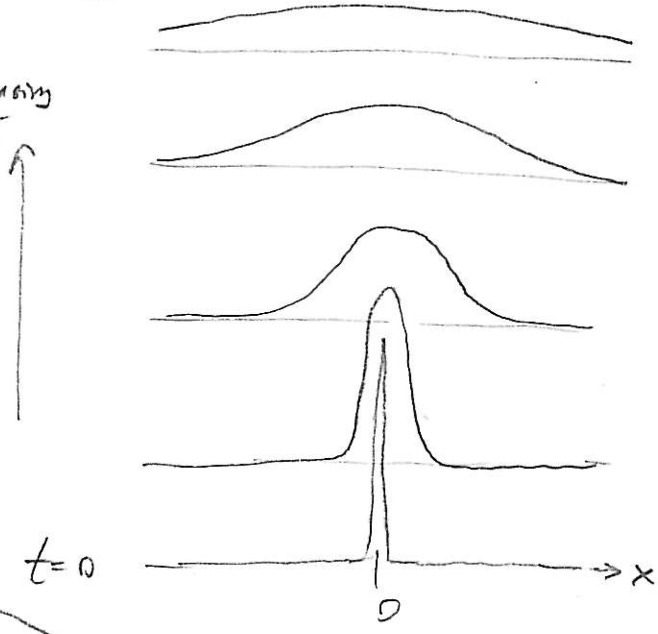
p. 345–6 #1 Math 46, HW8 Solns (2008) 5

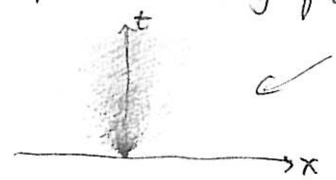
Normal (Gaussian) with growing variance.



larger k would make things happen faster with 6, ie rescale t axis.

sequence of graphs of $u(x,t;0)$ for increasing values of t.

Spacetime density plot:



darkness indicates size of $u(x,t;0)$

as $t \to 0^+$, $u(x,t;0)$ tends to $\delta(x)$, the delta distribution.

This is demonstrated by the derivation in Ex 6·6 (p. 345), and the realization that the only object $u(x,0;0)$ satisfying $f(0) = \int u(x,0;0)\, f(x)\, dx$ is the delta.

This is the 'filtration' property of δ:

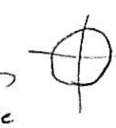
For any continuous func f:

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0)$$

Similarly $\int_{-\infty}^{\infty} \delta(x-\xi) f(x)\, dx = f(\xi)$

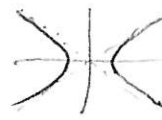
inner product against a δ distn. extracts func value at a single point.

#6 If ever confused remember $u_{xx} + u_{yy} = 0$ like $x^2 + y^2 = 1$



discriminant < 0 elliptic circle or ellipse

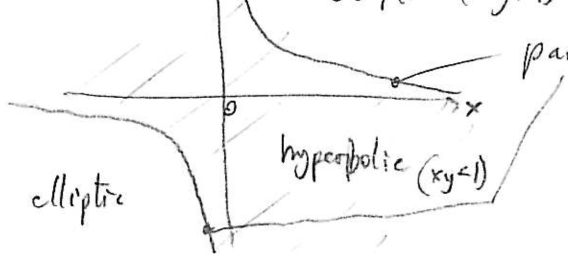
discriminant > 0 $u_{xx} - u_{yy} = 0$ like $x^2 - y^2 = 1$



hyperbolic hyperbola

To the question: $y u_{xx} - 2u_{xy} + x u_{yy} = 0$

discriminant $= b^2 - 4ac = 4 - 4yx = 0$ when $xy = 1$ or $y = \frac{1}{x}$. parabolic.



Closer to the origin, discrim. > 0 so hyperbolic

MATH 46 — HW 8 — SOLUTIONS

p. 345–346 #2 a. w.r.t. x, it's ODE in x with y const.

$\Rightarrow$ use Meth. Und. coeffs. gives

$$u = \underbrace{c_1(y)\sin x + c_2(y)\cos x}_{\text{unknown funcs.}} + \underbrace{6y}_{\text{particular soln.}}$$

d. $v = u_x$ so $v_t + v = 1$

$\Rightarrow v = 1 + a(x)e^{-t}$

integrate in x $\Rightarrow u = x + A(x)e^{-t} + B(t)$

e. $\frac{1}{2}(u^2)_t = x - t$ so $\frac{u^2}{2} = xt - \frac{t^2}{2} + a(x)$

$$u = \pm\sqrt{2xt - t^2 + a(x)}$$

#3 $u_{xt} = f(x,t) \xrightarrow{\int dt} u_x = \int_0^t f(x,s)\,ds + a(x)$

$\xrightarrow{\int dx}$

$$u = \int_0^x \int_0^t f(y,s)\,ds\,dy + A(x) + B(t)$$

Match ICs ($t=0$): $u(x,0) = A(x) + B(0) = g(x)$. $x > 0$. ①

BCs ($x=0$): $u(0,t) = A(0) + B(t) = h(t)$ $t > 0$. ②

subst. $t=0$ $\Rightarrow$ $A(0) + B(0) = h(0)$ [$= g(0)$ since told this] ③

so ① gives $A(x) = g(x) - B(0)$

② gives $B(t) = h(t) - A(0) \overset{\text{use ③}}{=} h(t) + B(0) - \underbrace{h(0)}_{\text{or } g(0).}$

So $u(x,t) = \int_0^x \int_0^t f(y,s)\,ds\,dy + g(x) + h(t) - g(0)$

p. 365–367 #1

$w = e^{u/k}$ so $u = k\ln w \Rightarrow \frac{du}{dw} = \frac{k}{w}$

Chain rule:

$$\begin{array}{c} u \\ | \\ w \\ / \ \backslash \\ x \quad t \end{array}$$

transform each term in PDE:

$$\frac{\partial u}{\partial t} = \frac{du}{dw}\frac{\partial w}{\partial t} = \frac{k}{w}w_t$$

$$\frac{\partial u}{\partial x} = \frac{du}{dw}\frac{\partial w}{\partial x} = \frac{k}{w}w_x$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(u_x) = k\underbrace{\frac{\partial}{\partial x}\left(\frac{1}{w}\right)}_{= w_x \cdot \frac{-1}{w^2}} w_x + \frac{k}{w}w_{xx}$$

so PDE becomes by substituting,

$$\frac{k}{w}w_t = \cancel{-\frac{k^2}{w^2}w_x^2} + \frac{k^2}{w}w_{xx} + \cancel{\frac{k^2}{w^2}w_x^2}$$

ie $w_t = k w_{xx}$ (linear) heat eqn for w.

ICs: $w(x,0) = e^{f(x)/k}$.

Solve heat eqn on $\mathbb{R}$: $\quad w(x,t) = \dfrac{1}{\sqrt{4\pi k t}} \displaystyle\int_{-\infty}^{\infty} e^{f(\xi)/k} \, \overbrace{e^{-\frac{(\xi-x)^2}{4kt}}}^{\text{fundamental solution.}} \, d\xi$

Transform back to u: $\quad u(x,t) = k \ln \left[\dfrac{1}{\sqrt{4\pi k t}} \displaystyle\int_{-\infty}^{\infty} e^{f(\xi)/k} \, e^{-\frac{(\xi-x)^2}{4kt}} \, d\xi \right]$

a closed-form expression!

#13 'Energy method', mult. by u & integrate: (or write $\frac{d}{dt}\int_\Omega u^2 dx$ & subst. the PDE)

$$\underbrace{\int_\Omega u u_t \, dx}_{\substack{\text{recognize as} \\ \frac{d}{dt}\frac{1}{2}\int_\Omega u^2 dx}} = \int_\Omega u \Delta u \, dx \overset{\text{Green's 1st Ident.}}{=} \underbrace{\int_{\partial\Omega} u \frac{\partial u}{\partial n} \, dA}_{\substack{\text{zero if} \\ u \text{ sat homog BCs.}}} - \int_\Omega \underbrace{\vec{\nabla} u \cdot \vec{\nabla} u}_{|\vec{\nabla} u|^2 \geq 0} \, dx$$

$\frac{d}{dt}\frac{1}{2}\int_\Omega u^2 dx$ call $E(t)$, notice ≥ 0.

so $\quad E'(t) \leq 0$. (if u sat. homog. BCs.)

But if $u := u_1 - u_2$ is difference of 2 solutions to given problem, u sat. PDE

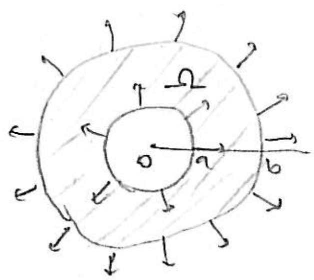
with homog. IC $\quad u(x,0) = 0 \quad x \in \Omega$

homog. BC $\quad u(x,t) = 0 \quad x \in \partial\Omega, t > 0$

IC forces $E(0) = 0$, so $E(t) = 0$ for all $t > 0$. $\Rightarrow u(x,t) \equiv 0 \quad x \in \Omega, t > 0$.

So $u_1 = u_2$ & the solution is unique.

#5



u = density.

total mass in annulus $= \displaystyle\int_0^{2\pi}\int_a^b u(r, \theta, t) \, r \, dr \, d\theta$. (ignore (assume no θ dep.)) (t: time.)

$= 2\pi \displaystyle\int_a^b r u(r,t) \, dr$

Net flux entering $= \displaystyle\int_{\partial\Omega} \vec{J} \cdot \vec{n} \, ds = 2\pi a J(a,t) - 2\pi b J(b,t)$ (J: radial component of $\vec{J}$.)

Key thing is flux acts over perimeter of circle, which is $\propto$ radius.

$\Rightarrow$ Cons. Law (integral form) is $\quad \dfrac{d}{dt} 2\pi \displaystyle\int_a^b r u(r,t) \, dr = 2\pi \underbrace{\left[a J(a,t) - b J(b,t) \right]}_{\text{Fund. thm. calculus: } -\int_a^b (r J(r,t))_r \, dr}$

$\hookrightarrow$ Differential form (get rid of integral signs since true for any a,b)

$$\frac{d}{dt} r u = -(rJ)_r$$

$$\Rightarrow \quad u_t = -\frac{1}{r}(rJ)_r \xrightarrow{\text{use } \vec{J} = -D\vec{\nabla} u \text{ ie radial part is } J = -Du_r} \underline{u_t = +\frac{D}{r}(r u_r)_r}$$

tells you $\Delta u = \dfrac{1}{r}(r u_r)_r$ for radial functions

$\left(\Delta u = \dfrac{1}{r}(r u_r)_r + \dfrac{1}{r^2} u_{\theta\theta} \text{ for polars} \right)$.

Barnett
5/22/08.

A)

$$u(x,t) = \int_{-\infty}^{\infty} u(x,t;\xi)\, f(\xi)\, d\xi = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} f(\xi)\, d\xi$$

(subst $f(\xi) = \sin \mu\xi$):

$$= \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4kt}} \sin \mu\xi \, d\xi$$

change var: $y = \xi - x$
ie $\xi = x + y$
$d\xi = dy$

$$= \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{4kt}} \sin(\mu x + \mu y)\, dy$$

$$= \underbrace{\sin \mu x}_{\substack{\text{original} \\ \text{func } f(x)}} \underbrace{\frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{4kt}} \cos \mu y \, dy}_{T(t)} + \cancel{\cos \mu x \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} \underbrace{e^{-\frac{y^2}{4kt}}}_{\text{even}} \underbrace{\sin \mu y}_{\text{odd}} \, dy}$$

zero. even odd symm

But using heat eqn. $u_t = k u_{xx}$ sub in $u = \sin \mu x \; T(t)$

$$\sin\mu x \; T'(t) = k\, T(t)\, (-\mu^2 \sin \mu x) \qquad \text{holds } \forall x$$

so $T' + k\mu^2 T = 0$ an ODE for $T(t)$, with initial condition

↳ soln. $\underline{T(t) = e^{-k\mu^2 t}}$ $T(0) = 1$ since $u(x,0) = \sin \mu x$

so we've done our tricky integral quickly!

EXTRA.

In fact this is the integral needed to derive the Fourier transform of the Gaussian:

we've shown above via equating $T(t)$:

$$\int_{-\infty}^{\infty} e^{-\frac{y^2}{4kt}} \cos \mu y \, dy = \sqrt{4\pi k t}\, e^{-k\mu^2 t}$$

substitute $y = x$, $\mu = \xi$, $\frac{1}{4kt} = a$

$$\underbrace{\int_{-\infty}^{\infty} e^{-ax^2} \cos \xi x \, dx} = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\xi^2}{4a}}$$

since $e^{i\xi x} = \cos \xi x + i \sin \xi x$
but $e^{-ax^2} \sin \xi x$ is odd symm (integral vanishes),

the LHS is equal to

$$\underbrace{\int_{-\infty}^{\infty} e^{-ax^2} e^{ix\xi}\, dx}_{\mathcal{F}(e^{-ax^2})(\xi)} = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\xi^2}{4a}}$$

So we used heat equation to prove a Fourier transform!

compare Table 6.2

... why good? note propagating heat eqn in time multiplies by a Gaussian in freq. (Fourier) space.

#11 $\quad u(x,t) = U(z)$ where $z = \frac{x}{\sqrt{kt}}$ happens to be dim'less.

Chain rule: $U - z - x, t$

so $\quad u_t = \frac{\partial}{\partial t}U(z) = \frac{dU}{dz}\frac{\partial z}{\partial t} = U'\cdot\left(-\frac{1}{2}\frac{x}{\sqrt{k}}t^{-3/2}\right)$

$$u_x = \frac{dU}{dz}\frac{\partial z}{\partial x} = U'\cdot\left(\frac{1}{\sqrt{kt}}\right).$$

$$u_{xx} = \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{kt}}U'\left(\frac{x}{\sqrt{kt}}\right)\right) = \frac{1}{\sqrt{kt}}\frac{dU'}{dz}\cdot\frac{\partial z}{\partial x} = \left(\frac{1}{\sqrt{kt}}\right)^2 U''$$

PDE becomes $\quad -\frac{1}{2}\frac{x}{\sqrt{k t^3}}U' = k\frac{1}{kt}U'' \quad$ ← want all in terms of z.

$\Rightarrow \quad -\frac{1}{2}zU' = U'' \qquad$ set $V = U'$

then $\quad \frac{V'}{V} = -\frac{1}{2}z \xrightarrow{\text{integrate}} \ln V = -\frac{1}{4}z^2 + c$

$$V(z) = ce^{-\frac{z^2}{4}}$$

$\Rightarrow \quad U(z) = \int V(z)\,dz = c\int_0^z e^{-z^2/4}\,dz + d \qquad$ match c, d to ICs.

for $z\to\infty$, Note $\int_0^\infty e^{-z^2/4}\,dz = \frac{1}{2}\sqrt{2\pi} = \sqrt{\frac{\pi}{2}}$ so total jump in u from $x=0$ to $x=\infty$ needs to be -1,

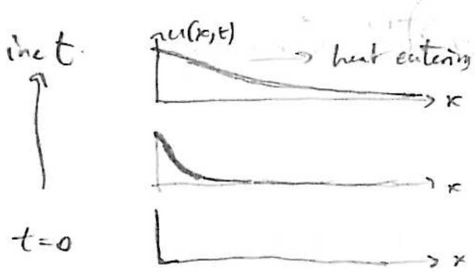
$\Rightarrow c = -\sqrt{\frac{\pi}{2}}$

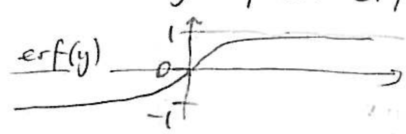
at $x=0$ $(z=0)$ $u=1$ so $d=1$.

$$\Rightarrow \quad U(z) = 1 - \underbrace{\sqrt{\frac{\pi}{2}}\int_0^z e^{-z^2/4}\,dz}$$

need to write as erf so change to e^{-y^2}
$y^2 = z^2/4$ ie $y = \frac{z}{2} = \frac{x}{2\sqrt{kt}}$

ie $\quad u(x,t) = 1 - \mathrm{erf}\left(\frac{x}{2\sqrt{kt}}\right) \quad$ using definition $\mathrm{erf}(y) := \frac{2}{\sqrt{\pi}}\int_0^y e^{-s^2}ds$ as in p. 195





(however defns of erf(·) vary.)

Regardless of your defn of erf,

$$u(x,t) = 1 - \sqrt{\frac{\pi}{2kt}}\int_0^x e^{-\frac{y^2}{4kt}}\,dx$$

p. 371–374 #5 $\quad$ Look up (p. 369) $\quad \Delta u = \frac{1}{r^2}(r^2u_r)_r$ + angular parts.

so $\quad \frac{1}{r^2}(r^2u_r)_r = 0 \quad$ for radially symm. solutions.

mult. by r^2 & integrate.

so $\quad r^2u_r = c$

$$u_r = cr^{-2}$$

$$u = -cr^{-1} + d = \frac{c}{r} + d \quad \text{for consts } c, d.$$

Alternative proof:

$r = \sqrt{x_1^2+x_2^2+x_3^2}$ so $u_{x_1} = \frac{du}{dr}\frac{\partial r}{\partial x_1} = u'(r)\frac{x_1}{r}$ so $u_{x_1x_1} = \frac{u'}{r} + u''\left(\frac{x_1}{r}\right)^2 - x_1u'\boxed{\frac{2x_1}{2r^3}}$ ← $\frac{\partial(r^{-1})}{\partial x_1}$

so $\frac{\partial r}{\partial x_1} = \frac{2x_1}{2r} = \frac{x_1}{r}$ etc.

so $\Delta u = \sum_{i=1}^3 u_{x_ix_i} = u'' + \frac{u'}{r}(3-1) = \frac{1}{r^2}(r^2u')'$ as above.

(#6) $-\Delta u = \lambda u$ mult. by u & integrate as in Energy method (4)

$$-\underbrace{\int_\Omega u\Delta u\,dx}_{\text{Green's 1st Identity}} = \lambda\int_\Omega u^2\,dx \qquad \text{so} \quad \lambda = \frac{\int_\Omega |\vec\nabla u|^2 dx}{\int_\Omega u^2 dx} > 0$$

if nontrivial.

note sign! $+\int_\Omega |\vec\nabla u|^2 dx - \int_{\partial\Omega} u\frac{\partial u}{\partial n}\,dA$ zero since $u=0$ on $\partial\Omega$.

p. 365-367 #13. Energy method (mult. by u & integrate) $\int_0^l u u_t\,dx = \int_0^l (u u_{xx} - u u^3)\,dx$

homog. BCs

$$\frac{1}{2}E'(t) = \frac{d}{dt}\frac{1}{2}\int_0^l u^2\,dx = -\underbrace{\int_0^l (u_x)^2\,dx}_{\geq 0} + \cancel{\left[u u_x\right]_0^l} - \underbrace{\int_0^l u^4\,dx}_{\geq 0}$$

so $E'(t) \leq 0$ but $E(t) \geq 0$ and IC give $E(0) = 0$

$\Rightarrow E(t) \equiv 0 \quad \forall t \quad \Rightarrow \quad u \equiv 0 \quad \forall t,\ x\in[0,l]$ uniquely zero, trivial soln.

(#3) a.

$$vLu = -v\nabla\cdot(p\nabla u) - quv$$
$$uLv = -\underbrace{u\nabla\cdot(p\nabla v)}_{} - quv$$

subtract & q's cancel, also these cancel

write using $\nabla\cdot(up\nabla v) = u\nabla\cdot(p\nabla v) + p\vec\nabla u\cdot\vec\nabla v$

so $vLu - uLv = \nabla\cdot\left[up\nabla v - vp\nabla u\right]$ integrate then use Divergence thm

$$\int_\Omega (vLu - uLv)\,dx = \int_\Omega \nabla\cdot\left[p(u\nabla v - v\nabla u)\right]dx \overset{\text{Div. thm}}{=} \int_{\partial\Omega} p\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dA \quad \text{QED.}$$

Learn the identity: $\nabla\cdot(a\nabla b) = \vec\nabla a\cdot\vec\nabla b + a\Delta b$