Directional Derivatives and the Gradient Vector Part 2

Lecture 25

February 28, 2007

Recall

Fact

- *If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector* $\mathbf{u} = \langle a, b \rangle$ *and*

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b.$$

- *If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by*

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Maximizing the Directional Derivative

Theorem

Suppose that f is a differentiable function of two (or three) variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(x, y)$ is $|\nabla f|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(x)$.

Example

Example

- If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.
- In what direction does f have the maximum rate of change? What is this maximum rate of change?

Example

Example

Suppose that the temperature at a point (x, y, z) in space is given by

$$T(x, y, z) = \frac{80}{1 + x^2 + 2y^2 + 3z^2},$$

where T is measured in degree Celsius and x, y, z in meters.

- In which direction does the temperature increase fastest at the point $(1, 1, -2)$?
- What is the maximum rate of increase?

Tangent Planes to Level Surfaces

Definition

- A level surface is a surface with equation

$$F(x, y, z) = k.$$

- Let $P(x_0, y_0, z_0)$ be a point on S and let C be any curve that lies on S and passes trough P.
- Recall that C is described by a continuous vector function

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle.$$

Tangent Planes to Level Surfaces

Fact

- *If x, y, and z are differentiable and F is also differentiable, we can apply the Chain Rule:*

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0;$$

- *Or*

$$\nabla F \cdot \mathbf{r}'(t) = 0.$$

- *The gradient vector at P, $\nabla F(x_0, y_0 z_0)$ is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ to any curve C on S that passes through P.*

The Tangent Plane

Definition

- We define **the tangent plane to the level surface** $F(x,y,z) = k$ **at** $P(x_0, y_0, z_0)$ as the plane passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$.
- It has equation

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(z_0, y_0, z_0)(z - z_0) = 0.$$

The Normal Line

Definition

- The **normal line** to S at P is the line passing through P and perpendicular to the tangent plane.
- The symmetric equations are

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Special Case

Definition

If the equation of the surface S is of the form $z = f(x, y)$, that is

$$F(x, y, z) = f(x, y) - z = 0$$

then the equation of the tangent plane becomes

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

Examples

- Find the tangent plane and normal line of the surface
$$F(x, y, z) = x^2 + y^2 + z - 9 = 0$$
at the point $P_0(1, 2, 4)$.
- Find the equation of the tangent plane at the point $(-2, 1, -3)$ to the ellipsoid
$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3.$$

Significance of the Gradient Vector

Fact

- *The gradient* ∇f *gives the direction of fastest increase of* f.
- *The gradient* Δf *is orthogonal to the level surface* S *of* f *through a point* P.