

Dartmouth College
Mathematics 25

Assignment 2
due Wednesday, October 7

1. Find all integer solutions to $11305x + 455y = 105$.
2. Let $m, n \in \mathbb{Z}$, and consider the lattice point (m, n) in the plane. We say that the point (m, n) is visible from the origin, $(0, 0)$, if and only if there are no other lattice points on the line segment between $(0, 0)$ and (m, n) . Prove that (m, n) is visible from the origin if and only if $\gcd(m, n) = 1$. You may, if you find it convenient, assume that $m, n \geq 0$, though this is not necessary.
3. Prove that there are infinitely many integers m so that $6m + 5$ is prime.
4. Our general theory tells us that there are infinitely many solutions to the Diophantine equation $4x + 5y = n$ for any integer n . Let $P(n)$ be the statement that $4x + 5y = n$ is solvable for nonnegative x and y . Show that $P(n)$ is true for all $n \geq 12$. Induction is one means to this end.
5. The Fibonacci numbers f_n are ubiquitous. They are defined recursively by $f_1 = f_2 = 1$, and for $n \geq 3$, $f_n = f_{n-1} + f_{n-2}$; thus the sequence begins $1, 1, 2, 3, 5, 8, 13, \dots$. Show that $0 \leq f_n < f_{n+1}$ for all $n \geq 2$. For $n \geq 3$, show that $\gcd(f_{n+2}, f_{n+1}) = 1$ and that the Euclid's algorithm takes exactly n steps to run.
6. On an island live 17 people all of whom have blue eyes. They are capable of perfect logic and live by two rules:
 - No person reveals the color of another person's eyes to that person.
 - If a person learns that their eyes are blue, they must jump into the island's volcano at midnight of the day they learn this fact.

A missionary comes to the island with the intent of converting the residents to the "true faith", but after months of effort fails to convert even a single person. Frustrated, he leaves, but just as he departs, the residents all gather at the beach and the missionary says, "at least one of you has blue eyes."

- (a) Describe what happens over the next year. Hint: This is an induction problem.
- (b) What, if anything, did the missionary tell the residents that they didn't already know?