Chapter 12: Infinite Sequences and Series

10/02/2006

Sequences

- A **sequence** is a list of numbers written in a definite order:

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

- The sequence is also denoted by

$$\{a_n\} \text{ or } \{a_n\}_{n=1}^{\infty}.$$

Examples

- $\{\sqrt{n}\}_{n=1}^{\infty}$, $a_n = \sqrt{n}$, $\{\sqrt{1}, \sqrt{2}, \ldots \sqrt{n}, \ldots\}$.

Examples

- $\{\sqrt{n}\}_{n=1}^{\infty}$, $a_n = \sqrt{n}$, $\{\sqrt{1}, \sqrt{2}, \ldots \sqrt{n}, \ldots\}$.

- $\left\{\frac{n}{n+1}\right\}$, $a_n = \frac{n}{n+1}$, $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots\right\}$.

Examples

- $\{\sqrt{n}\}_{n=1}^{\infty}$, $a_n = \sqrt{n}$, $\{\sqrt{1}, \sqrt{2}, \ldots \sqrt{n}, \ldots\}$.

- $\left\{\frac{n}{n+1}\right\}$, $a_n = \frac{n}{n+1}$, $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, \frac{n}{n+1}, \ldots\right\}$.

- $\left\{(-1)^{n+1}\frac{1}{n}\right\}$, $a_n = (-1)^{n+1}\frac{1}{n}$, $\left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \cdots, -1^{n+1}\frac{1}{n}, \ldots\right\}$.

- Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3126}, \dots\right\}.$$

- Find a formula for the general term a_n of the sequence

$$\left\{ \frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3126}, \dots \right\}.$$

- The **Fibonacci sequence** $\{f_n\}$ is defined recursively by the conditions

$$f_1 = 1 \; f_2 = 1 \; f_n = f_{n-1} + f_{n-2} \; n \geq 3.$$

The limit of a sequence

- A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

How to compute the limit of a sequence?

Theorem. *If* $\lim_{x\to\infty} f(x) = L$ *and* $f(n) = a_n$*, when* n *is an integer, then*

$$\lim_{n\to\infty} a_n = L.$$

- Example: $\lim_{n\to\infty} \frac{1}{n+1}$.

Definition

- $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$a_n > M \text{ whenever } n > N.$$

The Limit Laws

$$\begin{aligned}
\lim_{n\to\infty}(a_n \pm b_n) &= \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n \\
\lim_{n\to\infty} c a_n &= c \lim_{n\to\infty} a_n \\
\lim_{n\to\infty}(a_n b_n) &= \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n \\
\lim_{n\to\infty} \frac{a_n}{b_n} &= \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \text{ if } \lim_{n\to\infty} b_n \neq 0 \\
\lim_{n\to\infty} a_n^p &= [\lim_{n\to\infty} a_n]^p \text{if } p > 0 \text{ and } a_n > 0.
\end{aligned}$$

Examples

- Determine whether the sequence

$$\frac{8n-3}{4n+1}$$

converges or diverges. If it is convergent, find the limit.

Examples

- Determine whether the sequence

$$\frac{8n-3}{4n+1}$$

converges or diverges. If it is convergent, find the limit.

- Calculate

$$\lim_{n\to\infty} \frac{\ln n}{n}.$$

Example

- Find the values of r for which the sequence $\{r^n\}$ is convergent.

Monotone sequences

- A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$.
- It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$.
- It is called **monotonic** if it is either increasing or decreasing.

Bounded sequences

- A sequence $\{a_n\}$ is **bounded above** if there is a number M such that
$$a_n \leq M \text{ for all } n \geq 1.$$

- It is **bounded below** if there us a number m such that
$$m \leq a_n \text{ for all } n \geq 1.$$

- If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence.**

Monotonic Sequence Theorem

- Every bounded, monotonic sequence is convergent.

Series

- An **infinite series** is an expression obtained by adding the terms of an infinite sequence $\{a_n\}$. It is denoted

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n.$$

- **Partial sums** are expressions

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$