5/19/2011

~ SOLUTIONS ~

Math 46, Applied Math (Spring 2011): Midterm 2

2 hours, 50 points total, 6 questions. Heed the available numbers of points. Good luck!

1. [6 points] Use integration by parts to find a 2-term asymptotic expansion for $I(\varepsilon) = \int_0^\varepsilon e^{-1/t} dt$ in the *small* parameter $\varepsilon \to 0^+$.

Need to be able to integrate $e^{-1/t}$, ie want $\frac{1}{t^2} e^{-1/t} = \frac{d}{dt} e^{-1/t}$

$$\Rightarrow I(\varepsilon) = \int_0^\varepsilon \overbrace{t^2}^{u} \overbrace{\frac{1}{t^2} e^{-1/t}}^{v'} dt \overset{\text{by parts}}{=} -\int_0^\varepsilon \overbrace{2t}^{u'} \overbrace{e^{-1/t}}^{v} dt + \left[\overset{u}{t^2} \overset{v}{e^{-1/t}}\right]_0^\varepsilon$$

$$= \varepsilon^2 e^{-1/\varepsilon} - \int_0^\varepsilon 2t^3 \frac{1}{t^2} e^{-1/t} dt$$

$$= \varepsilon^2 e^{-1/\varepsilon} - \left[2t^3 e^{-1/t}\right]_0^\varepsilon + \underbrace{\int_0^\varepsilon 6t^2 e^{-1/t} dt}_{\text{remainder } R(\varepsilon)}$$

$$= \varepsilon^2 e^{-1/\varepsilon} - 2\varepsilon^3 e^{-1/\varepsilon} + \cdots$$

* Note if you tried $I(\varepsilon) = \int_0^\varepsilon \underset{u'}{1} \underset{v}{e^{-1/t}} dt = \left[t e^{-1/t}\right]_0^\varepsilon - \int_0^\varepsilon t \frac{1}{t^2} e^{-1/t} dt$, this is not smaller than $I(\varepsilon)$!

You get growing terms, so stop & try as above!

$R(\varepsilon) = o(\text{last term}) = o(\varepsilon^3 e^{-1/\varepsilon})$?

[BONUS: prove that the remainder term satisfies the needed condition for an asymptotic expansion]

* If little-o is to hold, $\lim_{\varepsilon \to 0^+} \frac{R(\varepsilon)}{\varepsilon^3 e^{-1/\varepsilon}} \overset{?}{=} 0$ both vanish so use l'Hôpital;

$$\lim_{\varepsilon \to 0^+} \frac{\int_0^\varepsilon 6t^2 e^{-1/t} dt}{\varepsilon^3 e^{-1/\varepsilon}} = \lim_{\varepsilon \to 0^+} \frac{6\varepsilon^2 e^{-1/\varepsilon}}{3\varepsilon^2 e^{-1/\varepsilon} + \frac{\varepsilon^3}{\varepsilon^2} e^{-1/\varepsilon}} = \lim_{\varepsilon \to 0^+} \frac{6\varepsilon}{3\varepsilon + 1} = 0 \quad \checkmark \text{ QED.}$$

* Instead you could bound $R(\varepsilon) \le 6\varepsilon^2 \int_0^\varepsilon e^{-1/t} dt = 6\varepsilon^2 I(\varepsilon) = 6\varepsilon^2 \left[(\varepsilon^2 - 2\varepsilon^3) e^{-1/\varepsilon} + R\right]$

so $R(\varepsilon) \le (1 - 6\varepsilon^2)^{-1} 6\varepsilon^2 (\varepsilon^2 - 2\varepsilon^3) e^{-1/\varepsilon} = O(\varepsilon^4 e^{-1/\varepsilon}) = o(\varepsilon^3 e^{-1/\varepsilon})$.

2. [6 points] Write the first 3 terms (i.e. trivial term plus two more) in the Neumann series for the solution of

$$\underbrace{u(t) = 12t^2}_{f(t)} + \lambda \underbrace{\int_0^t (t-s)u(s)\,ds}_{(Ku)(t)},$$

where $\lambda \in \mathbb{R}$ is some constant.

so $u - \lambda K u = f$.
ie $u = (1-\lambda K)^{-1} f$

Neumann series soln. $u = f + \lambda K f + \lambda^2 K^2 f + \cdots$

$(Kf)(t) = \int_0^t (t-s)\, 12 s^2\, ds = 12\left(t\frac{t^3}{3} - \frac{t^4}{4}\right) = t^4$

insert

$(K^2 f)(t) = K(Kf)(t) = \int_0^t (t-s)\, s^4\, ds = t\frac{t^5}{5} - \frac{t^6}{6} = \frac{t^6}{30}$

so $u(t) = 12t^2 + \lambda t^4 + \lambda^2 \frac{t^6}{30} + \cdots$

[easy one].

3. [9 points] Consider the integral operator $(Ku)(x) := \int_0^\pi \underbrace{\sin 2x}_{\alpha_1(x)} \underbrace{\sin y}_{\beta_1(y)}\, u(y)dy$ acting on functions on $(0,\pi)$. ie $\beta_1(x) = \sin x$

[3] (a) Give the general solution to $Ku(x) - 3u(x) = \sin x$, or explain why not possible. $n=1$ degenerate Fredholm.

Convert to lin. sys by $(\beta_i, \cdot)$

$$\sum_j c_j \alpha_j(x) - 3u(x) = \sin x \quad (*)$$

$$\sum_j (\beta_i, \alpha_j) c_j - 3c_i = f_i = (\beta_i, \sin x), \ \forall i$$

$n=1$ so $(\beta_1, \alpha_1) c_1 - 3c_1 = (\beta_1, \sin x) = \int_0^\pi \sin^2 x\, dx = \underbrace{\pi/2}_{f_1}$

$A = [a_{11}] = \int_0^\pi \sin 2x \sin x\, dx = 0$ by Fourier sine orthog.

$\Rightarrow (0-3)c_1 = \pi/2 \Rightarrow c_1 = -\pi/6$.

Use $(*)$: $u(x) = -\frac{1}{3}[\sin x - c_1 \alpha_1(x)] = -\frac{1}{3}[\sin x + \frac{\pi}{6}\sin 2x]$

[2] (b) Give the general solution to $Ku(x) = \sin x$, or explain why not possible.

$f(x)$ not in $\mathrm{Span}\{\alpha_j\} = \mathrm{Span}\{\sin 2x\}$

$\Rightarrow$ no solution! (Ku is always a multiple of $\sin 2x$, but our RHS is not.)

[2] (c) Give the general solution to $Ku(x) = \sin 2x$, or explain why not possible .

is in $\mathrm{Span}\{\sin 2x\} \Rightarrow$ solvable.

lin sys is $A\vec{c} = \vec{f}$ ie $\underset{[0]}{A} c_1 = f_1 = (\beta_1, \sin 2x) = (\sin x, \sin 2x) = 0$.

So $c_1 \in \mathbb{R}$ anything ... standard method isn't useful here! But... [tricky]

Writing original eqn, $\sin 2x \int_0^\pi \sin y\, u(y)\, dy = \sin 2x$ we have the only constraint on u is $(u, \sin x) = 1$, ie $c_1 = 1$. (Ie, u = any Fourier sine series with $b_1 = \frac{2}{\pi}$ coeff)

[2] (d) What are all eigenvalue(s) (with multiplicity) and eigenspace(s) of this operator?

are those of A matrix + $\lambda = 0$ w/ ∞-multiplicity (true since degenerate Fredholm).

But $A = [0]$ has only $\lambda = 0$, with corresponding eigfuncs $\alpha_1(x) = \sin 2x$

$\Rightarrow$ $\lambda = 0$ is only eigenvalue, ∞-multiplicity.

eigenspace $= c \sin 2x + \{$ all functions orthogonal to $\beta_1(x) = \sin x\}$

$= \{u, \int_0^\pi u(x) \sin x\, dx = 0\}$

this absorbs $c \sin 2x$ for $c \in \mathbb{R}$ anyway.

4. [10 points]

[8] (a) By converting to a Sturm-Liouville problem, find all positive eigenvalues and eigenfunctions of the operator K which acts as $(Ku)(x) := \int_0^1 k(x,y)u(y)dy$, with kernel

$$k(x,y) = \begin{cases} 1-x, & y < x \\ 1-y, & y > x \end{cases}$$

write $Ku = \lambda u$ to find spectrum

[Hint: you'll need to extract a boundary condition at each end.]

$$(Ku)(x) = \int_0^x (1-x)\, u(y)\,dy + \int_x^1 (1-y)\, u(y)\, dy = \lambda u(x) \quad (1)$$

$\frac{d}{dx}$, Leibniz. $\hookrightarrow$ $$-\int_0^x u(y)\,dy + \cancel{(1-x)u(x)} - \cancel{(1-x)u(x)} = \lambda u'(x) \quad (2)$$

d/dx $\hookrightarrow$ $-u(x) = \lambda u''(x)$. ODE: $u'' + \frac{1}{\lambda} u = 0$. , $r = \pm i\sqrt{\frac{1}{\lambda}}$

BCs? insert $x=0$ into (1): $0 + \int_0^1 (1-y)\,u(y)\,dy = \lambda u(0)$ not informative

$x=1$ into (1): $\cancel{(1-1)}_{0} \int_0^1 u(y)\,dy + 0 = \lambda u(1)$ Dirichlet at $x=1$.

Insert $x=0$ into (2): $-0 = \lambda u'(0)$ Neumann at $x=0$.

Gen SLP. soln: $u(x) = A\cos\left(\frac{1}{\sqrt{\lambda}}x\right) + \cancel{B}\sin\left(\frac{1}{\sqrt{\lambda}}x\right)$

$\hookrightarrow B=0$ to satisfy Neumann at $x=0$.

for vanishing at $x=1$ need $\cos\frac{1}{\sqrt{\lambda}} = 0$ ie $\frac{1}{\sqrt{\lambda}} = (n+\frac{1}{2})\pi$

$$\Rightarrow \lambda_n = \frac{1}{(n+1/2)^2\pi^2} \qquad n = 0, 1, \ldots$$

$$\phi_n(x) = \cos(n+\tfrac{1}{2})\pi x$$

[2] (b) Use the energy method on the SLP to show that there are no negative or zero eigenvalues. [If you couldn't get an SLP above, just demonstrate the energy method on the simplest SLP you can think of.]

Mult. by u & integrate $\underbrace{\int_0^1 u u''\,dx}_{\text{by parts}} + \frac{1}{\lambda}\int_0^1 u^2\,dx = 0$

$-\int_0^1 u'^2\,dx$

$$\text{so} \quad \lambda = \frac{\int_0^1 u^2\,dx}{\int_0^1 u'^2\,dx} = \frac{\text{positive}}{\text{positive}} > 0.$$

5. [10 points]

A 1D 2π-periodic image f is blurred by applying a Fredholm operator K with convolution kernel $k(x, y) = k(x - y)$, with even, 2π-periodic aperture function

$$k(s) = -\ln\left(2\sin\left|\frac{s}{2}\right|\right) = \underbrace{\cos s}_{k_1=1} + \underbrace{\frac{1}{2}}_{k_2=\frac{1}{2}}\cos 2s + \underbrace{\frac{1}{3}}_{k_3=\frac{1}{3}}\cos 3s + \cdots$$

, etc in cosine series for $k(s)$.

Recall that such an operator has eigenvalues $\lambda_n = \pi k_n$, $n = 0, 1, \ldots$, where k_n are the Fourier cosine coefficients of $k(s)$. $(k_0 = 0)$,

[3] (a) Given the image $f(x) = \sin 7x$ find the blurred image $g(x) = (Kf)(x)$:

Either use fact that convolution kernels have Fourier series as eigenfuncs,

or just compute: $g(x) = (Kf)(x) = \int_{-\pi}^{\pi}\left[\cos(x-y) + \cdots \frac{1}{7}\cos 7(x-y) + \cdots\right]\sin 7y\,dy$

orthog. to $\sin 7y$

$= \int_{-\pi}^{\pi}\left[\cdots + \frac{1}{7}(\cos 7x \cos 7y + \sin 7x \sin 7y) + \cdots\right]\sin 7y\,dy = \frac{1}{7}\sin 7x \int_{-\pi}^{\pi}\sin^2 7y\,dy = \frac{\pi}{7}\sin 7x$

Note you could have got via mult. by $\lambda_7 = \pi k_7 = \pi/7$ too.

[3] (b) Give a formula for the Fourier coefficients $(\hat{a}_n, \hat{b}_n)$ of the best reconstructed image $\hat{f}$ given those (A_n, B_n) of a measured blurry image g. Can all Fourier coefficients be reconstructed? (explain; you may assume no noise here)

By above reasoning (or see worksheet) $\left.\begin{array}{l} A_n = \pi k_n a_n \\ B_n = \pi k_n b_n \end{array}\right\}$ $n = 0, 1, 2, \ldots$ ← note!

So $\begin{cases} \hat{a}_n = \dfrac{A_n}{\pi k_n} = \dfrac{n}{\pi} A_n \\ \hat{b}_n = \dfrac{B_n}{\pi k_n} = \dfrac{n}{\pi} B_n \end{cases}$ $n = 1, 2, \ldots$

But $n=0$ has $k_0 = 0$ so cannot reconstruct this const. coefficient. (we 'regularize' by setting $\hat{a}_0 = 0$).

[2] (c) If noise of size $\underbrace{10^{-3}}_{\varepsilon}$ pollutes each Fourier coefficient of g, and a noise of size $\underbrace{0.1}_{E}$ (ie 10%) is acceptable in $\hat{f}$, how many coefficients should be reconstructed?

Recall noise in A_n or B_n gets multiplied by $\frac{n}{\pi}$ above.

So $\frac{n}{\pi}\varepsilon \le E$ ie $n \le \frac{\pi E}{\varepsilon} = \pi \frac{0.1}{10^{-3}} = 100\pi$ so 314×2 coeffs can be reconstructed.

This was a red herring (irrelevant).

[2] (d) The aperture function is unbounded, $\lim_{s \to 0} k(s) = \infty$. Is the aperture function in $L^2([-\pi, \pi])$? Prove it.

$\|k\|^2 := \int_{-\pi}^{\pi} k^2(s)\,ds = 2\int_0^{\pi} \ln^2(2\sin s/2)\,ds$ ← I don't know to do the integral directly!

One clue it's finite is $2\sin s/2 \approx s$ for $s \to 0^+$ & $\ln^2$ is integrable.

Parseval to the rescue: $\|k\|^2 = \sum_{n=0}^{\infty}\left(\sqrt{\frac{\pi}{2}}\,k_n\right)^2 = \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi}{2}\cdot\frac{\pi^2}{6} = \frac{\pi^3}{12} < \infty$

↑ since cosine basis should be normalized.

So it's in L^2.

6. [9 points] Short-answer questions.

[3] (a) Let K be a symmetric Fredholm operator with eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ and corresponding eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$. Either give the general solution to $Ku - \lambda_1 u = \phi_2$ or explain why not possible.

$\lambda = \lambda_1$ $\quad f = \phi_2$

Since K symm, eigfuncs are orthog, so $(f, \phi_1) = f_1 = 0$.

$\Rightarrow$ the (†) eqn for $i=1$ is $(\underbrace{\lambda_1 - \lambda_1}_{0}) c_1 = f_1 = 0$, c_1 anything.

Only other term is $i=2$ for $(\lambda_2 - \lambda_1) c_2 = f_2 = 1$, $c_2 = \dfrac{1}{\lambda_2 - \lambda_1}$

$\Rightarrow$ gen. soln. $u(x) = c\,\phi_1(x) + \dfrac{\phi_2(x)}{\lambda_2 - \lambda_1}$, $c \in \mathbb{R}$.

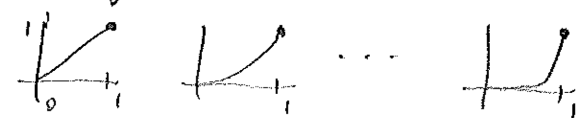
[3] (b) Let $\{f_n\}$ be a complete orthogonal set. Prove that no non-trivial function g can be added to the set whilst maintaining orthogonality of the resulting set.

Let g be a nontrivial function, suppose orthog. to all f_n's.

ie $(g, f_n) = 0$ By completeness (of $\{f_n\}$) $\Rightarrow g \equiv 0$, contradiction with non-trivial

[3] (c) Give an example of an interval and a sequence of functions that converge in L^2 but not uniformly on that interval (sketching may help.) Let's converge to zero, easiest.

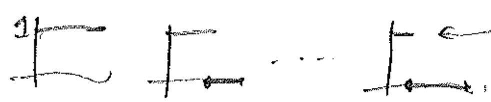
Example: $f_n = x^n$ on $(0,1)$:



max is always 1.

$\|f_n\|^2 = \int_0^1 x^{2n}\,dx = \dfrac{1}{2n+1} \to 0$.

Example: $f_n = \begin{cases} 1 & x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$ on $(0,1)$



$\|f_n\|^2 = \int_0^{1/n} 1^2\,dx = \frac{1}{n} \to 0$.

[BONUS: give an example as in (c) but the other way round, i.e. uniform but not L^2]

Requires unbounded interval, Example: $f_n = \begin{cases} 1/\sqrt{n} & |x| < n \\ 0 & \text{otherwise} \end{cases}$ on $\mathbb{R}$

$\|f_n\|^2 = \int_{-n}^{n} \frac{1}{n}\,dx = 2 \quad \forall n \not\to 0$.

But $\max_{x \in \mathbb{R}} |f_n(x)| = \frac{1}{\sqrt{n}} \to 0$, unif. convergent.