Math 3: Fall 2008

Exam 2 Solutions

1. The slope of the tangent line to the curve $y^3 + x^2y + x = 11$ through the point $(1, 2)$ is:

(a) $2/3$

(b) 1

(c) -2

(d) $-5/13$

(e) None of the above.

Answer: (d) Taking the derivative of both sides, we have $3y^2\frac{dy}{dx} + 2xy + x^2\frac{dy}{dx} + 1 = 0$. Thus, $\frac{dy}{dx}(3y^2 + x^2) = -2xy - 1$, and so

$$\frac{dy}{dx} = \frac{-2xy - 1}{3y^2 + x^2}$$

Thus, the slope of that tangent line at the point $(1, 2)$ is $\frac{-4-1}{12+1} = -5/13$

2. For the function $f(x) = x^2 - 2\ln x$, which item lists the intervals where the function is increasing?

(a) $(0, 1)$

(b) $(1, \infty)$

(c) $(-1, 0) \cup (1, \infty)$

(d) $(-\infty, 2)$

(e) $(0, \infty)$

Answer: (b) Notice that the domain of the function is $(0, \infty)$ because that is the domain of the natural log function. Now $f'(x) = 2x - \frac{2}{x} = \frac{2}{x}(x^2 - 1) = \frac{2}{x}(x - 1)(x + 1)$. Since our domain is $(0, \infty)$, we know that $f'(x) = 0$ only when $x = 1$. When $0 < x < 1$, we have $f'(x) = \frac{2}{x}(x - 1)(x + 1) < 0$, and when $x > 1$, we have $f'(x) = \frac{2}{x}(x - 1)(x + 1) > 0$. Thus, f is increasing on the interval $(1, \infty)$.

3. In the interval $[-2, 3]$, the value of the function $f(x) = x^3 - 3x - 1$ at its absolute maximum is:

(a) -3

(b) 1

(c) 17

(d) 29

(e) None of the above.

Answer: (c) We have $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1)$. Thus, $f'(x) = 0$ exactly when $x = -1$ and $x = 1$. These are the only critical points, and there are no singular points because f is differentiable on $(-2, 3)$. To find the value at the absolute maximum, we need only compare the values of f at the critical points and the endpoints. We have $f(-2) = -8+6-1 = -3$, $f(-1) = -1+3-1 = 1$, $f(1) = 1-3-1 = -3$, and $f(3) = 27-9-1 = 17$.

4. The derivative of $f(x) = x^{2\sin x}$ is

(a) $2\,x^{2\sin x}\left(\cos x \ln x + \dfrac{\sin x}{x}\right)$

(b) $(2\sin x)\,x^{2\sin x - 1}$

(c) $2\,x^{2\sin x}\cos x$

(d) $x^{2\cos x}$

(e) None of the above.

Answer: (a) We have $f(x) = x^{2\sin x} = (e^{\ln x})^{2\sin x} = e^{2(\ln x)(\sin x)}$. Thus,

$$\begin{aligned} f'(x) &= e^{2(\ln x)(\sin x)} \cdot \left(\frac{2}{x}\sin x + 2(\ln x)(\cos x)\right) \\ &= (e^{\ln x})^{2\sin x} \cdot 2 \cdot \left(\frac{1}{x}\sin x + \ln x \cos x\right) \\ &= x^{2\sin x} \cdot 2 \cdot \left(\ln x \cos x + \frac{\sin x}{x}\right) \\ &= 2\,x^{2\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right) \end{aligned}$$

5. Suppose that we apply Newton's method to approximate the root of the equation $3x^2 - e^x = 0$. If we start at $x_0 = 1$, then, after one iteration of the method, x_1 is:

(a) 1

(b) $\dfrac{3}{6-e}$

(c) $3 - e$

(d) $-\dfrac{3}{3-e}$

(e) None of the above.

Answer: (b) We have $f(x) = 3x^2 - e^x$, so $f'(x) = 6x - e^x$. Newton's Method gives

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{3-e}{6-e} = \frac{6-e}{6-e} - \frac{3-e}{6-e} = \frac{3}{6-e}$$

6. Consider the function $f(x) = |x-1| + 1$ on the interval $[0, 2]$. Using the Mean Value Theorem we can conclude:

(a) $f'(c) = 0$ for some c in the interval $(0, 2)$.

(b) $f'(c) = 2$ for some c in the interval $(0, 2)$.

(c) The function has a zero in the interval $[0, 2]$.

(d) The Mean Value Theorem does not apply because this function is not continuous in $[0, 2]$.

(e) The Mean Value Theorem does not apply because this function is not differentiable in $(0, 2)$.

Answer: (e) When $x \leq 1$ we have $x - 1 \leq 0$, so $f(x) = -(x-1)+1 = -x + 2$. When $x \geq 1$, we have $x - 1 \geq 0$, so $f(x) = (x-1) + 1 = x$. Thus $f'(x) = -1$ on the interval $(0, 1)$ and $f'(x) = 1$ on the interval $(1, 2)$. f is not differentiable at 1 because the left-side derivative is -1 but the right-side derivative is 1.

7. The solution to the initial value problem

$$\frac{dy}{dx} = -y \sin x, \quad y(0) = 2e$$

is:

(a) $y = 2e - 1 + \cos x$.

(b) $y = 2e^{1+\sin x}$.

(c) $y = 2e^{\cos x}$.

(d) $y = e^{\sin x} \cos x$.

(e) None of the above.

Answer: (c) We have $\frac{dy}{y} = -\sin x \; dx$, so integrating both sides gives $\ln|y| = \cos x + C$. Thus, $|y| = e^{\cos x + C} = e^C e^{\cos x}$ and so $y = \pm e^C e^{\cos x}$. Writing A for the contant $\pm e^C$, we have $y = Ae^{\cos x}$. Now using $y(0) = 2e$ we get $2e = Ae^{\cos 0} = Ae$, hence $A = 2$. Therefore, $y = 2e^{\cos x}$.

8. If you apply Euler's Method to the differential equation $y' = x^2 + y^2$ starting at the point $(0, 1)$ with a step size of 1, then the estimate to $y(2)$ you obtain is:

(a) 1

(b) 2

(c) 5

(d) 7

(e) None of the above.

Answer: (d) We have $x_0 = 0$, $y_0 = 1$, and $h = 1$. Euler's Method gives $x_1 = 0 + 1 = 1$ and $y_1 = 1 + 1 \cdot (0 + 1) = 2$. Using it again we get $x_2 = 1 + 1 = 2$ and $y_2 = 2 + (1 + 4) = 7$.

9. A suitable linearization approximates the value of $\sqrt{83}$ as:

(a) 9

(b) 83/9

(c) 19/2

(d) $9+\sqrt{2}$

(e) None of the above.

Answer: (e) We have $f(x)=\sqrt{x}$, and hence $f'(x)=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}}$. Notice that $\sqrt{81}=9$, so we base our linearization at $a=81$ giving

$$L(x)=f(81)+f'(81)(x-81)=9+\frac{1}{2\cdot 9}(x-81)=9+\frac{1}{18}(x-81)$$

We have $L(83)=9+\frac{1}{18}(83-81)=9+\frac{2}{18}=\frac{81}{9}+\frac{1}{9}=\frac{82}{9}$.

10. A ball is thrown up vertically and it reaches the highest point exactly 2 seconds after it was thrown. What was the initial velocity of the ball?

(a) 4.9 meters/second.

(b) 2 meters/second.

(c) 98 meters/second.

(d) 19.6 meters/second.

(e) None of the above.

Answer: (d) Using units of meters/second, we know that the position at time t is $s(t)=-4.9t^2+v_0t+s_0$, so the velocity at time t is $s'(t)=-9.8t+v_0$. If the ball reaches its highest point exactly 2 seconds after it was thrown, we know that 2 is a local max of $s(t)$, so $s'(2)=0$. Thus, $0=(-9.8)\cdot 2+v_0$, and hence $v_0=9.8\cdot 2=19.6$.

NON-MULTIPLE-CHOICE. SHOW ALL YOUR WORK.

1. Assume that the position of a particle moving along the x-axis is given as a function of time by $x(t) = t^3 - 3t + 5$.

(a) (5 pts) For what values of t is the particle at rest?

Answer: We have

$$x'(t) = 3t^2 - 3 = 3(t^2 - 1) = 3(t-1)(t+1)$$

Therefore, $x'(t) = 0$ if and only if either $t = -1$ or $t = 1$. Since $x'(t)$ is the velocity at time t, and being at rest means that the velocity is 0, it follows that the particle is at rest only at times $t = -1$ and $t = 1$.

(b) (5 pts) When is the particle moving to the left?

Answer: The particles is moving to the left when the velocity is negative, i.e. when $x'(t) < 0$.

- When $t < -1$, we have $t - 1 < 0$ and $t + 1 < 0$, hence $x'(t) = 3(t-1)(t+1) > 0$.
- When $-1 < t < 1$, we have $t - 1 < 0$ and $t + 1 > 0$, hence $x'(t) = 3(t-1)(t+1) < 0$.
- When $t > 1$, we have $t - 1 > 0$ and $t + 1 > 0$, hence $x'(t) = 3(t-1)(t+1) > 0$.

Thus, the particle is moving left exactly when $-1 < t < 1$.

(c) (5 pts) When is the particle speeding up?

Answer: The particle is speeding up when the velocity, $x'(t)$, and the acceleration, $x''(t)$, have the same sign. We have

$$x''(t) = 6t$$

so $x''(t) > 0$ when $t > 0$ and $x''(t) < 0$ when $t < 0$. Now both $x'(t) > 0$ and $x''(t) > 0$ when $t > 1$. Also, both $x'(t) < 0$ and $x''(t) < 0$ when $-1 < t < 0$. Therefore, the particle speeding up when $-1 < t < 0$ and when $t > 1$.

2a. (5 pts) Find

$$\int \frac{12x^7 + 4x^5 + 5}{x^2}\ dx$$

Answer: We have

$$\begin{aligned}
\int \frac{12x^7 + 4x^5 + 5}{x^2}\ dx &= \int (12x^5 + 4x^3 + \frac{5}{x^2})\ dx \\
&= \int (12x^5 + 4x^3 + 5x^{-2})\ dx \\
&= \frac{12}{6}x^6 + \frac{4}{4}x^4 + \frac{5}{-1}x^{-1} + C \\
&= 2x^6 + x^4 - \frac{5}{x} + C
\end{aligned}$$

2b. (5 pts) Solve the differential equation

$$y' = \frac{12x^7 + 4x^5 + 5}{x^2}$$

subject to the condition that $y(1) = 2$.

Answer: We know from part (a) that $y = 2x^6 + x^4 - \frac{5}{x} + C$ for some constant C. Now we also need $y(1) = 2$, hence

$$\begin{aligned}
2 &= y(1) \\
&= 2 \cdot 1^6 + 1^4 - \frac{5}{1} + C \\
&= 2 + 1 - 5 + C \\
&= -2 + C
\end{aligned}$$

Thus, we must have $C = 2 + 2 = 4$, and hence $y = 2x^6 + x^4 - \frac{5}{x} + 4$.

3. A radioactive substance decays at a rate proportional to the amount present. Suppose that we measure time in days, that we begin (at day 0) with 30 grams, and that after 40 days we have 25 grams. Let $y(t)$ be the amount of the substance present at time t.

(a) (5 pts) Find an equation for $y(t)$. (Your answer may involve natural logs.)

Answer: We know that $y(t) = y_0 e^{kt}$, and since we are given that $y_0 = 30$, we have $y(t) = 30e^{kt}$. We also have $y(40) = 25$, so $25 = 30e^{40k}$. Thus, $e^{40k} = \frac{25}{30} = \frac{5}{6}$, so $40k = \ln \frac{5}{6}$, and hence $k = \frac{1}{40} \ln \frac{5}{6}$. Therefore, our equation is:

$$y(t) = 30e^{\frac{1}{40} \cdot (\ln \frac{5}{6}) \cdot t}$$

(b) (5 pts) What is the half-life of the substance? (Your answer may involve natural logs.)

Answer: Half of the original amount is 15 grams, so we need to solve

$$15 = 30e^{\frac{1}{40} \cdot (\ln \frac{5}{6}) \cdot t}$$

Dividing by 30, we have $\frac{1}{2} = e^{\frac{1}{40} \cdot (\ln \frac{5}{6}) \cdot t}$, so $\ln \frac{1}{2} = \frac{1}{40} \cdot (\ln \frac{5}{6}) \cdot t$. Thus, the half-life is:

$$\frac{40 \cdot \ln \frac{1}{2}}{\ln \frac{5}{6}}$$

(c) (5 pts) What is $\lim_{t \to \infty} y(t)$?

Answer: We always have $y(t) > 0$ (because $e^x > 0$ for all x). Since the amount of our substance is cut in half for every period of time equal to our half-life, it follows that $y(t)$ gets arbitrarily small. Therefore, we must have $\lim_{t \to \infty} y(t) = 0$.

More formally, we can argue as follows. Notice that $\ln \frac{5}{6} < 0$, so $k = \frac{1}{40} \cdot \ln \frac{5}{6} < 0$ and hence $-k > 0$. Since $\lim_{t \to \infty} e^{-t} = \lim_{t \to \infty} \frac{1}{e^t} = 0$, it follows that

$$\lim_{t \to \infty} y(t) = \lim_{t \to \infty} 30e^{kt} = \lim_{t \to \infty} 30(e^{-t})^{-k} = 30 \cdot 0^{-k} = 0$$