1. (20) Let E be the solid lying in the first octant, inside the cone $z = \sqrt{x^2 + y^2}$ and below the plane $z = 2$. (That is, E is the set of (x, y, z) such that $x^2 + y^2 \geq z^2$, $x \geq 0$, $y \geq 0$ and $z \leq 2$.) We want to consider the triple integral

$$\iiint_E xyz\, dV.$$

(a) Express $\iiint_E xyz\, dV$ as an iterated integral in rectangular coordinates.

ANS: We can view E as the solid region between the surfaces $z = \sqrt{x^2 + y^2}$ and $z = 2$ which is above the planar region D in the xy-plane which is the quarter of the circle of radius 2 (centered at the origin) lying in the first quadrant. Hence the integral equals $\boxed{\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 xyz\, dz\, dy\, dx}$

(Your answer should be an iterated integral!)

(b) Express $\iiint_E xyz\, dV$ as an iterated integral in cylindrical coordinates.

ANS: $\boxed{\int_0^{\pi/2} \int_0^2 \int_r^2 r^3 \cos(\theta) \sin(\theta) z\, dz\, dr\, d\theta}$

(Your answer should be an iterated integral!)

(c) Express $\iiint_E xyz\, dV$ as an iterated integral in spherical coordinates.

ANS: $\boxed{\int_0^{\pi/4} \int_0^{\pi/2} \int_0^{2\sec(\phi)} \rho^5 \sin^3(\phi) \cos(\phi) \cos(\theta) \sin(\theta)\, d\rho\, d\theta\, d\phi.}$

(Your answer should be an iterated integral!)

(d) Evaluate $\iiint_E xyz\,dV$

ANS: The most reasonable approach seems to be cylindrical coordinates. Then

$$\int_0^{\pi/2}\int_0^2\int_r^2 r^3\cos(\theta)\sin(\theta)z\,dz\,dr\,d\theta = \Big(\int_0^{\pi/2}\sin(\theta)\cos(\theta)\,d\theta\Big)\Big(\frac{1}{2}\int_0^2(4-r^2)r^3\,dr\Big)$$

which, after letting $u=\sin(\theta)$ in the first integral, is

$$\begin{aligned}&= \frac{1}{2}\Big(\int_0^1 u\,du\Big)\Big(r^4-\frac{r^6}{6}\Big)_0^2\\ &= \frac{1}{2^2}\Big(2^4-\frac{2^6}{6}\Big)\\ &= 4-\frac{8}{3} = \boxed{\frac{4}{3}}.\end{aligned}$$

2. (10) Let L be a lamina occupying the region in the xy-plane inside the circle $x^2+y^2=a^2$ and above the x-axis. Suppose the density at a point in L is proportional to its distance to the x-axis.

(a) Find the mass of L.

ANS: The density is of the form $\rho(x,y)=ky$. Using polar coordinates, the mass is

$$\begin{aligned}M &= \iint_D ky\,dA = k\int_0^\pi r^2\sin\theta\,dr\,d\theta = k\frac{a^3}{3}\int_0^\pi\sin\theta\,d\theta\\ &= k\frac{a^3}{3}\Big(-\cos\theta\Big)\Big|_0^\pi = \boxed{\frac{2a^3k}{3}}.\end{aligned}$$

Mass =: ______________________

(b) You may assume, by symmetry, that the center of mass of the lamina L from the previous page occurs at $(0,\bar{y})$. Find the center of mass by computing $\bar{y}$.

ANS: Recall that $\bar{y}=\frac{M_x}{M}=\frac{1}{M}\iint_D y(ky)\,dA$. But

$$M_x = k\int_0^\pi r^3\sin^2\theta\,dr\,d\theta = k\frac{a^4}{4}\int_0^\pi\frac{1}{2}(1-\cos 2\theta)\,d\theta = \frac{a^4\pi k}{8}.$$

Hence

$$\bar{y} = \frac{\frac{a^4\pi}{8}}{\frac{2a^3}{3}} = \frac{a^4\pi 3}{8\cdot 2a^3} = \boxed{\frac{3\pi a}{16}}.$$

$\bar{y} =$: ____________________

3. (10) Let E be a solid with constant density which occupies the part of the first octant inside the sphere $x^2 + y^2 + z^2 = a^2$. You may assume, by symmetry, that the center of mass of E is of the form $(\bar{z}, \bar{z}, \bar{z})$. Find the center of mass by computing $\bar{z}$.

ANS: It seems easier to computer $\bar{z} = \frac{M_{xy}}{M}$. If the density is k, then the mass, M, is just k times the volume of an eighth of a sphere of radius a: $M = k\frac{1}{8}(\frac{4}{3}\pi a^3) = k\frac{\pi a^3}{6}$.

Next, computer M_{xy} using spherical coordinates:

$$\begin{aligned} M_{xy} &= \iiint_E kz\,dV = k\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho^3\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta \\ &= k\frac{a^4}{4}\Big(\frac{\pi}{2}\Big)\int_0^{\pi/2}\sin\phi\cos\phi\,d\phi = k\frac{\pi a^4}{8}\int_0^1 u\,du = \frac{\pi a^4 k}{16}. \end{aligned}$$

Therefore we have

$$\bar{z} = \frac{M_{xy}}{M} = \frac{\frac{\pi a^4}{16}}{\frac{\pi a^3}{6}} = \frac{\pi a^4}{16}\frac{6}{\pi a^3} = \boxed{\frac{3a}{8}}.$$

$\bar{z} =$: ____________________

4. (15) Let $\mathbf{T}$ be the transformation given by $x = u$ and $y = v(1+u^2)$. Let S be the rectangle in the uv-plane given by $0 \le u \le 3$ and $0 \le v \le 2$.

(a) Sketch the image R of S under the transformation $\mathbf{T}$.

(b) Evaluate $I = \displaystyle\iint_R \frac{y}{(1+x^2)^2}\,dA$.

ANS: The Jacobian is given by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ 2uv & 1+u^2 \end{vmatrix} = 1 + u^2 \ge 0.$$

Hence

$$\iint_R \frac{y}{(1+x^2)^2}\,dA = \iint_S \frac{v(1+u^2)}{(1+u^2)^2}|(1+u^2)|\,dA = \int_0^2\int_0^3 v\,du\,dv = 3\Big(\frac{v^2}{2}\Big)\Big|_0^2 = \boxed{6}.$$

$I =$: ____________________

5. (6) Find values or formulas for A, B and $f(x,y)$ so that the following equality holds.

$$\int_0^{\pi/4}\int_0^{4\sec\theta} r^2\cos\theta\,dr\,d\theta = \int_0^A\int_0^B f(x,y)\,dy\,dx.$$

ANS: $\boxed{A=4 \quad B=x \quad f(x,y)=x}$

$A =:$ __________

$B =:$ __________

$f(x,y) =:$ __________

6. (5) Let $\mathbf{T}$ be the transformation from uv-coordinates to xy-coordinates such that $x = u^2 - v^2$ and $y = 2uv$. Compute the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$.

ANS: $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = \boxed{4u^2+4v^2}$

$\dfrac{\partial(x,y)}{\partial(u,v)} =:$ __________

7. (5) Find the volume of the solid region bounded by the parabolic cylinder $x = z^2$ and the planes $x = z+2$, $y = 1$ and $y = 2$.

ANS: $V = \int_{-1}^2\int_{z^2}^{z+2}\int_1^2 1\,dy\,dx\,dz = \int_{-1}^2(z+2-z^2)\,dz = 2+4-\frac{8}{3}-(\frac{1}{2}-2+\frac{1}{3}) = \boxed{\frac{9}{2}}$.

ANS: __________

8. (8) Match the given equation to the object it describes in *spherical coordinates* by placing the appropriate letter in the blank space provided. (Some answers could occur more than once.)

______ $\phi = 3\pi/4$.

______ $\theta = \pi/3$.

______ $\phi = \pi$.

______ $\rho\sin\phi = 1$.

(A) A cylinder
(B) A cone or half-cone.
(C) A half-plane.
(D) A point.
(E) A ray or half-line.
(F) A sphere.
(G) A line.
(H) A plane.

ANS: BCEA

9. (8) Match the given equation to the object it describes in *cylindrical coordinates* by placing the appropriate letter in the blank space provided. (Some answers could occur more than once.)

______ $r\cos\theta = 1$.

______ $z = 2r$.

______ $r = 2\cos\theta$.

______ $r = 0$.

(A) A cylinder
(B) A cone or half-cone.
(C) A half-plane.
(D) A point.
(E) A ray or half-line.
(F) A sphere.
(G) A line.
(H) A plane.

ANS: HBAG

10. (8) Find values of formulas for A, B, C and $g(x, y, z)$ such that

$$\int_0^2 \int_0^{6-3y} \int_0^{\frac{1}{2}(6-z-3y)} xyz\,dx\,dz\,dy = \int_0^A \int_0^B \int_0^C g(x, y, z)\,dy\,dz\,dx.$$

ANS: $\boxed{A = 3 \quad B = 6 - 2x \quad C = \frac{1}{3}(6 - z - 2x) \quad g(x, y, z) = xyz}$

$A =$: ____________________

$B =$: ____________________

$C =$: ____________________

$g(x, y, z) =$: ____________________

11. (5) Evaluate the iterated integral $\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec\phi} \rho^3 \sin^2\phi\,d\rho\,d\phi\,d\theta$ by converting to cylindrical coordinates.

ANS: In rectangular coordinates, the integrand is $\rho\sin(\phi) = \sqrt{x^2 + y^2}$. The region is that inside the cone $z = \sqrt{x^2 + y^2}$ and below the plane $z = 1$. Hence the integral is equal to

$$\int_0^{2\pi} \int_0^1 \int_r^1 r^2\,dz\,dr\,d\theta = 2\pi \int_0^1 r^2 - r^3\,dr = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \boxed{\frac{\pi}{6}}.$$