# Ergodic Markov Chains

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# Definition

- A Markov chain is called an *ergodic chain* if it is possible to go from every state to every state (not necessarily in one move).
- Ergodic Markov chains are also called *irreducible*.
- A Markov chain is called a *regular* chain if some power of the transition matrix has only positive elements.

# Example

- Let the transition matrix of a Markov chain be defined by

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array} .$$

- Then this is an ergodic chain which is not regular.

# Example: Ehrenfest Model

- We have two urns that, between them, contain four balls.
- At each step, one of the four balls is chosen at random and moved from the urn that it is in into the other urn.
- We choose, as states, the number of balls in the first urn.

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array} .$$

# Regular Markov Chains

- Any transition matrix that has no zeros determines a regular Markov chain.

- However, it is possible for a regular Markov chain to have a transition matrix that has zeros.

- For example, recall the matrix of the Land of Oz

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{ccc} \mathsf{R} & \mathsf{N} & \mathsf{S} \end{array} \\ \begin{array}{c} \mathsf{R} \\ \mathsf{N} \\ \mathsf{S} \end{array} & \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \end{array} .$$

**Theorem.** *Let* $\mathbf{P}$ *be the transition matrix for a regular chain. Then, as* $n \to \infty$*, the powers* $\mathbf{P}^n$ *approach a limiting matrix* $\mathbf{W}$ *with all rows the same vector* $\mathbf{w}$*. The vector* $\mathbf{w}$ *is a strictly positive probability vector (i.e., the components are all positive and they sum to one).*

# Example

- For the Land of Oz example, the sixth power of the transition matrix **P** is, to three decimal places,

$$\mathbf{P}^6 = \begin{array}{c c} & \begin{array}{ccc} \mathsf{R} & \mathsf{N} & \mathsf{S} \end{array} \\ \begin{array}{c} \mathsf{R} \\ \mathsf{N} \\ \mathsf{S} \end{array} & \begin{pmatrix} .4 & .2 & .4 \\ .4 & .2 & .4 \\ .4 & .2 & .4 \end{pmatrix} \end{array} .$$

**Theorem.** *Let* $\mathbf{P}$ *be a regular transition matrix, let*

$$\mathbf{W} = \lim_{n \to \infty} \mathbf{P}^n ,$$

*let* $\mathbf{w}$ *be the common row of* $\mathbf{W}$*, and let* $\mathbf{c}$ *be the column vector all of whose components are 1. Then*

**(a)** $\mathbf{wP} = \mathbf{w}$*, and any row vector* $\mathbf{v}$ *such that* $\mathbf{vP} = \mathbf{v}$ *is a constant multiple of* $\mathbf{w}$*.*

**(b)** $\mathbf{Pc} = \mathbf{c}$*, and any column vector* $\mathbf{x}$ *such that* $\mathbf{Px} = \mathbf{x}$ *is a multiple of* $\mathbf{c}$*.*

# Definition: Fixed Vectors

- A row vector $\mathbf{w}$ with the property $\mathbf{wP} = \mathbf{w}$ is called a *fixed row vector* for $\mathbf{P}$.
- Similarly, a column vector $\mathbf{x}$ such that $\mathbf{Px} = \mathbf{x}$ is called a fixed column vector for $\mathbf{P}$.

# Example

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$$w_1 + w_2 + w_3 = 1$$

and

$$\begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} .$$

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$$
\begin{aligned}
w_1 + w_2 + w_3 &= 1 \ , \\
(1/2)w_1 + (1/2)w_2 + (1/4)w_3 &= w_1 \ , \\
(1/4)w_1 + (1/4)w_3 &= w_2 \ , \\
(1/4)w_1 + (1/2)w_2 + (1/2)w_3 &= w_3 \ .
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(1/4)w_1 + (1/2)w_2 + (1/2)w_3 &= w_3 \ .
\end{aligned}
$$

The solution is

$$
\mathbf{w} = (\,.4 \quad .2 \quad .4\,) \ ,
$$

# Another method

- Assume that the value at a particular state, say state one, is 1, and then use all but one of the linear equations from $\mathbf{wP} = \mathbf{w}$.

- This set of equations will have a unique solution and we can obtain $\mathbf{w}$ from this solution by dividing each of its entries by their sum to give the probability vector $\mathbf{w}$.

# Example (cont'd)

- Set $w_1 = 1$, and then solve the first and second linear equations from $\mathbf{wP} = \mathbf{w}$.

$$\begin{aligned} (1/2) + (1/2)w_2 + (1/4)w_3 &= 1 \ , \\ (1/4) + (1/4)w_3 &= w_2 \ . \end{aligned}$$

- We obtain

$$\begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 & 1 \end{pmatrix} \ .$$

# Equilibrium

- Suppose that our starting vector picks state $s_i$ as a starting state with probability $w_i$, for all $i$.
- Then the probability of being in the various states after $n$ steps is given by $\mathbf{wP}^n = \mathbf{w}$, and is the same on all steps.
- This method of starting provides us with a process that is called "stationary."

# Ergodic Markov Chains

**Theorem.** *For an ergodic Markov chain, there is a unique probability vector* $\mathbf{w}$ *such that* $\mathbf{wP} = \mathbf{w}$ *and* $\mathbf{w}$ *is strictly positive. Any row vector such that* $\mathbf{vP} = \mathbf{v}$ *is a multiple of* $\mathbf{w}$. *Any column vector* $\mathbf{x}$ *such that* $\mathbf{Px} = \mathbf{x}$ *is a constant vector.*

# The Ergodic Theorem

**Theorem.** *Let* $\mathbf{P}$ *be the transition matrix for an ergodic chain. Let* $\mathbf{A}_n$ *be the matrix defined by*

$$\mathbf{A}_n = \frac{\mathbf{I} + \mathbf{P} + \mathbf{P}^2 + \ldots + \mathbf{P}^n}{n+1} \ .$$

*Then* $\mathbf{A}_n \to \mathbf{W}$*, where* $\mathbf{W}$ *is a matrix all of whose rows are equal to the unique fixed probability vector* $\mathbf{w}$ *for* $\mathbf{P}$*.*

# Exercises

Which of the following matrices are transition matrices for regular Markov chains?

1. $\mathbf{P} = \begin{pmatrix} .5 & .5 \\ 1 & 0 \end{pmatrix}$.

2. $\mathbf{P} = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \\ 0 & 1/5 & 4/5 \end{pmatrix}$.

3. $\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Consider the Markov chain with general $2 \times 2$ transition matrix

$$\mathbf{P} = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} .$$

1. Under what conditions is $\mathbf{P}$ absorbing?

2. Under what conditions is $\mathbf{P}$ ergodic but not regular?

3. Under what conditions is $\mathbf{P}$ regular?

Find the fixed probability vector $\mathbf{w}$ for the matrices in the previous exercise that are ergodic.