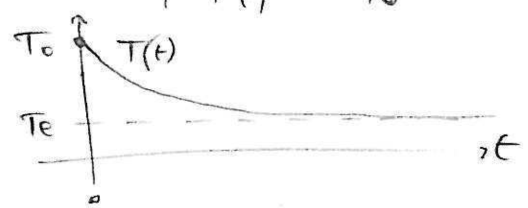
MATH 46 HW2 SOLUTIONS - — Spring '08. ① 4/12/07 Barnett & 4/9/08

p. 40-44 #5 a) $\begin{cases} T' = -h(T-T_e) \\ T(0) = T_0 \end{cases} \xrightarrow{\text{standard 1st-order lin. form}} T' + hT = hT_e$

so $e^{ht}T = \int hT_e e^{ht}dt + c$

so $T(t) = T_e + (T_0 - T_e)e^{-ht}$

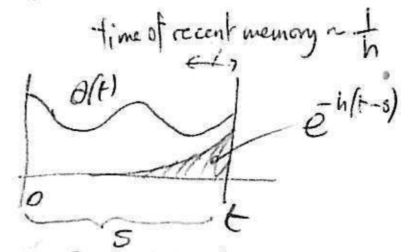


b) $T' + hT = h\theta(t) \xrightarrow{\text{solve}} T(t) = he^{-ht}\left[\int \theta(t)e^{ht}dt + c\right]$

to match IC can write integral starting at $t=0$
so it contributes nothing at $t=0$, so $ch = T_0$ gives IC.

$$T(t) = he^{-ht}\int_0^t \theta(s)e^{hs}ds + T_0e^{-ht}$$

$$= h\int_0^t \theta(s)e^{-h(t-s)}ds + T_0e^{-ht}$$

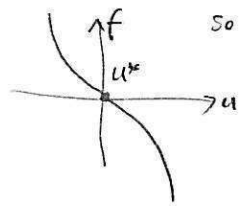


this is Duhamel's Principle:
value at t is a weighted average (convolution) of recent $\theta(t)$ values.

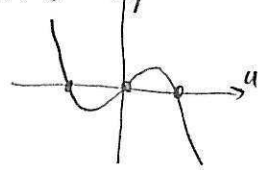
p. 52-54 #6 $f(u) = (\lambda - b)u - au^3 \longrightarrow f'(u) = \lambda - b - 3au^2$

a) $\lambda < b$ so $f'(0) < 0$ equilibrium points given by $f(u) = 0$; note $u^* = 0$ is always equilibrium.

so stable equilibrium at $u^* = 0$.



b) $\lambda > b$



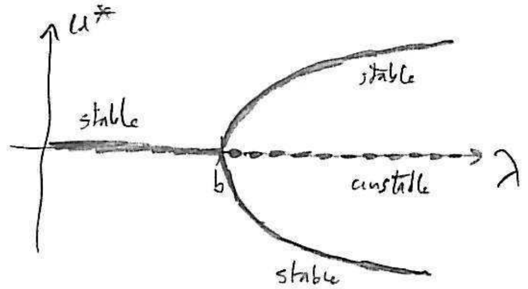
3 equilibrium points: $u^* = 0$ now has $f'(0) > 0$ so is unstable

solve for other u^* (root of f):

$\frac{f(u_*)}{u_*} = 0 = (\lambda - b) - au_*^2$ so $u^* = \pm\sqrt{\frac{\lambda - b}{a}}$

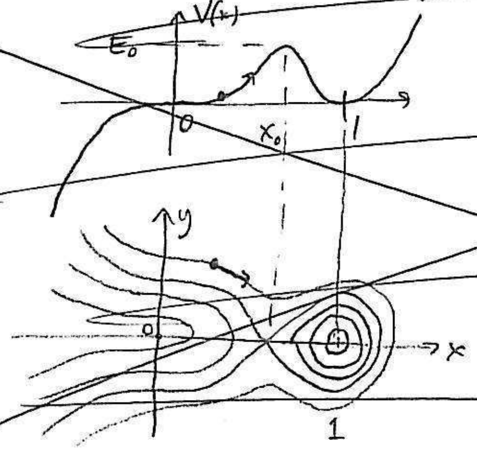
They have $f'(u^*) < 0$ so are stable.

c)



pitchfork bifurcation!

p.67 #1 Sketch $V(x)$ then get contours (level curves) of E from it.



oscillates when trapped in potential well around $x=1$. (When level curve closes on itself)

This happens only if $E < E_0 := \max\limits_{0<x<1} V(x)$

$V'(x) = 3x^2(1-x)^2 - 2x^3(1-x) = 0$

so $3(1-x) = 2x$, $x_0 = 3/5$

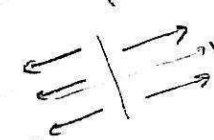
So $E_0 = V(x_0) = (\frac{3}{5})^3(\frac{2}{5})^2 = \frac{108}{3125}$.

p.67-68 #2 $A = \begin{pmatrix} 3 & b \\ 1 & 1 \end{pmatrix}$ eigvals $\begin{vmatrix} 3-\lambda & b \\ 1 & 1-\lambda \end{vmatrix} = 0$ so $3 - 4\lambda + \lambda^2 - b = 0$

ie $\lambda^2 - 4\lambda + 3 - b = 0$ so $\lambda = 2 \pm \sqrt{4-(3-b)} = 2 \pm \sqrt{1+b}$

$\boxed{b>3}$ roots: $\mathbb{C}\lambda$ — $\lambda_1 < 0 < \lambda_2$: saddle point, (unstable)

$\boxed{b=3}$ — $\lambda_1 = 0 < \lambda_2$: exp. growth in one direction, neutral in other. (unstable).



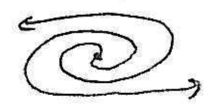
$\boxed{-1<b<3}$ — $0 < \lambda_1 < \lambda_2$: unstable node

$\boxed{b=-1}$ double — $0 < \lambda_1 = \lambda_2$: are there 2 L.I. eigenvectors in $\lambda=2$ eigenspace?

$A - \lambda I = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ which has $\dim \mathrm{Nul}(A-\lambda I) = 1$ ie only 1-dim eigenspace.

$\Rightarrow te^{\lambda t}$ behaviour, for $\lambda > 0$, unstable.

$\boxed{b<-1}$ — $\lambda = 2 \pm i\beta$: unstable spiral node



#6 $x' = y$
$y' = -\frac{1}{m}(kx + ay)$ ie $\vec{x}' = A\vec{x}$ with $A = \begin{pmatrix} 0 & 1 \\ -k/m & -a/m \end{pmatrix}$

eigvals. $-\lambda(-a/m - \lambda) + \frac{k}{m} = 0$ ie $m\lambda^2 + a\lambda + k = 0$, $\lambda = \frac{1}{2m}(-a \pm \sqrt{a^2 - 4mk})$
(same as char. eqn for 2nd order).

$\boxed{a=0}$: $\lambda = \pm i\sqrt{\frac{k}{m}}$ center, stable but not loc. asym. stable. (undamped SHM)

$\boxed{a^2-4mk=0}$: $\lambda = \frac{-a}{2m}$ twice stable node ($\alpha e^{\lambda t} + \beta t e^{\lambda t}$ type) ('critically damped')

$\boxed{a^2-4mk<0}$: $\lambda = \alpha \pm i\beta$ stable spiral. (damped osc.) $\boxed{a^2-4mk>0}$: stable node $0<\lambda_1<\lambda_2$ (overdamped)

p. 100–104 #3 $\quad e^{-t} = o(t^{-2})$ as $t \to \infty$

It is proved if $\lim_{t\to\infty} \frac{e^{-t}}{t^{-2}}$ exists and is zero, equivalently $\ln \frac{e^{-t}}{t^{-2}} \to -\infty$

$$\ln \frac{e^{-t}}{t^{-2}} = -t + 2\ln t \qquad \text{but } \ln t = o(t) \text{ since } \lim_{t\to\infty} \frac{\ln t}{t} = \frac{1/t}{1} = \frac{1}{t} \to 0$$

L'Hôpital's rule since both $\ln t$ and t diverge

$$\text{so } \lim_{t\to\infty} \frac{e^{-t}}{t^{-2}} = \lim_{t\to\infty} \left(-t + o(t)\right) = -\infty \qquad \text{QED -}$$

Alternative (simpler?):
$$\lim_{t\to\infty} \frac{e^{-t}}{t^{-2}} = \lim_{t\to\infty} \frac{t^2}{e^t} \overset{\text{l'Hôpital}}{=} \lim_{t\to\infty} \frac{2t}{e^t} \overset{\text{l'Hôpital}}{=} \lim_{t\to\infty} \frac{2}{e^t} = 0.$$

#5 d. $\quad \cos\varepsilon = 1 - \frac{\varepsilon^2}{2!} + \frac{\varepsilon^4}{4!} - \dots$ Taylor series.

$(1+x)^{-1} = 1 - x + O(x^2)$ use binomial for small ε^2.

$$\text{so } \frac{\varepsilon^{1/2}}{1-\cos\varepsilon} = \frac{\varepsilon^{1/2}}{\varepsilon^2/2 + O(\varepsilon^4)} = 2\varepsilon^{-3/2}\left(1 + O(\varepsilon^2)\right)^{-1} = 2\varepsilon^{-3/2}\left(1 + O(\varepsilon^2)\right)$$
$$= O(\varepsilon^{-3/2})$$

since $|e^{-x^2}| \le 1 \ \forall x$.

g. $$\left|\int_0^\varepsilon e^{-x^2} dx\right| \le \int_0^\varepsilon |e^{-x^2}| dx \le \int_0^\varepsilon 1\, dx = \varepsilon = O(\varepsilon).$$

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p. 52–54

#10 $\quad u' = f(u) = u^3 - u + h \quad$ we seek roots of f, which we call u^*, at various fixed h.

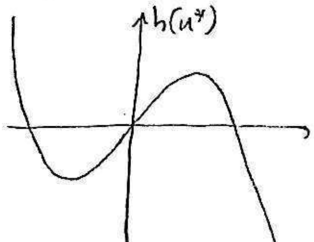
eg. $h = 0$ have $f(u) = u^3 - u$ so $u^* = 0, \pm 1$

At other $h \neq 0$ it would involve solving a cubic to find u^*.

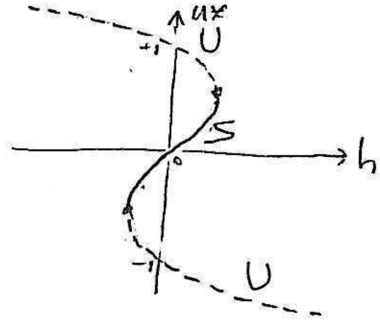
and $f'(u) = 3u^2 - 1$
so $u^* = 0$ has $f'(u^*) < 0$ stable,
$u^* = \pm 1$ has $f'(u^*) > 0$, unstable.

However we seek pairs (u^*, h) in the plane. Since $f(u)$ contains h additively this is much easier to do by fixing u^* and reading off h explicitly:

$f(u^*) = 0 \iff u^{*3} - u^* + h = 0 \iff h = -u^{*3} + u^*$ ie a cubic for $h(u^*)$



flip to usual axes



as usual, at the transition from U to S you get points where it's SS (semistable).

p.102-104 #1 Newton's 2nd Law: net force equals mass time acceleration:

$$m\ddot{y} = -a\underbrace{\dot{y}|\dot{y}|}_{\text{magnitude } \dot{y}^2 \text{ but with sign of } \dot{y}} - ky$$

dimensions

$$\begin{array}{c|cccc} & m & a & k & A \\ M & 1 & 1 & 1 & \\ L & & -1 & & 1 \\ T & & & -2 & \end{array}$$

4 params.

if damping small want to choose length & time scales <u>not</u> involving a, which is possible since crossing out this column leaves a 3x3 full rank matrix.

so $\begin{cases} y_c = A & \text{IC release distance.} \\ t_c = \sqrt{\frac{m}{k}} & \text{osc. period (undamped)} \end{cases}$

rescale:

$$m\frac{y_c}{t_c^2}\bar{y}'' = -a\frac{y_c^2}{t_c^2}\bar{y}'|\bar{y}'| - ky_c\bar{y}$$

substitute & simplify ... (use eg $t_c^2 k = m$)

get $\begin{cases} \bar{y}'' + \frac{Aa}{m}\bar{y}'|\bar{y}'| + \bar{y} = 0 \\ \text{with } \bar{y}(0) = 1, \quad \bar{y}'(0) = 0. \end{cases}$

so $\varepsilon = \frac{Aa}{m} \ll 1$ since damping small

notice the book gives $(\bar{y}')^2$ instead of $\bar{y}'|\bar{y}'|$ which will not always give resistive force opposed to velocity, as it should!

#2 Unperturbed is $u_0'' - u_0 = 0$ which has gen. soln. $c_1e^t + c_2e^{-t}$

It is <u>not oscillatory</u>!

The zeroth-order always inherits original ICs, viz. $\left.\begin{array}{l} u_0(0) = 1 \\ u_0'(0) = -1 \end{array}\right\}$ solve for c_1, c_2 get $c_1 = 0$, $c_2 = 1$.

Thus $u_0(t)$ is special in that it couples <u>only</u> into the <u>decaying</u> solution ~ this is unusual since generic case is to have $c_1 \neq 0$ where the <u>growing</u> dominates as $t \to \infty$ (expect perturbation to cause this too!)

Subst. $u = u_0 + \varepsilon u_1 \cdots$ into ODE:

$$u_0'' + \varepsilon u_1'' + \cdots - u_0 - \varepsilon u_1 - \cdots = \varepsilon t u_0 + \varepsilon^2 t u_1 + \cdots$$

$O(\varepsilon^1)$: $u_1'' - u_1 = tu_0 = te^{-t}$ note this decay rate coincides with a homogeneous solution $(u_1'' + u_1 = 0)$ so we'll need to include one higher power of t in Meth. Und. Coeffs.

Guess $At^2e^{-t} + Bte^{-t}$

(d/dt) $\hookrightarrow 2Ate^{-t} - At^2e^{-t} + Be^{-t} - Bte^{-t}$

(d/dt) $\hookrightarrow -2Ate^{-t} + At^2e^{-t} + (2A-B)e^{-t} + (2A-B)te^{-t} - Be^{-t}$

So ODE for u_1 becomes $\cancel{At^2e^{-t}} + (-4A+B)te^{-t} + (2A-2B)e^{-t} - \cancel{At^2e^{-t}} - \cancel{Bte^{-t}} = te^{-t}$

so $B = +A$ and $A = -\frac{1}{4}$

Gen. soln. for u_1 is $u_1(t) = -\frac{1}{4}(te^{-t} + t^2e^{-t}) + c_1e^{t} + c_2e^{-t}$

ICs for u_1 are (see worksheet) both <u>zero</u> $u_1(0) = 0$, $u_1'(0) = 0$ gives $c_1 = -1/8$, $c_2 = 1/8$

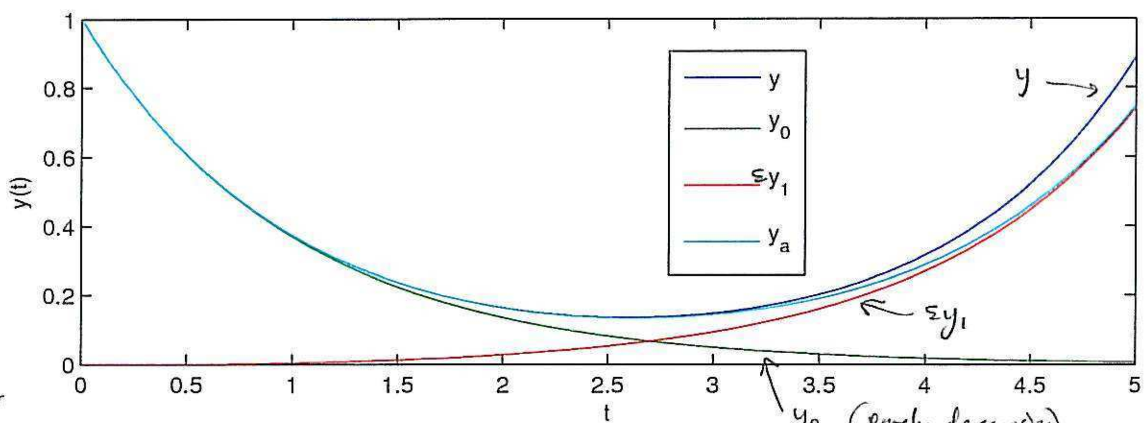
[p.100-104 #2 cont.]

Here's code to plot

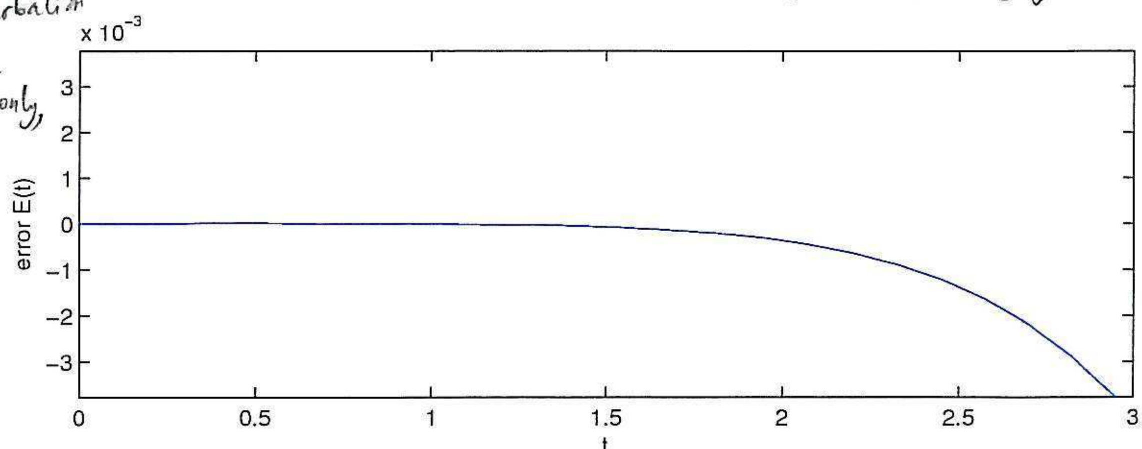
```
% y'' + y = eps.ty  ..... convert to 1st-order system y1'=y2, y2'=y1(1+eps.t)

eps = 0.04;
f = @(t,y) [y(2); y(1).*(1+eps*t)];      % construct a vector func to rep ODE
y0 = [1; -1];
tmax = 5;
[t,y] = ode45(f, [0, tmax], y0);
figure; plot(t, y(:,1), 'k-'); xlabel t; ylabel y(t);  % check your solution

y0 = exp(-t);                                          % zeroth-order
y1 = -(1+t).*t.*exp(-t) / 4  + (exp(t)-exp(-t))/8;     % first-order
ya = y0 + eps*y1;                                      % 2-term approximation
E = ya - y(:,1);                                       % error vs time

figure; subplot(2,1,1); % allows you to put multiple plots on one figure
plot(t, [y(:,1) y0 eps*y1 ya], '-');  % plot the 4 curves asked for
xlabel t; ylabel y(t); legend('y', 'y_0', 'y_1', 'y_a');
subplot(2,1,2); plot(t, E, '-'); xlabel t; ylabel('error E(t)');
T=3; v = max(abs(E(find(t<T)))); axis([0 T -v v]);    % trick: zoom in correctly
print -depsc2 p100_ex2.eps                             % make a printable file
```

this analytic formula is what the Method of Und. Coeffs gave us.

you can replace this line with the simpler axis([0 3 -4e-3 +4e-3]); if you want.



Note that the perturbation has broken the special condition of decaying only, so there's some growing e^{+t}, which of course rapidly dominates!



Notice how small the error is! We've done a great job even with going to only 1st-order in ε.

#8 a. $y'' + y = \varepsilon y y'^2$ ICs: $y(0) = 1$, $y'(0) = 0$

$\varepsilon = 0$ zeroth order $y'' + y = 0$ has $y_0(t) = A\cos t + B\sin t \xrightarrow{\text{match ICs}} A = 1, B = 0$

so $y_0 = \cos t$.

$\tau = \omega t$ with $\omega = 1 + \varepsilon\omega_1 + \cdots$

sub. in ODE: $\omega^2 y'' + y = \varepsilon y (\omega y')^2$ where now prime means $\frac{d}{d\tau}$.

$$\Rightarrow \left(1 + \varepsilon\omega_1 \cdots\right)^2 \left(y_0'' + \varepsilon y_1'' \cdots\right) + y_0 + \varepsilon y_1 \cdots = \varepsilon\left(y_0 + \varepsilon y_1 \cdots\right)\left(1 + \varepsilon\omega_1 \cdots\right)^2 \left(y_0' + \varepsilon y_1'\right)^2$$

We've already done $O(\varepsilon^0)$ which gave $y_0(\tau) = \cos\tau$

$O(\varepsilon^1)$: $2\omega_1 y_0'' + y_1'' + y_1 = y_0 y_0'^2$

sub y_0: $\underbrace{y_1'' + y_1}_{} = \underbrace{\cos\tau \sin^2\tau}_{} - 2\omega_1(-\cos\tau)$

Same differential operator (LHS) as for y_0, so homogeneous solns. are $\sin\tau$, $\cos\tau$.

want to extract Fourier components to try to cancel secular term (on-resonance driving eg. $\sin\tau$ or $\cos\tau$)

$$\cos\tau \sin^2\tau = \frac{1}{2}\left(\frac{1}{2i}\right)^2 \left(e^{i\tau} + e^{-i\tau}\right)\left(e^{i\tau} - e^{-i\tau}\right)^2$$
$$= -\frac{1}{8}\left[e^{3i\tau} + (1-2)e^{i\tau} + (-2+1)e^{-i\tau} + e^{-3i\tau}\right]$$
$$= -\frac{1}{4}\cos 3\tau + \frac{1}{4}\cos\tau$$

so choose $\omega_1 = -\frac{1}{8}$ to cancel secular $\cos\tau$ term in the driving.

Then use Meth. Und. Coeffs: $y_1'' + y_1 = -\frac{1}{4}\cos 3\tau$

$$\left.\begin{aligned} y_1(\tau) &= A\cos 3\tau \\ y_1'' &= -9A\cos 3\tau \end{aligned}\right\} \text{ so } -9A + A = -\tfrac{1}{4} \text{ so } A = \tfrac{1}{32}$$

Don't forget to match y_1 to its ICs ($y_1(0) = y_1'(0) = 0$): $y_1(\tau) = \frac{1}{32}\cos 3\tau + c_1\cos\tau + c_2\sin\tau$

so $c_1 = -\frac{1}{32}$, $c_2 = 0$

$y_1(\tau) = \frac{1}{32}\left(\cos 3\tau - \cos\tau\right)$

Put together: $\underline{y_a = \cos\tau + \frac{\varepsilon}{32}\left(\cos 3\tau - \cos\tau\right)\cdots}$ where $\underline{\tau = \left(1 - \frac{\varepsilon}{8} + \cdots\right)t}$

p. 100-104 #4

— binomial expansion

Recall $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$

holds for <u>any real n</u>, and any x. (although series won't converge if $|x|>1$ and $n \neq 0,1,2,3 \ldots$)

Use $\begin{cases} x = \varepsilon y = \varepsilon y_0 + \varepsilon^2 y_1 + \cdots \\ n = -3/2 \end{cases}$

So $F(y,\varepsilon) = 1 - \frac{3}{2}\left(\varepsilon y_0 + \varepsilon^2 y_1 + \cdots\right) + \frac{(-3/2)(-5/2)}{2!}\left(\varepsilon y_0 + \cdots\right)^2 + \cdots$

$= 1 - \frac{3}{2}\varepsilon y_0 + \left(-3/2\, y_1 + \frac{15}{8} y_0^2\right)\varepsilon^2 + O(\varepsilon^3)$.

$h'' = -(1+\varepsilon h)^{-2}$

$= -1 + 2\varepsilon h - 3\varepsilon^2 h^2 + O(\varepsilon^3)$ use binomial as above (now you see why we did it!)

Sub. $h = h_0 + \varepsilon h_1 + \varepsilon^2 h_2 + \dots$:

$$h_0'' + \varepsilon h_1'' + \varepsilon^2 h_2'' \dots = -1 + 2\varepsilon(h_0 + \varepsilon h_1 + \dots) - 3\varepsilon^2(h_0 + \dots)^2 + \dots$$

Compare powers of ε:

$O(\varepsilon^0)$: $h_0'' = -1 \xrightarrow{\text{integrate twice}}$ ie $h_0(t) = \frac{-t^2}{2} + at + b$ ICs give $a=1$, $b=0$

so $h_0(t) = t - \frac{t^2}{2}$

$O(\varepsilon^1)$: $h_1'' = 2h_0 = 2t - t^2$

integrate once so $h_1'(t) = t^2 - \frac{t^3}{3} + \cancel{c}$ zero by IC $h_1'(0) = 0$

integrate $\hookrightarrow$ $h_1(t) = \frac{t^3}{3} - \frac{t^4}{12} + \cancel{d}$ zero since $h_1(0) = 0$

$O(\varepsilon^2)$: $h_2'' = 2h_1 - 3h_0^2 = \frac{2}{3}t^3 - \frac{1}{6}t^4 - 3\left(t - \frac{t^2}{2}\right)^2$

$= -3t^2 + \frac{11}{3}t^3 - \frac{11}{12}t^4$

again integrate twice w/ both ICs zero so $h_2(t) = -\frac{1}{4}t^4 + \frac{11}{60}t^5 - \frac{11}{360}t^6$

Answer: $h(t) = \underbrace{t - \frac{t^2}{2}}_{\text{only use 2-term for max height.}} + \varepsilon\left(\frac{t^3}{3} - \frac{t^4}{12}\right) + \varepsilon^2\left(-\frac{t^4}{4} + \frac{11}{60}t^5 - \frac{11}{360}t^6\right) + \dots$

its derivative vanishes at $t = t_m$, ie $1 - t + \varepsilon\left(t^2 - \frac{t^3}{3}\right) = 0$ Subst. $t = 1 + \varepsilon\alpha + \dots$

So $\cancel{1} - \cancel{1} - \varepsilon\alpha + \varepsilon\left(1 + 2\varepsilon\alpha + \dots - \frac{1}{3} - \varepsilon\alpha \dots\right) = 0 \Rightarrow \alpha = 2/3$, $t_m = 1 + \frac{2}{3}\varepsilon + \dots$

Subst. t_m into $h(t)$ to get $h_{max} = h\left(1 + \frac{2}{3}\varepsilon + \dots\right) = 1 + \frac{2}{3}\varepsilon - \frac{1}{2} - \frac{2}{3}\varepsilon \dots = \frac{1}{2} + O(\varepsilon^2) \dots$