Universal Enveloping Algebras (and deformations)

Under bracket multiplication, Lie algebras are non-associative. The idea behind the construction of the universal enveloping algebra of some Lie algebra $\mathfrak{g}$ is to pass from this non-associative object to its more friendly unital associative counterpart $U\mathfrak{g}$ (allowing for the use of associative methods such as localization). The reverse is a very natural process: Any associative algebra A over the field k becomes a Lie algebra over k (called the *underlying* Lie algebra of A, and denoted A_L) with the Lie bracket

$$[a, b] = ab - ba.$$

That is, from an associative product, one can construct a Lie bracket by simply taking the commutator with respect to that associative product.

The construction of the universal enveloping algebra attempts to reverse this process: given a Lie algebra $\mathfrak{g}$ over k we find the "most general" unital associative k-algebra A such that the Lie algebra A_L contains $\mathfrak{g}$. This is the *universal enveloping algebra* $U_{\mathfrak{g}}$, and can be directly constructed via

$$U_{\mathfrak{g}} = (\mathfrak{g} \otimes \mathfrak{g}) / \langle a \otimes b - b \otimes a - [a, b] \rangle.$$

Note that the universal enveloping algebra construction is not *inverse* to the formation of A_L: if we start with an associative algebra A, then $U_{A_L} \supsetneq A$.

The beauty of this construction is that is preserves the representation theory: the representations of $\mathfrak{g}$ are in bijection with modules over $U_{\mathfrak{g}}$, and this bijection preserves equivalence, irreducibility, etc.. Moreover, any nice decomposition of $\mathfrak{g}$ can be extended to a nice decomposition of $U_{\mathfrak{g}}$. So, for example, fix a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+,$$

where $\mathfrak{h}$ is the maximal torus of the borel subalgebra $\mathfrak{b}$ and

$$\mathfrak{n}^+ = \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha$$

is the sum of the weight spaces associated to positive roots. Then

$$U\mathfrak{g} = U\mathfrak{n}^- \oplus U\mathfrak{h} \oplus U\mathfrak{n}^+, \quad U\mathfrak{b} = U\mathfrak{h} \oplus U\mathfrak{n}^+.$$

The construction of the group algebra for a given group is in many ways analogous to constructing the universal enveloping algebra for a given Lie algebra. Both constructions are universal and translate representation theory into module theory. Furthermore, just as group algebras carry natural comultiplications which turn them into Hopf algebras, universal enveloping algebras are very natural Hopf algebras. In fact, where Lie algebras fail to be associative, and therefore are never Hopf algebras, universal enveloping algebras are used in place.

1 Drinfel'd-Jimbo quantum groups

The *Drinfel'd-Jimbo quantum groups* are quasitriangular Hopf algebras that are "deformations" of the universal enveloping algebra algebras of a certain Lie algebra, frequently semisimple or affine, by some parameter q or $\bar{h}$ (one recovers $U_{\mathfrak{g}}$ by setting $q = 1$ or $\bar{h} = 0$).

1.1 From talk "Invariant trace for the category of representations of Lie Super algebras," Nathan Geer, Georgia Tech

$$U_h(\mathfrak{g}) \cong U(\mathfrak{g})[[h]] \qquad \text{as ???}$$

. Let $\mathcal{C}$ be the category of $U_h(\mathfrak{g})$-modules $\tilde{V}$ where $\tilde{V} = V[[h]]$, V is a $\mathfrak{g}$-module. $\mathcal{C}$ is a *ribbon category* ("R-matrix" – same as in the tantilizer stuff???). Ribbons are referring to unkinking knots (framed knots). Define Rib_e to be the category of colored ribbon graphs - links labeled with $U_h(\mathfrak{g})$-modules, given directions, and with elements of $\tilde{V}^*$ stuck in.

References

[Dx] J. Dixmier, *Enveloping algebras*, Graduate Studies in Mathematics 11, American Mathematical Society, Providence, RI, 1996.