

## MATH 23 SPRING 2006

# DIFFERENTIAL EQUATIONS

# FINAL EXAM

DUE DATE: TUESDAY, JUNE 6, 5PM

YOUR NAME (PLEASE PRINT):

**Instructions:** This is an open book, open notes exam. You are free to use a calculator or computer to check your answers, but you must justify all of your answers to receive credit.

You have until the due date to work on all **9** problems. However, the exam is not intended to take more than 3 to 4 hours to complete.

The Honor Principle requires that you neither give nor receive any aid on this exam. Additionally, the only resources you may consult while taking the exam are the course text book and your personal notes, exams and homework.

[illegible]

**Problem 1.** Consider the non-linear autonomous system

$$\begin{aligned}x_1' &= x_2 \\x_2' &= -2x_1 - 2x_2 + x_1^2 - x_2^2.\end{aligned}$$

- Find all of the critical points of this system and classify them as to stability and type.
- Show that the non-linear second order differential equation

$$y'' + (y')^2 + 2y' + 2y - y^2 = 0$$

can be transformed into the first order system above.

- Use parts (a) and (b) to describe the long term behavior of solutions to the initial value problem

$$\begin{aligned}y'' + (y')^2 + 2y' + 2y - y^2 &= 0 \\y(t_0) &= y_0 \\y'(t_0) &= y_0'\end{aligned}$$

when  $y_0$  and  $y_0'$  are sufficiently small.

**Problem 2.**

- Find an autonomous ordinary differential equation with an unstable critical point at  $y = 0$  and an asymptotically stable critical point at  $y = 2$ .
- Find an autonomous ordinary differential equation with a semi-stable critical point at  $y = 0$  and an unstable critical point at  $y = 2$ .

**Problem 3.** Find the general solution to the system

$$\mathbf{x}' = \begin{pmatrix} -4 & 1 \\ -1 & -2 \end{pmatrix} \mathbf{x}.$$

Classify the critical point at the origin as to stability and type, and sketch the phase portrait.

**Problem 4.** Let the function  $f$  be defined on the interval  $[0, L]$  by

$$f(x) = \begin{cases} L & 0 \leq x < L/4, \\ L/2 & L/4 \leq x < L/2, \\ 0 & L/2 \leq x \leq L. \end{cases}$$

- Find the Fourier cosine series for  $f$  with period  $2L$ . Sketch the graph of the function to which the Fourier series converges.
- Use part (a) to solve the boundary value problem

$$\begin{aligned} u_{xx} &= 9u_t, \quad 0 < x < 100, \quad t > 0 \\ u_x(0, t) &= u_x(100, t) = 0, \quad t > 0 \\ u(x, 0) &= \begin{cases} 100 & 0 \leq x < 25, \\ 50 & 25 \leq x < 50 \\ 0 & 50 \leq x \leq 100. \end{cases} \end{aligned}$$

[*Hint:* This boundary value problem corresponds to a certain type of heat conduction problem. Which one is it?]

**Problem 5.** Consider the nonlinear differential equation

$$t^2 \frac{dy}{dt} + 1 = t^2 y^2 - ty. \quad (1)$$

- Show that the substitution

$$y = \frac{1}{t} + \frac{1}{u}$$

transforms (1) into the linear equation

$$t^2 \frac{du}{dt} + tu = -t^2. \quad (2)$$

- Solve equation (2) and use this to find the general solution to (1).

**Problem 6.** This problem deals with the nonlinear autonomous system

$$\begin{aligned}\frac{dx}{dt} &= y^3 \\ \frac{dy}{dt} &= -x^3.\end{aligned}$$

- Find an equation satisfied by the trajectories of this system.
- Sketch several of the trajectories found in part (a). Be sure to indicate the direction of motion along these trajectories.
- Use part (b) to determine the type and stability of this system's critical point at the origin.

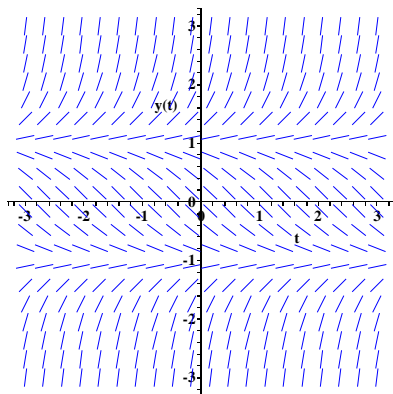
**Problem 7.** Find all solutions of the boundary value problem

$$\begin{aligned}u_{xx} + u_{yy} &= 0, \quad 0 < x < L, \quad -\infty < y < \infty \\ u(0, y) = u(L, y) &= 0, \quad -\infty < y < \infty\end{aligned}$$

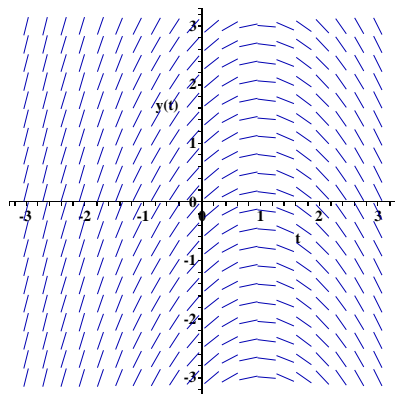
which are of the form  $u(x, y) = X(x)Y(y)$ .

**Problem 8.** Match the following differential equations with their direction fields (shown on the next page). You do not need to show any work.

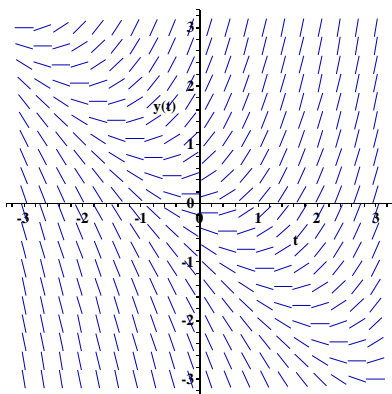
- $y' = -t + 1$
- $y' = y^2 - 1$
- $y' = -y$
- $y' = y + t$
- $y' = \sin(t - y)$
- $y' = (1 + y^2)^{-1}$



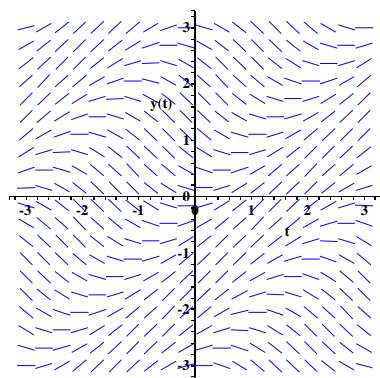
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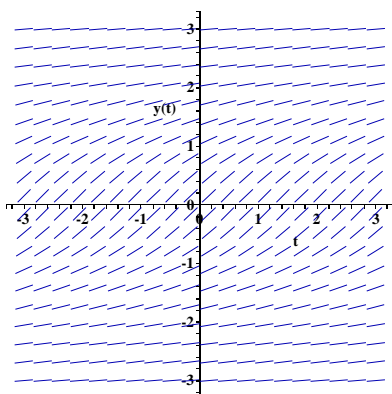
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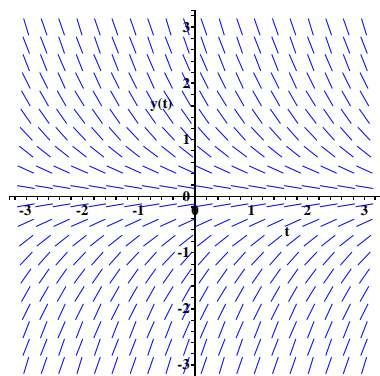
III



IV



V



VI

**Problem 9.** Consider the second order equation

$$xy'' + 2y' + xy = 0, \quad x > 0. \quad (3)$$

- a. Show that  $x = 0$  is a regular singular point of (3). Furthermore, show that (3) has solutions of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n$$

only if  $r = -1$  or  $r = 0$ .

- b. Show that when  $r = -1$  both  $a_0$  and  $a_1$  are free, and find the recursion relation satisfied by the remaining coefficients.
- c. Show that when we set  $a_0 = 1$  and  $a_1 = 0$  in (b) we get the solution

$$y_1(x) = \frac{\cos x}{x}$$

and when we set  $a_0 = 0$  and  $a_1 = 1$  we get the solution

$$y_2(x) = \frac{\sin x}{x}.$$