Dartmouth College

Mathematics 24

This will be the entire problem set due Wednesday, 10 Janurary. These are not necessarily standard problems. Each of you must hand in your best guess about the validity of each of these conjectures, as well as your reason for each assertion. Prove what you can, disprove what you can. Justify whatever conclusions you draw.

1. Let $S_n = \dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \cdots + \dfrac{1}{n(n+1)}$. Then it is easy to verify that:

$$S_1 = \frac{1}{1\cdot 2} = \frac{1}{2} \qquad S_2 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{2}{3}$$
$$S_3 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{3}{4} \qquad S_4 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} = \frac{4}{5}$$

   Based upon these four examples, one might *conjecture* that for all natural numbers n, $S_n = \dfrac{n}{n+1}$. Is the conjecture valid? Can you justify your assertion (i.e., proof or counterexample)?

2. Consider the polynomial $p(x) = x^2 + x + 41$, first studied by Euler (1707 – 1783). Consider the values of this polynomial for various nonnegative integer values of x. Recall that a prime p is a number divisible only by by ± 1 and $\pm p$. Note that $p(0) = 41$ is prime. So too is $p(1) = 43$. And also $p(2) = 47$, $p(3) = 53$, $p(4) = 61$, and $p(5) = 71$). Is $p(n)$ a prime number for all integers $n \geq 0$?

3. For positive integers n, the polynomial $x^n - 1$ is of great importance in mathematics; its roots are called roots of unity. Consider the following factorizations into "irreducible" polynomials over in the integers:

$$x^1 - 1 = x - 1, \qquad x^2 - 1 = (x-1)(x+1),$$
$$x^3 - 1 = (x-1)(x^2+x+1), \qquad x^4 - 1 = (x-1)(x+1)(x^2+1),$$
$$x^5 - 1 = (x-1)(x^4+x^3+x^2+x+1), \qquad x^6 - 1 = (x-1)(x+1)(x^2+x+1)(x^2-x+1),$$

   Note that the absolute value of the coefficients of the factors never exceed 1. Is this true in general?

4. What is wrong with the following proof using induction?

   Theorem: All horses have the same color.

   Proof: We establish this well-known fact by mathematical induction. Clearly, all horses in any set of 1 horse have the same color. This completes the base step of the induction. Now assume that all horses in any set of n horses have the same color. Consider a set of $n+1$ horses, labeled with the integers $1, 2, \ldots, n+1$. By the induction hypothesis, the horses $1, 2, \ldots, n$ all have the same color, as do the horses $2, 3, \ldots n+1$. Since the two sets have common members, namely $2, 3, \ldots, n$, all $n+1$ horses must have the same color.