Partial Fraction Decomposition

Take the following integrals using partial fraction decomposition.

1. $\displaystyle\int \frac{5x+1}{(2x+1)(x-1)}\,dx = \int \frac{1}{2x+1} + \frac{2}{x-1}\,dx = \boxed{\frac{1}{2}\ln(2x+1) + 2\ln(x-1) + C}$

$$\frac{5x+1}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$$

$$5x+1 = A(x-1) + B(2x+1)$$

$$\begin{cases} 5 = A+2B \\ 1 = -A+B \end{cases} \Rightarrow \quad 6=3B \Rightarrow 2=B \\ \qquad\qquad\qquad 1=A$$

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2. $\displaystyle\int \frac{2}{2x^2+3x+1}\,dx = \int \frac{4}{2x+1} - \frac{2}{x+1}\,dx = \frac{4}{2}\ln(2x+1) - 2\ln(x+1) + C$

$$\frac{2}{2x^2+3x+1} = \frac{2}{(2x+1)(x+1)} = \frac{A}{2x+1} + \frac{B}{x+1}$$

$$\boxed{2\ln(2x+1) - 2\ln(x+1) + C}$$

$$2 = A(x+1) + B(2x+1)$$

$$\begin{cases} 0 = A+2B \qquad B=-2 \\ 2 = A+B \qquad A=4 \end{cases}$$

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3. $\displaystyle\int \frac{z^2-5z+16}{(2z+1)(z-2)^2}\,dz = \frac{3}{2z+1} - \frac{1}{z-2} + \frac{2}{(z-2)^2} = \boxed{\frac{3}{2}\ln(2z+1) - \ln(z-2) - \frac{2}{(z-2)} + C}$

$$\frac{z^2-5z+16}{(2z+1)(z-2)^2} = \frac{A}{2z+1} + \frac{B}{z-2} + \frac{C}{(z-2)^2}$$

$$z^2+5z+16 = A(z^2-4z+4) + B(2z^2-3z-2) + C(2z+1)$$

$$\begin{array}{ll} 1 = A+2B & C=2 \\ -5 = -4A-3B+2C & B=-1 \\ 16 = 4A-2B+C & A=3 \end{array}$$

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4. $\displaystyle\int \frac{4x^3+2x^2-2x-1}{x^4-x^2}\,dx$

$$\frac{4x^3+2x^2-2x-1}{x^2(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{x-1}$$

$$4x^3+2x^2-2x-1 = A(x^3-x) + B(x^2-1) + C(x^3-x^2) + D(x^3+x^2)$$

$$\begin{cases} 4x^3 = (A+C+D)x^3 \\ 2x^2 = (B-C+D)x^2 \\ -2x = (-A)x \\ -1 = (-B)x \end{cases} \qquad \begin{array}{l} A=2 \\ B=1 \\ C=\frac{1}{2} \\ D=\frac{3}{2} \end{array} \qquad \int \frac{2}{x} + \frac{1}{x^2} + \frac{1/2}{x+1} + \frac{3/2}{x-1}\,dx$$

$$= \boxed{2\ln x - \frac{1}{x} + \frac{1}{2}\ln(x+1) + \frac{3}{2}\ln(x-1) + C}$$

The Process:

(1) Factor denominator
(2) Write as sum of fractions
(3) Solve for unknowns

Limitations

° degree of denominator must be greater than the degree of the numerator
° denominator factors into linear factors

Examples:

(1) $\displaystyle\int \frac{10x^2+2}{4x^3-4x^2+x}\,dx$

Factor denominator:

$$4x^3-4x^2+x = x(4x^2-4x+1)$$
$$= x \cdot = x(2x-1)^2$$

write as sum of fractions:

$$\frac{10x^2+2}{x(2x-1)^2} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{(2x-1)^2}$$

$$10x^2+2 = A(2x-1)^2 + Bx(2x-1) + Cx$$
$$10x^2+2 = A\cdot 4x^2 - A\cdot 4x + A + B\cdot 2x^2 - B\cdot x + Cx$$

$$10 = 4A + 2B$$
$$0 = -4A - B + C \implies$$
$$2 = A$$

$$10 = 4(2) + 2B \implies B=1$$
$$0 = -4(2) - (1) + C \implies C=9$$

$$\frac{10x^2+2}{x(2x-1)^2} = \frac{2}{x} + \frac{1}{2x-1} + \frac{9}{(2x-1)^2}$$

$$\int\left(\frac{2}{x} + \frac{1}{2x-1} + \frac{9}{(2x-1)^2}\right)dx = 2\ln x + \frac{\ln|2x-1|}{2} - \frac{9}{2(2x-1)} + C$$