# Lecture Notes

Math 3 / Fall 2012

Week 7

# Monotonicity

Suppose $f$ is defined on some interval $I$ (of any kind)

- We say that $f$ is **increasing on** $I$ when

$$a < b \Rightarrow f(a) < f(b) \qquad \text{for all } a, b \text{ in } I$$

- We say that $f$ is **decreasing on** $I$ when

$$a < b \Rightarrow f(a) > f(b) \qquad \text{for all } a, b \text{ in } I$$

- We say that $f$ is **nondecreasing on** $I$ when

$$a < b \Rightarrow f(a) \leq f(b) \qquad \text{for all } a, b \text{ in } I$$

- We say that $f$ is **nonincreasing on** $I$ when

$$a < b \Rightarrow f(a) \geq f(b) \qquad \text{for all } a, b \text{ in } I$$

# Monotonicity and derivatives

Suppose the function $f$ is continuous on the interval $[a, b]$, and differentiable on $(a, b)$.

- If $f'(x) > 0$ on $(a, b)$ then $f$ is increasing on $[a, b]$.
- If $f'(x) < 0$ on $(a, b)$ then $f$ is decreasing on $[a, b]$.
- If $f'(x) \geq 0$ on $(a, b)$ then $f$ is nondecreasing on $[a, b]$.
- If $f'(x) \leq 0$ on $(a, b)$ then $f$ is nonincreasing on $[a, b]$.

# Extrema and derivatives

- If $f$ changes from increasing to decreasing at $x$

  then $f$ has a local maximum at $x$
- If $f$ changes from decreasing to increasing at $x$

  then $f$ has a local minimum at $x$

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- If $f'$ changes from positive to negative at $x$

  then $f$ has a local maximum at $x$
- If $f'$ changes from negative to positive at $x$

  then $f$ has a local minimum at $x$

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- If $f'(x) = 0$ and $f''(x) < 0$

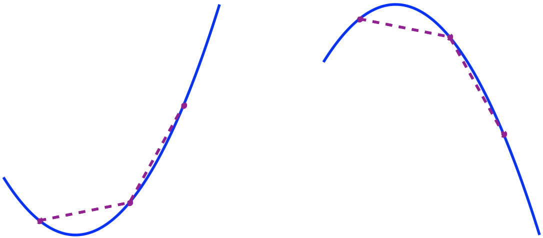
  then $f$ has a local maximum at $x$
- If $f'(x) = 0$ and $f''(x) > 0$

  then $f$ has a local minimum at $x$

# Concavity

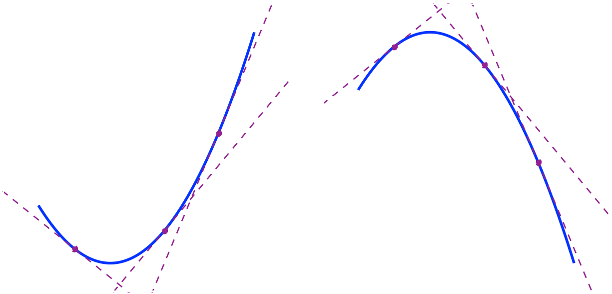
Suppose $f$ is defined on some interval $I$ (of any kind)

- We say that $f$ is **concave up on** $I$ if whenever we pick $a, b$ from $I$ the graph of $f$ on $[a, b]$ lies below the line segment joining $(a, f(a))$ to $(b, f(b))$
- We say that $f$ is **concave down on** $I$ if whenever we pick $a, b$ from $I$ the graph of $f$ on $[a, b]$ lies above the line segment joining $(a, f(a))$ to $(b, f(b))$



# Concavity and derivatives

- If $f'$ is increasing on $(a, b)$ then $f$ is concave up on $[a, b]$
- If $f'$ is decreasing on $(a, b)$ then $f$ is concave down on $[a, b]$



- If $f''$ is positive on $(a, b)$ then $f$ is concave up on $[a, b]$
- If $f''$ is negative on $(a, b)$ then $f$ is concave down on $[a, b]$

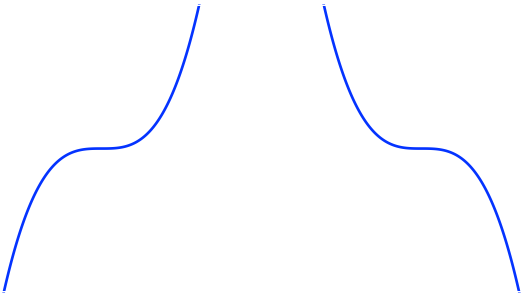
# Inflection points

Inflection points occur where a function $f$ changes concavity.
These occur where ...

... $f'$ has a local extremum

... $f''$ changes sign

... $f''(x) = 0$ and $f'''(x) \neq 0$



# Sketching functions

Sketch the following functions and identify <u>all</u> the salient features.

1. $2x^3 + x^2 - 20x$
2. $1 - x^4$
3. $x + \dfrac{1}{x}$
4. $\dfrac{(x-2)^2}{x+1}$
5. $\dfrac{x}{x^2+1}$
6. $\dfrac{x^2-1}{x^2+1}$
7. $\dfrac{x^2+1}{x^2-1}$

Checklist:

- Identify the domain of $f$
- Identify the roots of $f$
- Identify vertical/horizontal asymptotes of $f$
- Where is $f$ increasing/decreasing?
- Identify local extrema of $f$
- Where is $f$ concave up/down?
- Identify inflection points of $f$
- Any other exceptional behavior?

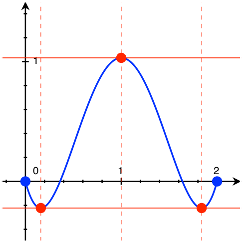
# Extreme value theorem

## Theorem

*Suppose $f$ is continuous on the* <u>*closed*</u> *interval $[a, b]$. Then there are numbers $x_{\min}$ and $x_{\max}$ in $[a, b]$ such that*

$$f(x_{\min}) \le f(x) \le f(x_{\max})$$

*for all numbers $x$ in $[a, b]$.*



# Fixed perimeter, maximal area

What are the largest possible area of a rectangle with an perimeter of $10\,\mathrm{m}^2$?

If $x, y$ are the side lengths of the rectangle:
Objective: maximize $A = xy$
Constraint: $2x + 2y = 10$
Bounds: $0 \leq x \leq 5$, $0 \leq y \leq 5$

| $x$ | $y$ | $A$ |
|---|---|---|
| 0 | 5 | 0 |
| 5/2 | 5/2 | 25/4 |
| 5 | 0 | 0 |

# Fixed area, minimal perimeter

What is the smallest possible perimeter of a rectangle with an area of $10\,\mathrm{m}^2$?

If $x, y$ are the side lengths of the rectangle:
Objective: minimize $P = 2x + 2y$
Constraint: $xy = 10$
Bounds: $0 < x < \infty$, $0 < y < \infty$

| $x$ | $y$ | $P$ |
|---|---|---|
| $0^+$ | $\infty$ | $\infty$ |
| $\sqrt{10}$ | $\sqrt{10}$ | $4\sqrt{10}$ |
| $\infty$ | $0^+$ | $\infty$ |

# Minimal distance

Find the minimal distance from $y = \sqrt{x}$ to the point $(4, 0)$.

Objective: minimize $d = \sqrt{(x-4)^2 + (y-0)^2}$
Constraint: $y = \sqrt{x}$
Bounds: $0 \le x < \infty$, $0 \le y < \infty$

Use square distance to avoid roots!

| $x$ | $y$ | $d^2$ |
|---|---|---|
| $\infty$ | $\infty$ | $\infty$ |
| $7/2$ | $\sqrt{7/2}$ | $15/4$ |
| $0$ | $0$ | $16$ |

# Inscribed rectangle

What is the largest area rectangle, with sides parallel to the coordinate axes, that can be inscribed inside the ellipse $4x^2 + y^2 = 1$.

If the top right corner has coordinates $(x, y)$:
Objective: maximize $A = 4xy$
Constraint: $4x^2 + y^2 = 1$
Bounds: $0 \leq x \leq 1/2$, $0 \leq y \leq 1$

| $x$ | $y$ | $A$ |
|---|---|---|
| $0$ | $1$ | $0$ |
| $1/\sqrt{8}$ | $1/\sqrt{2}$ | $1$ |
| $1/2$ | $0$ | $0$ |

# Manufacturing cans

To manufacture a cyclindrical can, the amount of tin used consists of three pieces: two squares whose sides are equal to the diameter of the can from which the top and bottom are cut, and one rectangle for the side of the can. What is the maximal can volume that can be made using $40\,\text{in}^2$ of tin.

Objective: minimize $V = \pi r^2 h$
Constraint: $40 = 8r^2 + 2\pi rh$
Bounds: $0 \leq h < \infty$, $0 < r \leq \sqrt{5}$

| $r$ | $h$ | $V$ |
|---|---|---|
| $0^+$ | $\infty$ | 0 |
| $\sqrt{5}$ | 0 | 0 |
| $\sqrt{5/3}$ | 3.29 | 17.21 |

# A preview of integration...

$\boxed{\text{Rate of change}}$ :: $\boxed{\text{Derivatives}}$

$\boxed{\text{Accumulation}}$ :: $\boxed{\text{Integrals}}$

# Accumulating water

Water pours into a tank at a rate of $3\,\text{ml/s}$. How much water is in the tank after one minute?

Easy answer: $3 \times 60 = 180\,\text{ml}$ assuming the tank was empty!

Hard answer: $\displaystyle\int 3\,dt = 3t + C$

So $3 \times 60 + C$ ml, where $C$ is the amount of water already in the tank at $t = 0$.

# Accumulating water

Water pours into a tank at a rate of $2te^{-t^2}$ ml/s. How much water is in the tank after one minute?

Easy answer: ??? ml!

Hard answer: $\int 2te^{-t^2}\,dt = -e^{-t^2} + C$

So $-e^{-3600} + C$ ml, where $-1 + C$ ml is the amount of water already in the tank at $t = 0$.

# Approximate answers

Let's use Euler's Method to evaluate $\int x\,dx$

We start with the initial value problem:

$$y' = x, \qquad y(0) = 0.$$

To approximate $y(1)$ with step size $\Delta x = 0.1$, we have:

| $n$ | 0 | 1 | 2 | 3 | 4 | $\cdots$ | 10 |
|---|---|---|---|---|---|---|---|
| $x_n$ | .0 | .1 | .2 | .3 | .4 | $\cdots$ | 1.0 |
| $y_n$ | .00 | .00 | .01 | .03 | .06 | $\cdots$ | .45 |

# Approximate answers

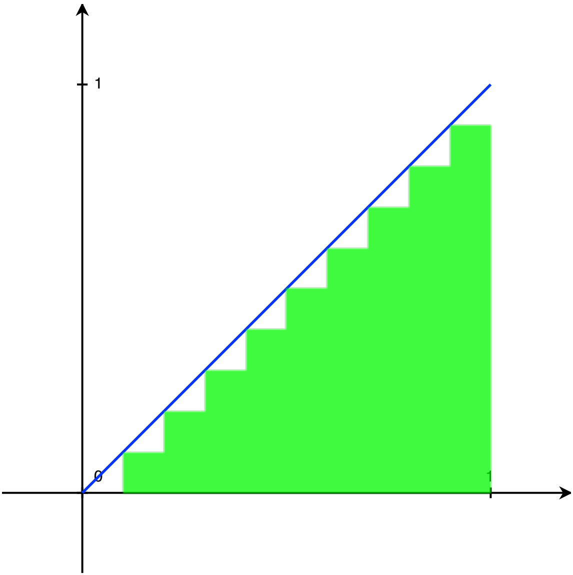
To approximate $y(1)$ with step size $\Delta x = 0.05$, we have:

| $n$ | 0 | 1 | 2 | 3 | 4 | $\cdots$ | 100 |
|---|---|---|---|---|---|---|---|
| $x_n$ | .00 | .05 | .10 | .15 | .20 | $\cdots$ | 1.00 |
| $y_n$ | .0000 | .0000 | .0025 | .0075 | .0150 | $\cdots$ | .4750 |

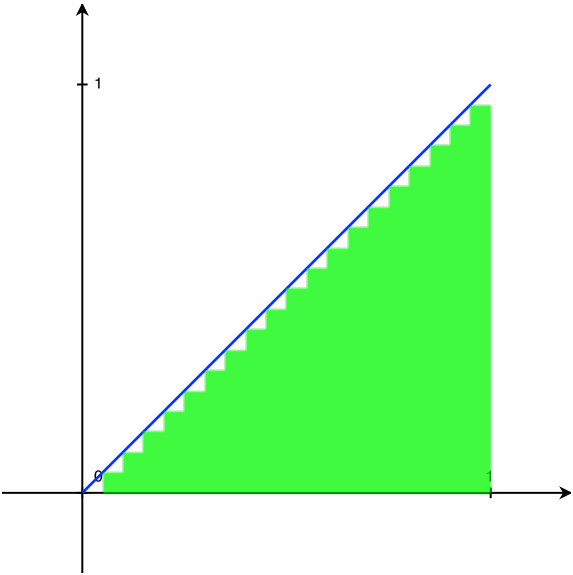
To approximate $y(1)$ with step size $\Delta x = 0.01$, we have:

| $n$ | 0 | 1 | 2 | 3 | 4 | $\cdots$ | 100 |
|---|---|---|---|---|---|---|---|
| $x_n$ | .00 | .01 | .02 | .03 | .04 | $\cdots$ | 1.00 |
| $y_n$ | .0000 | .0000 | .0001 | .0003 | .0006 | $\cdots$ | .4950 |

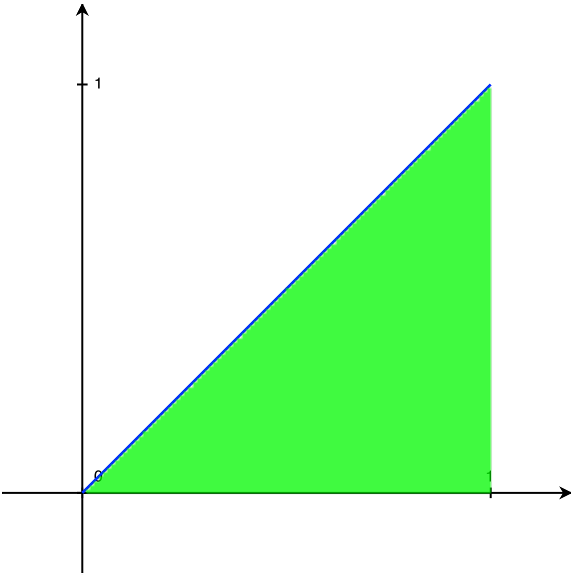
# Visualization $\Delta x = 0.1$



# Visualization $\Delta x = 0.05$



# Visualization $\Delta x = 0.01$



# A fundamental fact...

## Theorem

*If $F'(x) = f(x)$ then $F(b) - F(a)$ is the signed area between the graph $y = f(x)$, the $x$-axis, and the two vertical lines $x = a$ and $x = b$.*