Workshop Problems 7

Problem 1. Let H be a subspace of a finite-dimensional vector space V. Show that if $\dim H = \dim V$ then $H = V$.

Problem 2. Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix.

a. Show that $\operatorname{Col} AB \subset \operatorname{Col} A$. Conclude that $\operatorname{rank} AB \leq \operatorname{rank} A$.

b. Show that $\operatorname{Nul} B \subset \operatorname{Nul} AB$. Conclude that $\dim \operatorname{Nul} B \leq \dim \operatorname{Nul} AB$.

c. Use parts (a) and (b) together with the rank theorem to show that

$$\operatorname{rank} AB \leq \min \{\operatorname{rank} A, \operatorname{rank} B\}.$$

That is, show that $\operatorname{rank} AB \leq \operatorname{rank} A$ *and* $\operatorname{rank} AB \leq \operatorname{rank} B$.

Problem 3. Use the results of Problem 2 to show that if A and B are both $n \times n$ then

$$\dim \operatorname{Nul} AB \geq \max \{\dim \operatorname{Nul} A, \dim \operatorname{Nul} B\}.$$

That is, show that $\dim \operatorname{Nul} AB \geq \dim \operatorname{Nul} A$ *and* $\dim \operatorname{Nul} AB \geq \dim \operatorname{Nul} B$.

Problem 4. Let V and W be vector spaces and let $T : V \to W$ be a linear transformation. Denote the range of T by $T(V)$.

a. Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ spans V then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_p)\}$ spans $T(V)$.

b. Use part (a) to show that $\dim T(V) \leq \dim V$.