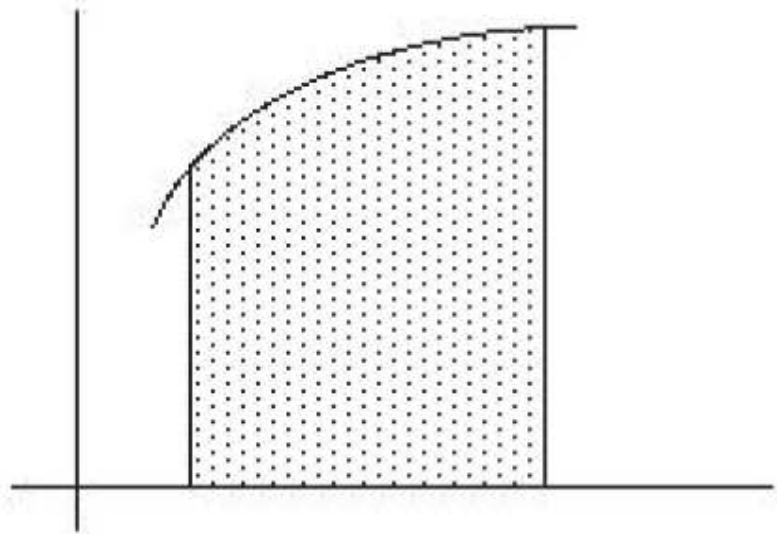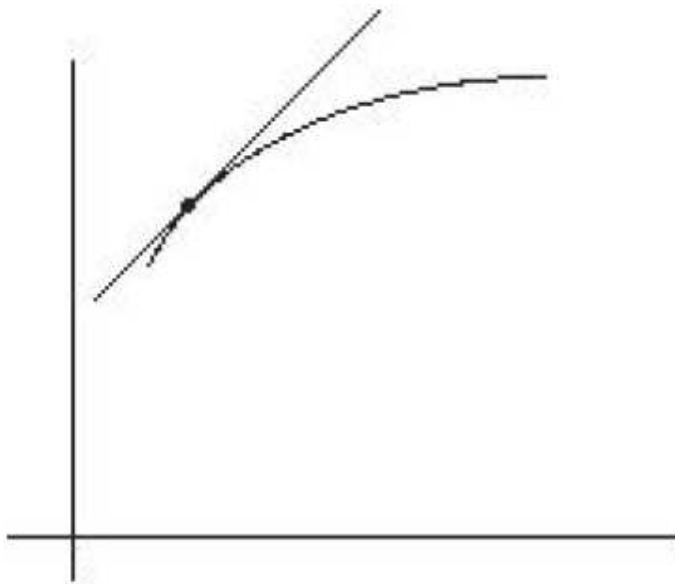# Modeling Accumulations: Introduction to the Issues

11/07/2005

The purpose of calculus is twofold:

- to find the slope of a curve at a point;

- and to find the area lying under a curve and above an interval of the x-axis.

The Area of a Circle

- We all know the formula A for the area of a circle:
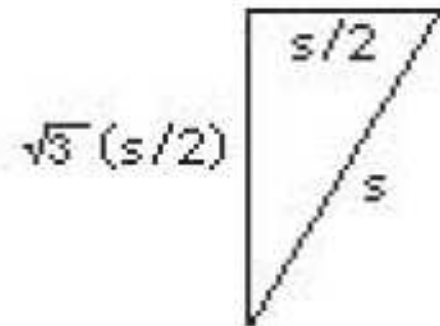
$$A = \pi r^2,$$

  where r is the radius and is the irrational number whose decimal expansion to 20 decimal places begins 3.14159265358979323846.

- What exactly is the area of a circle?

- What is *its definition*?

- How do we compute the area of a circle?

4
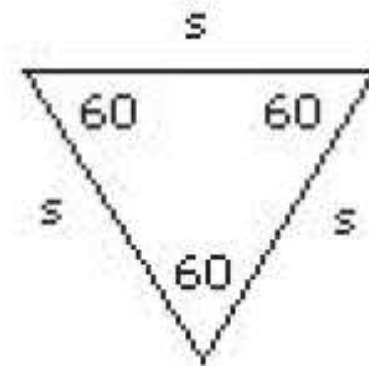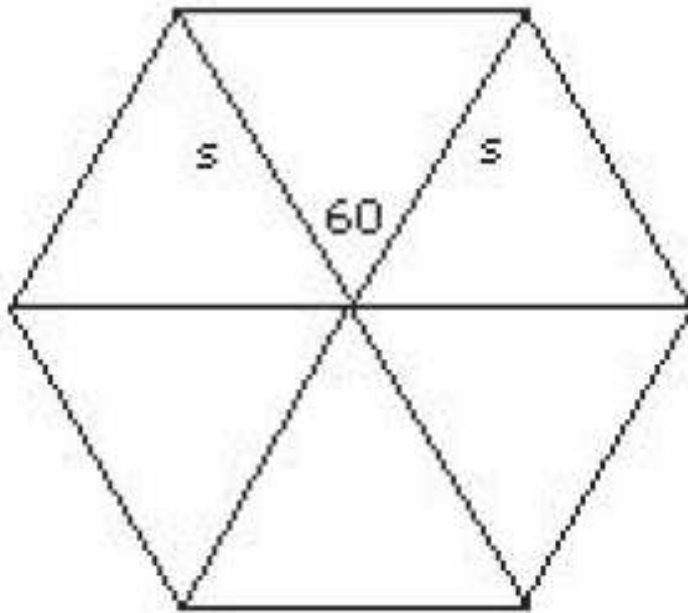
- It is composed of six congruent isoseles triangles, each with a $60 (= 360/6)$ degree central angle.

- The base angles of each triangle are also 60 degrees, and the third side has the same length as the other two.

- The area of one of the isoseles triangles is

$$\frac{1}{2} s \frac{\sqrt{3} s}{2}.$$

s

s/2

$\sqrt{3}$ (s/2)

s

s

Area =

$\sqrt{3}$ (3/2)(s)(s)

Perim = 6s

s/2

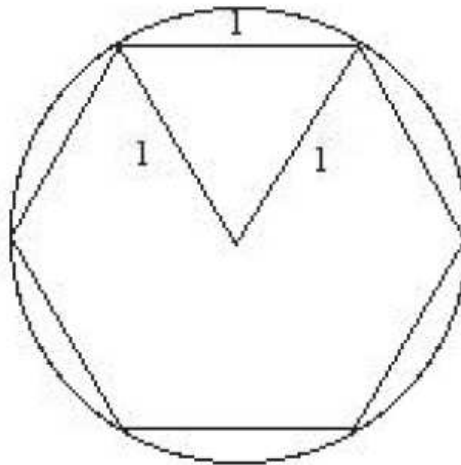$\sqrt{3}$ (s/2)

s
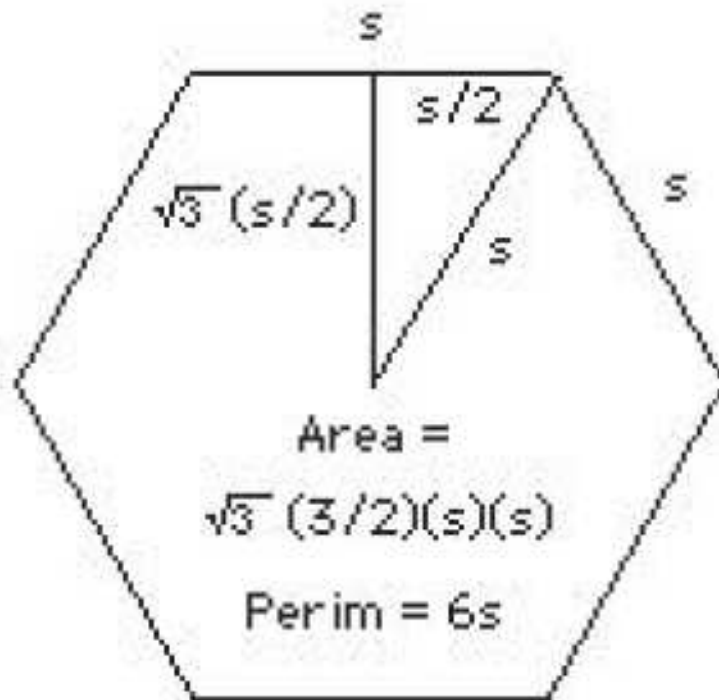
- The formula we have just derived tells us that the area of the hexagon is

$$\frac{\sqrt{3}}{3}\sqrt{3}s^2,$$

  or approximately 2.598.

- This does not give a very good approximation to the area of the circle.

- We could replace the hexagon with more completely filling shapes whose areas we can still calculate.

- One of the best ways to do this is to double the number of edges of the hexagon, thereby obtaining a regular 12-gon; and then to continue doubling repeatedly to obtain in succession a 24-gon, a 48-gon, a 96-gon, and so on.

s

1   1

y

new s

x   x

1−y

1   1
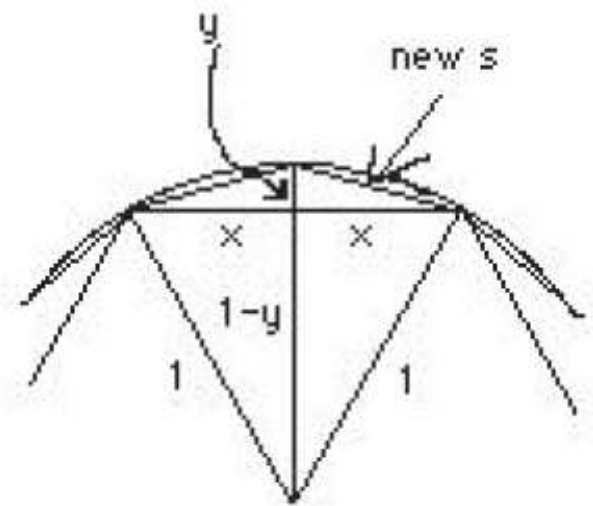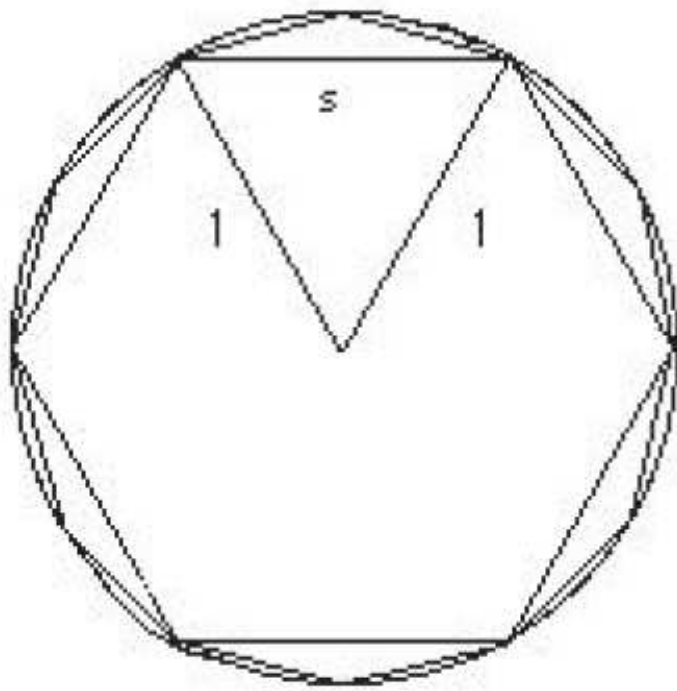
The area of the 12-gon

- The area of the 12-gon equals the area of the hexagon plus 12 times the area of a little triangle:

$$\frac{3}{2}\sqrt{3} + 12\frac{x}{y}.$$
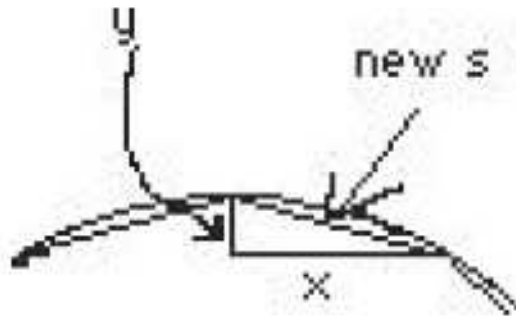
1. $x = s/2$ [To begin $s = 1$.]

2. $y = 1 - \sqrt{1 - x^2}$

3. $new\ s = \sqrt{x^2 + y^2}$

4. $new\ n = 2n$ [To begin $n = 6$.]

5. $new\ A = A + (new\ n)\frac{xy}{2}$ [To begin $A = \frac{3}{2}\sqrt{3}$.]

Areas of Regular Polygons	
sides	area
6	2.598076
12	3.000000
24	3.105829
48	3.132629
96	3.139350
192	3.141032
384	3.141452
768	3.141558
1536	3.141584
3072	3.141590
6144	3.141592

What is the Area of a Circle?

- It is reasonable to define the area of a unit circle to be the limit of the areas of the inscribed regular polygons that come from starting with a hexagon and doubling the number of sides at each successive stage.

- What is so special about these geometrical figures?

- Why not use any collection of shapes that are contained in the circle and fill it in the limit?

Another Calculation of the Area of a Circle

- We divide the interval $[0, 1]$ into n subintervals of equal length $h = 1/n$.

- The circle is the graph of the function

$$f(x) = \sqrt{1 - x^2}.$$
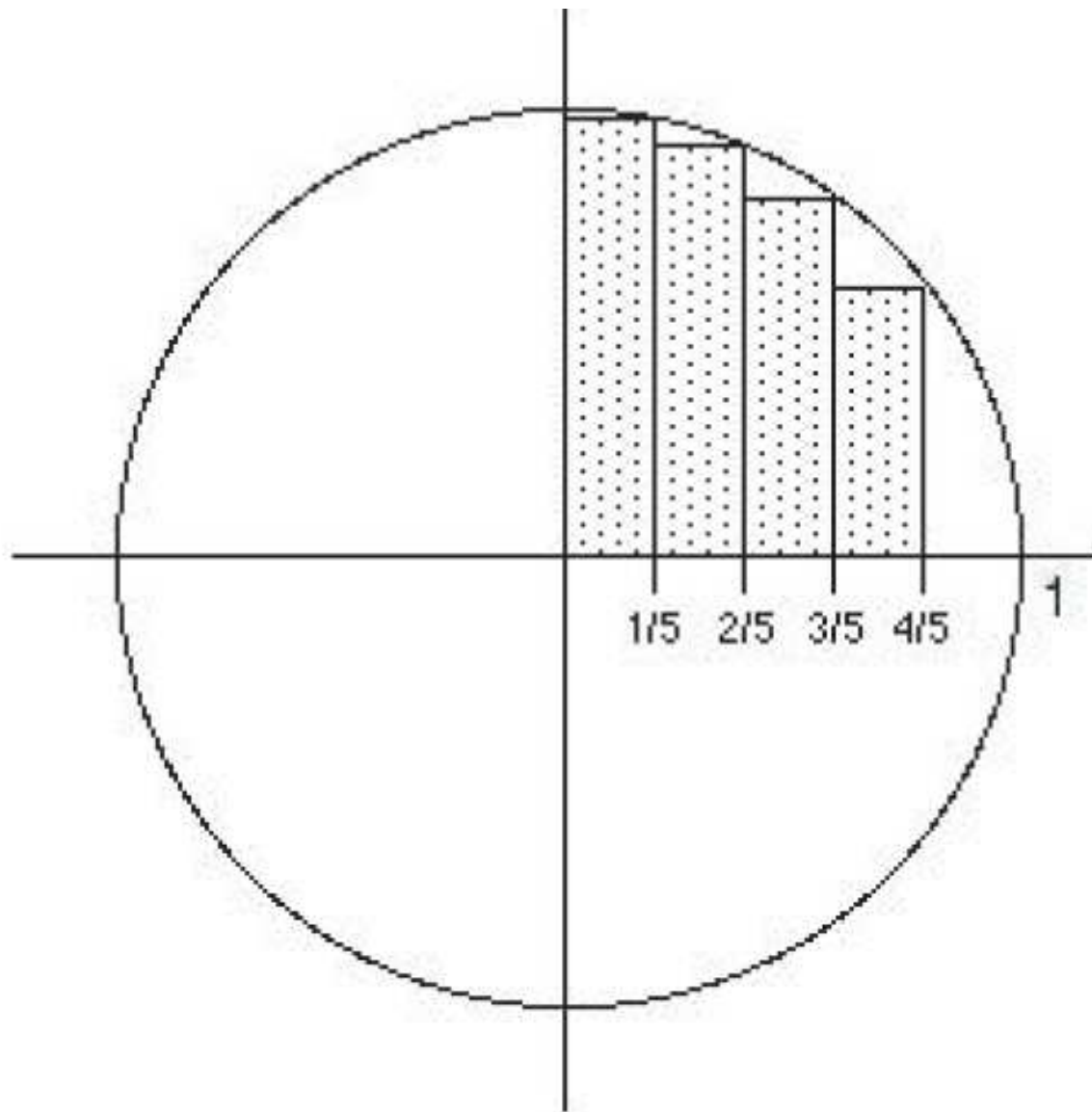
Approximating Area of Unit Circle with Rectangles

rectangles	sum of areas times 4
5	2.637049
500	3.137487
1000	3.139555
2000	3.140580
5000	3.141189

The Method of Accumulations

- The process of passing to the limit not only provides a calculational tool, but it gives a way to define what is meant by the area under the curve.

The Circumference of a Circle

Circumferences of Regular Polygons		
sides	perimeter	π
6	6.000000	3.000000
12	6.211656	3.105829
24	6.265260	3.132629
48	6.278700	3.139350
96	6.282066	3.141032
192	6.282906	3.141452
384	6.283116	3.141558
768	6.283170	3.141584
1536	6.283182	3.141590
3072	6.283182	3.141592
6144	6.283182	3.141593

The Volume of Water in a River
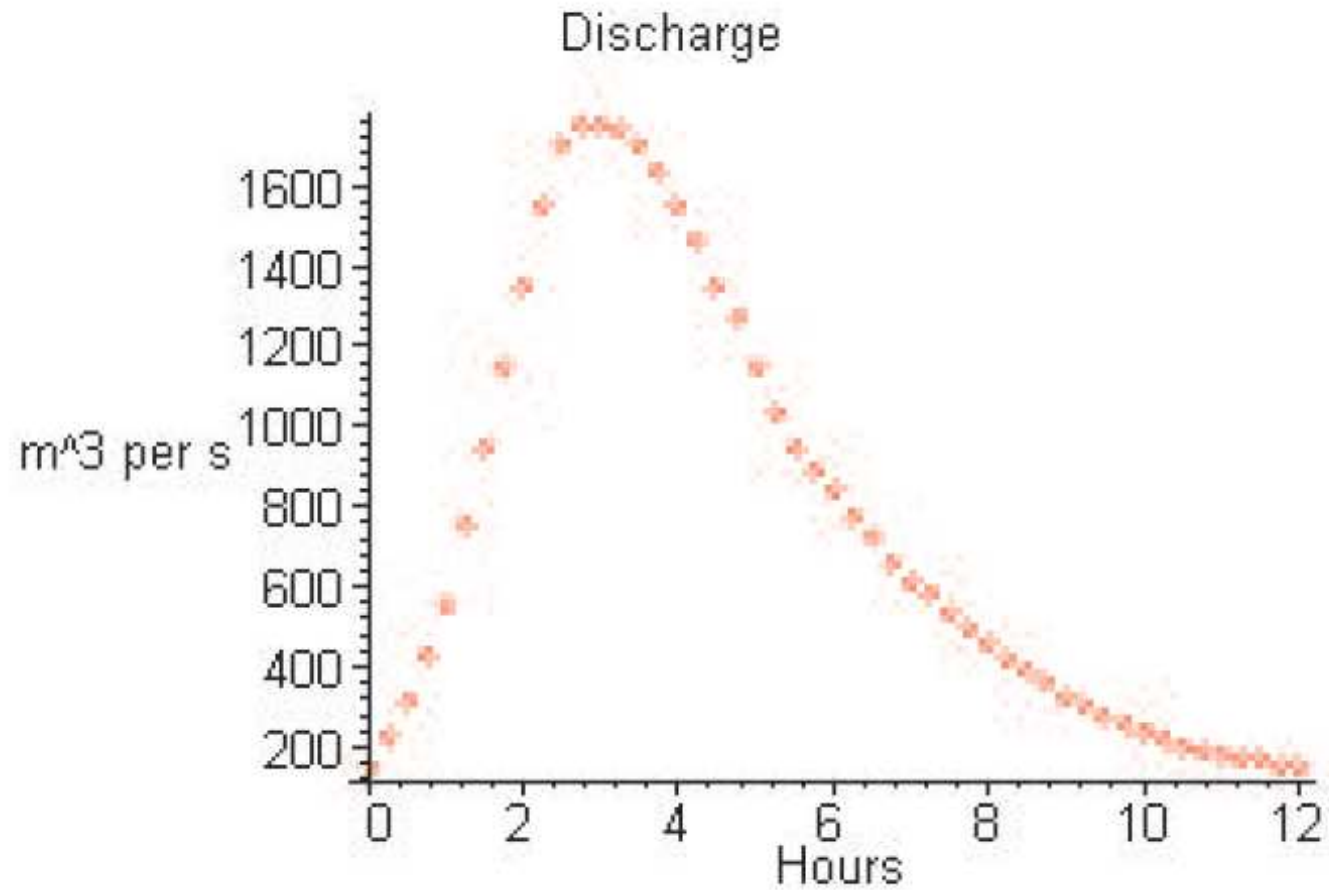
Discharge of a River Stream					
hours	m^3/s	hours	m^3/s	hours	m^3/s
0	150	4	1550	8	460
0.25	230	4.25	1460	8.25	423
0.5	310	4.5	1350	8.5	390
0.75	430	4.75	1270	8.75	365
1	550	5	1150	9	325
1.25	750	5.25	1030	9.25	300
1.5	950	5.5	950	9.5	280
1.75	1150	5.75	892	9.75	260
2	1350	6	837	10	233
2.25	1550	6.25	770	10.25	220
2.5	1700	6.5	725	10.5	199
2.75	1745	6.75	658	10.75	188
3	1750	7	610	11	180
3.25	1740	7.25	579	11.25	175
3.5	1700	7.5	535	11.5	168
3.75	1630	7.75	500	11.75	155
				12	150
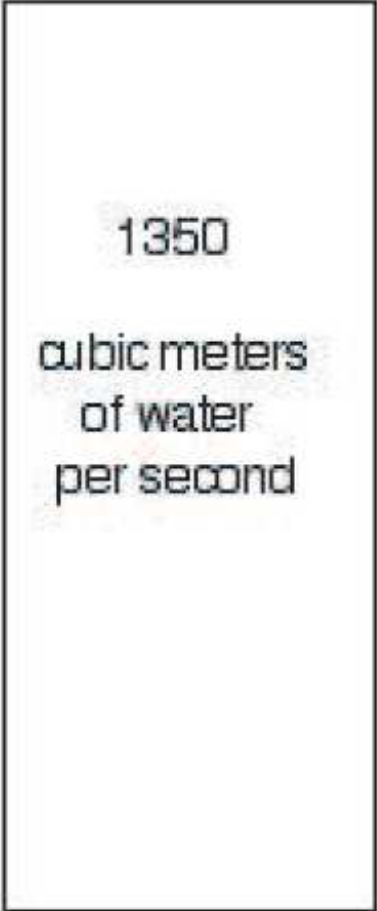
Volume Over Each Subinterval					
subint	m^3	subint	m^3	subint	m^3
0	135000	4	1395000	8	414000
0.25	207000	4.25	1314000	8.25	380700
0.5	279000	4.5	1215000	8.5	351000
0.75	387000	4.75	1143000	8.75	328500
1	495000	5	1035000	9	292500
1.25	675000	5.25	927000	9.25	270000
1.5	855000	5.5	855000	9.5	252000
1.75	1035000	5.75	802800	9.75	234000
2	1215000	6	753300	10	209700
2.25	1395000	6.25	693000	10.25	198000
2.5	1530000	6.5	652500	10.5	179100
2.75	1570500	6.75	592200	10.75	169200
3	1575000	7	549000	11	162000
3.25	1566000	7.25	521100	11.25	157500
3.5	1530000	7.5	481500	11.5	151200
3.75	1467000	7.75	450000	11.75	139500
				12	135000