CENTRALIZERS OF THE INFINITE SYMMETRIC GROUP

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CONTENTS

ABSTRACT. We review and introduce several approaches to the study of centralizer algebras of the infinite symmetric group S_∞. Our study is led by the double commutant relationships between finite symmetric groups and partition algebras; each approach produces a centralizer algebra that is contained in a partition algebra. Our goal is to incorporate invariants of S_∞, which ties our work to the study of symmetric functions in non-commuting variables. We resultantly explore sequence spaces as permutation modules, which yields families of non-unitary representations of S_∞.

Keywords: representation theory; partition algebras; diagram algebras; centralizer algebras; infinite symmetric group; symmetric functions; bounded operators; Banach spaces.

1. INTRODUCTION

Classical Schur-Weyl duality relates the representation theory of the general linear group $\mathrm{GL}_n(\mathbb{C})$ and the finite symmetric group S_k via their commuting actions on a common vector space. Namely, if $V \cong \mathbb{C}^n$, then the diagonal action of $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C})$ on the algebraic k-fold tensor product $V^{\otimes k}$ fully centralizes the permutation action of S_k on the tensor factors:

$$(1.1) \qquad \mathbb{C}S_k \cong \mathrm{End}_{\mathrm{GL}(V)}(V^{\otimes k}) \qquad \text{when } n = \dim(V) \geq k.$$

In [VT07], Tsilevich and Vershik extend this setting to study an infinite symmetric group S_∞ by creating an infinite tensor power $V^{\otimes \infty}$ on which $\mathrm{GL}(V)$ and S_∞ share commuting actions. In the following, we study the action of S_∞ on finite tensor powers by shifting the role of the symmetric group and generalizing V instead.

The group $\mathrm{GL}_n(\mathbb{C})$ naturally contains S_n as the set of $n \times n$ permutation matrices. By replacing $\mathrm{GL}_n(\mathbb{C})$ with its subgroup S_n, and calculating the centralizer of the diagonal action of S_n on $V^{\otimes k}$, where V has now become a permutation module, one acquires the *partition algebra* $P_k(n)$. The partition algebras arose independently in the work of Martin [Mar91, Mar94, Mar96, Mar00] and Jones [Jon94] as generalizations of the Temperley-Lieb algebras and the Potts model in statistical mechanics. Their work modernly serves to coalesce the study of several *tensor power centralizer algebras*, including the group algebras of the finite symmetric groups, the Temperley-Lieb algebras,

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the Brauer algebras, and the algebras of uniform block permutations; all of these are examples of *diagram algebras*, which we discuss in Section 2.

Sam and Snowden [SS13] provide a first approach for treating S_∞ as a finite tensor power centralizer algebra. Their work arrives amongst a recently rejuvenated effort to understand representation-theoretic stability of chains of groups, most salient being the chain of finite symmetric groups $S_1 \hookrightarrow S_2 \hookrightarrow S_3 \hookrightarrow \cdots$. Bowman, De Visscher, and Orellana [BDVO] use the representation-theoretic duality between S_n and $P_k(n)$ to study stability in *Kronecker coefficients*—decomposition numbers for symmetric group representations—when n is large relative to k. Church, Ellenberg, and Farb [CEF12] use category-theoretic methods to create corresponding chains of modules, each of whose structure is tied together into a single *FI-module*. Sam and Snowden's approach is related, but makes the additional connection back to Schur-Weyl duality, and treats many other examples of groups. The case of most interest here is their consideration of the action of S_∞ on a countable-dimensional vector space $V \cong \mathbb{C}^{(\mathbb{N})}$, as reviewed in Section 3.2. This gives rise to the *downwards* and *upwards partition categories*, whose homogeneous degree k components each form subalgebras of a partition algebra; we call these subalgebras *bottom-* and *top-propagating partition algebras*, respectively.

Another motivation for a treatment of S_∞ comes from connections between Hopf algebras and symmetric functions. An influential result of Gessel [Ges84] provides that the Hopf algebra structure of the *Solomon descent algebra* is in duality with the algebraic structure of the *quasi-symmetric functions*, and vice-versa. Malvenuto and Reutenauer [MR95] revisit this connection, in essence using the classical relationship between $\mathrm{GL}(V)$ and S_k in (1.1); they exploit the Hopf algebra structure of the tensor algebra

$$(1.2) \qquad T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}, \qquad \text{where } V \cong \mathbb{C}^n,$$

as well as its bi-module structure for $\mathrm{GL}(V)$ and $\bigoplus_{k\geq 0} \mathbb{C}S_k$, which restricts to the work of Gessel. Aguiar and Orellana generalize [MR95] in [AO08], drawing on the centralizer relationship between the complex reflection groups $C_r \wr S_n$, in the role of $\mathrm{GL}_n(\mathbb{C})$, and the subalgebra U_k of the partition algebra spanned by the *uniform block permutations*. The result is a graded Hopf algebra $U = \bigoplus_{k\geq 0} U_k$, which analogously contains the Hopf algebra of *symmetric functions in non-commuting variables* NCSym. However, there is a fragility in the centralizer relationships between $C_r \wr S_n$ and U_k, necessitating r and n be kept large relative to k; a study of U all at once thus suggests the presence of S_∞ at work in the background.

A first guess of how to illuminate the role of S_∞ in the work of [AO08] might be to let S_∞ act on $\mathbb{C}^{(\mathbb{N})}$ as in [SS13] discussed above. However, this particular generalization is missing the S_∞-invariant structure crucial to the application to symmetric functions. To see how, start from the finite-dimensional case, where V has basis $\{v_1, \ldots, v_n\}$ over $\mathbb{C}$. Then $V^{\otimes k}$ can be canonically identified with the space of homogeneous polynomials of degree k in non-commuting variables $v_1, \ldots, v_n$; the tensor algebra in (1.2) is isomorphic to the full polynomial ring. Then, the symmetric functions NCSym correspond to elements of $T(V)$ that are invariant under the action of S_n by permuting $v_1, \ldots, v_n$; in each homogeneous degree k, symmetric functions correspond to the invariant elements of $V^{\otimes k}$. For example,

$$(V)^{S_n} = \mathbb{C}\Big\{\sum_{i=1}^{n} v_i\Big\} \qquad \text{and} \qquad (V \otimes V)^{S_n} = \mathbb{C}\Big\{\sum_{i=1}^{n} v_i \otimes v_i, \ \sum_{i,j=1}^{n} v_i \otimes v_j\Big\}.$$

For a complete treatment of all symmetric functions, one passes to the case of countably many variables, where S_∞ acts on functions in non-commuting variables $\{v_i\}_{i\in\mathbb{N}} = \{v_1, v_2, \ldots\}$. No non-trivial S_∞-invariants exist in the tensor space $V^{\otimes k}$ if $V \cong \mathbb{C}^{(\mathbb{N})}$ as in [SS13]; $\mathbb{C}^{(\mathbb{N})}$ contains only finite linear combinations of its basis elements.

In Section 4, we present two other choices of vector space V in order to capture the invariant structure discussed above. In each case, a countable set $\{v_1, v_2, \dots\}$ is contained in V, and endomorphisms considered are determined by their images on that set. In Section 4.1, we let V be a Banach space of p-power summable sequences ℓ^p, and impart an action of S_∞. A metric is chosen so that V contains the desired S_∞-invariants, and we show that all desired invariants appear in subsequent tensor powers. Theorem 3 then states that the centralizer of the action of S_∞ on $\overline{V^{\otimes k}}$ inside the set of bounded maps is the same algebra U_k used in [AO08]. In Section 4.2, we instead let V be the Banach space of bounded sequences ℓ^∞, and describe suitable analogs to k-fold tensor powers and bounded maps. Again, each space contains all desired S_∞-invariants. Theorem 8 states that the centralizer algebra is again a subalgebra of the partition algebra, this time isomorphic to the top-propagating partition algebra arising in [SS13].

It is notable that the inclusion of non-trivial S_∞-invariants into our permutation modules comes at a cost in return. Namely, the representations studied in Section 4 are not unitary. The non-unitary representation theory of wild groups, of which S_∞ is an example, is largely intractable; we refer to [Oko97] for a survey of the representation theory of S_∞ and to [Kir94] for an exposition on tame and wild groups. By including the desired S_∞-invariants, we acquire representations which are reducible, by design, but not fully decomposable. This is reflected in the fact that the centralizer algebras of the actions of S_∞ are small in some sense; in particular, the double commutant property present in classical Schur-Weyl duality does not hold, as discussed in Remark 5. However, a further study of these non-unitary representations may still be made more manageable with leverage provided by the centralizer algebras calculated here.

2. Diagram algebras

A *set partition* of a set S is a set of pairwise disjoint subsets of S, called *blocks*, whose union is S. Fix $k \in \mathbb{N} = \{1, 2, \dots\}$, and denote

$$[k] = \{1, \dots, k\} \qquad \text{and} \qquad [k'] = \{1', \dots, k'\},$$

so that $[k] \cup [k'] = \{1, \dots, k, 1', \dots, k'\}$ is formally a set with $2k$ elements. To each set partition of $[k] \cup [k']$, we associate an equivalence class of graphs, called a $(k\text{-})diagram$, as follows. Consider the set of graphs with vertices $[k] \cup [k']$, and let two graphs be equivalent if they have the same connected components. To each diagram d associate the set partition of $[k] \cup [k']$ determined by the connected components of any of its representatives. For example,

(2.1)

and

are equivalent, and both represent diagrams for the set partition $\{\{1, 2, 1'\}, \{3\}, \{4, 2', 3', 4'\}\}$. Let D_k be the set of k-diagrams.

Let $\mathbb{C}(x)$ be the field of rational functions with complex coefficients in an indeterminate x. The product $d_1 * d_2$ of two diagrams d_1 and d_2 is defined as the concatenation of d_1 above d_2, where one identifies the bottom vertices of d_1 with the top vertices of d_2, and removes any components consisting only of middle vertices. This defines the *partition monoid*, which can be extended to an algebra as follows. If there are m middle components in the concatenation of d_1 and d_2 as before, let $d_1 d_2 = x^m d_1 * d_2$, and extend linearly. For example,

This product is associative and independent of the graphs chosen to represent the partition diagrams.

The *partition algebra* $P_k(x)$ is the span over $\mathbb{C}(x)$ of the set D_k of k-diagrams equipped with this product, where $P_0(x) = \mathbb{C}(x)$. The vector space $P_k(x)$ is an associative algebra with identity given by the diagram corresponding to $\{\{1, 1'\}, \ldots, \{k, k'\}\}$. The dimension of $P_k(x)$ is the number of set partitions of $2k$ elements, that is, the *Bell number* $B(2k)$.

Each partition algebra contains many important subalgebras, including group algebras of symmetric groups, Brauer algebras, and Temperley-Lieb algebras; see [HR05, Section 1]. Three specific subalgebras of $P_k(x)$ will be of interest in later sections: U_k, TP_k, and BP_k. The algebra U_k of *uniform block permutations* is spanned by the set of diagrams d satisfying

$$|B \cap [k]| = |B \cap [k']| \qquad \text{for every } B \in d.$$

For example,

More on the role of U_k can be found in Theorem 3 and Remark 6.

We say a block is *propagating* if it contains vertices in both the top and bottom of a diagram. The *top-propagating partition algebra* TP_k is the subalgebra spanned by diagrams for which all blocks containing top vertices are propagating,

$$(2.2) \qquad TP_k = \mathbb{C}\left\{d \in D_k \,\middle|\, \text{for every } B \in d\colon B \cap [k] \neq B\right\}.$$

Similarly, the *bottom-propagating partition algebra* BP_k is the subalgebra spanned by diagrams which have no blocks isolated to the bottom. For example,
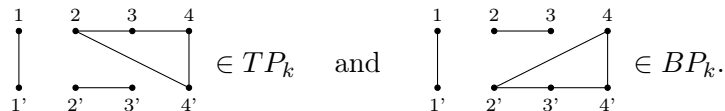
These algebras appear in Sections 3.2 and 4.2, respectively. Note that for each of U_k, TP_k, and BP_k, concatenation of diagrams never results in middle components, and therefore none of these algebras are dependent on the parameter x.

3. Vector spaces of finite or countable dimension as permutation modules

3.1. Finite symmetric group S_n and its action on $(\mathbb{C}^n)^{\otimes k}$.

Let V denote the n-dimensional permutation representation of the symmetric group S_n. That is, let V have basis $\{v_1, \ldots, v_n\}$, on which S_n acts by

$$\sigma \cdot v_i = v_{\sigma(i)} \qquad \text{for } \sigma \in S_n.$$

For $\mathbf{i} = (i_1, \ldots, i_k) \in [n]^k$, a k-tuple of integers in $\{1, \ldots, n\}$, and $\sigma \in S_n$, we let

$$v_{\mathbf{i}} = v_{i_1} \otimes \cdots \otimes v_{i_k} \in V^{\otimes k} \qquad \text{and} \qquad \sigma(\mathbf{i}) = (\sigma(i_1), \ldots, \sigma(i_k)).$$

Let S_n act diagonally on the basis $\{v_{\mathbf{i}}\}_{\mathbf{i} \in [n]^k}$ of $V^{\otimes k}$, that is,

$$\sigma \cdot v_{\mathbf{i}} = v_{\sigma(\mathbf{i})},$$

and extend this action linearly to $V^{\otimes k}$. Thus, $V^{\otimes k}$ becomes a module for S_n.

As in Section 2, arrange the vertices of a k-diagram reading $1, \ldots, k$ from left to right in the top row and $1', \ldots, k'$ from left to right in the bottom row. For each k-diagram d and each $2k$-tuple of integers $i_1 \ldots, i_k, i_{1'}, \ldots, i_{k'} \in [n]$, we define

$$(3.1) \qquad d^{(i_1, \ldots, i_k)}_{(i_{1'}, \ldots, i_{k'})} = \begin{cases} 1 & \text{if } i_\ell = i_m \text{ whenever vertices } \ell \text{ and } m \text{ are connected in } d, \\ 0 & \text{otherwise.} \end{cases}$$

For example,

$$\left(\begin{smallmatrix}1 & 2 \\ & \\ 1' & 2'\end{smallmatrix}\right)^{(3,7)}_{(3,5)} = 0 \qquad \text{and} \qquad \left(\begin{smallmatrix}1 & 2 \\ & \\ 1' & 2'\end{smallmatrix}\right)^{(3,7)}_{(3,3)} = \left(\begin{smallmatrix}1 & 2 \\ & \\ 1' & 2'\end{smallmatrix}\right)^{(4,4)}_{(4,4)} = 1.$$

The algebra $P_k(n)$ acts on $V^{\otimes k}$; namely, for each $d \in P_k(n)$ and $\mathbf{i} \in [n]^k$, we define

$$d \cdot v_{\mathbf{i}} = \sum_{\mathbf{j} \in [n]^k} d^{\mathbf{j}}_{\mathbf{i}} v_{\mathbf{j}},$$

and extend linearly. For example, we have the following identities in the action of $P_2(n)$ on $V^{\otimes 2}$

(3.2)
$$\square \cdot (v_i \otimes v_j) = \delta_{ij} v_i \otimes v_i, \qquad\qquad \times \cdot (v_i \otimes v_j) = v_j \otimes v_i,$$
$$\underline{} \cdot (v_i \otimes v_j) = \delta_{ij} \sum_{\ell=1}^{n} v_\ell \otimes v_\ell, \qquad \text{and} \qquad \diagup \cdot (v_i \otimes v_j) = \left(\sum_{\ell=1}^{n} v_\ell\right) \otimes v_i.$$

Theorem 1 ([Jon94]). *S_n and $P_k(n)$ generate full centralizers of each other in $\mathrm{End}(V^{\otimes k})$. In fact,*
 (1) *$P_k(n)$ generates $\mathrm{End}_{S_n}(V^{\otimes k})$, and when $n \geq 2k$, we have $P_k(n) \cong \mathrm{End}_{S_n}(V^{\otimes k})$;*
 (2) *S_n generates $\mathrm{End}_{P_k(n)}(V^{\otimes k})$.*

The main calculation in the proof of part (1) will be used in several settings later, so let us review. Let $A \in \mathrm{End}(V^{\otimes k})$ be given by the matrix $(A^{\mathbf{j}}_{\mathbf{i}})_{\mathbf{i},\mathbf{j} \in [n]^k}$ such that for each $\mathbf{i} \in [n]^k$,

$$A(v_{\mathbf{i}}) = \sum_{\mathbf{j} \in [n]^k} A^{\mathbf{j}}_{\mathbf{i}} v_{\mathbf{j}}.$$

If $\sigma \in S_n$, and $\sigma A = A \sigma$ in $\mathrm{End}(V^{\otimes k})$, then for each $\mathbf{i} \in [n]^k$,

(3.3) $$\sigma A(v_{\mathbf{i}}) = \sum_{\mathbf{j} \in [n]^k} A^{\mathbf{j}}_{\mathbf{i}} v_{\sigma(\mathbf{j})} \qquad \text{equals} \qquad A\sigma(v_{\mathbf{i}}) = \sum_{\mathbf{j} \in [n]^k} A^{\mathbf{j}}_{\sigma(\mathbf{i})} v_{\mathbf{j}} = \sum_{\mathbf{j} \in [n]^k} A^{\sigma(\mathbf{j})}_{\sigma(\mathbf{i})} v_{\sigma(\mathbf{j})},$$

since σ is a bijection of $[n]^k$. So for every $\sigma \in S_n$, we have

$$\sigma A = A\sigma \quad \text{if and only if} \quad A^{\sigma(\mathbf{j})}_{\sigma(\mathbf{i})} = A^{\mathbf{j}}_{\mathbf{i}} \quad \text{for every } \mathbf{i}, \mathbf{j} \in [n]^k.$$

Thus, $A \in \mathrm{End}_{S_n}(V^{\otimes k})$ if and only if the entries of $(A^{\mathbf{j}}_{\mathbf{i}})_{\mathbf{i},\mathbf{j} \in [n]^k}$ are uniform on S_n-orbits, which exactly describes those linear transformations coming from $P_k(n)$.

3.2. **Infinite symmetric group S_∞ and its action on $(\mathbb{C}^{(\mathbb{N})})^{\otimes k}$.** Embed $S_n \hookrightarrow S_{n+1}$ as the subgroup which fixes $n+1$. Then, let S_∞ be the direct limit of $\{S_n\}_{n \in \mathbb{N}}$, that is, the permutations of $\mathbb{N}$ which fix all but finitely many elements. Let V be a countable-dimensional vector space with basis $\{v_i\}_{i \in \mathbb{N}}$, that is,

$$V = \mathbb{C}\{v_i\}_{i \in \mathbb{N}} = \left\{ \sum_{i \in \mathbb{N}} a_i v_i \,\middle|\, a_i = 0 \text{ for all but finitely many } i \right\} \cong \mathbb{C}^{(\mathbb{N})}.$$

The same calculation as in (3.3) leads to the same conclusion; namely, $A \in \mathrm{End}_{S_\infty}(V^{\otimes k})$ if and only if the entries of its matrix representation $(A^{\mathbf{j}}_{\mathbf{i}})_{\mathbf{i},\mathbf{j} \in \mathbb{N}^k}$ with respect to $\{v_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{N}^k}$ are uniform on S_∞-orbits. So $\mathrm{End}_{S_\infty}(V^{\otimes k})$ is still spanned by endomorphisms coming from diagrams in D_k. However, endomorphisms of $V^{\otimes k}$ have images in $V^{\otimes k}$, that is, for every $\mathbf{i} \in \mathbb{N}^k$, the set $\{\mathbf{j} \in \mathbb{N}^k \mid A^{\mathbf{j}}_{\mathbf{i}} \neq 0\}$ is finite. So $\mathrm{End}_{S_\infty}(V^{\otimes k}) \cong TP_k$, the top-propagating partition algebra defined in (2.2).

In [SS13], Sam and Snowden study the endomorphisms of

$$T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$$

generated by k, ℓ-*diagrams* on $[k] \cup [\ell']$, where k and ℓ are not necessarily equal, that have no blocks isolated to the top row. Multiplication between a k_1, ℓ_1-diagram and a k_2, ℓ_2-diagram is defined as concatenation when $\ell_1 = k_2$ and is zero otherwise. No middle components arise in resolving concatenations because there are no blocks isolated to the top row of any diagram; so multiplication involves no parameter. Under this multiplication, the k, ℓ-diagrams for $k, \ell \in \mathbb{N}$ generate the *upwards partition algebra* UP[1]. The top-propagating partition algebra is exactly the degree-k homogeneous component of UP.

4. Sequence spaces as permutation modules

The study of symmetric functions in countably many variables requires to leave the finite-dimensional realm and instead consider vector spaces V that contain countable linearly independent subsets $\{v_i\}_{i \in \mathbb{N}}$. With an eye toward studying S_∞ invariants as mentioned in Section 1, we require $\sum_{i=1}^{\infty} v_i$ to be interpretable as an element of V. This rules out the possibility of $\{v_i\}_{i \in \mathbb{N}}$ being a *Hamel basis* of V as in Section 3.2, in which case every vector has a unique expression as a finite linear combination of the basis vectors $\{v_i\}_{i \in \mathbb{N}}$. We propose the following approach:

Choose a vector space V containing a countable linearly independent subset $\{v_i\}_{i \in \mathbb{N}}$ and a vector $\sum_{i=1}^{\infty} v_i$ that is invariant under $\mathbb{C}S_\infty$, which is considered as a subalgebra of an algebra of endomorphisms of V that are determined by their images on $\{v_i\}_{i \in \mathbb{N}}$.

4.1. **p-power summable sequences.** We recall definitions from Banach space theory that will be used throughout this section. A sequence $\{w_\ell\}_{\ell \in \mathbb{N}}$ in a normed vector space V is called a *Cauchy sequence*, if for every $\varepsilon > 0$ there is some $L \in \mathbb{N}$ such that for all integers $\ell, m > L$, we have $\|w_\ell - w_m\| < \varepsilon$. A normed vector space V is called a *Banach space* if every Cauchy sequence $\{w_\ell\}_{\ell \in \mathbb{N}}$ in V converges to some vector $w = \lim_{\ell \to \infty} w_\ell$ in V, meaning $\lim_{\ell \to \infty} \|w - w_\ell\| = 0$. A sequence $\{v_i\}_{i \in \mathbb{N}}$ in a Banach space V is called a *Schauder basis* if for every $v \in V$ there exist unique scalars $\{a_i\}_{i \in \mathbb{N}}$ such that

$$(4.1) \qquad v = \sum_{i=1}^{\infty} a_i v_i = \lim_{\ell \to \infty} \sum_{i=1}^{\ell} a_i v_i \qquad \text{meaning} \qquad \lim_{\ell \to \infty} \left\| v - \sum_{i=1}^{\ell} a_i v_i \right\| = 0;$$

qualitatively, a Schauder basis is a linearly independent set such that every element of V can be written uniquely as in (4.1). The basis $\{v_i\}_{i \in \mathbb{N}}$ is called *unconditional* if the convergence is always unconditional. If now A is a continuous endomorphism on V, and if $v = \sum_{i=1}^{\infty} a_i v_i$, then $\{A(v_i)\}_{i \in \mathbb{N}}$ determines $A(v)$ since

$$A(v) = A\left(\lim_{\ell \to \infty} \sum_{i=1}^{\ell} a_i v_i \right) = \lim_{\ell \to \infty} A\left(\sum_{i=1}^{\ell} a_i v_i \right) = \lim_{\ell \to \infty} \sum_{i=1}^{\ell} a_i A(v_i).$$

We therefore study the following special case of the above-mentioned approach:

Choose a Banach space V with a countable Schauder basis $\{v_i\}_{i \in \mathbb{N}}$ such that $\sum_{i=1}^{\infty} v_i$ converges, and study $\mathbb{C}S_\infty$ as a subalgebra of the algebra of continuous endomorphisms of V.

To this end, we consider L^p-spaces of sequences of the form

$$(4.2) \qquad V = L^p(\mathbb{N}, \mu) = \left\{ v = (a_1, a_2, \dots) \in \mathbb{C}^{\mathbb{N}} \,\middle|\, \|v\|^p = \sum_{i=1}^{\infty} |a_i|^p \mu_i^p < \infty \right\},$$

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[1] Note that here, we write all actions as left actions, whereas the actions in [SS13] are right actions; therefore our use of upwards and downwards is reversed.

where $1 \leq p < \infty$, and μ is a weighted counting measure on $\mathbb{N}$, which is determined by a sequence $(\mu_i)_{i \in \mathbb{N}}$ with $\mu_i > 0$ for all $i \in \mathbb{N}$ via $\mu(\{i\}) = \mu_i^p$. The space $\ell^\infty = L^\infty(\mathbb{N}, \mu)$ of bounded sequences will be dealt with in Section 4.2. The normed vector space (4.2) is a Banach space that has unconditional Schauder bases such as $\{v_i\}_{i \in \mathbb{N}}$ given by $v_i = (\delta_{ij})_{j \in \mathbb{N}}$, that is, $v_1 = (1, 0, 0, \ldots)$, $v_2 = (0, 1, 0 \ldots)$, etc. In particular, $v = (a_1, a_2, \ldots) \in V$ if and only if $\sum_{i \in \mathbb{N}} a_i v_i$ converges unconditionally to v in V. This allows to introduce the notation $\sum_{i \in \mathbb{N}} a_i v_i$ for arbitrary $(a_1, a_2, \ldots) \in \mathbb{C}^{\mathbb{N}}$ so that

$$V = \left\{ v = \sum_{i \in \mathbb{N}} a_i v_i \in \mathbb{C}^{\mathbb{N}} \,\middle|\, \|v\|^p = \sum_{i=1}^{\infty} |a_i|^p \mu_i^p < \infty \right\}.$$

In order to ensure that $\sum_{i \in \mathbb{N}} v_i = (1, 1, 1, \ldots) \in V$, we henceforth require that $(\mu_i)_{i \in \mathbb{N}} \in \ell^p$, that is,

$$\sum_{i=1}^{\infty} \mu_i^p < \infty.$$

We turn to the algebraic k-fold tensor product $V^{\otimes k} = (L^p(\mathbb{N}, \mu))^{\otimes k}$, and note that it carries a canonical cross norm so that its completion $\overline{V^{\otimes k}}$ is isomorphic to V [DF93, Chapter 7]; namely,

$$\overline{V^{\otimes k}} = L^p(\mathbb{N}^k, \mu^k) = \left\{ v = \sum_{\mathbf{i} \in \mathbb{N}^k} a_{\mathbf{i}} v_{\mathbf{i}} \in \mathbb{C}^{\mathbb{N}^k} \,\middle|\, \|v\|^p = \sum_{\mathbf{i} \in \mathbb{N}^k} |a_{\mathbf{i}}|^p \mu_{\mathbf{i}}^p < \infty \right\},$$

where $\sum_{\mathbf{i} \in \mathbb{N}^k} a_{\mathbf{i}} v_{\mathbf{i}}$ represents the function $v \colon \mathbb{N}^k \to \mathbb{C}$ given by $v(\mathbf{i}) = a_{\mathbf{i}}$, and for $\mathbf{i} = (i_1, \ldots, i_k) \in \mathbb{N}^k$,

$$v_{\mathbf{i}} = v_{i_1} \otimes \cdots \otimes v_{i_k} \qquad \text{and} \qquad \mu_{\mathbf{i}} = \prod_{\ell=1}^{k} \mu_{i_\ell} = \|v_{\mathbf{i}}\|.$$

In particular, $\{v_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{N}^k}$ is an unconditional Schauder basis of $\overline{V^{\otimes k}}$, and $V^{\otimes k}$ can be identified with the dense subset of linear combinations of vectors of the form

$$\sum_{\mathbf{i} = (i_1, \ldots, i_k) \in \mathbb{N}^k} \left(\prod_{\ell=1}^{k} a_{\ell i_\ell} \right) v_{\mathbf{i}} = \left(\sum_{i_1 \in \mathbb{N}} a_{1 i_1} v_{i_1} \right) \otimes \ldots \otimes \left(\sum_{i_k \in \mathbb{N}} a_{k i_k} v_{i_k} \right) \quad \text{with} \quad \sup_{1 \leq \ell \leq k} \sum_{i_\ell = 1}^{\infty} |a_{\ell i_\ell}|^p \mu_{i_\ell}^p < \infty.$$

We point out that there are several so-called *reasonable cross norms* on tensor products of Banach spaces, ranging from the injective to the projective one as introduced by Grothendieck, but refer to [DF93] for a detailed exposition.

A linear operator on a Banach space, say $A \colon \overline{V^{\otimes k}} \to \overline{V^{\otimes k}}$, is continuous if and only if it is *bounded*, meaning it maps bounded sets to bounded sets, which happens precisely if it has finite *operator norm*,

$$\|A\| = \sup_{v \in \overline{V^{\otimes k}} \colon \|v\| \leq 1} \|A(v)\|.$$

Moreover, the set of bounded operators

$$\mathcal{B}(\overline{V^{\otimes k}}) = \left\{ A \in \mathrm{End}(\overline{V^{\otimes k}}) \,\middle|\, \|A\| < \infty \right\}$$

is a Banach space with respect to the operator norm.

We let S_∞ act diagonally on the basis $\{v_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{N}^k}$ of $\overline{V^{\otimes k}}$ so that for $\sigma \in S_\infty$ and $\mathbf{i} = (i_1, \ldots, i_k) \in \mathbb{N}^k$,

$$\sigma \cdot v_{\mathbf{i}} = v_{\sigma(\mathbf{i})}, \qquad \text{where} \qquad \sigma(\mathbf{i}) = (\sigma(i_1), \ldots, \sigma(i_k)),$$

and we extend linearly. In other words, if $\sigma \in S_\infty$ and $v = \sum_{\mathbf{i} \in \mathbb{N}^k} a_{\mathbf{i}} v_{\mathbf{i}} \in \overline{V^{\otimes k}}$, then

$$\sigma \cdot v = \sum_{\mathbf{i} \in \mathbb{N}^k} a_{\mathbf{i}} v_{\sigma(\mathbf{i})} = \sum_{\mathbf{i} \in \mathbb{N}^k} a_{\sigma^{-1}(\mathbf{i})} v_{\mathbf{i}} \in \overline{V^{\otimes k}}.$$

Each $\sigma \in S_\infty$ gives rise to a continuous endomorphism since it fixes all but finitely many of the vectors $\{v_\mathbf{i}\}_{\mathbf{i} \in \mathbb{N}^k}$. Thus, $\mathbb{C}S_\infty$ can be regarded as a subalgebra of $\mathcal{B}(\overline{V^{\otimes k}})$. In view of the connection to symmetric polynomials, we define for each set partition π of $[k]$,

$$\pi_{(i_1, \ldots, i_k)} = \begin{cases} 1 & \text{if } i_\ell = i_m \text{ whenever } \ell \text{ and } m \text{ are in the same block of } \pi, \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$(4.3) \qquad m_\pi = \sum_{\mathbf{i} \in \mathbb{N}^k} \pi_\mathbf{i} v_\mathbf{i} \in \mathbb{C}^{\mathbb{N}^k}.$$

For example,

$$m_{\{\{1,2,3\}\}} = \sum_{i \in \mathbb{N}} v_i \otimes v_i \otimes v_i \qquad \text{and} \qquad m_{\{\{1,3\},\{2\}\}} = \sum_{i,j \in \mathbb{N}} v_i \otimes v_j \otimes v_i.$$

These elements correspond to the so-called *monomial symmetric functions* of degree k in NCSym, which form a basis for the homogeneous symmetric functions of degree k in non-commuting variables. The following proposition reveals that the finitely many functions defined by (4.3) are elements of $\overline{V^{\otimes k}}$, and that they yield a basis of the subspace $(\overline{V^{\otimes k}})^{S_\infty}$ of S_∞-invariant vectors. In particular, $\overline{V^{\otimes k}}$ encapsulates symmetric polynomials of degree k in countably many non-commuting variables.

Proposition 2. *The vector space $(\overline{V^{\otimes k}})^{S_\infty}$ has $\{m_\pi \mid \pi \text{ a set partition of } [k]\}$ as a basis.*

Proof. We first show that the number of orbits of the diagonal action of S_∞ on $\mathbb{N}^k$ equals the Bell number $B(k)$, that is, the number of set partitions of $[k]$. To this end, we associate to each $\mathbf{i} = (i_1, \ldots, i_k) \in \mathbb{N}^k$ a set partition $\pi(\mathbf{i})$ of $[k]$ via the equivalence relation $\ell \sim m$ if and only if $i_\ell = i_m$. For example,

$$\pi((1,2,2,3,1)) = \pi((3,1,1,2,3)) = \pi((4,1,1,2,4)) = \{\{1,5\},\{2,3\},\{4\}\}.$$

Note that $\pi(\mathbf{i}) = \pi(\mathbf{j})$ if and only if $\mathbf{i} = \sigma(\mathbf{j})$ for some $\sigma \in S_\infty$, which yields the claimed statement. Hence, if $v = \sum_{\mathbf{i} \in \mathbb{N}^k} a_\mathbf{i} v_\mathbf{i} \in \mathbb{C}^{\mathbb{N}^k}$ satisfies $a_{\sigma(\mathbf{i})} = a_\mathbf{i}$ for every $\mathbf{i} \in \mathbb{N}^k$ and $\sigma \in S_\infty$, then the set $\{|a_\mathbf{i}| \mid \mathbf{i} \in \mathbb{N}^k\}$ has at most $B(k)$ elements, and is therefore bounded so that

$$\|v\|^p = \sum_{\mathbf{i} \in \mathbb{N}^k} |a_\mathbf{i}|^p \mu_\mathbf{i}^p \leq \max_{\mathbf{i} \in \mathbb{N}^k} |a_\mathbf{i}|^p \sum_{(i_1, \ldots, i_k) \in \mathbb{N}^k} \left(\prod_{\ell=1}^k \mu_{i_\ell}^p \right) = \max_{\mathbf{i} \in \mathbb{N}^k} |a_\mathbf{i}|^p \left(\sum_{i=1}^\infty \mu_i^p \right)^k < \infty.$$

Thus, $(\overline{V^{\otimes k}})^{S_\infty}$ is $B(k)$-dimensional. Since all m_π are S_∞-invariant, they are elements of $\overline{V^{\otimes k}}$. The set $\{m_\pi \mid \pi \text{ a set partition of } [k]\}$ is now easily seen to be a linearly independent subset of $(\overline{V^{\otimes k}})^{S_\infty}$ with $B(k)$ elements, which completes the proof. $\square$

Theorem 3. *The centralizer of $\mathbb{C}S_\infty$ in $\mathcal{B}(\overline{V^{\otimes k}})$ is isomorphic to the finite-dimensional algebra U_k of uniform block permutations.*

Proof. Recall that every $A \in \mathcal{B}(\overline{V^{\otimes k}})$ is determined by its images on $\{v_\mathbf{i}\}_{\mathbf{i} \in \mathbb{N}^k}$, which we arrange in a matrix $(A_\mathbf{i}^\mathbf{j})_{\mathbf{i},\mathbf{j} \in \mathbb{N}^k} \in \mathbb{C}^{\mathbb{N}^k \times \mathbb{N}^k}$ such that for each $\mathbf{i} \in \mathbb{N}^k$,

$$A(v_\mathbf{i}) = \sum_{\mathbf{j} \in \mathbb{N}^k} A_\mathbf{i}^\mathbf{j} v_\mathbf{j} \in \overline{V^{\otimes k}}.$$

Similarly to (3.3), if $\sigma \in S_\infty$ and $A\sigma = \sigma A$ as elements in $\mathcal{B}(\overline{V^{\otimes k}})$, then for each $\mathbf{i} \in \mathbb{N}^k$,

$$\sigma A(v_\mathbf{i}) = \sum_{\mathbf{j} \in \mathbb{N}^k} A_\mathbf{i}^\mathbf{j} v_{\sigma(\mathbf{j})} \qquad \text{equals} \qquad A\sigma(v_\mathbf{i}) = \sum_{\mathbf{j} \in \mathbb{N}^k} A_{\sigma(\mathbf{i})}^{\sigma(\mathbf{j})} v_{\sigma(\mathbf{j})}.$$

Hence, A is in the centralizer of $\mathbb{C}S_\infty$ if and only if

$$(4.4) \qquad A^{\sigma(\mathbf{j})}_{\sigma(\mathbf{i})} = A^{\mathbf{j}}_{\mathbf{i}} \quad \text{for every } \mathbf{i}, \mathbf{j} \in \mathbb{N}^k \text{ and } \sigma \in \mathbb{S}_\infty.$$

Analogous to the proof of Proposition 2, one finds that the diagonal action of S_∞ on $\mathbb{N}^k \times \mathbb{N}^k$, given by $\sigma(\mathbf{i}, \mathbf{j}) = (\sigma(\mathbf{i}), \sigma(\mathbf{j}))$ for $\mathbf{i}, \mathbf{j} \in \mathbb{N}^k$, has a finite number of orbits indexed by set partitions of $[k] \cup [k']$. In particular, $(A^{\mathbf{j}}_{\mathbf{i}})_{\mathbf{i}, \mathbf{j} \in \mathbb{N}^k}$ has uniform entries on S_∞-orbits if and only if it is a linear combination of the finitely many diagram matrices defined by (3.1). It remains to show that the combinations that yield bounded operators correspond to elements of the algebra U_k. With this in mind, let A be in the centralizer of $\mathbb{C}S_\infty$ in $\mathcal{B}(\overline{V^{\otimes k}})$. For each $\mathbf{i}_0, \mathbf{j}_0 \in \mathbb{N}^k$ and $\sigma \in S_\infty$, we have

$$\|A(v_{\sigma(\mathbf{i}_0)})\| = \|A\sigma(v_{\mathbf{i}_0})\| = \|\sigma A(v_{\mathbf{i}_0})\| = \left\| \sum_{\mathbf{j} \in \mathbb{N}^k} A^{\mathbf{j}}_{\mathbf{i}_0} v_{\sigma(\mathbf{j})} \right\| \geq \|A^{\mathbf{j}_0}_{\mathbf{i}_0} v_{\sigma(\mathbf{j}_0)}\| = |A^{\mathbf{j}_0}_{\mathbf{i}_0}| \mu_{\sigma(\mathbf{j}_0)}.$$

In particular,

$$\|A\| \geq \frac{\|A(v_{\sigma(\mathbf{i}_0)})\|}{\|v_{\sigma(\mathbf{i}_0)}\|} \geq |A^{\mathbf{j}_0}_{\mathbf{i}_0}| \frac{\mu_{\sigma(\mathbf{j}_0)}}{\mu_{\sigma(\mathbf{i}_0)}}.$$

If now some $\ell \in [k]$ appears $L > 0$ times more often in $\mathbf{i}_0$ than in $\mathbf{j}_0$, and if $\sigma \in S_\infty$ fixes all entries in $\mathbf{i}_0$ and $\mathbf{j}_0$ different from ℓ, then

$$\frac{\mu_{\sigma(\mathbf{j}_0)}}{\mu_{\sigma(\mathbf{i}_0)}} = \left(\frac{\mu_\ell}{\mu_{\sigma(\ell)}} \right)^L \frac{\mu_{\mathbf{j}_0}}{\mu_{\mathbf{i}_0}}.$$

Since $(\mu_i)_{i \in \mathbb{N}} \in \ell^p$, we have $\mu_i \to 0$ for $i \to \infty$ so that this quotient can be made arbitrarily large. Hence, $A^{\mathbf{j}_0}_{\mathbf{i}_0} = 0$ whenever $\mathbf{i}_0$ and $\mathbf{j}_0$ are not rearrangements of each other. On the other hand, every matrix $(A^{\mathbf{j}}_{\mathbf{i}})_{\mathbf{i}, \mathbf{j} \in \mathbb{N}^k} \in \mathbb{C}^{\mathbb{N}^k \times \mathbb{N}^k}$ with this property that also satisfies (4.4) is a finite linear combination of diagram matrices $(d^{\mathbf{j}}_{\mathbf{i}})_{\mathbf{i}, \mathbf{j} \in \mathbb{N}^k}$ as in (3.1) that correspond to elements of U_k. Note that for every such $(d^{\mathbf{j}}_{\mathbf{i}})_{\mathbf{i}, \mathbf{j} \in \mathbb{N}^k}$ and $\mathbf{i}_0 \in \mathbb{N}^k$, the vector $d(v_{\mathbf{i}_0}) = \sum_{\mathbf{j} \in \mathbb{N}^k} d^{\mathbf{j}}_{\mathbf{i}_0} v_{\mathbf{j}}$ is either 0 or equal to $v_{\mathbf{j}_0}$ for some rearrangement $\mathbf{j}_0$ of $\mathbf{i}_0$, and $d(v_{\mathbf{i}_0}) = d(v_{\mathbf{i}_1}) \neq 0$ implies $\mathbf{i}_0 = \mathbf{i}_1$. In particular,

$$d \colon \overline{V^{\otimes k}} \to \overline{V^{\otimes k}} \qquad \text{given by} \qquad d\left(\sum_{\mathbf{i} \in \mathbb{N}^k} a_{\mathbf{i}} v_{\mathbf{i}} \right) = \sum_{\mathbf{i} \in \mathbb{N}^k} a_{\mathbf{i}} d(v_{\mathbf{i}})$$

is a well-defined projector of norm 1, which completes the proof. $\qquad \square$

Remark 4. If $p = 2$, then the norm on $\overline{V^{\otimes k}} = L^2(\mathbb{N}^k, \mu^k)$ originates from an inner product, and $\overline{V^{\otimes k}}$ is a Hilbert space. However, the action of S_∞ would be unitary only if μ^k was a multiple of the counting measure on $\mathbb{N}^k$, that is, if $\overline{V^{\otimes k}} \cong \ell^2$, in which case $(\overline{V^{\otimes k}})^{S_\infty}$ would be trivial. We refer to [Pic88, Oko97] and references therein for the study of unitary representations of S_∞.

Remark 5. The group algebra $\mathbb{C}S_\infty$ does not satisfy the double commutant property in $\mathcal{B}(\overline{V^{\otimes k}})$, that is, the centralizer of U_k in $\mathcal{B}(\overline{V^{\otimes k}})$ strictly contains $\mathbb{C}S_\infty$. For example, when $k = 1$, the centralizer of $\mathbb{C}S_\infty$ in $\mathcal{B}(\overline{V^{\otimes 1}}) = \mathcal{B}(V)$ is $U_1 = \mathbb{C}\{\mathrm{id}_V\}$; but the centralizer of $\mathbb{C}\{\mathrm{id}_V\}$ is all of $\mathcal{B}(V)$. This results from the fact that the action of S_∞ on $\mathcal{B}(V)$ is not semisimple. In fact, the S_∞-invariant subspace $\mathbb{C}\{\sum_{i \in \mathbb{N}} v_i\}$ does not have closed $\mathbb{C}S_\infty$-invariant complements so that all projection operators with range $\mathbb{C}\{\sum_{i \in \mathbb{N}} v_i\}$ are unbounded. In particular, $V = \overline{V^{\otimes 1}}$ is not fully decomposable as a module for $\mathbb{C}S_\infty$. A similar argument applies for $k > 1$.

Remark 6. Despite the delicate properties of the action of S_∞ on $\overline{V^{\otimes k}}$ mentioned in the previous remark, Proposition 2 frames the vector space $(\overline{V^{\otimes k}})^{S_\infty}$ of S_∞-invariant vectors as a natural module for the algebra U_k of uniform block permutations. In fact, in [AO08], Aguiar and Orellana study the combinatorial Hopf algebra of uniform block permutations, and find that the ring of symmetric functions in non-commuting variables naturally lives in their algebra. A priori, this may be surprising

since in their setting, U_k arises as the centralizer of the seemingly unrelated complex reflection group $C_r \wr S_n$ on a permutation-like module, as shown in [Tan97]. However, there is a subtlety in the centralizer relationship depending on the values of k and r. One can use the action of $C_r \wr S_n$ on $\mathbb{C}^n$ as defined in [AO08, Section 3.1], calculate the corresponding commutation conditions as in (3.3), and take the limit as $r, n \to \infty$ to obtain the same commutation conditions as in the proof of Theorem 3. In light of this observation, the results of Aguiar and Orellana connecting the Hopf algebra of uniform block permutations and the ring of symmetric functions in non-commuting variables appear natural.

4.2. **Bounded sequences .** In the following, we consider the Banach space of bounded sequences

$$\ell^\infty = \left\{ v = (a_1, a_2, \ldots) \in \mathbb{C}^{\mathbb{N}} \,\middle|\, \|v\|_\infty = \sup_{i \in \mathbb{N}} |a_i| < \infty \right\}.$$

Recall that ℓ^∞ is not *separable*, meaning that it has no countable dense subsets. In particular, ℓ^∞ has no countable Schauder bases. For example, if $\{v_i\}_{i \in \mathbb{N}}$ is given by $v_i = (\delta_{ij})_{j \in \mathbb{N}}$, then $\{\sum_{i=1}^\ell v_i\}_{\ell \in \mathbb{N}}$ is not a Cauchy sequence even though $(1, 1, 1, \ldots) \in \ell^\infty$. Therefore, we no longer use the notation $\sum_{i \in \mathbb{N}} a_i v_i$ for $(a_1, a_2, \ldots) \in \mathbb{C}^{\mathbb{N}}$. Following the previous section, we regard $(\ell^\infty)^{\otimes k}$ as a subspace of

$$\ell^\infty(\mathbb{N}^k) = \left\{ v = (a_{\mathbf{i}})_{\mathbf{i} \in \mathbb{N}^k} \in \mathbb{C}^{\mathbb{N}^k} \,\middle|\, \|v\|_\infty = \sup_{\mathbf{i} \in \mathbb{N}^k} |a_{\mathbf{i}}| < \infty \right\}$$

such that

$$v_{\mathbf{i}} = (\delta_{\mathbf{ij}})_{\mathbf{j} \in \mathbb{N}^k} = v_{i_1} \otimes \cdots \otimes v_{i_k}, \text{ where } \delta_{\mathbf{ij}} = \begin{cases} 1 & \text{if } \mathbf{i} = \mathbf{j} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $(\ell^\infty)^{\otimes k}$ is not dense in $\ell^\infty(\mathbb{N}^k)$ if $k > 1$. Again, $\{v_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{N}^k}$ is not a Schauder basis. Thus, we only consider operators A on $\ell^\infty(\mathbb{N}^k)$ which are determined by their associated matrix with entries $A_{\mathbf{i}}^{\mathbf{j}} = (A(v_{\mathbf{i}}))_{\mathbf{j}}$ for $\mathbf{i}, \mathbf{j} \in \mathbb{N}^k$. To this end, let

$$\mathcal{B}_{\mathrm{Mat}}(\ell^\infty(\mathbb{N}^k)) = \left\{ A = (A_{\mathbf{i}}^{\mathbf{j}})_{\mathbf{i}, \mathbf{j} \in \mathbb{N}^k} \in \mathbb{C}^{\mathbb{N}^k \times \mathbb{N}^k} \,\middle|\, \|A\|_{\mathrm{Mat}} = \sup_{\mathbf{j} \in \mathbb{N}^k} \left\{ \sum_{\mathbf{i} \in \mathbb{N}^k} |A_{\mathbf{i}}^{\mathbf{j}}| \right\} < \infty \right\}.$$

If $A = (A_{\mathbf{i}}^{\mathbf{j}})_{\mathbf{i}, \mathbf{j} \in \mathbb{N}^k} \in \mathcal{B}_{\mathrm{Mat}}(\ell^\infty(\mathbb{N}^k))$ and $v = (b_{\mathbf{i}})_{\mathbf{i} \in \mathbb{N}^k} \in \ell^\infty(\mathbb{N}^k)$, we define for each $\mathbf{j} \in \mathbb{N}^k$,

$$(A(v))_{\mathbf{j}} = \sum_{\mathbf{i} \in \mathbb{N}^k} A_{\mathbf{i}}^{\mathbf{j}} b_{\mathbf{i}},$$

which converges absolutely since

$$\sum_{\mathbf{i} \in \mathbb{N}^k} |A_{\mathbf{i}}^{\mathbf{j}}| |b_{\mathbf{i}}| \leq \sum_{\mathbf{i} \in \mathbb{N}^k} |A_{\mathbf{i}}^{\mathbf{j}}| \|v\|_\infty \leq \|A\|_{\mathrm{Mat}} \|v\|_\infty.$$

Hence, $A(v) \in \ell^\infty(\mathbb{N}^k)$, and A gives rise to an operator on $\ell^\infty(\mathbb{N}^k)$ with norm bounded by $\|A\|_{\mathrm{Mat}}$.

Lemma 7 ([KT34, Sec. 13 Satz 4]). *The norm $\|A\|_\infty$ of each $A \in \mathcal{B}_{\mathrm{Mat}}(\ell^\infty(\mathbb{N}^k))$ as an operator on $\ell^\infty(\mathbb{N}^k)$ equals $\|A\|_{\mathrm{Mat}}$, and $\mathcal{B}_{\mathrm{Mat}}(\ell^\infty(\mathbb{N}^k))$ is a subalgebra of $\mathcal{B}(\ell^\infty(\mathbb{N}^k))$.*

Proof. In order to show $\|A\|_\infty \geq \|A\|_{\mathrm{Mat}}$, note that for each $\mathbf{j} \in \mathbb{N}^k$ we can find $w = (b_{\mathbf{i}})_{\mathbf{i} \in \mathbb{N}^k}$ with $\|w\|_\infty = 1$ and

$$(A(w))_{\mathbf{j}} = \sum_{\mathbf{i} \in \mathbb{N}^k} A_{\mathbf{i}}^{\mathbf{j}} b_{\mathbf{i}} = \sum_{\mathbf{i} \in \mathbb{N}^k} |A_{\mathbf{i}}^{\mathbf{j}}|.$$

Hence, if $\{\mathbf{j}_\ell\}_{\ell \in \mathbb{N}}$ is a sequence in $\mathbb{N}^k$ such that $\sum_{\mathbf{i} \in \mathbb{N}^k} |A_{\mathbf{i}}^{\mathbf{j}_\ell}| \to \|A\|_{\mathrm{Mat}}$, then we can find a sequence of unit vectors $\{w_\ell\}_{\ell \in \mathbb{N}}$ in $\ell^\infty(\mathbb{N}^k)$ such that $\sup_{\ell \in \mathbb{N}} \|A(w_\ell)\|_\infty \geq \|A\|_{\mathrm{Mat}}$, showing that $\|A\|_\infty \geq \|A\|_{\mathrm{Mat}}$. As for the second statement, $\mathcal{B}_{\mathrm{Mat}}(\ell^\infty(\mathbb{N}^k))$ is easily seen to be a vector space, and it suffices

to show that it is closed under composition. If $A = (A_\mathbf{i}^\mathbf{j})_{\mathbf{i},\mathbf{j}\in\mathbb{N}^k}, B = (B_\mathbf{i}^\mathbf{j})_{\mathbf{i},\mathbf{j}\in\mathbb{N}^k} \in \mathcal{B}_{\mathrm{Mat}}(\ell^\infty(\mathbb{N}^k))$, let $C = (C_\mathbf{i}^\mathbf{j})_{\mathbf{i},\mathbf{j}\in\mathbb{N}^k}$ have entries

$$C_\mathbf{i}^\mathbf{k} = \sum_{\mathbf{j}\in\mathbb{N}^k} A_\mathbf{j}^\mathbf{k} B_\mathbf{i}^\mathbf{j},$$

which converge absolutely since

$$\sum_{\mathbf{j}\in\mathbb{N}^k} |A_\mathbf{j}^\mathbf{k}||B_\mathbf{i}^\mathbf{j}| \le \sum_{\mathbf{j}\in\mathbb{N}^k} |A_\mathbf{j}^\mathbf{k}|\|B\| \le \|A\|\|B\|.$$

Moreover, for each $\mathbf{k} \in \mathbb{N}^k$,

$$\sum_{\mathbf{i}\in\mathbb{N}^k} |C_\mathbf{i}^\mathbf{k}| \le \sum_{\mathbf{i}\in\mathbb{N}^k}\sum_{\mathbf{j}\in\mathbb{N}^k} |A_\mathbf{j}^\mathbf{k}||B_\mathbf{i}^\mathbf{j}| = \sum_{\mathbf{j}\in\mathbb{N}^k} |A_\mathbf{j}^\mathbf{k}| \sum_{\mathbf{i}\in\mathbb{N}^k} |B_\mathbf{i}^\mathbf{j}| \le \sum_{\mathbf{j}\in\mathbb{N}^k} |A_\mathbf{j}^\mathbf{k}|\|B\| \le \|A\|\|B\|.$$

Hence, $C \in \mathcal{B}_{\mathrm{Mat}}(\ell^\infty(\mathbb{N}^k))$. Unsurprisingly, C equals the composition of A and B since for every $v = (c_\mathbf{j})_{\mathbf{j}\in\mathbb{N}^k} \in \ell^\infty(\mathbb{N}^k)$ and $\mathbf{k} \in \mathbb{N}^k$,

$$(C(v))_\mathbf{k} = \sum_{\mathbf{i}\in\mathbb{N}^k} C_\mathbf{i}^\mathbf{k} c_\mathbf{i} = \sum_{\mathbf{i}\in\mathbb{N}^k}\sum_{\mathbf{j}\in\mathbb{N}^k} A_\mathbf{j}^\mathbf{k} B_\mathbf{i}^\mathbf{j} c_\mathbf{i} = \sum_{\mathbf{j}\in\mathbb{N}^k} A_\mathbf{j}^\mathbf{k} (B(v))_\mathbf{j} = (A(B(v)))_\mathbf{k}.$$

$\square$

We define a norm-preserving action of S_∞ on $\ell^\infty(\mathbb{N}^k)$ by

$$\sigma \cdot (a_\mathbf{i})_{\mathbf{i}\in\mathbb{N}^k} = (a_{\sigma^{-1}(\mathbf{i})})_{\mathbf{i}\in\mathbb{N}^k} \qquad \text{for } \sigma \in S_\infty,$$

so that for every $\mathbf{i} \in \mathbb{N}^k$,

$$\sigma \cdot v_\mathbf{i} = v_{\sigma(\mathbf{i})}.$$

Theorem 8. *The vector space $(\ell^\infty(\mathbb{N}^k))^{S_\infty}$ of S_∞-invariant elements in $\ell^\infty(\mathbb{N}^k)$ has the functions $\{m_\pi \mid \pi \text{ a set partition of } [k]\}$ defined by (4.3) as a basis. Moreover, the centralizer of $\mathbb{C}S_\infty$ in $\mathcal{B}_{\mathrm{Mat}}(\ell^\infty(\mathbb{N}^k))$ is isomorphic to the finite-dimensional bottom-propagating partition algebra BP_k.*

Proof. The first statement is proven in essentially the same way as Proposition 2. As for the second statement, we follow the proof of Theorem 3, and note that $A = (A_\mathbf{i}^\mathbf{j})_{\mathbf{i},\mathbf{j}\in\mathbb{N}^k} \in \mathcal{B}_{\mathrm{Mat}}(\ell^\infty(\mathbb{N}^k))$ is in the centralizer of $\mathbb{C}S_\infty$ if and only if it is a linear combination of the finitely many diagram matrices defined by (3.1). The claim now follows from the observation that if $A = (A_\mathbf{i}^\mathbf{j})_{\mathbf{i},\mathbf{j}\in\mathbb{N}^k}$ is such a linear combination, then $\{\mathbf{i} \in \mathbb{N}^k \mid A_\mathbf{i}^\mathbf{j} \ne 0\}$ is finite for every $\mathbf{j} \in \mathbb{N}^k$ if and only if A is a linear combination of matrices corresponding to diagrams with no blocks isolated to the bottom row, that is, diagrams in BP_k. $\square$

Remark 9. The case surrounding the countable-dimensional vector space in Section 3.2 can be reconsidered in a similar fashion. Namely, let $V = c_{00} \subset \ell^\infty$ be the set of all sequences $(a_1, a_2, \ldots) \in \mathbb{C}^\mathbb{N}$ whose support $\{i \in \mathbb{N} \mid a_i \ne 0\}$ is finite. Then, $V^{\otimes k}$ can be regarded as a subspace of $\ell^\infty(\mathbb{N}^k)$, and it has $\{v_\mathbf{i}\}_{\mathbf{i}\in\mathbb{N}^k}$ as a countable Hamel basis. Thus, every $A \in \mathrm{End}(V^{\otimes k})$ is uniquely determined by $\{A(v_\mathbf{i})\}_{\mathbf{i}\in\mathbb{N}^k}$, whose elements are arranged as a matrix $(A_\mathbf{i}^\mathbf{j})_{\mathbf{i},\mathbf{j}\in\mathbb{N}^k} \in \mathbb{C}^{\mathbb{N}^k \times \mathbb{N}^k}$ so that for $\mathbf{i},\mathbf{j} \in \mathbb{N}^k$ and $v = (b_\mathbf{i})_{\mathbf{i}\in\mathbb{N}^k} \in V^{\otimes k}$,

$$A(v_\mathbf{i}) = \sum_{\mathbf{j}\in\mathbb{N}^k} A_\mathbf{i}^\mathbf{j} v_\mathbf{j}, \qquad \text{giving} \qquad (A(v))_\mathbf{j} = \sum_{\mathbf{i}\in\mathbb{N}^k} A_\mathbf{i}^\mathbf{j} b_\mathbf{i}.$$

In particular, a diagram matrix $(d_\mathbf{i}^\mathbf{j})_{\mathbf{i},\mathbf{j}\in\mathbb{N}^k}$ as in (3.1) corresponds to an element of $\mathrm{End}(V^{\otimes k})$ if and only if $\{\mathbf{j} \in \mathbb{N}^k \mid d_\mathbf{i}^\mathbf{j} \ne 0\}$ is finite for every $\mathbf{i} \in \mathbb{N}^k$. This happens precisely if the corresponding diagram has no blocks isolated to its top row, for which reason $\mathrm{End}_{S_\infty}(V^{\otimes k}) \cong TP_k$.

Remark 10. As in Section 4.1, S_∞ is strictly contained in its double commutant; see Remark 5. Similarly to Remark 6, the vector space $(\ell^\infty(\mathbb{N}^k))^{S_\infty}$ becomes a natural module for the bottom-propagating partition algebra BP_k, and it has a basis consisting of elements which may be identified with monomial symmetric functions. Of course, the entire partition algebra $P_k(x)$ has a natural action on the set of set partitions of $[k]$ obtained by identifying each set partition π with the diagram $d = \pi \cup \{\{1'\}, \ldots, \{k'\}\}$; these diagrams form a left ideal in $P_k(x)$. One might expect an action of $P_k(x)$ on the basis $\{m_\pi \mid \pi \text{ a set partition of } [k]\}$ of $(\ell^\infty(\mathbb{N}^k))^{S_\infty}$. Recall that in Theorem 1, x must be specialized to the number n of basis vectors on which S_n acts. The transition $x \to \infty$ can be carried out rigorously only if middle components are avoided in diagram concatenations, in this case by restricting to BP_k.

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