1. (15) Complete the following ***definitions***.

(a) Suppose that f is analytic at z_0. Then f has a *zero of order* m at z_0 if ...

(b) Suppose that an analytic function f has an isolated singularity at z_0. Then the *residue of* f *at* z_0 is ...

(c) Suppose that an analytic function f has an isolated singularity at z_0. Then f has a *pole of order* m at z_0 if ...

2. (15)(a) State the Residue Theorem.

(b) Show that if $a > 0$ and if C any simple closed contour containing the points $|z| = a$ in its interior, then

$$\frac{1}{2\pi i}\int_C \frac{ze^z}{z^2+a^2}\,dz = \cos(a).$$

3. (10)(a) If C is the positively oriented circle of radius 1 centered at 0, then evaluate

$$\int_C z^2 \sin\left(\frac{1}{z}\right) dz$$

(b) Let C be a positively oriented simple closed contour containing the complex number a in its interior. Evaluate

$$\int_C \frac{z \sin^2(z)}{(z-a)^2}\, dz.$$

4. (15) Let $f(z) = \dfrac{1}{1-z} + \dfrac{1}{2-z}$. Find the Laurent expansion of f about $z = 0$ which is valid in the given region.

(a) $A_1 = \{\, z \in \mathbf{C} : 0 < |z| < 1 \,\}$.

(b) $A_2 = \{\, z \in \mathbf{C} : 1 < |z| < 2 \,\}$.

(c) $A_3 = \{\, z \in \mathbf{C} : 2 < |z| \,\}$.

5. (10) Suppose that f is analytic in $A = \{ z \in \mathbf{C} : 0 < |z - z_0| < R \}$ with $R > 0$. If f' is the derivative of f, then prove that $\operatorname{Res}(f'; z_0) = 0$.

6. (20)(a) State Liouville's Theorem.

(b) Suppose that f and g are entire functions with g not identically zero and with

$$|f(z)| \leq |g(z)| \quad \text{for all } z \in \mathbf{C}.$$

Let $h(z) = \dfrac{f(z)}{g(z)}$. Explain why h has only isolated singularities.

(c) What sort of isolated singularities can h have? Explain.

(d) What can you conclude about the relationship between f and g? Justify your assertions.

7. (15) Suppose that f is analytic is a domain D containing the closed disk $R := \{ z \in \mathbf{C} : |z| \leq 1 \}$, and that $|f(z)| \leq 1$ for all $z \in R$ with $f(0) = 0$. Let

$$g(z) := \begin{cases} \dfrac{f(z)}{z} & \text{if } z \neq 0, \text{ and} \\ f'(0) & \text{if } z = 0.) \end{cases}$$

(a) Prove that g is analytic in D.

(b) Show that $|f(z)| \leq |z|$ for all $z \in R$. (Hint: I suggest you apply the maximum modulus principle to g.)

(c) If f is as above, then show that we always have $|f'(0)| \leq 1$ and that $|f'(0)| = 1$ if and only if $f(z) = e^{i\theta} z$ for all $z \in R$.

NAME : ______________________________

Math 43

25 April 2002
Dana P Williams

- You have two hours to complete all seven problems.
- Only fully justified answers will receive full credit.
- There are two blank sheets at the end of the exam for scratch work.
- Problems 6d and 7c may be more challenging. Don't spend too much time on them.
- You are to work alone and neither receive from nor provide assistance to anyone else.

Problem	Points	Score
1	15	
2	15	
3	10	
4	15	
5	10	
6	20	
7	15	
Total	100	