MATH 46 HW5 — SOLUTIONS —

Barnett ①

(2009)

p. 148–150 #(2)

a. by parts

$$\underbrace{\int_\lambda^\infty \tfrac{1}{t}e^{-t}\,dt}_{u\ v'} = -\int_\lambda^\infty \underbrace{-\tfrac{1}{t^2}}_{u'}\underbrace{(-e^{-t})}_{v}\,dt + \underbrace{\left[\tfrac{1}{t}(-e^{-t})\right]_\lambda^\infty}_{uv}$$

by parts

$$= \frac{1}{\lambda}e^{-\lambda} - \left[\frac{1}{t^2}(-e^{-t})\right]_\lambda^\infty + \int_\lambda^\infty \frac{-2}{t^3}(-e^{-t})\,dt$$

by parts

$$\left[\frac{2}{t^3}(-e^{-t})\right]_\lambda^\infty - 2\int_\lambda^\infty \frac{-3}{t^4}(-e^{-t})\,dt$$

$$= \frac{1}{\lambda}e^{-\lambda} - \frac{1}{\lambda^2}e^{-\lambda} + \frac{2}{\lambda^3}e^{-\lambda} - 3!\int_\lambda^\infty \frac{e^{-t}}{t^4}\,dt$$

here $n=3$

in general you can see last term is $(-1)^{n-1}\dfrac{(n-1)!}{\lambda^n}e^{-\lambda}$

$$(-1)^n\, n!\int_\lambda^\infty \frac{e^{-t}}{t^{n+1}}\,dt =: r_n(\lambda)$$

$$= e^{-\lambda}\left[\frac{1}{\lambda} - \frac{1}{\lambda^2} + \frac{2}{\lambda^3} - \frac{3!}{\lambda^4} + \cdots + (-1)^{n-1}\frac{(n-1)!}{\lambda^n}\right] + r_n(\lambda)$$

Some of you also used induction, which is more rigorous.

b. This was easy: since $\left|\int_\lambda^\infty \frac{e^{-t}}{t^{n+1}}\,dt\right| \le \int_\lambda^\infty \frac{1}{t^{n+1}}\,dt = \left[-\frac{1}{n}t^{-n}\right]_\lambda^\infty = \frac{1}{n\lambda^n}$

since $e^{-t} < 1 \quad \forall t > 0$.

c. Needs a more elaborate estimate, change var $t = t' + \lambda$:

$$|r_n(\lambda)| \le n!\,e^{-\lambda}\int_0^\infty \frac{e^{-t'}}{(\lambda+t')^{n+1}}\,dt' \le n!\,e^{-\lambda}\frac{1}{\lambda^{n+1}}\boxed{\int_0^\infty e^{-t'}\,dt'}$$

since $\frac{1}{\lambda+t'} \le \frac{1}{\lambda}$ for $\forall t' > 0$.

Then $|r_n(\lambda)| \le \frac{n!}{n\lambda^n} = \frac{(n-1)!}{\lambda^n} \to 0$ for fixed n.

So $\dfrac{|r_n(\lambda)|}{\underbrace{(n-1)!\,e^{-\lambda}\lambda^{-n}}_{\text{comparison func}}} \le \dfrac{n}{\lambda} \to 0$ for $\lambda \to \infty$ and fixed n, so little-oh as claimed.

The point of this claim is that remainder $r_n(\lambda)$ is then little-oh of <u>last term</u> in expansion, making it a valid <u>asymptotic</u> expansion, see below (2.26) p. 99.

d. ratio between successive terms is $-\frac{n}{\lambda}$, which for <u>any</u> $\lambda > 0$ fixed, becomes eventually greater than '1' in size, leading to divergence (... for roughly $n > \lambda$).

e. Here's code to do what I asked concisely; you can avoid writing a loop using `cumsum` ('cumulative sum' command):

```
n = 1:20;
l = 10;                                  ← λ value
f = -factorial(n-1) ./ power(-l, n);     ← f is now the list -(-1)^n (n-1)!/λ^n
s = exp(-l) * cumsum(f);                 ← s is now successive approx's to Ei(λ).
ei = expint(l)                           ← numerical (matlab's own) 'exact' answer.
figure; semilogy(n, abs(s - ei)/ei, '+-'); xlabel n; ylabel('rel error');
```

minus sign handles $(-1)^n$

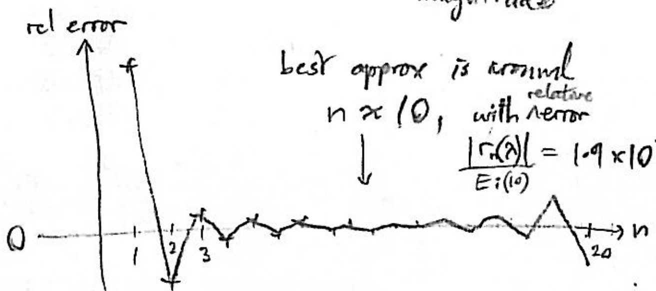
`./` needed since elementwise dividing two lists.

`n`: n values as x-coords.

`abs`: magnitude

`(s - ei)/ei`: relative error as y coords

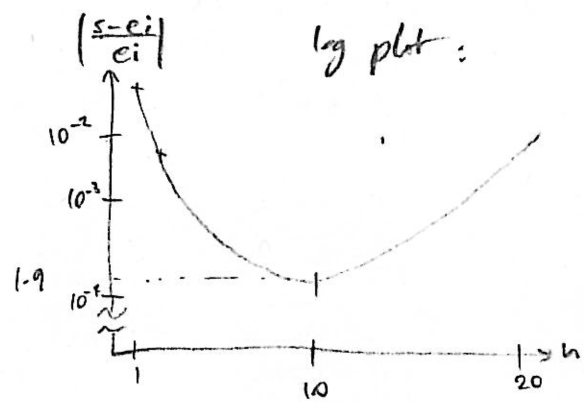
Gives (linear plot)



$Ei(10) = 4.156969 \times 10^{-6}$

$n = 10$ approx $= 4.156165 \times 10^{-6}$

log plot:



(use 'semilogy' command instead of 'plot'.)

A.) a) $\left\{ \frac{1}{\sqrt{2\pi}},\ \frac{1}{\sqrt{\pi}}\sin x,\ \frac{1}{\sqrt{\pi}}\sin 2x,\ \cdots\ ,\ \frac{1}{\sqrt{\pi}}\cos x,\ \frac{1}{\sqrt{\pi}}\cos 2x,\ \cdots \right\}$

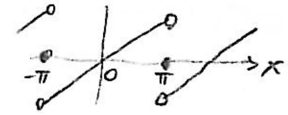
eg. check $\|1\| = \int_{-\pi}^{\pi} 1^2\,dx = 2\pi$ so $\left\|\frac{1}{\sqrt{2\pi}}\right\| = 1$.

b) $f(x) = x$ is odd symmetric so only $\sin nx$, $n = 1, 2, \cdots$ needed.

$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$ with $f_n(x) = \frac{1}{\sqrt{\pi}} \sin nx$

$$c_n = (f, f_n) = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} x \sin nx\, dx = \frac{1}{\sqrt{\pi}}\, 2(-1)^{n+1} \frac{\pi}{n} = \frac{2\sqrt{\pi}(-1)^{n+1}}{n}$$

c) at $x = \pi$ each $\sin nx = 0$ so Fourier converges to zero.



d) Parseval: $\sum_{n=1}^{\infty} c_n^2 = \|f\|^2 = \int_{-\pi}^{\pi} x^2\,dx = \left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2}{3}\pi^3$

$$\frac{4\pi}{1^2} + \frac{4\pi}{2^2} + \frac{4\pi}{3^2} + \cdots$$

Riemann zeta function

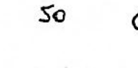
$$\Rightarrow \quad \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\|f\|^2}{4\pi} = \frac{\pi^2}{6} \quad [= \zeta(2)]$$

p.214-215 #1) squared norms of funcs $f_n(x) = \cos\frac{n\pi x}{\ell}$ are $\|f_n\|^2 = \int_0^{\ell} \cos^2\frac{n\pi x}{\ell}\,dx$

[see below for orthog. property].

$$= \begin{cases} \ell, & n = 0 \\ \ell/2, & \text{otherwise.} \end{cases}$$

$$\text{so} \quad c_n = \frac{(f_n, f)}{\|f_n\|^2} = \begin{cases} \frac{1}{\ell}\int_0^{\ell} f(x)\,dx, & n = 0 \\ \frac{2}{\ell}\int_0^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx, & \text{otherwise} \end{cases}$$



Given $f(x) = 1 - x$ we note its coefficients are $(-1)^n$ times those of $f(x) = x$ $\begin{pmatrix}\text{by symm.} \\ x \to 1-x\end{pmatrix}$

so $c_0 = \int_0^1 x\,dx = 1/2$

$\boxed{n>0}$: $c_n = 2(-1)^n \int_0^1 x \cos n\pi x\,dx$ $\xrightarrow{\text{by parts}}$ $-\int_0^1 \frac{\sin n\pi x}{n\pi}\,dx + \left[x\frac{\sin n\pi x}{n\pi}\right]_0^1 = \frac{1}{n^2\pi^2}\left[\cos n\pi x\right]_0^1$

$= \frac{-2}{n^2\pi^2}$ n odd, zero otherwise

ie $c_n = \frac{4}{n^2\pi^2}$ n odd, zero otherwise

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So on $[0,1]$, $1-x = \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n^2\pi^2} \cos n\pi x = \frac{1}{2} + \frac{4}{\pi^2}\left[\cos \pi x + \frac{1}{3^2}\cos 3\pi x + \dots\right]$

Orthog. set?

Addition formula: $\cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l} = \frac{1}{2^2}\left(e^{\frac{in\pi x}{l}} + e^{-\frac{in\pi x}{l}}\right)\left(e^{\frac{im\pi x}{l}} + e^{-\frac{im\pi x}{l}}\right)$

$$= \frac{1}{4}\left(e^{\frac{i(n+m)\pi x}{l}} + \underbrace{c.c.}_{\text{complex conjugate}}\right) + \frac{1}{4}\left(e^{\frac{i(n-m)\pi x}{l}} + c.c.\right)$$

$$= \frac{1}{2}\cos\frac{(n+m)\pi x}{l} + \frac{1}{2}\cos\frac{(n-m)\pi x}{l}$$

Now use $(k \in \mathbb{Z})$ $\int_0^l \cos\frac{k\pi x}{l}\,dx = \begin{cases} l, & k=0 \\ 0, & \text{otherwise} \end{cases}$ using by parts.

Then $(f_n, f_m) = \int_0^l \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,dx = \begin{cases} l & n=m=0 \\ l/2 & n=m\neq 0 \\ 0 & n\neq m \end{cases}$ for $n,m \geq 0$ ← orthog. property.

#3 $q(t) := (f+tg, f+tg) = \|f+tg\|^2 \geq 0$ for all real t
with equality iff $f=-tg$

So discriminant must be either <0 if $f \neq -tg$ for some t
$= 0$ if $f=-tg$ for some t } but positive discriminant cannot happen (ie never 2 real roots).

Expand: $q(t) = \underbrace{\|g\|^2}_{\text{"a"}} t^2 + \underbrace{2(f,g)}_{\text{"b"}} t + \underbrace{\|f\|^2}_{\text{"c"}}$ so discriminant $= \sqrt{b^2-ac} = |(f,g)|^2 - \|g\|^2\|f\|^2 \leq 0$ by above (for $at^2+2bt+c=0$)

so $|(f,g)|^2 \leq \|f\|^2\|g\|^2$ or $|(f,g)| \leq \|f\|\|g\|$ not in vector case ratio of LHS to RHS is $\cos\theta$, angle between.

#5 $f_1 = 1 = g_1$ since $(x,1)=0$ by symmetry
$f_2 = x = g_2$ $g_3 = x^2 - \frac{(x^2,x)}{\|x\|^2}x - \frac{(x^2,1)}{\|1\|^2}1$ since have opposite symmetry, vanishes
$f_3 = x^2$
$f_4 = x^3$ with $\|1\|^2 = \int_{-1}^1 1^2\,dx = 2$
$\int_{-1}^1 x^2 \cdot 1\,dx = \left[\frac{x^3}{3}\right]_{-1}^1 = \frac{2}{3}$

so $g_3(x) = x^2 - \frac{1}{3}$

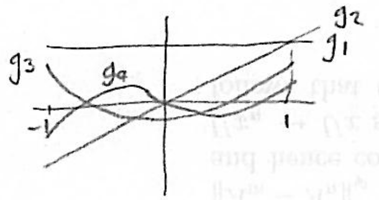
$(f_4, g_2) = (x^3, x) = \int_{-1}^1 x^4\,dx = \frac{2}{5}$
$\|g_2\|^2 = \|x\|^2 = \int_{-1}^1 x^2\,dx = \frac{2}{3}$ } so $g_4(x) = x^3 - \frac{3}{5}x$

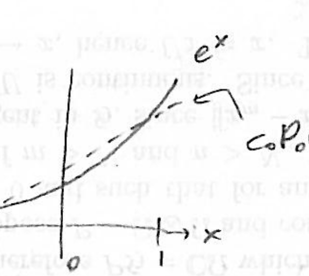
$$\begin{cases} g_1 = 1 & =: P_0 \\ g_2 = x & =: P_1 \\ g_3 = x^2 - \frac{1}{3} & =: P_2 \\ g_4 = x^3 - \frac{3}{5}x & =: P_3 \end{cases}$$

given $f(x) = e^x$

$$c_0 = \frac{(g_1, f)}{\|g_1\|^2} = \frac{\int_{-1}^{1} 1 \cdot e^x dx}{2} = \tfrac{1}{2}[e^x]_{-1}^{1} = \tfrac{1}{2}(e^1 - e^{-1})$$

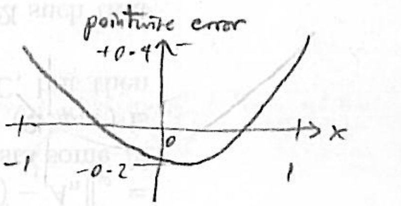
$$c_1 = \frac{(g_2, f)}{\|g_2\|^2} = \frac{\int_{-1}^{1} x e^x dx}{\int_{-1}^{1} x^2 dx} = \frac{3}{2}\left[-\int_{-1}^{1} e^x dx + [xe^x]_{-1}^{1}\right] = 3e^{-1}$$





$$c_0 P_0(x) + c_1 P_1(x) = \frac{e^1 - e^{-1}}{2} + 3e^{-1}x$$

Pointwise error $= f_{\text{approx}} - f = \frac{e^1 - e^{-1}}{2} + 3e^{-1}x - e^x$



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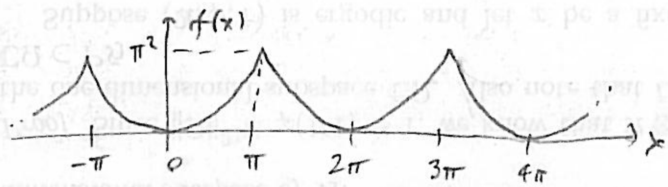


e^{t_c}

Even func. on $[-\pi, \pi]$ so only cos series needed.

$f_n(x) = \cos nx \quad n = 0, 1, 2 \dots$

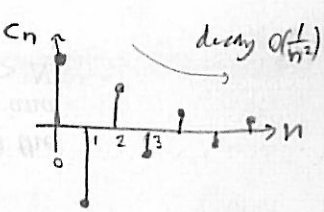
Average $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\frac{2\pi^3}{3} = \frac{\pi^2}{3}$

$n > 0$: $c_n = \underbrace{\frac{1}{\pi}}_{1/\|f_n\|^2}\int_{-\pi}^{\pi} x^2 \cos nx \, dx \longrightarrow -\frac{2}{n}\int x \sin nx \, dx + \frac{1}{n}\left[x^2 \sin nx\right]_{-\pi}^{\pi}$

$$= \frac{2}{n^2}\int(-\cos nx)\,dx - \frac{2}{n^2}\left[x(-\cos nx)\right]_{-\pi}^{\pi}$$

$= 0$ for $n \neq 0$ $\qquad 2\pi(-1)^{n-1}$

$$c_n = \frac{1}{\pi}\cdot\frac{2}{n^2}\cdot 2\pi(-1)^n = \frac{(-1)^n 4}{n^2}$$



So $f(x) = \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \dots\right]$

Choose $x = 0$ so $f(0) = 0^2 = 0$ and $\frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \dots + \frac{(-1)^{n+1}}{n^2} + \dots$

so $\cos nx = 1 \;\forall n$.

p. 224-226 #3 $\quad$ ODE $y'' + \lambda y = 0$ $\quad$ v. familiar gen. solution $\begin{cases} A + Bx, & \lambda = 0 \\ A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x & \lambda > 0 \end{cases}$

BCs $\quad$ values $y(0) = y(l)$ so $A\cos\sqrt{\lambda}l + B\sin\sqrt{\lambda}l = A \cdot 1 + \cancel{B \cdot 0}$

derivs. $y'(0) = y'(l)$ so $-A\sqrt{\lambda}\sin\sqrt{\lambda}l + B\sqrt{\lambda}\cos\sqrt{\lambda}l = B\sqrt{\lambda}$

so nontrivial only if $\sqrt{\lambda}l = 2\pi n$

When $\lambda = \lambda_n = \left(\frac{2n\pi}{l}\right)^2$ both A, B arbitrary $\Rightarrow$ a <u>2-dimensional</u> eigenspace for each λ_n, $n \geq 1$, spanned by $\{\cos\frac{2n\pi x}{l}, \sin\frac{2n\pi x}{l}\}$

When $\lambda = \lambda_0 = 0$, 1-dimensional eigenspace, eigenfunction = const. $\quad$ No negative eigenvalues.

MATH 46 HW 5 — SOLUTIONS (last page)

Barnett

p. 224-226 #4

$\lambda = 0$ eigval? $-y'' = 0$ so $y = Ax + B$ BCs give $y = A(1-x)$
so <u>yes</u>, efunc $y = 1 - x$.

$\lambda < 0$ eigval? Then $y'' - k^2 y = 0$ for some real k. so $y = Ae^{+kx} + Be^{-kx}$

$$\left.\begin{aligned} y(1) = 0 &: \quad Ae^{k} = -Be^{-k} \\ y(0) + y'(0) = 0 &: \quad A + B + Ak - Bk = 0 \end{aligned}\right\} \quad A\frac{1+k}{1-k} = -B = Ae^{2k}$$

We get the trivial soln $A = B = 0$ if $\underbrace{\frac{1+k}{1-k} \neq e^{2k}}$ note many of you didn't prove no such k.
take logs: $\ln(1+k) - \ln(1-k) \neq 2k$

Taylor expand the LHS: $\left(k + \frac{k^2}{2} + \frac{k^3}{3} + \cdots\right) - \left(-k + \frac{k^2}{2} - \frac{k^3}{3} + \cdots\right) = 2\underbrace{\left(k + \frac{k^3}{3} + \frac{k^5}{5} + \cdots\right)}$ radius of convergence

all coeffs > 0 so always > 0 $\forall |k| < 1$
For $k > 1$, LHS < 0 so cannot equal e^{2k}.

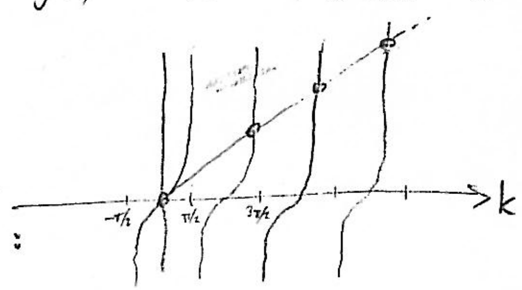
Therefore only trivial soln possible; no negative eigenvalues.

$\lambda > 0$? $y'' + k^2 y = 0$ k real so $y = A\sin kx + B\cos kx$

$y(0) + y'(0)$: $B + kA = 0$ and $y(1) = A\sin k + B\cos k = 0$

so $-\frac{B}{A} = \underbrace{k = \tan k}$ transcendental eqn (no analytic soln)

graphical proof that ∞ many eigvals:



#6 $\lambda = 0$: $y'' = 0$ so $y = Ax + B$

$$\left.\begin{aligned} @x = 0 &: \quad B - aA = 0 \\ @x = l &: \quad Al + B + bA = 0 \end{aligned}\right) \text{sub in.}$$

so $A(l + a + b) = 0$ $\Rightarrow$ $-l = a + b$ $\underset{\text{(iff)}}{\Longleftrightarrow}$ nontrivial soln. ($\lambda = 0$ eigval.)