# Math 22 Workshop IV
# Due May 8, 2006

INSTRUCTIONS: You are encouraged to work on these problems in a group of three to four people, but you may work alone if you insist. Each group should write up a single solution and turn their paper in *in class* on Monday.

1. Let $D : \mathbb{P}_4 \to \mathbb{P}_4$ be the linear transformation given by

$$D\big(p(t)\big) = (1-t^2)p''(t) - 2t\,p'(t) + 20p(t).$$

(a) Find the matrix ${}_\beta[D]_\beta$ for $D$ relative to the basis $\beta = \{\,1, t, t^2, t^3, t^4\,\}$ for $\mathbb{P}_4$.

(b) Use your answer above and coordinates to find a basis for the kernel and range of $D$.

(c) Using your answers above, show that there is, up to a scalar multiple, only one polynomial $p$ of degree at most 4 which is a solution to the differential equation

$$(1-t^2)p'' - 2tp + 20p = 0.$$

(d) Use your analysis above to produce a polynomial $q$ of degree at most 4 so that the differential equation

$$(1-t^2)p'' - 2tp + 20p = q$$

has no polynomial solution of degree at most 4.

You may have learned the following result in high school.

**Theorem 1** *If $p(t)$ is a polynomial and if $a$ is a real number such that $p(a) = 0$, then there is a polynomial $q$ such that $p(t) = (t-a)q(t)$.*

In the next exercise, we want to give a proof of this for polynomials of degree at most 3.

2. Let $a \in \mathbf{R}$ and define $T : \mathbb{P}_3 \to \mathbf{R}$ by $T(p) := p(a)$. Let $\beta = \{\,1, t, t^2, t^3\,\}$ be the usual basis for $\mathbb{P}_3$, and let $\gamma = \{\,1\,\}$ be the standard basis for $\mathbf{R}$.

(a) Use the matrix ${}_\gamma[T]_\beta$ of $T$ relative to the bases $\beta$ and $\gamma$ to find a basis for the kernel and range of $T$.

(b) Show directly (*without* using Theorem 1 above) that every polynomial in your basis for the kernel of $T$ is divisible by $t-a$. Conclude that every polynomial in the kernel of $T$ is divisible by $t-a$.

(c) Note that part (b) gives a proof of Theorem 1 for polynomials of degree at most 3.