Homework 2

1) Page 53 #6

$$u' = (\lambda - b)u - au^3 \quad = f(u)$$

a, b are fixed positive constants.

λ can vary.

1st We need to find the critical pts.

ie when $u(\lambda - b) - au^3 = 0$

$$u[(\lambda - b) - au^2] = 0$$

$u^* = 0$ & $u^* = \pm\sqrt{\dfrac{\lambda - b}{a}}$

a) if $\lambda < b$ $\sqrt{\dfrac{\lambda - b}{a}}$ is not real. $\Rightarrow$ there is only one real critical pt.

$$f'(u) = \lambda - b - 3au^2$$

$$f'(0) = \lambda - b < 0$$

$\Rightarrow$ $u^* = 0$ is a stable critical pt.

(b) if $\lambda > b$ $\rightarrow \lambda - b > 0$

$u^* = 0$ $u^* = \pm\sqrt{\dfrac{\lambda - b}{a}}$ are equilibrium.

$$f'\left(\pm\sqrt{\frac{\lambda - b}{a}}\right) = \lambda - b - 3a\left(\pm\sqrt{\frac{\lambda - b}{a}}\right)^2$$

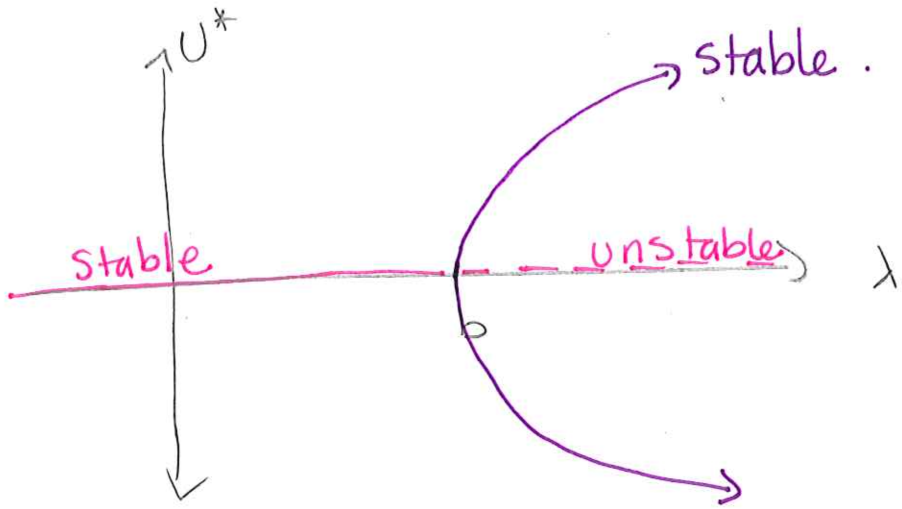
$$= \lambda - b - 3a\frac{(\lambda - b)}{a}$$

$$= -2(\lambda - b) < 0$$

$u^* = \pm\sqrt{\frac{\lambda - b}{a}}$ are stable.

$f'(0) = \lambda - b > 0 \Rightarrow u^* = 0$ is unstable.

c)



2) page 54 #10.

$u' = u^3 - u + h = f(u)$

if $h=0$ the roots are $u(u^2-1)=0$

$u=0 \quad u=\pm 1$

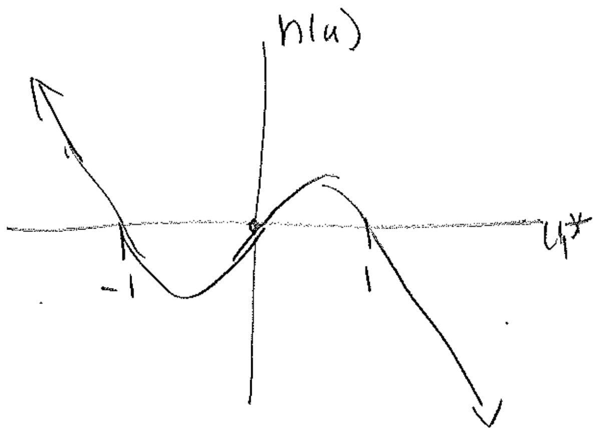
$f'(u) = 3u^2 - 1$

$f'(0) = -1 < 0 \Rightarrow u^* = 0$ is stable.

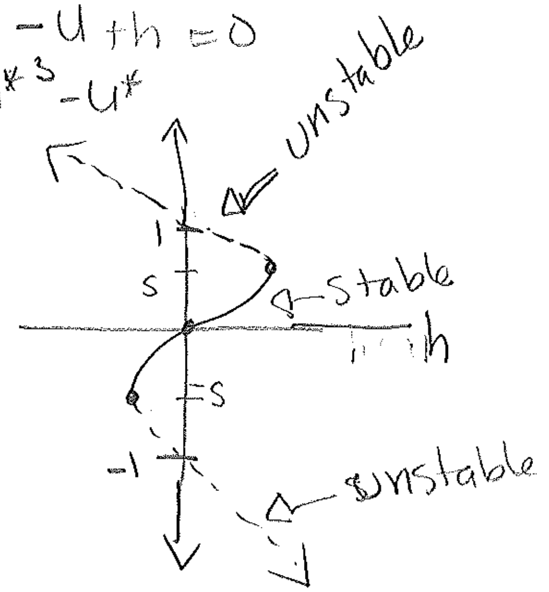
$f'(\pm 1) = 3 - 1 = 2 > 0 \Rightarrow U^* = \pm 1$ unstable

note we care about $f(u) = 0 \Rightarrow U^3 - U + h = 0$

We can rewrite this as $h(u^*) = u^{*3} - u^*$



flip axis $\longrightarrow$



$U^* = S$ is the pt of semi stable.

3) Page 100 #1 magnitude is $(y')^2$

$$\begin{cases} m y'' + k y + a y' |y'| = 0 \\ y(0) = A \quad y'(0) = 0 \end{cases}$$ k, a are constants.

Fundamental units.

$[m] = M$

$[k] = \frac{M}{T^2}$ $[a] = \frac{M}{L}$ $[A] = L$

Dimension Matrix

$$\begin{array}{c} \\ M \\ L \\ T \end{array} \begin{array}{c} \begin{array}{cccc} m & a & A & k \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \end{array}$$

Possible time scales $t_c = \sqrt{\frac{a}{k m^2}}$, $t_c = \sqrt{\frac{m}{k}}$

If damping is small we want the length and time scale to not involve a.

$\Rightarrow t_c = \sqrt{\frac{m}{k}}$ $y_c = \sqrt{\frac{m}{k}}$

take $\bar{t} = \frac{t}{t_c}$ $\bar{y} = \frac{y}{y_c}$

$\frac{dy}{dt} = \frac{y_c}{t_c}\frac{d\bar{y}}{d\bar{t}}$ and $\frac{d^2y}{dt^2} = \frac{y_c}{t_c^2}\frac{d^2\bar{y}}{d\bar{t}^2}$

$\Rightarrow$ The ODE becomes.

$$m\frac{y_c}{t_c^2}\bar{y}'' + k y_c \bar{y} + a\frac{y_c^2}{t_c^2}\bar{y}'|\bar{y}'| = 0$$

$$\to \frac{m}{t_c^2}\bar{y}'' + k\bar{y} + \frac{a y_c}{t_c^2}\bar{y}'|\bar{y}'| = 0$$

$$\frac{m}{\left(\sqrt{\frac{m}{k}}\right)^2}\bar{y}'' + k\bar{y} + \frac{a y_c}{\left(\sqrt{\frac{m}{k}}\right)^2}\bar{y}'|y'| = 0$$

$$k\bar{y}'' + k\bar{y} + \frac{a k y_c}{m}\bar{y}'|\bar{y}'| = 0.$$

$$\Rightarrow \bar{y}'' + \bar{y} + \frac{a y_c}{m}\bar{y}'|\bar{y}'| = 0$$ let $\varepsilon = \frac{a y_c}{m}$ this is small since a is small.

now initial conditions

$y(0) = A \rightarrow \quad y_c\,\bar{y}(0) = A \Rightarrow \quad y_c = A.$

So $\bar{y}(0) = 1$

$y'(0) = \frac{y_c}{t_c}\bar{y}'(0) = 0 \rightarrow \bar{y}'(0) = 0.$

new equation is

$$\begin{cases} \bar{y}'' + \bar{y} + \varepsilon\bar{y}'|y'| = 0 \\ \bar{y}'(0) = 0 \quad \bar{y}(0) = 1 \end{cases}$$ where $\varepsilon = \frac{aA}{m}$

4) $u'' - u = \varepsilon t u$

$u(0) = 1$

$u'(0) = -1$

(a) The unperturbed problem is

$$\begin{cases} u'' - u = 0 \\ u(0) = 1 \quad u'(0) = -1 \end{cases}$$

Find the solution.

$r^2 = \pm 1 \Rightarrow u(t) = Ae^{t} + Be^{-t}$

$u(0) = A + B = 1$

$u'(0) = A - B = -1$

$2A = 0 \Rightarrow A = 0$

$\Rightarrow B = 1$

$u(t) = e^{-t} \Rightarrow$ solution is decaying.

The general solution is

$u_1(t) = Ae^{t} + Be^{-t}$

Since the unperturbed problem has a decaying solution and it is the only way to get a decaying solution we expect the solution to grow under perturbation.

(b) Use regular perturbation to solve.

$u(t) = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \cdots$

Plug into ODE.

$u_0'' + \varepsilon u_1'' + \varepsilon^2 u_2'' + \cdots - u_0 - \varepsilon u_1 - \varepsilon^2 u_2 + \cdots = \varepsilon t(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots)$

$u_0(0) + \varepsilon u_1(0) + \varepsilon^2 u_2(0) + \cdots = 1$

$u_0'(0) + \varepsilon u_1'(0) + \varepsilon^2 u_2'(0) + \cdots = -1$

Collect equations. (We only need two terms)

ε^0: $u_0'' - u_0 = 0$

$u_0(0) = 1 \quad u_0'(0) = -1$

$\rightarrow u_0 = e^{-t}$

ε^1: $u_1'' - u_1 = t u_0 = t e^{-t} \qquad u_1(0) = 0 \quad u_1'(0) = 0.$

homogeneous solution

$u_1^h = A e^{-t} + B e^{t}$

Particular solution (Note: We have to be careful of secular terms)

$u_1^p = C t e^{-t} + D t^2 e^{-t}$

Plug u_1^p into equation, to solve for C & D

$(u_1^p)' = 2D t e^{-t} - D t^2 e^{-t} + C e^{-t} - C t e^{-t}$

$(u_1^p)'' = -2D t e^{-t} + D t^2 e^{-t} + (2D - C) e^{-t} + (2D - C) t e^{-t} - C e^{-t}$

So ODE becomes:

$(-4D + \cancel{C}) t e^{-t} + \cancel{D t^2 e^{-t}} + (2D - 2C) e^{-t} - (\cancel{C t e^{-t}} + \cancel{D t^2 e^{-t}}) = t e^{-t}$

$-4D = 1 \rightarrow D = -1/4$

$2D - 2C = 0 \rightarrow C = D = -1/4$

$\rightarrow u_1(t) = -\frac{1}{4}(t e^{-t} + t^2 e^{-t}) + A e^{-t} + B e^{t}$

$u_1(0) = A + B = 0$

$u'(0) = -\frac{1}{4} \quad -A + B = 0$ $\Big\} \Rightarrow A = -\frac{1}{8}, \; B = 1/8$

$u_1(t) = -\frac{1}{4}(t e^{-t} + t^2 e^{-t}) + -\frac{1}{8} e^{-t} + \frac{1}{8} e^{t}$

So our 2term approximation is

$$u(t) \approx e^{-t} + \varepsilon\left(\frac{1}{8}e^{t} - \frac{1}{8}e^{-t} - \frac{t}{4}e^{-t} - \frac{t^2}{4}e^{-t}\right)$$

c) Let $v = u'$

Then $v' = u'' = u(t\varepsilon + 1)$

For Matlab we need to write as a vector equation

$$Y' = F(Y, t)$$

here $Y = \begin{bmatrix} u \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} y(1) \\ y(2) \end{bmatrix}}_{\text{matlab form.}}$

Then $\begin{bmatrix} y(1) \\ y(2) \end{bmatrix}' = \begin{bmatrix} y(2) \\ y(1).*(t\varepsilon+1) \end{bmatrix}$

See attached matlab code. It generates the plots.

5) Page 100 #3

$$e^{-t} = o(1/t^2)$$

$$\lim_{t \to \infty} \frac{e^{-t}}{1/t^2} = \lim_{t \to \infty} \frac{t^2}{e^t} \overset{LH}{=} \lim_{t \to \infty} \frac{2t}{e^t} \overset{LH}{=} \lim_{t \to \infty} \frac{2}{e^t}$$

$$= 0.$$

(Used L'Hopital's rule twice to prove the statement)

6) Page 101 #4.

$$f(y,\varepsilon) = \frac{1}{(1+\varepsilon y)^{3/2}} \qquad y = y_0 + y_1\varepsilon + \cdots$$

Expand f in powers of ε up to $O(\varepsilon^2)$

Recall: $(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$

$$= \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k$$

So $f(y,\varepsilon) = \frac{1}{(1+\varepsilon y)^{3/2}} = 1 + 3/2\,\varepsilon y + \frac{3/2(3/2-1)}{2!}\varepsilon^2 y^2$

$$= 1 + 3/2\,\varepsilon y + \frac{3}{8}\varepsilon^2 y^2$$

now $y = y_0 + \varepsilon y_1 + \cdots$

$\Rightarrow f(y,\varepsilon) = 1 + 3/2\,\varepsilon(y_0 + \varepsilon y_1 + \cdots) + 3/8\,\varepsilon^2(y_0 + \varepsilon y_1 + \cdots)^2$

$\Rightarrow$ up to $O(\varepsilon^2)$

$$f(y,\varepsilon) = 1 + 3/2\,\varepsilon y_0 + 3/2\,\varepsilon^2 y_1 + 3/8\,\varepsilon^2 y_0^2$$

7) Page 101 # 5 d, g

d) Goal: show $\frac{\sqrt{\varepsilon}}{1-\cos\varepsilon} = O(\varepsilon^{-3/2})$ as $\varepsilon \to 0^+$.

$\Rightarrow$ Want to show $\exists M$ st

$$\left| \frac{\sqrt{\varepsilon}}{1-\cos\varepsilon} \frac{1}{\varepsilon^{-3/2}} \right| \leq M \quad \forall \varepsilon < \varepsilon_0.$$

$$\left| \frac{\sqrt{\varepsilon}}{1-\cos\varepsilon} \frac{1}{\varepsilon^{-3/2}} \right| = \left| \frac{\varepsilon^2}{1-\cos\varepsilon} \right| = \left| \frac{\varepsilon^2}{1-\underbrace{\left(1+\frac{\varepsilon^2}{2}+\frac{\varepsilon^4}{4!}+\cdots\right)}_{\text{MaClaurin series of cos.}}} \right|$$

$$= \left| \frac{\varepsilon^2}{\frac{\varepsilon^2}{2}-\frac{\varepsilon^4}{4!}+\cdots} \right| = \left| \frac{1}{\underbrace{\frac{1}{2}-\frac{\varepsilon^2}{4!}+\cdots}_{\leq 1/2 \text{ but } > \frac{1}{4}}} \right| \leq 4$$

$$\Rightarrow \frac{\sqrt{\varepsilon}}{1-\cos\varepsilon} = O\left(\varepsilon^{-3/2}\right) \text{ as } \varepsilon \to 0^+.$$

9) Goal show $\int_0^\varepsilon e^{-x^2}\,dx = O(\varepsilon)$ as $\varepsilon \to 0$.

ie need to find an M st

$$\left| \frac{\int_0^\varepsilon e^{-x^2}\,dx}{\varepsilon} \right| \le M$$

<u>Way 1</u> $\max_{x \in [0,\varepsilon]} e^{-x^2} = 1$

$$\Rightarrow \int_0^\varepsilon e^{-x^2}\,dx \le 1(\varepsilon)$$

$$\Rightarrow \left| \frac{\int_0^\varepsilon e^{-x^2}\,dx}{\varepsilon} \right| \le \frac{\varepsilon}{\varepsilon} = 1 \Rightarrow \text{"Big O" relationship.}$$

<u>Way 2</u>: $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} + \cdots$

$$\int_0^\varepsilon e^{-x^2}\,dx \sim x - \frac{x^3}{3} + \frac{x^5}{2!5} + \cdots \Big|_0^\varepsilon$$

$$\Rightarrow \left| \frac{\int_0^\varepsilon e^{-x^2}\,dx}{\varepsilon} \right| = \left| \frac{\varepsilon - \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{2!5} + \cdots}{\varepsilon} \right|$$

$$\le 1$$

Page 10 | #8a

$y'' + y = \varepsilon y(y')^2$

$y(0) = 1 \quad y'(0) = 1$

we know we should use Poincaré - Lindstedt. by directions but aslo because ε with a nonlinear lower order term.

$\Rightarrow$ Let $\tau = \omega t$ where $\omega = 1 + \omega_1 \varepsilon + \varepsilon^2 \omega_2 + \cdots$

Guess soln. $y(t) = y_0(\tau) + \varepsilon y_1(\tau) + \varepsilon^2 y_2(\tau) + \cdots$

Now the equation becomes

$$\omega^2 \frac{d^2 y}{d\tau^2} + y(\tau) = \varepsilon \omega^2 y(\tau) \left(\frac{dy}{d\tau}\right)^2$$

expanding

$$(1 + \omega_1 \varepsilon + \omega_2 \varepsilon^2 + \cdots)^2 (y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \cdots) + y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$
$$= \varepsilon (1 + \omega_1 \varepsilon + \omega_2 \varepsilon^2 + \cdots)^2 (y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots)(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots)^2$$

Collect equations

ε^0: $\left.\begin{matrix} y_0'' + y_0 = 0 \\ y_0(0) = 1 \quad y_0'(0) = 1 \end{matrix}\right\} \rightarrow y_0(\tau) = \cos(\tau)$

ε^1: $2\omega_1 y_0'' + y_1'' + y_1 = y_0 (y_0')^2$

$$y_1'' + y_1 = -2\omega_1(-\cos(\tau)) + \cos\tau(\sin^2\tau)$$
$$= +2\omega_1 \cos\tau + \cos\tau (1 - \cos^2\tau)$$
$$= 2\omega_1 \cos\tau + \cos\tau - \left(\frac{3}{4}\cos\tau + \frac{1}{4}\cos(3\tau)\right) \quad \text{from class}$$
$$= 2\omega_1 \cos\tau + \frac{1}{4}\cos\tau - \frac{1}{4}\cos(3\tau)$$

Take $\omega_1 = -1/8$ to avoid secular term.

Now solve

$$y_1'' + y_1 = -\frac{1}{4}\cos 3\tau$$

$$y_1(0) = 0 \qquad y_1'(0) = 0.$$

homogeneous $y_1^h = C_1 \cos\tau + C_2 \sin\tau$

Particular $y_1^p = A\cos 3\tau + B\sin 3\tau$

Plugin $\Rightarrow$ $-9A\cos 3\tau - 9B\sin 3\tau + A\cos 3\tau + B\sin 3\tau = \frac{-1}{4}\cos 3\tau$

$\Rightarrow B = 0$ $\quad$ $-9A + A = -1/4$

$-8A = -1/4$

$\Rightarrow A = +1/32$

$$y_1(\tau) = C_1\cos\tau + C_2\sin\tau + 1/32\cos(3\tau)$$

$$y_1(0) = C_1 + 1/32 = 0 \Rightarrow C_1 = -1/32$$

$$y_1'(0) = C_2 = 0$$

$$\Rightarrow y_1(\tau) = \frac{1}{32}(\cos(3\tau) - \cos\tau)$$

$$\Rightarrow y(\tau) = \cos\tau + \frac{\varepsilon}{32}(\cos(3\tau) - \cos\tau) + \ldots$$

where $\tau = 1 + \frac{-1}{8}\varepsilon + \ldots$

Page 102 #11

$$\frac{1}{(1+x)^k} = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

$h(t) = h_0 + \varepsilon h_1(t) + \varepsilon^2 h_2(t)$

$$\frac{d^2h}{dt^2} = -\left(1 + 2\varepsilon h + \frac{-2(-2-1)}{2!}(\varepsilon h)^2 + \frac{-2(-2-1)(-2-2)}{3!}(\varepsilon h)^3 + \cdots\right)$$

$$= -1 + 2\varepsilon h - 3\varepsilon^2 h^2 + O(\varepsilon^3)$$

Now plug in the regular perturbation series.

$$h_0'' + \varepsilon h_1'' + \varepsilon^2 h_2'' + \cdots = -1 + 2\varepsilon(h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \cdots) - 3\varepsilon^2(h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \cdots)^2$$

Collect equations by powers of ε.

ε^0: $h_0'' = -1 \Rightarrow h_0(t) = \frac{-t^2}{2} + at + b$.

$h_0(0) = 0 \quad h_0'(0) = 1$

$\Rightarrow h_0(t) = \frac{-t^2}{2} + t$.

ε^1: $h_1'' = 2h_0 = \frac{-t^2}{2} + t$

$\Rightarrow h_1' = \frac{-t^3}{6} + \frac{t^2}{2} + C \qquad h_1'(0) = 0 \Rightarrow C = 0$.

$h_1(t) = -\frac{t^4}{24} + \frac{t^3}{6} + D \qquad h_1(0) = 0 \Rightarrow D = 0$.

$\Rightarrow h_1(t) = \frac{-t^4}{24} + \frac{t^3}{6}$

ε^2: $h_2'' = 2h_1 - 3h_0^2 = \frac{2}{3}t^3 - \frac{1}{6}t^4 - 3\left(t - \frac{t^2}{2}\right)^2$

$\Rightarrow h_2(t) = -\frac{t^4}{4} + \frac{11}{60}t^5 - \frac{11}{360}t^6$

Answer:

$$h(t) = \underbrace{t - t^2/2 + \varepsilon\left(t^3/3 - t^4/12\right)}_{\text{only use this}} + \varepsilon^2\left(\frac{-t^4}{4} + \frac{11}{60}t^5 - \frac{11}{360}t^6\right) + \cdots$$

Now find t_{max}

$h'(t) = 1 - t + \varepsilon(t^2 - t^3/3) + \ldots$

Set equal to zero.

let $t = 1 + \varepsilon\alpha + \ldots$

Plug in to

$1 - t + \varepsilon(t^2 - t^3/3) = 0.$

$1 - (1 + \varepsilon\alpha + \ldots) + \varepsilon\left((1 + \varepsilon\alpha + \ldots)^2 - \frac{(1 + \varepsilon\alpha + \ldots)^3}{3}\right) = 0$

$-\varepsilon\alpha + \ldots + \varepsilon\left[(1 + \varepsilon\alpha + \ldots)^2 - \frac{(1 + \varepsilon\alpha + \ldots)^3}{3}\right] = 0.$

ε^0: There is no equation.

$-\alpha + 1 - 1/3 = 0 \rightarrow \alpha = 2/3$

$\rightarrow t_{max} = 1 + 2/3\,\varepsilon + \ldots$

$h_{max} = h(t_{max}) = h(1 + 2/3\,\varepsilon + \ldots) =$

$= 1 + 2/3\,\varepsilon - 1/2 - 2/3\,\varepsilon + \ldots = 1/2 + O(\varepsilon^2)$