Lecture Notes

Math 3 / Fall 2012

Week 2

Exponential functions

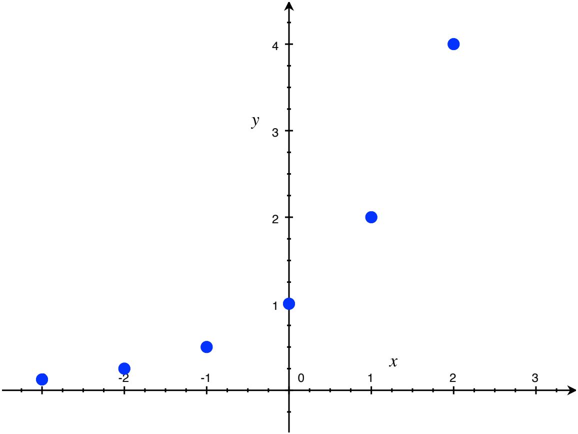
For any fixed base $b > 0$:

- $b^0 = 1$ and $b^1 = b$
- $b^{x+y} = b^x b^y$ and $b^{x-y} = b^x / b^y$
- $b^{xy} = (b^x)^y$ and $b^{x/y} = (b^x)^{1/y} = (b^{1/y})^x$

The **exponential function** b^x is the only <u>continuous</u> function with all these properties

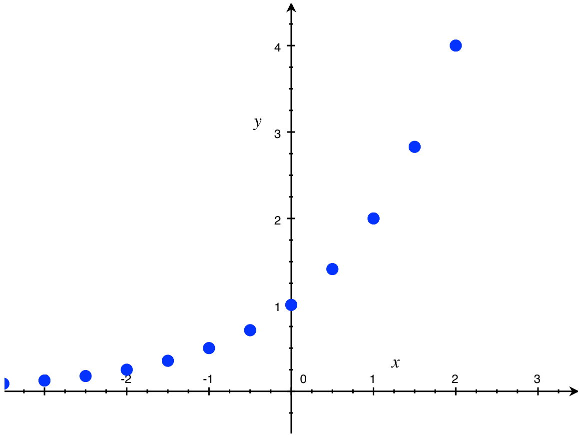
Exponential functions

$$f(n) = 2^n$$



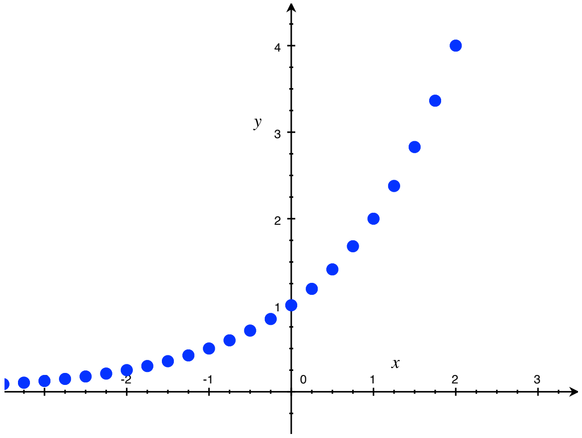
Exponential functions

$$f(n/2) = (2^{1/2})^n = (\sqrt{2})^n$$



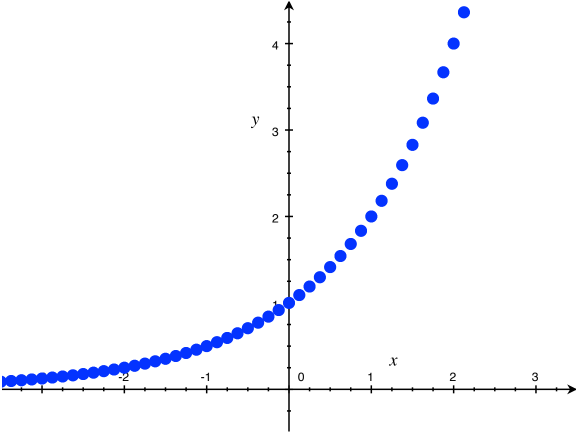
Exponential functions

$$f(n/4) = (2^{1/4})^n = (\sqrt[4]{2})^n$$



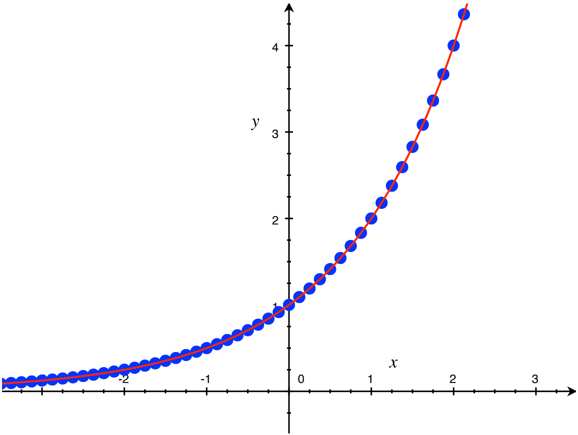
Exponential functions

$$f(n/8) = (2^{1/8})^n = (\sqrt[8]{2})^n$$



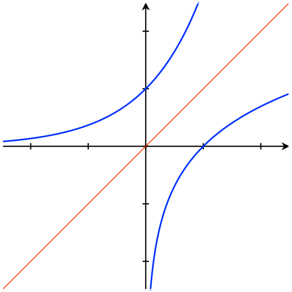
Exponential functions

$$f(x) = 2^x$$



Logarithm functions

For any fixed base $b > 0$, the **logarithm function** $\log_b x$ is the inverse of b^x



$$b^{\log_b x} = x = \log_b b^x$$

Laws of logarithms

For any fixed base $b > 0$, $b \neq 1$:

- $\log_b(1) = 0$ and $\log_b(b) = 1$
- $\log_b(xy) = \log_b(x) + \log_b(y)$

  and $\log_b(x/y) = \log_b(x) - \log_b(y)$
- $\log_b(x^p) = p \log_b(x)$

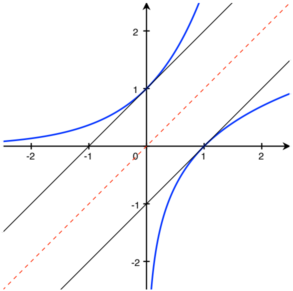
  and $\log_b(\sqrt[p]{x}) = \log_b(x)/p$

Natural exponential and logarithm

The irrational number

$$e = 2.718281828459045235360287471352662497 8\ldots$$

is such that the slope of e^x when $x = 0$ is 1

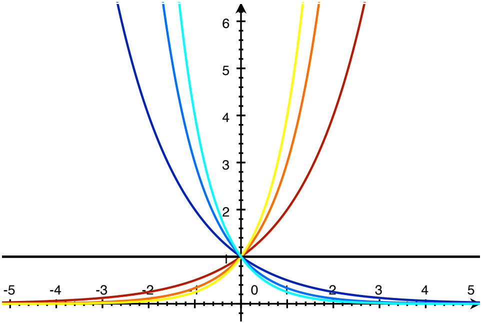


- The **natural exponential** is $\exp(x) = e^x$
- The **natural logarithm** is $\ln(x) = \log_e(x)$

Change of base for exponentials

$$b^x = e^{x \ln b}$$

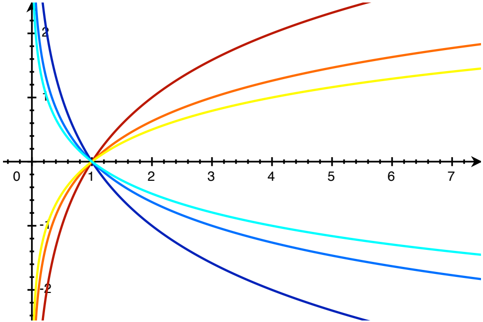
- If $b > 1$ then the exponential b^x is a horizontal scaling of e^x
- If $b < 1$ then the exponential b^x is a horizontal scaling of e^x reflected across the y-axis



Change of base for logarithms

$$\log_b x = \frac{\ln x}{\ln b}$$

- If $b > 1$ then the logarithm $\log_b x$ is a vertical scaling of $\ln x$
- If $b < 1$ then the logarithm $\log_b x$ is a vertical scaling of $\ln x$ reflected across the x-axis



Domain and range

For any base $b > 0$, $b \neq 1$:

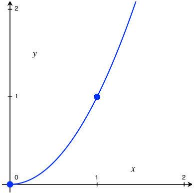
- The domain of b^x is
$$(-\infty, \infty)$$
- The range of b^x is
$$(0, \infty)$$
- Horizontal asymptote $y = 0$
- Goes through $(0, 1)$

- The domain of $\log_b x$ is
$$(0, \infty)$$
- The range of $\log_b x$ is
$$(-\infty, \infty)$$
- Vertical asymptote $x = 0$
- Goes through $(1, 0)$

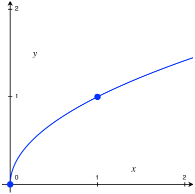
Power functions

$$x^p = e^{p \ln x} \qquad \text{when } x > 0$$

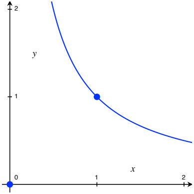
$p > 1$



$0 < p < 1$



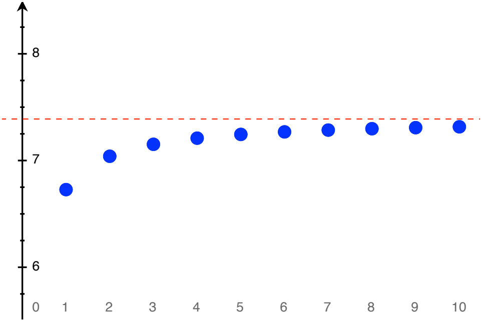
$p < 0$



Continuously compound interest

$$A(t) = A_0\left(1+\frac{r}{n}\right)^{nt} \quad \longrightarrow \quad A_0 e^{tr} \text{ as } n \to \infty$$

- $A(t)$ is the amount of money accumulated after t years
- A_0 is the initial amount you borrow or deposit
- r is the annual rate of interest
- n is the number of times the interest is compounded per year



Exponential models

If data fits an exponential model then log-data fits a linear model

$$N = N_0 R^t \quad \longrightarrow \quad \ln N = \ln N_0 + t \ln R$$

t	N	$\ln N$
1	613	6.42
2	746	6.61
3	909	6.81
4	1113	7.01
5	1361	7.22
6	1659	7.41
7	2028	7.61

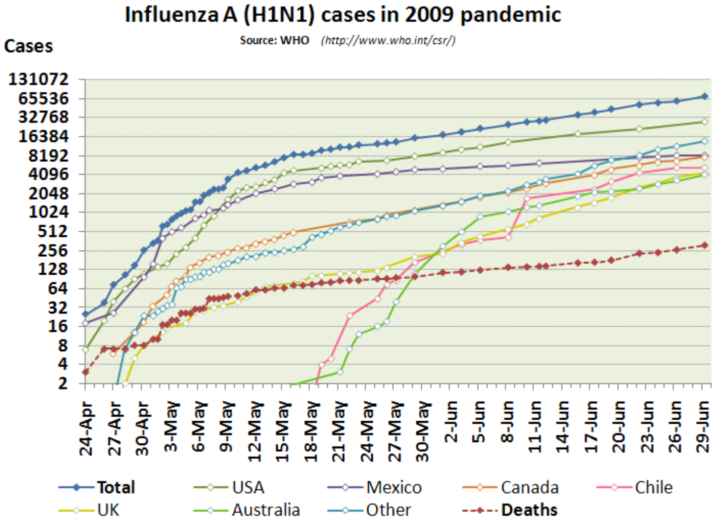
- Least squares best fit:

$$\ln N = 6.216 + 0.199t$$

- $N_0 = e^{6.216} = 500$
- $R = e^{0.199} = 1.22$
- Exponential model:

$$N = 500 \cdot 1.22^t$$

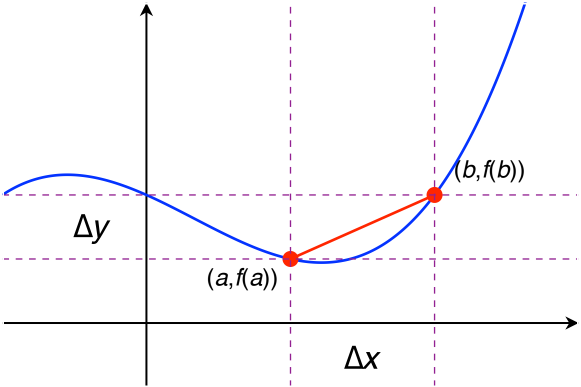
Semi-log graphs



Average rate of change

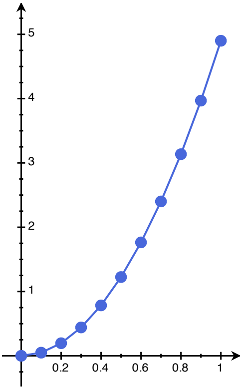
The **average rate of change** of the function f from a to b is

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$



Falling object

time (s)	distance (m)
0.0	0.000
0.1	0.049
0.2	0.196
0.3	0.441
0.4	0.784
0.5	1.225
0.6	1.764
0.7	2.401
0.8	3.136
0.9	3.969
1.0	4.900



Falling object: average speed

time (s)	distance (m)	speed (m/s)
0.0	0.000	0.49
0.1	0.049	1.47
0.2	0.196	2.45
0.3	0.441	3.43
0.4	0.784	4.41
0.5	1.225	5.39
0.6	1.764	6.37
0.7	2.401	7.35
0.8	3.136	8.33
0.9	3.969	9.31
1.0	4.900	—

- Average speed from 0.2 to 0.3: $\frac{0.441-0.196}{0.3-0.2} = 2.45$

Falling object: average acceleration

time (s)	distance (m)	speed (m/s)	accel (m/s^2)
0.0	0.000	0.49	9.8
0.1	0.049	1.47	9.8
0.2	0.196	2.45	9.8
0.3	0.441	3.43	9.8
0.4	0.784	4.41	9.8
0.5	1.225	5.39	9.8
0.6	1.764	6.37	9.8
0.7	2.401	7.35	9.8
0.8	3.136	8.33	9.8
0.9	3.969	9.31	—
1.0	4.900	—	—

Average velocity and acceleration

$$\begin{aligned} \text{Distance:} \quad & s(t) & & \text{(???)} \\ \text{Velocity:} \quad & v(t) = \frac{s(t+h) - s(t)}{h} = 9.8t + 0.49 & & \text{(linear)} \\ \text{Acceleration:} \quad & a(t) = \frac{v(t+h) - v(t)}{h} = 9.8 & & \text{(constant)} \end{aligned}$$

Question: What kind of function has linear rate of change?

Instantaneous velocity and acceleration

$$\text{Distance:} \quad s(t) = 4.9t^2$$

$$\text{Velocity:} \quad v(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} = 9.8t$$

$$\text{Acceleration:} \quad a(t) = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h} = 9.8$$

Answer: Quadratic functions have linear rate of change!

Difference quotient and derived functions

The **difference quotient** (with increment h) of f at x is

$$\frac{f(x+h)-f(x)}{h}$$

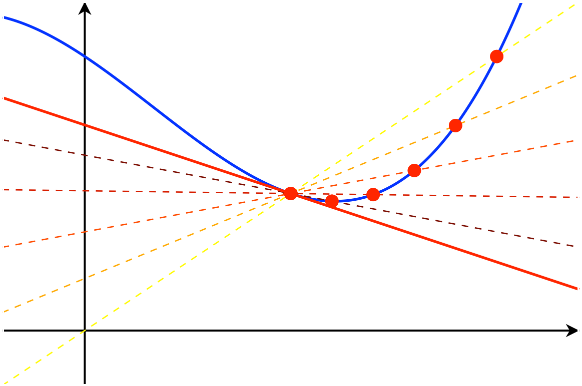
This is the **average rate of change** of f from x to $x+h$

The limit as $h \to 0$ is the **derivative** of f at x

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

This is the **instantaneous rate of change** of f at x

Difference quotient and derived functions



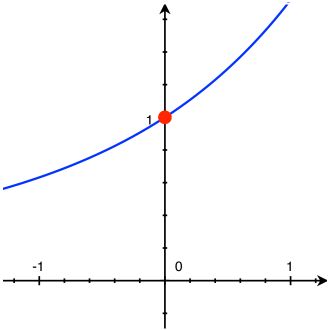
The **derivative** is the slope of the **tangent line**

Natural exponential

x	e^x	$\dfrac{e^{x+h} - e^x}{h}$
-0.5	0.61	0.64
-0.4	0.67	0.71
-0.3	0.74	0.78
-0.2	0.82	0.86
-0.1	0.91	0.95
0.0	1.00	1.05
0.1	1.11	1.16
0.2	1.22	1.28
0.3	1.35	1.42
0.4	1.49	1.57
0.5	1.65	1.73

Natural exponential

$$\frac{e^{x+h} - e^x}{h} = \frac{e^x e^h - e^x}{h} = e^x \frac{e^h - 1}{h}$$



As $h \to 0$ the difference quotient

$$\frac{e^h - 1}{h}$$

approaches the slope of the exponential function at $x = 0$

Therefore

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

because of the definition of e!

Natural exponential

$$\lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x$$

$\boxed{\text{The derivative of } e^x \text{ is } e^x}$

Binomial formulas

$$\begin{aligned}
(A+B)^1 &= A + B \\
(A+B)^2 &= A^2 + 2AB + B^2 \\
(A+B)^3 &= A^3 + 3A^2B + 3AB^2 + B^3 \\
(A+B)^4 &= A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4 \\
(A+B)^5 &= A^5 + 5A^4B + 10A^3B^2 + 10A^2B^3 + 5AB^4 + B^5
\end{aligned}$$

$$(A+B)^n = \sum_{m=0}^{n} \binom{n}{m} A^m B^{n-m}$$

Differences of like powers

$$
\begin{aligned}
A^2 - B^2 &= (A - B)(A + B) \\
A^3 - B^3 &= (A - B)(A^2 + AB + B^2) \\
A^4 - B^4 &= (A - B)(A^3 + A^2B + AB^2 + B^3) \\
A^5 - B^5 &= (A - B)(A^4 + A^3B + A^2B^2 + AB^3 + B^4)
\end{aligned}
$$

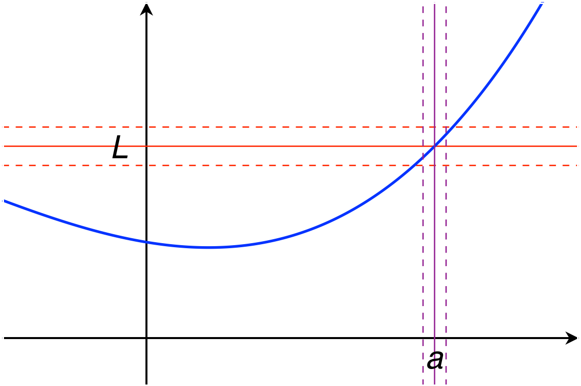
$$
A^{n+1} - B^{n+1} = (A - B)\sum_{m=0}^{n} A^m B^{n-m}
$$

Limits

We write

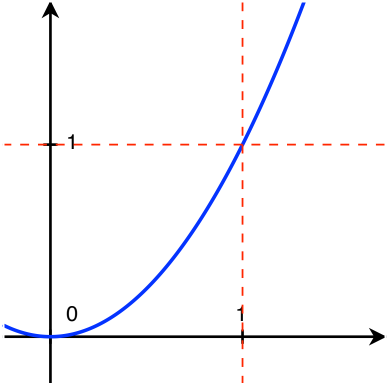
$$\lim_{x \to a} f(x) = L$$

when the values $f(x)$ can be made arbitrarily close to the number L whenever x is sufficiently close to a but $x \neq a$



Limits

$$\boxed{\text{The value } f(a) \text{ is not relevant in evaluating } \lim_{x \to a} f(x)}$$



Limits

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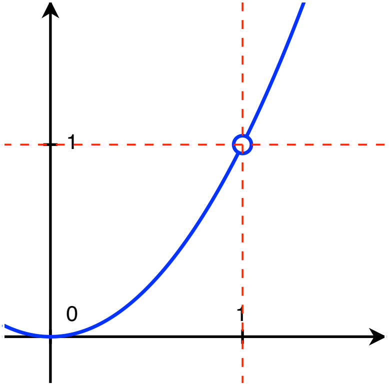

Limits

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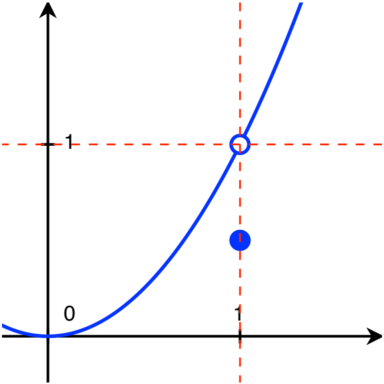
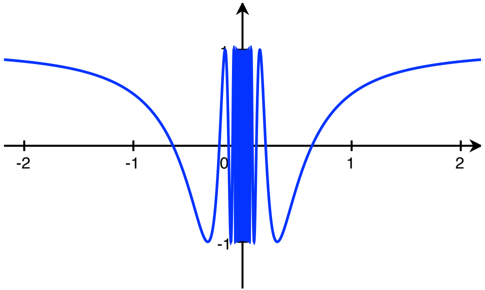


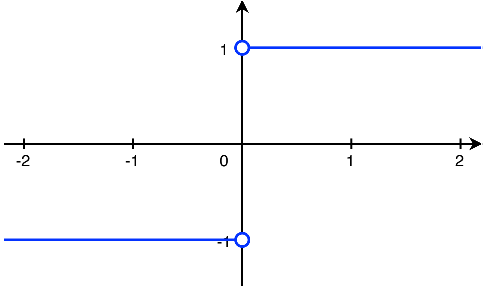
When the limit does not exist...

$$f(x) = \cos\left(\frac{1}{x}\right)$$



When the limit does not exist...

$$f(x) = \frac{x}{|x|}$$



One-sided limits

We write

$$\lim_{x \to a^+} f(x) = L$$

when the values $f(x)$ can be made arbitrarily close to the number L whenever x is sufficiently close to a but $x > a$

We write

$$\lim_{x \to a^-} f(x) = L$$

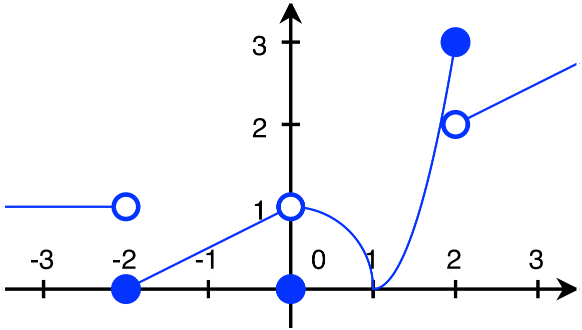
when the values $f(x)$ can be made arbitrarily close to the number L whenever x is sufficiently close to a but $x < a$

If <u>both</u> $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

Then $\lim_{x \to a} f(x) = L$

Piecewise defined functions

$$f(x) = \begin{cases} 1 & \text{if } x < -2 \\ (x+2)/2 & \text{if } -2 \le x < 0 \text{ or if } 2 < x \\ 0 & \text{if } x = 0 \\ \sqrt{1-x^2} & \text{if } 0 < x < 1 \\ 3(x-1)^2 & \text{if } 1 \le x \le 2 \end{cases}$$



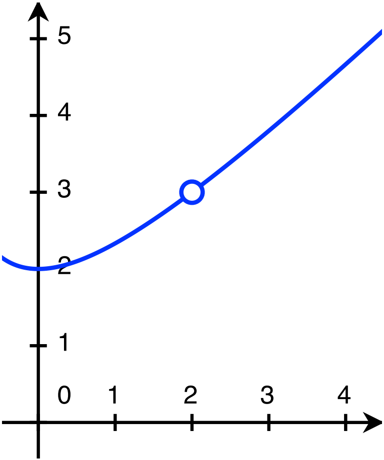
Algebra of limits

Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$
Then:

- $\lim_{x \to a} f(x) + g(x) = L + M$
- $\lim_{x \to a} f(x) - g(x) = L - M$
- $\lim_{x \to a} f(x) \cdot g(x) = L \cdot M$
- $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ provided $M \neq 0$

Limit tricks...

$$\lim_{x\to 2}\frac{x^3-8}{x^2-4}=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}=\lim_{x\to 2}\frac{x^2+2x+4}{x+2}=3$$



Limit tricks...

$$\begin{aligned}
\lim_{x\to 2} \frac{\sqrt{6x}-\sqrt{12}}{x-2} &= \lim_{x\to 2} \frac{\sqrt{6x}-\sqrt{12}}{x-2} \cdot \frac{\sqrt{6x}+\sqrt{12}}{\sqrt{6x}+\sqrt{12}} \\
&= \lim_{x\to 2} \frac{6x-12}{x-2} \cdot \frac{1}{\sqrt{6x}+\sqrt{12}} \\
&= \lim_{x\to 2} 6 \cdot \frac{1}{\sqrt{6x}+\sqrt{12}} = 6 \cdot \frac{1}{\sqrt{12}+\sqrt{12}} = \frac{3}{\sqrt{12}}
\end{aligned}$$

$$\boxed{\text{Multiply by the conjugate}}$$

Limit tricks...

$$\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{(x^2 + 2xh + h^2) - x^2}{h} = \lim_{h\to 0} 2x + h = 2x$$

$$\boxed{\text{The derivative of } x^2 \text{ is } 2x}$$

Composition and limits

If

$$\lim_{x \to A} f(x) = B \quad \text{and} \quad \lim_{x \to B} g(x) = C$$

then

$$\lim_{x \to A} g(f(x)) = C$$

Example

$$\lim_{x \to 2} \sin\left(\frac{x-1}{x+1}\right) = \lim_{x \to 1/3} \sin(x) = \sin(1/3)$$

Limits at infinity

We write

$$\lim_{x \to \infty} f(x) = L$$

when the values $f(x)$ can be made arbitrarily close to the number L whenever x is sufficiently large ("close to ∞")

We write

$$\lim_{x \to -\infty} f(x) = L$$

when the values $f(x)$ can be made arbitrarily close to the number L whenever x is sufficiently small ("close to $-\infty$")

Limits at infinity correspond to **horizontal asymptotes**

More limit tricks...

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x-5}{\sqrt{x^2+3x+9}} &= \lim_{x\to\infty} \frac{x(3-5/x)}{\sqrt{x^2(1+3/x+9/x^2)}} \\
&= \lim_{x\to\infty} \frac{x}{\sqrt{x^2}} \cdot \frac{3-5/x}{\sqrt{1+3/x+9/x^2}} \\
&= \lim_{x\to\infty} \frac{3-5/x}{\sqrt{1+3/x+9/x^2}} \\
&= \frac{3-0}{\sqrt{1+0+0}} = 3
\end{aligned}$$

Factor out dominant terms

Rational functions

Suppose we have:

- a polynomial of degree n: $p(x) = ax^n + (\text{lower degree terms})$
- a polynomial of degree m: $q(x) = bx^m + (\text{lower degree terms})$

Then we have:

- if $n < m$ then $\displaystyle\lim_{x\to\infty} \frac{p(x)}{q(x)} = 0$
- if $n > m$ then $\displaystyle\lim_{x\to\infty} \frac{p(x)}{q(x)} = \pm\infty$ (same sign as a/b)
- if $n = m$ then $\displaystyle\lim_{x\to\infty} \frac{p(x)}{q(x)} = \frac{a}{b}$