Homework 2 - Math 71 - Sketch of Solutions to Some Problems

* $\frac{33}{11}$ σ m-cycle. Prove σ^i m-cycle $\Leftrightarrow$ i and m relatively prime

$\Leftarrow$ $\exists A, B$ $\quad Ai + Bm = 1$

$$\therefore \sigma = (\sigma^i)^A (\sigma^m)^B = (\sigma^i)^A$$

σ^i is a product of disjoint cycles

$$\sigma^i = c_1 \cdots c_r, \quad r \geq 1$$

where c_k is a cycle of length $r_k > 1$ and $\sum r_k \leq m$

$$\therefore \sigma = (\sigma^i)^A = c_1^A \cdots c_r^A$$

$\therefore r = 1$ and $\sigma = c_1^A$ $\quad \therefore c_1$ is m-cycle (why?)

$\therefore \sigma^i = c_1$ an m-cycle

$\Rightarrow$ σ^i m-cycle. Suppose $p \mid i$, $p \mid m$ $\therefore i = pi'$ and $m = pm'$

$$(\sigma^i)^{m'} = \sigma^{im'} = \sigma^{i'pm'} = (\sigma^m)^{i'} = e$$

But $m = |\sigma^i|$ $\therefore m \mid m'$ and $m' \mid m$ $\therefore m = m'$ so $p = 1$. $\therefore$ i and m are relatively prime

$\frac{33}{16}$ Consider any m-cycle in S_n

$$(k_1\ k_2 \cdots k_m)$$

k_1 can be any one of n number

k_2 can be any one of $n-1$ numbers

$\vdots$

k_m can be any one of $n-m+1$ numbers

But $(k_1 k_2 \cdots k_m) = (k_2 \cdots k_m k_1) = \cdots$

So we have counted each m-cycle m times

$\therefore$ The number is $\dfrac{n(n-1)\cdots(n-m+1)}{m}$

$\frac{35}{5}$ Clearly $\mathbb{F}$ finite $\Rightarrow GL(2, \mathbb{F})$ finite.

Now consider the injection $\mathbb{F}^* \to GL(2, \mathbb{F})$ given by

$a \to \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$. $\therefore$ $\mathbb{F}^{\times}$ and hence $\mathbb{F}$ is finite.

40/2 Suppose $|x| = n$ Then $(\varphi(x))^n = \varphi(x^n) = e$

If $(\varphi(x))^k = e$, $\varphi(x^k) = e = \varphi(e)$ $\therefore x^k = e$

$\therefore k \geq n$ so $|\varphi(x)| = n$. Similarly $|\varphi(x)| = n \Rightarrow |x| = n$

40/4 How many elements are there of order 4?

40/7 Elements of order 2?

* 40/23 $\sigma(g) = g \iff g = 1$ and $\sigma^2 = id$

Define $\theta(g) = g^{-1}\sigma(g)$

θ is one-one: $\theta(g) = \theta(g')$

$$g^{-1}\sigma(g) = g'^{-1}\sigma(g')$$

$$g'g^{-1} = \sigma(g'g^{-1})$$

$$\therefore g'g^{-1} = 1 \qquad \therefore g = g'$$

$\therefore$ θ one-one. But θ is then onto since G is a finite set. $\therefore$ θ is a bijection $\therefore$ every $g \in G$ can be written $g = \theta(x) = x^{-1}\sigma(x)$ for some $x \in G$

$$\begin{aligned} \sigma(g) &= \sigma(x^{-1}\sigma(x)) \\ &= \sigma(x^{-1})\,\sigma^2(x) \\ &= \sigma(x^{-1})\,x \\ &= (x^{-1}\sigma(x))^{-1} \\ &= g^{-1} \end{aligned}$$

So σ is the map $g \to g^{-1}$. But σ is a homomorphism

Now show $\sigma(g) = g^{-1}$ a homo. $\Rightarrow G$ abelian