Math 46 : HW 9 solution (half-length) 2009. ① 6/3/09.

Zhiliang Zhang's solution to #10:

$$\int_{-\infty}^{\infty} |u(x)|^2 dx = \int_{-\infty}^{\infty} u(x) \overbrace{\overline{u(x)}}^{\text{expand as conjugate of inverse FT of } \hat{u}} dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x) \int_{-\infty}^{\infty} \overline{\hat{u}(\xi)}\, \overbrace{\overline{e^{-ix\xi}}}^{= e^{\overline{-ix\xi}} = e^{+ix\xi}} \, d\xi\, dx$$

since $x, \xi \in \mathbb{R}$, $\bar{i} = -i$.

change order of integration

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{u}(\xi)} \underbrace{\int_{-\infty}^{\infty} u(x) e^{ix\xi} dx}_{= \hat{u}(\xi) \text{ by defn.}} d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{u}}(\xi)\, \hat{u}(\xi)\, d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{u}(\xi)|^2 d\xi$$

QED.

v. nice! (no convolution of S-fns required!)

p. 395-398 #4

$$u(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi} e^{-a|\xi|} d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{(a-ix)\xi} d\xi + \frac{1}{2\pi}\int_{0}^{\infty} e^{(-a-ix)\xi} d\xi$$

$$= \frac{1}{2\pi}\left[\frac{1}{a-ix} \underbrace{\left[e^{(a-ix)\xi}\right]_{-\infty}^{0}}_{1} + \frac{1}{-a-ix} \underbrace{\left[e^{(-a-ix)\xi}\right]_{0}^{\infty}}_{-1} \right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{a-ix} + \frac{1}{a+ix}\right] = \frac{1}{2\pi} \frac{a+ix+a-ix}{(a-ix)(a+ix)} = \frac{1}{\pi}\frac{a}{a^2+x^2}$$

#5 b. $\mathcal{F}(e^{iax}u)(\xi) := \int_{-\infty}^{\infty} e^{ix\xi} e^{iax} u(x)\, dx = \int_{-\infty}^{\infty} e^{ix(\xi+a)} u(x) dx =: \hat{u}(\xi+a)$

c. $\mathcal{F}(u(x+a))(\xi) := \int_{-\infty}^{\infty} e^{ix\xi} u(x+a)\, dx \underset{(y=x+a)}{\overset{\text{change var.}}{=}} \int_{-\infty}^{\infty} e^{i(y-a)\xi} u(y)\, dy =: e^{-ia\xi}\hat{u}(\xi)$

#7

$$u_t - cu_x - u_{xx} = 0 \quad \text{FT in } x$$

$$\hat{u}_t(\xi,t) - c(-i\xi)\hat{u}(\xi,t) - (-i\xi)^2\hat{u}(\xi,t) = 0 \quad \text{an ODE in time. (fixed } \xi)$$

So $\hat{u}(\xi,t) = A(\xi)\, e^{-(ic\xi + \xi^2)t}$ matching ICs $(t=0)$ gives $A(\xi) = \hat{f}(\xi)$.

$$\hat{u}(\xi,t) = \underbrace{\boxed{e^{-ict\xi}}}_{\text{using \#5 b. this gives a translation in real space of } -ct \text{ ie: } x \leftarrow x+ct} \hat{f}(\xi)e^{-\xi^2 t}$$

just what you'd expect from 'drift' term cu_x.

so $$u(x,t) = \big(f * \underbrace{\mathcal{F}^{-1}(e^{-\xi^2 t})}_{\text{already done on p. 392 (choose } k=1)}\big)(x+ct) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x+ct-y)^2}{4t}} f(y)\,dy$$

A). $$\mathcal{F}(\delta(x-a))(\xi) := \int_{-\infty}^{\infty} e^{ix\xi}\,\delta(x-a)\,dx \overset{\text{sifting}}{=} e^{ia\xi}$$

Inversion formula for this is

$$\delta(x-a) = \mathcal{F}^{-1}(e^{ia\xi})(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi}e^{ia\xi}\,d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi(a-x)}\,d\xi$$

integral representation of δ-func. (convergence not discussed)

$$\mathcal{F}(e^{ikx})(\xi) = \int_{-\infty}^{\infty} e^{ikx}e^{i\xi x}\,dx = \int_{-\infty}^{\infty} e^{ix(k+\xi)}\,dx = 2\pi\,\delta(k+\xi)$$

$\hat{u}(\xi)$, $-k$, 0, ξ

#10 $$\frac{1}{2\pi}\int \hat{u}(\xi)\,\overline{\hat{u}(\xi)}\,d\xi = \frac{1}{2\pi}\int\int e^{i\xi x}u(x)\,dx \int e^{-i\xi y}\,\overline{u}(y)\,dy\,d\xi$$

$$= \int\int u(x)\,\overline{u(y)}\,\boxed{\frac{1}{2\pi}\int e^{i\xi(x-y)}\,d\xi}\,dx\,dy$$

$\to \delta(x-y)$ using B. results.

$$= \int u(x)\underbrace{\int \overline{u(y)}\,\delta(x-y)\,dy}_{\overline{u(x)}}\,dx$$

$\overline{u(x)}$ by sifting property

$$= \int |u(x)|^2\,dx$$

QED.

#11 Find a $\hat{u}(\xi)$ that gives the required integral : $u(x) = e^{-|x|} \overset{FT}{\longleftrightarrow} \hat{u}(\xi) = \frac{2}{1+\xi^2}$

so $$\int_{-\infty}^{\infty}\frac{d\xi}{(1+\xi^2)^2} = \frac{1}{4}\int_{-\infty}^{\infty}|\hat{u}(\xi)|^2 d\xi \overset{\text{Plancherel}}{=} \frac{2\pi}{4}\int_{-\infty}^{\infty}|u(x)|^2dx = \frac{\pi}{2}\int_{-\infty}^{\infty}e^{-2|x|}dx \overset{\text{even symm.}}{=} \pi\int_0^{\infty}e^{-2x}dx$$

$$= \pi/2$$

neat, hey?

#15 Use results of p. 393 : $\hat{u}(\xi,y) = c(\xi)\,e^{-|\xi|y}$

so $$\hat{u}_y(\xi,y) = \underbrace{d(\xi)}_{\text{by ICs } \hat{u}_y(\xi,0)=\hat{f}(\xi), \text{ must have } d(\xi)=\hat{f}(\xi)}\,e^{-|\xi|y}$$

for some other func. $d(\xi)$.

so $$u_y(x,y) = \big(f * \mathcal{F}^{-1}[e^{-|\xi|y}]\big)(x) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\eta)\,d\eta}{(x-\eta)^2+y^2}$$

$-inx$ using p. 393.

Integrate wrt. y (by recognising the derivative). : $$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\ln\left[(x-\eta)^2+y^2\right]f(\eta)\,d\eta + C$$

doesn't dep. on x since that would mess up the PDE $u_{xx}+u_{yy}=0$.

Note that you can't do this by integrating wrt. x.