HW 9 ①

15.6 #10

$f(x,y,z) = \sqrt{x+yz}$

(a) $\nabla f(x,y,z) = \langle \frac{1}{2}(x+yz)^{-1/2}, \frac{1}{2}(x+yz)^{-1/2}(z), \frac{1}{2}(x+yz)^{-1/2}(y) \rangle$

(b) $\nabla f(1,3,1) = \langle \frac{1}{2}(\frac{1}{2}), \frac{1}{2}(\frac{1}{2})(1), \frac{1}{2}(\frac{1}{2})3 \rangle$

$= \langle \frac{1}{4}, \frac{1}{4}, \frac{3}{4} \rangle$.

(c) $D_u f(1,3,1) = ?$ when $\vec{u} = \langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \rangle$

note : $\|\vec{u}\| = 1$

$D_u f(1,3,1) = \nabla f(1,3,1) \cdot \vec{u}$

$= \langle \frac{1}{4}, \frac{1}{4}, \frac{3}{4} \rangle \cdot \langle \frac{2}{7}, \frac{3}{7}, 6/7 \rangle$

$= \frac{2}{28} + \frac{3}{28} + \frac{18}{28} = \frac{23}{18}$

#24 max rate of change $= \|\nabla f(1,1,-1)\|$

$\nabla f(1,1,-1) = ?$

$\nabla f(x,y,z) = \langle \frac{1}{z}, \frac{1}{z}, \frac{-(x+y)}{z^2} \rangle$

$\nabla f(1,1,-1) = \langle -1, -1, -2 \rangle$

$\therefore$ max rate of change of $f = \| \langle -1, -1, -2 \rangle \|$

$= \sqrt{1+1+4}$

$= \sqrt{6}.$

& it occurs in the dirn of $\langle -1, -1, -2 \rangle$

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$y = x^2 - z^2$

$x^2 - z^2 - y = 0$

Let $F(x, y, z) = x^2 - z^2 - y$

Surface is given by $F(x, y, z) = 0$

normal vector to the tgt plane at

$\nabla F(4, 7, 3)$

$= \langle F_x(4, 7, 3), F_y(4, 7, 3), F_z(4, 7, 3) \rangle$

$= \langle 8, -1, -6 \rangle$

(a) Eqn of the tgt plane : $8(x-4) - 1(y-7) - 6(z-3) = 0$

or

$8x - y - 6z = 7.$

(b) Eqn of normal line :

$$\left. \begin{aligned} x &= 4 + 8t \\ y &= 7 - t \\ z &= 3 - 6t \end{aligned} \right\} \quad t \text{ scalar.}$$

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$F(x,y,z) = x^2+z^2-y$

normal vector to $x^2+z^2-y=0$ is $\nabla F(x,y,z)$

$= \langle 2x, -1, 2z \rangle$

Normal vector to $x+2y+3z=1$

is $\langle 1, 2, 3 \rangle$

We want to find (x,y,z) s.t.

$\cancel{(x,y,z)}$ $\langle 2x, -1, 2z \rangle = c\langle 1,2,3 \rangle$

c constant.

$2x = c$

$-1 = 2c \Rightarrow c = -1/2$

$2z = 3c$

$x = -1/4$

$z = -3/4$

pt should be on $x^2+z^2-y=0$

& hence $y = \frac{1}{16} + \frac{9}{16}$

$= \frac{10}{16}$

$= \cancel{5/8}$ $5/8$

So the pt is $\left(-1/4, 5/8, -3/4\right)$

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$f(x,y) = x^3y + 12x^2 - 8y$

$f_x = 3x^2y + 24x$

$f_y = x^3 - 8$

critical pts :

$3x^2y + 24x = 0$

$x^3 - 8 = 0$

$\Rightarrow x^3 = 8 \Rightarrow x = \sqrt[3]{8} = 2$

$12y + 48 = 0$

$y = -4$

critical pt $= (2, -4)$

$f_{xx} = 6xy + 24$ $\quad$ $f_{xx}(2,-4) = -48 + 24 = -24$

$f_{xy} = 3x^2$ $\quad$ $f_{xy}(2,-4) = 12$

$f_{yy} = 0$ $\quad$ $f_{yy}(2,-4) = 0$

$D(2,-4) = f_{xx} f_{yy} - f_{xy}^2$

$= -(12)^2 = -144 < 0$

So $(2,-4)$ is a saddle pt.

8 $f(x,y) = e^{4y-x^2-y^2}$

$f_x = -2x e^{4y-x^2-y^2}$

$f_y = (4-2y) e^{4y-x^2-y^2}$

Critical pt: $-2x = 0, \quad (4-2y) = 0$

$\Rightarrow x = 0 \qquad y = 2$

$(0,2)$ is the only critical pt.

$f_{xx} = (-2x)^2 e^{4y-x^2-y^2} + e^{4y-x^2-y^2}(-2)$

$= (4x^2-2) e^{4y-x^2-y^2}$

$f_{xy} = -2x e^{4y-x^2-y^2}(4-2y) + e^{4y-x^2-y^2} * 0$

$= -2x(4-2y) e^{4y-x^2-y^2}$

$f_{yy} = (4-2y) e^{4y-x^2-y^2}(4-2y) + e^{4y-x^2-y^2}(-2)$

$= ((4-2y)^2 - 2) e^{4y-x^2-y^2}$

$D(0,2) = ((-2) e^{8-4})((0-2) e^{4})$

$- 0$

$= 4e^8 > 0$

$f_{xx}(0,2) = -2e^4 < 0$

Hence ~~this~~ function has a local max at $(0,2)$

& value $f(0,2) = e^4$

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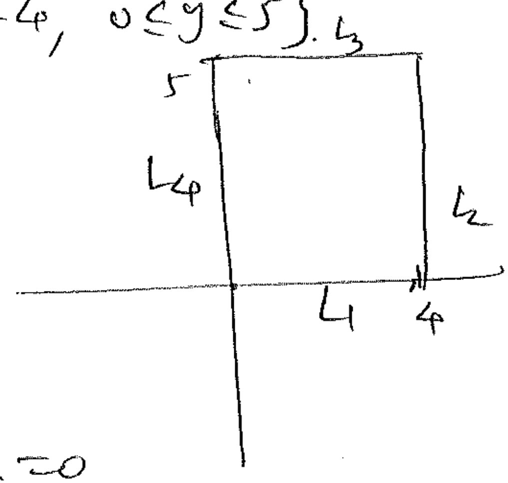
$f(x,y) = 4x + 6y - x^2 - y^2$

$D = \{ (x,y) \,/\, 0 \le x \le 4,\ 0 \le y \le 5 \}$.



$f_x = 4 - 2x$

$f_y = 6 - 2y$

critical pt: $4 - 2x = 0$

$6 - 2y = 0$

so pt $(2, 3)$

$f(2,3) = 8 + 18 - 4 - 9 = 13$

Along L_1: $y = 0$ $f(x,0) = 4x - x^2$, $0 \le x \le 4$

critical pt on L_1: $4 - 2x = 0$

$\Rightarrow x = 2$

$f(0,0) = 0$ $f(2,0) = 4$ $f(4,0) = 0$

on L_2: $x=4$,

$$f(4,y) = 16+6y-16-y^2$$
$$= 6y-y^2 \qquad 0 \le y \le 5$$

critical pt on L_2: $6-2y=0$

$y=3$

$\cancel{f(0}$ $f(4,3) = 9$ $\quad f(4,5) = 5$

on L_3: $y=5$ $\quad f(x,5) = 4x+30-x^2-25$

$$= 4x-x^2+5$$

$0 \le x \le 4$

critical pt on L_3: $4-2x=0$

$x=2$

$f(0,5) = 5$ $\quad f(2,5) = 9$

on L_4: $x=0$, $\quad f(0,y) = 6y-y^2 \quad 0 \le y \le 5$

critical pt on L_4: $6-2y=0$

$y=3$

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$f(0,3) = 9$

So abso max value of f on D is 13

& abso min value of f is 0

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distance from origin

$$d = \sqrt{x^2+y^2+z^2}$$

where $y^2 = 9+xz$

minimize d with $y^2 = 9+xz$

let $f = d^2 = x^2+y^2+z^2$

$f(x,z) = x^2+9+xz+z^2$

$f_x = 2x+z = 0$ $\Rightarrow$ $2x+z=0$

$f_z = x+2z = 0$ $\quad$ $2x+4z=0$

$-3z=0 \Rightarrow z=0$

$x=0$

Hence $(0,0)$ is the only critical pt.

$D(\cdot)$ $f_{xx} = 2$

$f_{xz} = 1$

$f_{zz} = 2$

$D(0,0) = 4-1 = 3 > 0$

$f_{xx}(0,0) = 2 > 0$ & hence f has a minimum at $(0,0)$

$\downarrow$

$\left(f(0,0) = 9 = d^2 \right.$ $\quad$ $x=0,\ z=0 \Rightarrow y^2 = 9$

$\left. \Rightarrow d = 3 \right)$ $\quad$ $\Rightarrow y = \pm\sqrt{9} = \pm 3$

So closest to origin are two pts $(0, \pm 3, 0)$.

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#4.

$$f(x,y) = 4x + 6y$$

$$g(x,y) = x^2+y^2 = 13.$$

$$\nabla f = \lambda \nabla g$$

$$\langle 4, 6\rangle = \lambda \langle 2x, 2y\rangle$$

$$2\lambda x = 4$$
$$2\lambda y = 6.$$

$$\Rightarrow \quad x = \frac{2}{\lambda} \quad \& \quad y = 3/\lambda .$$

$$\frac{4}{\lambda^2} + \frac{9}{\lambda^2} = 13$$

$$\frac{13}{\lambda^2} = 13$$

$$\lambda^2 = 1 \quad \Rightarrow \lambda = \pm 1$$

pts: $\cancel{\left(\frac{2}{\lambda}, \frac{3}{\lambda}\right)}$, $\cancel{(-2}$

$$(2, 3) \quad (-2, -3).$$

$$f(2,3) = 26, \quad f(-2,-3) = -26.$$

So max value $= 26$

min value $= -26$

6.

$$f(x,y) = e^{xy} \qquad \underbrace{x^3+y^3}_{g(x,y)} = 16.$$

$$\nabla f = \langle ye^{xy}, \; xe^{xy} \rangle$$

$$\nabla g = \langle 3x^2, \; 3y^2 \rangle$$

$$\nabla f = \lambda \nabla g$$

$$ye^{xy} = \lambda 3x^2$$

$$xe^{xy} = \lambda 3y^2$$

If ~~Now~~ $x \neq 0$ & $y \neq 0$

we get $\lambda = \dfrac{ye^{xy}}{3x^2} = \dfrac{xe^{xy}}{3y^2}$

$$\Rightarrow \quad 3y^3 e^{xy} = 3x^3 e^{xy}$$

$$\Rightarrow \quad x^3 = y^3$$

$$\Rightarrow \quad x = y$$

But $x^3+y^3 = 16 \Rightarrow \quad 2y^3 = 16,$

$$\Rightarrow \quad y^3 = 8 \qquad \Rightarrow y = 2$$

$$\Rightarrow x = 2$$

pt $(2,2)$.

If $x=0 \Rightarrow y=0$ & $(0,0)$ is not on $x^3+y^3=16$

So we get only one pt $(2,2)$.

We can choose pts on $x^3+y^3=16$ such that $f(x,y)$ is arbitrarily close to 0 $(\neq 0)$. So fun f does not have min on $x^3+y^3=16$.

It has max at $(2,2)$ & max value $f(2,2)=e^4$.

8.

$f(x,y,z) = 8x-4z$ $\qquad \underbrace{x^2+10y^2+z^2}_{g(x,y,z)}=5$

$\nabla f = \langle 8, 0, -4\rangle$

$\nabla g = \langle 2x, 20y, 2z\rangle$

$\nabla f = \lambda \nabla g \Rightarrow$ $2\lambda x = 8$
$20\lambda y = 0$
$2\lambda z = -4$

$$\Rightarrow \quad x = \frac{4}{\lambda}$$

$$y = 0$$

$$z = -2/\lambda.$$

$$x^2 + 10y^2 + z^2 = 5$$

$$\Rightarrow \quad \frac{16}{\lambda^2} + \frac{4}{\lambda^2} = 5$$

$$\Rightarrow \quad \frac{20}{\lambda^2} = 5$$

$$\Rightarrow \quad \lambda^2 = 4 \Rightarrow \lambda = \pm 2$$

pts: $(2, 0, -1)$, $(-2, 0, 1)$

$f(2,0,-1) = 20$ & $f(-2,0,1) = -20$

max value

min value.