Taylor and Maclaurin Series (cont'd)

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Recall

- The **Taylor series of the function** f **at** a is

$$f(x) = f(a) + \frac{f^{'}(a)}{1!}(x-a) + \frac{f^{''}(a)}{2!}(x-a)^2 + \frac{f^{'''}(a)}{3!}(x-a)^3 + \cdots$$

- The **Maclaurin series**

$$f(x) = f(0) + \frac{f^{'}(0)}{1!}x + \frac{f^{''}(0)}{2!}x^2 + \cdots$$

When does a function equals its Taylor series?

- Consider the partial sums of the Taylor series:

$$T_n(x) = f(a) + \frac{f^{'}(a)}{1!}(x-a) + \frac{f^{''}(a)}{2!}(x-a)^2 + \cdots \frac{f^{n}(a)}{n!}(x-a)^n.$$

- T_n is called the n**th-degree Taylor polynomial of** f **at** a**.**

- Then f equals its Taylor series if $f(x) = \lim_{n\to\infty} T_n(x)$.

The remainder of the Taylor series

- Let $R_n(x) = f(x) - T_n(x)$
- If
$$\lim_{n\to\infty} R_n(x) = 0$$
for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

Useful formulas

- **(Taylor's inequality)** If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

  for $|x-a| \leq d$.

- Often we use the following fact:

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$

  for every real number x.

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- Find the Taylor series for $f(x) = e^x$at $a = 1$.
- Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

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- Evaluate $\int e^{-x^2} \mathrm{d}x$ as an infinite series.

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- Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{5^n}{3^n n!}$$