1. MULTIPLE CHOICE: For each of the following questions, please circle the correct answer. You do not need to show any work. (3 points each)

(a) Let $\epsilon > 0$, and let $X_1, \ldots, X_n$ be an independent trials process with $E[X_i] = \mu$ and $V(X_i) = \sigma^2$ for each i and sum S_n. What is the value of $\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| \le \epsilon\right)$?

i. 1

ii. ∞

iii. σ^2/ϵ^2

iv. 0

v. The probability does not tend to a limit

(b) If $c \in \mathbb{R}$ is any constant and X is any random variable, which of the following must be correct?

i. $V(cX) = cV(X)$

ii. $V(c+X) = V(X)$

iii. $V(c+X) = c + V(X)$

iv. $V(cX) = V(c)V(X)$

v. None of the above

(c) Consider a Bernoulli trials process with probability $p = 1/5$ of success. If X is the number of trials until the first success, what is $V(X)$?

i. 5

ii. 1/5

iii. 4

iv. 20

v. None of the above

(d) Which of the following is a good approximation for the probability that the number of successes among n independent Bernoulli trials is within 2 standard deviations of the expected number?

i. $NA(2)$

ii. $2NA(2)$

iii. $.5 - NA(2)$

iv. $NA(2) - NA(-2)$

v. $NA(4)$

(e) Let $X_1, \ldots X_n$ be independent random variables with sum S_n and average A_n. Let $\epsilon > 0$. Which of the following can you approximate using the Central Limit Theorem?

i. $\mathbb{P}(A_n \geq \epsilon)$

ii. $\mathbb{P}(S_n \leq \epsilon)$

iii. $\mathbb{P}(-\epsilon \leq A_n \leq 2\epsilon)$

iv. All of the above

v. None of the above

(f) Let X have a Poisson distribution with parameter λ. Which of the following is true?

i. $E[X] = 1/\lambda$

ii. $V(X) = 1/\lambda$

iii. $V(X) = \lambda$

iv. We do not have enough information to calculate $V(X)$

v. None of the above

MC: a) i b) ii c) iv
d) ii e) iv f) iii

2. SHORT ANSWER: Answer each of the following questions. You do not need to show any work. (3 points each)

(a) Let X be a random variable representing the roll of a die. What is the value of $E[X|\text{roll was odd}]$?

Answer: 3

(b) Consider a Bernoulli trials process with $n = 100$ and $p = 1/5$. What does Chebyshev's inequality give as a bound for $\mathbb{P}(|X - 20| \geq 8)$?

Answer: $\frac{1}{4}$

(c) Give an approximation for the probability that if a die is rolled 12 times, the sum is equal to exactly 42.

Answer: $\frac{1}{\sqrt{70\pi}}$

(d) Suppose Professor Trelawney always gives the same 3-question true/false exam. If all answers are equally likely, how many exams must she grade, on average, before she has seen all possible strings of responses? (TFT is a possible response, and it is distinct from TTF.)

Answer: $8\sum_{k=1}^{8}\frac{1}{k}$

(e) Suppose you have a box with 50 black socks, 36 white socks, and 34 brown socks. If you select 10 socks from the box at random, what is the expected number of white socks you end up with?

Answer: $36\frac{10}{120} = 3$

(f) Explain briefly why it is necessary to standardize a sum if we want to apply the Central Limit Theorem (i.e. why does the limit of the distributions of the sums have to be the *standard* normal density?).

Answer: If $S_n = X_1 + \cdots + X_n$ where $E[X_i] = \mu$, $V(X_i) = \sigma^2$, then $E[S_n] = n\mu$ and $V(S_n) = n\sigma^2$. Both of these tend to ∞ as $n \to \infty$.

3. Prove that if you have a Bernoulli trials process with probability p of success in each trial, the expected number of trials until the first success is $\frac{1}{p}$.

$$E[X] = \sum_{k=1}^{\infty} k\, p(1-p)^{k-1} =$$

$$= p \sum_{k=1}^{\infty} k(1-p)^{k-1}$$

We have

$$\sum_{k=1}^{\infty} kx^{k-1} = \left(\sum_{k=1}^{\infty} x^k\right)' = \left(\frac{x}{1-x}\right)' = \frac{1}{(1-x)^2}$$

So

$$E[X] = p\frac{1}{p^2} = \frac{1}{p}.$$

4. A coin is tossed until the first time that the total number of heads reaches 59. Estimate the probability that more than 100 tosses are required for this to happen.

$S_{100} = X_1 + \cdots + X_{100}$

where $X_i = \begin{cases} 1 & \text{if } i^{th} \text{ toss} = H \\ 0 & \text{if } i^{th} \text{ toss} = T \end{cases}$

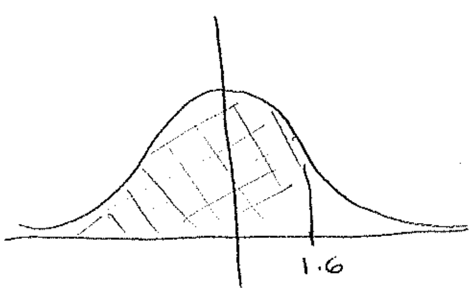
So $E[S_{100}] = \frac{1}{2}(100) = 50$

$V(S_{100}) = \frac{1}{4}(100) = 25$

$\mathbb{P}$(more than 100 tosses before 59 heads)

$= \mathbb{P}(S_{100} < 59) = \mathbb{P}(S_{100} \leq 58)$

$= \mathbb{P}\left(S_{100}^* \leq \frac{58-50}{5}\right) = .5 + NA(1.6)$



5. Recall that students taking their Defense Against the Dark Arts O.W.L.s are given 5 questions, and must answer at least 4 correctly in order to pass the exam. Draco Malfoy finds a cheat sheet containing 100 questions, and knows that each of the 5 questions he'll be asked will be chosen randomly from that sheet (each is equally likely).

   (a) If he goes into the exam knowing the answer to 85 of the 100 questions, should he expect to pass? (Show an explicit calculation for the expected number of questions to which he knows the answers – the expected value should be in closed form. That is, it **should not** contain a sum.)

   Hypergeometric w/ $N = 100$, $k = 85$, $m = \cancel{85}\ 5$

   From HW (question about yellow balls in urn), expected value of hypergeometric is

   $$\frac{km}{N} = \frac{85(5)}{100} = 4.25$$

   So yes, he should expect to pass.

   (b) What is the minimum number of questions of the 100 in the list to which Draco must know the answer if he wants his expected number of correctly answered questions to be at least 4?

   Need $\frac{km}{N} \geq 4$

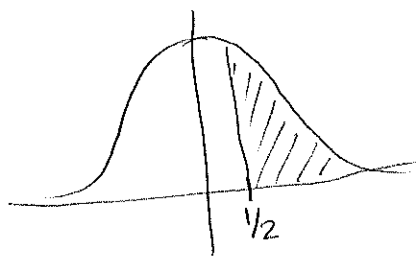
   So $\frac{5k}{100} \geq 4 \Rightarrow k \geq 80$

6. A drunk is walking on a number line (i.e. a line that looks like $\cdots -2, -1, 0, 1, 2, \ldots$), starting at position 0 and taking a step to the left or to the right with probability 1/2 at each move.

   (a) Estimate the probability that after 100 moves, he is 5 steps or farther to the right of where he started.

$E[S_{100}] = 0 \qquad V(S_{100}) = 100$

$\mathbb{P}(S_{100} \geq 5) = \mathbb{P}\left(S_{100}^{*} \geq \frac{5-0}{10}\right) = .5 - NA(.5)$



   (b) What is the smallest value of k that will guarantee a probability of at least .99 that the drunk is (strictly) less than k steps away from his starting position after 100 moves?

By Chebyshev's inequality,

$\mathbb{P}(|S_{100} - 0| < k) \geq .99$

$= 1 - \mathbb{P}(|S_{100}| \geq k)$ so want $\mathbb{P}(|S_{100}| \geq k) \leq .01$.

$\mathbb{P}(|S_{100}| \geq k) \leq \frac{100}{k^2}$ so need

$\frac{100}{k^2} = .01 \Rightarrow k^2 = 10000$, so $k = 100$.

7. Alice and Bob are playing the dice game: Alice rolls a 6-sided die (with sides $1, 2, 3, 4, 5, 6$), Bob rolls a 4-sided die (with sides $1, 2, 3, 4$), and whenever Bob's roll is at least as high as Alice's, she pays him 1 dollar. Whenever Alice's roll is higher than Bob's, Bob pays her 1 dollar.

   (a) Suppose they play the game 60 times. Let W_{60} be Alice's winnings at the end of the 60 games. What is $E[W_{60}]$?

   We saw before (in class) that $\mathbb{P}(A \text{ wins}) = \frac{14}{24}$.

   so $E[W60] = \left(\frac{14}{24}\cdot 1 - \frac{10}{24}\cdot(-1)\right)60 = 60\left(\frac{4}{24}\right) =$

   $= 10.$

   (b) What is $V(W_{60})$?

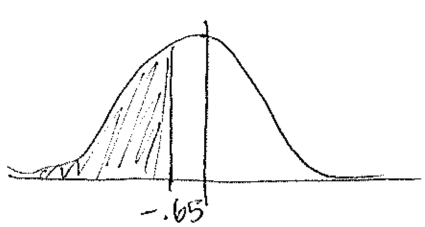
   $E[X_i^2] = 1$ so $V(X_i) = 1 - (4/24)^2$

   $\Rightarrow V(W60) = 60\left(1 - (4/24)^2\right)$

   (c) Estimate the probability that after 60 games, Alice ends up with no more than 5 dollars.

   $\mathbb{P}(W60 \le 5) = \mathbb{P}\left(W60^* \le \frac{5-10}{\sqrt{60(1-(\frac{4}{24})^2)}}\right)$

   $= .5 - NA\left(\frac{5}{\sqrt{60(1-(4/24)^2)}}\right)$



Bonus Question (3 pts.): You have two one-hour fuses: lighting one end of a fuse will cause it to burn down to the other end in exactly one hour's time. You know nothing else about the fuses; in particular you don't know how long any segment of a fuse will burn, only that an entire fuse takes one hour (the fuse doesn't necessarily burn evenly). Can you use the two fuses to tell when exactly 45 minutes have passed? (Explain.)

Yes.
Light both ends of 1 fuse, and 1 end of the other. When the first fuse burns down, 30 mins have passed. Light the remaining end. The remaining fuse will burn out in 15 minutes.