Math 11 Section 3
Wednesday, September 24, 2008

Example 1: A sphere in $\mathbb{R}^3$ has equation

$$x^2 + 2x + y^2 - 4y + z^2 = 20.$$

Find the center and radius of the sphere.

Solution:

Complete the square:

$$x^2 + 2x + 1 + y_2 - 4y + 4 + z^2 = 20 + 1 + 4$$

$$(x + 1)^2 + (y - 2)^2 + z^2 = 25$$

$$(x - (-1))^2 + (y - 2)^2 + (z - 0)^2 = 5^2.$$

On the left we see the square of the distance between the point $(-1, 2, 0)$ and the point (x, y, z), and on the right we see the square of 5. Hence the solution consists of all points (x, y, z) whose distance from $(-1, 2, 0)$ is 5.

The center of the sphere is $(-1, 2, 0)$ and its radius is 5.

Example 2: We may identify[1] a point, such as $(3, 2, 4)$, with the *position vector* of that point, $\langle 3, 2, 4\rangle$.[2] For example, we say that if we add the displacement of an object to its initial position, we get its final position:

$$\vec{p}_{init} + \vec{d} = \vec{p}_{fin}.$$

A moving object starts at the point $(3, 2, 4)$ and moves with constant velocity $(1, 2, 2)$. Units are in meters and seconds.

(a.) The speed of the object is the length (or norm) of the velocity vector, and the direction of motion of the object is a unit vector (a vector of length 1) with the same direction as the velocity vector. Find the speed and direction of motion of the object.

Solution:

[1]When mathematicians say we "identify" two things, we mean we consider them to be the same.

[2]Actually, the way I think of it is this: The vector is the triple of numbers $(3, 2, 4)$. A point and an arrow are two different geometric representations, or pictures, of the vector.

The speed is the length of the velocity vector,

$$|\langle 1,2,2\rangle| = \sqrt{1^2+2^2+2^2} = 3,$$

or 3 meters per second.

The direction is a unit vector in the direction of the velocity vector $\vec{v} = \langle 1,2,2\rangle$. Since $\vec{v}$ has length 3, we can get a unit vector in the same direction by multiplying $\vec{v}$ by the scalar $\frac{1}{3}$. The direction of motion is

$$\vec{u} = \frac{1}{|\vec{v}|}\,\vec{v} = \frac{1}{3}\langle 1,2,2\rangle = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right\rangle.$$

(b.) What is the object's displacement after 1 second? After 4 seconds? After t seconds?

Solution:

In one second the object, whose speed is 3 meters per second, moves 3 meters in the direction of the vector $\vec{u} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right\rangle$, so its displacement is

$$3\vec{u} = \langle 1,2,2\rangle.$$

Notice that this is numerically the same as the velocity vector $\vec{v}$. Its units are meters rather than meters per second. We can view the displacement over one second as the velocity times the elapsed time, or 1 second times $\vec{v}$ meters per second.

In 4 seconds the object (moving at constant velocity) has 4 times the displacement, or

$$\langle 4,8,8\rangle.$$

Numerically, this is $4\vec{v}$.

In t seconds the object has t times the displacement, or

$$\langle t, 2t, 2t\rangle,$$

which numerically is $t\vec{v}$.

(c.) What is the object's position after 1 second? After 4 seconds? After t seconds?

Solution:

We get final position by adding displacement to initial position, so the position after 1, 4, and t seconds respectively is

$$(3, 2, 4) + \langle 1, 2, 2\rangle = (4, 4, 6),$$

$$(3, 2, 4) + \langle 4, 8, 8\rangle = (7, 10, 12),$$
$$(3, 2, 4) + \langle t, 2t, 2t\rangle = (3 + t,\ 2 + 2t,\ 4 + 2t).$$

Example 2 continued:
(extra challenges)

(d.) A moving object starts at the point $\vec{p}_{init}$ and moves with constant velocity $\vec{v}$. What is its position after t seconds?

Solution:

Using the same reasoning as before, we see the object's displacement after t seconds is $t\vec{v}$, so its position after t seconds is

$$\vec{p} = \vec{p}_{init} + t\vec{v}.$$

Thinking of position as a function of time, we may write this as

$$\vec{p}(t) = \vec{p}_{init} + t\vec{v}.$$

We will see a lot of functions like this later.

(e.) A moving object starts at the point $(3, 2, 4)$ and moves so that its velocity t seconds after it begins to move is $(t, 2t, 2t)$. What is the object's position after it has been moving for 2 seconds?

Solution:
If the object's velocity at time t is $\vec{v} = \langle t, 2t, 2t, \rangle$, then its speed is

$$|\vec{v}| = |\,\langle t, 2t, 2t\rangle\,| = 3t$$

and its direction of motion is

$$\frac{1}{|\vec{v}|}\,\vec{v} = \left\langle \frac{1}{3}, \frac{2}{3}\ \frac{2}{3} \right\rangle.$$

We can find the distance the object travels over the first two seconds by integrating the speed:

$$\int_0^2 3t\,dt = \left.\frac{3t^2}{2}\right|_{t=0}^{t=2} = 6.$$

Since the object is always traveling in the same direction, it travels 6 meters in the direction of the unit[3] vector $\left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$, so its displacement is

$$6\left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle = \langle 2, 4, 4 \rangle\,.$$

Hence its position after 2 seconds is

$$(3, 2, 4) + (2, 4, 4) = (5, 6, 8).$$

[3] It's important to notice that this is a vector of length 1.