x_3$$

Thus, the $\vec{x}$ is

$$\begin{pmatrix} -2x_3 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

and the eigenvector is

$$\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

6. (**10 Points**) The point x_0 is an ordinary point of the ODE

$$(2x-1)(x^2-1)y'' + xy' + y = 0.$$

Determine a lower bound for the radius of convergence of the series solution to this ODE centered at $x_0 = 0$.

$$y'' + \frac{x}{(2x-1)(x^2-1)}y' + \frac{1}{(2x-1)(x^2-1)}y = 0.$$

At $x = \frac{1}{2}$ and $x = 1$, $x = -1$, $P(x) = 0$. The minimum distance from $x_0 = 0$ is $\frac{1}{2}$. Therefore the power series solution converges at least for $|x| < \frac{1}{2}$

7. (**15 Points**) Consider an Euler equation

$$x^2y'' - 5xy' + 9y = 0, x > 0.$$

(a) (**5 Points**) Find a fundamental set of solutions for this equation on $x > 0$.
Let

$$y = x^r, y' = rx^{r-1}, y'' = r(r-1)x^{r-2}$$

Then, the differential equation becomes

$$r(r-1)x^r - 5rx^r + 9x^r = 0 \to r(r-1) - 5r + 9 = 0 \to r^2 - 6r + 9 = 0$$

So

$$(r-3)^2 = 0$$

Therefore, a fundamental set of solutions is

$$y_1 = x^3, y_2 = x^3 \ln x$$

(b) (**5 Points**) Compute the Wronskian of the fundamental set of solutions you found in the previous part and verify that it does not vanish on $x > 0$.

The Wronskian is

$$W = \begin{vmatrix} x^3 & x^3 \ln x \\ 3x^2 & 3x^2 \ln x + x^2 \end{vmatrix} = 3x^5 \ln x + x^5 - 3x^5 \ln x = x^5 \neq 0 \text{ for } x > 0$$

(c) (**2 Points**) Write the general solution to this equation.
and the general solution is

$$y = c_1 x^3 + c_2 x^3 \ln x$$

(d) (**3 Points**) Solve the initial value problem with $y(1) = 2$ and $y'(1) = 1$

$$y = c_1 x^3 + c_2 x^3 \ln x$$

$$y' = 3c_1 x^2 + c_2 3x^2 \ln x + c_2 x^2$$

The initial conditions yield

$$y(1) = c_1 = 2$$

$$y'(1) = 3 \cdot 2 \cdot 1^2 + c_2 \cdot 1^2 = 1 \to c_2 = -5$$

Therefore

$$y = 2x^3 - 5x^3 \ln x$$

8. (**10 Points**) By letting $z = y'$, rewrite the following system of differential equations

$$\begin{aligned} x' &= \frac{1}{2}x - 3y - 3t \\ y'' &= y' + 5x + y - 2\sin(t) \end{aligned}$$

in the form of

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \\ g_3(t) \end{pmatrix},$$

where $\mathbf{A}$ is a 3×3 matrix.
Since $z = y'$, $z' = y''$ and the second equation becomes

$$z' = z + 5x + y - 2\sin(t)$$

Therefore

$$\begin{aligned} x' &= \frac{1}{2}x - 3y - 3t \\ y' &= z \\ z' &= z + 5x + y - 2\sin(t) = 5x + y + z - 2\sin(t). \end{aligned}$$

In a matrix form

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -3 & 0 \\ 0 & 0 & 1 \\ 5 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} -3t \\ 0 \\ -2\sin t \end{pmatrix}$$