Supplementary Homework for Math 43

In our *proof* of the Cauchy-Goursat Theorem we needed to show that the intersection $\bigcap \Delta_n$ was a single point. Here's an outline of a proof.

If F is a closed and bounded subset of $\mathbf{C}$, then we define the *diameter* of F by

$$\operatorname{dia} F := \max\{\, |z-w| : z, w \in F\,\}.$$

Recall that if $z_0 = x_0 + iy_0$ and $a > 0$, then

$$N_a(z_0) = \{\, z \in \mathbf{C} : |z - z_0| < a\,\}$$

is the ball centered at z_0. We also define

$$B_a(z_0) := \{\, x + iy \in \mathbf{C} : |x - x_0| \le a \text{ and } |y - y_0| \le a\,\}$$

to be the closed box centered at z_0.

Supplementary problem 1: Let $F_1 \supset F_2 \supset F_3 \supset \dots$ be a nested sequence of nonempty closed and bounded subsets of $\mathbf{C}$ with $\operatorname{dia} F_n \le d_n := 2^{-n}L$ for some constant L. Then there is a unique point common to all the F_n. (That is, the intersection $\bigcap F_n$ consists of a single point.)

I suggest you proceed as follows.

1. Let $z_n \in F_n$, and set $B_n := B_{2d_n}(z_n)$. Show that $F_n \subset B_n$ and that $B_{n+1} \subset B_n$.

2. Use problem 10 on page 122 of our text (see problem 11 too), to show that there is a unique point $z = x + iy$ common to all the boxes B_n.

3. If $N_\epsilon(z)$ is a neighborhood of z and if $2^{-n}L < \epsilon$, then $F_n \subset B_n \subset N_\epsilon(z)$.

4. Conclude that z is an accumlation point of each F_n, and, since each F_n is closed, that $z \in F_n$.