

Littlewood-Richardson rule

Let μ, ν, λ be partitions such that $\mu, \nu \subset \lambda$. A tableau T of shape λ/μ of weight λ is a filling of μ/λ with ν_1 1's, ν_2 2's, \dots , such that the row filling are weakly increasing and the column fillings are strictly decreasing. We can derive a word $w(T) = a_1 a_2 \dots a_N$, where $N = |\lambda/\mu|$, by reading off the numbers in T rights to left, top to bottom. For example, if

$$\mu = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}, \quad \nu = \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & \square & & \\ \hline \end{array}, \quad \lambda = \begin{array}{|c|c|c|c|c|} \hline \square & \square & \square & \square & \square \\ \hline \square & \square & \square & & \\ \hline \square & \square & & & \\ \hline \end{array},$$

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 3 & & \\ \hline 1 & 3 & & \\ \hline \end{array},$$

has word $w(T) = 2 \ 1 \ 1 \ 1 \ 3 \ 2 \ 3 \ 1$.

A word $w = a_1 a_2 \dots a_N$ is a *lattice permutation* if for each $1 \leq r \leq N$ and $2 \leq i \leq n$, there are no more occurrences of i than that of $i - 1$. So $2 \ 1 \ 1 \ 1 \ 3 \ 2 \ 3 \ 1$ is not a lattice permutation, but $1 \ 2 \ 1 \ 1 \ 3 \ 2 \ 3 \ 1$ is. In the above example, the only tableau of shape λ/μ of weight ν which is also generates a lattice permutation is the one generating the word $1 \ 1 \ 1 \ 1 \ 2 \ 2 \ 3 \ 3$.

Theorem 1 (Littlewood-Richardson rule). *Let s_λ be the Schur function indexed by the partition λ . Then*

$$s_\mu s_\lambda = \sum_{\substack{\mu, \nu \subset \lambda \\ |\lambda| = |\mu| + |\nu|}} c_{\mu, \nu}^\lambda s_\lambda$$

where $c_{\mu, \nu}^\lambda$ is the number of tableau of shape λ/μ of weight ν which generates a lattice permutation.

Example.

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} = \begin{array}{l} 0 \cdot (2, 1, 1, 1, 1) \\ + 1 \cdot (2, 2, 2) \\ + 1 \cdot (4, 1, 1) \end{array} + \begin{array}{l} + 1 \cdot (2, 2, 1, 1) \\ + 2 \cdot (3, 2, 1) \\ + 1 \cdot (4, 2) \end{array} + \begin{array}{l} + 1 \cdot (3, 1, 1, 1) \\ + 1 \cdot (3, 3) \\ + 0 \cdot (5, 1) \end{array}$$

The interesting factor in here $2 \cdot (3, 2, 1)$, represented by

$$\begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 1 & \\ \hline 2 & & \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & 2 & \\ \hline 1 & & \end{array}.$$

Incidentally, $c_{\mu, \nu}^\lambda$ is also the number of ways to build λ from μ by laying down boxes from the first row of ν into rows 1 through $\ell(\mu) + 1$, then from the second row of ν into rows $f_1 + 1$ through $\ell(\mu) + 2$ (where f_1 is the first row with boxes from ν_1 in it), then from the third row of ν into rows $f_2 + 1$ through $\ell(\mu) + 3$, and so on, so that at each step you have a partition.

Special Cases

Horizontal strips: If $\nu = (n)$, then each $c_{\mu\nu}^\lambda$ is at most 1, since there is only one way to fill in a bunch of boxes with 1's. Moreover, λ/μ must be a *horizontal strip* (a skew shape where every column has at most one box), since if there were more than one box in any column, we would have to fill with only weakly decreasing numbers.

Vertical strips: If $\nu = (1^n)$, then, again, each $c_{\mu\nu}^\lambda$ is at most 1, and is nonzero exactly when λ/μ is a length n *vertical strip* (a skew shape where every row has at most one box).

Rectangles: Let $m \leq n$ and s, t be non-negative integers. Then each $c_{(s^m)(t^n)}^\lambda$ is 1 if $\lambda \in \mathcal{T}(s^m, t^n)$, and is zero otherwise, where $\mathcal{T}(s^m, t^n)$ is the set of partitions λ with length $\leq m+n$ such that

$$\begin{aligned}\lambda_{m+1} &= \lambda_{m+2} = \cdots = \lambda_n = t, \\ \lambda_m &\geq \max(s, t), \\ \lambda_i + \lambda_{m+n-i+1} &= s + t, \quad i = 1, \dots, m.\end{aligned}$$

In other words, $\mathcal{T}(s^m, t^n)$ is the set of partitions made of stacking (s^m) on top of (t^n) , carving a corner out of (t^n) , rotating it 180° and gluing it to the end of (s^m) . For example,

The diagram shows the product of two partitions: a 2x2 square (representing (2^2)) and a 2x3 rectangle (representing (3^2)). This is equal to the sum of six partitions, each consisting of a 2x2 square stacked on top of a 2x3 rectangle, with a 2x2 square carved out of the bottom-right corner of the 2x3 rectangle and rotated 180° to be attached to the right side of the 2x2 square. The six partitions represent different ways to carve out a 2x2 corner from the 2x3 rectangle and rotate it to be attached to the right side of the 2x2 square, resulting in partitions of length 6.

References

- [Mac] I. G. Macdonald *Symmetric functions and Hall polynomials*, 2nd ed., Oxford University Press, 1995.