Classical Lie Algebras

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1 A generic finite dimensional complex semisimple Lie algebra

We follow from treatment in [Hum] and [Ser].

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. A *weight* is an element of $\mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$, the dual of $\mathfrak{h}$. If M is a $\mathfrak{g}$-module and $\mu \in \mathfrak{h}^*$, the *μ-weight space* of M is

$$M_\mu = \{m \in M \mid hm = \mu(h)m, \text{ for all } h \in \mathfrak{h}\} \qquad \text{and} \qquad M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu.$$

Notice that $\mathfrak{g}$ is a $\mathfrak{g}$-module under the adjoint action:

$$\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g} \qquad \text{given by} \qquad \mathrm{ad}_x(y) = [x, y].$$

Under the adjoint action, $\mathfrak{g}_0 = \{x \in \mathfrak{g} \mid [h, x] = 0 \text{ for all } h \in \mathfrak{g}\} = \mathfrak{h}$, and $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right), \quad \mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{g}\},$$

where $R = \{\alpha \in \mathfrak{h}^* \mid \alpha \neq 0, \mathfrak{g}_\alpha \neq 0\}$ is the set of *roots*.

The set R is the root system of $\mathfrak{g}$, whose structure is explored in depth in [Hum, Ch. 8-10] and [Ser, Ch. 5]. In particular, the roots of $\mathfrak{g}$ span $\mathfrak{h}^*$, and we choose a basis $\{\alpha_1, \ldots, \alpha_n\} \subseteq R$

for $\mathfrak{h}^*$, called the set of *simple roots* (see [Hum, Ch.10, §1], [Ser, Ch.5, §8]). This basis is chosen to have the property that it generates R by integral linear combinations which have either all non-negative or all non-positive coefficients. Let R^+ be the set *positive roots*, the roots which are non-negative combinations of the simple roots, and let $R^- = \{-\alpha \mid \alpha \in R^+\}$ be the set of *negative roots*, so $R = R^+ \cup R^-$. This fixes the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \qquad \text{where} \quad \mathfrak{n}^+ = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n}^- = \bigoplus_{\alpha \in R^-} \mathfrak{g}_\alpha. \tag{1.1}$$

The *Killing form* is the symmetric bilinear form $\langle , \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defined by

$$\langle x, y \rangle = \mathrm{Tr}(\mathrm{ad}_x \mathrm{ad}_y).$$

The properties of this form are explored in [Hum, Ch. 5]. In particular, $\langle , \rangle$ is ad-invariant, meaning that $\langle [x,y], z \rangle = -\langle y, [x,z] \rangle$ for $x, y, z \in \mathfrak{g}$, and nondegenerate on $\mathfrak{g}$ and $\mathfrak{h}$. Therefore, the map

$$\begin{array}{ccc} \mathfrak{h} & \longrightarrow & \mathfrak{h}^* \\ h & \mapsto & \langle h, \cdot \rangle \\ h_\mu & \mapsto & \mu \end{array} \qquad \text{is an isomorphism,} \tag{1.2}$$

where h_μ is the unique element of $\mathfrak{h}$ such that

$$\langle h_\mu, h \rangle = \mu(h) \quad \text{for all } h \in \mathfrak{h}. \tag{1.3}$$

Abusing notation, define a form $\langle , \rangle : \mathfrak{h}^* \otimes \mathfrak{h}^* \to \mathbb{C}$ by

$$\langle \lambda, \mu \rangle = \langle h_\lambda, h_\mu \rangle. \tag{1.4}$$

So $\langle , \rangle : \mathfrak{h}^* \otimes \mathfrak{h}^* \to \mathbb{C}$ is also symmetric, bilinear, and nondegenerate on $\mathfrak{h}^*$, and

$$\langle \lambda, \mu \rangle = \mu(h_\lambda) = \lambda(h_\mu). \tag{1.5}$$

By investigating structure the Killing form may provide, we can build a basis for $\mathfrak{g}$ which corresponds to the triangular decomposition in (1.1). To this end, first notice that if $h \in \mathfrak{h}$, $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, then

$$\alpha(h)\langle x, y \rangle = \langle [h,x], y \rangle = -\langle x, [h,y] \rangle = -\beta(h)\langle x, y \rangle.$$

So

$$\text{if } \alpha \neq -\beta, \text{ then } \langle x, y \rangle = 0, \tag{1.6}$$

i.e. since the form $\langle , \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ is nondegenerate, the subspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal precisely when $\alpha + \beta \neq 0$. Also, by (1.3),

$$\langle h, [x,y] \rangle = \langle [h,x], y \rangle = \alpha(h)\langle x, y \rangle = \langle h, h_\alpha \rangle \langle x, y \rangle = \langle h, h_\alpha \langle x, y \rangle \rangle,$$

so, again, since $\langle , \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ is nondegenerate,

$$[x, y] = \langle x, y \rangle h_\alpha, \qquad \text{for } x \in \mathfrak{g}_\alpha,\ y \in \mathfrak{g}_\beta. \tag{1.7}$$

Now let $\{h_1, \ldots, h_n\}$ be a basis for $\mathfrak{h}$ and let $\{h_1^*, \ldots, h_n^*\}$ be the dual basis with respect to $\langle , \rangle : \mathfrak{h} \otimes \mathfrak{h} \to \mathbb{C}$ (i.e.$\langle h_i, h_j^* \rangle = \delta_{ij}$). For each $\alpha \in R^+$, choose $b_\alpha \in \mathfrak{g}_\alpha$ and the corresponding $b_\alpha^* \in \mathfrak{g}_{-\alpha}$ satisfying $\langle b_\alpha, b_\alpha^* \rangle = 1$. Then

$$\{h_1, \ldots, h_n, b_\alpha, b_\alpha^* \mid \alpha \in R^+\} \tag{1.8}$$

is a basis of $\mathfrak{g}$ is dual basis

$$\{h_1^*, \ldots, h_n^*, b_\alpha^*, b_\alpha \mid \alpha \in R^+\}. \tag{1.9}$$

2 The four families

Each finite dimensional complex simple Lie algebras belongs to one of four families, $\{A_r \mid r \geq 1\}$, $\{B_r \mid r \geq 3\}$, $\{C_r \mid r \geq 3\}$, or $\{D_r \mid r \geq 4\}$, where r is the *rank* of each algebra, or is one of five exceptional Lie algebras, G_2, F_4, E_6, E_7, or E_8. The *classical* Lie algebras are those belonging to the four families.

Each Lie algebra will be represented faithfully as subsets of

$$\mathfrak{gl}_n = \{n \times n \text{ matrices with complex entries }\}$$

with multiplication given by the commutator product, or *Lie bracket*:

$$[x, y] = xy - yx.$$

Let V be a $\mathfrak{g}$-module of dimension ℓ with orthonormal basis $\{v_1, \ldots, v_\ell\}$. To ease calculations, we treat Lie algebras $\mathfrak{gl}_n$, $\mathfrak{sl}_n$, $\mathfrak{so}_{2n}$, $\mathfrak{so}_{2n+1}$, and $\mathfrak{sp}_{2n}$ as their isomorphic images in $\mathrm{End}(V)$. Furthermore, we consider an alternate ad-invariant, nondegenerate, symmetric bilinear form to the Killing form given by

$$\mathrm{Tr}(,) \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{C} \qquad \text{given by} \qquad \mathrm{Tr}(x, y) = \mathrm{Tr}(xy), \tag{2.1}$$

where on the right hand side, x and y are considered as their image in $\mathrm{End}(V)$. It can be shown (see [Hum, ???])

$$\langle x, y\rangle = \begin{cases} 2(n+1)\mathrm{Tr}(x, y) & \text{in type } \mathfrak{sl}_{n+1},\ \mathfrak{sp}_{2n} \\ (2n-1)\mathrm{Tr}(x, y) & \text{in type } \mathfrak{so}_{2n+1}, \\ 2(n-1)\mathrm{Tr}(x, y) & \text{in type } \mathfrak{so}_{2n}. \end{cases}$$

As we will see, $\mathfrak{gl}_n$ has nontrivial center, and is therefore not semisimple. The Killing form is degenerate on the center of $\mathfrak{gl}_n$, while the trace form is nondegenerate on $\mathfrak{gl}_n$ and its cartan, and so we will use the trace form throughout for $\mathfrak{gl}_n$.

Let $E_{ij} \in \mathrm{End}(V)$ be defined by

$$E_{ij} v_\ell = \delta_{j\ell} v_i.$$

Type $\mathfrak{gl}_{n+1}$ Let $\ell = n + 1$. The general linear algebra $\mathfrak{gl}_{n+1} \cong \mathfrak{gl}(V) = \mathrm{End}(V)$ has basis

$$\{b_{ij}, h_i \mid 1 \leq i \neq j \leq n+1\},$$

where

$$b_{ij} = E_{ij} \quad \text{and} \quad h_i = E_{ii}.$$

The dual basis with respect to $\mathrm{Tr}(,)$ is given by

$$b^*_{ij} = b_{ji} \quad \text{and} \quad h^*_i = h_i.$$

The triangular decomposition is given by $\mathfrak{h} = \langle h_i \mid 1 \leq i \leq n+1\rangle$, $\mathfrak{n}^+ = \langle b_{ij} \mid 1 \leq i < j \leq n+1\rangle$, and $\mathfrak{n}^- = \langle b^*_{ij} \mid b_{ij} \in \mathfrak{n}^+\rangle$. As noted above, $\mathfrak{gl}_n$ has nontrivial center, generated by $\sum_{i=1}^{n+1} h_i$, and is therefore not semisimple.

Type $\mathfrak{sl}_{n+1}$: Let $\ell = n+1$. The special linear algebra $\mathfrak{sl}_\ell \cong \mathfrak{sl}(V) = \{x \in \mathrm{End}(V) \mid \mathrm{tr}(x) = 0\}$ has basis

$$\{b_{ij}, h_k \mid 1 \leq i \neq j \leq n+1, 1 \leq k \leq n\},$$

where

$$b_{ij} = E_{ij} \text{ and } h_k = E_{kk} - E_{k+1\ k+1}.$$

Thus the dual basis with respect to $\mathrm{Tr}(,)$ is given by

$$b_{ij}^* = b_{ji} \quad \text{and} \quad h_k^* = E_{11} + \cdots + E_{kk} - \frac{k}{n+1}\left(E_{11} + \cdots + E_{n+1\ n+1}\right).$$

The triangular decomposition is given by $\mathfrak{h} = \langle h_i \mid 1 \leq i \leq n\rangle$, $\mathfrak{n}^+ = \langle b_{ij} \mid 1 \leq i < j \leq n+1\rangle$, and $\mathfrak{n}^- = \langle b_{ij}^* \mid b_{ij} \in \mathfrak{n}^+\rangle$.

Type $\mathfrak{so}_{2n+1}$ and $\mathfrak{so}_{2n}$: Let $\ell = 2n$ or $2n+1$. The orthogonal algebra $\mathfrak{so}(V) \cong \mathfrak{so}_\ell$ is the subalgebra of $\mathfrak{sl}(V)$ for which the inner product on V is ad-invariant :

$$\mathfrak{so}(V) = \{x \in \mathfrak{sl}(V) | \langle xu, v\rangle + \langle u, xv\rangle = 0 \text{ for all } u, v \in V\}.$$

Thus $\mathfrak{so}(V)$ has basis

$$\{b_{ij} \mid 1 \leq i < j \leq \ell\},$$

where

$$b_{ij} = E_{ij} - E_{ji}.$$

For $i \geq j$, write $b_{ij} = (\delta_{ij} - 1)b_{ji}$. So $\mathfrak{so}(V)$ has bracket

$$[b_{i_1,j_1}, b_{i_2,j_2}] = \begin{cases} b_{i_1,j_2}, & j_1 = i_2, \\ b_{j_1,i_2}, & i_1 = j_2, \\ -b_{i_1,i_2}, & j_1 = j_2, \\ -b_{j_1,j_2}, & i_1 = i_2, \\ 0, & \text{otherwise.} \end{cases},$$

and dual basis (with respect to $\mathrm{Tr}(,)$) defined by

$$b_{ij}^* = b_{ij}.$$

Let $\mathfrak{h}$ be generated by $\{b_{2i-1,2i} \mid 1 \leq i \leq n\}$.

Type $\mathfrak{sp}_{2n}$: When $\ell = 2n$ is even, the symplectic algebra $\mathfrak{sp}(V) \cong \mathfrak{sp}_\ell$ is the subalgebra of $\mathfrak{sl}(V)$ for which the skew-symmetric form $J(,)$ on V, given by the matrix

$$J = \left(\begin{array}{c|c} 0 & -\mathbb{1} \\ \hline \mathbb{1} & 0 \end{array}\right),$$

is ad-invariant:

$$\mathfrak{sp}(V) = \{a \in \mathfrak{sl}(V) | J(au, v) + J(u, av) = 0 \text{ for all } u, v \in V\},$$

where $J(u,v) = u^t J v$. Then $\mathfrak{sp}(V)$ has basis (in $\mathrm{End}(V)$)

$$\left\{ \begin{array}{rcll} E_{ii} - E_{r+i,r+i} & = & A_i, & 1 \le i \le n, \\ E_{ij} - E_{r+j,r+i} & = & B_{i,j}, & 1 \le i \ne j \le n, \\ E_{i,r+i} & = & C_i, & 1 \le i \le n, \\ E_{i,r+j} + E_{j,r+i} & = & D_{ij}, & 1 \le i < j \le n, \\ E_{r+i,i} & = & E_i, & 1 \le i \le n, \\ E_{r+i,j} + E_{r+j,i} & = & F_{ij}, & 1 \le i < j \le n. \end{array} \right\}.$$

With respect to $\mathrm{Tr}(,)$, $\mathfrak{sp}(V)$ has dual basis

$$A_i^* = A_i, \quad B_{i,j}^* = B_{j,i},$$

$$C_i^* = 2E_i, \quad D_{ij}^* = F_{ij},$$

$$E_i^* = 2C_i, \quad \text{and} \quad F_{ij}^* = D_{ij}.$$

The triangular decomposition is given by $\mathfrak{h} = \langle A_i \mid 1 \le i \le n \rangle$, $\mathfrak{n}^+ = \langle C_i, D_{ij}, B_{i<j} \mid i = ... \rangle$, and $\mathfrak{n}^- = \langle b_i^* \mid b_i \in \mathfrak{n}^+ \rangle$. The skew symmetric form evaluated on the basis of V is

$$J(v_i, v_j) = \begin{cases} 1, & \text{if } j = i + r, \\ -1, & \text{if } j = i - r, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\varepsilon_1, \dots, \varepsilon_\ell$ be an orthonormal basis of the vector space $\mathbb{R}^\ell$. Then

$$\mathfrak{h}^* = \begin{cases} \mathbb{R}^\ell, & \mathfrak{g} = \mathfrak{gl}_\ell, \mathfrak{so}_{2\ell+1}, \mathfrak{so}_{2\ell}, \text{ or } \mathfrak{sp}_{2\ell} \\ \{\, a_1\varepsilon_1 + \cdots + a_\ell\varepsilon_\ell \mid a_i \in \mathbb{R}, a_1 + \cdots + a_\ell = 0 \,\}, & \mathfrak{g} = \mathfrak{sl}_\ell. \end{cases}$$

The roots for the classical Lie algebras can be expressed in terms of the ε_i:

$$R = \begin{cases} \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \le i < j \le n+1\}, & \text{for } \mathfrak{gl}_{n+1}, \\ \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \le i < j \le n+1\}, & \text{for } \mathfrak{sl}_{n+1}, \\ \{\pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \le i < j \le n\} \cup \{\pm\varepsilon_i \mid 1 \le i \le n\}, & \text{for } \mathfrak{so}_{2n+1}, \\ \{\pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \le i < j \le n\} \cup \{\pm 2\varepsilon_i \mid 1 \le i \le n\}, & \text{for } \mathfrak{sp}_{2n}, \\ \{\pm(\varepsilon_i \pm \varepsilon_j) \mid 1 \le i < j \le n\}, & \text{for } \mathfrak{so}_{2n}, \end{cases}$$

In each case where $\mathfrak{g}$ is semisimple, the roots span $\mathfrak{h}^*$. We choose as the basis $\{\alpha_i\}_{i=1\dots n}$, the set of *simple roots*,

$$\begin{array}{rcll} \alpha_i & = & \varepsilon_i - \varepsilon_{i+1}, & 1 \le i \le n-1, \\ \alpha_n & = & \begin{cases} \varepsilon_n - \varepsilon_{n+1} & \text{for type } \mathfrak{sl}_{n+1}, \mathfrak{gl}_{n+1} \\ \varepsilon_n & \text{for type } \mathfrak{so}_{2n+1} \\ 2\varepsilon_n & \text{for type } \mathfrak{sp}_{2n} \\ \varepsilon_{n-1} + \varepsilon_n & \text{for type } \mathfrak{so}_{2n} \end{cases} & \end{array}$$

Then the positive roots (those roots arising from nonnegative integral linear combinations of the simple roots) are

$$R^+ = \begin{cases} \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n+1\}, & \text{for } \mathfrak{sl}_{n+1}, \mathfrak{gl}_{n+1} \\ \{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\} \cup \{\varepsilon_i \mid 1 \le i \le n\}, & \text{for } \mathfrak{so}_{2n+1}, \\ \{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\} \cup \{2\varepsilon_i \mid 1 \le i \le n\}, & \text{for } \mathfrak{sp}_{2n}, \\ \{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\}, & \text{for } \mathfrak{so}_{2n}. \end{cases}$$

The finite dimensional irreducible $\mathfrak{g}$-modules are indexed by the *dominant integral weights*, denoted $P^+ \subseteq \mathfrak{h}^*$. In the case where $\mathfrak{g}$ is semisimple, the *fundamental weights* $\omega_i \in \mathfrak{h}^*$ form a $\mathbb{Z}_{\geq 0}$ basis for P^+, i.e.

$$P^+ = \sum_{i=1}^{n} \mathbb{Z}_{\geq 0}\omega_i.$$

Again, when $\mathfrak{g}$ is semisimple, the fundamental weights have the property that

$$\langle \omega_i, \alpha_j \rangle = \delta_{i,j}, \qquad \text{i.e.} \quad \omega_i(h_{\alpha_j}) = \delta_{i.j},$$

so are given by

$$\begin{aligned}
\omega_i &= \varepsilon_1 + \cdots + \varepsilon_i \\
&\quad - \tfrac{i}{n+1}(\varepsilon_1 + \cdots + \varepsilon_{n+1}) && 1 \leq i \leq n, && \text{for } \mathfrak{sl}_{n+1},
\end{aligned}$$

$$\begin{aligned}
\omega_i &= \varepsilon_1 + \cdots + \varepsilon_i, && 1 \leq i \leq n-2, \\
\omega_{n-1} &= \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{n-1} - \varepsilon_n), && && \text{for } \mathfrak{so}_{2n}, \\
\omega_n &= \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{n-1} + \varepsilon_n),
\end{aligned}$$

$$\begin{aligned}
\omega_i &= \varepsilon_1 + \cdots + \varepsilon_i, && 1 \leq i \leq n-1, && \text{for } \mathfrak{so}_{2n+1}, \\
\omega_n &= \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n),
\end{aligned}$$

$$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i, \qquad 1 \leq i \leq n, \qquad \text{for } \mathfrak{sp}_{2n}.$$

Thus the dominant integral weights are given by

$$\lambda = \begin{array}{l} \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \\ -\frac{|\lambda|}{n+1}(\varepsilon_1 + \cdots + \varepsilon_{n+1}) \end{array}, \qquad \text{for } \mathfrak{sl}_{n+1},$$

$$\text{where} \quad \begin{array}{c} \lambda_i \in \mathbb{Z}, \\ 0 \leq \lambda_n \leq \cdots \leq \lambda_1, \\ |\lambda| = \lambda_1 + \cdots + \lambda_n, \end{array}$$

$$\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n, \qquad \text{for } \mathfrak{so}_{2n},$$

$$\text{where} \quad \begin{array}{c} \lambda_i \in \mathbb{Z} \ \text{ for all } i \ \text{ or } \lambda_i \in \mathbb{Z} + \frac{1}{2} \ \text{ for all } \ i, \\ 0 \leq |\lambda_n| \leq \lambda_{n-1} \leq \cdots \leq \lambda_1, \end{array}$$

$$\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n, \qquad \text{for } \mathfrak{so}_{2n+1},$$

$$\text{where} \quad \begin{array}{c} \lambda_i \in \mathbb{Z} \ \text{ for all } i \ \text{ or } \lambda_i \in \mathbb{Z} + \frac{1}{2} \ \text{ for all } \ i, \\ 0 \leq \lambda_n \leq \cdots \leq \lambda_1 \end{array}$$

$$\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n, \qquad \text{for } \mathfrak{sp}_{2n},$$

$$\text{where} \quad \begin{array}{c} \lambda_i \in \mathbb{Z}, \\ 0 \leq \lambda_n \leq \cdots \leq \lambda_1. \end{array}$$

In the case where $\mathfrak{g} = \mathfrak{gl}_{n+1}$, the finite dimensional irreducible $\mathfrak{g}$-modules are indexed by

$$P^+ = \sum_{i=1}^{n} \mathbb{Z}_{\geq 0}\omega_i \quad + \quad \mathbb{Z}\omega_{n+1},$$

where

$$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i, \qquad \text{for } 1 \leq i \leq n+1.$$

So the dominant integral weights for $\mathfrak{gl}_{n+1}$ are given by

$$\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_{n+1}\varepsilon_{n+1}, \quad \text{where } \lambda_i \in \mathbb{Z},\ \lambda_1 \geq \cdots \geq \lambda_{n+1}.$$

Finally, define

$$\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha = \sum_{j=1}^{n} \omega_j. \tag{2.2}$$

Specifically, in each of these cases, this element is given by

$$\rho = \frac{1}{2}\sum_i (y - 2i)\varepsilon_i, \qquad \text{where } y = \begin{cases} n+1, & \text{if } \mathfrak{g} = \mathfrak{sl}_n, \\ 2n, & \text{if } \mathfrak{g} = \mathfrak{so}_{2n}, \\ 2n+2, & \text{if } \mathfrak{g} = \mathfrak{sp}_{2n}, \\ 2n+1, & \text{if } \mathfrak{g} = \mathfrak{so}_{2n+1}. \end{cases} \tag{2.3}$$

2.1 Partitions

It is convenient at times to represent weights as partitions. Identify each dominant integral weight λ with the configuration of boxes with $|\lambda_i|$ boxes in for i. In the case of $\mathfrak{g} = \mathfrak{so}_{2n}$, if $\lambda_r < 0$, mark the boxes with a minus sign. For example, if $\lambda = 3\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3 - \varepsilon_4$, the associated partition is

$$\begin{array}{|c|c|c|} \hline & & \\ \hline & \\ \cline{1-2} & \\ \cline{1-2} - \\ \cline{1-1} \end{array} \; .$$

When we refer to changing a weight by adding or subtracting a box in row i, this specifically means changing λ_i by ± 1.

If B is the box in row i and column j of λ, the *content* of B is

$$c(B) = (j - i). \tag{2.4}$$

For example, if we fill in the boxes in the above λ with their respective contents, we get

$$\begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline -1 & 0 \\ \cline{1-2} -2 & -1 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \; .$$

Let $L(\lambda)$ be the finite dimensional irreducible highest-weight $\mathfrak{g}$-module of weight λ, where $\lambda \in P^+$, i.e. the $\mathfrak{g}$-module generated by highest weight vector v_λ^+ of weight λ with action

$$hv_\lambda^+ = \lambda(h)v_\lambda^+ \quad \text{and} \quad b_\alpha v_\lambda^+ = 0 \quad \text{for } \alpha \in R^+, h \in \mathfrak{h}.$$

Since $\mathfrak{g}$ is semisimple, every finite dimensional $\mathfrak{g}$-module M decomposes as the direct sum of simple modules:

$$M = \bigoplus_{\lambda \in P^+} c_\lambda L(\lambda).$$

In particular, for all $\mu \in P^+$,

$$L(\mu) \otimes L(\omega_1) = \bigoplus_{\lambda \in \Lambda(\mu)} L(\lambda), \tag{2.5}$$

where

$$\Lambda(\mu) = \begin{cases} \mu^+ = \left\{ \begin{array}{c} \text{partitions of height} \le n \\ \text{obtained by adding a box to } \mu \end{array} \right\} & \text{for } \mathfrak{sl}_{n+1},\ \mathfrak{gl}_n \\[2ex] \mu^\pm = \left\{ \begin{array}{c} \text{partitions of height} \le n \\ \text{obtained by adding a box to } \mu \end{array} \right\} & \text{if } \begin{array}{l} \mathfrak{g} = \mathfrak{so}_{2n},\ \mathfrak{sp}_{2n}, \text{ or} \\ \mathfrak{g} = \mathfrak{so}_{2n+1} \text{ and } \lambda_n \le 0 \end{array} \\[2ex] \mu^\pm \cup \{\mu\} & \text{for } \mathfrak{so}_{2n+1} \text{ and } \lambda_n > 0. \end{cases}$$

Note that if $\mathfrak{g} = \mathfrak{so}_{2n}$, then there is a possibility of adding or removing a negative box, and the removal of a box from row n when $\mu_n = \frac{1}{2}$ or 0 will result in $\lambda_n = -\frac{1}{2}$ or -1, respectively.

Littlewood-Richardson stuff Let μ, ν, λ be partitions such that $\mu, \nu \subset \lambda$. A tableau T of shape λ/μ of weight ν is a filling of λ/μ with ν_1 1's, ν_2 2's, $\ldots$, such that the row fillings are weakly increasing and the column fillings are strictly increasing. We can derive a word $w(T) = a_1 a_2 \cdots a_N$, where $N = |\lambda/\mu|$, by reading off the numbers in T rights to left, top to bottom. For example, if

$$\lambda = \ydiagram{5,3,2}, \quad \mu = \ydiagram{1,1}, \quad \nu = \ydiagram{4,2,2},$$

then one tableau is

$$T = \begin{ytableau} \none & 1 & 1 & 1 & 2 \\ \none & 2 & 3 \\ 1 & 3 \end{ytableau},$$

and this tableau has word $w(T) = 2\ 1\ 1\ 1\ 3\ 2\ 3\ 1$.

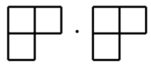
A word $w = a_1 a_2 \cdots a_N$ is a *lattice permutation* if for each $1 \le r \le N$ and $2 \le i \le n$, there are no more occurrences of i than of $i-1$. So 2 1 1 1 3 2 3 1 is not a lattice permutation, but 1 2 1 1 3 2 3 1 is. In the above example, the only tableau of shape λ/μ of weight ν which also generates a lattice permutation is the one generating the word 1 1 1 1 2 2 3 3.

Theorem 2.1 (Littlewood-Richardson rule)**.** *Let s_λ be the Schur function indexed by the partition λ. Then*

$$s_\mu s_\lambda = \sum_{\substack{\mu, \nu \subset \lambda \\ |\lambda| = |\mu| + |\nu|}} c^\lambda_{\mu,\nu} s_\lambda$$

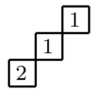
where $c^\lambda_{\mu\nu}$ is the number of tableau of shape λ/μ of weight ν which generates a lattice permutation.

For example,

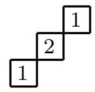


$$
\begin{aligned}
= \quad & 0\cdot(2,1,1,1,1) && +1\cdot(2,2,1,1) && +1\cdot(3,1,1,1)\\
& +1\cdot(2,2,2) && +2\cdot(3,2,1) && +1\cdot(3,3)\\
& +1\cdot(4,1,1) && +1\cdot(4,2) && +0\cdot(5,1)
\end{aligned}
$$

The interesting factor in here $2\cdot(3,2,1)$, represented by



and



.

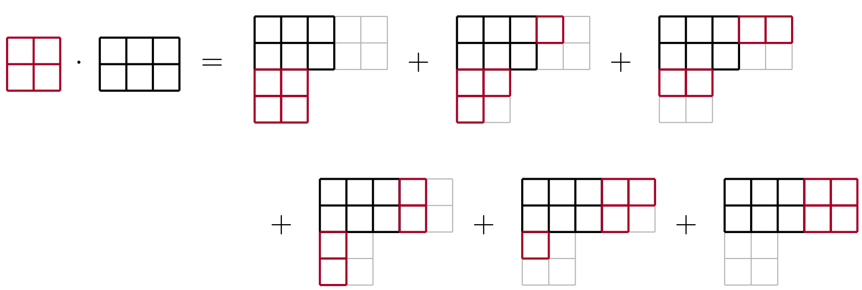
Example (Horizontal strips)**.** *If $\nu=(a)$, $a\in\mathbb{Z}_{>0}$, then each $c^{\lambda}_{\mu\nu}$ is at most 1, since there is only one way to fill in a bunch of boxes with 1's. Moreover, λ/μ must be a* horizonal strip *(a skew shape where every column has at more one box), since if there were more than one box in any column, we would have to fill with only weakly decreasing numbers.*

Example (Vertical strips)**.** *If $\nu=(1^a)$, then, again, each $c^{\lambda}_{\mu\nu}$ is at most 1, and is nonzero exactly when λ/μ is a length a* vertical strip *(a skew shape where every row has at more one box).*

Example (Rectangles)**.** *Let $p\geq q$ and a,b be non-negative integers. Then each $c^{\lambda}_{(a^p)(b^q)}$ is 1 if $\lambda\in\mathcal{T}(a^p,b^q)$, and is zero otherwise, where $\mathcal{T}(a^p,b^q)$ is the set of partitions λ with length $\leq p+q$ such that*

$$
\begin{aligned}
&\lambda_{p+1}=\lambda_{p+2}=\cdots=\lambda_q=t,\\
&\lambda_p\geq max(a,b),\\
&\lambda_i+\lambda_{p+q-i+1}=a+b,\quad i=1,\dots,p.
\end{aligned}
$$

In other words, $\mathcal{T}(a^p,b^q)$ is the set of partitions made of stacking (a^p) on top of (b^q), carving a corner out of (b^q), rotating it 180° and gluing it to the end of (a^p). For example,



2.2 The Casimir element and the operator t

If $\mathfrak{g}$ is semisimple, the *Casimir element* of $\mathfrak{g}$ is

$$\kappa=\sum_i b_ib_i^*,$$

where if $\{b_i\}$ is a basis for $\mathfrak{g}$, then $\{b_i^*\}$ is the dual basis to $\{b_i\}$ with respect to the Killing form, i.e. $\langle b_i^*,b_j\rangle=\delta_{ij}$. When $\mathfrak{g}=\mathfrak{gl}_n$, which is not semisimple, we write $\kappa=\sum_i b_ib_i^*$, where $\{b_i^*\}$ is the dual basis to $\{b_i\}$ with respect to the trace form. In either case, if M and N are $\mathfrak{g}$-modules, κ acts on $M \otimes N$ by

$$\Delta(\kappa) = (\kappa \otimes 1) + (1 \otimes \kappa) + 2t, \qquad \text{where} \quad t = \sum_i b_i \otimes b_i^*. \tag{2.6}$$

The Casimir is central in $U\mathfrak{g}$, so it acts on each irreducible component of a $\mathfrak{g}$-module M as a scalar.

Theorem 2.2.

(a) *Let $\mathfrak{g}$ be $\mathfrak{sl}_{n+1}$, $\mathfrak{so}_{2n}$, $\mathfrak{so}_{2n+1}$, or $\mathfrak{sp}_{2n}$, and let λ, μ, and ν be dominant integral weights for $\mathfrak{g}$. As in equation (2.3), let*

$$\rho = \frac{1}{2}\sum_i (y-2i)\varepsilon_i, \qquad \textit{where} \quad y = \begin{cases} n+1, & \textit{if } \mathfrak{g} = \mathfrak{sl}_n, \\ 2n, & \textit{if } \mathfrak{g} = \mathfrak{so}_{2n}, \\ 2n+2, & \textit{if } \mathfrak{g} = \mathfrak{sp}_{2n}, \\ 2n+1, & \textit{if } \mathfrak{g} = \mathfrak{so}_{2n+1}. \end{cases}$$

(i) *The Casimir element κ acts on a $\mathfrak{g}$-module $L(\lambda)$ of highest weight λ by the constant*

$$\langle \lambda, \lambda + 2\rho \rangle.$$

(ii) *If $L(\lambda)$ is a submodule of $L(\mu) \otimes L(\nu)$, then t acts on the $L(\lambda)$ isotypic component of $L(\mu) \otimes L(\nu)$ by the constant*

$$t_{\mu\nu}^{\lambda} = \tfrac{1}{2}(\langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \nu, \nu + 2\rho \rangle). \tag{2.7}$$

(b) *Let $\mathfrak{g} = \mathfrak{gl}_{n+1}$, and let λ, μ, and ν be dominant integral weights for $\mathfrak{g}$. Let*

$$\delta = n\varepsilon_1 + (n-1)\varepsilon_2 + \cdots + \varepsilon_n = \sum_{i=1}^{n} (n+1-i)\varepsilon_i.$$

(i) *The Casimir element κ acts on a $\mathfrak{g}$-module $L(\lambda)$ of highest weight λ by the constant*

$$\langle \lambda, \lambda + 2\delta \rangle - (n+1)|\lambda|.$$

(ii) *If $L(\lambda)$ is a submodule of $L(\mu) \otimes L(\nu)$, then t acts on the $L(\lambda)$ isotypic component of $L(\mu) \otimes L(\nu)$ by the constant*

$$t_{\mu\nu}^{\lambda} = \tfrac{1}{2}\Big(\langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \nu, \nu + 2\rho \rangle - (n+1)\big(|\lambda| - |\mu| - |\nu|\big)\Big). \tag{2.8}$$

Proof.

(a) First, take $\mathfrak{g}$ to be semisimple.

(i) As at the end of Section **??**, choose a basis of $\mathfrak{g}$ compatible with the triangular decomposition, $\{h_i, b_\alpha, b_\alpha^* \mid i = 1, \ldots, n, \alpha \in R^+\}$, where $h_i \in \mathfrak{h}$, $b_\alpha \in \mathfrak{g}_\alpha$, $b_\alpha^* \in \mathfrak{g}_{-\alpha}$, and $(b_\alpha)^* = b_\alpha^*$. So we can write

$$\kappa = \sum_{i=1}^{n} h_i h_i^* + \sum_{\alpha \in R^+} b_\alpha b_\alpha^* + b_\alpha^* b_\alpha.$$

The module $L(\lambda)$ is the $\mathfrak{g}$-module generated by highest weight vector v_λ^+ of weight λ, i.e.

$$hv_\lambda^+ = \lambda(h)v_\lambda^+ \quad \text{and} \quad b_\alpha v_\lambda^+ = 0 \quad \text{for } \alpha \in R^+, h \in \mathfrak{h}.$$

Recall from (2.2)

$$\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha,$$

and so

$$\begin{aligned}
\kappa v_\lambda^+ &= \left(\sum_{i=1}^n h_i h_i^* + \sum_{\alpha\in R^+}(b_\alpha b_\alpha^* + b_\alpha^* b_\alpha)\right) v_\lambda^+ \\
&= \left(\sum_{i=1}^n h_i \lambda(h_i^*) + \sum_{\alpha\in R^+}([b_\alpha, b_\alpha^*] + 2b_\alpha^* b_\alpha)\right) v_\lambda^+ \\
&= \left(\sum_{i=1}^n h_i \langle h_\lambda, h_i^*\rangle + \sum_{\alpha\in R^+}(\langle b_\alpha, b_\alpha^*\rangle h_\alpha + 2b_\alpha^* b_\alpha)\right) v_\lambda^+ && \text{by (1.3), (1.7)} \\
&= \left(h_\lambda + \sum_{\alpha\in R^+}(h_\alpha + 0)\right) v_\lambda^+ = \left(\lambda(h_\lambda) + \sum_{\alpha\in R^+}\lambda(h_\alpha)\right) v_\lambda^+ \\
&= \left(\langle\lambda,\lambda\rangle + \left\langle \lambda, \sum_{\alpha\in R^+}\alpha\right\rangle\right) v_\lambda^+ \quad = \quad \langle \lambda, \lambda+2\rho\rangle v_\lambda^+,
\end{aligned}$$

since $\lambda(h_\mu) = \langle \lambda, \mu\rangle$ by definition of $\langle,\rangle : \mathfrak{h}^* \otimes \mathfrak{h}^* \to \mathbb{C}$ (see (1.4)).

(ii) Since κ acts on $L(\mu)\otimes L(\nu)$ by $(\kappa \otimes 1_{L(\nu)}) + (1_{L(\mu)} \otimes \kappa) + 2t$, it follows directly from part (i) that t acts on the $L(\lambda)$ isotypic component of $L(\mu)\otimes L(\nu)$ by

$$\tfrac{1}{2}(\langle\lambda,\lambda+2\rho\rangle - \langle\mu,\mu+2\rho\rangle - \langle\nu,\nu+2\rho\rangle$$

as desired.

(b) (i) In the case where $\mathfrak{g} = \mathfrak{gl}_n$, recall $\{E_{ij} \mid 1 \le i,j \le n\}$ forms a basis with dual basis $\{E_{ji} \mid 1 \le i,j \le n\}$ with respect to the trace form. So

$$\begin{aligned}
\kappa = \sum_{1\le i,j\le n} E_{ij}E_{ji} &= \sum_{i=1}^n E_{ii}E_{ii} + \sum_{1\le i<j\le n} E_{ij}E_{ji} + E_{ji}E_{ij} \\
&= \sum_{i=1}^n E_{ii}E_{ii} + \sum_{1\le i<j\le n}([E_{ij},E_{ji}] + 2E_{ji}E_{ij}) \\
&= \sum_{i=1}^n E_{ii}E_{ii} + \sum_{1\le i<j\le n}(E_{ii} - E_{jj} + 2E_{ji}E_{ij})
\end{aligned}$$

The module $L(\lambda)$ is the $\mathfrak{g}$-module generated by highest weight vector v_λ^+ of weight λ, i.e.

$$E_{ii}v_\lambda^+ = \lambda_i v_\lambda^+ \quad \text{and} \quad E_{ij}v_\lambda^+ = 0 \quad \text{for } i<j.$$

So

$$\begin{aligned}
\kappa v_\lambda^+ &= \left(\sum_{i=1}^n E_{ii}E_{ii} + \sum_{1\le i<j\le n} (E_{ii} - E_{jj} + 2E_{ij}E_{ji}) \right) v_\lambda^+ \\
&= \left(\sum_{i=1}^n \lambda_i^2 + \sum_{1\le i<j\le n} \lambda_i - \lambda_j + 0 \right) v_\lambda^+ \\
&= \left(\langle \lambda, \lambda \rangle + \sum_{i=1}^n \big((n-i) - (i-1)\big)\lambda_i \right) v_\lambda^+ = \left(\langle \lambda, \lambda \rangle + \sum_{i=1}^n \big(2n+2-2i\big)\lambda_i - (n+1)\lambda_i \right) v_\lambda^+ \\
&= \left(\langle \lambda, \lambda \rangle + \langle \lambda, 2\delta \rangle - (n+1)|\lambda| \right) v_\lambda^+
\end{aligned}$$

(ii) Again, since κ acts on $L(\mu) \otimes L(\nu)$ by $(\kappa \otimes 1_{L(\nu)}) + (1_{L(\mu)} \otimes \kappa) + 2t$, it follows directly from part (i) that t acts on the $L(\lambda)$ isotypic component of $L(\mu) \otimes L(\nu)$ by

$$\tfrac{1}{2}\Big(\langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \nu, \nu + 2\rho \rangle - (n+1)\big(|\lambda| - |\mu| - |\nu|\big) \Big)$$

as desired.

□

$$\begin{aligned}
\kappa_{L(\omega_1)} &= \langle \omega_1, \omega_1 + 2\rho \rangle = \begin{cases} n - \frac{1}{n} & \mathfrak{g} = \mathfrak{sl}_n \\ 2n-1 & \mathfrak{g} = \mathfrak{so}_{2n} \\ 2n+1 & \mathfrak{g} = \mathfrak{sp}_{2n} \\ 2n & \mathfrak{g} = \mathfrak{so}_{2n+1} \end{cases} \\
\kappa_{L(\omega_1)} &= \langle \omega_1, \omega_1 + 2\delta \rangle - n = n - 1, \text{ if } \mathfrak{g} = \mathfrak{gl}_n.
\end{aligned} \tag{2.9}$$

Check:
In type $\mathfrak{sl}_n$, $\langle \omega_1, \omega_1 + 2\rho \rangle = \langle \varepsilon_1 - \frac{1}{n}(\varepsilon_1 + \cdots + \varepsilon_n), \varepsilon_1 - \frac{1}{n}(\varepsilon_1 + \cdots + \varepsilon_n) + \sum_{i=1}^n (n+1-2i)\varepsilon_i \rangle = 1 - \frac{1}{n} + n - 1 = n - \frac{1}{n}$.
In types $\mathfrak{so}_{2n}, \mathfrak{sp}_{2n}, \mathfrak{so}_{2n+1}$, $\langle \omega_1, \omega_1 + 2\rho \rangle = \langle \varepsilon_1, \varepsilon_1 + \sum_{i=1}^n (y - 2i)\varepsilon_i \rangle = 1 + y - 2 = y - 1$.
In type $\mathfrak{gl}_n$, $\langle \omega_1, \omega_1 + 2\delta \rangle = \langle \varepsilon_1, \varepsilon_1 + \sum_{i=1}^n 2n - 2i)\varepsilon_i \rangle = 1 + 2(n-1) - n = n - 1$.

From equation (2.5), we know exactly how $L(\mu) \otimes L(\omega_1)$ decomposes. In fact, we can express the constants in Theorem 2.2(b) in terms of contents of boxes added or removed in the associated partitions. Recall, from (2.4), if B is the box in row i and column j of λ, the *content* of B is

$$c(B) = (j - i).$$

See Section 2.1 for a discussion of this construction.

Theorem 2.3. *Let $\mathfrak{g}$ be $\mathfrak{gl}_{n+1}$, $\mathfrak{sl}_{n+1}$, $\mathfrak{so}_{2n}$, $\mathfrak{so}_{2n+1}$, or $\mathfrak{sp}_{2n}$, let μ be a dominant integral weight for $\mathfrak{g}$, and let $V = L(\omega_1)$. If $L(\lambda)$ has a nontrivial isotypic component in $L(\mu) \otimes L(\omega_1)$, write $t^\lambda_{\mu\omega_1}$ to mean the constant by which t acts on this component.*

1. *If $\mathfrak{g} = \mathfrak{gl}(V)$, then*
$$t^\lambda_{\mu\omega_1} = c(B),$$
where B is the box added to obtain λ from μ.

2. *If $\mathfrak{g} = \mathfrak{sl}(V)$, then*

$$t^{\lambda}_{\mu\omega_1} = c(B) - \frac{|\mu|}{n+1},$$

where B is the box added to obtain λ from μ.

3. *If $\mathfrak{g} = \mathfrak{so}(V)$, then*

$$t^{\lambda}_{\mu\omega_1} = \begin{cases} c(B) & \text{if } B \text{ is added,} \\ -c(B) - (n-1) & \text{if } B \text{ is removed,} \end{cases}$$

where B is the box added or removed to obtain λ from μ.

4. *If $\mathfrak{g} = \mathfrak{sp}(V)$, then*

$$t^{\lambda}_{\mu\omega_1} = \begin{cases} c(B) & \text{if } B \text{ is added,} \\ -c(B) - (n+1) & \text{if } B \text{ is removed,} \end{cases}$$

where B is the box added or removed to obtain λ from μ.

Proof.

Case $\mathfrak{g} = \mathfrak{gl}(V)$. Write $\mu = \mu_1\varepsilon_1 + \cdots + \mu_{n+1}\varepsilon_{n+1}$. Adding a box to μ in the i^{th} row is equivalent to adding ε_i. Note that $\omega_1 = \varepsilon_1$. So

$$\lambda = \mu + (\varepsilon_i - \varepsilon_1 + \omega_1).$$

Recall $\delta = \sum_{i=1}^{n}(n+1-i)\varepsilon_i$.
Thus

$$\begin{aligned} 2t^{\lambda}_{\mu\omega_1} &= \big(\langle \mu + \varepsilon_i - \varepsilon_1 + \omega_1, \mu + \varepsilon_i - \varepsilon_1 + \omega_1 + 2\delta\rangle - \langle \mu, \mu + 2\delta\rangle - \langle \omega_1, \omega_1 + 2\delta\rangle\big) \\ &\quad - (n+1)\big(|\lambda| - |\mu| - |\omega_1|\big) \\ &= \Big(2\langle \mu, \omega_1\rangle + 2\langle \varepsilon_i - \varepsilon_1, \mu\rangle + 2\langle \varepsilon_i - \varepsilon_1, \omega_1\rangle + \langle \varepsilon_i - \varepsilon_1, \varepsilon_i - \varepsilon_1 + 2\delta\rangle\Big) - 0 \\ &= 2\big(\mu_1 + \mu_i - \mu_1 - 1 + 1 + (n+1-i) - n\big) \\ &= 2(\mu_i + 1 - i) \end{aligned}$$

A box added to row i of μ is in position $(i, \mu_i + 1)$ and has content$(\mu_i + 1) - i$, so

$$t^{\lambda}_{\mu\omega_1} = c(B).$$

Case $\mathfrak{g} = \mathfrak{sl}(V)$. Write $\mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n - \frac{|\mu|}{n+1}(\varepsilon_1 + \cdots + \varepsilon_{n+1})$. Adding a box to μ in the i^{th} row is equivalent to adding $\varepsilon_i - \frac{1}{n+1}(\varepsilon_1 + \cdots + \varepsilon_{n+1})$. Note that $\omega_1 = \varepsilon_1 - \frac{1}{n+1}(\varepsilon_1 + \cdots + \varepsilon_{n+1})$. So

$$\lambda = \mu + (\varepsilon_i - \varepsilon_1 + \omega_1).$$

Recall $2\rho = \sum_{i=1}^{n} \omega_i = \sum_{j=1}^{n+1}(n+2-2j)\varepsilon_j$.
Thus

$$\begin{aligned}
2t^{\lambda}_{\mu\omega_1} &= \big(\langle \mu+\varepsilon_i-\varepsilon_1+\omega_1, \mu+\varepsilon_i-\varepsilon_1+\omega_1+2\rho\rangle - \langle \mu, \mu+2\rho\rangle - \langle \omega_1, \omega_1+2\rho\rangle\big) \\
&= \Big(2\langle \mu, \omega_1\rangle + 2\langle \varepsilon_i-\varepsilon_1, \mu\rangle + 2\langle \varepsilon_i-\varepsilon_1, \omega_1\rangle + \langle \varepsilon_i-\varepsilon_1, \varepsilon_i-\varepsilon_1+2\rho\rangle\Big) \\
&= 2\left(\left(\mu_1 - \frac{|\mu|}{n+1}\right) + (\mu_i-\mu_1) + 1 + \frac{1}{2}\Big((1+n+2-2i) - (-1+n+2-2)\Big)\right) \\
&= 2\left(\mu_i - \frac{|\mu|}{n+1} - 1 + 2 - i\right) \qquad = \qquad 2\big((\mu_i+1)-i\big) - 2\frac{|\mu|}{n+1}
\end{aligned}$$

The content of a box added from row i of μ is $(\mu_i+1)-i$, and so

$$t^{\lambda}_{\mu\omega_1} = c(B) - \frac{|\mu|}{n+1}.$$

Case 2: $\mathfrak{g} = \mathfrak{so}(V)$. Adding or removing a box from μ in the i^{th} row is equivalent to adding or subtracting ε_i, i.e. $\lambda = \mu \pm \varepsilon_i$. Recall $\omega_1 = \varepsilon_1$, and let $y = n$, so $2\rho = \sum_i (y-2i)\varepsilon_i$. Thus

$$\begin{aligned}
2t^{\lambda}_{\mu\nu} &= \big(\langle \lambda, \lambda+2\rho\rangle - \langle \mu, \mu+2\rho\rangle - \langle \omega_1, \omega_1+2\rho\rangle\big) \\
&= \big(\langle \mu\pm\varepsilon_i, \mu\pm\varepsilon_i+2\rho\rangle - \langle \mu, \mu+2\rho\rangle - \langle \varepsilon_1, \varepsilon_1+2\rho\rangle\big) \\
&= \big(\pm 2\langle \varepsilon_i, \mu\rangle + \langle \varepsilon_i, \varepsilon_i \pm 2\rho\rangle - \langle \varepsilon_1, \varepsilon_1+2\rho\rangle\big) \\
&= \pm 2\mu_i + \Big(1 \pm (y-2i) - (1+y-2)\Big) \\
&= \begin{cases} 2(\mu_i+1-i), & \text{if a box is added,} \\ -2\left(\mu_i - i + y - 1\right), & \text{if a box is removed,} \end{cases} \\
&= \begin{cases} 2c(b), & \text{if } b \text{ is added,} \\ 2(-c(b)-(n-1)), & \text{if } b \text{ is removed,} \end{cases}
\end{aligned}$$

since a box added would be in column μ_i+1 and row i, where a box removed would be in column μ_i and row i.

Case 3: $\mathfrak{g} = \mathfrak{sp}(V)$. Just as in case 2, $\lambda = \mu \pm \varepsilon_i$ and $\omega_1 = \varepsilon_1$. However, in this case $2\rho = \sum_i (y-2i)\varepsilon_i$ if we let $y = n+2$. So

$$\begin{aligned}
2t^{\lambda}_{\mu\nu} &= \begin{cases} 2(\mu_i+1-i) & \text{if a box is added,} \\ -2\left(\mu_i - i + y + 1\right) & \text{if a box is removed,} \end{cases} \\
&= \begin{cases} 2c(b) & \text{if } b \text{ is added,} \\ 2(-c(b)-(n+1)) & \text{if } b \text{ is removed,} \end{cases}
\end{aligned}$$

□

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