Your name:

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Math 11 Fall 2011, Homework 2, due Wednesday Oct 5
Please show your work. No credit is given for solutions without justification.

(1) Let $P = (2,0,0)$, $Q = (0,3,0)$, and $R = (0,0,4)$.

- (a) Sketch the three points and the vectors $\vec{PQ}$, $\vec{PR}$, and $\vec{QR}$ on the axes below.
- (b) Find the equation for the plane which passes through these three points.
- (c) Calculate cosine of the angle between the plane in (a) and the plane $x = y$.

(b) $\vec{PQ} = \langle -2,3,0 \rangle$, $\vec{PR} = \langle -2,0,4 \rangle$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ -2 & 3 & 0 \\ -2 & 0 & 4 \end{vmatrix} = \langle 12, 8, 6 \rangle$$ ← normal vector to plane

$12x + 8y + 6z = d$. Plug in $(x,y,z) = (2,0,0)$

$12 \cdot 2 + 0 + 0 = d \Rightarrow d = 24$

$$\boxed{12x + 8y + 6z = 24}$$ can divide by 2

or $6x + 4y + 3z = 12$

(c) Plane $x = y$ is $1x - 1y + 0z = 0$.

Normal vectors:

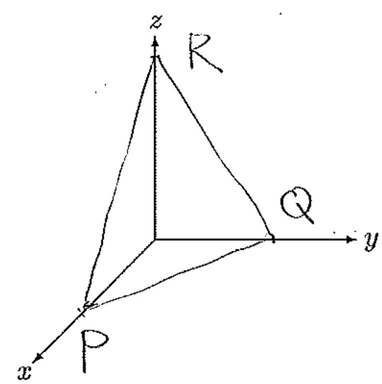
$n_1 = \langle 6, 4, 3 \rangle$

$n_2 = \langle 1, -1, 0 \rangle$

$n_1 \cdot n_1 = 36 + 16 + 9 = 61$

$n_2 \cdot n_2 = 1 + 1 = 2$

$n_1 \cdot n_2 = 6 - 4 + 0 = 2$



$$\cos\theta = \frac{n_1 \cdot n_2}{\|n_1\| \, \|n_2\|} = \frac{2}{\sqrt{2}\sqrt{61}} = \boxed{\sqrt{\frac{2}{61}}}$$

(2) Let $\mathbf{r}(t) = \langle 2t^2+1, 2t^2-1, t^3\rangle$.

(a) Calculate the arc length of $\mathbf{r}(t)$ from $0 \le t \le 2$.

(b) Derive and equation for the tangent line to $\mathbf{r}(t)$ at the point where $t = 1$.

(a) $\bar{r}'(t) = \langle 4t, 4t, 3t^2\rangle$ velocity

$\|\bar{r}'(t)\| = \sqrt{(16t^2+16t^2+9t^4)}$

$= \sqrt{32t^2+9t^4} = |t|\sqrt{32+9t^2}$ speed

Arc length formula

$$S = \int_0^2 t\sqrt{32+9t^2}\,dt$$

$|t| = t$ because $t \ge 0$.

u-substitution: $u = 32+9t^2$, $du = 18t\,dt$.

$$S = \int \frac{1}{18} u^{1/2}\,du = \frac{1}{27} u^{3/2} = \frac{1}{27}(32+9t^2)^{3/2}\Big|_0^2$$

$$\boxed{S = \frac{1}{27}\left(68^{3/2} - 32^{3/2}\right)}$$

Can be simplified to $\frac{1}{27}(136\sqrt{17} - 128\sqrt{2})$ not necessary

(b) $\bar{r}'(1) = \langle 4,4,3\rangle$ tangent vector

Line with direction $\langle 4,4,3\rangle$ through the point $\bar{r}(1) = \langle 3,1,1\rangle$ has equation

$$\boxed{\langle x,y,z\rangle = \langle 3,1,1\rangle + t\langle 4,4,3\rangle}$$

Alternatively:

$$\boxed{\begin{aligned} x &= 3+4t \\ y &= 1+4t \\ z &= 1+3t \end{aligned}}$$

(3) Consider the parameterization $\mathbf{r}(t) = \langle R\sin(t), R\cos(t)\rangle$, where $R > 0$.

(a) What curve is this? What is its path for $t \geq 0$? (where does it start, and what direction does it go?)

(b) Calculate the curvature using the formula

$$\kappa(t) = \frac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||}.$$

(c) Calculate the curvature using the formula

$$\kappa(t) = \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3}.$$

[Hint: a curve in two dimensions has the same curvature as when it thought of as sitting in three dimensions. Rewrite the curve as $\langle x(t), y(t), z(t)\rangle$ first.]

(a) Circular path (center $(0,0)$, radius R) starting at $\langle x, y\rangle = \langle 0, R\rangle$ going clockwise.

(b) $\bar{r}'(t) = \langle R\cos t, -R\sin t\rangle$

$\|\bar{r}'(t)\| = R \quad \Rightarrow \quad \vec{T}(t) = \dfrac{r'}{\|r'\|} = \langle \cos t, -\sin t\rangle$

$\vec{T}'(t) = \langle -\sin t, -\cos t\rangle$

$\|\vec{T}(t)\| = 1.$

$$\boxed{K = \frac{\|T'\|}{\|r'\|} = \frac{1}{R}}$$

(c) $\bar{r}'(t) = \langle R\cos t, -R\sin t\rangle$

$\bar{r}''(t) = \langle -R\sin t, -R\cos t\rangle$

$$\bar{r}' \times \bar{r}'' = \begin{vmatrix} i & j & k \\ R\cos t & -R\sin t & 0 \\ -R\sin t & -R\cos t & 0 \end{vmatrix} = \langle 0, 0, -R^2\cos^2 t - R^2\sin^2 t\rangle$$

$= \langle 0, 0, -R^2\rangle$

Length $= R^2$

$$\boxed{K = \frac{\|r' \times \bar{r}''\|}{\|r'\|^3} = \frac{R^2}{R^3} = \frac{1}{R}}$$