HW7

5.2 $f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \pmod 1 \quad = A\bar{x} \pmod 1$

Since the matrix is symmetric & $J = Df = A$

$A = P^{-1} D P \Rightarrow A^n = P^{-1} D^n P$ where $D = \mathrm{diag}(\lambda_j)$

& $A^T = P^{-1} D P \Rightarrow A^{Tn} = P^{-1} D^n P$

$\Rightarrow A^n A^{Tn} = P^{-1} D^{2n} P$

So the eigenvalues of $J_n J_n^T = \lambda_j^{2n}$

$\Rightarrow r_k^n = \sqrt{\lambda_k^{2n}} = \lambda_k^n$

$L_k = \lim\limits_{n \to \infty} (\lambda_k^n)^{1/n} = \lambda_k$

So. We need the eigenvalues of A.

$0 = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - 1 = \lambda^2 - \lambda - 1$

$\lambda = \dfrac{1 \pm \sqrt{1^2 - 4(-1)}}{2} = \dfrac{1}{2} \pm \dfrac{\sqrt{5}}{2}$

$\Rightarrow L_1 = \dfrac{1+\sqrt{5}}{2} \qquad L_2 = \dfrac{1-\sqrt{5}}{2}$

$h_1 = \ln L_1 \qquad h_2 = \ln L_2$

These #'s are exactly $1/2$ those of the cat map because The cat map is A^2

A. $B(x,y) = \begin{cases} (x/2,\ 2y \pmod 1) & y \geq 1/2 \\ (\frac{x+1}{2},\ 2y \pmod 1) & y < 1/2 \end{cases}$

$$J = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow J^n = \begin{pmatrix} (1/2)^n & 0 \\ 0 & 2^n \end{pmatrix}$$

$$r_1^n = 2^n \qquad r_2^n = (1/2)^n$$

$$L_1 = \lim_{n\to\infty} (2^n)^{1/n} = 2$$

$$L_2 = \lim_{n\to\infty} ((1/2)^n)^{1/n} = 1/2$$

$$h_1 = \ln 2 \qquad h_2 = \ln 1/2 = -\ln 2$$

The sum of the Lyapunov exponents is 0.
$\Rightarrow$ Map preserves area.

T7.2

(a) $\lambda = 3$ is a double eigenvalue.

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 \\ 3x_2 \end{pmatrix}$$

$\Rightarrow 3x_1 + x_2 = 3x_1 \Rightarrow x_2 = 0$

$\Rightarrow \therefore$ the eigenvector is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(b) $y' = 3y \Rightarrow y = Ce^{3t}$

$$x' = 3x + y$$

$$x' - 3x = y = e^{3t}$$

$$\frac{d}{dt}\left(e^{-3t}x\right) = 1$$

$$e^{-3t}x = t + C$$

$$x(t) = te^{3t} + Ce^{3t}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} te^{3t} + C_2 e^{3t} \\ C_1 e^{3t} \end{bmatrix}$$

7.2 $x'' + 3x' - 4x = 0$

a) $x_1 = x$

$x_2 = x_1'$

$x_2' = x'' = -3x' + 4x = -3x_2 + 4x_1$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} x_2 \\ -3x_2 + 4x_1 \end{bmatrix}$$

b) equilibrium is $(0,0)$.

$$Df = \begin{bmatrix} 0 & 1 \\ 4 & -3 \end{bmatrix}$$

eigenvalues $\begin{vmatrix} -\lambda & 1 \\ 4 & -3-\lambda \end{vmatrix} = -\lambda(-3-\lambda) - 4 = 0$

$\Rightarrow \lambda^2 + 3\lambda - 4 = 0$

$(\lambda + 4)(\lambda - 1) = 0$

$\lambda = -4, 1 \Rightarrow (0,0)$ is a saddle pt.

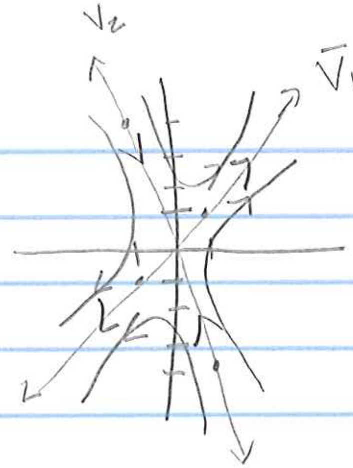
eigenvectors

$$\begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{cases} x_2 = x_1 \\ 4x_1 \end{cases}$$

$\lambda_1 = 1 \quad \bar{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -4\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$x_2 = 4x_1 \Rightarrow \bar{v}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \quad \lambda_2 = -4$



T7.9 $\quad x' = -x^3$

to determine stability we look at the solution. (We are lucky it is separable)

$$\frac{x'}{x^3} = -1$$

$$\Rightarrow \frac{1}{-2} x^{-2} = -t + C$$

$$\Rightarrow x^{-2} = C + 2t$$

$$\Rightarrow \frac{1}{C+2t} = x^2 \quad \Rightarrow x = \pm \frac{1}{\sqrt{2t+C}}$$

as $t \to \infty$ $\quad x \to 0 \quad \Rightarrow$ 0 is asymptotically stable.

Note: The solution is unique (choice of + or − depends on initial condition.

$x_0 = x(0) = 1$

$$\Rightarrow x(0) = \overset{\text{NOTICE}}{\downarrow}{+} \frac{1}{\sqrt{C}} = 1 \Rightarrow C = 1$$

So $x(t) = \dfrac{1}{\sqrt{1+2t}}$

T7.5

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow$ $y = c$ - constant

$x' = c \Rightarrow x = ct + d$

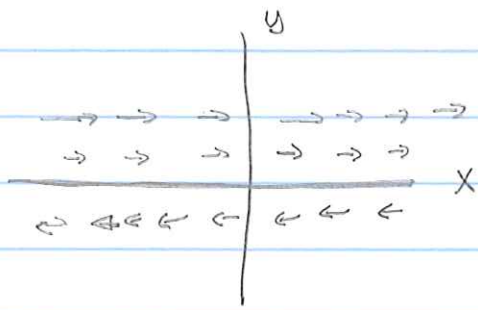
$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ct \\ c \end{pmatrix}$ is a solution.

if C is positive $x \to \infty$.

This does not contradict Thm 7.12 because

$\lambda = 0$ is a double eigenvalue. $\Rightarrow$ not distinct.

Phaseplot



7.3

$x' = 2x - y$

$y' = x^2 + 4y$

1st find equilibria

$\Rightarrow 2x - y = 0 \Rightarrow y = 2x$

$x^2 + 4y = x^2 + 8x = 0$

$\Rightarrow x(x+8) = 0 \Rightarrow x = 0, x = -8$

equilibria are $(0,0), (-8,-16)$

For stability look at Jacobian

$$Df = \begin{bmatrix} 2 & -1 \\ 2x & 4 \end{bmatrix}$$

$$Df(0,0) = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$$ eigenvalues $(2-\lambda)(4-\lambda) = 0$

$\Rightarrow \lambda = 4, 2.$

$\Rightarrow$ $(0,0)$ is unstable.

$$Df((-8,-16)) = \begin{bmatrix} 2 & -1 \\ -16 & 4 \end{bmatrix}$$

eigenvalues $(2-\lambda)(4-\lambda) - 16 = 0$

$\lambda^2 - 4\lambda - 2\lambda + 8 - 16 = 0$

$\lambda^2 - 6\lambda - 8 = 0$

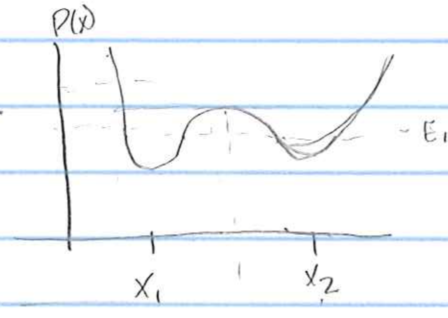
$$\lambda = \frac{6 \pm \sqrt{36 - 4(-8)}}{2}$$ $36 + 32 = 68$ $\quad 4\overline{)68}^{\,17}$

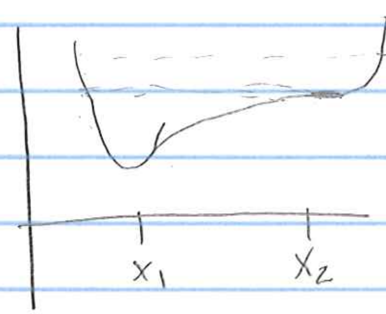
$= 3 \pm \sqrt{17}$

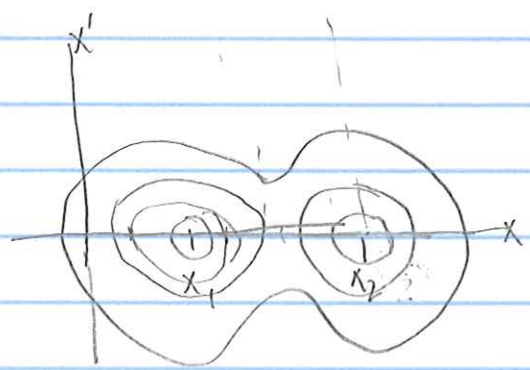
$\lambda_1 < 0 \quad \lambda_1 > 0 \Rightarrow$ saddle

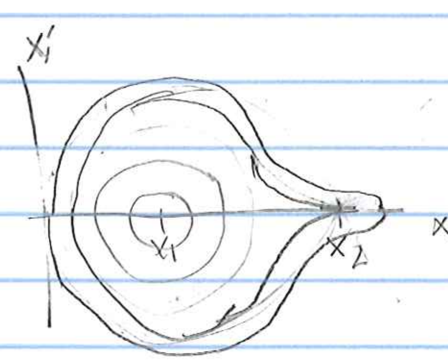
& unstable.

7.10









both x_1 & x_2 periodic