Math 118. Combinatorics.

Problem Set 3. Due on Friday, 2/18/11.

1. Prove that the number of maps $f : [n] \to [n]$ such that $f(1) = 1$ and $f(i) \leq 1 + \max\{f(j) : j < i\}$ is the Bell number B_n.

2. Let $\pi \in \mathcal{S}_n$ be random (chosen from the uniform distribution). Fix $1 \leq k \leq n$. What is the probability that in the disjoint cycle decomposition of π, the length of the cycle containing 1 is k?

3. Let $A_d(x)$ denote the dth Eulerian polynomial. Show that every zero of $A_d(x)$ is real.
Hint: Recall the formula proved in class relating $A_{d+1}(x)$, $A'_d(x)$ and $A_d(x)$.

4. Using only combinatorial arguments and the definitions of $\sec x$ and $\tan x$ in terms of Euler numbers, prove that
$$\frac{d}{dx} \sec^2 x = 2 \sec^2 x \tan x.$$

5. A cycle in a permutation is said to be *up-down* if, when written with its smallest element first, say $(b_1, b_2, \dots)$, we have that $b_1 < b_2 > b_3 < \dots$. Let Δ_n denote the set of permutations of $[n]$ that can be written as a product of up-down cycles. For example, $(1,5,2,7)(3)(4,8,6)(9) \in \Delta_9$, but $(1,3,5)(2,4)(6) \notin \Delta_6$. Prove that
$$|\Delta_n| = E_{n+1}.$$

6. Let $k < n/2$. Find a bijection f from the set of k-element subsets of $[n]$ to the set of $(n-k)$-element subsets of $[n]$ with the property that for every k-element subset S, $S \subseteq f(S)$.

7. (*) Let $\mathcal{C}_n \subset \mathcal{S}_n$ denote the subset of cyclic permutations, i.e., those that consist of one cycle of length n. Let $D(\pi)$ denote the descent set of π. Prove that for every $I \subseteq [n-1]$,
$$|\{\pi \in \mathcal{C}_{n+1} \,:\, D(\pi) \cap [n-1] = I\}| = |\{\sigma \in \mathcal{S}_n \,:\, D(\sigma) = I\}|.$$