An Example of a Non-linear Differential Equation

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In class we mentioned the following theorem, whose proof the interested reader can find in Section 2.8 of [1].

Theorem. 1. *Consider the initial value problem (IVP)*

$$y' = f(t, y), \qquad y(t_0) = y_0. \tag{1}$$

If f and $\partial f/\partial y$ are both continuous on a disk centered at (t_0, y_0) then (1) has a unique solution defined on some interval $t_0 - h < t < t_0 + h$, $h > 0$.

Several points should be made here. While this theorem does give us an effective way of determining if a solution to an IVP exists, it gives us no way of determining this solution nor does it give any information on the interval of definition of that solution. This should be compared to the analogous fact for IVPs involving linear first order ODEs (Theorem 2.4.1 of [1]). If f is linear then we can write down an explicit solution, and the interval of definition of that solution can be determined *from f and t_0 alone.* The moral is that the behavior of solutions to non-linear differential equations can be drastically different than that of linear equations, as the following example is meant to illustrate.

Problem 1. Consider the (non-linear) IVP

$$\frac{dy}{dx} = (1-2x)y^2 \tag{2}$$

$$y(0) = y_0. \tag{3}$$

a. Show that this initial value problem has a unique solution for any choice of y_0.

b. How does the interval of definition of the solution depend on y_0?

Theorem 1 allows us to deal with part (a) without actually solving the equation. In the notation of that theorem we have

$$\begin{aligned} f(x,y) &= (1-2x)y^2 \\ \frac{\partial f}{\partial y} &= 2(1-2x)y. \end{aligned}$$

Since these functions are continuous *everywhere*, the initial value problem consisting of (2) and *any* initial condition $y(x_0) = y_0$ has a unique solution. In particular, the IVP of the problem (when $x_0 = 0$) will always have a unique solution.

Since Theorem 1 gives no explicit information on the interval of definition of the solutions, to answer the question of part (b) we must actually solve the IVP. The equation 2 is separable with general solution

$$y = \frac{1}{x^2 - x + c} \qquad (c \in \mathbb{R}) \qquad \text{or} \qquad y \equiv 0.$$

If $y_0 \neq 0$ the initial condition gives

$$y_0 = y(0) = \frac{1}{c}$$

so that $c = 1/y_0$ and the solution to the IVP is

$$y = \frac{1}{x^2 - x + 1/y_0} \qquad (y_0 \neq 0), \qquad y \equiv 0 \qquad (y_0 = 0).$$

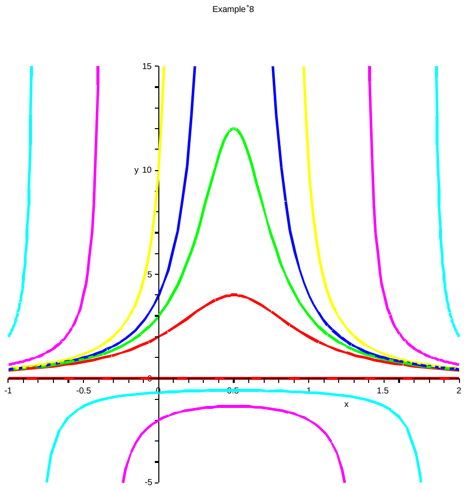

Figure 1: Several solutions to the IVP of Problem 1

Now we need to analyze the domains of our solutions. According to the quadratic formula, the roots of the equation $x^2 - x + 1/y_0 =$ are

$$r_1 = \frac{1 - \sqrt{1 - 4/y_0}}{2} \quad \text{and} \quad r_2 = \frac{1 + \sqrt{1 - 4/y_0}}{2} \tag{4}$$

and at these points (if they are real) our solution curve has a vertical asymptote. Since we require solutions to be defined on intervals, and the interval of definition must contain the initial x-value $x_0 = 0$ we have the following cases.

Case 1: $1 - 4/y_0 < 0$. In this case the denominator of our solution function has no real roots, and so the solution is defined everywhere. We note (for later use) that the condition $1 - 4/y_0 < 0$ is equivalent to $0 < y_0 < 4$.

Case 2: $1 - 4/y_0 = 0$. In this case the roots of the quadratic equation are the same and the solution function has a vertical asymptote only at $x = 1/2$. Therefore, the domain of definition of the solution is $-\infty < x < 1/2$. As above, we note that $1 - 4/y_0 = 0$ is equivalent to $y_0 = 4$.

Case 3: $1 - 4/y_0 > 0$. In this case the two roots r_1, r_2 (which correspond to vertical asymptotes) are distinct. The total domain of the solution function is therefore $-\infty < x < r_1$, $r_1 < x < r_2$, $r_2 < x < \infty$, and the domain of the solution is the interval that contains the initial value $x_0 = 0$. So the question is: which of these intervals contains 0? The answer depends on whether or not r_1 is positive (notice that r_1 is never 0). If $r_1 > 0$ then the domain of definition is $-\infty < x < r_1$. If $r_1 < 0$ then the domain of definition is $r_1 < x < r_2$, since $r_2 > 0$. It remains only to convert the inequalities $r_1 > 0$ and $r_1 < 0$ into conditions on y_0. This is easily done and we find that $r_1 > 0$ if and only if $y_0 > 4$ and $r_1 < 0$ if and only if $y_0 < 0$.

We summarize these cases as our final solution.

- If $y_0 < 0$ then the solution to the IVP is defined on $r_1 < x < r_2$, where r_1, r_2 are as in equation (4).
- If $0 \leq y_0 < 4$ then the solution is defined for all x.
- If $4 \leq y_0$ then the solution is defined for $x < r_1$, where r_1 is as in equation (4).

Figure 1 show several solutions plotted for various values of y_0. Since $y(0) = y_0$, the value of y_0 for a given curve is where that curve crosses the y-axis. One should compare the visual behavior of the solutions with that predicted by our computations above.

References

[1] Boyce, W. E.; DiPrima, R. C., Elementary Differential Equations and Boundary Value Problems, 8th Ed., John Wiley & Sons, Inc. (2005)