Math 8 Winter 2002 Exam 2 Practice Problems

Instructions: This will be a closed book, closed notes exam. Use of calculators will be permitted only for arithmetic calculations - no higher functions, symbolic manipulation or graphing. For the problems which are not multiple choice, you must justify your answers to get credit for the problem. The problems are meant to help you practice for the exam. They are inno way a comprehensive set of problems - you may well see completely different types of problems on the exam.

1. Find an equation of the line through the point $(1, 2, 3)$ perpendicular to the plane $x - 3y + 5z = 1$.

Solution:

The normal to the plane is $(1, -3, 5)$ so the equation of the line is $\vec{\mathbf{r}}(t) = (1, 2, 3) + t(1, -3, 5)$.

2. Are the lines $\vec{\mathbf{r}}_1(t) = (1, 2, 3) + t(4, -4, 6)$ and $\vec{\mathbf{r}}_2(t) = (3 + s, 2s, 6 + 3s)$ skew, parallel or intesecting?

Solution:

The vector parallel to $\vec{\mathbf{r}}_1$ is $(4, -4, 6)$ while the vector parallel to $\vec{\mathbf{r}}_2$ is $(1, 2, 3)$. As they are not multiples of one another, the lines are not parallel, so they must be skew or intersect. To see if they interset, we attempt to find a point of intersection by solving the following equations:

$$\begin{aligned} 1 + 4t &= 3 + s \\ 2 - 4t &= 2s \\ 3 + 6t &= 6 + 3s \end{aligned}$$

The second equation yields that $s = 1 - 2t$. Plugging this into the first equation and solving for t yields $t = \frac{1}{2}$. Thus, plugging $\{s = 0, t = \frac{1}{2}\}$ into the equations, we see this is a solution to all three. Thus, the lines intersect.

3. Find the volume of the parallelepiped determined by the vectors $\vec{\mathbf{u}} = (1, 2, 3)$, $\vec{\mathbf{v}} = (2, 0, 1)$ and $\vec{\mathbf{w}} = (3, 0, 4)$.

Solution:

This is $|\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})| = |(1,2,3) \cdot (0,-5,0)| = 10$.

4. Find the vector parametric equations for the line of intersection between the planes $x + 2y + 3z = 1$ and $2x + 5y + 4z = 3$.

Solution:

Solving the first equation for z yields $z = 1 - x - 2y$. Plugging into thesecond yields, $2x + 5y + 4(1 - x - 2y) = 3$ or $-2x - 3y = -1$. We can solve this for y, yielding $y = \frac{1-2x}{3}$. Thus, the equations for the line of intersection are:

$$
\begin{aligned}
x &= t \\
y &= \frac{1-2t}{3} \\
z &= 1 - t - \frac{2-4t}{3} = \frac{1}{3} - \frac{7}{3}t
\end{aligned}
$$

5. Choose a real number a such that the distance from the plane $ax+2y+z = 2$ to the point $(1,2,3)$ is 1.

Solution:

The distance from $(1,2,3)$ to $ax + 2y + z = 2$ is

$$\frac{|a \cdot 1 + 2 \cdot 2 + 1 \cdot 2 - 2|}{\sqrt{a^2 + 4 + 1}} = \frac{|a + 4|}{\sqrt{a^2 + 5}}$$

Setting this equal to 1 and solving for a yields $a = \frac{-11}{8}$.

6. Find the equation of the plane spanned by the intersecting lines $\vec{\mathbf{r}}_1(t) = (1,1,1) + t(1,2,3)$ and $\vec{\mathbf{r}}_2(t) = (0,-1,-2) + t(0,0,1)$.

Solution:

The lines cross at $(0,-1,2)$ and the cross product of the vectors parallel to the lines is $(2,-1,0)$ so the equation of the plane is $2x - y = 1$.

7. Find the arclength of the curve $\vec{\mathbf{r}}(t) = (t, \frac{t^3}{3}+4, \frac{\sqrt{2}}{2}t^2)$ from $t = 0$ to $t = 1$.

Solution:

The arclength is given by

$$\begin{aligned}
\int_0^1 |\vec{\mathbf{r}}'(t)|dt &= \int_0^1 |(1, t^2, \sqrt(2)t)|\; dt \\
&= \int_0^1 \sqrt{1+t^4+2t^2}\; dt \\
&= \int_0^1 \sqrt{(1+t^2)^2}\; dt \\
&= \int_0^1 (1+t^2)\; dt \\
&= 1 + \frac{1}{3} = \frac{4}{3}
\end{aligned}$$

8. A toy plane is launched from the point $(0, 0)$ in the xy-plane at an angle of $\frac{\pi}{4}$ from the x-axis with an initial speed of $\frac{75\sqrt{2}}{2}$m/s. After the launch, it provides itself with an acceleration, $\vec{\mathbf{a}}(t) = (1, t)$ and is also affected by gravity (assume, for this problem, that the force due to gravity has a magnitude of 10m/s).

   (a) Find an equation describing the position of the plane over time.

   (b) When, if ever, does the plane hit the ground again?

Solution:

The total acceleration of the plane is

$$\vec{\mathbf{a}}(t) = (1, t - 10)$$

Thus, $b\vec{f}v(t) = (t + C_1, \frac{t^2}{2} - 10t + C_2)$. Using the given information, we see $\vec{\mathbf{v}}(0) = (27\cos(\frac{\pi}{4}), 27\sin(\frac{\pi}{4})) = (150, 150)$. So, $C_1 = C_2 = 150$. Integrating again yields,

$$\vec{\mathbf{r}}(t) = (\frac{t^2}{2} + 150t + D_1, \frac{t^3}{6} - 5t^2 + 150t + D_2)$$

Since $\vec{\mathbf{r}}(0) = \vec{0}$, we have,

$$\vec{\mathbf{r}}(t) = (\frac{t^2}{2} + 150t, \frac{t^3}{6} - 5t^2 + 150t)$$

The plane hits the ground again when the y coordinate equals zero or

$$t(\frac{t^2}{6} - 5t + 150)$$

or when $t = 0, 15$. Since $t = 0$ is when the plane takes off, the plane hits the ground again at $t = 15$.