Lecture Notes

Math 3 / Fall 2012

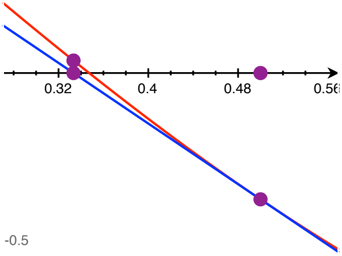
Week 5

Newton's method

0. Guess a number x_0 close to a root of f
1. Find the x-intercept x_1 of the tangent line

$$y = f'(x_0)(x - x_0) + f(x_0)$$

2. Repeat step #1 with x_1 instead of x_0

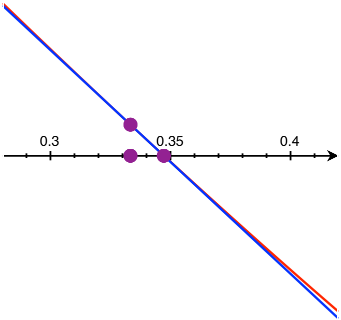


Newton's method

0. Guess a number x_0 close to a root of f
1. Find the x-intercept x_{i+1} of the tangent line

$$y = f'(x_i)(x - x_i) + f(x_i)$$

2. Repeat step 1 with x_{i+1} instead of x_i



Newton's method

$$f(x) = x^3 - 3x + 1 \qquad f'(x) = 3x^2 - 3$$

$$x_0 = 0.5 \quad \text{(initial guess)}$$
$$x_1 = x_0 - f(x_0)/f'(x_0) = 0.333\,333\,333\,333\,333\,333\,333\,333$$
$$x_2 = x_1 - f(x_1)/f'(x_1) = 0.347\,222\,222\,222\,222\,222\,222\,222$$
$$x_3 = x_2 - f(x_2)/f'(x_2) = 0.347\,296\,353\,163\,867\,969\,746\,452$$
$$x_4 = x_3 - f(x_3)/f'(x_3) = 0.347\,296\,355\,333\,860\,695\,843\,756$$
$$x_5 = x_4 - f(x_4)/f'(x_4) = 0.347\,296\,355\,333\,860\,697\,703\,433$$
$$x_6 = x_5 - f(x_5)/f'(x_5) = 0.347\,296\,355\,333\,860\,697\,703\,433$$

Newton's method: analysis

Theorem

Suppose that f is a twice differentiable function with a root r in the interval $[a, b]$ and suppose that

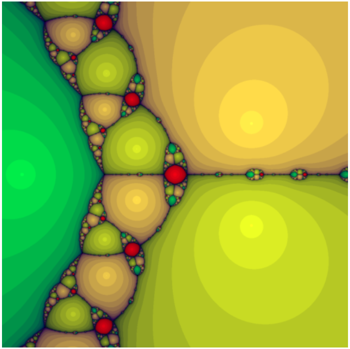
$$M \geq \left| \frac{f''(x)}{2f'(x)} \right| \quad \text{for every } x \text{ in } [a, b].$$

If $x_{i+1} = x_i - f(x_i)/f'(x_i)$ are successive steps in Newton's method in $[a, b]$ then

$$|x_{i+1} - r| \leq M|x_i - r|^2.$$

Newton's method: drawbacks

The choice of starting point x_0 is important but difficult to evaluate...



Newton's method: drawbacks

If r is a "double root" ($f(r) = 0$ and $f'(r) = 0$) the method may still work but slowly.

$$f(x) = x^2 - 2x + 1 \qquad f'(x) = 2x - 2$$

$$\begin{aligned} x_0 &= 0.50000 & x_5 &= 0.98438 \\ x_1 &= 0.75000 & x_6 &= 0.99219 \\ x_2 &= 0.87500 & x_7 &= 0.99609 \\ x_3 &= 0.93750 & x_8 &= 0.99805 \\ x_4 &= 0.96875 & x_9 &= 0.99902 \end{aligned}$$

Newton's method: drawbacks

If $f'(r)$ is undefined the method may not work at all...

$$f(x) = x^{1/3} \qquad f'(x) = \frac{1}{3}x^{-2/3}$$

$$\begin{array}{ll} x_0 = 1/2 & x_5 = -16 \\ x_1 = -1 & x_6 = 32 \\ x_2 = 2 & x_7 = -64 \\ x_3 = -4 & x_8 = 128 \\ x_4 = 8 & x_9 = -256 \end{array}$$

Linear approximation

The **linearization** of a function f at a is the linear function tangent to f at a:

$$L(x) = f(a) + f'(a)(x - a)$$

The linearization is a very good approximation to f near a

Because:

$$\begin{aligned}
\lim_{x \to a} \frac{f(x) - L(x)}{x - a} &= \lim_{x \to a} \frac{f(x) - (f(a) + f'(a)(x - a))}{x - a} \\
&= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} - f'(a) \frac{x - a}{x - a} \right) \\
&= f'(a) - f'(a) = 0
\end{aligned}$$

Differentials and error

Say you have an angle measured at $\alpha = 0.587 \pm 2.2 \times 10^{-2} \ldots$
What is $\sin(\alpha)$?
To deal with the error, we use the linearization of $\sin(x)$ at 0.587

$$\sin(0.587 + h) \approx \sin(0.587) + \cos(0.587)h = 0.554 + 0.833h$$

Therefore $\sin(\alpha) = 0.554 \pm 1.8 \times 10^{-2}$

$$\boxed{\Delta f \approx f' \, \Delta x \quad \text{or} \quad df = f' \, dx}$$

Polynomial approximation

1. The **linear approximation**

$$P_1(x) = f(a) + f'(a)(x-a)$$

   is the unique linear polynomial with $P_1(a) = f(a)$ and $P_1'(a) = f'(a)$

2. The **quadratic approximation**

$$P_2(x) = f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2$$

   is the unique quadratic polynomial with $P_2(a) = f(a)$, $P_2'(a) = f'(a)$, and $P_2''(a) = f''(a)$

3. The **cubic approximation**

$$P_3(x) = f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2 + \tfrac{1}{6}f'''(a)(x-a)^3$$

   is the unique cubic polynomial with $P_3(a) = f(a)$, $P_3'(a) = f'(a)$, $P_3''(a) = f''(a)$, and $P_3'''(a) = f'''(a)$

Polynomial approximation

The n-**th degree Taylor approximation**

$$P_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

is the unique linear function with $P_n(a) = f(a)$, $P_n'(a) = f'(a)$, $\ldots$, $P_n^{(n-1)}(a) = f^{(n-1)}(a)$, and $P_n^{(n)}(a) = f^{(n)}(a)$

Theorem

If f has an n-th derivative at a then

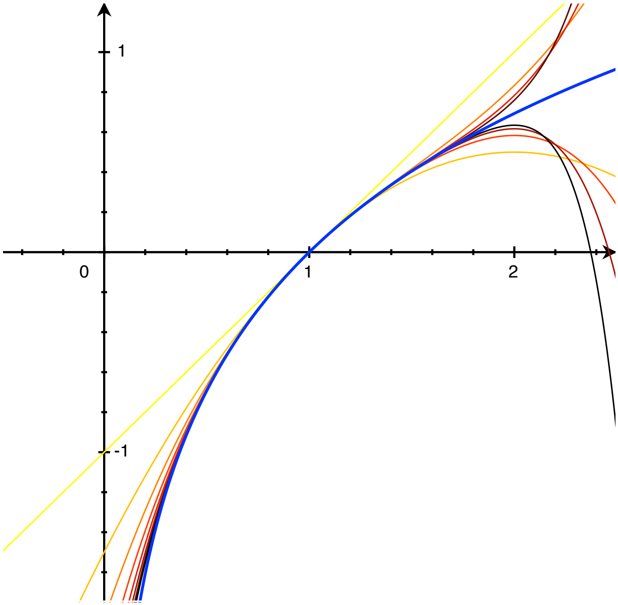
$$\lim_{x \to a} \frac{f(x) - P_n(x)}{(x-a)^n} = 0.$$

Taylor polynomial for $\ln x$ at $x = 1$

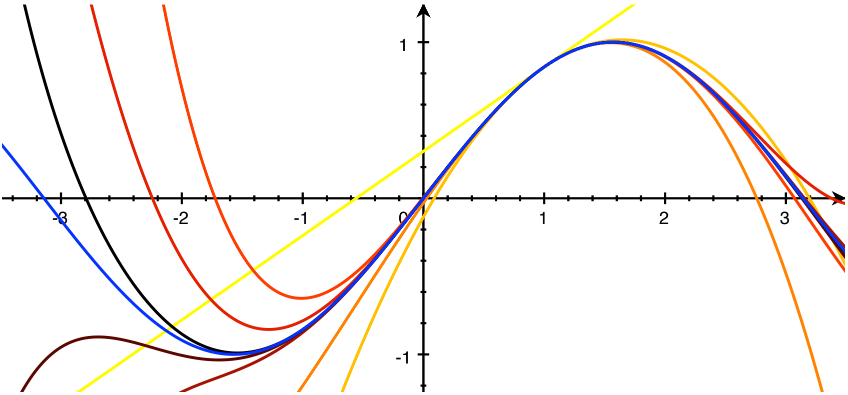
$$P(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 + \cdots$$

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$f^{(n)}(1)/n!$
0	$\ln x$	0	0
1	x^{-1}	1	1
2	$-x^{-2}$	-1	$-1/2$
3	$2x^{-3}$	2	$1/3$
4	$-6x^{-4}$	-6	$-1/4$
5	$24x^{-5}$	24	$1/5$

Taylor polynomial for $\ln x$ at $x = 1$



Taylor polynomial for $\sin x$ at $x = 1$



Taylor's theorem

The quality of approximation for the n-th degree Taylor polynomial $P_n(x)$ can be estimated using Taylor's Theorem...

Theorem

Suppose the $(n+1)$*-th derivative* $f^{(n+1)}(x)$ *exists and that* $|f^{(n+1)}(x)| \leq B$ *for all* x *in* $(a-\varepsilon, a+\varepsilon)$. *Then*

$$|f(x) - P_n(x)| \leq \frac{B}{(n+1)!}|x-a|^{n+1}$$

for all x *in* $(a-\varepsilon, a+\varepsilon)$.

Approximating e^x

Since the derivative of e^x is always e^x, the n-th degree Taylor polynomial for e^x is

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

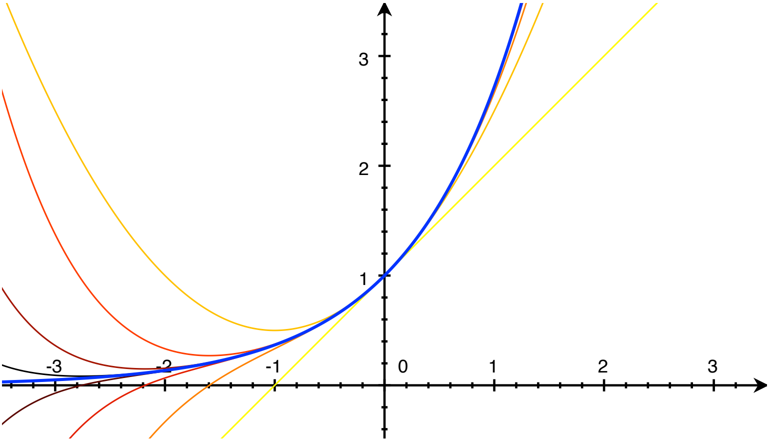
Taylor's theorem says that

$$\text{if } |x| \leq 1 \text{ then:} \quad |e^x - P_n(x)| \leq \frac{3}{(n+1)!}|x|^{n+1}$$

Since $3/6! < 5 \times 10^{-2}$, we see that

$$e \approx P_5(1) = 1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \tfrac{1}{120} = \frac{163}{60} = 2.7166\ldots$$

Approximating e^x



Antiderivative problem

$$\boxed{\text{Find a function } F(x) \text{ such that } F'(x) = f(x)}$$

Example

Find y such that $y' = 3x^2 - 1$.

Is $y = x^3 - x$?

Or is $y = x^3 - x + 2$?

If $F(x)$ is an antiderivative of $f(x)$ then so is every vertical translate $F(x) + C \ldots$

- A **particular antiderivative** of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$
- The **general antiderivative** of $f(x)$ is the family of functions $F(x) + C$ where $F'(x) = f(x)$ and C is an arbitrary constant

$$\boxed{\text{The general antiderivative of } f(x) \text{ is denoted } \int f(x)\, dx}$$

Power rule for antiderivatives

For every exponent $p \neq -1$:

$$\int x^p \, dx = \frac{x^{p+1}}{p+1} + C$$

Example

$$\int \left(x^3 - 5x^2 + \frac{1}{\sqrt{x}} \right) dx = \frac{x^4}{4} - \frac{5}{3}x^3 + 2\sqrt{x} + C$$

Antiderivatives of x^{-1}

$$\int \frac{1}{x}\,dx = \ln(x) + C \quad \text{when } x > 0!$$

$$\int \frac{1}{x}\,dx = \ln(-x) + C \quad \text{when } x < 0$$

Since $\frac{d}{dx}\left[\ln|x|\right] = \frac{1}{x}$, we know $\ln|x|$ is an antiderivative of x^{-1}...
so is $\ln|x| + 3$ and so is

$$\begin{cases} \ln(x) + 3 & \text{if } x > 0 \\ \ln(-x) + 5 & \text{if } x < 0 \end{cases}$$

The constant of integration can be chosen arbitrarily on each side of a discontinuity!!!

More antiderivatives...

$$
\begin{aligned}
\frac{d}{dx}\left[e^x\right] &= e^x &\Rightarrow&& \int e^x\,dx &= e^x + C \\
\frac{d}{dx}\left[\sin x\right] &= \cos x &\Rightarrow&& \int \cos x\,dx &= \sin x + C \\
\frac{d}{dx}\left[\cos x\right] &= -\sin x &\Rightarrow&& \int \sin x\,dx &= -\cos x + C \\
\frac{d}{dx}\left[\tan x\right] &= \sec^2 x &\Rightarrow&& \int \sec^2 x\,dx &= \tan x + C \\
\frac{d}{dx}\left[\sqrt{1-x^2}\right] &= -\frac{x}{\sqrt{1-x^2}} &\Rightarrow&& \int \frac{x}{\sqrt{1-x^2}}\,dx &= -\sqrt{1-x^2} + C
\end{aligned}
$$

Patterns for derivatives and antiderivatives

$$\frac{d}{dx}\left[\ln|f(x)|\right] = \frac{f'(x)}{f(x)} \quad \Rightarrow \quad \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| + C = \ln|\sec x| + C$$

Patterns for derivatives and antiderivatives

$$\frac{d}{dx}\left[(f(x))^2\right] = 2f(x)f'(x) \quad \Rightarrow \quad \int f(x)f'(x)\,dx = \frac{1}{2}(f(x))^2 + C$$

$$\int \sin x \cdot \cos x\,dx = \frac{1}{2}\sin^2 x + C$$

$$\int \frac{\ln x}{x}\,dx = \frac{1}{2}(\ln x)^2 + C$$

Initial value problems

Example

A particle goes back and forth along a line segment with acceleration $7.3\cos(\pi t)\,\mathrm{m/s^2}$. Knowing that the particle was at point $-2.1\,\mathrm{m}$ with velocity $1.7\,\mathrm{m/s}$ at time $t = 10\,\mathrm{s}$, find the position of the particle at an arbitrary time t.

We need to solve the IVP

$$y''(t) = 7.3\cos(\pi t), \quad y'(10) = 1.7, \quad y(10) = -2.1.$$

Initial value problems

We need to solve the IVP

$$y''(t) = 7.3\cos(\pi t), \quad y'(10) = 1.7, \quad y(10) = -2.1.$$

So

$$y'(t) = \int y''(t)\,dt = \frac{7.3}{\pi}\sin(\pi t) + C = 2.3\sin(\pi t) + C$$

and since $y'(10) = 1.7$ we must have $y'(t) = 2.3\sin(\pi t) + 1.7$
So

$$y(t) = \int y'(t)\,dt = -0.7\cos(\pi t) + 1.7t + C$$

and since $y(10) = -2.1$ we must have $C = -18.4$
Therefore $y(t) = -0.7\cos(\pi t) + 1.7t - 18.4$

Initial value problems

Example

A particle walks along a line segment with constant acceleration $3\,\mathrm{m/s^2}$. Knowing that the particle was at point $-2\,\mathrm{m}$ with velocity $4\,\mathrm{m/s}$ at time $t = 1\,\mathrm{s}$, find the intervals over which the particle is moving away from the reference point 0.

We know that:

$$y''(t) = 3 \quad y'(t) = 3t + 1 \quad y(t) = \frac{3}{2}t^2 + t - \frac{9}{2}$$

We need to find where the signs of y and y' agree... or when

$$y(t)y'(t) = \frac{1}{2}(3t+1)(3t^2+2t-9) \geq 0$$

The particle is moving away from 0 on $[-2.10, -0.33]$ and $[1.43, \infty)$

Leaning ladder

A ladder 10 feet long is resting against a wall. If the bottom of the ladder is sliding away from the wall at a rate of 1 foot per second, how fast is the top of the ladder moving down when the bottom of the ladder is 8 feet from the wall?

- Quantities:
  - $x =$ bottom of ladder
  - $y =$ top of ladder
  - $10 =$ length of ladder
- Relations: $x^2 + y^2 = 10^2$
- Currently: $x = 8$ and $dx/dt = 1$
- Wanted: dy/dt

Inflating balloon

You are inflating a balloon at a rate of $300\,\text{in}^3/\text{min}$. How is the diameter of the balloon changing when the balloon is $6\,\text{in}$ across?

- Quantities:
  - $w =$ diameter of ballon
  - $V =$ volume of balloon
- Relations: $V = \pi w^3/6$
- Currently: $w = 6$ and $dV/dt = 300$
- Wanted: dw/dt

Leaking tank

A tank of water in the shape of a cone is leaking water at a constant rate of $2\,\text{ft}^3/\text{s}$. The base radius of the tank is 5 ft and the height of the tank is 14 ft.

(a) At what rate is the depth of the water in the tank changing when the depth of the water is 6 ft?

(b) At what rate is the radius of the top of the water in the tank changing when the depth of the water is 6 ft?

- Quantities:
  - $h =$ depth of water
  - $r =$ radius of water top
  - $V =$ volume of water
- Relations: $V = \pi r^2 h/3$ and $r/h = 5/14$
- Currently: $h = 6$ and $dV/dt = -2$
- Wanted: dh/dt and dr/dt

Shadow problem

A light is on the top of a 12 ft tall pole and a 6 ft tall person is walking away from the pole at a rate of 2 ft/s.

(a) At what rate is the tip of the shadow moving away from the pole when the person is 20 ft from the pole?

(b) At what rate is the tip of the shadow moving away from the person when the person is 20 ft from the pole?

- Quantities:
  - $12 =$ height of pole
  - $6 =$ height of person
  - $x =$ distance of person
  - $y =$ distance of shadow
- Relations: $y/12 = (y - x)/6$
- Currently: $x = 20$
- Wanted: $\dfrac{dy}{dt}$ and $\dfrac{d}{dt}[x - y]$

Angle of sight

A ball is lifted up from the ground 10 ft away from you. From your perspective, the motion of the ball is such that to in order to follow it with your eyes, you need to constantly increase the angle of your head at a rate of 0.5 rad/min. How fast is the ball moving when it is 35 ft from the ground?

- Quantities:
  - $5 =$ height of your eyes
  - $h =$ height of ball
  - $\theta =$ angle from the ground
- Relations: $\tan\theta = (h-5)/10$
- Currently: $h = 35$ and $d\theta/dt = 1/2$
- Wanted: $\dfrac{dh}{dt}$