Directional Derivatives and the Gradient Vector Part 2

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Recall

- If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

- If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Maximizing the Directional Derivative

- Suppose that f is a differentiable function of two (or three) variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(x, y)$ is $|\nabla f|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(x)$.

Example

- If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.

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- If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.
- In what direction does f have the maximum rate of change? What is this maximum rate of change?

Example

- Suppose that the temperature at a point (x, y, z) in space is given by
$$T(x, y, z) = \frac{80}{1 + x^2 + 2y^2 + 3z^2},$$
where T is measured in degree Celsius and x, y, z in meters.

- In which direction does the temperature increase fastest at the point $(1, 1, -2)$?

- What is the maximum rate of increase?

Tangent Planes to Level Surfaces

- A level surface is a surface with equation

$$F(x, y, z) = k.$$

- Let $P(x_0, y_0, z_0)$ be a point on S and let C be any curve that lies on S and passes trough P.

- Recall that C is described by a continuous vector function

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle.$$

- If x, y, and z are differentiable and F is also differentiable, we can apply the Chain Rule:

$$\frac{\partial F}{\partial x}\frac{\mathsf{d}x}{\mathsf{d}t} + \frac{\partial F}{\partial y}\frac{\mathsf{d}y}{\mathsf{d}t} + \frac{\partial F}{\partial z}\frac{\mathsf{d}z}{\mathsf{d}t} = 0;$$

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- The gradient vector at P, $\nabla F(x_0, y_0 z_0)$ is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ to any curve C on S that passes through P.

The Tangent Plane

- We define **the tangent plane to the level surface $F(x, y, z) = k$ at** $P(x_0, y_0, z_0)$ as the plane passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$.

- It has equation

$$F_x(x_0, y_0, z_0)(x-x_0)+F_y(x_0, y_0, z_0)(y-y_0)+F_z(z_0, y_0, z_0)(z-z_0) = 0.$$

The Normal Line

- The **normal line** to S at P is the line passing through P and perpendicular to the tangent plane.

- The symmetric equations are

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Special Case

- If the equation of the surface S is of the form $z = f(x, y)$, that is

$$F(x, y, z) = f(x, y) - z = 0$$

then the equation of the tangent plane becomes

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

Examples

- Find the tangent plane and normal line of the surface

$$F(x, y, z) = x^2 + y^2 + z - 9 = 0$$

at the point $P_0(1, 2, 4)$.

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at the point $P_0(1, 2, 4)$.

- Find the equation of the tangent plane at the point $(-2, 1, -3)$ to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3.$$

Significance of the Gradient Vector

- The gradient ∇f gives the direction of fastest increase of f.
- The gradient Δf is orthogonal to the level surface S of f through a point P.