Your name:

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**Math 11 Fall 2011, Homework 1, due Wed Sep 28**

*Please show your work. No credit is given for solutions without justification.*

(1) Sketch a picture of the pyramid with vertices

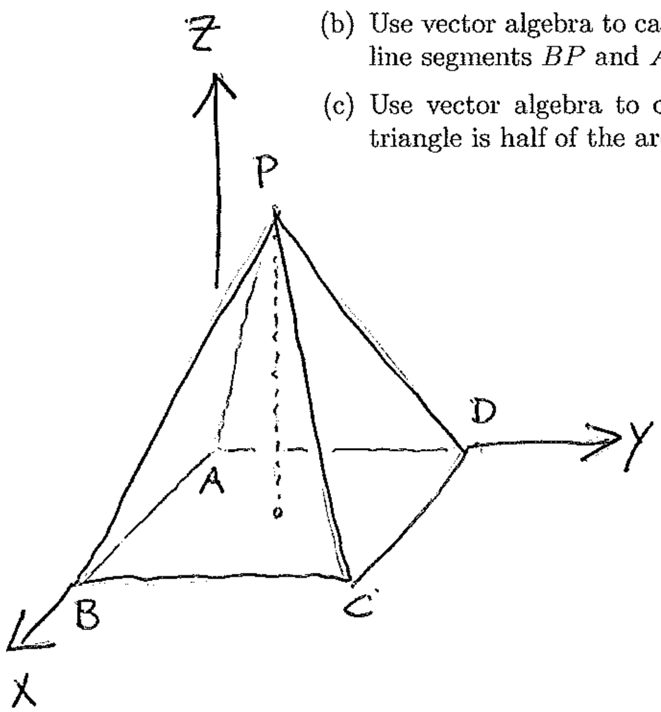
$$A = (0,0,0), B = (2,0,0), C = (2,2,0), D = (0,2,0), P = (1,1,3)$$

and answer the following questions.

(a) Derive a parametric equation for the straight line through points $B$ and $P$.

(b) Use vector algebra to calculate the cosine of the angle $BPA$ (i.e., the angle between line segments $BP$ and $AP$ at point $P$.)

(c) Use vector algebra to calculate the area of triangle $BPA$. (Hint: the area of a triangle is half of the area of a parallelogram.)




a) $\overrightarrow{BP} = \overrightarrow{OP} - \overrightarrow{OB} = \langle -1, 1, 3\rangle$

line:

$$\langle x,y,z\rangle = \overrightarrow{OB} + t\,\overrightarrow{BP} = \langle 2,0,0\rangle + t\langle -1,1,3\rangle = \langle 2-t, t, 3t\rangle$$

$$\boxed{\begin{cases} x = 2-t \\ y = t \\ z = 3t \end{cases}}$$

b) $\overrightarrow{AP} = \overrightarrow{OP} = \langle 1,1,3\rangle$

$\overrightarrow{AP}\cdot\overrightarrow{AP} = 1+1+9 = 11$

$\overrightarrow{BP}\cdot\overrightarrow{BP} = 1+1+9 = 11$

$\overrightarrow{AP}\cdot\overrightarrow{BP} = -1+1+9 = 9$

$$\cos\theta = \frac{\overrightarrow{AP}\cdot\overrightarrow{BP}}{\|\overrightarrow{AP}\|\,\|\overrightarrow{BP}\|} = \frac{9}{\sqrt{11}\sqrt{11}} = \boxed{\frac{9}{11}}$$

c) $\overrightarrow{AP}\times\overrightarrow{BP} = \begin{vmatrix} i & j & k \\ 1 & 1 & 3 \\ -1 & 1 & 3 \end{vmatrix} = \langle 0, -6, 2\rangle$

Length $= \sqrt{6^2+2^2} = \sqrt{40}$

= area pllgm.

Area triangle ABP

$$= \frac{\sqrt{40}}{2} = \frac{\sqrt{40}}{\sqrt{4}} = \boxed{\sqrt{10}}$$

(2) For the vectors $\mathbf{v} = \langle 1, 0, -1\rangle$ and $\mathbf{w} = \langle 0, 1, -1\rangle$ calculate the components of the decomposition

$$\mathbf{w} = \mathbf{w}_{\parallel} + \mathbf{w}_{\perp}$$

where $\mathbf{w}_{\parallel}$ is the projection of $\mathbf{w}$ along $\mathbf{v}$ while $\mathbf{w}_{\perp}$ is the component of $\mathbf{w}$ perpendicular to $\mathbf{v}$.

Formula for $\bar{W}_{\parallel}$:

$$\bar{W}_{\parallel} = \left(\frac{\bar{V}\cdot\bar{W}}{\bar{V}\cdot\bar{V}}\right)\bar{V}$$

$$\bar{V}\cdot\bar{W} = 0+0+1 = 1 \qquad \left(\frac{\bar{V}\cdot\bar{W}}{\bar{V}\cdot\bar{V}}\right) = \frac{1}{2}$$

$$\bar{V}\cdot\bar{V} = 1+0+1 = 2$$

$$\bar{W}_{\parallel} = \frac{1}{2}\bar{V} = \boxed{\left\langle \frac{1}{2}, 0, -\frac{1}{2}\right\rangle}$$

$$\bar{W}_{\perp} = \bar{W} - \bar{W}_{\parallel} = \langle 0, 1, -1\rangle - \left\langle \frac{1}{2}, 0, -\frac{1}{2}\right\rangle$$

$$= \boxed{\left\langle -\frac{1}{2}, 1, -\frac{1}{2}\right\rangle}$$

(3) For each of the following equalities indicate whether it is true or false for all vectors **a** and **b** in $\mathbb{R}^3$. Use vector algebra to justify your answers.

(a) $(\mathbf{a}+\mathbf{b}) \times (\mathbf{a}-\mathbf{b}) = \mathbf{a} \times \mathbf{a} - \mathbf{b} \times \mathbf{b}$

(b) $(\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}-\mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b}$

(c) $\|\mathbf{a}+\mathbf{b}\|^2 + \|\mathbf{a}-\mathbf{b}\|^2 = 2\|\mathbf{a}\|^2 + 2\|\mathbf{b}\|^2$.

Both cross & dot product "distribute", therefore you can "foil".

(a) $(\bar{a}+\bar{b}) \times (\bar{a}-\bar{b}) \overset{\text{foil}}{=} \bar{a}\times\bar{a} - \bar{a}\times\bar{b} + \bar{b}\times\bar{a} - \bar{b}\times\bar{b}$

Now use $\bar{b}\times\bar{a} = -\bar{a}\times\bar{b}$ to get

$$\ldots = \bar{a}\times\bar{a} - 2(\bar{a}\times\bar{b}) + \bar{b}\times\bar{b}.$$

So the equality is $\boxed{\text{false}}$.

REMARK. We can further simplify using

$$\bar{a}\times\bar{a} = \bar{0}, \quad \bar{b}\times\bar{b} = \bar{0}$$

Then $(\bar{a}+\bar{b})\times(\bar{a}-\bar{b}) = 2(\bar{a}\times\bar{b})$.

(b) Similar: $(\bar{a}+\bar{b})\cdot(\bar{a}-\bar{b}) =$

$$= \bar{a}\cdot\bar{a} - \bar{a}\cdot\bar{b} + \bar{b}\cdot\bar{a} - \bar{b}\cdot\bar{b}$$

$$= \bar{a}\cdot\bar{a} - \bar{b}\cdot\bar{b} \quad \text{because } \bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a}$$

$\boxed{\text{true.}}$

(c) $\|a+b\|^2 = (\bar{a}+\bar{b})\cdot(\bar{a}+\bar{b}) = \bar{a}\cdot\bar{a} + 2\bar{a}\cdot\bar{b} + \bar{b}\cdot\bar{b}$

$\|a-b\|^2 = (\bar{a}-\bar{b})\cdot(\bar{a}-\bar{b}) = \bar{a}\cdot\bar{a} - 2\bar{a}\cdot\bar{b} + \bar{b}\cdot\bar{b}$

In both cases use $\bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a}$.

$\boxed{\text{true}}$ Now add them to get

$$\ldots = 2(\bar{a}\cdot\bar{a}) + 2(\bar{b}\cdot\bar{b}) = 2\|a\|^2 + 2\|b\|^2$$