Math 73/103 Midterm

1. (15) Give precise statements (no proofs necessary on this problem) of Littlewood's Three Principles:

(a) Every Lebesgue measurable set is almost a disjoint union of intervals.

ANS: If $m(E) < \infty$ and $\epsilon > 0$, then there is a finite set of disjoint open intervals $\{ I_i \}_{i=1}^n$ such that $m(E \Delta F) < \epsilon$.

(b) Every sequence of Lebesgue measurable functions that converges almost everywhere is nearly uniformly convergent.

ANS: Let $(X, \mathfrak{M}, \mu)$ be a finite measure space. Suppose that $\{f_n\}$ is a sequence of measurable functions converging to f almost everywhere. If $\epsilon > 0$, then there is a set $E \subset X$ such the $\mu(X \setminus E) < \epsilon$ and such that $\{f_n\}$ converges to f uniformly on E.

(c) Every Lebesgue measurable function is nearly continuous.

ANS: Suppose that f is a Lebesgue measurable function on $[a, b]$ and that $\epsilon > 0$. Then there is a closed set $K \subset [a, b]$ such that $f|_K$ is continuous and $m([a, b] \setminus K) < \epsilon$.

2. Let $(\mathbf{R}, \mathfrak{M}, m)$ be Lebesgue measure. Recall that $E \in \mathfrak{M}$ if and only if $E + y \in \mathfrak{M}$ for all $y \in \mathbf{R}$, and that $m(E) = m(E + y)$.

(a) Let $f \in \mathcal{L}^1(m)$ and $y \in \mathbf{R}$. Define $g(x) = f(x - y)$. Show that $g \in \mathcal{L}^1(m)$ and that

$$\int_{\mathbf{R}} f(x)\, dm(x) = \int_{\mathbf{R}} f(x - y)\, dm(x).$$

ANS: Since $g^{-1}(V) = f^{-1}(V) + y$, we see that g is measurable whenever f is.

Now assume that $f \geq 0$. If $f = \mathbb{I}_E$, then

$$\int_{\mathbf{R}} g(x)\, dm(x) = m(E + y) = m(E) \quad \text{for all } E \in \mathfrak{M}.$$

Thus

$$\int_{\mathbf{R}} f(x)\, dm(x) = \int_{\mathbf{R}} g(x)\, dm(x) \tag{1}$$

for any characteristic function, and by linearity, for any non-negative measurable simple function. Letting $\{\phi_n\}$ be non-negative simple functions converging monotonically up to f, the Monotone Convergence Theorem implies that (1) holds for all $f \in L^+(m)$. In particular, it holds for $|f|$ if $f \in \mathcal{L}^1(m)$, and we conclude that $g \in \mathcal{L}^1(m)$. After writing $f = \mathrm{Re}(f)^+ - \mathrm{Re}(f)^- + i\big(\mathrm{Im}(f)^+ - \mathrm{Im}(f)^-\big)$, we have that

$$\int_{\mathbf{R}} f(x)\,dm(x) = \int_{\mathbf{R}} g(x)\,dm(x),$$

as claimed.

(b) If $f \in \mathcal{L}^1(m)$, let $\lambda_y(f) \in \mathcal{L}^1(m)$ be given by $\lambda_y(f)(x) = f(x-y)$. Show that $y \mapsto \lambda_y(f)$ is continuous from $\mathbf{R}$ to $L^1(m)$ in the sense that if $y_n \to y$ in $\mathbf{R}$, then $\|\lambda_{y_n}(f) - \lambda_y(f)\|_1 \to 0$.

ANS: We have $\lambda_y(f) \in \mathcal{L}^1(m)$ by part (a). Also $\|\lambda_y(f)\|_1 = \|f\|_1$ and it follows that $\|\lambda_y(f) - \lambda_s(f)\|_1 = \|\lambda_{y-s}(f) - f\|_1$. Thus it will suffice to see that if $y_n \to 0$, then $\|\lambda_{y_n}(f) - f\|_1 \to 0$. However, if g is continuous and vanishes off a closed bounded interval $[a,b]$, then g is uniformly continuous. Hence given $\epsilon > 0$, there is a $0 < \delta < 1$ such that $|y| < \delta$ implies $\|\lambda_y(g) - g\|_\infty < \epsilon$. But since $\lambda_y(g) - g$ must vanish off $[a-1, b+1]$, we then have $\|\lambda_y(g) - g\|_1 < \epsilon(b-a+2)$.

Now if $f \in \mathcal{L}^1(m)$ and $\epsilon > 0$, then we can find a continuous function g, vanishing off a bounded interval, such that $\|f-g\|_1 < \epsilon/3$. Suppose that $y_n \to y$. By the above, there is a N such that $n \geq N$ implies $\|\lambda_{y_n}(g) - g\|_1 < \epsilon/3$. Thus if $n \geq N$, we have

$$\begin{aligned}\|\lambda_{y_n}(f) - f\|_1 &\leq \|\lambda_{y_n}(f-g)\|_1 + \|\lambda_{y_n}(g) - g\|_1 + \|g - f\|_1 \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.\end{aligned}$$

This suffices.

(Hint: in part (a) start with characteristic functions. In part (b), you can reduce to the case where $y = 0$, and the conclusion is not so hard if f is continuous and vanishes off a bounded interval.)

3. Recall that if X is a topological space, then $\mathfrak{B}(X)$ is the σ-algebra of Borel sets in X. Show that $\mathfrak{B}(\mathbf{R}^2) = \mathfrak{B}(\mathbf{R}) \otimes \mathfrak{B}(\mathbf{R})$.

ANS: Let $\mathfrak{N} = \{\, E \subset \mathbf{R} : E \times X \in \mathfrak{B}(\mathbf{R}^2)\,\}$. Since $\mathfrak{B}(\mathbf{R}^2)$ is a σ-algebra, so is $\mathfrak{N}$. Clearly, $\mathfrak{N}$ contains all open sets. Hence $\mathfrak{N} \supset \mathfrak{B}(\mathbf{R})$ and $E \times \mathbf{R} \in B(\mathbf{R}^2)$ for all Borel sets E. By symmetry, $\mathbf{R} \times F \in \mathfrak{B}(\mathbf{R}^2)$ for all Borel sets F. Hence $\mathfrak{B}(\mathbf{R}^2)$ contains every Borel rectangle $E \times F = (E \times \mathbf{R}) \cap (\mathbf{R} \times F)$, and we must have $\mathfrak{B}(\mathbf{R}) \otimes \mathfrak{B}(\mathbf{R}) \subset \mathfrak{B}(\mathbf{R}^2)$. (So far, our arguments would work with any topological space in place of $\mathbf{R}$.)

But the open rectangles are a base for the (product) topology on $\mathbf{R}^2$ and since the topology is second countable, every open set $V \subset \mathbf{R}^2$ is a *countable* union of open rectangles. But every open rectangle is in $\mathfrak{B}(\mathbf{R}) \otimes \mathfrak{B}(\mathbf{R})$ so, $\mathfrak{B}(\mathbf{R}) \otimes \mathfrak{B}(\mathbf{R})$ contains every open set. But then $\mathfrak{B}(\mathbf{R}^2) \subset \mathfrak{B}(\mathbf{R}) \otimes \mathfrak{B}(\mathbf{R})$. This completes the proof.

4. Suppose that f and g are functions from $\mathbf{R}$ to $\mathbf{R}$ with f Lebesgue measurable and g Borel. Which of $g \circ f$ and $f \circ g$ must be Lebesgue measurable? Why? (You need not deal with the other case.)

ANS: Recall that f Lebesgue measurable just means $f^{-1}(V) \in \mathfrak{M}$ for all open sets $V \subset \mathbf{R}$. This implies that $f^{-1}(B) \in \mathfrak{M}$ for all Borel sets $B \in \mathfrak{B}(\mathbf{R})$. Saying that g is Borel means that $g^{-1}(V) \in \mathfrak{B}(\mathbf{R})$ for all open sets V. But if V is open in $\mathbf{R}$, then $(g \circ f)^{-1}(V) = f^{-1}\big(g^{-1}(V)\big)$. Since $g^{-1}(V) \in \mathfrak{B}(\mathbf{R})$, we have $(g \circ f)^{-1}(V) \in \mathfrak{M}$. That is, $g \circ f$ is always Lebesgue measurable.

5. Suppose that $f \in \mathcal{L}^1(\mathbf{R}, \mathfrak{M}, m)$ and that f is also continuous. Is it necessarily true that $\lim_{x\to\infty} f(x) = 0$?

ANS: No. Note that by the Monotone Convergence Theorem and the fact that the Riemann integral coincides with the Lebesgue integral on bounded sets, a non-negative improperly Riemann integrable function is in $\mathcal{L}^1$. Now draw a picture with little triangles of width 4^{-n} and height 2^n centered at each point $n \geq 1$.

6. Carefully state the Monotone Convergence Theorem and Fatou's Lemma for non-negative functions. What happens if drop the hypothesis that each $f_n \geq 0$? Justify your assertions.

ANS: Both are false w/o the positivity assumption. Consider $f_n \equiv -\frac{1}{n}$.

7. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is Lebesgue measurable.

(a) Show that $F : (\mathbf{R}^2, \mathfrak{M} \otimes \mathfrak{M}) \to (\mathbf{R}^2, \mathfrak{B}(\mathbf{R}^2))$ given by $F(x, y) = \big(f(x), y\big)$ is measurable. (This just means that $F^{-1}(V) \in \mathfrak{M} \otimes \mathfrak{M}$ when V is open in $\mathbf{R}^2$.)

ANS: Since $\mathbf{R}^2$ is second countable, it suffices to check that $F^{-1}(V) \in \mathfrak{M} \otimes \mathfrak{M}$ for any V in a countable basis for the topology on $\mathbf{R}^2$. But $\mathbf{R}^2$ has a countable basis of open rectangles of the form $U \times V$ where U and V are open intervals (with, say, rational endpoints). But $F^{-1}(U \times V) = f^{-1}(U) \times V \in \mathfrak{M} \otimes \mathfrak{M}$. Hence F is measurable.

(b) Show that

$$G(f) = \{\, \big(x, f(x)\big) \in \mathbf{R}^2 : x \in \mathbf{R} \,\}$$

is in $\mathfrak{M} \otimes \mathfrak{M}$.

ANS: Let $\Delta = \{\, (x, x) \in \mathbf{R}^2 : x \in \mathbf{R} \,\}$. Then Δ is closed (and hence Borel). Thus if F is as in part (a), $F^{-1}(\Delta) \in \mathfrak{M} \otimes \mathfrak{M}$. But $G(f) = F^{-1}(\Delta)$.

(c) Show that for almost all y,

$$m\big(\{\, x \in \mathbf{R} : f(x) = y \,\}\big) = 0.$$

ANS: By part (b), $\mathbb{I}_{G(f)}$ is a $\mathfrak{M} \times \mathfrak{M}$-measurable function on $\mathbf{R}^2$. Applying Tonelli's Theorem, we have

$$m \times m(G(f)) = \int_{\mathbf{R}} m(G(f)_x)\, dm(x) = \int_{\mathbf{R}} m(G(f)^y)\, dm(y). \tag{2}$$

But $G(f)_x = \{f(x)\}$ has Lebesgue measure zero for all x. It then follows from (2) that

$$G(f)^y = \{\, x \in \mathbf{R} : f(x) = y \,\}$$

has Lebesgue measure zero for m-almost all y.

(Hint: all these parts are connected. Any if you were to use something like Tonelli or Fubini's Theorem, you should carefully explain how.)