1.3.10) Write a vector equation that is equivalent to the given system of equations:

$$\begin{array}{ccccccc} 4x_1 & + & x_2 & + & 3x_3 & = & 9 \\ x_1 & - & 7x_2 & - & 2x_3 & = & 2 \\ 8x_1 & + & 6x_2 & - & 5x_3 & = & 15 \end{array}$$

Solution:

$$x_1 \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 15 \end{bmatrix}$$

1.3.12) Determine if $\mathbf{b}$ is a linear combination of a_1, a_2, and a_3.

$$a_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \quad a_2 = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} \quad a_3 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$$

Solution:

To see if $\mathbf{b}$ is a linear combination of a_1, a_2, and a_3 we must find a solution to the equation

$$\mathbf{b} = x_1a_1 + x_2a_2 + x_3a_3$$

Rewrite as a augmented matrix

$$\begin{bmatrix} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{bmatrix}$$

Now we will row reduce:

$R_2 : 2R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 5 & 4 & 1 \\ 2 & 5 & 8 & -7 \end{bmatrix}$$

$R_3 : R_3 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 5 & 4 & 1 \\ 0 & 5 & 4 & -11 \end{bmatrix}$$

$R_3 : -R_2 + R_3$

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & -12 \end{bmatrix}$$

Since $0 \neq -12$ we have that $\mathbf{b}$ is not a linear combination of a_1, a_2, and a_3.

1.3.26) Let $A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}$, let $\mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$, and let W be the set of all linear combinations of the columns of A.

a. Is $\mathbf{b}$ in W

b. Show that the third column of A is in W

Solution: a. First let

$$a_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad a_2 = \begin{bmatrix} 0 \\ 8 \\ -2 \end{bmatrix} \quad a_3 = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$$

We want to find x_1, x_2, and x_3 such that

$$\mathbf{b} = x_1 a_1 + x_2 a_2 + x_3 a_3$$

As in 1.3.12 we will consider the Augmented matrix $B = [a_1, a_2, a_3, \mathbf{b}]$ and row reduce. Now

$$B = \begin{bmatrix} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 3 \end{bmatrix}$$

$R_1 : (1/2)R_1$

$$\begin{bmatrix} 1 & 0 & 3 & 5 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 3 \end{bmatrix}$$

$R_2 : R_1 + R_2$

$$\begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 8 & 8 & 8 \\ 1 & -2 & 1 & 3 \end{bmatrix}$$

$R_2 : (1/8)R_2$
$R_3 : R_3 - R_1$

$$\begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & -2 & -2 \end{bmatrix}$$

$R_3 : R_3 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we can see that there are an infinite number of solutions. Let $x_3 = t$

Then

$$x_3 = t$$
$$x_2 = 1 - t$$
$$x_2 = 5 - 3t$$

is a solution.

In particular if $t = 0$ we get

$$\mathbf{b} = 5a_1 + 1a_2 + 0a_3$$

b. a_3 is in W since a_3 =$0a_1$+ $0a_2$+$1a_3$

1.3.30) Let $\mathbf{v}$ be the center of mass of a system of point masses located at $\mathbf{v_1}, ..., \mathbf{v_k}$ as in Exercise 29. Is $\mathbf{v}$ in $Span\{\mathbf{v_1}, ..., \mathbf{v_k}\}$?

Solution: From exercise 29 we know

$$\mathbf{v} = (1/m)[m_1\mathbf{v_1} + \cdots + m_k\mathbf{v_k}]$$

where

$$m = m_1 + \cdots + m_k$$

hence $\mathbf{v}$ is a linear combination of $\mathbf{v_1}, ..., \mathbf{v_k}$ which implies $\mathbf{v} \in Span\{\mathbf{v_1}, ..., \mathbf{v_k}\}$.

1.4.26) Let $\mathbf{u} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}$,$\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$. It can be shown that $3\mathbf{u} - 5\mathbf{v} - \mathbf{w} = \mathbf{0}$. Use this fact to find x_1 and x_2 tat satisfy the equation

$$\begin{bmatrix} 7 & 2 \\ 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}$$

.

Solution:

$3\mathbf{u} - 5\mathbf{v} - \mathbf{w} = \mathbf{0}$ implies $3\mathbf{u} - 5\mathbf{v} = \mathbf{w}$ this gives

$$\begin{aligned} x_1 &= 3 \\ x_2 &= -5 \end{aligned}$$

satisfy the equation.

1.5.6) Find the solution set to the homogeneous system

$$\begin{array}{rcrcrcl} x_1 & + & 3x_2 & - & 5x_3 & = & 0 \\ x_1 & + & 4x_2 & - & 8x_3 & = & 0 \\ -3x_1 & - & 7x_2 & + & 9x_3 & = & 0 \end{array}$$

Solution:

Let A be the matrix of coefficients of the system and row reduce the augmented matrix $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$ to echelon form

$$\begin{bmatrix} 1 & 3 & -5 & 0 \\ 1 & 4 & -8 & 0 \\ -3 & -7 & 9 & 0 \end{bmatrix}$$

$R_2 : R_2 - R_1$
$R_3 : R_3 + 3R_1$

$$\begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -6 & 0 \end{bmatrix}$$

$R_1 : R_1 - 3R_2$
$R_3 : R_3 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since x_3 is a free variable we have nontrival solutions.

Our Solutions are:

$$\begin{aligned} x_3 &= t \\ x_2 &= 3t \\ x_1 &= -4t \end{aligned}$$

1.5.26) Suppose $A\mathbf{x} = \mathbf{b}$ has a solution. Explain why the solution is unique precisely when $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Solution: Suppose $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, and assume $\mathbf{a_1}$ and $\mathbf{a_2}$ are solutions to $A\mathbf{x} = \mathbf{b}$. We will show $\mathbf{a_1} = \mathbf{a_2}$. By assumption we have $A\mathbf{a_1} = \mathbf{b}$ and $A\mathbf{a_2} = \mathbf{b}$. By Theorem 1.4.5 we have

$$A(\mathbf{a_1} - \mathbf{a_2}) = A\mathbf{a_1} - A\mathbf{a_2} = \mathbf{b} - \mathbf{b} = \mathbf{0}$$

Hence $\mathbf{a_1} - \mathbf{a_2}$ is a solution to $A\mathbf{x} = \mathbf{0}$. So by assumption $\mathbf{a_1} - \mathbf{a_2} = \mathbf{0}$ or $\mathbf{a_1} = \mathbf{a_2}$. Hence $A\mathbf{x} = \mathbf{b}$ has a unique solution.

Suppose $A\mathbf{x} = \mathbf{b}$ has a unique solution say $\mathbf{a}$, and let $\mathbf{c}$ be a solution to $A\mathbf{x} = \mathbf{0}$. Then by Theorem 1.4.5 we have

$$A(\mathbf{a} + \mathbf{c}) = A\mathbf{a} + A\mathbf{c} = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

Hence $\mathbf{a} + \mathbf{c}$ is a solution to $A\mathbf{x} = \mathbf{b}$ and since we assumed that $\mathbf{a}$ was a unique solution to $A\mathbf{x} = \mathbf{b}$ we must have $\mathbf{a} + \mathbf{c} = \mathbf{a}$ or $\mathbf{c} = \mathbf{0}$. Hence $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

1.7.6) Determine if the columns of the matrix A form a linearly independent set, where

$$A = \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$$

Solution: We know from the chapter that the columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. So we will row reduce the augmented matrix

$$B = \begin{bmatrix} -4 & -3 & 0 & 0 \\ 0 & -1 & 4 & 0 \\ 1 & 0 & 3 & 0 \\ 5 & 4 & 6 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$
$R_2 : -R_2$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ -4 & -3 & 0 & 0 \\ 5 & 4 & 6 & 0 \end{bmatrix}$$

$R_3 : R_3 + 4R_1$
$R_4 : R_4 - 5R_1$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & -3 & 12 & 0 \\ 0 & 4 & -9 & 0 \end{bmatrix}$$

$R_3 : R_3 + 3R_2$
$R_4 : R_4 - 4R_1$

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 \end{bmatrix}$$

From here we can see $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, hence the columns of A are linearly independent.

1.7.16) Determine whether the vectors are linearly independent.

$$\begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} , \begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} = (2/3) \begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix}$$

So the vectors are linearly dependent.

1.7.36) If $\mathbf{v_1}, \ldots, \mathbf{v_4}$ are in $\mathbb{R}^4$ and $\mathbf{v_3}$ is not a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_4}$, then $\{\mathbf{v_1}, \ldots, \mathbf{v_4}\}$ is linearly independent.

Solution: False.

Let $\mathbf{v_1} = \mathbf{v_2} = \mathbf{v_4} = \mathbf{0}$ and $\mathbf{v_3} = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$

The any linear combination of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_4}$ is the $\mathbf{0}$ vector hence $\mathbf{v_3} \notin Span\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_4}\}$ but $\mathbf{v_1}, \ldots, \mathbf{v_4}$ is not linearly independent by theorem 9 since it contains the zero vector.