1

Flatland

In 1884 an extraordinary individual named A Square succeeded in publishing his memoirs. Actually an intermediary by the name of Abbott published them for him--A Square himself was in prison for heresy at the time. A Square was extraordinary not because he had such an odd name, but rather because he had such a descriptive and accurate name. For you see, A Square was a square.

Now you might be wondering just where A Square lived. After all, you wouldn't expect to find a two-dimensional square living in a three-dimensional universe such as ours. You might allow for a slightly thickened square, say a creature with the dimensions of a sheet of paper, but certainly not a completely flat individual like A Square. Anyhow, A Square didn't live in our three-dimensional universe. He lived in Flatland, a two-dimensional universe resembling a giant plane.

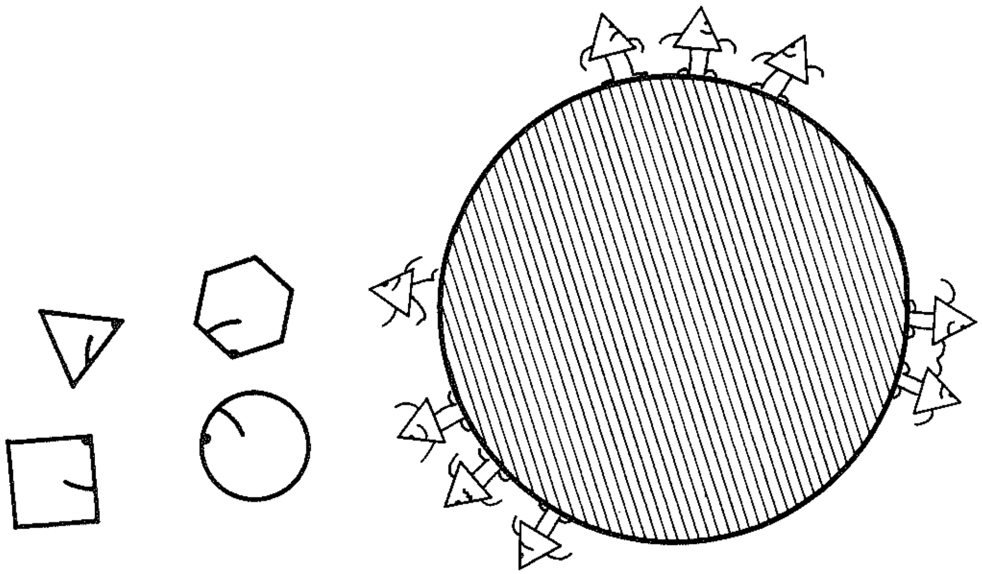

Figure 1.1: Flatlanders move freely in a "plane", while Astrians are confined to the edge of their disk-shaped planet Astria.

Flatland also happens to be the title under which A Square's memoirs were published. It's now available in paperback, and I recommend it highly. In 1907 C. H. Hinton published a similar book, *An Episode of Flatland*. The chief difference between these books is that the residents of Flatland proper can move freely about their two-dimensional universe, whereas the inhabitants of Hinton's world are constrained by gravity to living on the circular edge of their disk-shaped planet Astria (Figure 1.1). For the full story on the lore of Astria, see A. K. Dewdney's *The Planiverse*.

Getting back to the subject at hand, the Flatlanders all thought that Flatland was a giant plane, what we Spacelanders would call a

Euclidean plane. To be accurate, I should say that they *assumed* that Flatland was a plane, since nobody ever gave the issue any thought. Well, almost nobody. Once a physicist by the name of A Stone had proposed an alternative theory, something about Flatland having a finite area, yet having no boundary. He compared Flatland to a circle. For the most part people didn't understand him. It was obvious that a circle had a finite circumference and no endpoints, but what did that have to do with Flatland, which obviously had an infinite area? At least part of the problem was linguistic: The only word for "plane" was the word for "Flatland" itself, so to express the idea that Flatland was *not* a plane, one was trapped into stating that "Flatland is not Flatland". Needless to say, this theory attracted few disciples.

A Square, though, was among the few. He was particularly intrigued by the idea that a person could set out in one direction and come back from the opposite direction, without ever having turned around. He was so intrigued that he wanted to try it out. The Flatlanders were for the most part a timid lot, and few had ever travelled more than a day or two's journey beyond the outlying farms of Flatsburgh. A Square reasoned that if he were willing to spend a month tromping eastward through the woods, he might just have a shot at coming back from the west.

He was delighted when two friends volunteered to go with him. The friends, A Pentagon and A Hexagon, didn't believe any of A Square's theories--they just wanted to keep him out of trouble. To this end they insisted that A Square buy up all the red thread he

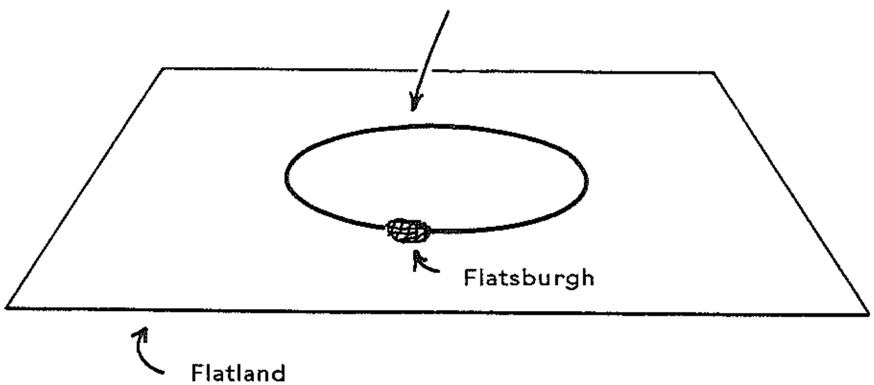

Figure 1.2: Even A Square's companions thought they had veered in a circle.

could find in Flatsburgh. The idea was that they would lay out a trail of red thread behind them, so that after they had travelled for a month and given up, they could then find their way back to Flatsburgh.

As it turned out, the thread was unnecessary. Much to A Square's delight--and A Pentagon's and A Hexagon's relief--they returned from the west after three weeks of travel. Not that this convinced anyone of anything. Even A Pentagon and A Hexagon thought that they must have veered slightly to one side or the other, bending their route into a giant circle in the plane of Flatland (Figure 1.2). A Square had no reply to their theory, but this did little to dampen his enthusiasm. He was ready to try it again!

By now red thread was in short supply in Flatland, so this time A Square laid out a trail of blue thread to mark his route. He set

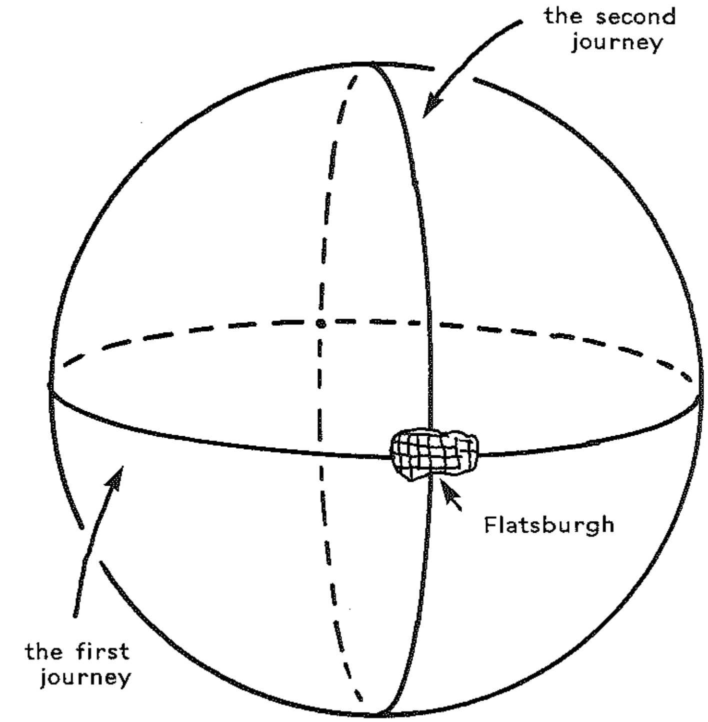

Figure 1.3: The two threads ought to cross, even if Flatland were a "hypercircle" (i.e. a sphere).

out to the north, and, sure enough, returned two weeks later from the south. Again everyone assumed that he had simply veered in a circle, and counted him lucky for getting back at all.

A Square was mystified that his journey was so much shorter this time, but something else bothered him even more: he had never come across the red thread they laid out on the first journey. The physicists of Flatland were equally intrigued. They confirmed that even if Flatland were a so-called "hypercircle" as A Stone had suggested, the two threads would still cross (Figure

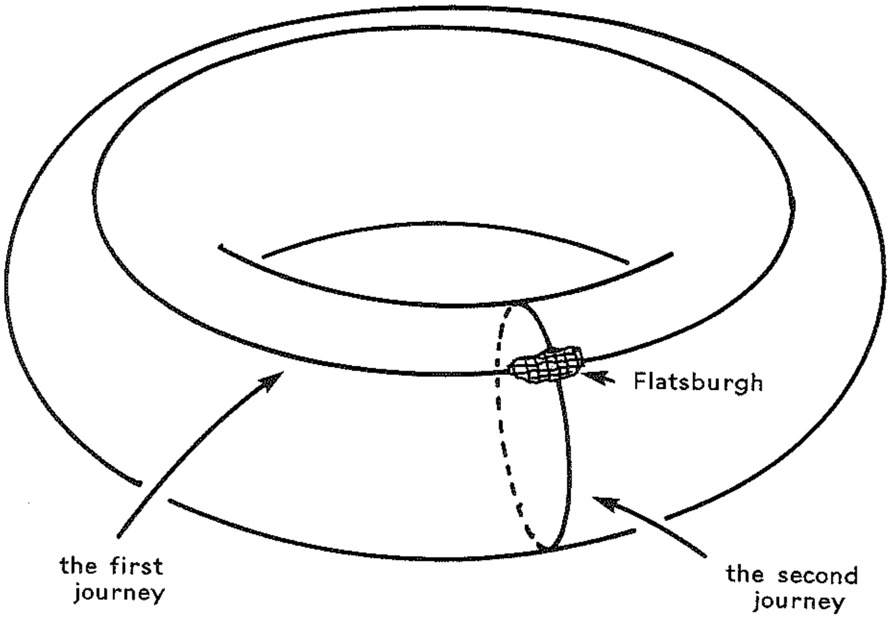

Figure 1.4: Spacelanders sometimes visualize a torus as the surface of a doughnut.

1.3). There was, of course, the possibility that the red thread had broken for one reason or another. To investigate this possibility, the scientists formed two expeditions: one party retraced the red thread, the other retraced the blue. Both threads were found to be intact.

The Mystery of the Nonintersecting Threads remained a mystery for quite a few years. Some of the bolder Flatlanders even took to retracing the threads periodically as a sort of pilgrimage. The first hint of a resolution came when a physicist proposed that Flatland should be regarded neither as a "Flatland" (i.e. a plane) nor as a hypercircle, but as something he called a "torus". At

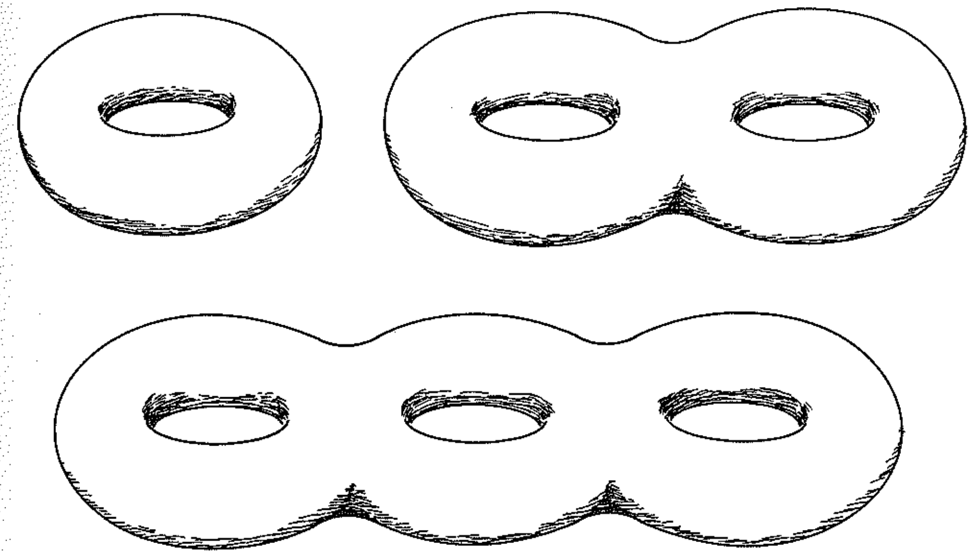

Figure 1.5: Some possible shapes for Flatland.

first no one had any idea what he was talking about. Gradually though, people agreed that this theory resolved the Mystery of the Nonintersecting Threads, and everyone was happy about that. So for many years Flatland was thought to be a torus (Figure 1.4).

Until one day somebody came up with yet another theory on the "shape" of Flatland. This theory explained the Mystery of the Nonintersecting Threads just as well as the torus theory did, but it gave a different overall view of Flatland.

And this new theory was just the first of many. For the next few months people were constantly coming up with new possibilities for the shape of Flatland (Figure 1.5). Soon a vast Universal Survey was undertaken to map all of Flatland and thereby determine its true shape once and for all . . .

(STORY TO BE CONTINUED IN CHAPTER 4)

As Spacelanders we have three dimensions available for drawing pictures like Figure 1.5, so it's easy for us to understand how a two-dimensional universe can close back on itself. The Flatlanders inhabiting such a universe would have a much tougher time. To sympathize with their feelings, try imagining yourself in each of the following situations.[1]

1. You are on an expedition to a distant galaxy in search of intelligent life. When you reach the galaxy you head for the most hospitable-looking planet you can find, only to discover that you're back on Earth.
2. You are an astronomer. You seem to be observing the exact same object in two different locations in the sky.
3. You are a radio astronomer searching for signals from extraterrestrials. You have detected a faint signal coming from a distant galaxy. Once you tune it in you recognize it as a broadcast of the old TV show "Father Knows Best".

Each of the above situations leads you to suspect that space is built differently than you thought it was. That is, space seems to have a different shape than the obvious one you assumed it had.

[1] Warning: These situations are designed to stimulate the imagination. Don't worry about technical complications!

Not that you have any idea what this actual shape is!

In fact, no one knows what the shape of the real universe is. But people do know a fair amount about what the *possible* shapes are. These possible shapes are the topic of this book. Such a possible shape is called a **three-dimensional manifold**, or **three-manifold** for short. (Similarly, a two-dimensional shape for Flatland is called a **two-dimensional manifold**, or, more commonly, a **surface**.) At this point your conception of a three-manifold is probably pretty vague. Don't worry: we'll start seeing some examples in Chapter 2. The main thing now is to realize that our universe might conceivably close back on itself, just as the various surfaces representing Flatland close back on themselves.

This book centers on a series of examples of three-manifolds. Rather than developing an extensive theory of these manifolds, you'll come to know each of them in a visual and intuitive way. Obviously this is not an easy task. Imagine the difficulties A Square would have in communicating to A Hexagon the true nature of a torus. A Square cannot draw a definitive picture of a torus, being confined to two dimensions as he is. Similarly, we cannot draw a definitive picture of any three-manifold.

There is some hope, though. You can use tricks to define various three-manifolds, and as you work with them over a period of time you'll find your intuition for them growing steadily. The human mind is remarkably flexible in this regard. *Just be sure to read slowly and give things plenty of time to digest.* At most a chapter, and often as little as a single exercise, will be plenty for one sitting.

This book provides not a series of answers, but rather a series of questions designed to lead you to your own intuitive understanding of three-manifolds. Prepare your imagination for a workout!

2

Gluing

A popular video game pits two players in biplanes in aerial combat on a TV screen. An interesting feature of the game is that when a biplane flies off one edge of the screen it doesn't crash, but rather it comes back from the opposite edge of the screen. Mathematically speaking, the screen's edges have been "glued" together. (The gluing is purely abstract: there is no need to *physically* connect the edges.) A square or rectangle whose opposite edges are abstractly glued in this fashion is called a **torus** or, more precisely, a **flat two-dimensional torus.** There is a connection between this flat two-dimensional torus and the doughnut-surface torus of Chapter 1, but for the time being *you should forget the doughnut surface entirely.*

edges (unlike the square from which it was made, which does have edges).

The second complication is more interesting. It turns out that there are surfaces which are infinitely long, yet have only a finite area. A typical example is a doughnut surface with a so called "cusp" (Figure 3.8). The cusp is an infinitely long tube which gets narrower as it goes. The first centimeter of cusp has a surface area of 1 square centimeter (cm^2), the next centimeter of cusp has an area of $\frac{1}{2}$ cm^2, the next an area of $\frac{1}{4}$ cm^2, and so on. Thus, the total surface area of the cusp is $1 + \frac{1}{2} + \frac{1}{4} + \ldots = 2$ cm^2. What's important is not that the area of the cusp is precisely 2 cm^2, but that it's finite. For once you add in the (finite) area of the rest of the surface, you find that the surface as a whole has a finite area even though it's infinitely long. By convention a surface is classified as closed or open according to its distance across rather than its area, so the doughnut surface with a cusp is called open in spite of its finite area. After the following exercise we won't encounter any more cusps in this book.

Exercise 3.13. What would happen to A Square if he tried to take a trip down a cusp? □

This book deals mainly with closed manifolds, so from now on

"manifold" will mean "closed manifold",

unless explicitly stated otherwise.

4

Orientability

When our story left off in Chapter 1, our hero A Square and his fellow Flatlanders had just embarked on a Universal Survey of all of Flatland. The excitement was immense as the first survey party set out. But this excitement was nothing compared to the chaos that followed its return!

An old farmer living in an outlying agricultural district was the first one to run into the returning surveyors. He was going around a bend in the road and the surveyors were coming from the opposite direction. Fortunately no one was hurt in the collision. The farmer was a little annoyed that the surveyors didn't have the courtesy to keep to the proper side of the road, but his anger was quickly overcome by his joy at seeing them back safely, and by his interest in hearing their tales of adventure. He accompanied them into town.

As they approached the Flatsburgh City Limits, one of the surveyors noticed that the "Welcome to Flatsburgh" sign, the one that announced the Rotary Club meetings and all, had been replaced by a backwards version of the same thing. A "hgrubstalF ot emocleW" sign, as it were.

"Those kids, always up to mischief," he chuckled.

"What's that you say?" asked the farmer, not sure he had heard properly.

"Oh, nothing. I was just amused at what some kids had done to that sign."

The farmer had no idea why the surveyor was so amused by the sign, but he decided not to make an issue of it.

The further the surveyors got into Flatsburgh, the more bewildered they became. *All* the signs were written backwards, and everyone, not just the old farmer, had taken to walking on the wrong side of the road. It was as if all of Flatsburgh had been mysteriously transformed into its mirror image while they were gone. Flatlanders in general tend to be superstitious, and for the surveyors this mirror reversal of Flatsburgh did not bode well.

Not that the citizens of Flatsburgh were any happier with the situation! They insisted that nothing the least bit unusual had happened in Flatsburgh while the surveyors were gone. It was the surveyors they found to be unusual, with their backwards writing and strange ideas about how Flatsburgh had been somehow transformed. In fact, they found the surveyors to be downright creepy, and, except for relatives and close friends, no one even

Figure 4.1: A mirror reversed Earth.

wanted to go near them.

As you might suspect, the Universal Survey was called off, and for the next three years no one went more than shouting distance from the civilized parts of Flatland . . .

(STORY TO BE CONCLUDED IN CHAPTER 5)

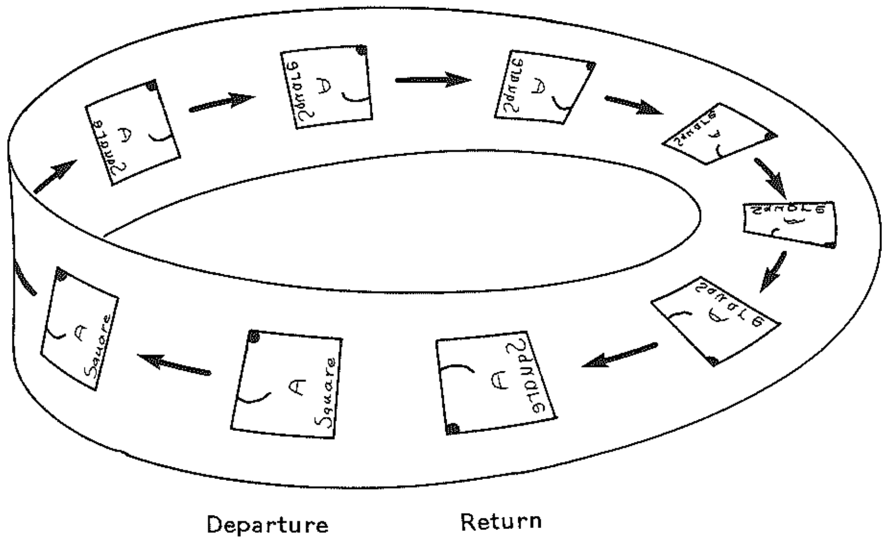

Figure 4.2: When A Square travels around a Möbius strip he comes back as his mirror image.

Exercise 4.1. Write a story in which you travel across the universe to an apparently distant galaxy, only to discover that you've made a complete trip around the universe and returned to our own galaxy. When you find the Earth, you're startled to see that it looks like Figure 4.1. What do you see when you land? Describe a walk through your hometown. What do people think of you? □

In the Flatland story, each surveyor came back to Flatland as his own mirror image. To see just how this occured, study Figure 4.2, which shows a swath of territory similar to the one the

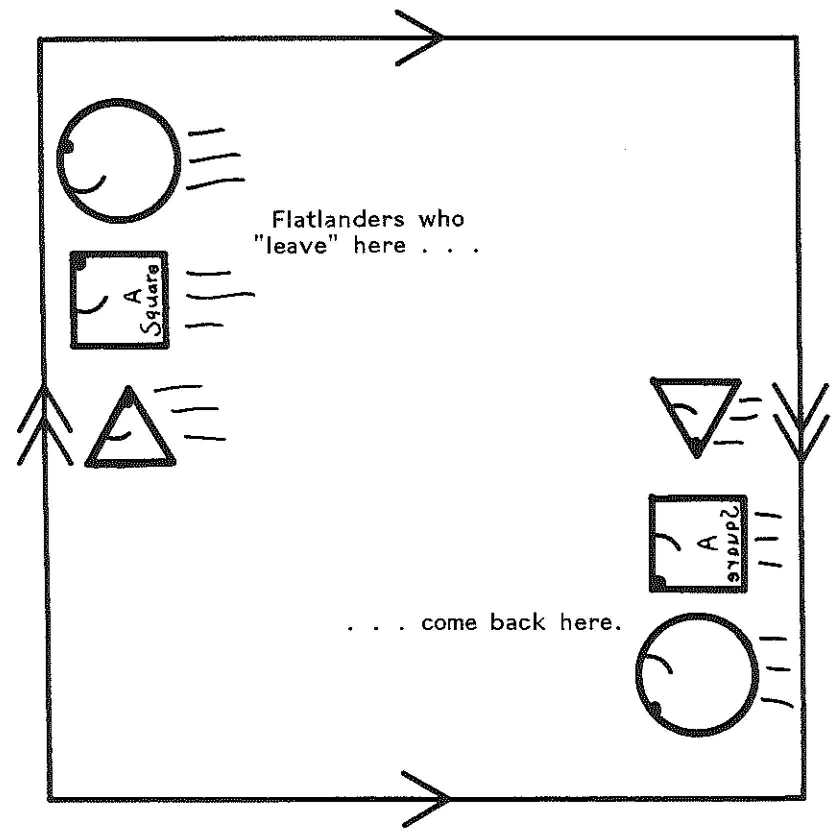

Figure 4.3: Glue the edges of this square so that the arrows match up and you'll get a Klein bottle. A Flatlander travelling off to the left comes back from the right as his mirror image.

surveyors traversed. This swath of territory is a **Möbius strip.** [A true Möbius strip has zero thickness. If you mistakenly imagine it to have a slight thickness--like a Möbius strip made from real paper--then you'll run into problems with A Square returning from his journey on the opposite side of the paper from which he

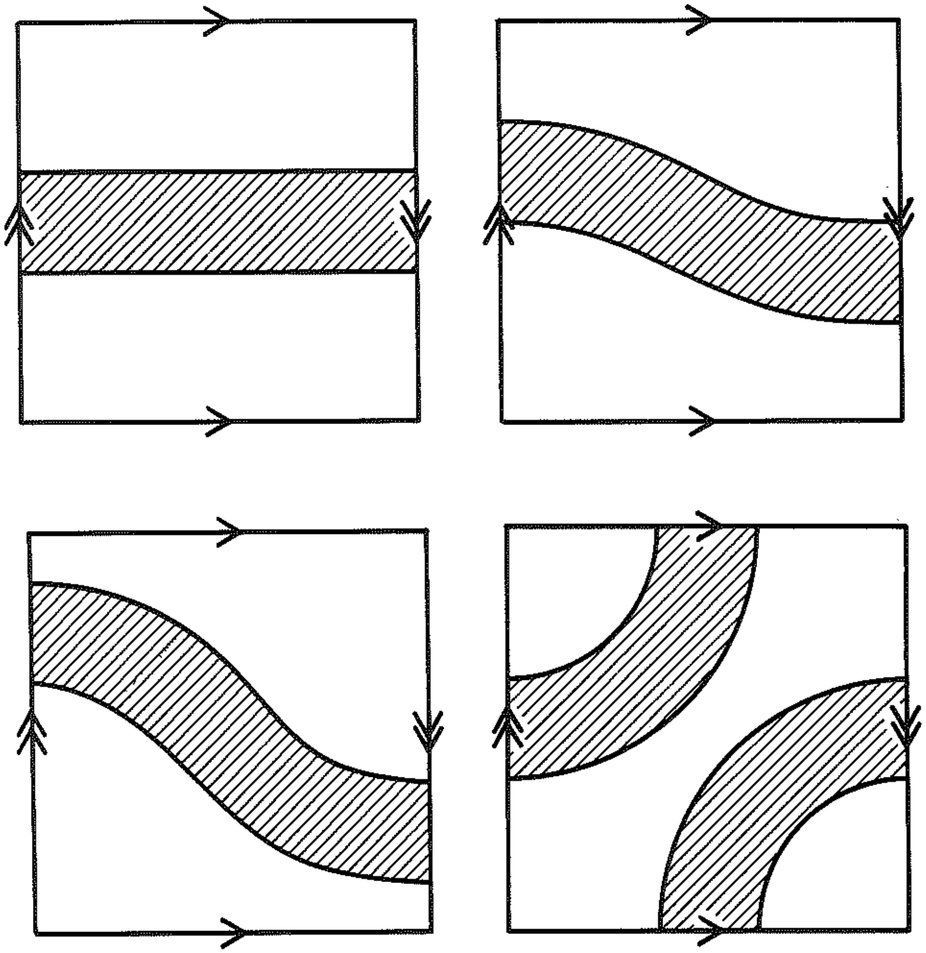

Figure 4.4: A Klein bottle contains many Möbius strips.

started. As long as the Möbius strip is truly two-dimensional (i.e. no thickness) this problem does not arise.]

The question is, in what sort of surface could a Flatlander traverse a Möbius strip? A **Klein bottle** is one example. You can make a Klein bottle from a square in almost the same way we made a

flat torus from a square. Only now the edges are to be glued so that the arrows shown in Figure 4.3 match up. As with the flat torus, I don't mean that these gluings should actually be carried out in three-dimensional space; I mean only that a Flatlander heading out across one edge comes back from the opposite one. The top and bottom edges are glued *exactly* as in the flat torus: when a Flatlander crosses the top edge he comes back from the bottom edge and that's all there is to it. The left and right edges, though, are glued with a "flip". When a Flatlander crosses the left edge he comes back from the right edge, but he comes back mirror reversed. A Klein bottle contains many Möbius strips (see Figure 4.4).

Exercise 4.2. Which of the positions in Figure 4.5 constitute a winning three-in-a-row in Klein bottle tic-tac-toe? □

Exercise 4.3. Imagine the chessboard in Figure 2.6 to be glued to form a Klein bottle rather than a torus. Which black pieces does the white knight threaten now? Which black pieces threaten it? □

There's a nice way to analyze positions in Klein bottle tic-tac-toe and chess. For example, say there's a Klein bottle tic-tac-toe game in progress. The position is as shown on the left side of Figure 4.6 and it's X's turn to move. Rather than hastily taking the upper right hand square, X pauses to carefully analyze the situation. He notes that the board's top edge is glued to its bottom

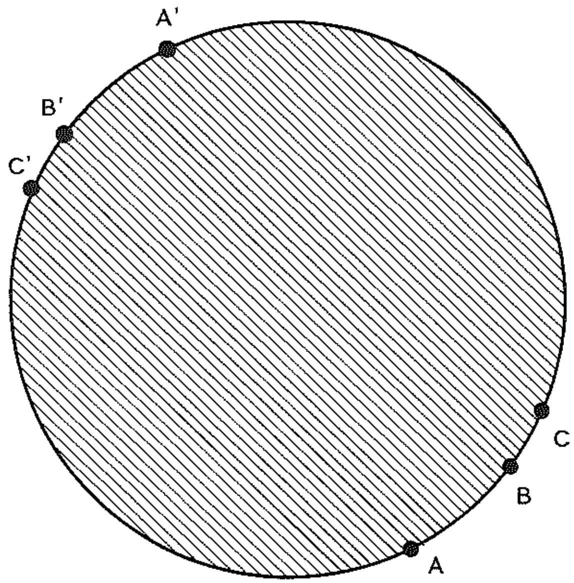

Figure 4.14: Topologically, a projective plane is a disk with opposite boundary points glued.

Exercise 4.17. Is orientability a local or a global property? Is it topological or geometrical? □

5

Connected Sums

Conclusion of the Flatland story:

The mirror-reversed surveyors adapted to their new condition more quickly than most had expected. The hardest part was learning to write properly, but even this became routine after a while. And with their increased competence came a greater acceptance on the part of the community. Things returned to normal.

In fact, as the years went by people even got a little adventurous. Almost every week somebody or another was heading out on an expedition. There were, of course, occasional incidents of explorers coming back mirror-reversed, but this was no longer a disaster. The reversed explorers were quickly rehabilitated. Besides, the reversal incidents were limited to those who passed through a certain "Reversing Region". The rest of Flatland seemed harmless enough, and trips there became quite common.

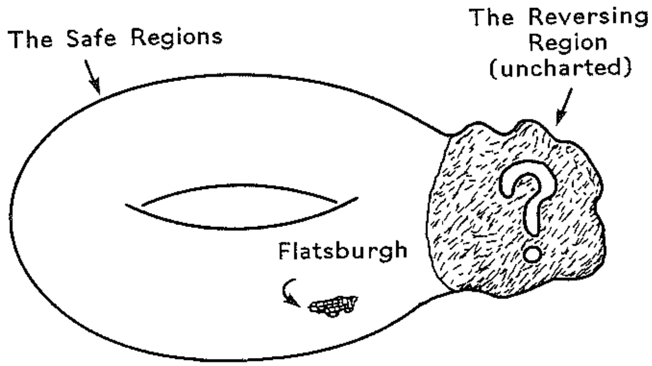

Figure 5.1: The safe regions of Flatland were charted first.

To protect travellers from accidental reversal, the Reversing Region was marked with clumps of stones spaced ten paces apart along its boundary. Once this was done, even the most timid Flatlanders enjoyed travelling about in the safe regions.

It wasn't long before an official survey of the safe regions was undertaken. The surveyors found that the safe regions resembled a doughnut surface (Figure 5.1), in accordance with one of the earliest proposed theories on the shape of Flatland.

Others were quick to point out, however, that this didn't mean that Flatland as a whole was a doughnut surface. For example, if the Reversing Region also resembled a doughnut surface, then Flatland would be a two-holed doughnut surface (Figure 5.2).

Curiosity overcame fear, and a survey of the Reversing Region was begun. Only the boldest of the surveyors volunteered for the job. It wasn't that they were afraid of getting reversed--that was common enough by now. They were afraid of getting reversed

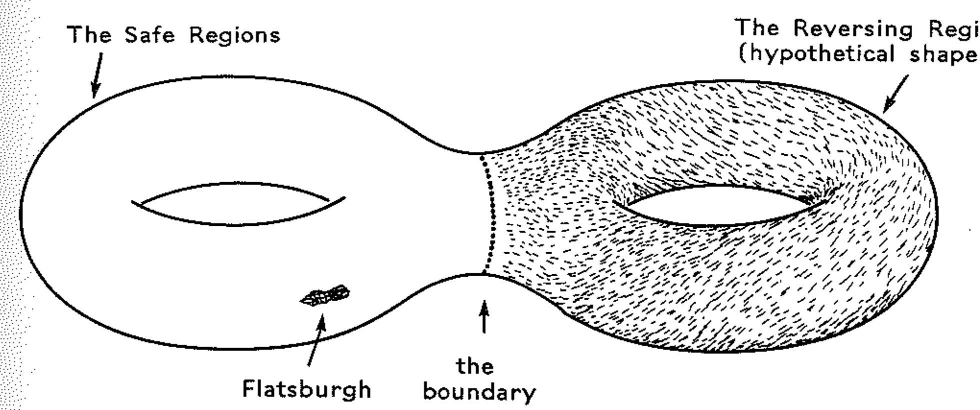

Figure 5.2: People were quick to point out that if the safe region and the Reversing Region each resembled a doughnut surface, then Flatland as a whole would be a two-holed doughnut surface.

twice! The popular consensus was that a second reversal would result in certain death. (There was a minority opinion that a second reversal would simply restore the victim to his original state, but this opinion didn't sell as well in the newspapers.)

The survey was divided into two stages. The purpose of the first stage was to get a rough idea of just how big the Reversing Region was, and to divide it into sectors to be mapped in detail during the second stage.

The first stage went smoothly, even though three of the surveyors came back mirror-reversed. But these reversed surveyors were brave enough to go back into the Reversing Region to help with the detailed surveying of the second stage. In fact, they even drew lots to see who was the bravest and would go back in first!

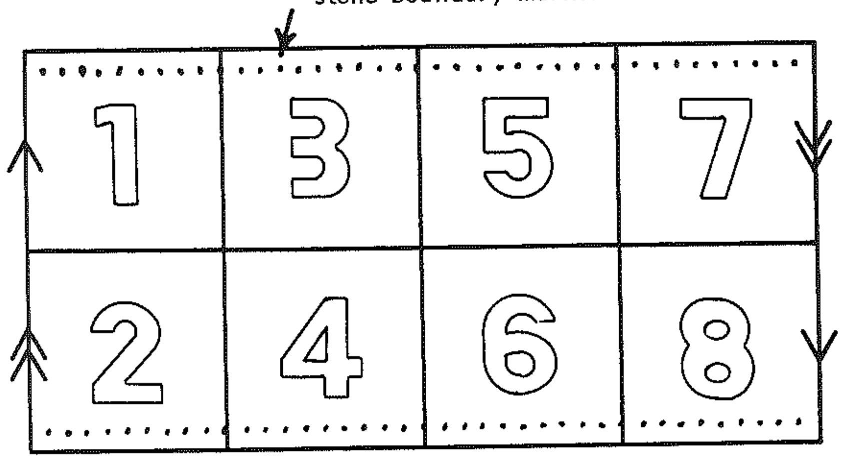

Figure 5.3: How the eight sectors pieced together.

After the completion of the first stage the Reversing Region was divided into eight sectors. During the second stage a separate team was sent to each sector to map it in detail. The whole operation had an air of Russian roulette, with each team wondering whether they were the ones mapping the dangerous sector that did the reversing. To everyone's surprise--and relief--all eight teams reported their respective sectors to be perfectly normal!

It was only when they compiled, consolidated, and compared the data from the different sectors that things got mysterious. They found the sectors connected up as shown in Figure 5.3. The mysterious thing was that Sector 1 connected to Sector 8, not Sector 7; it was Sector 2 that connected to Sector 7! They connected in such a confusing way!

Eventually confusion gave way to enlightenment. The Flatlanders realized that the mirror-reversal phenomenon wasn't so mysterious after all. It was simply that the space of Flatland connected up with itself in such a way that anyone taking a trip around the Reversing Region would come back with his left side where his right side was, and his right side where his left side was. The Flatlanders had discovered the Möbius strip!

This was an immense intellectual achievement. But it was a very practical achievement as well. The reversed surveyors were all sent on a trip around the Reversing Region to restore them to their original condition. Thereafter the Reversing Region was used mainly for pranks and other amusements.

Thus, the Universal Survey was complete: the surveyors had established beyond a doubt that Flatland consists of two regions, one a Mobius strip, and the other resembling a torus. The Flatlanders lived happily and peacefully forever after.

THE END

> *NOTE: All surfaces in this chapter will be considered topologically, so you may bend and twist them however you like!*

The Shape of Space

How to Visualize Surfaces and Three-Dimensional Manifolds

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