Your name:

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Math 11 Fall 2011, Homework 5, due Wed Oct 26

Please show your work. No credit is given for solutions without justification.

(1) Choose the correct answer. *On multiple choice problems on homework we do not grade your work, but if you do not show relevant work you will not get credit, even if you select the correct solution.*

(a) Suppose that $\nabla f_{(1,2)} = \langle 2, -1 \rangle$ and that $\mathbf{c}(t) = \langle t^2, 2t^3 \rangle$. Find the value of the derivative of the composite function

$$\frac{d}{dt} f(\mathbf{c}(t))$$

at $t = 1$.

(A) 1 (B) -1 (C) 2 (D) -2 (E) 3 (F) -3

$$\frac{d}{dt} f(\bar{c}(t)) = \nabla f \cdot \bar{c}'(t) = \langle 2,-1 \rangle \cdot \langle 2,6 \rangle = 4 - 6 = -2$$

$$\bar{c}'(t) = \langle 2t, 6t^2 \rangle, \quad \bar{c}'(1) = \langle 2,6 \rangle$$

$$t = 1 \text{ at } \bar{c}(1) = \langle 1,2 \rangle.$$

(b) Use Lagrange multipliers to find the maximum value of xy subject to the constraint $2x^2 + 3y^2 = 7$. If you eliminate λ from Lagrange's equations, which equation in x, y results?

(A) $x = y$ (B) $2x = 3y$ (C) $3x = 2y$ (D) $x^2 = y^2$ (E) $2x^2 = 3y^2$ (F) $3x^2 = 2y^2$

$f(x,y) = xy$ needs to optimized

$g(x,y) = 2x^2 + 3y^2 = 7$ is the constraint.

Lagrange's condition $\nabla f = \lambda \nabla g$ leads to

$f_x = \lambda g_x$, or $y = 4x\lambda$

$f_y = \lambda g_y$, or $x = 6y\lambda$

Eliminate λ: $\lambda = \dfrac{y}{4x} = \dfrac{x}{6y} \Rightarrow 6y^2 = 4x^2.$

$3y^2 = 2x^2.$

(c) Consider the function $f(x, y) = 3x^2 + y^2 - 3xy$. What type of point is $P = (0, 0)$?

(A) P is not a critical point
(B) P is a local maximum, but not an absolute maximum
(C) P is a local minimum, but not an absolute minimum
(D) P is an absolute maximum
(E) P is an absolute minimum
(F) P is a saddle point

$$f_x = 6x - 3y \qquad f_x(0,0) = 0$$
$$f_y = 2y - 3x \qquad f_y(0,0) = 0$$

$P = (0,0)$ is a critical point, and it is the <u>only</u> critical point: $f_x = 0$, $f_y = 0$ has no other solutions.

$$f_{xx} = 6, \quad f_{xy} = -3, \quad f_{yy} = 2$$

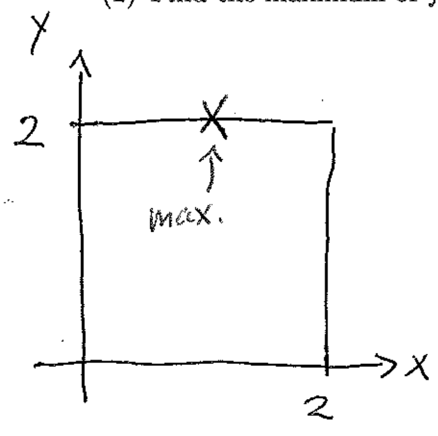
$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6 & -3 \\ -3 & 2 \end{vmatrix} = 6 \cdot 2 - (-3)^2 = 3.$$

Since discriminant $D > 0$ point P is a local max or local min.

Since $f_{xx} > 0$ (and $f_{yy} > 0$) it is <u>local</u> <u>min</u>.

Because there are <u>no</u> <u>other</u> critical points, the point P is necessarily a gl<u>oba</u>l min.

(2) Find the maximum of $f(x,y) = y^2 + xy - x^2$ on the square $0 \le x \le 2$, $0 \le y \le 2$.



Points inside the square:

$$f_x = y - 2x = 0$$

$$f_y = 2y + x = 0$$

The only critical point is $(0,0)$, which is on the boundary.

Now check the four edges, one at a time.

i) $y = 0$, $0 \le x \le 2$: $f(x,0) = -x^2$

This is maximal at $x = 0$, with $f(0,0) = 0$.

ii) $y = 2$, $0 \le x \le 2$: $f(x,2) = 4 + 2x - x^2$

Derivative: $2 - 2x = 0$ at $x = 1$.

Then $f(x,2)$ is maximal at $x = 1$: $f(1,2) = 5$

iii) $x = 0$, $0 \le y \le 2$: $f(0,y) = y^2$.

This is maximal at $y = 2$, $f(0,2) = 4$

iv) $x = 2$, $0 \le y \le 2$: $f(2,y) = y^2 + 2y - 4$

Derivative $2y + 2 = 0$ at $y = -1$.

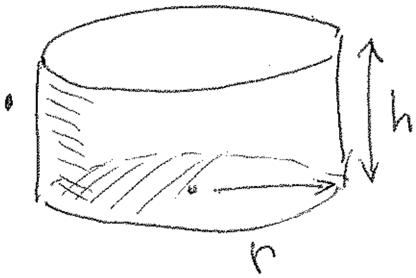
$f(2,y)$ is maximal if $y = 2$ maximal,

so $f(2,2) = 4$.

The absolute maximum for interior + edges is

$f(1,2) = 5$.

(3) Find the radius r and height h of a cylindrical can with a disk at the bottom but not at the top, with volume $V = 16\pi$ and minimal surface area. Note: volume equals area of base times height, while surface area equals surface area of the base plus $2\pi rh$ for the side of the can.



surface area

$$S = \text{base} + 2\pi rh = \pi r^2 + 2\pi rh$$

volume

$$V = \text{area} \times \text{height} = \pi r^2 h$$

Want minimal surface area S, so we try to minimize

$$f(r,h) = \pi r^2 + 2\pi rh.$$

Constraint $V = 16\pi$ means that $g(r,h) = 16\pi$, where

$$g(r,h) = \pi r^2 h$$

Lagrange's equations:

$$f_r = \lambda g_r \qquad 2\pi r + 2\pi h = 2\pi rh \cdot \lambda$$

$$f_h = \lambda g_h \qquad 2\pi r = \pi r^2 \cdot \lambda$$

Eliminate λ:

$$\cancel{\lambda = \frac{2\pi rh}{2\pi r + 2\pi h} = \frac{2\pi r}{\pi r^2}} \leftarrow$$ see next page.

$$\cancel{\Rightarrow \frac{rh}{r+h} = \frac{2}{r} \Rightarrow r^2 h = 2r + 2h}$$

Eliminate λ:

$$\lambda = \frac{2\pi r + 2\pi h}{2\pi r h} = \frac{r+h}{rh}$$ assuming $r \neq 0$, $h \neq 0$ which is OK

$$\lambda = \frac{2\pi r}{\pi r^2} = \frac{2}{r}$$

$$\Rightarrow \quad \frac{r+h}{rh} = \frac{2}{r} \quad \Rightarrow \quad r^2 + rh = 2rh$$

$$r^2 = rh$$

$$\boxed{r = h}$$

To find r, h use the constraint value:

$$V = 16\pi$$
$$\pi r^2 h = 16\pi$$
$$r^2 h = 16$$

With $r = h$ this becomes $r^3 = 16$, $r = 2\sqrt[3]{2}$

$h = 2\sqrt[3]{2}$.