3.5: Issues in Curve Sketching (cont'd)
and
4.1: Modeling Accumulations

Mathematics 3
Lecture 21
Dartmouth College

February 19, 2010

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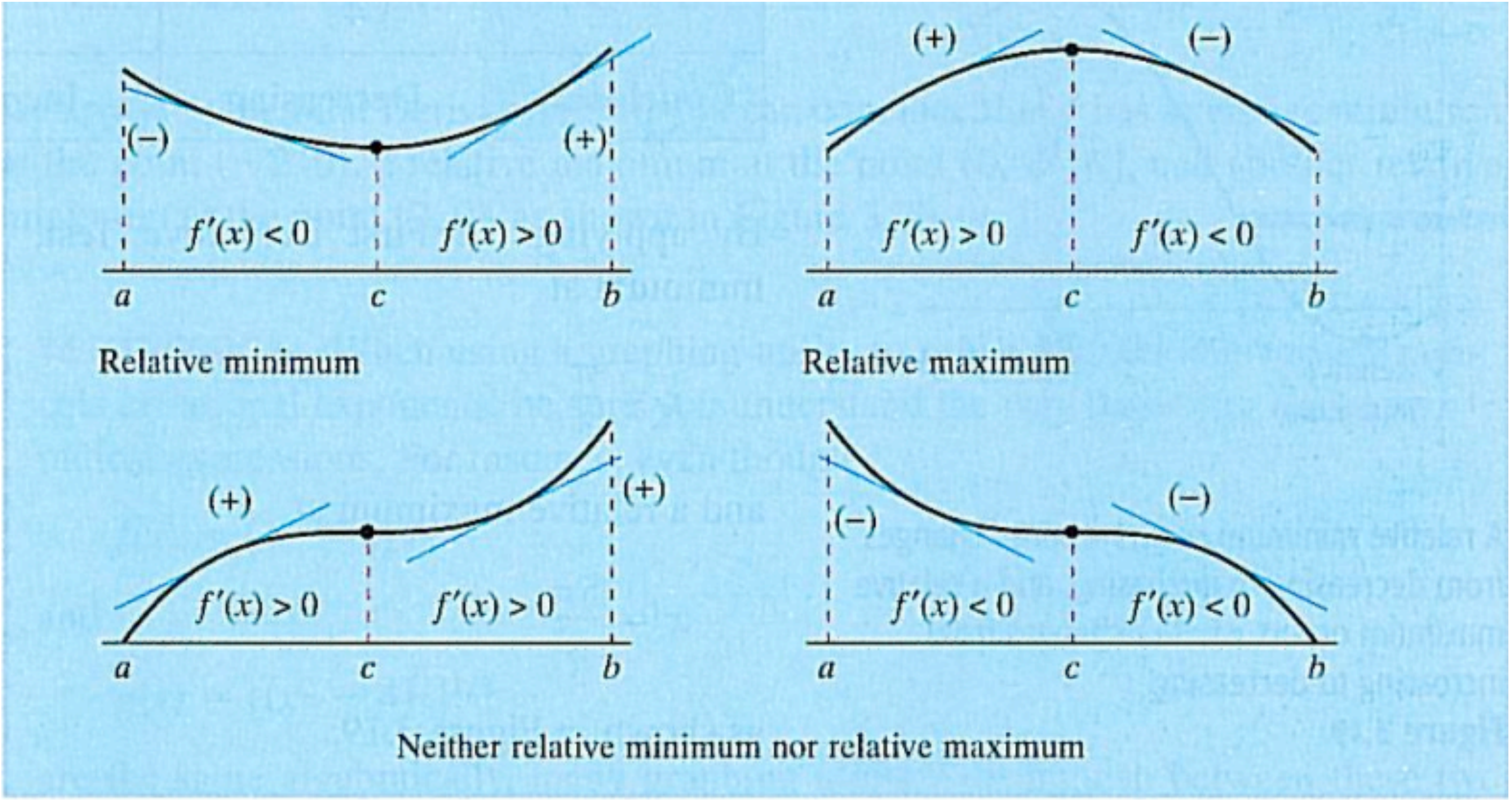
The First Derivative Test for Local Extrema



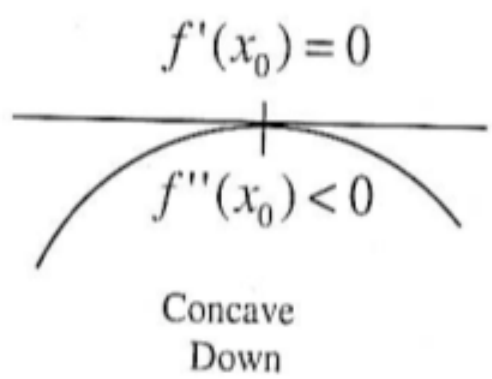
The Second Derivative Test for Local Extrema

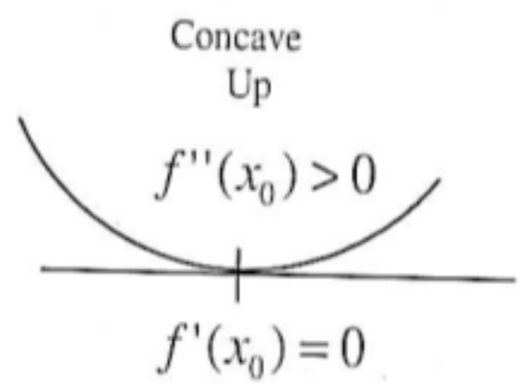
Theorem 3 (p. 274) Let f be a function such that the second derivative f'' exists on an open interval I containing x_0.

1. If $f'(x_0) = 0$ and $f''(x_0) > 0$, then $f(x_0)$ is a local minimum.

2. If $f'(x_0) = 0$ and $f''(x_0) < 0$, then $f(x_0)$ is local maximum.

3. If $f'(x_0) = 0$ and $f''(x_0) = 0$ the test **fails**. Use the First Derivative Test to decide...





Guidelines for Sketching a Curve

Properties to look for	Explanation	$f(x) = \dfrac{2x^2}{x^2-1}$
1. Domain	**All** x where $f(x)$ is **defined**	$x \neq \pm 1$
2. x- and y-intercepts	**x-intercepts:** $f(x) = 0$ **y-intercepts:** $f(0)$	$x = 0$ $y = 0$
3. Symmetries	**even:** $f(-x) = f(x)$ **odd:** $f(-x) = -f(x)$ **periodic:** $f(x+p) = f(x)$	even
4. Asymptotes	**horizontal:** $y = \lim\limits_{x\to\pm\infty} f(x)$ **vertical:** $x = a$ **if** $\lim\limits_{x\to a^{\pm}} f(x) = \pm\infty$	$y = 2$ $x = -1$ **and** $x = 1$
5. Increases or Decreases (I/D-Test)	**increases:** $f'(x) > 0$ **decreases:** $f'(x) < 0$	$(-\infty, -1)$ **and** $(-1, 0)$ $(0, 1)$ **and** $(1, \infty)$
6. Local Maxima and Minima (1st or 2nd Derivative Test)	**maximum:** f' from $+$ to $-$ at $x = c$ **minimum:** f' from $-$ to $+$ at $x = c$	$x = 0$ none
7. Concavity and Inflections (Concavity Test)	concave **upward:** $f''(x) > 0$ concave **downward:** $f''(x) < 0$ **inflection** point: f'' changes **sign**	$(-\infty, -1)$ **and** $(1, \infty)$ $(-1, 1)$ none
8. Sketch the Curve		

NB: This handout is posted in the (Documents) section of Blackboard.

Example 1

Analyze the graph of the function $f(x) = \dfrac{2x^2 - 18}{x^2 - 4}$.

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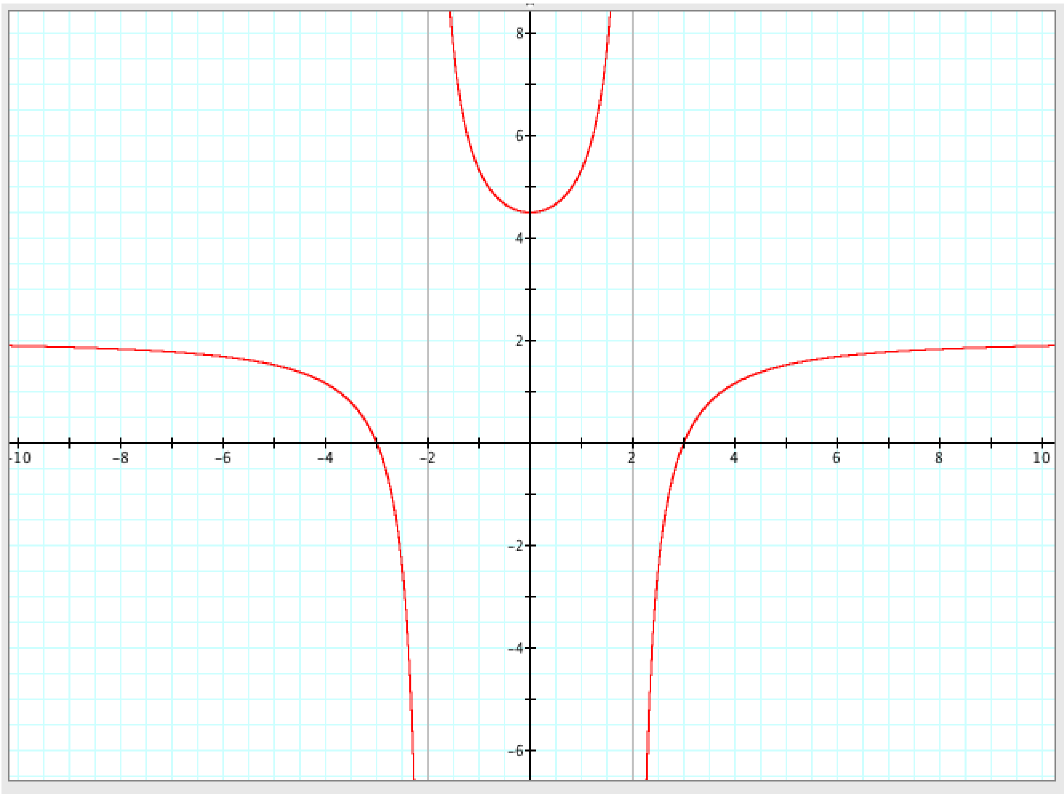

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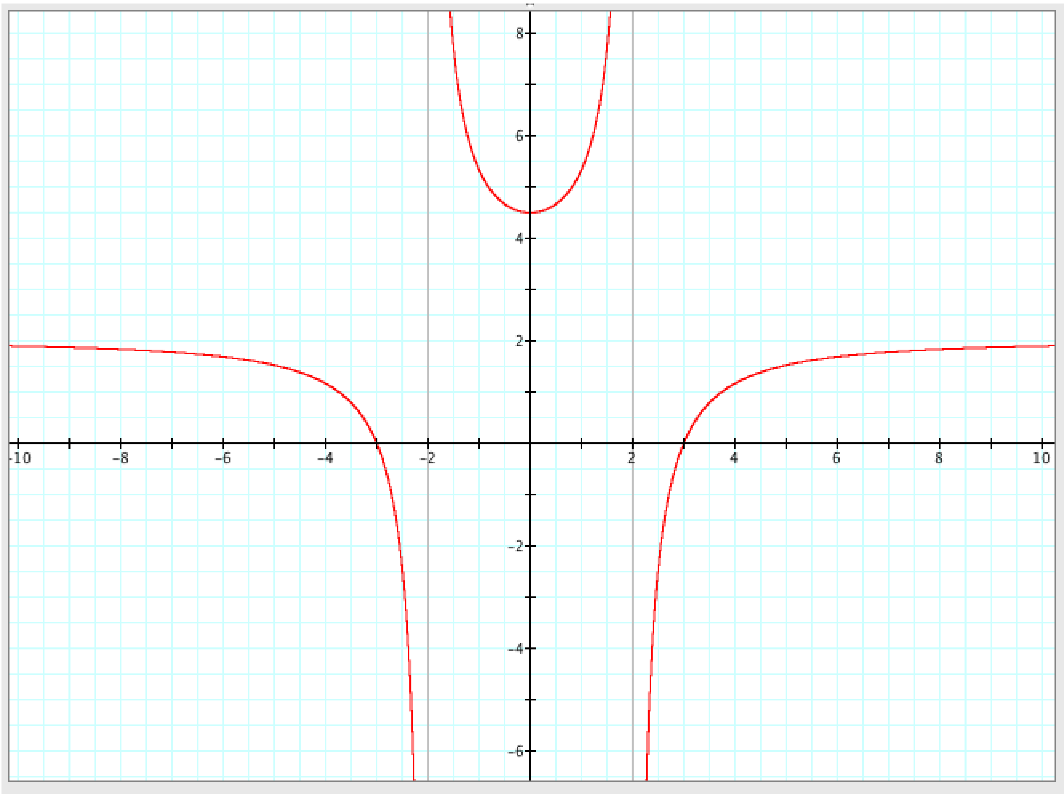

Now for a reason why graphing calculators aren't always best...

Example 2

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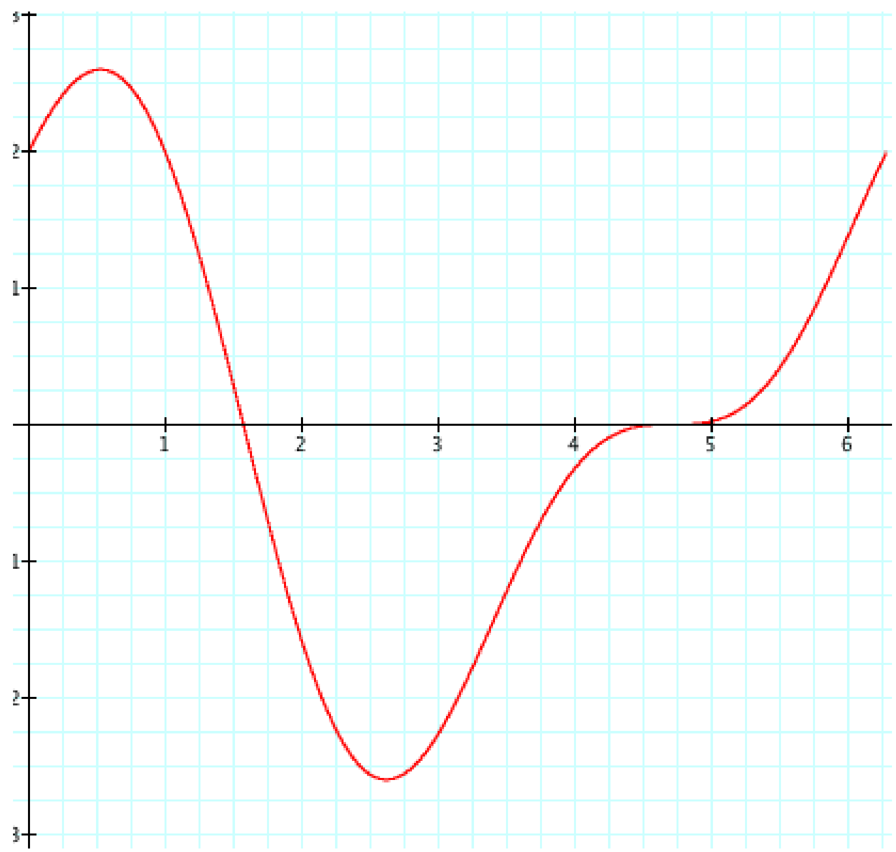
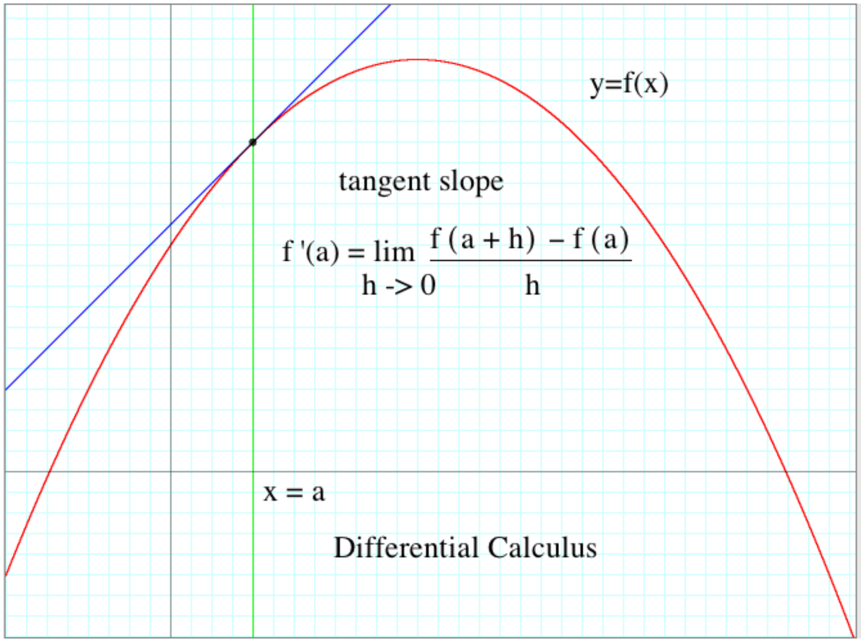
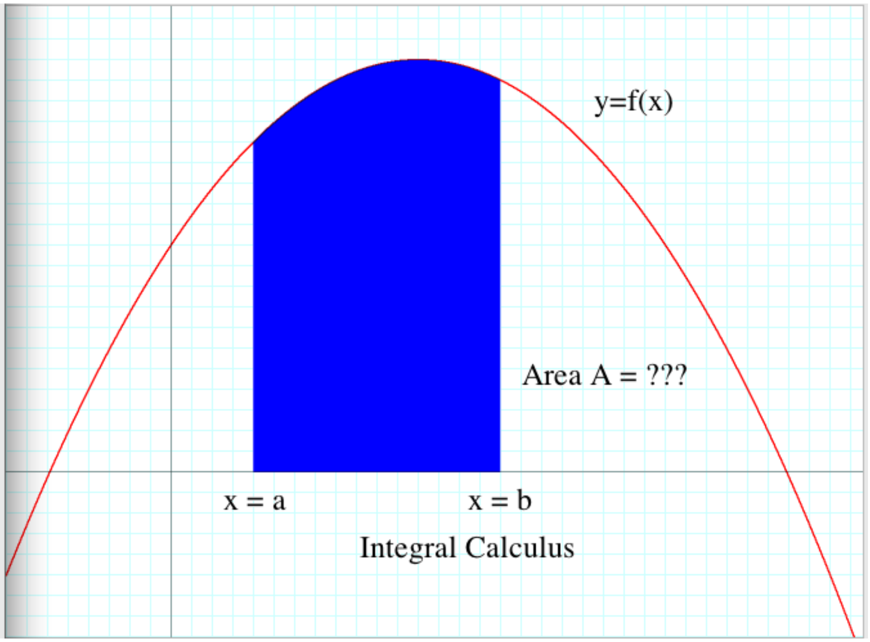


Two Main Purposes of The Calculus

1.) Find the SLOPE of a tangent line to a curve $y = f(x)$ at a point $x = a$.

2.) Find the AREA under a curve $y = f(x)$ over the interval $a \leq x \leq b$.





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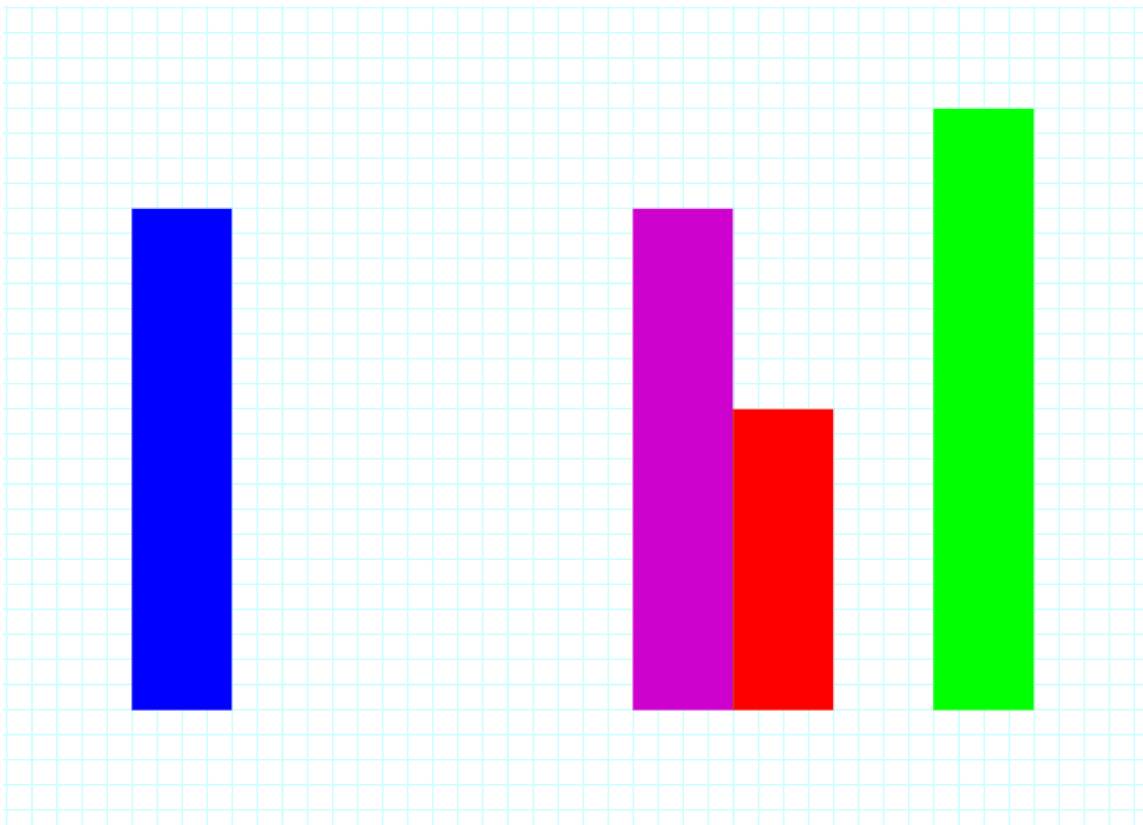

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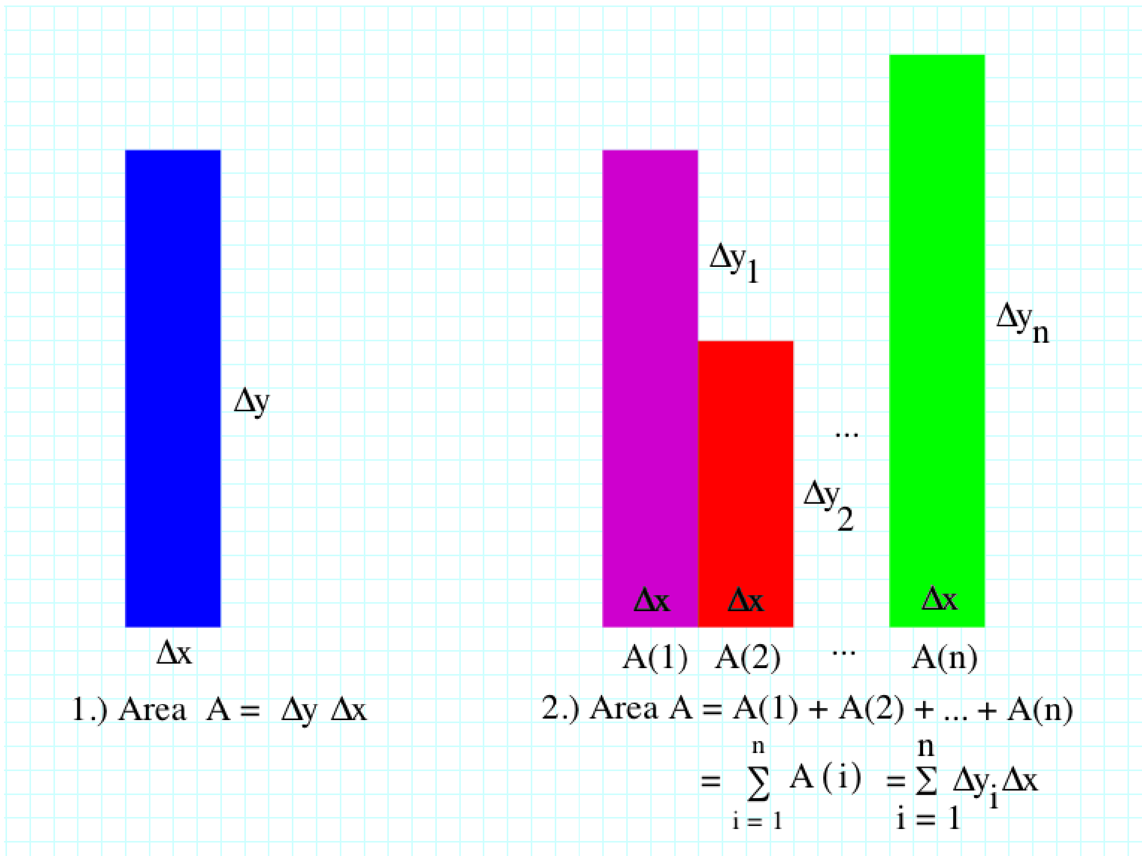
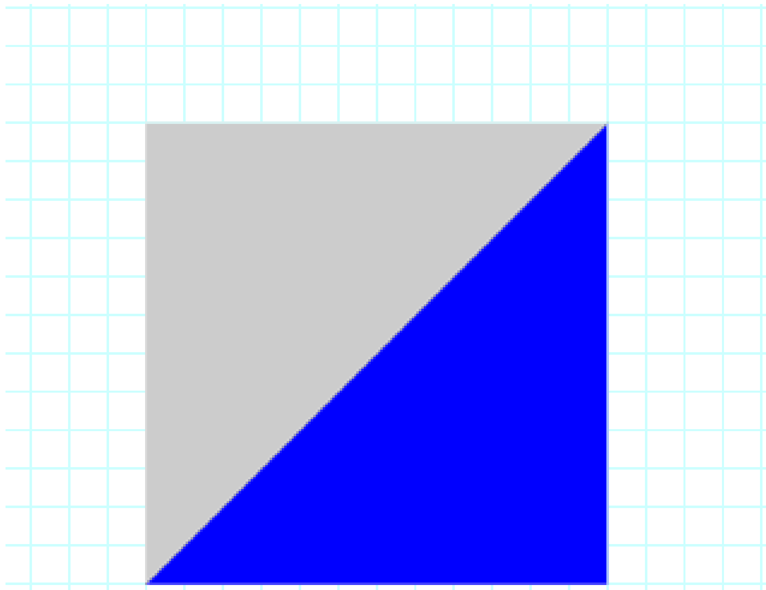


The Area of a Triangle

These facts allow us to compute the areas of triangles.

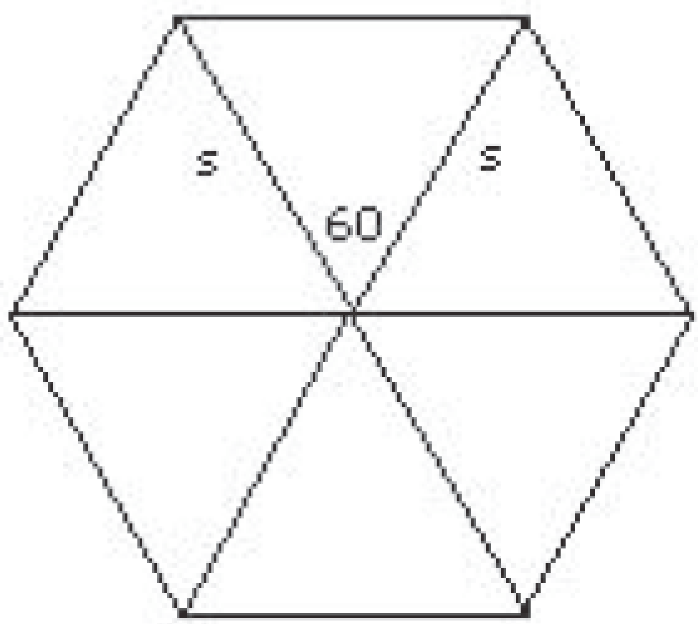


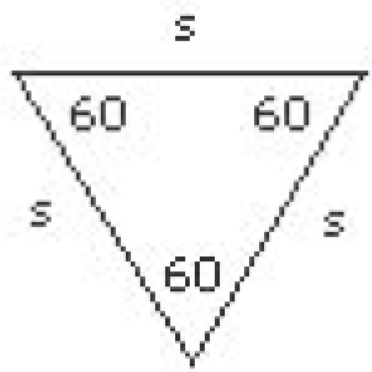
$$\text{Area} = \frac{1}{2}(\text{base}) \times (\text{height})$$

The Area of a Regular Hexagon

We use our formula for triangles to compute the area of a hexagon.

Let $s =$ the length of a side.





The Area of a Regular Hexagon

- It is composed of six congruent isoseles triangles, each with a $60°(= 360°/6)$ degree central angle.

- The base angles of each triangle are also $60°$ degrees, and the third side has the same length as the other two.

- The area of one of the isoseles triangles is

$$A_{\Delta} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}s\frac{\sqrt{3}s}{2} = \frac{\sqrt{3}}{4}s^2.$$

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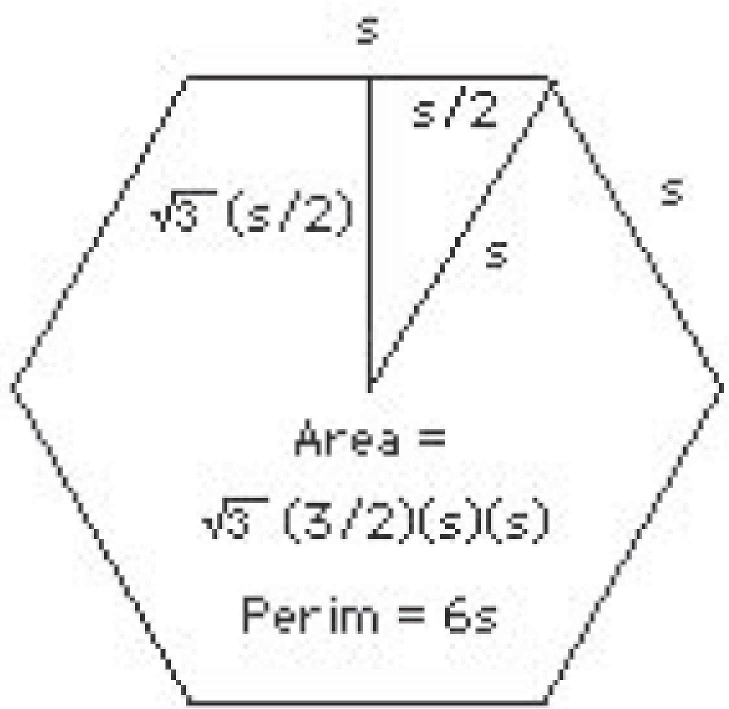
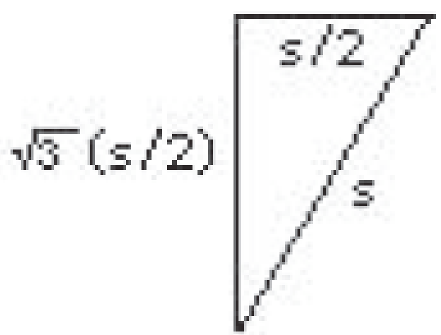


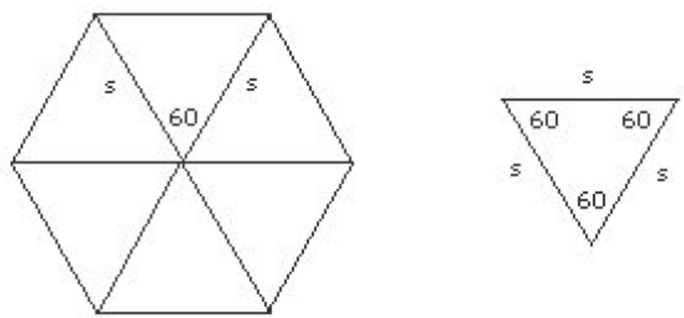


The Area of a Regular Hexagon

Thus, the area of the hexagon is six times the areas of the triangles:

$$A = 6 \times A_\Delta = 6\left(\frac{\sqrt{3}}{4}s^2\right) = \frac{3\sqrt{3}}{2}s^2$$

When $s = 1$ we get approximately $A = 2.59808$.



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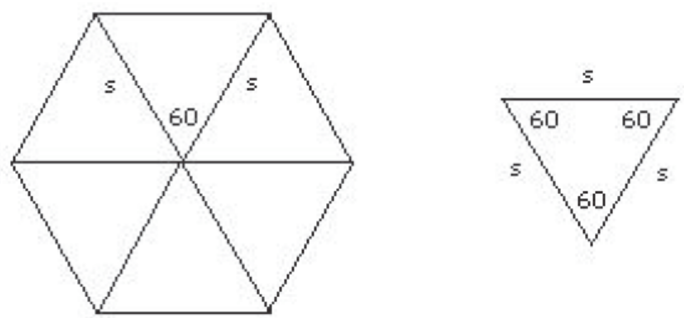

NOTE: Since *every* polygon can be divided into triangles, we can (define and) compute their areas!

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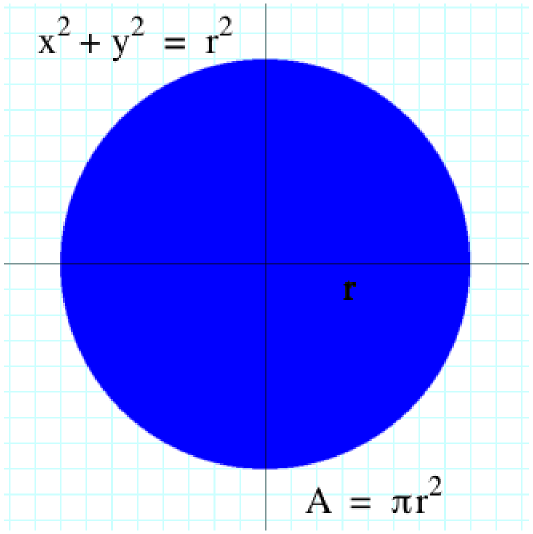

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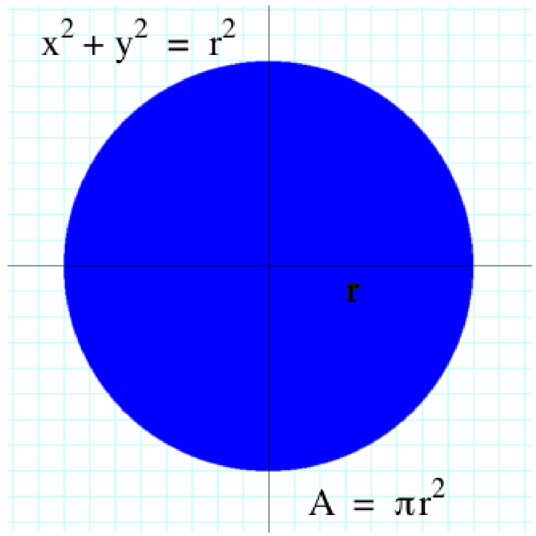

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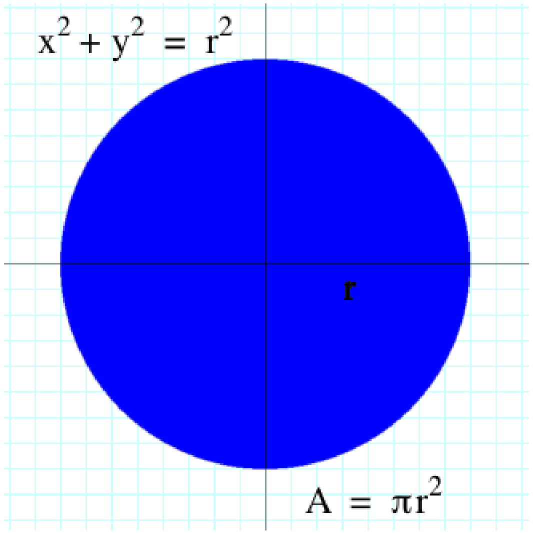

where π is the irrational number $\pi = 3.14159...$

What exactly is the (precise definition of the) "area" of a circle?

Moreover, how do we "prove" the above formula is actually correct?

Brief Ancient History of The Area of a Circle

The ancient Babylonians calculated the area of a circle by taking 3 times the square of its radius, which gave a value of $\pi = 3$. One Babylonian tablet (ca. 1900 - 1680 B.C.) indicates a value of 3.125 for π, which is a closer approximation.

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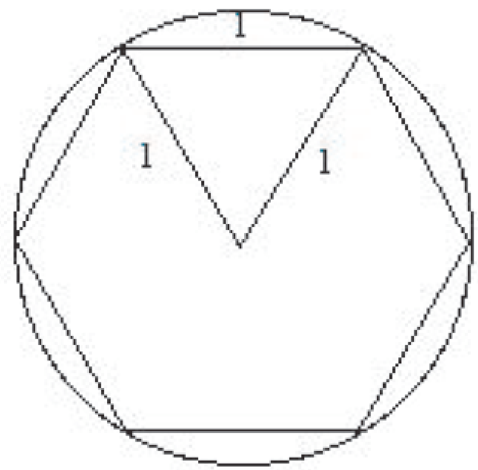
Liu Hui (approx 250 A.D.) was a Chinese mathematician, who wrote a commentary of the *Nine Chapters on the Mathematical Art*, used a very ingenious method for finding the area of a circle.

Archimedes and the Area of a Circle

One can approximate the area of the circle with an inscribed hexagon:



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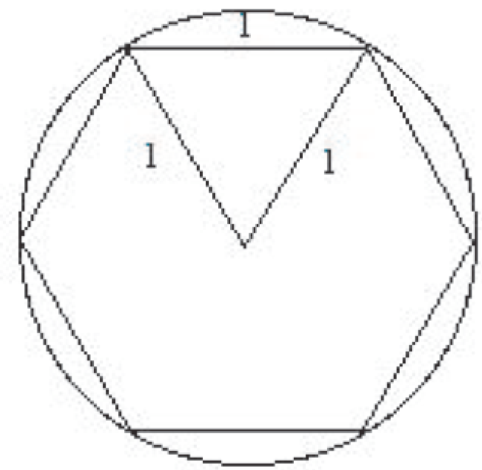

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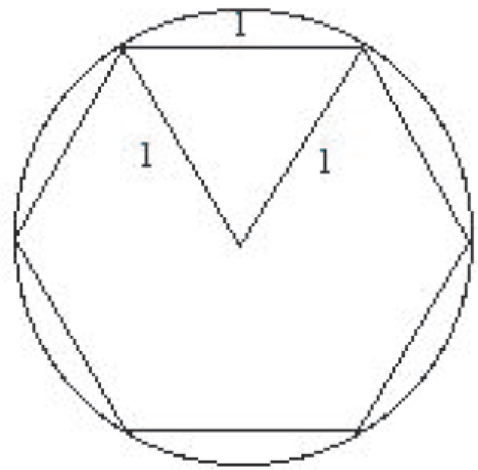

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Archimedes tried to approximate the area of the circle better using inscribed (and circumscribed polygons) with MORE sides. See the website here and our Math 3 applet here.

Areas of Regular Polygons	
sides	area
6	2.598076
12	3.000000
24	3.105829
48	3.132629
96	3.139350
192	3.141032
384	3.141452
768	3.141558
1536	3.141584
3072	3.141590
6144	3.141592

What is the Area of a Circle?

It is reasonable to define (precisely) the area of a unit circle to be the limit of the areas of the inscribed (or circumscribed) regular polygons

$$P_1, P_2, P_3, \cdots$$

that come from starting with a hexagon (or triangle) and increasing the number of sides at each successive stage, i.e,

$$A = \lim_{n\to\infty} \mathrm{Area}(P_n).$$

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What is so special about these geometrical figures?

Why not use any collection of shapes that are contained in the circle and fill it in the limit?

Another Calculation of the Area of a Circle

We divide the interval $[0, 1]$ into n subintervals of equal length

$$h = \Delta x = \frac{1}{n}.$$

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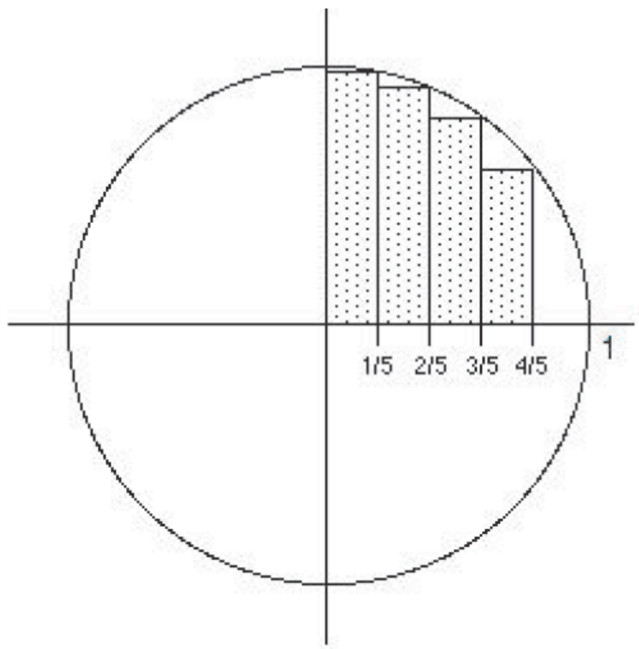
The (top of the) circle is the graph of the function $f(x) = \sqrt{1 - x^2}$.



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Approximating Area of Unit Circle with Rectangles

rectangles	sum of areas times 4
5	2.637049
500	3.137487
1000	3.139555
2000	3.140580
5000	3.141189

$$\text{Area} \quad A = \lim_{n\to\infty} (\text{Sum of the areas of n rectangles})$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

The Method of Accumulations

The process of passing to the limit not only provides a useful calculational tool, but it gives a **precise** way to define what is meant by the area under the curve $y = f(x)$ over an interval $[a, b]$.

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The process of passing to the limit not only provides a useful calculational tool, but it gives a **precise** way to define what is meant by the area under the curve $y = f(x)$ over an interval $[a, b]$.

Then (later) we will find the most **AMAZING** thing will happen:

Area is related to antiderivatives!!!

BTW: There is another method for estimating areas by throwing darts at a dartboard. This is called a Monte Carlo Simulation.

A hint for Monday's exam from Spiked Math