Linear Algebra Final Exam

1:00–3:00, Sunday, June 2
Bradley 102

1 Let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be a linear transformation with the property that $T \circ T \circ T = 0$ (we'll refer to $T \circ T \circ T$ as T^3 for the rest of this problem).

(a) What exactly does this mean? That is, what is the practical upshot when it comes to plugging in vectors to T^3?

Solution

It means that $T^3(\mathbf{v}) = \mathbf{0}$ for every vector $\mathbf{v} \in \mathbf{R}^3$.

(b) Suppose that $\mathbf{x} \in \mathbf{R}^3$ is such that $T^2(\mathbf{x}) = T(T(\mathbf{x})) \neq \mathbf{0}$. If $\mathbf{z} = cT^2(\mathbf{x})$, then what is $T(\mathbf{z})$? Let $\mathbf{y} = c_1T(\mathbf{x}) + c_2T^2(\mathbf{x})$. What is $T^2(\mathbf{y})$?

Solution

Well,

$$T(\mathbf{z}) = T(cT^2(\mathbf{x})) = cT^3(\mathbf{x}) = \mathbf{0}$$

and

$$T^2(\mathbf{y}) = T^2(c_1T(\mathbf{x}) + c_2T^2(\mathbf{x})) = c_1T^3(\mathbf{x}) + c_2T^3(T(\mathbf{x})) = \mathbf{0}.$$

(c) Let's say

$$\mathbf{b}_1 = \mathbf{x} \quad \mathbf{b}_2 = T(\mathbf{x}), \quad \text{and} \quad \mathbf{b}_3 = T^2(\mathbf{x}).$$

Show that $\mathbf{b}_1$ is not a linear combination of $\mathbf{b}_2$ and $\mathbf{b}_3$.

Solution

From what we've seen in (b), any linear combination $\mathbf{y}$ of $\mathbf{b}_2$ and $\mathbf{b}_3$ has the property that $T^2(\mathbf{y}) = \mathbf{0}$. Since $T^2(\mathbf{b}_1) = T^2(\mathbf{x}) \neq \mathbf{0}$, $\mathbf{b}_1$ can't be a linear combination of $\mathbf{b}_2$ and $\mathbf{b}_3$.

(d) Explain why the set $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for $\mathbf{R}^3$. (**Hint:** Some of the work you've already done might help.)

Solution

We know that $T(\mathbf{b}_2) = T^2(\mathbf{x}) \neq \mathbf{0}$ and $T(\mathbf{b}_3) = \mathbf{0}$, so $\mathbf{b}_2$ is not a multiple of $\mathbf{b}_1$. And we showed in (c) that $\mathbf{b}_1$ is not a linear combination of $\mathbf{b}_2$ and $\mathbf{b}_3$, so the whole set must be linearly independent. Since we are working in $\mathbf{R}^3$, any linearly independent set of 3 vectors is a basis.

(e) Find the $\mathcal{B}$-matrix for the linear transformation T. (This can be done with very little work).

Solution

Since

$$T(\mathbf{b}_1) = \mathbf{b}_2, \quad T(\mathbf{b}_2) = \mathbf{b}_3 \quad \text{and} \quad T(\mathbf{b}_3) = \mathbf{0},$$

the matrix must be

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

2 Let

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 2 & 5 \end{pmatrix}.$$

(a) Find the eigenvalues of A. (**Hint:** $\lambda^2 - 9\lambda + 18 = (\lambda - 3)(\lambda - 6)$)

Solution

The characteristic polynomial is

$$\det(A-\lambda I) = (3-\lambda)((4-\lambda)(5-\lambda)-2) = (3-\lambda)(\lambda^2-9\lambda+18) = (3-\lambda)(\lambda-3)(\lambda-6).$$

Therefore the eigenvalues of A are 3 and 6.

(b) Find bases for the eigenspaces of A.

Solution

First let's handle $\lambda = 3$. We need to solve $(A - 3I)\mathbf{v} = \mathbf{0}$ by reducing the matrix

$$A - 3I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This shows that the eigenspace for $\lambda = 3$ is

$$\operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

For $\lambda = 6$, we row reduce

$$A - 6I = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

This shows that the eigenspace for $\lambda = 3$ is

$$\operatorname{span}\left\{ \begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \end{pmatrix} \right\}.$$

(c) Write down an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Briefly explain yourself.

Solution

We have a basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \end{pmatrix} \right\}$$

for $\mathbf{R}^3$ consisting of eigenvectors for A. The matrix P is the change of coordinates from $\mathcal{B}$ to the standard basis, and D is the diagonal matrix with the eigenvalues on the diagonal. Thus if

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & -1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

then $A = PDP^{-1}$.

3

(a) Suppose $T : V \longrightarrow V$ is a linear transformation, and that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for V. If the $\mathcal{B}$-matrix for T is

$$A = \begin{pmatrix} 2 & 3 & 5 \\ 7 & 11 & 13 \\ 17 & 19 & 23 \end{pmatrix},$$

then what is $T(2\mathbf{b}_1 + 4\mathbf{b}_3)$?

Solution

Calculate $T(2\mathbf{b}_1 + 4\mathbf{b}_3) = 16\mathbf{b}_1 + 58\mathbf{b}_2 + 110\mathbf{b}_3$.

(b) Explain why the image (range) of a linear transformation $T : V \longrightarrow W$ is a subspace of W.

Solution

The image of T is the set $\mathrm{Im}(T) = \{T(\mathbf{v}) \mid \mathbf{v} \in V\}$. If $\mathbf{x}, \mathbf{y} \in \mathrm{Im}(T)$ and $c, d \in \mathbf{R}$, then $\mathbf{x} = T(\mathbf{v})$ for some $\mathbf{v} \in V$ and $\mathbf{y} = T(\mathbf{u})$ for some $\mathbf{u} \in V$. Then

$$c\mathbf{x} + d\mathbf{y} = cT(\mathbf{v}) + dT(\mathbf{u}) = T(c\mathbf{v} + d\mathbf{u}).$$

Since V is a vector space, $c\mathbf{v} + d\mathbf{u} \in V$, so $c\mathbf{x} + d\mathbf{y} \in \mathrm{Im}(T)$.

(c) Is the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 8 \\ 0 & 0 & 13 \end{pmatrix}$$

diagonalizable? Explain.

Solution

Yes. It is upper triangular, so we can get the eigenvalues from the diagonal. Since there are three distinct eigenvalues and we're working with $\mathbf{R}^3$, there will be a basis for $\mathbf{R}^3$ consisting of eigenvectors of A, which is the same as saying that A is diagonalizable.

(d) If the column space of a 8×4 matrix A is 3-dimensional, then what is the dimension of the null space?

Solution

There are 3 pivot columns, leaving 1 nonpivot column, so the dimension of $\mathrm{Nul}(A)$ is 1.

4 Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 5 & 5 \\ 0 & 0 & 3 & 3 \end{pmatrix}$$

(a) Find a basis (which I will refer to as $\mathcal{B}$) for $\mathrm{Nul}(A)$.

Solution

First row reduce A:

$$A = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 5 & 5 \\ 0 & 0 & 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We see from this that $\mathrm{Nul}(A)$ has a basis

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

(b) Let $V = \mathrm{Nul}(A)$. Then we can define a linear transformation

$$T : V \longrightarrow \mathbf{R}^3$$

by $T(\mathbf{v}) = A\mathbf{v}$. Write down the matrix for T in terms of the basis $\mathcal{B}$ of V and the standard basis $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbf{R}^3$.

Solution

The matrix we want will be $M = [T(\mathbf{b}_1), T(\mathbf{b}_2)]$. Since both $\mathbf{b}_1$ and $\mathbf{b}_2$ are in $\mathrm{Nul}(A)$, $T(\mathbf{b}_1) = T(\mathbf{b}_2) = \mathbf{0}$, so the matrix is

$$M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$