1. (**10 Points**) Identify all of the equilibrium solutions to the following ODE and determine whether they are stable, unstable or semistable.

$$y' = \frac{1}{2}(y-2)^2(1-y). \tag{1}$$

Please justify your answer.

2. (**15 Points**) The function $Y(t) = -\frac{7}{50}\cos(t) + \frac{1}{50}\sin(t)$ is a particular solution to the second-order linear ODE:

$$y'' + y' - 6y = \cos(t), \; -\infty < t < \infty \tag{2}$$

Find the solution $y = \phi(t)$ of Equation 2 which satisfies $\phi(0) = 3$ and $\phi'(0) = 1$.

3. (**10 Points**)

(a) (**5 Points**) Verify that the functions $y_1(t) = e^t$ and $y_2(t) = t$ form a fundamental set of solutions to the homogeneous ODE

$$(1-t)y'' + ty' - y = 0$$

on the interval $-\infty < t < 1$.

(b) (**5 Points**) Suppose $p(t)$ and $q(t)$ are continuous functions on the interval $-5 < t < 3$. Is it possible for the functions $f(t) = t^2 e^t$ and $g(t) = te^{-t}$ to form a fundamental set of solutions for the second-order linear ODE

$$y'' + p(t)y' + q(t)y = 0$$

on the interval $-5 < t < 3$? Please explain and justify your answer.

4. (**15 Points**) Find an explicit solution to the IVP

$$(y^2 + 2y) + 2x(1 + y)y' = 0, \ y(1) = 2$$

on the interval $x > 0$. Please remember to show all of your work.

5. (**10 Points**) Find the longest interval in which the solution of the initial value problem

$$(t^2 + 4t + 2)y'' + \sin(t)y' + y = \cos(t), \quad y(1) = \pi,\ y'(1) = 2\pi \tag{3}$$

is certain to exist. Please explain and justify your answer.

6. (**15 Points**) Let $P(t)$ denote the total number of students (at a small liberal arts college in New England) who have heard a certain rumor at time t. Suppose that P follows the logistic differential equation

$$\frac{dP}{dt} = 0.008P(M - P), \tag{4}$$

and that at $t = 0$, 10 students out of $M = 1,000$ students have heard the rumor. At what time t will 50% of the students have heard the rumor? Remember to justify your answer.

7. (**15 Points**) Consider the second-order linear ordinary differential equation

$$(1-x)y'' + xy' - y = (1-x)^2 x^2 e^x,\ 0 < x < 1 \tag{5}$$

The functions $y_1(x) = e^x$ and $y_2(x) = x$ form a fundamental set of solutions to the associated homogeneous differential equation and have $W(y_1, y_2)(x) = (1-x)e^x$. Use the method of variation of parameters to find a solution to Equation 5.

8. (**10 Points**) Use the method of undetermined coefficients to find the general solution of the second-order differential equation

$$y'' + 4y' = 2\cos(2t). \tag{6}$$