Maxima and Minima

Lecture 26

March 2, 2007

Definition

Local Maximum and Local Minimum

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  when (x, y) is near (a, b).

- The number $f(a, b)$ is called a **local maximum value**.
- A function f has a **local minimum** at (a, b) if

$$f(x, y) \geq f(a, b)$$

  when (x, y) is near (a, b) and $f(a, b)$ is called a **local minimum value**.

Absolute Maximum and Absolute Minimum

Definition

If

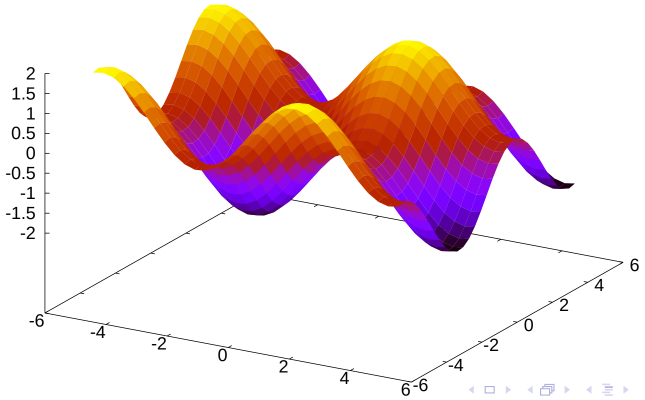
$$f(x, y) \leq f(a, b)$$

(or $f(x, y) \geq f(a, b)$) for all points (x, y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a, b).

Examples

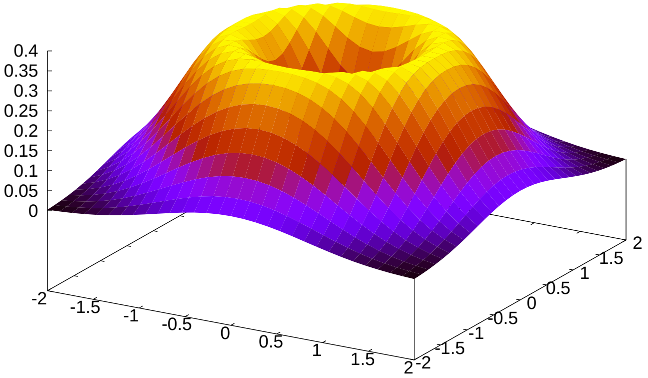
Example

$f(x, y) = \sin x + \sin y.$



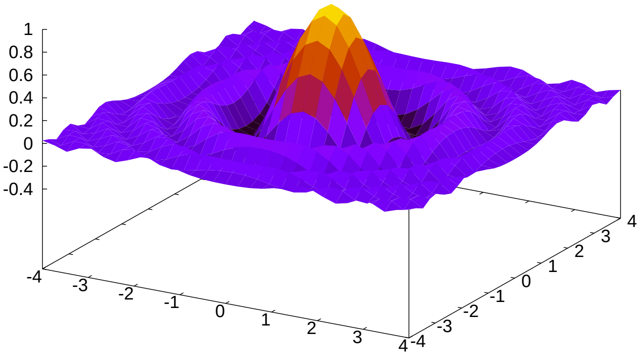
Example

$$f(x,y) = (x^2 + y^2)e^{-x^2-y^2}$$



Example

$f(x,y) = \cos(x^2+y^2)/(1+x^2+y^2)$



Critical Points

Fact

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

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Definition

A point (a, b) is called a **critical point** (or **stationary point**) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Example

Example

Find the critical points and extreme values of the function

$$f(x, y) = x^2 + y^2 - 4x - 4y + 10.$$

Example

Find the extreme values of $f(x, y) = y^2 - x^2$.

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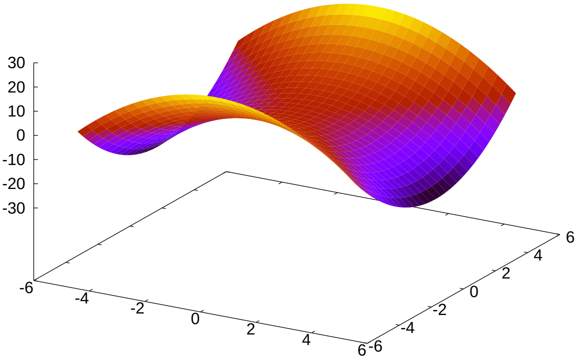

Theorem

Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2.$$

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1. *If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.*
2. *If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.*

Second Derivative Test

Theorem

Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2.$$

1. *If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.*
2. *If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.*
3. *If $D < 0$, then $f(a, b)$ is not a local maximum or minimum. In this case the point (a, b) is called a* ***saddle point*** *of f.*

Important

Fact

If $D = 0$, the test gives no information: f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point of f.

Example

The function f has continuous second derivatives, and a critical point at $(1, 2)$. Suppose that $f_{xx}(1, 2) = 1$, $f_{xy}(1, 2) = 4$ and $f_{yy}(1, 2) = 18$. Then the point $(1, 2)$ is

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2. a local minimum
3. a saddle point
4. cannot be determined.

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4. cannot be determined.

What if $f_{yy}(1,2) = 16$?

Example

Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.