Math 71 Homework 6

231/7 D division ring, $Z(D)$ center. $Z(D)$ is subring containing

1. Show inverses exist in $Z(D)$: Let $a \neq 0$ in $Z(D)$

$\forall r \in D$, $ar^{-1} = r^{-1}a$ $\therefore (ar^{-1})^{-1} = (r^{-1}a)^{-1}$

$ra^{-1} = a^{-1}r$ $\therefore a^{-1} \in Z(D)$.

231/8 Let $q = a + bi + cj + dk \in Z(\mathbb{H})$ $qi = iq$

$ai - b - ck + dj = ai - b + ck - dj$ $\therefore c = d = 0$

$q = a + bi$ Consider $qj = jq$ $\therefore b = 0$

$\therefore Z(\mathbb{H}) = \mathbb{R} \subseteq \mathbb{H}$.

231/11 $x^2 - 1 = 0$ $\therefore (x-1)(x+1) = 0$

231/14 (a) $x \neq 0$ $x^n = 0$ $\therefore x x^{n-1} = 0$. $\therefore x$ is zero divisor

(c) $(1+x)(1 - x + x^2 - \cdots \pm x^{n-1}) = 1$

(d) u unit with inverse v. x nilpotent with $x^n = 0$.

$(u+x)(v - v^2x + v^3x^2 - \cdots \pm v^n x^{n-1}) = 1$

231/15 First $x = -x$: $x + x = (x+x)^2$

$2x = (2x)^2 = 4x^2 = 4x$ $\therefore 2x = 0$

$\therefore x = -x$. Now

$a + b = (a+b)^2 = a^2 + ab + ba + b^2 = a + ab + ba + b$

$\therefore ab = -ba = ba$.

146/7 D_8

147/30 $n = 2^3 \cdot 3 \cdot 7$ $N_7 = 1, 8$ Simple implies $N_7 \neq 1$ $\therefore N_7 = 8$

$\exists$ 8 cyclic groups of order 7 whose intersection is $\{e\}$

$\therefore$ there are 48 elements of order 7

238/3 (i) $\Rightarrow$ $(\sum a_n x^n)(\sum b_n x^n) = 1$ for some $\sum b_n x^n$ formal power series

(fps) $\therefore a_0 b_0 + (a_0 b_1 + a_1 b_0)x + \cdots = 1$ $\therefore a_0 b_0 = 1$

$\Leftarrow$ Given $\bar{a}_0 = a_0^{-1}$ Want to find fps $\sum b_n x^n$ such that

$(\sum a_n x^n)(\sum b_n x^n) = 1$. Define $b_0, b_1, \ldots$ inductively

$a_0 b_0 = 1$ so $b_0 = \bar{a}_0$ $\quad a_0 b_1 + a_1 b_0 = 0$ so $b_1 = \bar{a}_0(-a_1 b_0)$

etc.

238/4 ~~$\Leftarrow$~~ $\Rightarrow$ Given two fps $\sum a_n X^n$, $\sum b_n X^n$ whose product is zero. Assume both non-zero and write them as $a_n X^n + a_{n+1} X^{n+1} + \cdots$ with $a_n \neq 0$ and $b_m X^m + b_{m+1} X^{m+1} + \cdots$ with $b_m \neq 0$. $\therefore \left(\sum a_n X^n\right)\left(\sum b_n X^n\right) = a_n b_m X^{n+m} + \cdots = 0$

$\therefore a_n b_m = 0 \quad \therefore a_n = 0$ or $b_m = 0$ contradiction

247/1 Suppose $\varphi : 2\mathbb{Z} \to 3\mathbb{Z}$ isom. $\varphi(2) = 3k$ for some k

$\varphi(4) = \varphi(2+2) = \varphi(2) + \varphi(2) = 3k + 3k = 6k$

$\varphi(4) = \varphi(2 \cdot 2) = \varphi(2)\varphi(2) = 9k^2$ not equal

247/2 $\exists$ two units in $\mathbb{Z}[x]$: 1 and -1

$\exists$ infinitely many units in $\mathbb{Q}[x]$: $\mathbb{Q}^*$

248/4 Suppose $\varphi : \mathbb{Z} \to \mathbb{Z}_{30}$ ~~isomorphism~~ and $\varphi(1) = k$

φ is ring homo. $\Leftrightarrow \varphi(1)\varphi(1) = \varphi(1) \Leftrightarrow k^2 \equiv k \bmod 30$

$k = 0, 1, 6, 10, 15, 16, 21, 25$

249/13
$$a + bi \leftrightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$