Improper Integrals

09/29/2006

Lecture 5

Improper Integrals

- If the interval of integration of f is infinite or f has an infinite discontinuty in $[a, b]$, then the definite integral of f is called an *improper integral.*

Type I: Infinite intervals

- If $\int_a^t f(x)\mathrm{d}x$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\mathrm{d}x = \lim_{t\to\infty} \int_a^t f(x)\mathrm{d}x,$$

  provided this limit exists (as a finite number).

- If $\int_t^b f(x)\mathrm{d}x$ exists for every number $t \leq b$, then

$$\int_{-\infty}^{b} f(x)\mathrm{d}x = \lim_{t\to-\infty} \int_t^b f(x)\mathrm{d}x$$

provided this limit exists (as a finite number).

- The improper intervals $\int_a^\infty f(x)\mathrm{d}x$ and $\int_{-\infty}^b f(x)\mathrm{d}x$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

- If $\int_t^b f(x)\mathrm{d}x$ exists for every number $t \leq b$, then

$$\int_{-\infty}^{b} f(x)\mathrm{d}x = \lim_{t \to -\infty} \int_{t}^{b} f(x)\mathrm{d}x$$

  provided this limit exists (as a finite number).

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- If both $\int_a^\infty f(x)\mathrm{d}x$ and $\int_{-\infty}^b f(x)\mathrm{d}x$ are convergent, then we define

$$\int_{\infty}^{\infty} f(x)\mathrm{d}x = \int_{-\infty}^{a} f(x)\mathrm{d}x + \int_{a}^{\infty} f(x)\mathrm{d}x.$$

Examples

- Determine whether the following integrals are convergent or divergent:

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- Is the area under the curve $y = \frac{\ln x}{x^2}$ from $x = 1$ to $x = \infty$ finite? If so, what is its value?

- For what values of p is the integral

$$\int_1^\infty \frac{1}{x^p} \mathrm{d}x$$

convergent?

Type 2: Discontinuous Integrands

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- If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\mathrm{d}x = \lim_{t \to b^-} \int_a^t f(x)\mathrm{d}x$$

  if this limit exists.

- If $f$is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\mathrm{d}x = \lim_{t \to a^+} \int_t^b f(x)\mathrm{d}x$$

  if the limit exists.

- If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\mathrm{d}x$ and $\int_c^b f(x)\mathrm{d}x$ are convergent, then we define

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x.$$

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- $\int_0^1 \frac{\mathsf{d}x}{\sqrt{x}}$
- $\int_0^1 \frac{1}{1-x}\mathsf{d}x$
- $\int_0^3 \frac{\mathsf{d}x}{(x-1)^{2/3}}$