# 4.9: Arc Length (cont'd)
# and
# 4.10: Inverse Trig Functions

Mathematics 3

**The FINAL Lecture! ☺**

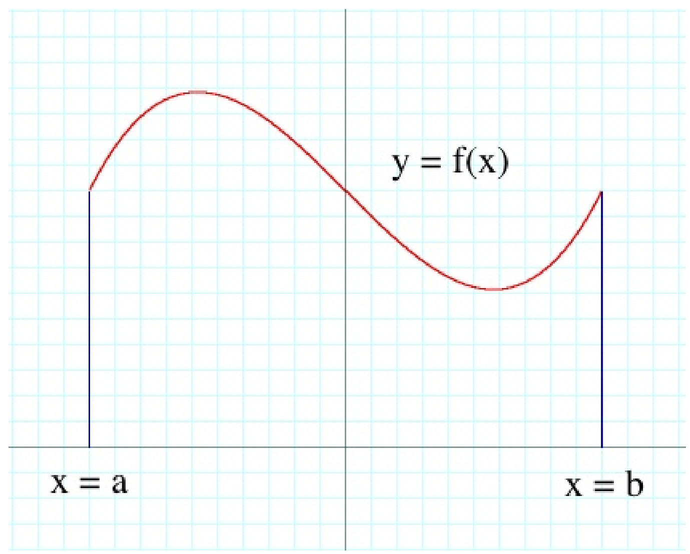
Dartmouth College

**March 08, 2010**

# The Arc Length Formula

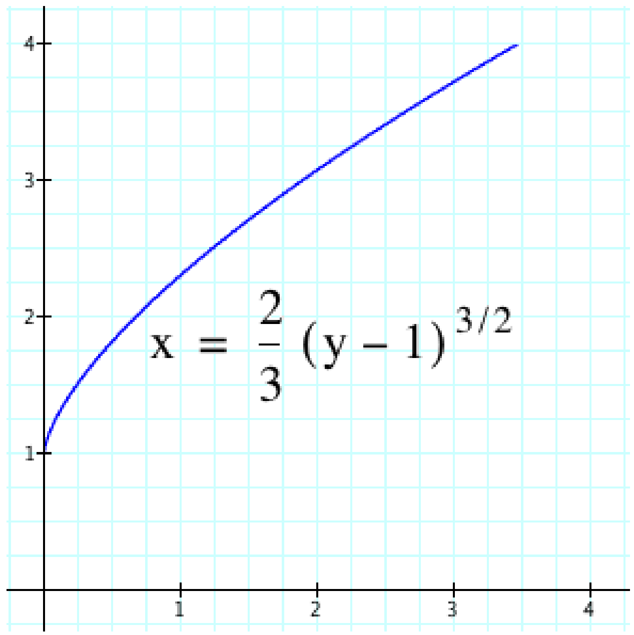
The integral formula to compute the length $L$ of the graph of a (differentiable) function $y = f(x)$ between $x = a$ and $x = b$ is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$



# Example 1

Determine the arc length of $x = \frac{2}{3}(y-1)^{3/2}$ over $1 \leq y \leq 4$.

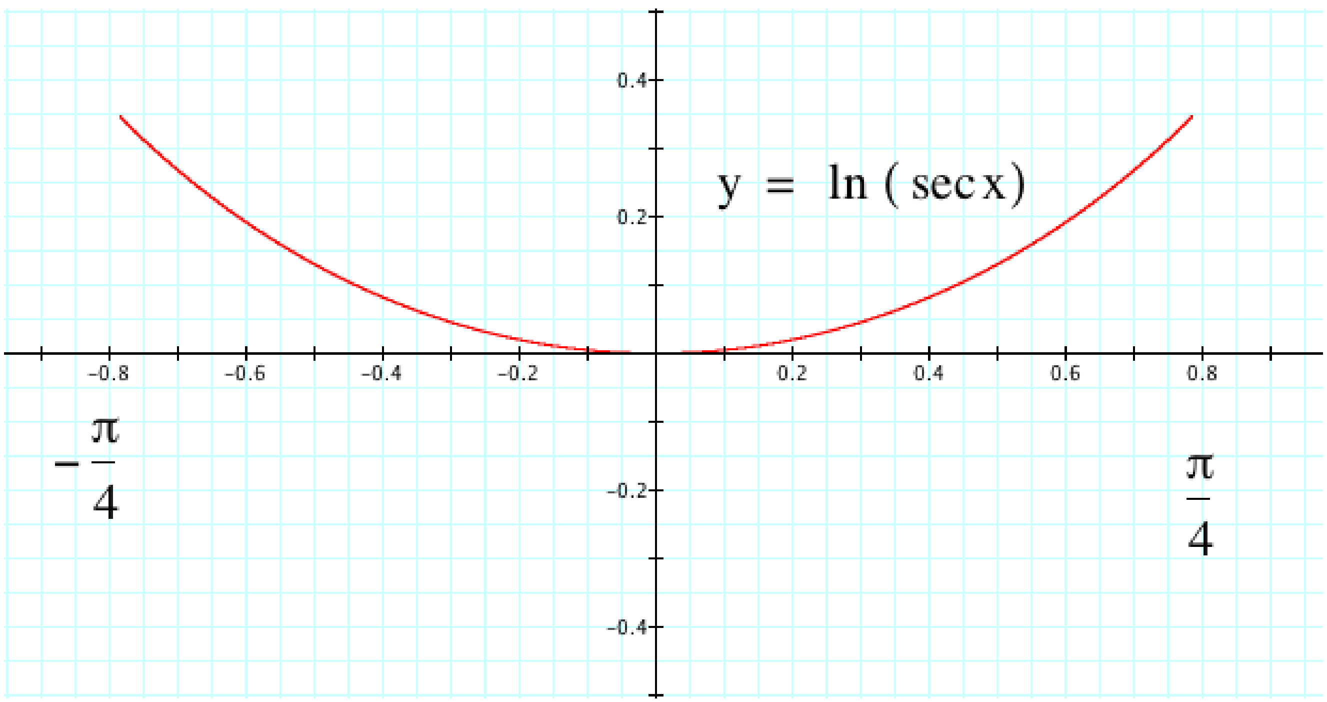


# Example 2

Find the arc length of the curve

$$y = \ln(\sec x)$$

between $x = -\pi/4$ and $x = \pi/4$.



**Integrals of the Six Basic Trigonometric Functions**

$$\int \sin u \, du = -\cos u + C \qquad \int \cos u \, du = \sin u + C$$

$$\int \tan u \, du = -\ln|\cos u| + C \qquad \int \cot u \, du = \ln|\sin u| + C$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + C \qquad \int \csc u \, du = -\ln|\csc u + \cot u| + C$$

# We need a few more derivative/integral formulas to complete basic calculus...

For example, we know that

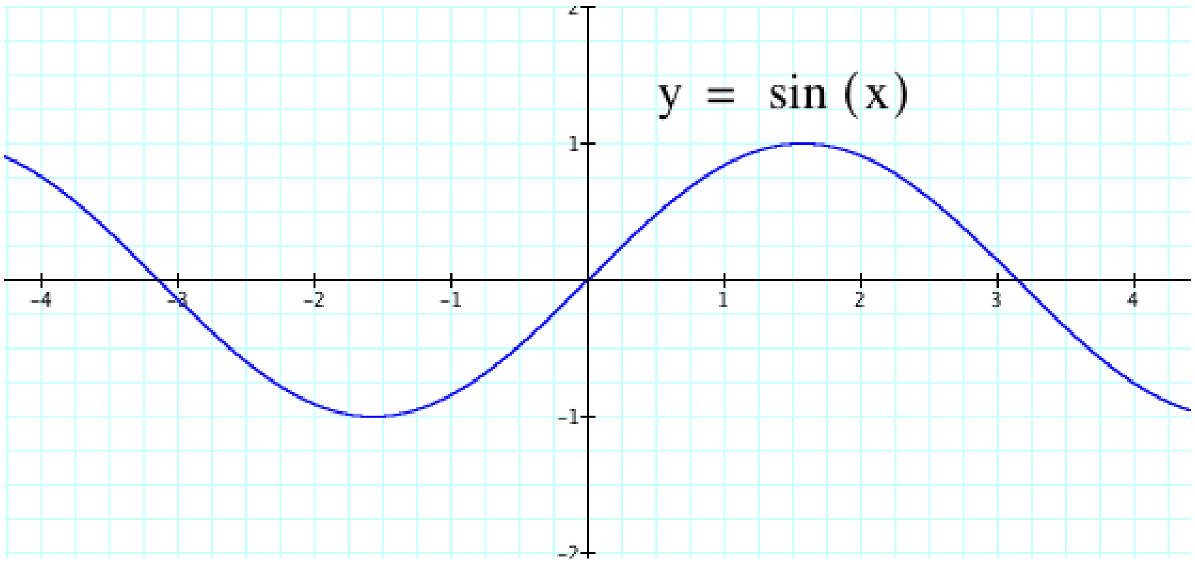
$$\int (9 + x^2)\, dx = 9x + \frac{x^3}{3} + C$$

but what about

$$\int \frac{1}{9 + x^2}\, dx = \quad ???$$

# Inverse Trigonometric Functions

**Problem:** The function $y = \sin(x)$ is **not** invertible...



# Inverse Trigonometric Functions

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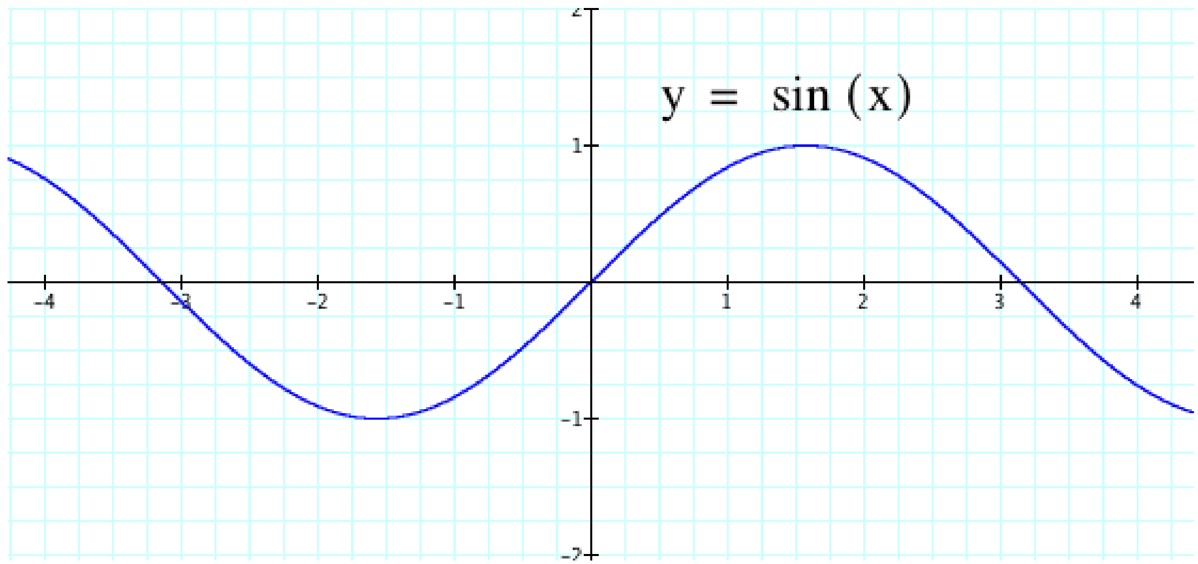
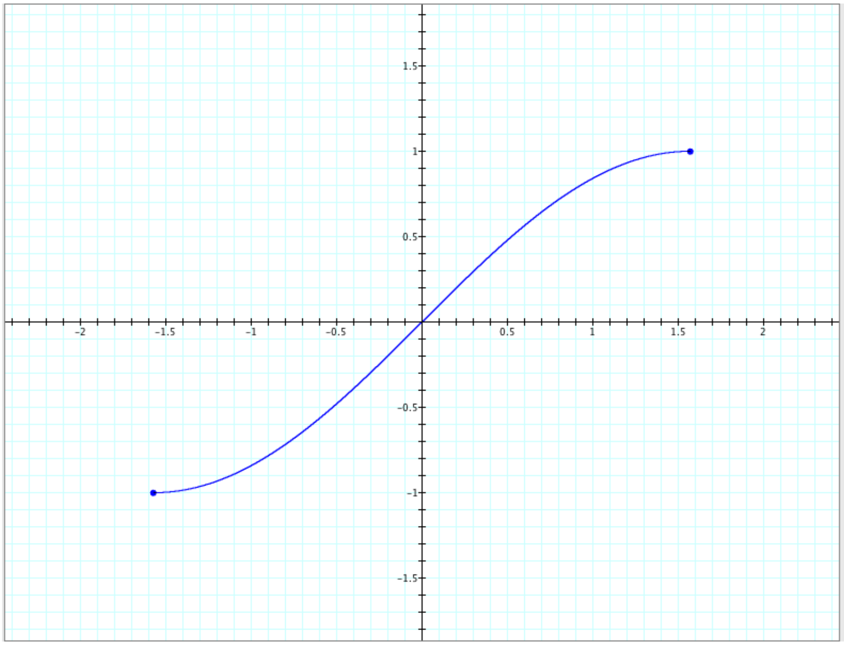
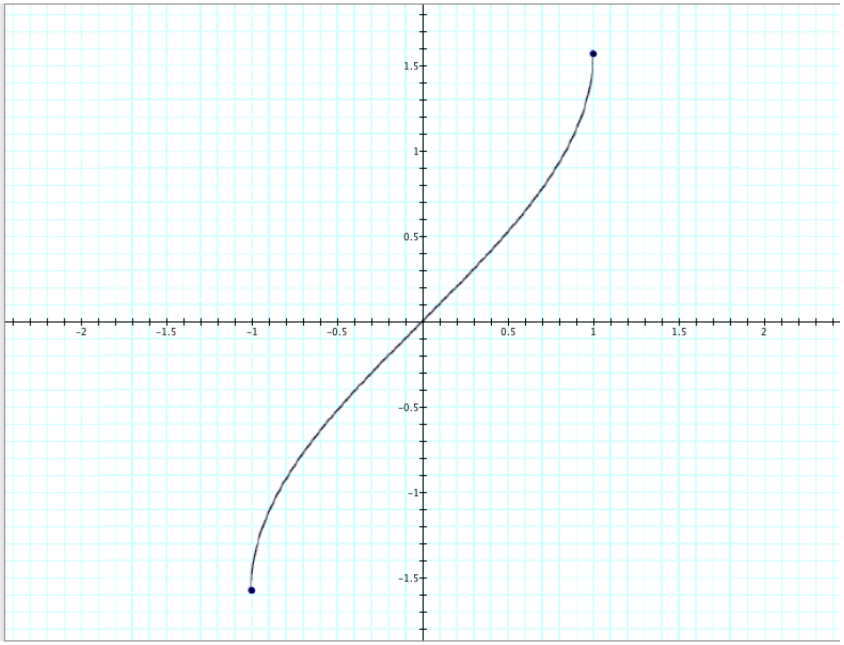



But, if we *restrict* the domain to $-\pi/2 \leq x \leq \pi/2$ we get an invertible function...

# The Arcsine (Inverse Sine) Function

For $-1 \leq x \leq 1$, $y = \arcsin(x)$ if and only if $\sin(y) = x$, $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.



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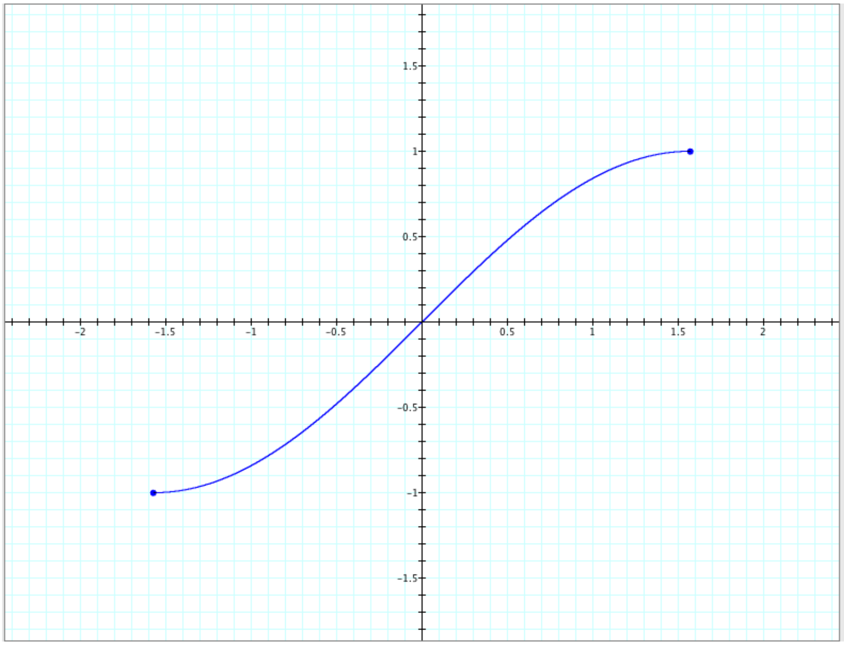


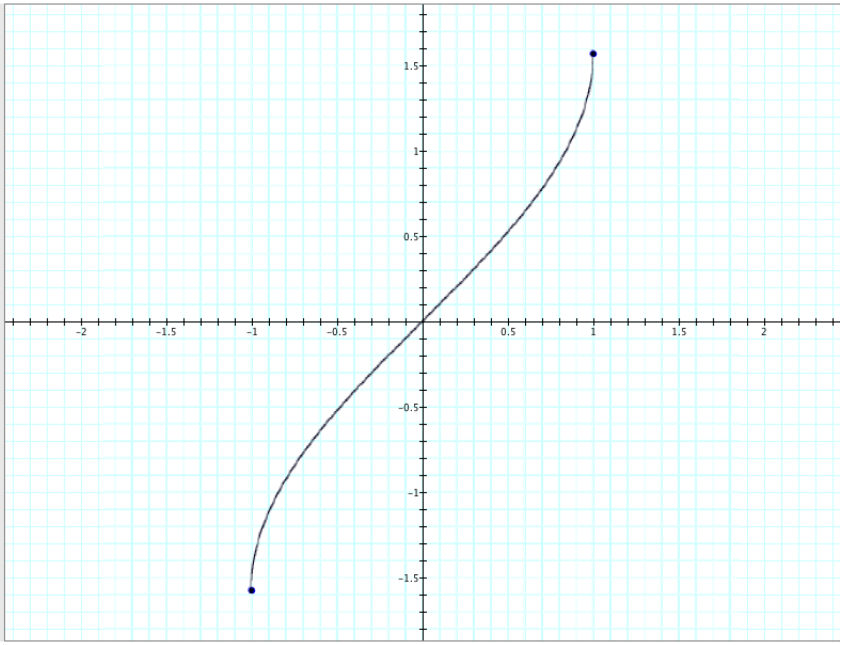


$y = \arcsin(x)$ is the angle $y$ (rads) between $-\pi/2$ and $\pi/2$ whose sine is $x$.

# Example 3

Evaluate the following:

a.) $\arcsin(-\frac{1}{2})$

b.) $\sin(\sin^{-1}(\frac{1}{3}))$

c.) $\arcsin(\sin(\frac{2\pi}{3}))$

# Derivative of the Arcsine Function

We will use facts about inverse functions and their derivatives:

$$
\begin{aligned}
y &= \arcsin(x) = \sin^{-1}(x) \\
\sin(y) &= x \\
\frac{d}{dx}\sin(y) &= \frac{d}{dx}(x) \\
\cos(y)\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-\sin^2(y)}} = \frac{1}{\sqrt{1-x^2}}
\end{aligned}
$$

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\end{aligned}$$

$$\frac{d}{dx}\arcsin(x) = \frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$$

# Example 4

Evaluate the following:

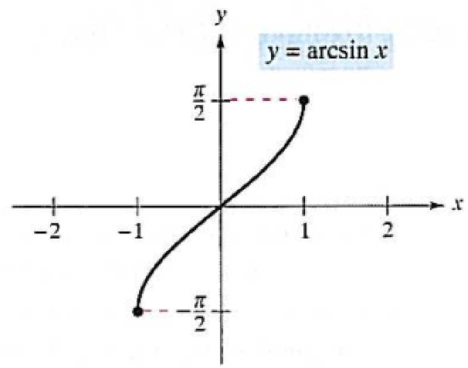
a.) $\dfrac{d}{dx}\arcsin(2x)$

b.) $\dfrac{d}{dx}\sin^{-1}(e^{2x})$

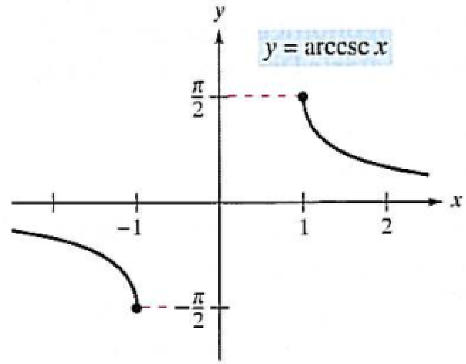
c.) Differentiate $y = \arcsin(x) + x\sqrt{1-x^2}$ and simplify.

d.) $\displaystyle\int \frac{dx}{\sqrt{4-x^2}}$

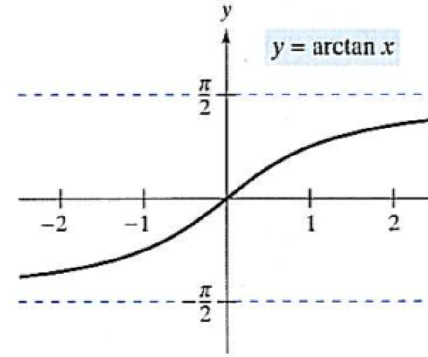
# The Six Inverse Trig Functions



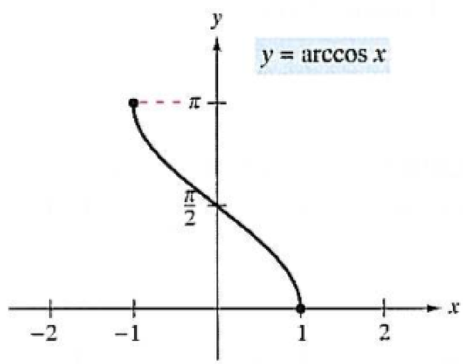

Domain: $[-1, 1]$
Range: $[-\pi/2, \pi/2]$



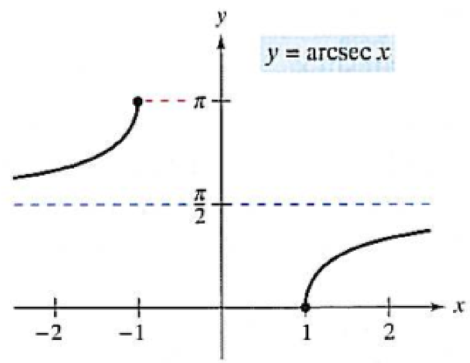

Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[-\pi/2, 0) \cup (0, \pi/2]$



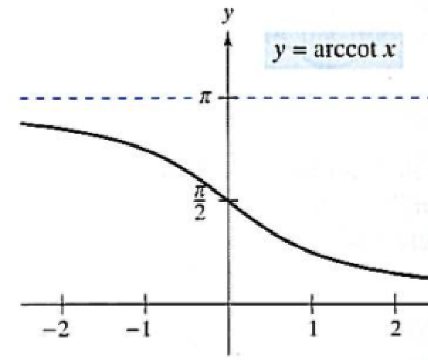

Domain: $(-\infty, \infty)$
Range: $(-\pi/2, \pi/2)$




Domain: $[-1, 1]$
Range: $[0, \pi]$




Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[0, \pi/2) \cup (\pi/2, \pi]$




Domain: $(-\infty, \infty)$
Range: $(0, \pi)$

# Derivatives of Inverse Trig Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

Two of these give important integral formulas:

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + C = \arcsin(x) + C$$

$$\int \frac{dx}{1+x^2} = \tan^{-1}(x) + C = \arctan(x) + C$$

## Example 5

Evaluate the following:

a.) $\tan(\sec^{-1}(\sqrt{5}/2))$.

b.) Solve $\arctan(2x - 5) = \dfrac{\pi}{4}$ for $x$.

c.) If $f(x) = x\cot^{-1}(\sqrt{x})$ find $f'(x)$.

d.) Find the general solution to $9y' + x^2y' = 1$.

**Basic Differentiation Rules for Elementary Functions**

1. $\frac{d}{dx}[cu] = cu'$
2. $\frac{d}{dx}[u \pm v] = u' \pm v'$
3. $\frac{d}{dx}[uv] = uv' + vu'$
4. $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}[c] = 0$
6. $\frac{d}{dx}[u^n] = nu^{n-1}u'$
7. $\frac{d}{dx}[x] = 1$
8. $\frac{d}{dx}[|u|] = \frac{u}{|u|}(u'),\ u \neq 0$
9. $\frac{d}{dx}[\ln u] = \frac{u'}{u}$
10. $\frac{d}{dx}[e^u] = e^u u'$
11. $\frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$
12. $\frac{d}{dx}[a^u] = (\ln a)a^u u'$
13. $\frac{d}{dx}[\sin u] = (\cos u)u'$
14. $\frac{d}{dx}[\cos u] = -(\sin u)u'$
15. $\frac{d}{dx}[\tan u] = (\sec^2 u)u'$
16. $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$
17. $\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$
18. $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
19. $\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$
20. $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$
21. $\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$
22. $\frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$
23. $\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}}$
24. $\frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$

**Basic Integration Rules ($a > 0$)**

1. $\int kf(u)\,du = k\int f(u)\,du$

2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$

3. $\int du = u + C$

4. $\int u^n du = \dfrac{u^{n+1}}{n+1} + C, \quad n \neq -1$

5. $\int \dfrac{du}{u} = \ln|u| + C$

6. $\int e^u\,du = e^u + C$

7. $\int a^u\,du = \left(\dfrac{1}{\ln a}\right)a^u + C$

8. $\int \sin u\,du = -\cos u + C$

9. $\int \cos u\,du = \sin u + C$

10. $\int \tan u\,du = -\ln|\cos u| + C$

11. $\int \cot u\,du = \ln|\sin u| + C$

12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$

13. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$

14. $\int \sec^2 u\,du = \tan u + C$

15. $\int \csc^2 u\,du = -\cot u + C$

16. $\int \sec u \tan u\,du = \sec u + C$

17. $\int \csc u \cot u\,du = -\csc u + C$

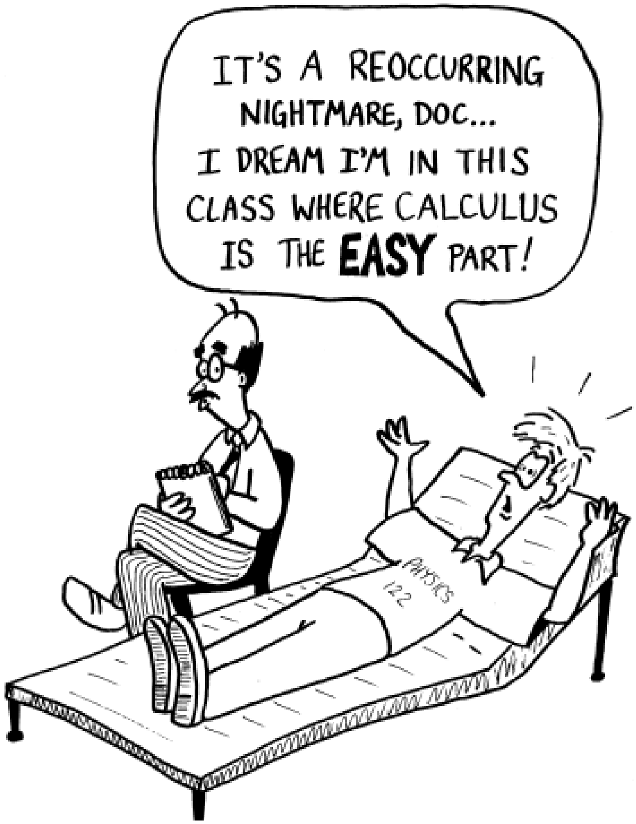
18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \arcsin\dfrac{u}{a} + C$

19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\arctan\dfrac{u}{a} + C$

20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\operatorname{arcsec}\dfrac{|u|}{a} + C$

**Hope you have had FUN learning Calculus this term!** ☺

**NB:** The Final Exam is on Sunday at 8:00 am in this room.




Cartoon by Geoff Draper