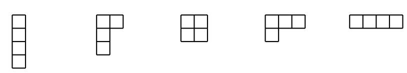
Research Summary – Zajj Daugherty – February 11, 2012

My research is in combinatorics and representation theory. Specifically I use combinatorial techniques to study algebras that are generalizations of the group algebra of the symmetric group or braid groups, Brauer algebras, and Hecke algebras.

In combinatorial representation theory, we use combinatorial data, like partitions or tableaux, to classify the representation theory of algebraic objects (Lie algebras, reflection groups, braid groups, etc.). This approach originated in the study of the *symmetric group* S_k, the group of permutations of k objects, as depicted in Figure 1 (A). A powerful way to study an algebraic object is through its representation theory: all of the the ways it can act on a vector space in a way that respects the algebraic structure. We call the vector space, together with the action of the algebraic object, a *module*. Every algebra has infinitely many modules, but in many cases we can build most of them from a more manageable set of *simple modules*. Like building molecules from atoms, we can build the algebra itself from its simple modules. The simple S_k-modules are in bijection with *partitions* of k, which we draw as k boxes pushed up and left into a corner. For example, the partitions of 4 are

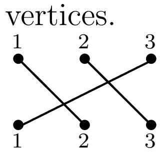


so there are five simple S_4-modules. The dimension of each module and the action of S_4 are also encoded in its partition. So we can decode huge amounts of information about S_k by building combinatorial objects, partitions.

Figure 1. Centralizer algebras arising via formal sums of diagrams, all with multiplication given by concatenation.

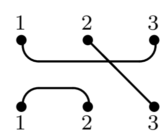
(A) The group algebra of the symmetric group can be represented as planar permutations of k vertices.

$k = 3$:



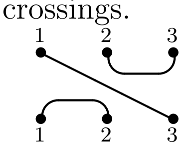
(B) The Brauer algebra can be represented as bijections on $2k$ vertices.

$k = 3$:



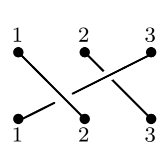
(C) The Temperley-Lieb algebra can be represented as bijections on $2k$ vertices with no crossings.

$k = 3$:



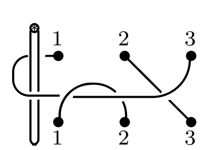
(D) The Hecke algebra can be represented as braids on k strands.

$k = 3$:



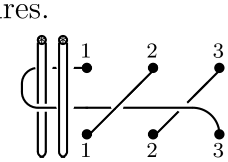
(E) The affine BMW algebra can be represented by tangles on k strands in a space with one puncture.

$k = 3$:



(F) The two-boundary Hecke algebra can be represented as braids on k strands in a space with two punctures.

$k = 3$:



In 1901, Schur connected S_k to the general linear group, $\mathrm{GL}_n(\mathbb{C})$, the group of invertible $n \times n$ matrices. The key observation was that they both act on the vector space

$$(\mathbb{C}^n)^{\otimes k} = \underbrace{\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n}_{k \text{ factors}},$$

(S_k by permuting factors, $\mathrm{GL}_n(\mathbb{C})$ by transforming coordinate-wise) and that these actions commute (their group-algebras *centralize* each other, a slightly stronger result). Amazingly, this enables us to calculate many simple $\mathrm{GL}_n(\mathbb{C})$-modules by pairing them up with simple S_k-modules. This pairing of simple modules using the centralizer property is called *Schur-Weyl duality.*

By replacing S_k with other groups or algebras and varying the space playing the role of $(\mathbb{C}^n)^{\otimes k}$, several advances have been made in the development of *tensor power centralizer algebras*, algebras of operators which preserve symmetries in a tensor space. Striking examples include:

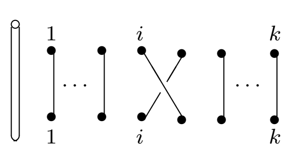
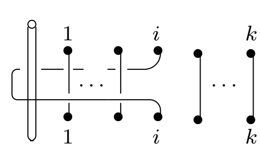
(1) the *Brauer algebras* centralize the action of symplectic and orthogonal groups on tensor space $(\mathbb{C}^n)^{\otimes k}$ (see Figure 1 (B), [Br]);
(2) the *degenerate affine Hecke algebra of type A* centralizes the action of the complex simple Lie algebra $\mathfrak{sl}_n$ on $L(\lambda) \otimes (\mathbb{C}^n)^{\otimes k}$, where $L(\lambda)$ is the irreducible $\mathfrak{sl}_n$-module indexed by a partition λ ([AS]);
(3) the *degenerate affine Wenzl algebra* centralizes the action of symplectic and orthogonal Lie algebras on $L(\lambda) \otimes (\mathbb{C}^n)^{\otimes k}$, where $L(\lambda)$ is the irreducible $\mathfrak{so}_n$- or $\mathfrak{sp}_{2n}$-module indexed by a partition λ ([AMR]).

A paper of Orellana and Ram [OR] suggested a unified approach to tensor power centralizer algebras, giving a beautiful treatment of the *affine and cyclotomic Hecke and Birman-Murakami-Wenzl (BMW) algebras* as quotients of the affine braid group.

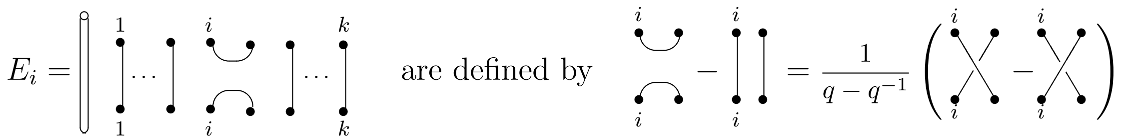
Many of these examples also have diagrammatic depictions, some of which are illustrated in Figure 1. The connection to diagrams, braids and tangles provides applications to knot theory, and therefore to molecular biology, particle physics, and quantum computing. In some cases, Schur-Weyl duality appears secondarily; the diagrams themselves motivate the problem and centralizer relationships with Lie algebras and quantum groups reveal rich mathematical structure.

1. Affine and degenerate affine BMW algebras

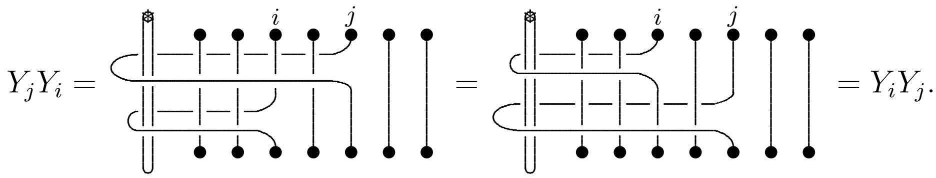
The affine BMW algebra W_k models tangles in a space with one puncture (see Figure 1 (E)). This algebra is a quotient of the group algebra of the affine braid group, and is generated over a commutative ring C by invertible elements

$T_i =$  and $Y_i =$ 

where elements

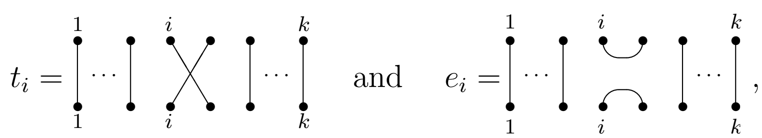


for some $q \in C$ (with additional relations). Pictorially, we can see that the Y_is pairwise commute:

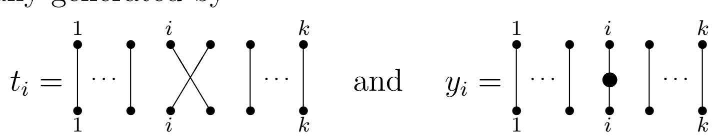


In fact, the set of Laurent polynomials $C[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]$ forms a subalgebra of W_k.

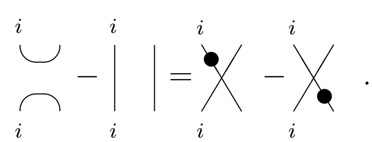
The degenerate version of this algebra, the *degenerate affine BMW algebra* $\mathcal{W}_k$, was introduced by Nazarov in his study of the action of Jucys-Murphy elements on the Brauer algebra in [Na]. It contains the Brauer algebra, generated over a commutative ring C by elements



and a polynomial ring $C[y_1, \ldots, y_k]$, with additional relations. A diagrammatical depiction of $\mathcal{W}_k$ has not been established; however, we conjecture that $\mathcal{W}_k$ can be diagrammatically generated by

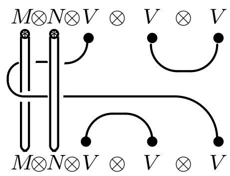


where e_i is defined by



In joint work with Ram and Virk in [DRV, DRV2], we have brought the two algebras together by expressing them totally analogously as quotients of the group algebra of the affine braid group (as in [OR]) and the degenerate affine braid algebra (a new definition in [DRV]), respectively. By treating these two algebras completely in tandem, we are able to make many connections across the literature, and to leverage insights for either algebra to uncover beautiful and surprising results.

From the perspective of representation theory, the similarities between W_k and $\mathcal{W}_k$ are born from the fact that both arise naturally in the context of Schur-Weyl duality for orthogonal and symplectic quantum groups and Lie algebras, respectively:



$\in \mathrm{End}_{U_q\mathfrak{sl}_2}\,(M \otimes N \otimes V^{\otimes 3})$

FIGURE 2. An element of the two-boundary Temperley-Lieb algebra, both as a tangle (with no crossings) and as an operator on tensor space.

both act on tensor space of the form $M \otimes V^{\otimes k}$, where $V = \mathbb{C}^n$ and M is any finite-dimensional simple module. This is the perspective driving my work with A. Ram and R. Virk in [DRV, DRV2] and subsequent papers; from here, we have been unifying the theory of these two algebras, with many projects remaining.

In [DRV] we showed that centers of the affine and degenerate affine BMW algebras are certain rings of symmetric functions, subrings either of the polynomial ring $C[y_1, \ldots, y_k]$ or of the Laurent polynomial ring $C[Y_1^{\pm 1}, \ldots, Y_k^{\pm 1}]$. An amazing consequence quite deeply ties this work to algebraic geometry, projective representation theory of the symmetric group, the BKP hierarchy of differential equations, representations of Lie superalgebras, and twisted Gel'fand pairs. Specifically, we may be able to harness the similarities between W_k and $\mathcal{W}_k$ in order to make exciting progress concerning K-theory of symplectic Grassmannians.

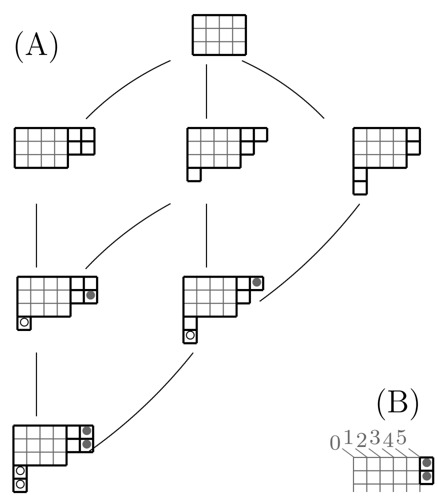
2. Two-boundary centralizer algebras

Centralizer algebras also have motivations in statistical mechanics. The Temperley-Lieb algebra arises in studying loop models and spin chains, but also as a tensor power centralizer algebra. Recent work in statistical mechanics uncovered yet another generalization, and another potential use of Schur-Weyl duality (see [GN]). Specifically, a connection was discovered between the two-boundary Temperley-Lieb algebra (drawn in Figure 2) and a quotient of the affine Hecke algebra of type C. Since the two-boundary Temperley-Lieb algebra is the centralizer of the quantum group $U_h\mathfrak{sl}_2$ on tensor space $M \otimes N \otimes (\mathbb{C}^2)^{\otimes k}$, where M and N are simple $U_h\mathfrak{sl}_2$-modules, this connection opened the community's eyes to the possibility of constructing affine Hecke algebra type-C modules explicitly using Schur-Weyl duality tools.

Motivated by this insight, I laid the groundwork for studying two-boundary centralizer algebras in general in my thesis and accompanying paper, [Da, Da2]. In particular, I studied the centralizer of the action of a specific family or Lie algebras, $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ or $\mathfrak{gl}_n(\mathbb{C})$ on $M \otimes N \otimes V^{\otimes k}$, where M and N are finite-dimensional simple modules indexed by rectangles and V is the standard module:

$$M = L((a^p)) = L\left(\overset{a}{\boxed{}}\, p\right), \quad N = L((b^q)) = L\left(\overset{b}{\boxed{}}\, q\right), \quad V = L(\square) = \mathbb{C}^n.$$

FIGURE 3. (A) is a combinatorial picture of a four-dimensional simple $\mathcal{H}_{2,k}$-module; the basis is indexed by downward-moving paths, and the action is a function of the contents of the boxes added at each step. This module conjecturally corresponds to a simple $\mathbb{H}$-module indexed by the skew shape in (B).



I defined the *degenerate extended two-boundary Hecke algebra* $\mathcal{H}_{2,k}^{\text{ext}}$, which has a large subalgebra of the centralizer of the $\mathfrak{g}$-action on $M \otimes N \otimes V^{\otimes k}$ as a quotient.

I also calculated the representation theory of $\mathcal{H}_{2,k}^{\text{ext}}$, showing that its simple modules are in bijection with certain partitions. I also showed that the bases of these modules and the actions of $\mathcal{H}_{2,k}^{\text{ext}}$ thereon can be calculated explicitly using the combinatorics of Young tableaux. For example, a diagram corresponding to a simple four-dimensional $\mathcal{H}_{2,k}^{\text{ext}}$-module that arises from the action on tensor space with $M = L(3^4)$ and $N = L(2^2)$ is depicted in Figure 3 (A).

While the degenerate affine Hecke algebra of type A is isomorphic to a centralizer algebra, no such relationship is known for the type C counterpart. However, one subalgebra of $\mathcal{H}_{2,k}^{\text{ext}}$, the *degenerate two-boundary Hecke algebra* $\mathcal{H}_{2,k}$, is strikingly similar to $\mathbb{H}$, the degenerate affine Hecke algebra of type C. This observation suggests the possibility of studying representations of type C Hecke algebras using Schur-Weyl duality techniques.

I have already begun investigation into many similarities between $\mathbb{H}$ and $\mathcal{H}_{2,k}$, including constructing parallels between their centers, central characters, and representation theory. Furthermore, in preliminary work with Ram in [DR], we have also defined the two-boundary Hecke algebra (pictured as a diagram algebra in Figure 1 (F)) as the two-boundary analog to the affine Hecke algebra of type A, or, alternatively, the quantum analog to the degenerate two-boundary Hecke algebra. In this quantized version, there is an isomorphism between the affine Hecke algebra of type C and the two-boundary Hecke algebra through a realization of both algebras as diagram algebras. In [Lu], Lusztig studies the correspondence between the affine algebras and their degenerate versions; because of this correspondence, we expect an isomorphism between the degenerate affine Hecke algebra of type C and the degenerate two-boundary Hecke algebra.

References

[AMR] S. Ariki, A. Mathas and H. Rui, *Cyclotomic Nazarov-Wenzl algebras*, Nagoya Math. J. **182** (2006), 47–134.

[AS] T. Arakawa and T. Suzuki, *Duality between $\mathfrak{sl}_n(\mathbb{C})$ and the Degenerate Affine Hecke Algebra*, J. Algebra **209** (1998), 288–304.

[Br] R. Brauer, *On algebras which are connected with the semisimple continuous groups*, Ann. of Math. **38** (1937), 857–872.

[Da] Z. Daugherty, *Degenerate two-boundary centralizer algebras*, Doctoral dissertation, University of Wisconsin–Madison, June 2010.

[Da2] Z. Daugherty, *Degenerate two-boundary centralizer algebras*, to appear in Pacific J. of Math.. arXiv:1007.3950v1

[DRV] Z. Daugherty, A. Ram, and R. Virk, *Affine and graded BMW algebras: The center*, preprint (submitted). arXiv:1105.4207

[GN] J. de Gier and A. Nichols, *The two-boundary Temperley-Lieb algebra*, J. Algebra **321** (2009) 1132–1167.

[Lu] G. Lusztig, *Cuspidal local systems and graded Hecke algebras I*, Inst. Hautes Etudes Sci. Publ. Math. **67** (1988), 145–202.

[Na] M. Nazarov, *Youngs Orthogonal Form for Brauers Centralizer Algebra*, J. Algebra **182** (1996), 664–693.

[OR] R. Orellana and A. Ram, *Affine braids, Markov traces and the category $\mathcal{O}$*, in *Proceedings of the International Colloquium on Algebraic Groups and Homogeneous Spaces Mumbai 2004*, V.B. Mehta ed., Tata Institute of Fundamental Research, Narosa Publishing House, Amer. Math. Soc. (2007) 423-473.

PAPERS IN PROGRESS

[Da3] Z. Daugherty *Centralizer properties of the graded Hecke algebra of type C*

[DR] Z. Daugherty, A. Ram, *Two-boundary affine Hecke algebras.*

[DRV2] Z. Daugherty, A. Ram, and R. Virk, *Affine and degenerate affine BMW algebras: Actions of tensor space*. See *Notes on Tanatalizer algebras*, available from http://ms.unimelb.edu.au/∼ram/notes.html.

[DR] Z. Daugherty and A. Ram, *Representation theory and graded versions of affine and degenerate affine BMW algebras.*