Math 22: Linear Algebra
Long Homework Assignment 2
Due Wednesday, May 22

Constructing New Vector Spaces

In this homework you will explore a number of ways to construct new vector spaces from old ones. We'll also look at how to get a new basis from the old ones, and study what happens with linear transformations.

1 Direct Sum

Let V and W both be vector spaces. Our first construction produces a bigger vector space in which V and W can live side by side in perfect harmony. We define

$$V \oplus W = \{(\mathbf{v}, \mathbf{w}) \mid \mathbf{v} \in V, \mathbf{w} \in W\}.$$

The set $V \oplus W$ is a vector space, called the *direct sum* of V and W.

Problem 1 First, let's check that this *is* a vector space.

(a) Define a sum and scalar product for this set. What is $(\mathbf{v}, \mathbf{0}) + (\mathbf{0}, \mathbf{w})$?

Solution Let's say $(\mathbf{a}, \mathbf{b}) + (\mathbf{x}, y) = (\mathbf{a} + \mathbf{x}, \mathbf{b} + y)$ and $c(\mathbf{x}, \mathbf{y}) = (c\mathbf{x}, c\mathbf{y})$. This implies that $(\mathbf{v}, \mathbf{0}) + (\mathbf{0}, \mathbf{w}) = (\mathbf{v}, \mathbf{w})$.

(b) Check that $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$ for any $a, b \in \mathbb{R}$ and $\mathbf{x} \in V \oplus W$.

Solution Let $\mathbf{x} = (\mathbf{v}, \mathbf{w})$. Then we can calculate

$$\begin{aligned} (a+b)\mathbf{x} = (a+b)(\mathbf{v}, \mathbf{w}) &= ((a+b)\mathbf{v}, (a+b)\mathbf{w}) \\ &= (a\mathbf{v} + b\mathbf{v}, a\mathbf{w} + b\mathbf{w}) \\ &= (a\mathbf{v}, a\mathbf{w}) + (b\mathbf{v}, b\mathbf{w}) \\ &= a(\mathbf{v}, \mathbf{w}) + b(\mathbf{v}, \mathbf{w}) \quad = \quad a\mathbf{x} + b\mathbf{x}. \end{aligned}$$

(c) Check that $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$ for any $a \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in V \oplus W$.

Solution This is similar: if $\mathbf{x} = (\mathbf{u}, \mathbf{v})$ and $\mathbf{y} = (\mathbf{w}, \mathbf{z})$, then

$$\begin{aligned} a(\mathbf{x}+\mathbf{y}) &= a(\mathbf{u}+\mathbf{w}, \mathbf{v}+\mathbf{z})) &&= (a(\mathbf{u}+\mathbf{w}), a(\mathbf{v}+\mathbf{z})) \\ &&&= (a\mathbf{u}+a\mathbf{w}, a\mathbf{v}+a\mathbf{z}) \\ &&&= (a\mathbf{u}, a\mathbf{v}) + (a\mathbf{w}, a\mathbf{z}) \\ &&&= a(\mathbf{u}, \mathbf{v}) + b(\mathbf{w}, \mathbf{z}) \qquad = a\mathbf{x} + a\mathbf{y}. \end{aligned}$$

(d) What is the zero vector in $V \oplus W$? Check its properties.

Solution The zero vector is $(\mathbf{0}, \mathbf{0})$, because

$$(\mathbf{v}, \mathbf{w}) + (\mathbf{0}, \mathbf{0}) = (\mathbf{v}+\mathbf{0}, \mathbf{w}+\mathbf{0}) = (\mathbf{v}, \mathbf{w}).$$

The next thing to study is linear transformations.

Problem 2 Let $S : V \to V'$ and $T : W \to W'$ both be linear transformations. Define $S \oplus T : V \oplus W \to V' \oplus W'$ (in other words, I'm not telling you what this transformation is – you have to come up with a reasonable definition on your own), and check that $S \oplus T$ is a linear transformation.

Solution We should define $S \oplus T(\mathbf{v}, w) = (S(\mathbf{v}), T(\mathbf{w})$. To check it is a linear transformation, we calculate

$$\begin{aligned} S \oplus T(a(\mathbf{u}, \mathbf{v}) + b(\mathbf{w}, \mathbf{z})) &= S \oplus T((a\mathbf{u}+b\mathbf{w}, a\mathbf{v}+b\mathbf{z})) \\ &= (S(a\mathbf{u}+b\mathbf{w}), T(a\mathbf{v}+b\mathbf{z})) \\ &= (aS(\mathbf{u})+bS(\mathbf{w}), aT(\mathbf{v})+bT(\mathbf{z})) \\ &= (aS(\mathbf{u}), aT(\mathbf{v})) + (bS(\mathbf{w}), bT(\mathbf{z})) \\ &= aS \oplus T(\mathbf{u}, \mathbf{v}) + bS \oplus T(\mathbf{w}, \mathbf{z}). \end{aligned}$$

The vector space $V \oplus W$ comes with some natural linear transformations, namely

$$i_V : V \to V \oplus W \qquad \text{and} \qquad i_W : W \to V \oplus W,$$

given by the formulas $i_V(\mathbf{v}) = (\mathbf{v}, \mathbf{0})$ and $i_W(\mathbf{w}) = (\mathbf{0}, \mathbf{w})$.

Problem 3

(a) Check that i_V is a linear transformation (notice that the same proof will show that i_W is a linear transformation too; you can feel free to assume that i_W is a linear transformation from here on).

Solution Simply calculate

$$i_V(a\mathbf{x} + b\mathbf{y}) = (a\mathbf{x} + b\mathbf{y}, \mathbf{0}) = a(\mathbf{x}, \mathbf{0}) + b(\mathbf{y}, \mathbf{0}) = ai_V(\mathbf{x}) + bi_V(\mathbf{y}).$$

(b) Show that a linear transformation $T : V \oplus W \to X$ is determined by the compositions $T \circ i_V$ and $T \circ i_W$. (In other words, suppose $S : V \oplus W \to X$ and $T : V \oplus W \to X$ are both linear transformations, and assume that $S \circ i_V(\mathbf{x}) = T \circ i_V(\mathbf{x})$ for every $\mathbf{x} \in V$ and $S \circ i_W(\mathbf{x}) = T \circ i_W(\mathbf{x})$ for every $\mathbf{x} \in W$. Show that $S(\mathbf{x}) = T(\mathbf{x})$ for every $\mathbf{x} \in V \oplus W$.

Solution We have already seen that any vector $(\mathbf{v}, \mathbf{w}) \in V \oplus W$ can be written $(\mathbf{v}, \mathbf{w}) = i_V(\mathbf{v}) + i_W(\mathbf{w})$. This means that

$$\begin{aligned} S(\mathbf{v}, \mathbf{w}) &= S(i_V(\mathbf{v}) + i_W(\mathbf{w})) \\ &= (S \circ i_V)(\mathbf{v}) + (S \circ i_W)(\mathbf{w}) \\ &= (T \circ i_V)(\mathbf{v}) + (T \circ i_W)(\mathbf{w}) \\ &= T(i_V(\mathbf{v}) + i_W(\mathbf{w})) \\ &= T(\mathbf{v}, \mathbf{w}). \end{aligned}$$

In other words, $S \circ i_V(\mathbf{x}) = T \circ i_V(\mathbf{x})$ for every $\mathbf{x} \in V$ and $S \circ i_W(\mathbf{x}) = T \circ i_W(\mathbf{x})$ for every $\mathbf{x} \in W$, then S and T are the same function.

(c) Show that if $S_V : V \to X$ and $S_W : W \to X$ are linear transformations, then there is a unique linear transformation $T : V \oplus W \to X$ such that $S_V = T \circ i_V$ and $S_W = T \circ i_W$. Write down an explicit formula for T in terms of S_V and S_W.

Solution We have already shown in (b) that if there is such a function, then there is only one (i.e., it is unique). Let's define

$$T(\mathbf{v}, \mathbf{w}) = S_V(\mathbf{v}) + S_W(\mathbf{w}).$$

It is not hard to check that T is a linear transformation.

(d) Suppose Z is some other vector space with linear transformations $j_V : V \to Z$ and $j_W : W \to Z$, and which has the property:

(∗) if $S_V : V \to X$ and $S_W : W \to X$ are linear transformations, then there is a unique linear transformation $T : Z \to X$ such that $S_V = T \circ j_V$ and $S_W = T \circ j_W$.

Show that Z is isomorphic to $V \oplus W$. (Hint Use the formal properties to get linear transformations $V \oplus W \longrightarrow X$ and $X \longrightarrow V \oplus W$. Then use uniqueness property to show that these maps are inverses of one another.)

Solution This is probably the most challenging problem on this assignment. We have to do two things:

- Find linear transformations $S : V \oplus W \longrightarrow Z$ and $T : Z \longrightarrow V \oplus W$, and
- Show that S and T are inverses of one another.

Let's start on the first thing. Since we have

$$V \xrightarrow{i_V} V \oplus W \xleftarrow{i_W} W,$$

the property (∗) of Z says that there is a unique linear transformation $T : Z \longrightarrow V \oplus W$ that fits into the diagram

$$\begin{tikzcd} V \arrow[d, "j_V"'] \arrow[dr, "i_V"] & \\ Z \arrow[r, "T"] & V \oplus W. \\ W \arrow[u, "j_W"] \arrow[ur, "i_W"'] & \end{tikzcd}$$

Likewise, since we have $V \xrightarrow{j_V} Z \xleftarrow{j_W} W$, there is a unique linear transformation $S : V \oplus W \longrightarrow Z$ which fits in the diagram

$$\begin{tikzcd}
V \arrow[d, "i_V"'] \arrow[dr, "j_V"] & \\
V \oplus W \arrow[r, "S"] & Z. \\
W \arrow[u, "i_W"] \arrow[ur, "j_W"'] &
\end{tikzcd}$$

This defines our linear transformations S and T. It remains to show that they are inverses. Notice that

$$T \circ S \circ i_V = T(S \circ i_V) = T \circ j_V = i_V \quad \text{and} \quad T \circ S \circ i_W = T(S \circ i_W) = T \circ j_W = i_W.$$

In terms of the diagram, we see that $T \circ S$ must be the unique (i.e., only) linear transformation that fits into the diagram

$$\begin{tikzcd}
V \arrow[d, "i_V"'] \arrow[dr, "i_V"] & \\
V \oplus W \arrow[r, "T \circ S"] & V \oplus W. \\
W \arrow[u, "i_W"] \arrow[ur, "i_W"'] &
\end{tikzcd}$$

But it is easy to see that the identity transformation also fits into the diagram. The conclusion? It must be that $T \circ S = \mathrm{Id}_{V \oplus W}$. Exactly the same reasoning will show that $S \circ T = \mathrm{Id}_Z$, and this completes the proof!

Part (d) shows that the property proved in (c) characterizes the direct sum *up to isomorphism.*

The last thing to do in this section is to find a basis for the direct sum.

Problem 4

(a) Suppose $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis for V and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m\}$ is a basis for W. Show that the set

$$i_V(\mathcal{B}) \cup i_W(\mathcal{C}) = \{i_V(\mathbf{b}_1), i_V(\mathbf{b}_2), \ldots, i_V(\mathbf{b}_n), i_W(\mathbf{c}_1), i_W(\mathbf{c}_2), \ldots, i_W(\mathbf{c}_m)\}$$

is a basis for $V \oplus W$.

Solution We have to show that the set spans $V \oplus W$. Let $(\mathbf{v}, \mathbf{w}) \in V \oplus W$. Then we can write

$$\mathbf{v} = a_1\mathbf{b}_1 + \cdots + a_n\mathbf{b}_n \quad \text{and} \quad \mathbf{w} = z_1\mathbf{c}_1 + \cdots + z_m\mathbf{c}_m.$$

Therefore,

$$\begin{aligned}
(\mathbf{v}, \mathbf{w}) &= i_V(\mathbf{v}) + i_W(\mathbf{w}) \\
&= i_V(a_1\mathbf{b}_1 + \cdots + a_n\mathbf{b}_n) + i_W(z_1\mathbf{c}_1 + \cdots + z_m\mathbf{c}_m) \\
&= a_1 i_V(\mathbf{b}_1) + \cdots a_n i_V(\mathbf{b}_n) + z_1 i_W(\mathbf{c}_1) + \cdots + z_m i_W(\mathbf{c}_m),
\end{aligned}$$

which shows that $(\mathbf{v}, \mathbf{w})$ is a linear combination of the vectors in $i_V(\mathcal{B}) \cup i_W(\mathcal{C})$. In other words, the given set of vectors spans $V \oplus W$. Next we have to show that $i_V(\mathcal{B}) \cup i_W(\mathcal{C})$ is a linearly independent set. Suppose

$$a_1 i_V(\mathbf{b}_1) + \cdots a_n i_V(\mathbf{b}_n) + z_1 i_W(\mathbf{c}_1) + \cdots + z_m i_W(\mathbf{c}_m) = (\mathbf{0}, \mathbf{0}).$$

We need to show that all the a's and all the z's are zero. Well, we can combine the left hand side to get

$$\begin{aligned}
i_V(a_1\mathbf{b}_1 + \cdots + a_n\mathbf{b}_n) + i_W(z_1\mathbf{c}_1 + \cdots + z_m\mathbf{c}_m) &= (\mathbf{0}, \mathbf{0}) \\
(a_1\mathbf{b}_1 + \cdots + a_n\mathbf{b}_n, z_1\mathbf{c}_1 + \cdots + z_m\mathbf{c}_m) &= (\mathbf{0}, \mathbf{0})
\end{aligned}$$

which means that

$$a_1\mathbf{b}_1 + \cdots + a_n\mathbf{b}_n = \mathbf{0} \quad \text{and} \quad z_1\mathbf{c}_1 + \cdots + z_m\mathbf{c}_m = \mathbf{0}.$$

Since $\mathcal{B}$ and $\mathcal{C}$ are bases, they are linearly independent sets, and so all the a's and all the z's must be zero, which is what we needed to show.

(b) If V is n-dimensional and W is m-dimensional, then what is the dimension of $V \oplus W$?

Solution It is $(n + m)$-dimensional.

2 Sets of Linear Transformations

If V and W are vector spaces, then we denote the set of all linear transformations from V to W by

$$\mathrm{Hom}(V, W) = \{T : V \to W \,|\, T \text{ is a linear transformation}\}.$$

You are going to show that this is a vector space.

Problem 5

(a) Define addition and scalar multiplication on the set $\mathrm{Hom}(V, W)$ (again, it is up to you to come up with a reasonable way to add these transformations and multiply them by scalars).

Solution If $S, T \in \mathrm{Hom}(V, W)$, then let's define $S + T$ to be the linear transformation $\mathbf{v} \mapsto S(\mathbf{v}) + T(\mathbf{v})$. Similarly, if $c \in \mathbb{R}$, then define (cT) to be the linear transforamtion $\mathbf{v} \mapsto c \cdot T(\mathbf{v})$.

(b) Check that for $c, d \in \mathbb{R}$ and $\mathbf{x} \in \mathrm{Hom}(V, W)$, $c(d\mathbf{x}) = (cd)\mathbf{x}$.

Solution This is almost stupid. Remembering that $\mathbf{x}$ is a linear transformation, we see that we have to show that, no matter what vector we plug in on the left and right hand side, we get the same answer (i.e., the left and right hand sides are different descriptions of the *same function*). To do this, we calculate

$$\overbrace{(c(d\mathbf{x}))}^{\substack{\text{linear}\\ \text{transformation}}}(\mathbf{v}) = \underbrace{c \cdot (d \cdot \overbrace{\mathbf{x}(\mathbf{v})}^{\text{vector}}) = (cd) \cdot \overbrace{\mathbf{x}(\mathbf{v})}^{\text{vector}}}_{\text{true for vectors}} = \overbrace{((cd)\mathbf{x})}^{\substack{\text{linear}\\ \text{transformation}}}(\mathbf{v}).$$

(c) Check that for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathrm{Hom}(V, W)$, $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$.

Solution This uses the same kind of reasoning: we plug in a vector $\mathbf{v} \in V$ to both sides, and find that the results are the same because addition of vectors is associative.

Problem 6 Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for V and let $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m\}$ be a basis for W.

(a) If $\mathbf{v} \in V$ then $\mathbf{v}$ has a unique expression of the form

$$\mathbf{v} = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n.$$

Define $T_{ij}(\mathbf{v}) = a_i\mathbf{c}_j$. Show that $T_{ij} : V \to W$ is a linear transformation, so $T_{i,j} \in \mathrm{Hom}(V, W)$.

Solution Let's say $\mathbf{v} = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n$ and $\mathbf{w} = z_1\mathbf{b}_1 + z_2\mathbf{b}_2 + \cdots + z_n\mathbf{b}_n$. Then

$$T_{ij}(s\mathbf{v}+t\mathbf{w}) = T_{ij}(\cdots+(sa_i+tz_i)b_i+\cdots) = (sa_i+tz_i)c_i = sT_{ij}(\mathbf{v})+tT_{ij}(\mathbf{w}).$$

(b) Show that the set $\mathcal{D} = \{T_{ij} \,|\, 1 \leq i \leq n \;\text{ and }\; 1 \leq j \leq m\}$ is linearly independent.

Solution Suppose $\sum_{i,j} a_{ij}T_{ij} = 0$ where each $a_{ij} \in \mathbb{R}$. This means that

$$\sum_{i,j} a_{ij}T_{ij}(\mathbf{v}) = \mathbf{0}$$

for any vector $\mathbf{v} \in V$. We have to show that each coefficient $a_{ij} = 0$. Since the equation above is true for any vector $\mathbf{v}$, let's try plugging in $\mathbf{b}_i$. It's easy to see that $T_{ij}(\mathbf{b}_i) = \mathbf{c}_j$ and $T_{kj}(\mathbf{b}_i) = \mathbf{0}$ if $i \neq k$. So most of the sum disappears and we are left with only the terms with first index equal to i:

$$a_{i1}\mathbf{c}_i + a_{i2}\mathbf{c}_2 + \cdots + a_{im}\mathbf{c}_m = \mathbf{0}.$$

Since $\mathcal{C}$ is a basis, it is a linealy independent set. This means that all the coeffients a_{ij} are equal to zero. Since i can be any number we like, this proves the result.

(c) Show that the set $\mathcal{D} = \{T_{ij} \,|\, 1 \leq i \leq n \;\text{ and }\; 1 \leq j \leq m\}$ spans $\mathrm{Hom}(V, W)$.

Solution Let $T : V \longrightarrow W$ be a linear transformation. We have to show that T is a linear combination of the T_{ij}. Well, since $\mathcal{C}$ spans W, we can write

$$T(\mathbf{b}_i) = a_{i1}\mathbf{c}_1 + a_{i2}\mathbf{c}_2 + \cdots + a_{im}\mathbf{c}_m.$$

Let's write $S : V \longrightarrow W$ for the linear transformation

$$S(\mathbf{v}) = \sum_{i,j} a_{ij} T_{ij}(\mathbf{v}).$$

Then we have

$$S(\mathbf{b}_i) = \sum_{i,j} a_{ij} T_{ij}(\mathbf{b}_i) = a_{i1}\mathbf{c}_1 + a_{i2}\mathbf{c}_2 + \cdots + a_{im}\mathbf{c}_m = T(\mathbf{b}_i)$$

for each i. Since T is determined by its values on the basis $\mathcal{B}$, we conclude that $S = T$. In other words, T is a linear combination of the T_{ij}.

Problem 7 If V is n-dimensional and W is m-dimensional, then what is the dimension of $\mathrm{Hom}(V, W)$?

Solution The vector space $\mathrm{Hom}(V, W)$ is nm-dimensional.

We should also discuss linear transformations of $\mathrm{Hom}(V, W)$. Suppose $S : U \to V$ is a linear transformation and $T \in \mathrm{Hom}(V, W)$. Then we have a picture like this:

$$\begin{tikzcd} U \arrow[d, "S"'] & \\ V \arrow[r, "T"] & W. \end{tikzcd}$$

These linear transformations together give us a linear transformation $T \circ S : U \to W$. Define

$$S^* : \mathrm{Hom}(V, W) \to \mathrm{Hom}(U, W) \qquad \text{by} \qquad S^*(T) = T \circ S.$$

This is known as the linear transformation *induced* by S. Notice that something a little unexpected happens here: the homomorphism induced by $S : U \to V$ goes in the "wrong" direction. Constructions that reverse the direction of functions are called *contravariant*.

On the other hand, suppose $S : W \to X$ is a linear transformation and again $T \in \mathrm{Hom}(V, W)$. Then we have a picture like this:

$$\begin{tikzcd} V \arrow[r, "T"] & W \arrow[d, "S"] \\ & X. \end{tikzcd}$$

These linear transformations together give us a linear transformation $S \circ T : V \to X$. Define

$$S_* : \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, X) \qquad \text{by} \qquad S_*(T) = S \circ T.$$

This time the induced linear transformation goes the "right" way; this kind of construction is called *covariant*.

Problem 8

(a) Check that S^* is a linear transformation.

Solution We have to show that $S^*(aF+bG) = aS^*(F)+bS^*(G)$. To do this, we'll show that the right and left hand sides give us the same result when we plug in any vector $\mathbf{u} \in U$. This is a simple calculation

$$\begin{aligned} S^*(aF+bG)(\mathbf{u}) &= (aF+bG) \circ S(\mathbf{u}) \\ &= (aF+bG)(S(\mathbf{u})) \\ &= aF(S(\mathbf{u})+bG(S(\mathbf{u})) \\ &= a(F \circ S)(\mathbf{u}) + b(G \circ S)(\mathbf{u}) \\ &= (aS^*F + bS^*G)(\mathbf{u}). \end{aligned}$$

(b) Check that S_* is a linear transformation.

Solution This is very similar, and I'll leave it to you.

3 Quotient Vector Spaces

This is the last basic method of producing new vector spaces from old ones. In this situation, we have $V \subseteq W$ is a vector subspace, and we want to ignore V completely.

We start by talking about adding *sets* rather than just vectors. If $A, B \subseteq W$, then

$$A + B = \{\mathbf{a} + \mathbf{b} \,|\, \mathbf{a} \in A, \mathbf{b} \in B\};$$

in other words, the sum of these two sets is the set of all possible sums of a vector in A with a vector in B. We can also take the scalar multiple of a set: for $c \in \mathbb{R}$,

$$cA = \{c\mathbf{a} \mid \mathbf{a} \in A\}.$$

Remember from Chapter 1 that the solution sets of linear systems can be written down as the translation of a subspace by any chosen element of the solution set. We'll work with similar kinds of subsets of W. If $\mathbf{x} \in W$, then we can look at the set

$$\mathbf{x} + V = \{\mathbf{x} + \mathbf{v} \mid \mathbf{v} \in V\}.$$

The set of all such gadgets is denoted W/V.

Problem 9 Suppose $\mathbf{x}, \mathbf{y} \in W$.

(a) Show that $(\mathbf{x} + V) + (\mathbf{y} + V) = (\mathbf{x} + \mathbf{y}) + V$.

Solution If $\mathbf{u}$ is a vector in the left hand side, then $\mathbf{u} = \mathbf{x} + \mathbf{v}_1 + \mathbf{y} + \mathbf{v}_2$ for some vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$. But this can be simplified to $\mathbf{u} = \mathbf{x} + \mathbf{y} + (\mathbf{v}_1 + \mathbf{v}_2)$, which is a vector in the right hand side. Going in the other direction,

$$(\mathbf{x} + \mathbf{y}) + \mathbf{v} = \mathbf{x} + \mathbf{v} + \mathbf{y} + \mathbf{0},$$

which is a vector in the left hand side.

(b) Show that $c(\mathbf{x} + V) = c\mathbf{x} + V$.

Solution If $\mathbf{u}$ is a vector in the left hand side, then $\mathbf{u} = \mathbf{x} + \mathbf{v}$, so $c\mathbf{u} = c\mathbf{x} + c\mathbf{v}$. Now $c\mathbf{v}$ is in V because V is a subspace, and therefore $c\mathbf{u}$ is in the right hand side. It is just as easy to go in the reverse direction.

Maybe it's hard to see what needs to be done here. In (a), the left hand side is the set of all vectors that can be written in the form $\mathbf{x} + \mathbf{v}_1 + \mathbf{y} + \mathbf{v}_2$, where $\mathbf{v}_1, \mathbf{v}_2 \in V$. The right hand side is a set of vectors with a slightly different description. You need to show that these two sets are the same.

Problem 10 It is not too hard to see that W/V is a vector space with the given operations.

(a) What is the zero vector? What is the additive inverse of $\mathbf{x} + V$?

Solution The zero vector is $\mathbf{0} + V$, and the additive inverse of $\mathbf{x} + V$ is $-\mathbf{x} + V$.

(b) Define a function

$$Q : W \to W/V \qquad \text{by} \qquad Q(\mathbf{x}) = \mathbf{x} + V.$$

Show that Q is a linear transformation.

Solution Using Problem 9, we simply calculate

$$\begin{aligned} Q(a\mathbf{x} + b\mathbf{y}) &= a\mathbf{x} + b\mathbf{y} + V \\ &= (a\mathbf{x} + V) + (b\mathbf{y} + V) \\ &= a(\mathbf{x} + V) + b(\mathbf{y} + V) \\ &= aQ(\mathbf{x}) + bQ(\mathbf{y}). \end{aligned}$$

Finally, let's think about a basis. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis for V. We know that we can add more vectors, say $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m$ to $\mathcal{B}$ to get a basis for W. Write $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m\}$, so $\mathcal{B} \cup \mathcal{C}$ is a basis for W.

Problem 11

(a) Show that $Q(\mathcal{C}) = \{Q(\mathbf{c}_1), Q(\mathbf{c}_2), \ldots, Q(\mathbf{c}_m)\}$ is a basis for W/V.

Solution First we show that $Q(\mathcal{C})$ spans W/V. Let $\mathbf{x} + V \in W/V$. Then $\mathbf{x} \in W$, so we can write

$$\mathbf{x} = \overbrace{a_1\mathbf{b}_1 + \cdots + a_n\mathbf{b}_n}^{\text{in } V\text{, call it } \mathbf{v}} + \overbrace{z_1\mathbf{c}_1 + \cdots + z_m\mathbf{c}_m}^{\text{call this } \mathbf{y}},$$

and so

$$\begin{aligned} \mathbf{x} + V &= \{\mathbf{x} + \mathbf{w} \mid \mathbf{w} \in V\} \\ &= \{\mathbf{y} + \mathbf{v} + \mathbf{w} \mid \mathbf{w} \in V\} \\ &= \{\mathbf{y} + (\mathbf{v} + \mathbf{w}) \mid (\mathbf{v} + \mathbf{w}) \in V\} \\ &= \mathbf{y} + V, \end{aligned}$$

which, I hope, clarifies what Q does a little bit. Anyway, we can now see that

$$\mathbf{x}+V=\mathbf{y}+V=z_1\mathbf{c}_1+\cdots+z_m\mathbf{c}_m+V=z_1(\mathbf{c}_1+V)+\cdots+z_m(\mathbf{c}_m+V),$$

so $Q(\mathcal{C})$ does span W/V. To show that the set $Q(\mathcal{C})$ is linearly independent, suppose

$$\overbrace{z_1(\mathbf{c}_1+V)+\cdots+z_m(\mathbf{c}_m+V)}^{\mathbf{y}+V}=\mathbf{0}+V.$$

Then it must be that $\mathbf{y}+\mathbf{v}=\mathbf{0}+\mathbf{w}$ for some $\mathbf{v},\mathbf{w}\in V$. In other words, $\mathbf{y}=\mathbf{w}-\mathbf{v}$, which is in V, so $\mathbf{y}\in V$. Therefore, $\mathbf{y}$ is a linear combination of the $\mathbf{b}_i$'s which means that

$$\begin{aligned} z_1\mathbf{c}_1+\cdots+z_m\mathbf{c}_m &= a_1\mathbf{b}_1+\cdots+a_n\mathbf{b}_n \\ z_1\mathbf{c}_1+\cdots+z_m\mathbf{c}_m-a_1\mathbf{b}_1-\cdots-a_n\mathbf{b}_n &= \mathbf{0}. \end{aligned}$$

Since $\mathcal{B}\cup\mathcal{C}$ is a basis for W, this means that each $z_i=0$.

(b) If W is m-dimensional and V is n-dimensional, then what is the dimension of W/V?

Solution It is $(m-n)$-dimensional.