Math 71 Sketch of Solutions of Homework 1 Problems

21/3 $\underline{a}, \underline{b}, \underline{c} \in \mathbb{Z}_n$ a, b, c integers ≥ 0 and $\leq n-1$

$a + b = qn + r, \quad 0 \leq r < n \quad \therefore \underline{a} + \underline{b} = \underline{r}$

$r + c = pn + s \quad 0 \leq s < n$ so ~~[illegible]~~

~~[illegible]~~ $(\underline{a} + \underline{b}) + \underline{c} = \underline{s}$ Show $a + b + c = ln + s$ (same s as before) Similarly $\underline{a} + (\underline{b} + \underline{c}) = \underline{t}$ and $a + b + c = un + t \quad \therefore l = u$ and $s = t$

$\therefore \underline{s} = \underline{t}$

22/19c Suffice to consider $x^a x^b$ where $a = -k$, $k > 0$ and $b > 0$

$$x^a x^b = \underbrace{x^{-1} \cdots x^{-1}}_{k} \underbrace{x \cdots x}_{b}$$

Cases (i) $b > k \quad x^a x^b = x^{b-k} = x^{a+b}$

(ii) $b = k \quad x^a x^b = e = x^{a+b}$

(iii) $b < k \quad x^a x^b = (x^{-1})^{k-b} = x^{b-k} = x^{a+b}$

22/25 $1 = (ab)^2 = abab$ multiply on left by a, on right by b

$ab = a^2 bab^2 = ba$

23/31 We count the elements of $t(G)$. Every $g \in t(G)$ is paired with $g^{-1} \in t(G)$ $(g \neq g^{-1})$. Cannot have $\{g, g^{-1}\} = \{h, h^{-1}\}$ if $g \neq h$ or h^{-1}

$\therefore t(G)$ has even no. of elements

$\therefore G - t(G)$ has an even no. of elements (since $|G|$ is even) But $e \in G - t(G)$ $\therefore \exists$ another element $g_0 \in G - t(G)$ $\therefore g_0 = g_0^{-1}$ $\therefore g_0^2 = e$ $\therefore g_0$ has order 2

27/3 $(sr^i)(sr^i) = s^2 r^{-i} r^i = e \quad \therefore sr^i$ has order 2 ~~[illegible]~~

equivalently, sr^i is a reflection and $\therefore$ has order 2

Then s and sr have order 2. Let $G' \leq D_{2n}$ be generated by s and sr. $s(sr) = r \in G'$ $\therefore r^i \in G'$

$\therefore sr^i \in G'$ $\therefore G' = D_{2n}$

28/4 $r^k r^i = r^i r^k$

$r^k(sr^i) = sr^{-k}r^i = sr^{i-k} = sr^{i+k} = (sr^i)r^k$

reason $1 = r^{2k} = r^k r^k$ so $r^k = r^{-k}$

Suppose z commutes with all elements

If $z = sr^i$ then

$(sr^i)r = r(sr^i) = sr^{i-1}$

$\therefore r^{i+1} = r^{i-1}$ $\therefore r = r^{-1}$ impossible

$\therefore z$ cannot be sr^i

Now suppose $z = r^i$

$r^i s = sr^i = r^{-i}s$ $\therefore r^i = r^{-i}$

$\therefore i = k$ by EX. 33.