(#7) $(x+1)^3 = \varepsilon x$ zeroth order $(x_0+1)^3 = 0$ by setting $\varepsilon = 0$ $\Rightarrow x_0 = -1$ three times.

What powers of ε needed in first order correction? Try general power α:

subst. $x = \underset{-1}{x_0} + \varepsilon^\alpha x_1 + \varepsilon^\beta x_2 \cdots$ into eqn

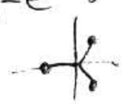
you can see usual $\alpha = 1$, fails to match.

$$(\cancel{-1} + \varepsilon^\alpha x_1 + \varepsilon^\beta x_2 \cancel{+1})^3 = \varepsilon(-1 + \varepsilon^\alpha x_1 + \varepsilon^\beta x_2 + \cdots)$$

$$\Rightarrow \quad \varepsilon^{3\alpha} x_1^3 + 3\varepsilon^{2\alpha+\beta} x_1^2 x_2 \cdots = -\varepsilon + \varepsilon^{1+\alpha} x_1 \cdots$$

requires $\alpha = \frac{1}{3}$ in order to match the ε on RHS.

$O(\varepsilon)$: $x_1^3 = -1$ so $x_1 = -1, -e^{\pm\frac{2\pi}{3}i}$ cube roots of -1



Now require $2\alpha + \beta = 1 + \alpha$ ie $\beta = \frac{2}{3}$ to match next term on RHS

$O(\varepsilon^{4/3})$: $3x_1^2 x_2 = x_1$ ie $x_2 = \frac{1}{3x_1} = -\frac{1}{3}, -\frac{1}{3}e^{\pm\frac{2\pi}{3}i}$

So expansions of the 3 roots are:

$$\begin{cases} -1 - \varepsilon^{1/3} - \frac{1}{3}\varepsilon^{2/3} \ \ldots \\ -1 - \varepsilon^{1/3} e^{\frac{2\pi}{3}i} - \frac{1}{3}\varepsilon^{2/3} e^{\frac{2\pi}{3}i} \ \ldots \\ -1 - \varepsilon^{1/3} e^{-\frac{2\pi}{3}i} - \frac{1}{3}\varepsilon^{2/3} e^{-\frac{2\pi}{3}i} \end{cases}$$

only asked for this one!

HW3 Math 46 Solutions S09

(#14) $\frac{dy}{dt} = e^{-\varepsilon/y}$ $y(0) = 1$ for $\varepsilon \ll 1$

leading order $\varepsilon = 0$ so $y_0' = 1$, $y_0(t) = 1 + t$,

Taylor expand

$$1 - \frac{\varepsilon}{y} + \frac{1}{2!}\left(\frac{\varepsilon}{y}\right)^2 - \cdots$$

subst. pert. expansion $y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 \cdots$

and use $\frac{1}{y} = \frac{1}{y_0}\left(1 + \varepsilon\frac{y_1}{y_0} + \cdots\right)^{-1} = \frac{1}{y_0}\left(1 - \varepsilon\frac{y_1}{y_0} \cdots\right.$

via Binomial expansion $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$

$$1 - \frac{\varepsilon}{y_0}\left(1 - \varepsilon\frac{y_1}{y_0} \cdots\right) + \frac{\varepsilon^2}{2}\frac{1}{y_0^2}\left(1 + O(\varepsilon) \cdots\right) - \cdots$$

$$= 1 - \varepsilon\frac{1}{1+t} + \varepsilon^2\left(\frac{y_1}{(1+t)^2} + \frac{1}{2(1+t)^2}\right) + O(\varepsilon^3) \cdots$$

So ODE becomes. $y_0' + \varepsilon y_1' + \varepsilon^2 y_2' \cdots = 1 - \varepsilon \frac{1}{1+t} + \frac{\varepsilon^2}{(1+t)^2}\left(y_1 + \frac{1}{2}\right) \cdots$

we've done $O(\varepsilon^0)$

so $O(\varepsilon^1)$: $y_1' = -\frac{1}{1+t}$ with IC $y_1(0) = 0$ by using pert. series in IC.

$O(\varepsilon^2)$: $y_2' = \frac{y_1 + 1/2}{(1+t)^2}$ with IC $y_2(0) = 0$

Can solve for $y_1(t) = -\ln(1+t)$ so $y_2' = \frac{1/2 - \ln(1+t)}{(1+t)^2}$ probably easy to integrate.

Exact? Soln. $e^{1/y} dy = dt$ will involve $Ei(\cdot)$ the exponential integral, not v. revealing.

#16 $y'' = \varepsilon t y$ $y(0) = 0$, $y'(0) = 1$ $y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 \cdots$

$y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' \cdots = \varepsilon t y_0 + \varepsilon^2 t y_1 \cdots$

$O(\varepsilon^0)$: $y_0'' = 0$ so $y_0(t) = t$.

$O(\varepsilon^1)$: $y_1'' = t y_0 = t^2$ so $y_1(t) = \frac{1}{12}t^4 + At + B$ with $\left.\begin{matrix} y_1(0) = 0 \\ y_1'(0) = 0 \end{matrix}\right\}$ so $A = B = 0$

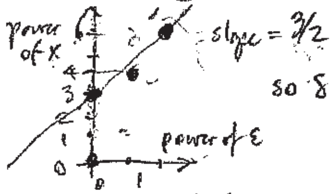
$O(\varepsilon^2)$: $y_2'' = t y_1 = \frac{1}{12}t^5$ so $y_2(t) = \frac{1}{504}t^7$

$y_a = t + \varepsilon \frac{t^4}{12} + \varepsilon^2 \frac{t^7}{504}$

$r(y, \varepsilon) := y_a'' - \varepsilon t y_a = \cancel{\varepsilon t^2} + \cancel{\frac{\varepsilon^2}{12}t^5} - \cancel{\varepsilon t^2} - \cancel{\frac{\varepsilon^3}{12}t^5} - \varepsilon^3 \frac{t^8}{504} = -\frac{\varepsilon^3 t^8}{504}$

since t^8 is unbounded on $[0, \infty)$, y_a does not uniformly satisfy the ODE on $[0, t)$.

p. 111–112 #1 a. zeroth order regular roots $\cancel{+}x^3 + 8 = 0$ ie $x = 2, 2e^{\frac{2\pi}{3}i}, 2e^{-\frac{2\pi}{3}i}$



so $\delta = \varepsilon^{-2/3}$. $y = \frac{x}{\delta}$: $\varepsilon^2 \delta^6 y^6 - \varepsilon \delta^4 y^4 - \delta^3 y^3 + 8 = 0$.

$\varepsilon^{-2} y^6 - \varepsilon^{-5/3} y^4 - \varepsilon^{-2} y^3 + 8 = 0$ mult. by ε^2

$y^6 - \varepsilon^{1/3} y^4 - y^3 + 8\varepsilon^2 = 0$

ε graphical method for dominant balancing..

leading order ($\varepsilon = 0$) $y^6 - y^3 = 0$ so $y = \underline{0 \text{ (thrice)}}, 1, e^{\frac{2\pi}{3}i}, e^{-\frac{2\pi}{3}i}$ ← these duplicate the 3 regular roots.

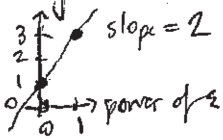
so all six roots are $x = 2, 2e^{\frac{2\pi}{3}i}, 2e^{-\frac{2\pi}{3}i}, \varepsilon^{-2/3}, \varepsilon^{-2/3} e^{\frac{2\pi}{3}i}, \varepsilon^{-2/3} e^{-\frac{2\pi}{3}i}$

#2 $\varepsilon x^3 + x - 2 = 0$ regular pert. for root $x_0 = 2$ use $x = x_0 + \varepsilon x_1 + \cdots$

$\varepsilon(x_0 + \varepsilon x_1 \cdots)^3 + x_0 + \varepsilon x_1 - 2 = 0$

$O(\varepsilon^1)$: $x_0^3 + x_1 = 0$ so $x_1 = -x_0^3 = -8$ so $x = 2 - 8\varepsilon + \cdots$

singular root



$\delta = \varepsilon^{-1/2}$

$$\varepsilon^{-1/2} y^3 + \varepsilon^{-1/2} y - 2 = 0$$
mult by $\varepsilon^{1/2}$
$$y^3 + y - 2\varepsilon^{1/2} = 0$$
now use regular pert.

as suggested, $y = y_0 + \varepsilon^{1/2} y_1 + \cdots$ so $(y_0 + \varepsilon^{1/2} y_1 - \ldots)^3 + y_0 + \varepsilon^{1/2} y_1 - 2\varepsilon^{1/2} = 0$

$y_0 = 0, i, -i$

copy of regular roots.

$O(\varepsilon^{1/2})$: $3y_0^2 y_1 + y_1 - 2 = 0$

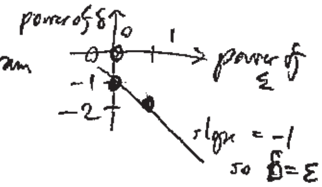
$\Rightarrow$ $y_1 = -1$ for both roots.

Summary: roots $x = \begin{cases} 2 - 8\varepsilon + O(\varepsilon^2) \\ \varepsilon^{-1/2} i - 1 + O(\varepsilon^{1/2}) \\ -\varepsilon^{-1/2} i - 1 + O(\varepsilon^{1/2}). \end{cases}$

p. 121-123 #1 a. $\varepsilon y'' + 2y' + y = 0$ $\quad y(0) = 0 \quad y(1) = 1$

outer $2y_0' + y_0 = 0$ so $y_0(x) = Ce^{-x/2}$ so if match @ $x=1$, $C = e^{1/2}$

Inner [bdry layer at $x=0$] $\frac{\varepsilon}{\delta^2} Y'' + 2\frac{1}{\delta} Y' + Y = 0$ balance via diagram



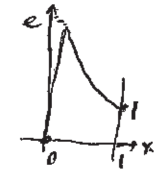
$\varepsilon^{-1} Y'' + 2\varepsilon^{-1} Y' + Y = 0$ $\quad \xi = \frac{x}{\delta}$

mult by ε: $Y'' + 2Y' + \cancel{\varepsilon Y} = 0$

$Y_i(\xi) = Ae^{-2\xi} + B$
$= A(1 - e^{-2\xi})$, since BC $y(0) = 0$.

$\lim_{\xi \to \infty} Y_i(\xi) = A = \lim_{x \to 0} y_0(x) = e^{1/2} = c_m$

So $y_u(x) = y_0(x) + y_i(x) - c_m = e^{1/2}(e^{-x/2} + 1 - e^{-2x/\varepsilon} - 1)$
$= e^{1/2}(e^{-x/2} - e^{-2x/\varepsilon})$

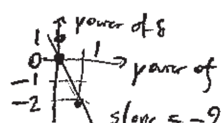


f. $\varepsilon y'' + xy' - xy = 0$ $\quad y(0) = 0 \quad y(1) = e$ $\quad$ outer $y' = y$ so $y_0 = Ce^x$ so $C = 1$ by $y(1) = 1$.

inner (BL @ $x=0$): use $\frac{x}{\delta} = \xi$

$\frac{\varepsilon}{\delta^2} Y'' + \xi Y' - \delta Y = 0$

$Y'' + \xi Y' - \cancel{\varepsilon^{1/2} Y} = 0$



so using $V = Y_i'$, $\frac{dV}{d\xi} = -\xi V \Rightarrow \int \frac{dV}{V} = -\int \xi \, d\xi$

$\Rightarrow \ln V = -\frac{1}{2}\xi^2 + C$ ie $V(\xi) = Ae^{-\frac{1}{2}\xi^2}$ ie $Y_i(\xi) = A\int e^{-\xi^2/2} d\xi + B$

A boundary layer at $x=0$ with the 'erf' shape

BC@$\xi = 0$: $y = 0$ gives $B = 0$

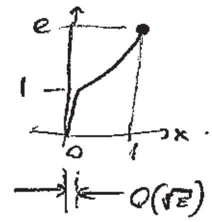
ie $Y_i(\xi) = A\int_0^\xi e^{-s^2/2} ds$.

To take lim we use gaussian integral (eg see p. 148) $\int_0^\infty e^{-s^2/2}\,ds = \frac{1}{2}\sqrt{2\pi} = \sqrt{\frac{\pi}{2}}$

So $c_m = \sqrt{\frac{\pi}{2}}A = \lim_{\xi\to\infty} Y_i(\xi) = \lim_{x\to 0} y_o(x) = C = 1$. ie $A = \sqrt{\frac{2}{\pi}}$

$\Rightarrow y_u = \underbrace{\sqrt{\frac{2}{\pi}}\int_0^{x/\sqrt{\varepsilon}} e^{-s^2/2}\,ds}_{\text{can be written}} \underbrace{-1}_{-c_m} + e^x = \underline{\operatorname{erf}\left(\frac{x}{\sqrt{2\varepsilon}}\right) - 1 + e^x}$

$\sqrt{2}\int_0^{\frac{x}{\sqrt{2}\sqrt{\varepsilon}}} e^{-t^2}\,dt = \sqrt{\frac{\pi}{2}}\operatorname{erf}\left(\frac{x}{\sqrt{2\varepsilon}}\right)$



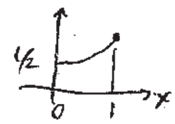
h. $\varepsilon y'' - (2-x^2)y = -1 \qquad y'(0) = 0 \quad y(1) = 1$

outer $(2-x^2)y_o = 1$ so $y_o(x) = \frac{1}{2-x^2}$ $\quad y_o(1) = 1$ and $y_o'(0) = \left.\frac{2x}{(2-x^2)^2}\right|_{x=0} = 0$

so outer satisfies both BCs ~ no inner is needed.

(you may solve for bdry layers but find their coeffs are necessarily zero).

$y_u(x) = \frac{1}{2-x^2}$ is uniform approx.



i. $\varepsilon y'' - b(x)y' = 0 \qquad y(0) = \alpha \quad y(1) = \beta \qquad \alpha \neq \beta, \quad b(x) > 0$

relative signs tell you bdry layer at $x=1$ will be well-behaved by not at $x=0$.

so $\xi = \frac{1-x}{\delta}$: inner will be

$\frac{\varepsilon}{\delta^2}Y_i'' + \frac{b(1-\delta\xi)}{\delta}Y_i' = 0$ — in $\lim \delta\to 0$, $b(1)$, for leading order. balance with $\delta = \varepsilon$.

so outer is $b(x)y_o' = 0$ ie $y_o' = 0$ ie $y_o(x) = \alpha$ since match at $y(0) = \alpha$.

use $V = Y'$ so $\cancel{\varepsilon^{-1}}V' + \cancel{\varepsilon^{-1}}b(1-\varepsilon\xi)V = 0$

ie $V(\xi) = Ae^{-b(1)\xi}$

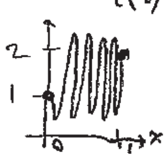
$Y_i(\xi) = Ae^{-b(1)\xi} + B$ — needs to match. outer in intermediate scale around $x=1$.

$\lim_{\xi\to\infty} Y_i(\xi) = B = \lim_{x\to 1} y_o(x) = \alpha$ so $B = \alpha = c_m$ $\qquad$ match @ $x=1$: $A + B = \beta$ so $A = \beta - \alpha$

$y_u(x) = \underbrace{\cancel{\alpha}}_{y_o} - \underbrace{\cancel{\alpha}}_{c_m} + (\beta-\alpha)e^{-b(1)\frac{1-x}{\varepsilon}} + \alpha$.

#2 $\varepsilon u'' + u = 0$: $u = A\sin\frac{x}{\sqrt{\varepsilon}} + B\cos\frac{x}{\sqrt{\varepsilon}}$ ie high frequency osc. at period $O(\sqrt{\varepsilon})$.

$u(0) = 1$ forces $B = 1$ so $u(1) = 2$ gives $2 = A\sin\frac{1}{\sqrt{\varepsilon}} + \cos\frac{1}{\sqrt{\varepsilon}}$ ie $A = \frac{2-\cos(\varepsilon^{-1/2})}{\sin(\varepsilon^{-1/2})}$



Failure since in outer there's only 1 term which must vanish, same with inner since can't balance against anything. There is no separation of scales.

#3 $\quad \varepsilon y'' + y' + ae^{y} = 0 \qquad y(0) = 0 \qquad y(1) = 0$

outer : $\quad \frac{dy_0}{dx} e^{-y_0} = -a \qquad$ ie $\int e^{-y_0} dy_0 = -ax + c$

ie $\quad e^{-y_0} = ax + c \quad$ or $\quad y_0 = -\ln(ax + c)$

either BL@ $x=0$ so $y_0(1) = 0$ so $a + c = 1$, $c = 1 - a$

or BL@ $x = 1$ so $c = 1$.

inner : try BL@ $x=0$, $\xi = \frac{x}{\delta}$ $\qquad \frac{\varepsilon}{\delta^2} Y'' + \frac{1}{\delta} Y' + ae^{Y} = 0 \qquad \delta = \varepsilon.$

so $\quad Y_i'' + Y_i' + \underbrace{\varepsilon a e^{Y_i}}_{\to 0} = 0 \qquad$ ie $\quad Y_i(\xi) = Ae^{-\xi} + B$

well-behaved (BL @ $x=1$ would not exist since signs of y'' & y' are the same).

Outer must then have $c = 1 - a$ so $y_0(x) = -\ln(ax - a + 1) = -\ln(a(x-1) + 1)$

When $\boxed{a < +1}$ then the argument $a(x-1) + 1$ is always positive for $x \in (0,1)$.

If this doesn't hold, outer diverges (ln of zero) and would get interior layer not bdry layer.

#4 $\quad \varepsilon u'' \underbrace{- (2x+1) u'}_{\text{always } < 0} + 2u = 0 \qquad u(0) = 1 , \quad u(1) = 0$

so. BL @ $x=1$ facing backward for correct decay in BL.

outer: $\quad (2x+1) u_0' = 2u$

$\int \frac{du}{u} = \int \frac{2}{2x+1} dx \qquad$ ie $\quad \ln|u| = \ln|x + 1/2| + c \qquad$ ie $\quad u(x) = c(x + 1/2)$

match $y_0(0) = 1$. $\qquad$ so $c = 2$ $\quad$ ie $\quad y_0(x) = 1 + 2x$.

inner : $\quad \xi = \frac{1-x}{\delta} \qquad \frac{\varepsilon}{\delta^2} U'' + \underbrace{\frac{1}{\delta}(2[1 - \delta\xi] + 1) U'}_{\frac{3}{\delta} - 2\xi} + 2U = 0$

$\delta = \varepsilon$ to balance first 2 terms, leave others small.

$U'' + 3U' + O(\varepsilon) = 0 \quad$ ie $\quad U_i(\xi) = Ae^{-\xi/3} + B = A(1 - e^{-\xi/3})$ $\quad \Leftarrow B = -A$ since $U_i(0) = 0$ follows from $u(1) = 0$.

Match $\quad \lim_{\xi \to \infty} U_i(\xi) = A = \lim_{x \to 1} u_0(x) = 3 = c_m.$

So $\quad u_u = 1 + 2x + 3(1 - e^{-\xi/3}) - 3 \quad = \quad \underline{1 + 2x - 3e^{-\frac{1-x}{3\varepsilon}}}$