Hopf Algebras

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Big idea: The Hopf algebra structure is essentially what one needs in order to guarantee that tensor products of modules are also modules.

Definition. A *bialgebra* B over a field k is a structure which is both a *unital* (has identity) associative algebra and a *coalgebra* (has a map $B \to B \times B$) over k, such that these structures are compatible. Compatibility just means that the *comultiplication* and the *counit* ε are both unital algebra homomorphisms, or equivalently, that the multiplication and the unit of the algebra both be coalgebra morphisms (these are equivalent in that they are expressed by the same diagram).

A *Hopf algebra* is a bialgebra with an *anitpode*—a map which generalizes the inversion map on a group

$$\begin{tikzcd}
& B \arrow[dl, "\Delta"'] \arrow[d, "\varepsilon"] \arrow[dr, "\Delta"] & \\
B \otimes B \arrow[d, "S\otimes\mathrm{id}"'] & k \arrow[dd, "\eta"] & B \otimes B \arrow[d, "\mathrm{id}\otimes S"] \\
B \otimes B \arrow[dr, "\mathrm{mult}"'] & & B \otimes B \arrow[dl, "\mathrm{mult}"] \\
& B &
\end{tikzcd}$$

commutes, where $\eta : k \to B$ is the unit map $\eta(c) = c \cdot 1$.

Example: group algebras A super-simple example of these are group algebras with normal multiplication, comultiplication given by

$$g \xmapsto{\Delta} g \otimes_k g,$$

and antipode

$$g \xmapsto{S} g^{-1}.$$

Example: enveloping algebras and quantum groups. Let $\mathfrak{g}$ be a Lie algebra over k. The enveloping algebra $U\mathfrak{g}$ of $\mathfrak{g}$ is the algebra generated by $\mathfrak{g}$ with the relations $xy - yx = [x, y]$ for all $x, y \in \mathfrak{g}$. If V is a $\mathfrak{g}$-module (i.e. a $U\mathfrak{g}$-module), then this action extends to $V \otimes V$ by $x \cdot (v_1 \otimes v_2) = xv_1 \otimes v_2 + v_1 \otimes xv_2$, for all $x \in \mathfrak{g}$, $v_1, v_2 \in V$. This reflects the Hopf algebra structure on $U\mathfrak{g}$ given by

$$\begin{array}{rrcl} \textit{counit} & x & \overset{\varepsilon}{\longmapsto} & 0, \\ \textit{comultiplication} & x & \overset{\Delta}{\longmapsto} & x \otimes 1 + 1 \otimes x, \text{ and} \\ \textit{antipode} & x & \overset{S}{\longmapsto} & -x, \end{array}$$

for $x \in \mathfrak{g}$. Similarly, if $U = U_h\mathfrak{g}$ is the Drinfeld-Jimbo quantum group, then U is also a Hopf algebra. A little more about this is below in the section about ribbon Hopf algebras. Fun fact, though, is that in either case, ε is the representation corresponding to the trivial module.

Example: polynomial functions on a Lie group. Let $B = \mathcal{O}(G)$ be the algebra of complex-valued polynomial functions on a complex Lie group G, and identify $\mathcal{O}(G \times G)$ with $B \otimes B$. Then, B is a Hopf algebra with

$$\begin{array}{rl} \textit{counit} & \varepsilon(f) = f(1), \\ \textit{comultiplication} & (\Delta(f))(g_1, g_2) = f(g_1 g_2), \text{ and} \\ \textit{antipode} & (S(f))(g) = f(g^{-1}). \end{array}$$

If $\mathfrak{g}$ is the Lie algebra of G, then the above two examples are dual to one another: Define a bilinear form $\mathcal{O}(G) \otimes U\mathfrak{g} \to \mathbb{C}$ by $\langle f, x \rangle = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} f(\exp(tx))$. Then,

$$\begin{array}{rcl} \langle fg, x \rangle & = & \langle f \otimes g, \Delta(x) \rangle, \\ \langle 1, x \rangle & = & \varepsilon(x), \\ \langle \Delta(f), x \otimes y \rangle & = & \langle f, xy \rangle, \\ \varepsilon(f) & = & \langle f, e \rangle, \\ \langle S(f), x \rangle & = & \langle f, S(x) \rangle, \\ \langle f^*, x \rangle & = & \overline{\langle f, S(x)^* \rangle}. \end{array}$$

Why topologists care. Let G be a topological group. There are the two familiar maps

$$G \times G \xrightarrow{\nabla} G \quad \text{(multiplication)} \qquad \text{and} \qquad G \xrightarrow{\Delta} G \times G \quad \text{(the diagonal in } G\text{)}.$$

Thus a functor

$$F : \{\text{top spaces}\} \to \{\text{vector spaces}\}$$

which takes cross products into tensor products gives me a vector space $F(G)$ and maps

$$F(G) \otimes F(G) \to F(G) \qquad \text{and} \qquad F(G) \to F(G) \otimes F(G).$$

So if functors like F let us in some sense see the structure of topological spaces reflected in the category of vector spaces, then what they turn groups into Hopf algebras. If F "linearizes" the category of topological spaces somehow, then this says that "the linearization of a group is Hopf algebra."

Quasitriangular and ribbon Hopf algebras

Let U be a Hopf algebra with coproduct Δ, counit ε, and antipode S. For $x \in U$, write

$$\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)} \qquad \text{and} \qquad \Delta^{\mathrm{op}}(x) = \sum_x x_{(2)} \otimes x_{(1)}.$$

A *quasitriangular Hopf algebra* $(U, \mathcal{R})$ is a Hopf algebra U with an invertible element

$$\mathcal{R} = \sum_{\mathcal{R}} R_1 \otimes R_2 \qquad \text{in} \quad U \otimes U$$

such that if

$$\mathcal{R}_{12} = \sum_{\mathcal{R}} R_1 \otimes R_2 \otimes 1, \qquad \mathcal{R}_{13} = \sum_{\mathcal{R}} R_1 \otimes 1 \otimes R_2, \qquad \mathcal{R}_{23} = \sum_{\mathcal{R}} 1 \otimes R_1 \otimes R_2,$$

then for $x \in U$,

$$\mathcal{R}\Delta(x)\mathcal{R}^{-1} = \Delta^{\mathrm{op}}(x), \quad (\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \quad \text{and} \quad (\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}. \tag{1}$$

By [Dr, Prop. 2.1], the element u in U defined by

$$u = \sum_{\mathcal{R}} S(R_2)R_1 \qquad \text{satisfies} \qquad uxu^{-1} = S^2(x) \quad \text{for all } x \in U. \tag{2}$$

A *ribbon Hopf algebra* $(U, \mathcal{R}, v)$ is a quasitriangular Hopf algebra $(U, \mathcal{R})$ with an invertible element v satisfying

$$v \in Z(U), \quad v^2 = uS(u), \quad S(v) = v, \quad \varepsilon(v) = 1, \quad \Delta(v) = (\mathcal{R}_{21}\mathcal{R})^{-1}(v \otimes v), \tag{3}$$

where $\mathcal{R}_{21} = \sum_{\mathcal{R}} R_2 \otimes R_1$.

Examples. *Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra, and $U_h\mathfrak{g}$ be the Drinfel'd-Jimbo quantum group corresponding to $\mathfrak{g}$. Both*

$$U = U\mathfrak{g} \text{ with } \mathcal{R} = 1 \otimes 1 \text{ and } v = 1, \qquad \text{and} \qquad U = U_h\mathfrak{g} \quad \text{with} \quad v = e^{-h\rho}u, \tag{4}$$

are ribbon Hopf algebras (see [LR, Corollary (2.15)]).

If M and N are U-modules then $M \otimes N$ is a U-module with action given by

$$x(m \otimes n) = \sum_x x_{(1)}m \otimes x_{(2)}n, \qquad \text{where} \qquad \Delta(x) = \sum_x x_{(1)} \otimes x_{(2)}, \tag{5}$$

for $x \in U$. The *trivial* U-module is

$$\mathbf{1} = \mathbb{C}1, \qquad \text{with} \quad x1 = \varepsilon(x)1, \quad \text{for } x \in U. \tag{6}$$

If V is a U-module then the *dual module* is

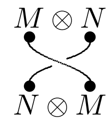
$$V^* = \mathrm{Hom}(V, \mathbb{C}) \qquad \text{with} \qquad (x\varphi)(m) = \varphi(S(x)m), \quad \text{for } x \in U, \tag{7}$$

and, if $e_1, \ldots, e_n$ is a basis of V, and $e^1, \ldots, e^n$ is the dual basis in V^* then

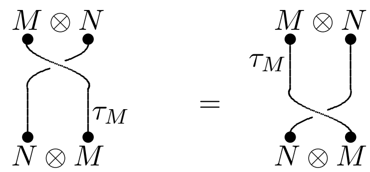
$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{\mathrm{coev}} & V \otimes V^* \\ 1 & \longmapsto & \sum_i e_i \otimes e^i \end{array} \qquad \text{and} \qquad \begin{array}{ccc} V^* \otimes V & \xrightarrow{\mathrm{ev}} & \mathbf{1} \\ \varphi \otimes m & \longmapsto & \varphi(m) \end{array} \tag{8}$$

are U-module homomorphisms. For U-modules M and N, the map

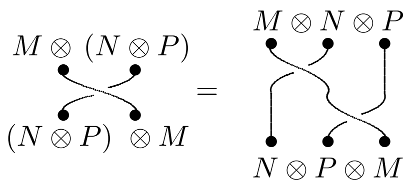
$$\check{R}_{MN}\colon\quad \begin{array}{rcl} M\otimes N & \longrightarrow & N\otimes M \\ m\otimes n & \longmapsto & \displaystyle\sum_{\mathcal{R}} R_2 n\otimes R_1 m \end{array} \tag{9}$$

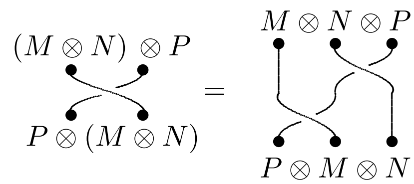


is a U-module isomorphism. For U modules M, N, P and a U-module isomorphism $\tau_M\colon M\to M$,



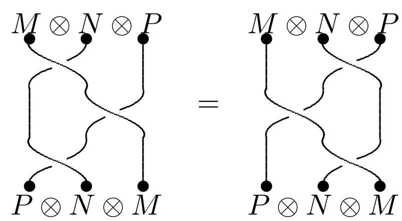
$$\check{R}_{MN}(\mathrm{id}_N\otimes\tau_M)=(\tau_M\otimes\mathrm{id}_N)\check{R}_{MN},$$





$$\check{R}_{M,N\otimes P}=(\check{R}_{MN}\otimes\mathrm{id}_P)(\mathrm{id}_N\otimes\check{R}_{MP})\qquad\qquad \check{R}_{M\otimes N,P}=(\mathrm{id}_M\otimes\check{R}_{NP})(\check{R}_{MP}\otimes\mathrm{id}_N),$$

which, together, imply the braid relation



$$(\check{R}_{MN}\otimes\mathrm{id}_P)(\mathrm{id}_N\otimes\check{R}_{MP})(\check{R}_{NP}\otimes\mathrm{id}_M)=(\mathrm{id}_M\otimes\check{R}_{NP})(\check{R}_{MP}\otimes\mathrm{id}_N)(\mathrm{id}_P\otimes\check{R}_{MN}).$$

If M is a U-module and

$$C_M\colon\quad \begin{array}{rcl} M & \longrightarrow & M \\ m & \longmapsto & vm \end{array}\qquad\text{then}\qquad C_{M\otimes N}=(\check{R}_{MN}\check{R}_{NM})^{-1}(C_M\otimes C_N), \tag{10}$$

by the last identity in (3).

Remark 1. [Dr, Prop. 3.1] Let $(U,\mathcal{R})$ be a quasitriangular Hopf algebra. Then $\mathcal{R}$ satisfies the *quantum Yang-Baxter equation*,

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23}=\mathcal{R}_{12}(\Delta\otimes\mathrm{id})(\mathcal{R})=(\Delta^{\mathrm{op}}\otimes\mathrm{id})(\mathcal{R})\mathcal{R}_{12}=\mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{11}$$

Since $\mathcal{R}=(\varepsilon\otimes\mathrm{id}\otimes\mathrm{id})(\Delta\otimes\mathrm{id})(\mathcal{R})=(\varepsilon\otimes\mathrm{id}\otimes\mathrm{id})\mathcal{R}_{13}\mathcal{R}_{23}=(\varepsilon\otimes\mathrm{id})(\mathcal{R})\cdot\mathcal{R}$,
and $\mathcal{R}=(\mathrm{id}\otimes\mathrm{id}\otimes\varepsilon)(\mathrm{id}\otimes\Delta)(\mathcal{R})=(\mathrm{id}\otimes\mathrm{id}\otimes\varepsilon)\mathcal{R}_{13}\mathcal{R}_{12}=(\mathrm{id}\otimes\varepsilon)(\mathcal{R})\cdot\mathcal{R}$,
it follows that

$$(\varepsilon\otimes\mathrm{id})(\mathcal{R})=1\qquad\text{and}\qquad(\mathrm{id}\otimes\varepsilon)(\mathcal{R})=1. \tag{12}$$

Since $\mathcal{R}(S\otimes\mathrm{id})(\mathcal{R})=(m\otimes\mathrm{id})(\mathrm{id}\otimes S\otimes\mathrm{id})(\mathcal{R}_{13}\mathcal{R}_{23})=(m\otimes\mathrm{id})(\mathrm{id}\otimes S\otimes\mathrm{id})(\Delta\otimes\mathrm{id})(\mathcal{R})=(\varepsilon\otimes\mathrm{id})(\mathcal{R})=1$, it follows that $(S\otimes\mathrm{id})(\mathcal{R})=\mathcal{R}^{-1}$, and applying this relation to the quasitriangular Hopf algebra $(U^{\mathrm{op}},\mathcal{R}_{21})$ with antipode S^{-1} gives $(\mathrm{id}\otimes S^{-1})(\mathcal{R})=\mathcal{R}^{-1}$, and thus $(S\otimes S)(\mathcal{R})=(\mathrm{id}\otimes S)(S\otimes\mathrm{id})(\mathcal{R})=(\mathrm{id}\otimes S)(\mathrm{id}\otimes S^{-1})(\mathcal{R})=\mathcal{R}$. Summarizing,

$$(S\otimes\mathrm{id})(\mathcal{R})=(\mathrm{id}\otimes S^{-1})(\mathcal{R})=\mathcal{R}^{-1}\qquad\text{and}\qquad(S\otimes S)(\mathcal{R})=\mathcal{R}. \tag{13}$$

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