Math 68. Algebraic Combinatorics. Fall 2011.

Topics for the final project (updated on 11/27/11)

You can work together in pairs or groups of three. At the end of the term, each group will present their project giving a short talk. Here is a list of possible topics. If you have your own topic in mind you are welcome to discuss it with me.

1. *[Taken by Dan D.]* **Domino tilings of Aztec diamonds.** (1-2 people) Counting the number of domino tilings of a given shape is a difficult problem in general. However, for the so-called Aztec diamond, the number of domino tilings has a very simple formula.
   - [Aig] Pages 44–50.
2. *[Taken by Dan M.]* **Random walks in $\mathbb{Z}^d$.** (1-2 people) The probability that a random walk in $\mathbb{Z}^d$ returns to the origin is 1 for $d = 1, 2$, but strictly less than 1 for $d \geq 3$. In other words, you shouldn't get drunk unless you move in at most two dimensions.
   - [Aig] Pages 85–89.
3. *[Taken by Ed]* **Walks in graphs.** (1-2 people) The number of walks of given length between two vertices of a graph can be expressed in terms of the eigenvalues of its adjacency matrix. This is a nice connection of combinatorics and linear algebra.
   - [St] Section 1.
4. Present any of **Matousek's** *Thirty-three miniatures* [Mat]. (1-3 people) These are beautiful applications of linear algebra to solve combinatorial problems, and most are only 4 pages long!
5. **The Gessel-Viennot method.** (1-3 people) This is a remarkable formula to enumerate n-tuples of nonintersecting lattice paths. The answer is given by a determinant of binomial coefficients, and the proof is based on the combinatorics of involutions.
   - [Aig] Section 5.4.
   - [EC1] Section 2.7.
   - I. Gessel and G. Viennot, Binomial determinants, paths, and hook length formulae, *Advances in Math.* 58 (1985), 300–321.
6. **The descent number and the major index.** (1-2 people) We say that i is a *descent* of a permutation $\pi \in \mathcal{S}_n$ if $\pi_i > \pi_{i+1}$. The *major index* of π, denoted $\mathrm{maj}(\pi)$, is defined as the sum of all descents in π. For example, $\mathrm{maj}(12\cdots n) = 0$ and $\mathrm{maj}(n\cdots 21) = 1 + 2 + \cdots + (n-1)$. On the other hand, an *inversion* of π is a pair (i, j) such that $i < j$ and $\pi_i > \pi_j$. The *inversion number* of π, denoted $\mathrm{inv}(\pi)$, is the number of inversions of π.

   The goal of this project is to show that for any value k, the number of permutations $\pi \in \mathcal{S}_n$ with $\mathrm{maj}(\pi) = k$ is the same as the number of permutations $\pi \in \mathcal{S}_n$ with $\mathrm{inv}(\pi) = k$. In other words, the major index maj is equidistributed with the number of inversions inv, that is,
$$\sum_{\pi \in \mathcal{S}_n} q^{\mathrm{maj}(\pi)} = \sum_{\pi \in \mathcal{S}_n} q^{\mathrm{inv}(\pi)}.$$

A stronger version of this is the fact that the joint distribution of maj and inv is symmetric, that is,

$$\sum_{\pi\in\mathcal{S}_n} q^{\mathrm{maj}(\pi)}t^{\mathrm{inv}(\pi)} = \sum_{\pi\in\mathcal{S}_n} q^{\mathrm{inv}(\pi)}t^{\mathrm{maj}(\pi)}.$$

- [EC1] Proposition 1.4.6.
- D. Foata and M.-P. Schutzenberger, Major index and inversion number of permutations, *Math. Nach.* 83 (1978), 143–159.

7. **A combinatorial proof of the unimodality of the Gaussian polynomials.** (2-4 people) In class we will discuss an algebraic proof that the q-binomial coefficients are unimodal. A direct combinatorial proof was given by Kathy O'Hara.
   - D Zeilberger, Kathy O'Hara's constructive proof of the unimodality of the Gaussian Polynomials, *The American Mathematical Monthly*, Vol. 96, No. 7, 590–602.
8. **Viennot's geometric construction of the RSK correspondence.** (2-3 people) In class we will discuss the RSK algorithm, which gives a correspondence between permutations and pairs of standard Young tableaux. A beautiful geometric description of this correspondence is due to Viennot.
   - Section 3.6 of [Bruce Sagan, *The Symmetric group*, Springer, second edition, 2001].
9. *[Taken by Noah]* **Increasing and decreasing subsequences of permutations.** (1-2 people) This theory is an application of the Robinson-Schensted correspondence (or RSK algorithm).
   - [BS] Section 5.
10. **The transfer-matrix method.** (2-3 people) An application of counting walks in graphs to other problems in enumerative combinatorics.
    - [EC1] Section 4.7.
11. Read a paper from a combinatorics journal and present it in class. You are encouraged to talk with me to pick a suitable paper. Here are some interesting journals that you can find in the library or online.
    - *Journal of Combinatorial Theory A*,
    - *Electronic Journal of Combinatorics*,
    - *Journal of Algebraic Combinatorics*,
    - *European Journal of Combinatorics*,
    - *Annals of Combinatorics*,
    - *Discrete Mathematics.*