Math 68. Algebraic Combinatorics. Fall 2007.

Topics for the final project (updated on 11/1/07)

Each student should work on a project throughout the quarter. It is encouraged that you work together in groups of two or three students. At the end of the quarter, each group will present their project giving a short talk. Here is a list of possible topics. If you have your own topic in mind you are more than welcome to discuss it with me.

1. **Domino tilings of Aztec diamonds.** Counting the number of domino tilings of a given shape is a difficul problem in general. However, for the so-called Aztec diamond, the number of domino tilings has a very simple formula.
   - [Aig] Pages 44–50.

2. **Random walks in $\mathbb{Z}^d$.** The probability that a random walk in $\mathbb{Z}^d$ returns to the origin is 1 for $d = 1, 2$, but strictly less than 1 for $d \geq 3$. In other words, you shouldn't get drunk unless you move in at most two dimensions.
   - [Aig] Pages 85–89.

3. **The Gessel-Viennot method.** This is a beautiful formula to enumerate n-tuples of nonintersecting lattice paths. The answer is given by a determinant of binomial coefficients, and the proof is based on the combinatorics of involutions.
   - [Aig] Section 5.4.
   - [EC1] Section 2.7.
   - I. Gessel and G. Viennot, Binomial determinants, paths, and hook length formulae, *Advances in Math.* 58 (1985), 300–321.

4. **The descent number and the major index.** We say that i is a *descent* of a permutation $\pi \in \mathcal{S}_n$ if $\pi_i > \pi_{i+1}$. The *major index* of π, denoted $\mathrm{maj}(\pi)$, is defined as the sum of all descents in π. For example, $\mathrm{maj}(12 \cdots n) = 0$ and $\mathrm{maj}(n \cdots 21) = 1+2+\cdots+(n-1)$. On the other hand, an *inversion* of π is a pair (i, j) such that $i < j$ and $\pi_i > \pi_j$. The *inversion number* of π, denoted $\mathrm{inv}(\pi)$, is the number of inversions of π.

   The goal of this project is to show that for any value k, the number of permutations $\pi \in \mathcal{S}_n$ with $\mathrm{maj}(\pi) = k$ is the same as the number of permutations $\pi \in \mathcal{S}_n$ with $\mathrm{inv}(\pi) = k$. In other words, the major index maj is equidistributed with the number of inversions inv, that is,

   $$\sum_{\pi \in \mathcal{S}_n} q^{\mathrm{maj}(\pi)} = \sum_{\pi \in \mathcal{S}_n} q^{\mathrm{inv}(\pi)}.$$

   A stronger version of this is the fact that the joint distribution of maj and inv is symmetric, that is,

   $$\sum_{\pi \in \mathcal{S}_n} q^{\mathrm{maj}(\pi)} t^{\mathrm{inv}(\pi)} = \sum_{\pi \in \mathcal{S}_n} q^{\mathrm{inv}(\pi)} t^{\mathrm{maj}(\pi)}.$$

   - D. Foata and M.-P. Schutzenberger, Major index and inversion number of permutations, *Math. Nach.* 83 (1978), 143–159.

5. **Combinatorial proofs of the Lagrange Inversion Formula.** In class we will see an analytic proof of the Lagrange Inversion Formula. However, it is possible to give a combinatorial proof, interpreting the coefficients as counting plane forests.
   - [EC2] Section 5.4.
6. **Walks in graphs.** The number of walks of given length between two vertices of a graph can be expressed in terms of the eigenvalues of its adjacency matrix. This is a nice connection of combinatorics and linear algebra.
   - [St] Section 1.
7. **The transfer-matrix method.** An application of counting walks in graphs to other problems in enumerative combinatorics.
   - [EC1] Section 4.7.
8. **Viennot's geometric construction of the RSK correspondence.** In class we will discuss the RSK algorithm, which gives a correspondence between permutations and pairs of standard Young tableaux. A beautiful geometric description of this correspondence is due to Viennot.
   - Section 3.6 of [Bruce E. Sagan, *The Symmetric group*, Springer, second edition, 2001].
9. **Increasing and decreasing subsequences of permutations.** This theory is an application of the Robinson-Schensted correspondence (or RSK algorithm).
   - [BS] Section 5.
10. **A combinatorial proof of the unimodality of the Gaussian polynomials.** In class we discussed an algebraic proof that the q-binomial coefficients are unimodal. A direct combinatorial proof was given by Kathy O'Hara.
    - D Zeilberger, Kathy O'Hara's constructive proof of the unimodality of the Gaussian Polynomials, *The American Mathematical Monthly*, Vol. 96, No. 7, 590–602.
11. Read a paper from a combinatorics journal and present it in class. You are encouraged to talk with me to pick a suitable paper. Here are some interesting journals that you can find in the library or online.
    - *Journal of Combinatorial Theory A*,
    - *Electronic Journal of Combinatorics*,
    - *Journal of Algebraic Combinatorics*,
    - *European Journal of Combinatorics*,
    - *Annals of Combinatorics*,
    - *Discrete Mathematics.*