Second Midterm Practice Problems

DISCLAIMER: In no sense should this collection of problems be construed as representative of the actual exam. These are simply some problems left over from the preparation of the exam or from previous exams which should serve to indicate the general level of expectation.

Problem 1. Find the power series representations of two linearly independent solutions to

$$(x-1)^2 y'' - 2y = 0$$

about the regular point $x_0 = 0$.

Problem 2. Find the general solution to the system

$$\begin{aligned} x_1' &= 3x_1 + 4x_2 \\ x_2' &= 3x_1 + 2x_2 \end{aligned}$$

and use it to find the specific solution satisfying the initial conditions $x_1(0) = x_2(0) = 1$.

Problem 3. Show that $x = 0$ is an ordinary point of

$$xy'' + (\sin x)y' + xy = 0.$$

and find the first three nonzero terms in each of two linearly independent power series solutions of the form $y = \sum a_n x^n$. What are the radii of convergence of these series?

Problem 4. This problem concerns the differential equation

$$x^2 y'' + (3x-1)y' + y = 0. \tag{1}$$

a. Show that if

$$y = x^r \sum_{n=0}^{\infty} a_n x^n, a_0 \neq 0$$

solves (1), then $r = 0$.

b. Substitute $y = \sum_{n=0}^{\infty} a_n x^n$ into (1) and show that if $a_0 = 1$ then $a_n = n!$ for all n.

c. Show that the series solution in part (b) converges only when $x = 0$. Does this violate the general theory we discussed for series solutions near singular points? Explain.

Problem 5. Consider two interconnected tanks of salt water. Tank 1 initially contains 100 liters of salt water with a concentration of 30 grams per liter. Tank 2 initially contains 150 liters of fresh water. The solution in each tank is kept well-mixed and is pumped into the other tank at a rate of $r > 0$ liters per minute. The two tank system is closed: other than the transfer between the two tanks, no salt water is added or removed.

a. Let $Q_1(t)$ and $Q_2(t)$ denote the amount of salt in tanks 1 and 2, respectively, at time t. A system of differential equations modeling the flow of salt between the tanks has the form

$$\begin{aligned} \frac{dQ_1}{dt} &= -k_1 Q_1 + k_2 Q_2 \\ \frac{dQ_2}{dt} &= k_1 Q_1 - k_2 Q_2. \end{aligned}$$

Determine the values of k_1 and k_2 in terms of r.

b. Determine the limiting amount (as $t \to \infty$) of salt in each tank. [*Note:* Intuition suggests that you're answer should *not* depend on r.]