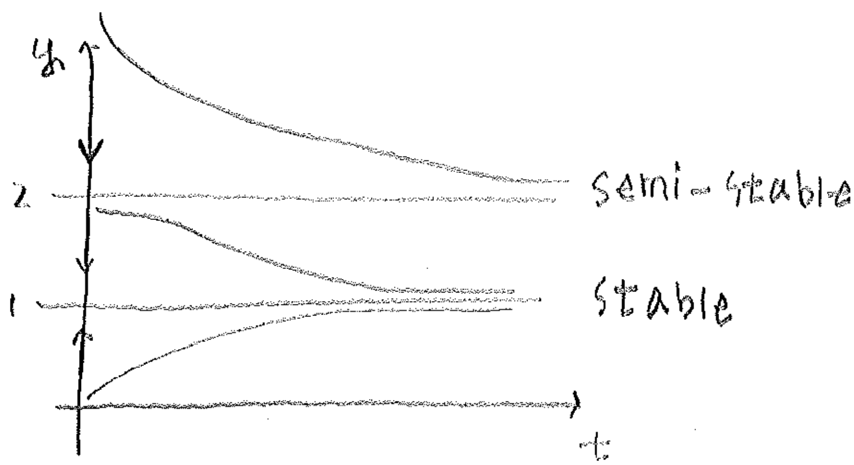
1. **(10 Points)** Identify all of the equilibrium solutions to the following ODE and determine whether they are stable, unstable or semistable.

$$y' = \frac{1}{2}(y-2)^2(1-y). \tag{1}$$

Please justify your answer.

$y=2$: semi-stable

$y=1$: stable



2. **(15 Points)** The function $Y(t) = -\frac{7}{50}\cos(t) + \frac{1}{50}\sin(t)$ is a particular solution to the second-order linear ODE:

$$y'' + y' - 6y = \cos(t), \; -\infty < t < \infty \tag{2}$$

Find the solution $y = \phi(t)$ of Equation 2 which satisfies $\phi(0) = 3$ and $\phi'(0) = 1$.

① Homogeneous Solution

$$y'' + y' - 6y = 0$$

$$\lambda^2 + \lambda - 6 = 0 \qquad (\lambda+3)(\lambda-2) = 0$$

$$\lambda = -3, 2$$

$$y_1 = e^{-3t} \qquad y_2 = e^{2t}$$

② general solution

$$y = \phi(t) = \alpha e^{-3t} + \beta e^{2t} - \frac{7}{50}\cos(t) + \frac{1}{50}\sin(t)$$

$$\phi'(t) = -3\alpha e^{-3t} + 2\beta e^{2t} + \frac{7}{50}\sin t + \frac{1}{50}\cos t$$

$$\phi(0) = \alpha + \beta - \frac{7}{50} = 3 \qquad \alpha + \beta = \frac{157}{50}$$

$$\Rightarrow$$

$$\phi'(0) = -3\alpha + 2\beta + \frac{1}{50} = 1 \qquad -3\alpha + 2\beta = \frac{49}{50}$$

$$2\alpha + 2\beta = \frac{314}{50} \qquad \frac{53}{50} + \beta = \frac{157}{50} \Rightarrow \beta = \frac{104}{50} = \frac{52}{25}$$

$$-\quad -3\alpha + 2\beta = \frac{49}{50}$$

$$5\alpha = \frac{265}{50} \qquad \alpha = \frac{53}{50}$$

$$\therefore \boxed{y(t) = \frac{53}{50}e^{-3t} + \frac{52}{25}e^{2t} - \frac{7}{50}\cos t + \frac{1}{50}\sin t}$$

3. **(10 Points)**

(a) **(5 Points)** Verify that the functions $y_1(t) = e^t$ and $y_2(t) = t$ form a fundamental set of solutions to the homogeneous ODE

$$(1-t)y'' + ty' - y = 0$$

on the interval $-\infty < t < 1$.

$$\left.\begin{aligned} y_1 &= e^t \\ y_1' &= e^t \\ y_1'' &= e^t \end{aligned}\right) \quad (1-t)e^t + te^t - e^t = 0. \quad y_1 = e^t \text{ is a sol.}$$

$$\left.\begin{aligned} y_2 &= t \\ y_2' &= 1 \\ y_2'' &= 0 \end{aligned}\right) \quad (1-t)0 + t - t = 0. \quad y_2 = t \text{ is a sol.}$$

$$W = \begin{vmatrix} e^t & t \\ e^t & 1 \end{vmatrix} = e^t - te^t = (1-t)e^t \neq 0 \text{ on } -\infty < t < 1$$

$\Rightarrow$ y_1 & y_2 is a pair of basic sol.

(b) **(5 Points)** Suppose $p(t)$ and $q(t)$ are continuous functions on the interval $-5 < t < 3$. Is it possible for the functions $f(t) = t^2e^t$ and $g(t) = te^{-t}$ to form a fundamental set of solutions for the second-order linear ODE

$$y'' + p(t)y' + q(t)y = 0$$

on the interval $-5 < t < 3$? Please explain and justify your answer.

$$W = \begin{vmatrix} t^2e^t & te^{-t} \\ 2te^t + t^2e^t & e^{-t} - te^{-t} \end{vmatrix}$$

$$= t^2e^t(e^{-t} - te^{-t}) - te^{-t}(2te^t + t^2e^t)$$

$$= t^2 - t^3 - 2t^2 - t^3 = -2t^3 - t^2 = -t^2(2t - 1)$$

$W = 0$ at $t = 0$ or $\frac{1}{2}$ on interval $-5 < t < 3$

Therefore, $f(t)$ and $g(t)$ cannot form a fundamental set of sol.

4. **(15 Points)** Find an explicit solution to the IVP

$$(y^2+2y)+2x(1+y)y'=0,\ y(1)=2$$

on the interval $x>0$. Please remember to show all of your work.

$M = y^2+2y \qquad N = 2x(1+y)$

$M_y = 2y+2 = N_x = 2(1+y) = 2y+2 \Rightarrow \underline{\text{Exact eq}}$

Let $\begin{cases} E_x = y^2+2y \\ E_y = 2x(1+y) \end{cases} \rightsquigarrow E = \int y^2+2y\,dx = xy^2+2xy+K(y)$

$\downarrow$

$E_y = 2xy+2x+K'(y)$

$K'(y)=0 \to K(y)=C$

$\therefore E(x,y) = xy^2+2xy+C.$

$\frac{dE}{dx}=0 \to E=C_1 \qquad xy^2+2xy=C.$

From $y(1)=2 \Rightarrow 1\cdot 2^2+2\cdot 1\cdot 2 = C \qquad \underline{C=8}$

$xy^2+2xy-8=0$ (quadratic eq)

$$y=\frac{-2x\pm\sqrt{4x^2-4x(-8)}}{2x}=\frac{-2x\pm\sqrt{4x^2+32x}}{2x}$$

$y(1)=\frac{-2\pm\sqrt{36}}{2}=\frac{-2\pm 6}{2}$ $\begin{cases} + \to 2 \\ - \to \cancel{-4} \end{cases}$ $\Rightarrow \boxed{y=\frac{-2x+\sqrt{4x^2+32x}}{2x}}$

5. **(10 Points)** Find the longest interval in which the solution of the initial value problem

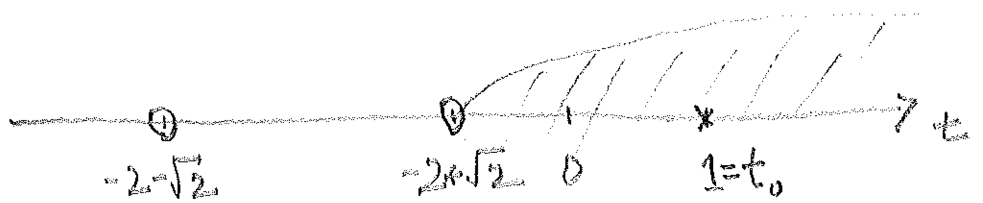
$$(t^2+4t+2)y''+\sin(t)y'+y=\cos(t), \quad y(1)=\pi,\ y'(1)=2\pi \tag{3}$$

is certain to exist. Please explain and justify your answer.

$$y'' + \underbrace{\frac{\sin t}{t^2+4t+2}}_{p(t)} y' + \underbrace{\frac{1}{t^2+4t+2}}_{q(t)} y = \underbrace{\frac{\cos t}{t^2+4t+2}}_{g(t)}$$

$p(t)$, $q(t)$, & $g(t)$ are continuous except $t = \frac{-4 \pm \sqrt{16-8}}{2}$

$= -2 \pm \sqrt{2}$



By the Existence & Uniqueness theorem

there exists a unique solution on $t > -2+\sqrt{2}$

6. **(15 Points)** Let $P(t)$ denote the total number of students (at a small liberal arts college in New England) who have heard a certain rumor at time t. Suppose that P follows the logistic differential equation

$$\frac{dP}{dt} = 0.008P(M - P), \tag{4}$$

and that at $t = 0$, 10 students out of $M = 1,000$ students have heard the rumor. At what time t will 50% of the students have heard the rumor? Remember to justify your answer.

$\frac{dp}{dt} = 0.008\, p\,(1000 - p)$, $\quad p(0) = 10$

$\frac{1}{p(1000-p)} \frac{dp}{dt} = 0.008$

Partial fraction

$\frac{1}{p(1000-p)} = \frac{A}{p} + \frac{B}{1000-p}$

$= \frac{1000A - Ap + Bp}{p(1000-p)}$

$1000A = 1$ $\quad -A + B = 0$ $\quad B = A = \frac{1}{1000}$

$A = \frac{1}{1000}$

$\Rightarrow \left(\frac{1}{p} + \frac{1}{1000-p}\right)\frac{dp}{dt} = 8$

$\int \frac{1}{p} + \frac{1}{1000-p}\, dp = \int 8\, dt$

$\ln P - \ln 1000 - P = 8t + C$

$\ln \frac{P}{1000-P} = 8t + C$

$\frac{P}{1000-P} = Ce^{8t}$

$P = (1000 - P)\, Ce^{8t}$

$P(1 + Ce^{8t}) = 1000\, Ce^{8t}$

$P = \frac{1000\, C e^{8t}}{1 + Ce^{8t}}$

$P(0) = \frac{1000\, C}{1 + C} = 10$

$(1 + C)10 = 1000\, C$

$990\, C = 10$ $\quad C = \frac{1}{99}$

$\boxed{\therefore P(t) = \frac{1000\, e^{8t}}{99 + e^{8t}}}$

50% of student = 500

$500 = \frac{1000\, e^{8t}}{99 + e^{8t}}$

$t = \frac{\ln 99}{8}$

7. **(15 Points)** Consider the second-order linear ordinary differential equation

$$(1-x)y'' + xy' - y = (1-x)^2 x^2 e^x,\ 0 < x < 1 \tag{5}$$

The functions $y_1(x) = e^x$ and $y_2(x) = x$ form a fundamental set of solutions to the associated homogeneous differential equation and have $W(y_1, y_2)(x) = (1-x)e^x$. Use the method of variation of parameters to find a solution to Equation 5.

$$y'' + \frac{x}{1-x} y' - \frac{1}{1-x} y = (1-x) x^2 e^x$$

$$Y_p = \alpha(x) y_1(x) + \beta(x) y_2(x)$$

$$\alpha(x) = -\int \frac{y_2 \cdot g}{W(y_1, y_2)} = -\int \frac{x \cancel{(1-x)} x^2 \cancel{e^x}}{\cancel{(1-x)} \cancel{e^x}} = -\int x^3 dx = -\frac{1}{4} x^4$$

$$\beta(x) = \int \frac{y_1 \, g}{W(y_1, y_2)} = \int \frac{e^x \cancel{(1-x)} x^2 \cancel{e^x}}{\cancel{(1-x)} \cancel{e^x}} = \int x^2 e^x dx$$

$$= e^x (2 - 2x + x^2)$$

Integration by parts

$$\therefore\ y(x) = \alpha e^x + \beta x - \frac{1}{4} x^4 e^x + e^x (2 - 2x + x^2) x$$

8. **(10 Points)** Use the method of undetermined coefficients to find the general solution of the second-order differential equation

$$y'' + 4y' = 2\cos(2t). \tag{6}$$

① Homogeneous sol.

$y'' + 4y' = 0$

$r^2 + 4r = 0$

$r = 0, -4$

$y_1 = 1 \quad y_2 = e^{-4t}$

② Particular sol.

$Y = A\cos 2t + B\sin 2t$

$Y' = -2A\sin 2t + 2B\cos 2t$

$Y'' = -4A\cos 2t - 4B\sin 2t$

$-4A\cos 2t - 4B\sin 2t + 4(-2A\sin 2t + 2B\cos 2t) = 2\cos 2t$

$(-4A + 8B)\cos 2t + (-4B - 8A)\sin 2t = 2\cos 2t$

$-4A + 8B = 2 \qquad \Rightarrow -4A - 16A = 2 \quad A = -\frac{1}{10}$

$-8A - 4B = 0 \quad B = -2A \qquad B = \frac{1}{5}$

$\Rightarrow Y = -\frac{1}{10}\cos 2t + \frac{1}{5}\sin 2t$

$\Rightarrow$ General sol

$y(t) = \alpha + \beta e^{-4t} - \frac{1}{10}\cos 2t + \frac{1}{5}\sin 2t.$