HW 6 Solutions M31 F11

1.12 The subgroup $H = \{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R}, ad \neq 0 \}$ of $GL(2, \mathbb{R})$ is not normal.

Proof Consider $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in H$, $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{R})$.

Observe that $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$. If H were normal, then

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

would have to be in H. However,

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

which is not in H, because the lower left entry is nonzero.

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

HW 6 Solutions   M31F11

(11.5) Let G be a group, $H \leq G$, $K \trianglelefteq G$. Then $H \cap K \trianglelefteq H$.

Proof: If $m \in H \cap K$, we need to show that for any $h \in H$, we have

$$hmh^{-1} \in H \cap K.$$

But since $m \in H$, and subgroups are closed under multiplication, it follows that $hmh^{-1} \in H$ for any $h \in H$.

Also, since $H \leq G$ and $K \trianglelefteq G$, the fact that $m \in K$ implies that $hmh^{-1} \in K$ also.

Hence $hmh^{-1} \in H \cap K$ for any $h \in H$, $m \in H \cap K$, so $H \cap K \trianglelefteq H$. ⊠

8) If $N \trianglelefteq G$ and $H \leq G$, then $NH \leq G$.

Proof We need to show that $NH = \{nh : n \in N, h \in H\}$ is closed under multiplication and inverses.

If $n_1h_1, n_2h_2 \in NH$, then

$$n_1h_1n_2h_2 = n_1h_1h_2h_2^{-1}n_2h_2$$

$$= n_1h_1h_2n_3$$

since $N \trianglelefteq G$. By using the same trick of multiplying by an element and its inverse,

$$n_1h_1h_2n_3 = n_1h_1h_2n_3(h_1h_2)^{-1}(h_1h_2)$$

$$= n_1n_4(h_1h_2) \in NH$$

since N, H are closed under multiplication. Hence NH is also closed under multiplication.

To see that NH is closed under inverses, observe that

$$(nh)^{-1} = h^{-1}n^{-1} = h^{-1}n^{-1}hh^{-1} = n_2h^{-1} \in NH,$$

since $N \trianglelefteq G$ and N & H are both closed under inverses.

Since NH is closed under multiplication and inverses, it is a subgroup of G. ⊠

(7) a) We know $H = \langle 4 \rangle \triangleleft \mathbb{Z}_{24}$ because $\mathbb{Z}_{24}$ is abelian, & all subgroups of abelian groups are normal by the Corollary to Theorem 11.2.

b) Claim $\mathbb{Z}_{24}/\langle 4 \rangle \cong \mathbb{Z}_4$.

Proof By the Fundamental Theorem (Section 13), it will suffice to find an epimorphism

$$\phi : \mathbb{Z}_{24} \longrightarrow \mathbb{Z}_4$$

such that $\ker \phi = \langle 4 \rangle$. We can take ϕ to be the identity projection: $\phi(m) = m \pmod 4$.

We know this map is a homomorphism since $4 \mid 24$, as discussed in class.

Since 1, 2, 3, 4 represent different elements of $\mathbb{Z}_{24}$, this map ϕ is onto; and clearly $\langle 4 \rangle$ is the kernel of ϕ. $(\phi(m) = 0 \iff 4 \mid m \iff m \in \langle 4 \rangle)$

Therefore, the Fundamental Theorem tells us that

$$\mathbb{Z}_{24}/\langle 4 \rangle \cong \mathbb{Z}_4$$

as claimed. $\boxtimes$

(7)c) Since ϕ is an isomorphism between $\mathbb{Z}_{24}/\langle 4\rangle$ and $\mathbb{Z}_4$, the order of $\langle 4\rangle + 14$ in $\mathbb{Z}_{24}/\langle 4\rangle$ is the same as $o(\phi(14))$ in $\mathbb{Z}_4$ by Theorem 12.5.

Hence, since $\phi(14) = 2$ has order 2, so does $\langle 4\rangle + 14$. ☒

You could also observe that 4 divides $14+14$, and hence $\langle 4\rangle + 14$ has order 2.

HW 6 Solutions

(1) If G is abelian and $|G| = 33$, then G is cyclic.

Proof: By Theorem 11.7, since 3 & 11 are primes which divide 33, G has a subgroup of size 3, A, and another of size 11, B. Theorem 10.5 tells us that these subgroups are cyclic. Let $A = \langle x \rangle$ and $B = \langle y \rangle$.

We proved in Exercise 4.23 that when G is abelian, $o(xy) = o(x)o(y)$ if $(o(x), o(y)) = 1$. Since $o(x) = 3$, $o(y) = 11$, and $(3,11) = 1$, it follows that $xy \in G$ has order 33. Since $\langle xy \rangle \leq G$ and $|\langle xy \rangle| = |G| = 33$, it follows that $\langle xy \rangle = G$, so G is cyclic. ☒

HW 6 Solutions M31 F11

(15.7) a) $\mathbb{C}^\times / \mathcal{U} \cong (\mathbb{R}^+, \cdot)$

Proof To use the Fundamental Theorem, we define a function $\phi: \mathbb{C}^\times \to (\mathbb{R}^+, \cdot)$ by $\phi(x+iy) = x^2+y^2$. We want to show that ϕ is an epimorphism, and that $\ker \phi = \mathcal{U}$.

The last statement follows from the observation that the identity in $(\mathbb{R}^+, \cdot)$ is 1, and the definition of $\mathcal{U} = \{x+iy : x^2+y^2 = 1\}$. So it remains to show that ϕ is an epimorphism. We know that ϕ is onto because if $r \in \mathbb{R}^+$, then $\phi(\sqrt{r}+0i) = (\sqrt{r})^2 = r$, and all positive reals have square roots.
To see that ϕ is a homomorphism, we must show that

$$\phi\big((x+iy)(z+iw)\big) = \phi(x+iy)\,\phi(z+iw).$$

But,
$$\begin{aligned}
\phi((x+iy)(z+iw)) &= \phi\big((xz-yw + i(xw+yz)\big) \\
&= (xz-yw)^2 + (xw+yz)^2 \\
&= x^2z^2 - 2xzyw + y^2w^2 \\
&\quad + x^2w^2 + 2xwyz + y^2z^2 \\
&= x^2z^2 + y^2w^2 + x^2w^2 + y^2z^2 \\
&= (x^2+y^2)(z^2+w^2) = \phi(x+iy)\,\phi(z+iw).
\end{aligned}$$

$\boxtimes$

(7.7) b) $\mathbb{C}^\times / \mathbb{R}^+ \cong \mathcal{U}$

Proof Observe that $\mathbb{R}^+$ is embedded in $\mathbb{C}^\times$ as the elements of the form $\{r+0i : r \in \mathbb{R}^+\}$. Since the multiplicative identity of $\mathbb{C}^\times$ is $1+0i \in \mathbb{R}^+$,

$$(r+0i)^{-1} = (\tfrac{1}{r} + 0i) \in \mathbb{R}^+,$$

we have $\mathbb{R}^+ \leq \mathbb{C}^\times$. Since $\mathbb{C}^\times$ is abelian, every subgroup is normal, so $\mathbb{C}^\times/\mathbb{R}^+$ is a group as claimed.

Again, to use the Fundamental Theorem, we must define a function

$$\phi : \mathbb{C}^\times \to \mathcal{U}.$$

Let $\phi(x+iy) = \frac{1}{\sqrt{x^2+y^2}}(x+iy)$. We need to show that $\ker \phi = \mathbb{R}^+$ and that ϕ is an epimorphism. To see that $\ker \phi = \mathbb{R}^+$, observe that

$\phi(x+iy) = 1+0i \iff y=0$ and $x>0$, because

$\sqrt{x^2} = |x|$, and $x/|x| = 1 \iff x > 0$

To see that ϕ is onto, observe that if $x+iy \in \mathcal{U}$, then $\frac{1}{\sqrt{x^2+y^2}} = 1$, so $\phi(x+iy) = x+iy$ in this case. Hence every element of $\mathcal{U}$ is

over

13.7 b) cont'd | mapped onto by itself.

To see that ϕ is a homomorphism, consider

$$\phi((x+iy)(z+iw)) = \phi(xz - yw + i(yz + xw))$$

$$= \frac{1}{\sqrt{(xz-yw)^2+(yz+xw)^2}}\left((xz-yw)+i(yz+xw)\right)$$

$$= \frac{1}{\sqrt{(x^2+y^2)(z^2+w^2)}}\left(xz-yw+i(yz+xw)\right)$$

$$= \frac{1}{\sqrt{x^2+y^2}}(x+iy)\frac{1}{\sqrt{z^2+w^2}}(z+iw)$$

$$= \phi(x+iy)\,\phi(z+iw).$$

Thus ϕ is a homomorphism, as claimed.

Hence, by the Fundamental Theorem,

$$\mathbb{C}^*/\mathbb{R}^+ \cong U. \quad \boxtimes$$

(13.13)

a) Let $K = \{e, (12)(34), (13)(24), (14)(23)\}$.

Then $K \trianglelefteq S_4$.

Proof We use Theorem 11.1 (iii):

$(123)K = \{(123), (134)(2), (1)(243), (142)(3)\}$

$K(123) = \{(123), (1)(243), (142)(3), (134)(2)\}$

$(132)K = \{(132), (1)(234), (124)(3), (143)(2)\}$

$K(132) = \{(132), (143)(2), (1)(234), (124)(3)\}$

$(1)(23)(4)K = \{(23), (1342), (1243), (14)(2)(3)\}$

$K(23) = \{(23), (1243), (1342), (14)(2)(3)\}$

$(12)K = \{(12), (34), (1324), (1423)\}$

$K(12) = \{(12), (34), (1423), (1324)\}$

$(13)K = \{(13), (1234), (1)(24)(3), (1432)\}$

$K(13) = \{(13), (1432), (24), (1234)\}$

and of course $Ke = eK$.

over

HW 6 Solutions M31P11

(13.13) a) Since the right & left cosets agree for K, Theorem 11.1 (iii) tells us that $K \triangleleft S_4$.

b) To see that $S_4/K \cong S_3$, observe that each element of S_3 is in a different coset of K. Hence, define $\phi : S_4/K \to S_3$ by $\phi(Ka) = a$ (where $a \in S_3$; equivalently, $a \in S_4$ is a perm. that fixes 4.)

We must check that ϕ is a homomorphism: that is, that $\phi(Ka \cdot Kb) = \phi(Ka)\,\phi(Kb)$.

But, by definition, $Ka \cdot Kb = Kab$, so

$$\phi(Ka \cdot Kb) = ab = \phi(Ka)\,\phi(Kb).$$

Clearly ϕ is 1-1 and onto by construction; hence ϕ is an isomorphism. ☒

---

there might well be a clever way of doing this, rather than relying on computations! I hope so, actually!

HW 6 Solutions   M 31 F 11

(13.16) Let $A \triangleleft G$, $B \triangleleft H$. Then $A \times B \triangleleft G \times H$, and

$$\frac{G \times H}{A \times B} \cong G/A \times H/B.$$

Proof To see that $A \times B \triangleleft G \times H$, observe that if $g \in G$, $h \in H$, $a \in A$, $b \in B$, then $(g,h)(a,b)(g,h)^{-1} = (g,h)(a,b)(g^{-1},h^{-1}) = (gag^{-1}, hbh^{-1}) \in A \times B$ since $A \triangleleft G$, $B \triangleleft H$.
Again, we use the Fundamental Theorem to show the isomorphism.
Define $\phi : G \times H \longrightarrow G/A \times H/B$ by

$$\phi(g,h) = (Ag, Bh).$$

Then
$$\begin{aligned}\phi((g_1,h_1)(g_2,h_2)) &= \phi(g_1g_2, h_1h_2)\\ &= (Ag_1g_2, Bh_1h_2)\\ &= (Ag_1, Bh_1)(Ag_2, Bh_2)\\ &= \phi(g_1,h_1)\,\phi(g_2,h_2).\end{aligned}$$

Thus ϕ is a homomorphism.

Clearly, ϕ is onto: if $(Ag, Bh) \in G/A \times H/B$, then $(Ag, Bh) = \phi(g,h)$ is in the image of ϕ.

To see that $\ker\phi = A \times B$, observe that

$\phi(g,h) = (A,B) \iff g \in A$ and $h \in B$. (Recall that (A,B) is the identity in $G/A \times H/B$.) Hence the Fund Thm gives us the isom. $\boxtimes$

HW 6 Solutions M31F11

Presentations

1. We know that 9, 10, and 11 will have primitive roots, since $9=3^2$, $10=2\cdot5$, and 11 is prime. However, $12=3\cdot4=3\cdot2^2$ will not have primitive roots.

Observe that $U(9)=\{1,2,4,5,7,8\}$

$U(10)=\{1,3,7,9\}$

$U(11)=\{1,2,3,4,5,6,7,8,9,10\}$

Calculations will show that $U(9)=\langle 2\rangle=\langle 5\rangle$ (note that $5\cdot2\equiv1 \bmod 9$, so $5=2^{-1}$ in this group), but $o(4)=3=o(7)$ and $o(8)=2$. Thus 2 & 5 are the only primitive roots of 9.

Similarly, $U(10)=\langle 3\rangle=\langle 7\rangle$ (since $3\cdot7=21\equiv1 \pmod{10}$ if one generates so must the other.) but $o(9)=2$, so we again have only 2 primitive roots.

Finally, 11 has four primitive roots: 2, 6, 7, 8.

Note that $U(11)=\langle 2\rangle=\{2,4,8,5,10,9,7,3,6,1\}$

$=\langle 6\rangle$ since $6=2^{-1} \pmod{11}$.

Also, we know that $o(10)=2$ since $10^2=100\equiv1$, so 10 is not a prim. root.

However, $\langle 3\rangle=\{1,3,9,5,4\}=\langle 4\rangle=\langle 9\rangle=\langle 5\rangle$ (since all non-identity elements of a cyclic group of prime order are generators).

Moreover, $\langle 7\rangle=\{1,7,5,2,3,10,4,6,9,8\}=\langle 8\rangle$. ☒

Presentations

(2) To show that we could distinguish between the two effects, it suffices to show that $\bar{K}$ and $\bar{L}$ don't form groups under the operations

$$a(1+t_1^2)K \cdot a(1+t_2^2)K = a(1+t_1^2)(1+t_2^2)K$$

Observe that $(1+t_1^2)(1+t_2^2) = 1+t_1^2+t_2^2+(t_1t_2)^2$.

However, since $(t_1+t_2)^2 = t_1^2+t_2^2+2t_1t_2$, this operation is only binary if $t_1t_2=2$, which does not need to hold for all $t_1, t_2 \in \mathbb{Z}$.
For example, we could take $t_1=1, t_2=4$.

Another reasonable group operation to try here would have been

$$a(1+t_1^2)K \cdot a(1+t_2^2)K = a^2(1+t_1^2)(1+t_2^2)K.$$

This is clearly not a binary operation on $\bar{K}$, since

$$a^2(1+t_1^2)(1+t_2^2) = a(a+at_1^2)(1+t_2^2)$$
$$= a(a+a(t_1^2+t_2^2)+a(t_1t_2)^2),$$

which is not of the form $a(1+t^2)$ unless $a=1$, and we assumed $a>1$.
Thus, in either case, K is not a gp.