Math 12, Fall 2007

Lecture 2

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Outline

1. Recap and overview
   - Last class
   - Quick review of reading topics
2. Further discussion
   - Examples
   - Group Work
3. Summary
4. Next class

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Coordinates in three space

- (x, y, z) coordinates to denote points
- Planes: $\alpha x + \beta y + \gamma x + \delta = 0$
- Spheres: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$

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Vectors in three space

- Vectors have magnitude and direction
- $< x, y, z >$ coordinates to denote vectors
- We intentionally confuse points and vectors
- Vector operations: both numeric and geometric

Concepts from reading

Basic Properties

- Multiplication of vectors, $\vec{u} =< a, b, c >, \vec{w} =< d, e, f >$
  1. dot product:
  $$\vec{u} \cdot \vec{v} = ad + be + cf$$
  2. cross product:
  $$\vec{u} \times \vec{v} =< bf - ce, cd - af, ae - bd >$$
- Geometric meaning:
  1. The dot product measures angles:
  $$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos(\theta)$$
  2. The cross product measures area: $|\vec{u} \times \vec{v}|$ is the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$.

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Why is this true?

- Properties of the dot product: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- Prove:
$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta)$$
- Prove: Two vectors are orthogonal if and only if their dot product is zero

Components and Projections

$$comp_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|}$$

$$proj_{\vec{u}}\vec{v} = (comp_{\vec{u}}\vec{v})\frac{\vec{u}}{|\vec{u}|} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|^2}\vec{u}$$

Sample problems

- Find dot products
- Find projections, components
- Find cross products

Properties of the cross product

- $\vec{u} \times \vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$
- If θ denotes the angle between $\vec{u}$ and $\vec{v}$ then

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin(\theta)$$

- Two nonzero vectors are parallel if and only if their cross product is zero.
- $|\vec{u} \times \vec{v}|$ is equal to the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$.

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Outline

Examples: dot product

1. Find all vectors that are perpendicular to $\vec{u} =< 1, 2 - 2 >$
2. If a force, $\vec{F}$, moves an object from point P to point Q, the work done by this force is $W = \vec{F} \cdot \vec{D}$ where $D = \vec{PQ}$. Gravity acts on a box positioned at the top of a 45 degree incline. The box moves 3 m down the ramp, how much work is done?

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Examples: cross product

1. Torque is defined to be the cross product of the position and force vectors:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

A wrench 30cm long lies along the positive y-axis and grips a bolt at the origin. A force is applied in the direction $< 0, 3, -4 >$ at the end of the wrench. Find the magnitude of the force needed to supply 100 J of torque to the bolt.

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Problems to work on

1. Let $\vec{v} = 5\vec{j}$ and let $\vec{u}$ be a variable vector in the xy-plane whose tip lies on the circle of radius 3. Find the maximum and minimum values of the length of the vector $\vec{u} \times \vec{v}$. In what direction does $\vec{u} \times \vec{v}$ point?
2. Prove that $\vec{u} \times \vec{v}$ is perpendicular to $\vec{u}$ and $\vec{v}$

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Summary

- dot product: measures angle, projections, components, work
- cross product: measures volume/area, torque, cross product is perpendicular to components

Work for next class

- Reading: 13.5
- Practice problems: 13.5 # 2,4,20,23
- f07hw3