Derivations

Let $\mathfrak{g}$ be a Lie algebra. A *derivation* of $\mathfrak{g}$ is a linear mapping $D : \mathfrak{g} \to \mathfrak{g}$ satisfying the product rule:

$$D([x,y]) = [Dx,y] + [x,Dy],$$

for al $x, y \in \mathfrak{g}$. The set $\mathfrak{d}$ of derivations of $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$ (i.e. $[\mathfrak{d},\mathfrak{d}] \subset \mathfrak{g}$). Each element x in $\mathfrak{g}$ defines a map $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ by $\mathrm{ad}_x(y) = [x,y]$. The map $x \to \mathrm{ad}_x$ is a Lie algebra homomorphism of $\mathfrak{g}$ into $\mathfrak{d}$. The image of this map is the set of *inner derivations* of $\mathfrak{g}$. A *characteristic ideal* of $\mathfrak{g}$ is a vector subspace which is stable under every derivation. If $\mathfrak{a}$ and $\mathfrak{b}$ are characteristic ideals of $\mathfrak{g}$, then $[\mathfrak{a},\mathfrak{b}]$ is also a characteristic ideal.

Fun fact: If $\mathfrak{g}$ is finite dimensional, then $(\mathrm{Aut}(\mathfrak{g}))_L = \mathfrak{d}$.

Semidirect products Let R be a ring and G be a group which acts on R by automorphisms. Recall that the *semidirect product* $R \rtimes G$, is the algebra

$$R \rtimes G = \{\sum_{g \in G} r_g g \mid r_g \in R\}$$

with multiplication given by $(r_1 g_1)(r_2 g_2) = r_1 g_1(r_2) g_1 g_2$. So $\mathbb{C}(G \rtimes H) = \mathbb{C}G \rtimes H$. In the setting of Lie algebras, the semidirect product $\mathfrak{d} \rtimes \mathfrak{g}$, is $\mathfrak{g} \oplus \mathfrak{d}$ with bracket $[D,x] = D(x)$ for $D \in \mathfrak{d}, x \in \mathfrak{g}$. In other (SAT reminiscent) words,

$$\text{Der is to Lie as Aut is to Grp.}$$

(B-KM Lie algebras arise as semidirect products of $\mathfrak{g}$ with $\mathfrak{d}$?)

So, of course, the semidirect product $\mathfrak{a} \rtimes \mathfrak{b}$ of two Lie algebras can be defined when there is a homomorphism $b \mapsto D_b$ of $\mathfrak{b}$ into the derivations of $\mathfrak{a}$. So

$$[(a,b),(a',b')] = \left([b,b']\ ,\ [a,a'] + (D_b(a') - D_{b'}(a))\right).$$

This semidirect product is a Lie algebra, with $\mathfrak{a}$ as an ideal and $\mathfrak{b}$ as a Lie subalgebra.

References

[Dx] J. Dixmier, *Enveloping algebras*, Graduate Studies in Mathematics 11, American Mathematical Society, Providence, RI, 1996.