1. (12) Complete the following ***definitions***.

   (a) A sequence $\{p_n\}_{n=1}^{\infty}$ in a metric space (E, d) is called *Cauchy* if ...

   ANS: for all $\epsilon > 0$ there is a $N \in \mathbf{N}$ such that $n, m \geq N$ implies $d(p_n, p_m) < \epsilon$.

   (b) A point p in a metric space E is called a *cluster point* of a subset S of E if ...

   ANS: for all $r > 0$, the open ball $D_r(p)$ in E contains infinitely many points of S.

   (c) A subset S of a metric space E is called *compact* if ...

   ANS: every open cover of S has a finite subcover.

2. (5) Show that in a metric space (E, d), every Cauchy sequence is bounded.

   ANS: Let $\{p_n\}_{n=1}^{\infty}$ be a Cauchy sequence in E. We need to see that the set $S = \{p_1, p_2, \cdots\}$ is contained in some ball. Let $N \in \mathbf{N}$ be such that $n, m \geq N$ implies $d(p_n, p_m) < 1$. Let $M = \max\{d(p_1, p_N), \ldots, d(p_{N-1}, p_N), 1\} + 1$. Then $S = \{p_1, p_2, \cdots\} \subset D_M(p_N)$.

3. (18) For each statement, circle either **T** for "True" or **F** for "False". Give a short reason or example to justify your answer.

   T F (a) The intersection of a family of compact subsets of a metric space E is compact.

   ANS: This is true. Compact sets are closed as is the intersection of closed sets. Hence the intersection is a closed subset of any of the sets in the collection and a closed subset of a compact set is compact.

   T F (b) The union of a family of compact subsets of a metric space E is compact.

   ANS: This is false. Any set is the union of its points, each of which is compact. Since we showed $(0, 1)$ is not compact, this is a non-compact union of compact sets.

   T F (c) Let $\{F_i\}_{i=1}^{\infty}$ be a collection of closed nonempty subsets of $\mathbf{R}^n$. Suppose that $F_{i+1} \subset F_i$ for all i and that F_1 is bounded. Then

   $$\bigcap_{i=1}^{\infty} F_i \neq \emptyset.$$

   ANS: This is true. A closed bounded subset of $\mathbf{R}^n$ is compact. Thus the F_i are nested compact subsets in the compact space F_1. Hence the intersection is nonempty by the Nested Sets Theorem.

T **F** (d) Let S be an infinite subset of a complete metric space (E, d). Then S has a cluster point in E.

ANS: This is false. The infinite set $\mathbf{N}$ in $\mathbf{R}$ has no cluster points.

T **F** (e) A closed and bounded subset of any metric space is compact.

ANS: This is false. Consider an infinite set E with the discrete metric. It is not covered by finitely many closed 1/2-balls.

Alternatively, consider $S = E = (0, 1)$ with the usual metric. Then S is closed and bounded in E, but it is not compact.

T **F** (f) A complete subspace of any metric space is closed.

ANS: This is true. Let S be a complete subspace of E. Suppose that $\{p_n\}$ is a sequence in S converging to $p \in E$. Then $\{p_n\}$ is Cauchy and must converge in S. Hence $p \in S$ and S is closed.

Math 63 — Exam I — Take Home Portion

Problems #1 through #3 are to be completed in class on Tuesday. The remaining problems are to be turned in at the beginning of class on Friday. Your solutions are to be fully justified and *neatly* written on *one side only* of $8\frac{1}{2}'' \times 11''$ paper with smooth edges and stapled in the upper left-hand corner. Please start each problem on a separate page.

On the take-home portion only, you may refer to your text and class notes. Otherwise, you are to work alone and neither receive assistance from nor provide assistance to anyone else.

COMMENTS: Problems #8 and #7 are a little more challenging than the others. I am not expecting everyone, or maybe anyone, to get both of them completely correct. Also, remember than on an "if and only if" problem, you can get a lot of credit for doing half the problem correctly even if you can't get the other half.

4. (8) Let S be a subspace of a metric space (E, d), and let U be a subset of S. Give a proof of the following assertion used in lecture: U is open in S if and only if there is an open set V in E such that $U = S \cap V$.[1]

ANS: Suppose V is open in E and $U = S \cap V$. Let $a \in U$. Then $a \in V$ and there is an $r > 0$ such that $D^E_r(a) = \{\, p \in E : d(p, a) < r \,\}$ is contained in V. But $D^E_r(a) \cap S = D^S_r(p) = \{\, p \in S : d(p, a) < r \,\}$. Hence $D^S_r(p) \subset U$, and U is open in S.

[1]This is straightforward if $U = D^S_r(a) := \{\, p \in S : d(p, a) < r \,\}$.

Now assume U is open in S. For each $a \in U$, there is a $r_a > 0$ such that $D^S(a) \subset U$. Then

$$U = \bigcup_{a \in U} D^S_{r_a}(a).$$

Let

$$V := \bigcup_{a \in U} D^E_{r_a}(a).$$

Then V is open in E, and

$$V \cap E = \bigcup_{a \in U} D^E_{r_a}(a) \cap S = \bigcup_{a \in U} D^S_{r_a}(a) = U$$

as required.

5. (8) Recall that a metric space E is called *sequentially compact* if every sequence in E has a convergent subsequence. Prove that E is sequentially compact if and only if every infinite subset of E has a cluster point.[2]

ANS: Suppose that E is sequentially compact. If S is an infinite subset, then proceeding as on page 11 of the text, there is a sequence $\{p_n\}_{n=1}^{\infty}$ in S such that $p_i \neq p_j$ if $i \neq j$. By assumption, $\{p_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{p_{n_k}\}_{k=1}^{\infty}$. Let p be the limit of this subsequence. It will suffice to see that p is a cluster point of S. Let $r > 0$, Then there is a $N \in \mathbf{N}$ such that $k \geq N$ implies $p_{n_k} \in D_r(p)$. Since the p_{n_k} are all distinct, there are infinitely many points of S in $D_r(p)$. Since r was arbitrary, p is a cluster point.

Now, conversely, assume that every infinite set S in E has a cluster point in E. Let $\{p_n\}_{n=1}^{\infty}$. Now the proof proceeds exactly as in the proof of Corollary 1 on page 56 of the text.

6. (10) (a) Suppose that $\{a_n\}_{n=1}^{\infty}$ be a sequence in $\mathbf{R}^2$. Let $a_n = (b_n, c_n)$ and $a = (b, c)$. Show that $a_n \to a$ in $\mathbf{R}^2$ if and only if $b_n \to b$ and $c_n \to c$ in $\mathbf{R}$.

(b) Show that $[a, b] \times [a, b]$ is sequentially compact.

ANS: (a) Note that $\|(x, y)\| = \sqrt{x^2 + y^2} \leq |x|$. Suppose that $a_n \to a$. Let $\epsilon > 0$. Then there is an $N \in \mathbf{N}$ such that $n \geq N$ implies $\|a_n - a\| < \epsilon$. But then by our observation above, $|b_n - b| < \epsilon$. Since $\epsilon > 0$ was arbitrary, we've shown $b_n \to b$. The proof that $c_n \to c$ is similar.

Now suppose that $b_n \to b$ and $c_n \to c$. Let $\epsilon > 0$. Find $N_1 \in \mathbf{N}$ such that $n \geq N_1$ implies $|b_n - b| < \epsilon/\sqrt{2}$. Similarly, let $N_2 \in \mathbf{N}$ be such that $n \geq N_2$ implies $|c_n - c| < \epsilon/\sqrt{2}$. Let $N = \max\{N_1, N_2\}$. Then if $n \geq N$, we have

$$\|a_n - a\| = \sqrt{(b_n - b)^2 + (c_n - c)^2} < \sqrt{\frac{\epsilon^2}{2} + \frac{\epsilon^2}{2}} = \epsilon.$$

Hence $a_n \to a$.

[2]The proof of Corollary 1 on page 56 might be useful.

(b) Now assume that $\{a_n\} = \{(b_n, c_n)\}$ is a sequence in $[a, b]^2$. Since $\{b_n\}$ is a sequence in $[a, b]$ and since $[a, b]$ is compact, there is a subsequence $\{\, b_{n_k} \,\}$ such that $b_{n_k} \to d$ in $[a, b]$. But $\{c_{n_k}\}$ is a also a sequence in $[a, b]$. Hence it has a convergent subsequence $\{c_{n_{k_l}}\}$ say converging to e in $[a, b]$. But then, using part (a), $\{a_{n_{k_l}}\}$ converges to (d, e). But $\{a_{n_{k_l}}\}$ is a subsequence of $\{a_n\}$, and we're done.

7. (7) A metric space is called *totally bounded* if for every $\epsilon > 0$, E is the union of finitely many closed ϵ-balls. Show that E is totally bounded if and only if every sequence in E has a Cauchy subsequence.

ANS: Suppose that E is totally bounded and that $\{p_n\}_{n=1}^{\infty}$ is a sequence in E. Since E is the union of finitely many closed 1-balls, there is a closed 1-ball B_1 containing infinitely many p_n. Choose n_1 such that $p_{n_1} \in B_1$. Since E is the union of finitely many closed 1/2-balls, so is B_1. Hence there is a closed 1/2-ball B_2 such that $B_1 \cap B_2$ contains infinitely many p_n. Choose $n_2 > n_1$ such that $p_{n_2} \in B_1 \cap B_2$. Continuing in this way, we get a subsequence $\{p_{n_k}\}_{k=1}^{\infty}$ and closed $\frac{1}{k}$-balls B_k such that

$$p_{n_k} \in B_1 \cap \cdots \cap B_k.$$

In particular, we have $p_{n_k} \in B_m$ for all $k \geq m$. It will suffice to see that $\{p_{n_k}\}$ is Cauchy.

Let $\epsilon > 0$ and choose N such that $\frac{1}{N} < \frac{\epsilon}{2}$. Let q_N be the center of the closed $\frac{1}{N}$-ball B_N. Then if $k, m \geq N$, we have p_{n_k} and p_{n_m} in B_N. Hence

$$d(p_{n_k}, p_{m_k}) \leq d(p_{n_k}, q_N) + d(q_N, p_{n_m}) \leq \frac{2}{N} < \epsilon.$$

Hence $\{p_{n_k}\}$ is Cauchy. This completes the first half of the proof.

Conversely, suppose every sequence in E has a Cauchy subsequence. Suppose to the contrary that E is not totally bounded. Then there is an $\epsilon > 0$ such that no finite collection of closed ϵ-balls covers E. For any $p \in E$, let $B(p) = \{\, q \in E : d(q, p) \leq \epsilon \,\}$ be the closed ϵ-ball centered at p. If $p_1 \in E$, then $E \neq B(p_1)$, so there is a $p_2 \in E \setminus B_1$. But E can't be the union $B(p_1) \cup B(p_2)$ of the two closed ϵ-balls, hence there is a p_3 in the complement of $B(p_1) \cup B(p_2)$. Continuing in this way we get a sequence $\{p_n\}_{n=1}^{\infty}$ such that

$$p_{n+1} \notin \bigcup_{i=1}^{n} B(p_i).$$

By assumption, $\{p_n\}_{n=1}^{\infty}$ has a Cauchy subsequence $\{p_{n_k}\}_{k=1}^{\infty}$. Choose $N \in \mathbf{N}$ such that $k, m \geq N$ implies that $d(p_{n_k}, p_{n_m}) < \epsilon$. Note that this means

$$p_{n_{N+1}} \in B(p_N).$$

But at the same time $n_{N+1} > N$ implies $p_{n_{N+1}} \notin B(p_N)$. This is a contradiction and finishes the proof.

8. (7) Show that a metric space that is both totally bounded and complete is compact.[3]

ANS: Suppose to the contrary that $\mathcal{U} = \{U_i\}$ of E by open sets with no finite cover. Since E can be covered by finitely many closed 1 balls, there is one, say S_1 that can't be covered by finitely many U_i. But we can cover S_1 by finitely many closed 1/2-balls (since E has such a cover), so there is a closed 1/2-ball B such that $S_2 = B \cap S$ can't be covered by finitely many U_i. Note that if $p, q \in S_2$ and a is the center of B, then $d(p,q) \leq d(p,a) + d(a,q) \leq 1$.

Continuing in this way we get closed subsets nonempty sets S_n such that $S_{n+1} \subset S_n$, S_n is contained in a closed $1/n$-ball with center a_n, *and S_n is not covered by finitely many U_i*. Note that if $p, q \in S_n$, then $d(p,q) \leq d(p,a_n) + d(a_n,q) < 2/n$.

Chose $p_n \in S_n$. I claim that $\{p_n\}$ is Cauchy. Let $\epsilon > 0$ and find $N \in \mathbf{N}$ such that $2/N < \epsilon$. Then if $n, m \geq N$, we have p_n and p_m in S_N. Thus by the above, $d(p_n, p_m) \leq 2/N < \epsilon$ as required.

Since E is complete, there is a $p \in E$ such that $p_n \to p$. Then $p \in U_{i_0}$ for some i_0. Since U_{i_0} is open, there is an $r > 0$ such that $D_r(p) \subset U_{i_0}$. Let N be such that $4/N < r$ and such that $d(p_N, p) < r/2$. Suppose that $q \in S_N$, then

$$d(q,p) \leq d(q,p_N) + d(p_N,p) < 2/N + r/2 < r/2 + r/2 = r.$$

But this shows that $S_N \subset D_r(p) \subset U_{i_0}$. This contradicts the fact S_n is not contained in any finite collection of U_i.

Hence E must be compact as claimed.

[3] I suggest you mimic the idea of the proof of the theorem on pages 55-56 of the text: Assume that there is a cover $\mathcal{U} = \{U_i\}$ of E by open sets with no finite cover. Since E is the union of closed balls of radius 1, there is a closed 1-ball not covered by finitely many U_i. But this ball is covered by finitely many closed 1/2-balls, etc.

NAME : ______________________

Math 63

29 January 2013
Dana P Williams

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Problem	Points	Score
1	12	
2	5	
3	18	
4	8	
5	8	
6	10	
7	7	
8	7	
Total	75	