Math 123 Homework Assignment #3
Friday, May 6

Part I:

1. Suppose that P and Q are projections in $B(H)$. We say that $P \perp Q$ if $P(H) \perp Q(H)$ and that $P \leq Q$ if $P(H) \subset Q(H)$.

(a) Show that the following are equivalent.

(i) $P \perp Q$.

(ii) $PQ = QP = 0$.

(iii) $P + Q$ is a projection.

(b) Show that the following are equivalent.

(i) $P \leq Q$.

(ii) $PQ = QP = P$.

(iii) $Q - P$ is a projection.

(Hint: Note that PQP is a positive operator. Also $PQP = PQ(PQ)^*$ so that $PQP = 0$ if and only if $PQ = QP = 0$.)

2. Let $\pi : A \to B(\mathcal{H})$ be an irreducible representation of a C^*-algebra A. Suppose that $\pi(A) \cap \mathcal{K}(\mathcal{H}) \neq \{ 0 \}$. Show that $\pi(A) \supset \mathcal{K}(\mathcal{H})$. ("If the range of an irreducible representation contains one nonzero compact operator, then it contains them all.") If you want a hint, look over the proof of Proposition BA.

ANS: By assumption, $\mathrm{id} : \pi(A) \to B(\mathcal{H})$ is irreducibe. If $\pi(A) \bigcap \mathcal{K}(\mathcal{H}) \neq \{ 0 \}$, then the restriction of id to the *ideal* $\pi(A) \bigcap \mathcal{K}(\mathcal{H})$ is non-zero, and hence irreducible. But then $\pi(A) \bigcap \mathcal{K}(\mathcal{H})$ is an irreducible C^*-subalgebra of $\mathcal{K}(\mathcal{H})$, and is therefore all of $\mathcal{K}(\mathcal{H})$.

3. Complete the proof of Lemma AZ. That is, show that $\mathrm{CCR}(A)$ is the largest CCR ideal in A in the sense that if J is any CCR ideal in A, then $J \subset \mathrm{CCR}(A)$.

Part II:

4. Suppose that π is a non-degenerate representation of A on $\mathcal{H}$. Let $\{ e_\lambda \}$ be an approximate identity for A. Show that $\pi(e_\lambda)$ converges to I in the strong operator topology; that is, prove that $\lim_\lambda \pi(e_\lambda)\xi = \xi$ for all $\xi \in \mathcal{H}$. Conclude that $S = \{ \pi(x)\xi : x \in A, \xi \in \mathcal{H} \}$ is dense in $\mathcal{H}$. (The point is that *a priori* all we are given is that S *spans* a dense subset of $\mathcal{H}$.)

> **ANS**: Let $\xi \in \mathcal{H}$ and $\epsilon > 0$ be given. By assumption there are vectors $\xi_1, \dots, \xi_n \in \mathcal{H}$ and elements $x_1, \dots, x_n \in A$ such that $\|\xi - \sum_{i=1}^n \pi(x_i)\xi_i\| < \epsilon$. On the other hand, $e_\lambda x_i \to x_i$ for each i. Thus $\pi(e_\lambda x_i) \to \pi(x_i)$ in norm. Choose $\lambda_0 \in \Lambda$ so that $\lambda \ge \lambda_0$ implies that $\| \sum_{i=1}^n \pi(x_i)\xi_i - \sum_{i=1}^n \pi(e_\lambda x_i)\xi_i\| < \epsilon$. Then
>
> $$\|\xi - \pi(e_\lambda)\xi\| \le \Big\|\xi - \sum_{i=1}^n \pi(x_i)\xi_i\Big\| + \Big\|\sum_{i=1}^n \pi(x_i)\xi_i - \sum_{i=1}^n \pi(e_\lambda x_i)\xi_i\Big\| + \Big\|\pi(e_\lambda)\Big(\sum_{i=1}^n \pi(x_i)\xi_i - \xi\Big)\Big\| < 3\epsilon.$$

5. Let $\{ J_\alpha : 0 \le \alpha \le \alpha_0 \}$ be a composition series for a *separable* C^*-algebra A. Show that α_0 is countable. (Recall that α_0 is called countable if $\{ \alpha : 0 \le \alpha < \alpha_0 \}$ is countable. Also, for each $\alpha < \alpha_0$ notice that you can find $a_\alpha \in J_{\alpha+1}$ such that $\|a_{\alpha+1} - a\| \ge 1$ for all $a \in J_\alpha$.)

> **ANS**: See Remark 8.12 in my book on crossed products.

6. Suppose that $\{ J_\alpha : 0 \le \alpha \le \alpha_0 \}$ is a composition series for a C^*-algebra A. A nondegenerate representation π if A is said to *live on the subquotient* $J_{\alpha+1}/J_\alpha$ if π is the canonical extension to A of a representation π' of $J_{\alpha+1}$ such that $\ker \pi' \supset J_\alpha$. That is, π' must be of the form $\pi' = \rho \circ q_\alpha$ where $q_\alpha : J_{\alpha+1} \to J_{\alpha+1}/J_\alpha$ is the natural map, and ρ is a nondegenerate representation of $J_{\alpha+1}/J_\alpha$. Show that every *irreducible* representation of A lives on a subquotient so that the spectrum of A can be identified with the disjoint union of the spectra of the subquotients $J_{\alpha+1}/J_\alpha$ for $\alpha < \alpha_0$.

Part III:

7. Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{ e_n \}_{n=1}^\infty$. Let S be the *unilateral shift* operator $S \in B(\mathcal{H})$ defined by $S(e_n) = e_{n+1}$ for all n. Finally, let A be the unital C^*-algebra generated by S (i.e., $A = C^*(S)$), and let $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$.

(a) Show that $S^*S - SS^* = P$, where P is the rank-one projection onto $\mathbb{C}e_1$.

(b) Show that A is irreducible, and that $\mathcal{K}(\mathcal{H}) \subseteq A$.

(c) Show that if $\alpha \in \mathbb{T}$, then there is a unitary operator U in $B(\mathcal{H})$ such that $USU^* = \alpha S$.

(d) Show that the quotient $A/\mathcal{K}(\mathcal{H})$ is $*$-isomorphic to $C(\mathbb{T})$.

(e) Conclude that A is GCR, but not CCR.

(f) Describe the (equivalence classes) of irreducible representations of A.

Suggestions: In part (b), show that e_1 is cyclic for the identity representation of A. Now observe that if V is a closed invariant subspace for A, then either $e_1 \in V$ or $e_1 \in V^\perp$. In part (d), notice that the image of S in the quotient is unitary (hence normal), generates, and has spectrum $\mathbb{T}$.

ANS: Look at Example A.31 in my book with Iain Raeburn on Morita equivalence.

8. Let $\mathcal{H}$ be a *separable* infinite dimensional Hilbert space. Recall that $T \in B(\mathcal{H})$ is said to be *below* if there is an $\epsilon > 0$ such that $|T\xi| \geq \epsilon|\xi|$ for all $\xi \in \mathcal{H}$.

(a) Show that if $T \in B(\mathcal{H})_{\text{s.a.}}$ is bounded from below, then T has a bounded inverse.

(b) If $T \in B(\mathcal{H})_{\text{s.a.}}$ and $\epsilon > 0$, then define

$$M_\epsilon = \overline{\operatorname{span}}\{\, f(T)\xi : \xi \in \mathcal{H},\ f \in C\big(\sigma(T)\big), \text{ and } f(\lambda) = 0 \text{ if } |\lambda| \leq \epsilon \,\}.$$

Show that $|T\xi| \geq \epsilon|\xi|$ for all $\xi \in M_\epsilon$, and that $TM_\epsilon = M_\epsilon$.

(c) Show that if $T \in B(\mathcal{H})_{\text{s.a.}}$ is not compact, then there is an $\epsilon > 0$ so that M_ϵ is infinite dimensional. In particular, conclude that there is a partial isometry $V \in B(\mathcal{H})$ such that V^*TV has a bounded inverse.

(d) Show that $\mathcal{K}(\mathcal{H})$ is the only non-zero proper closed ideal in $B(\mathcal{H})$.

(e) Assuming that any C^*-algebra has irreducible representations, conclude that $B(\mathcal{H})$ is not a GCR algebra.

ANS: See example A.32 in my book on Morita equivalence (written with Iain Raeburn). (a) If T is bounded from below, then $T\mathcal{H}$ is complete and therefore closed. Furthermore, $\ker T = \{\,0\,\}$. If $T = T^*$, then $T\mathcal{H}^\perp = T^*\mathcal{H}^\perp = \ker T = \{\,0\,\}$. Therefore T is a bounded bijection from $\mathcal{H}$ onto $\mathcal{H}$, and T^{-1} is bounded by the Closed Graph Theorem.

(b) Let $J_\epsilon = \{\, f \in C\big(\sigma(T)\big) : f(\lambda) = 0 \text{ if } |\lambda| \leq \epsilon \,\}$. Notice that if $f \in J_\epsilon$ and $g(\lambda) = \lambda$ for all $\lambda \in \sigma(T)$, then $g^2|f|^2 \geq \epsilon^2|f|^2$. It follows that $T^2 f(T)^* f(T) \geq \epsilon^2 f(T)^* f(T)$. Thus,

$$\begin{aligned} |Tf(T)\xi|^2 &= \langle T^2 f(T)^* f(T)\xi, \xi\rangle \\ &\geq \epsilon^2 \langle f(T)^* f(T)\xi, \xi\rangle \\ &= \epsilon^2 |f(T)\xi|^2. \end{aligned}$$

Now let $\xi \in M_\epsilon$. Let $\{ f_\lambda \}$ be an approximate identity in J_ϵ. Then we see that $f_\lambda(T)\xi \to \xi$. (Approximate ξ by $\sum_{i=1}^n g_i(T)\xi_i$ with $g_i \in J_\epsilon$ and $\xi_i \in \mathcal{H}$.) Thus

$$|T\xi|^2 = \lim_\lambda |Tf_\lambda(T)\xi| \geq \epsilon^2 \lim_\lambda |f_\lambda(T)\xi|^2 = \epsilon^2|\xi|^2.$$

This proves that T is bounded below on M_ϵ. But since we have $TM_\epsilon \subseteq M_\epsilon$ by construction, we have $TM_\epsilon = M_\epsilon$ by part (a).

(c) Let P_n be the projection onto $M_{\frac{1}{n}}$. Define

$$f_n(\lambda) = \begin{cases} 0 & \text{if } 0 \leq \lambda \leq \frac{1}{n}, \\ 2(\lambda - \frac{1}{n}) & \text{if } \frac{1}{n} \leq \lambda < \frac{2}{n}, \\ \lambda & \text{if } \lambda \geq \frac{2}{n}. \end{cases}$$

Then $f_n \in J_{\frac{1}{n}}$ and $f_n \to g$ uniformly on $\sigma(T)$, where $g(\lambda) = \lambda$ for all $\lambda \in \sigma(T)$. Thus $f_n(T) \to T$ and $P_n f_n(T) = f_n(T)$. If each $M_{\frac{1}{n}}$ were finite dimensional, then P_n, and hence $f_n(T)$, would be finite rank. Then T would be compact.

So choose ϵ so that $\dim M_\epsilon = \aleph_0 = \dim \mathcal{H}$. Then there is a partial isometry $V : \mathcal{H} \to \mathcal{H}$ such that $V\mathcal{H} = M_\epsilon$. Then V^*TV is bounded below on $\mathcal{H}$ and has a bounded inverse by part (a).

(d) Let I be a non-zero (closed) ideal in $B(\mathcal{H})$. Since $I \cap \mathcal{K}(\mathcal{H})$ is an ideal in $\mathcal{K}(\mathcal{H})$ we must have $\mathcal{K}(\mathcal{H}) \subseteq I$ since $\mathcal{K}(\mathcal{H})$ is simple. If $I \neq \mathcal{K}(\mathcal{H})$, then I contains a non-compact operator T. Since I is a C^*-algebra, and is therefore the span of its self-adjoint elements, we may assume that T is self-adjoint. Now it follows from part (c) that I contains an invertible element, and hence that $I = B(\mathcal{H})$ as required.

(e) It follows from the previous part that the Calkin algebra $\mathcal{C}(\mathcal{H}) = B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is simple. If $B(\mathcal{H})$ were GCR, then $CCR(\mathcal{C}(\mathcal{H})) \neq \{ 0 \}$. Since $\mathcal{C}(\mathcal{H})$ is simple, it follows that $\mathcal{C}(\mathcal{H})$ is CCR. Thus if π is an irreducible representation of $\mathcal{C}(\mathcal{H})$ and e is the identity element of $\mathcal{C}(\mathcal{H})$, then $\pi(e) = I_{\mathcal{H}_\pi}$ is a compact operator. This forces $\mathcal{H}_\pi$ to be finite dimensional, and since the simplicity of $\mathcal{C}(\mathcal{H})$ implies that π is an $*$-isomorphism of $\mathcal{C}(\mathcal{H})$ into $B(\mathcal{H}_\pi)$ (onto actually), $\mathcal{C}(\mathcal{H})$ must be finite dimensional as well. But one can easily find infinitely many orthogonal infinite dimensional projections $\{ P_n \}$ in $B(\mathcal{H})$ (when $\mathcal{H}$ is infinite dimensional). The images of the P_n in $\mathcal{C}(\mathcal{H})$ are clearly independent. This contradiction completes the proof.