Trigonometric Integrals

September 22, 2006

Strategy for Evaluating $\int \sin^m x \cos^n x \mathbf{d}x$

Strategy for Evaluating $\int \sin^m x \cos^n x \mathbf{d}x$

- If $n = 2k+1$, save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine. Then substitute $u = \sin x$.

- **Examples:** $\int \sin^2 x \cos^3 x \mathrm{d}x$.

Strategy for Evaluating $\int \sin^m x \cos^n x \mathbf{d}x$

- If $n = 2k+1$, save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine. Then substitute $u = \sin x$.

- **Examples:** $\int \sin^2 x \cos^3 x \mathrm{d}x$.

  $\int 20 \cos^3(15x)$

Strategy for Evaluating $\int \sin^m x \cos^n x \mathbf{d}x$

- If $n = 2k + 1$, save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine. Then substitute $u = \sin x$.

- **Examples:** $\int \sin^2 x \cos^3 x \mathrm{d}x$.

  $\int 20 \cos^3(15x)$

- If $m = 2k + 1$, save one sine factor amd use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine. Then substitute $u = \cos x$.

- **Examples:** $\int \sin^3 x \mathrm{d}x$;

- If the powers of both sine and cosine are even, use the half-angle identities

$$\begin{aligned}
\sin^2 x &= \frac{1}{2}(1 - \cos 2x) \\
\cos^2 x &= \frac{1}{2}(1 + \cos 2x)
\end{aligned}$$

- If the powers of both sine and cosine are even, use the half-angle identities

$$\begin{aligned}\sin^2 x &= \frac{1}{2}(1-\cos 2x)\\ \cos^2 x &= \frac{1}{2}(1+\cos 2x)\end{aligned}$$

- **Examples:** $\int_0^\pi \sin^2 x\mathbf{d}x$;

- If the powers of both sine and cosine are even, use the half-angle identities

$$\begin{aligned} \sin^2 x &= \frac{1}{2}(1 - \cos 2x) \\ \cos^2 x &= \frac{1}{2}(1 + \cos 2x) \end{aligned}$$

- **Examples:** $\int_0^\pi \sin^2 x \mathbf{d}x$;

  $\int \sin^4(4x)\mathrm{d}x$.

Strategy for Evaluating $\int \tan^m x \sec^n x \mathbf{d}x$

- $\int \tan x \mathrm{d}x = \ln|\sec x| + C$
- $\int \sec x \mathrm{d}x = \ln|\sec x + \tan x| + C$.

Strategy for Evaluating $\int \tan^m x \sec^n x \mathbf{d}x$

- $\int \tan x \mathrm{d}x = \ln|\sec x| + C$
- $\int \sec x \mathrm{d}x = \ln|\sec x + \tan x| + C$.
- If $n = 2k$, save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$. Then substitue $u = \tan x$.
- **Examples:** $\int \tan^4 x \sec^4 x \mathrm{d}x$

- If $m = 2k+1$, save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$. Then substitute $u = \sec x$.

- **Examples:** $\int \tan^3 x \mathrm{d}x$

Other Examples

- $\int_0^{\pi/2} \cos x \cos(\sin x) \mathrm{d}x$

Other Examples

- $\int_0^{\pi/2} \cos x \cos(\sin x) \mathsf{d}x$
- $\int \sec^3 x \mathsf{d}x$.

Other trigonometric integrals

- To evaluate $\int \sin mx \cos nx \mathrm{d}x$; $\int \cos mx \cos nx \mathrm{d}x$; $\int \sin mx \sin nx \mathrm{d}x$ use the identities:

$$\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$
$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$
$$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)].$$

- **Example:** $\int \sin 3x \cos 5x \mathrm{d}x$

Trigonometric substitution

- **The problem:** evaluate integrals of the form $\int \sqrt{a^2 - x^2}\mathrm{d}x$.

Trigonometric substitution

- **The problem:** evaluate integrals of the form $\int \sqrt{a^2 - x^2}\mathrm{d}x$.

- *The inverse substituion:*

$$\int f(x)\mathrm{d}x = \int f(g(t))g'(t)\mathrm{d}t \quad \text{if } x = g(t)$$

- For $\sqrt{a^2 - x^2}$ use the substitution $x = a\sin\theta$, $-\pi/2 \le \theta \le \pi/2$ and the identity $1 - \sin^2\theta = \cos^2\theta$.

- **Example:** $\int x^3\sqrt{9 - x^2}\mathrm{d}x$.

- For $\sqrt{a^2 - x^2}$ use the substitution $x = a \sin\theta$, $-\pi/2 \leq \theta \leq \pi/2$ and the identity $1 - \sin^2\theta = \cos^2\theta$.

- **Example:** $\int x^3\sqrt{9 - x^2}\mathrm{d}x$.

- For $\sqrt{a^2 + x^2}$ use the substitution $x = a \tan\theta$, $-\pi/2 < \theta < \pi/2$ and the identity $1 + \tan^2\theta = \sec^2\theta$.

- **Example:**

$$\int \frac{\mathrm{d}x}{\sqrt{4 + x^2}}.$$

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}} \mathrm{d}x$$