

Conditional Probability

6/30/2006

Example

Three candidates A , B , and C are running for office. We decided that A and B have an equal chance of winning and C is only $1/2$ as likely to win as A . Let A be the event “ A wins,” B that “ B wins,” and C that “ C wins.” Hence, we assigned probabilities $P(A) = 2/5$, $P(B) = 2/5$, and $P(C) = 1/5$.

Suppose that before the election is held, A drops out of the race. What are the values for $P(B|\tilde{A})$ and $P(C|\tilde{A})$?

Definition

Let $\Omega = \{\omega_1, \omega_2, \dots, \omega_r\}$ be the original sample space with distribution function $m(\omega_j)$ assigned. Suppose we learn that the event E has occurred.

- If a sample point ω_j is not in E , we want $m(\omega_j|E) = 0$.
- For ω_k in E , we should have the same relative magnitudes that they had before we learned that E had occurred:

$$m(\omega_k|E) = c m(\omega_k).$$

Definition ...

But we must also have

$$\sum_E m(\omega_k|E) = c \sum_E m(\omega_k) = 1 .$$

Thus,

$$c = \frac{1}{\sum_E m(\omega_k)} = \frac{1}{P(E)} .$$

Definition ...

Definition 1. *The conditional distribution given E is the distribution on Ω defined by*

$$m(\omega_k|E) = \frac{m(\omega_k)}{P(E)}$$

for ω_k in E , and $m(\omega_k|E) = 0$ for ω_k not in E .

Then, for a general event F ,

$$P(F|E) = \sum_{F \cap E} m(\omega_k|E) = \sum_{F \cap E} \frac{m(\omega_k)}{P(E)} = \frac{P(F \cap E)}{P(E)} .$$

We call $P(F|E)$ the *conditional probability of F occurring given that E occurs*.

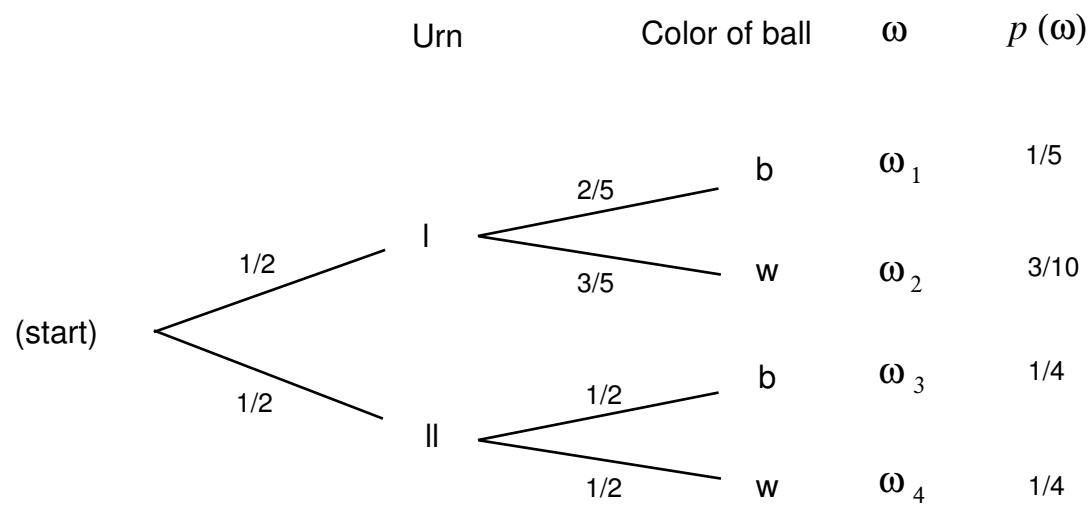
Example

Let us return to the example of rolling a die. Recall that F is the event $X = 6$, and E is the event $X > 4$. Note that $E \cap F$ is the event F . So, the above formula gives

$$\begin{aligned} P(F|E) &= \frac{P(F \cap E)}{P(E)} \\ &= \frac{1/6}{1/3} \\ &= \frac{1}{2} . \end{aligned}$$

Example

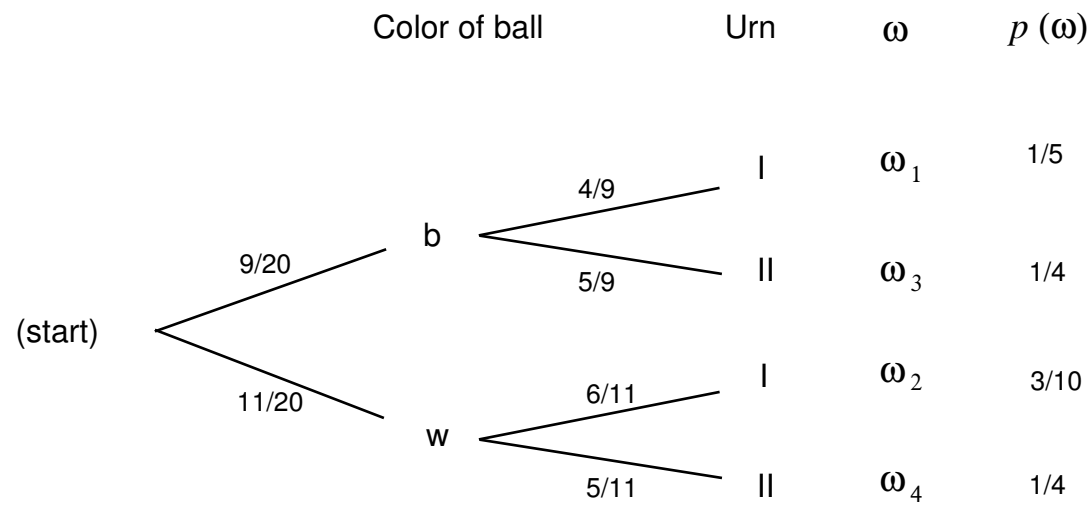
We have two urns, I and II. Urn I contains 2 black balls and 3 white balls. Urn II contains 1 black ball and 1 white ball. An urn is drawn at random and a ball is chosen at random from it. We can represent the sample space of this experiment as the paths through a tree.



- Let B be the event “a black ball is drawn,” and I the event “urn I is chosen.” Then the branch weight $2/5$, which is shown on one branch in the figure, can now be interpreted as the conditional probability $P(B|I)$.
- What is $P(I|B)$?

Bayes Probabilities

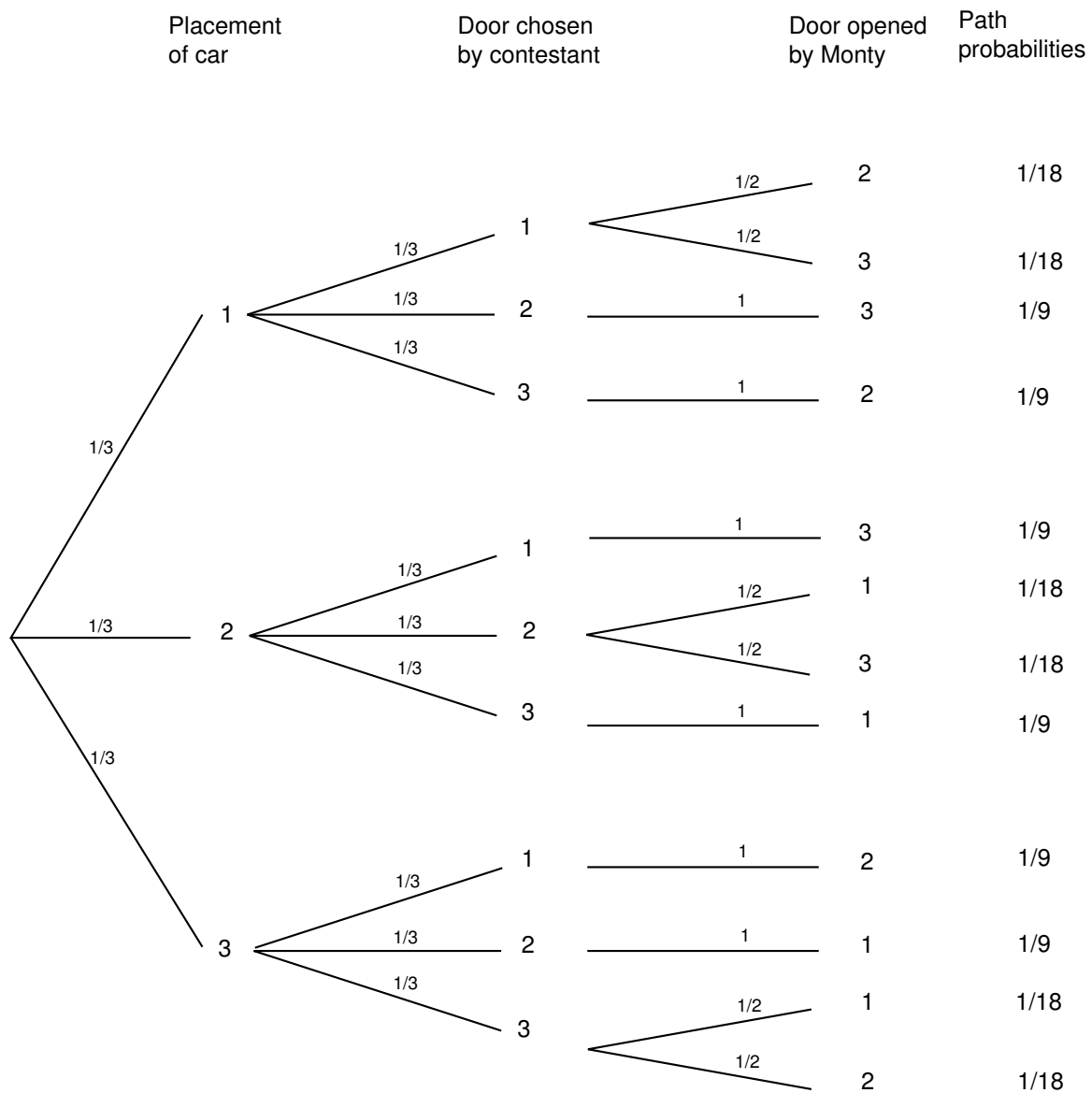
We have just calculated the *inverse probability* that a particular urn was chosen, given the color of the ball. Such an inverse probability is called a *Bayes probability*.



The Monty Hall problem

Suppose you're on Monty Hall's *Let's Make a Deal!* You are given the choice of three doors, behind one door is a car, the others, goats. You pick a door, say 1, Monty opens another door, say 3, which has a goat. Monty says to you "Do you want to pick door 2?" Is it to your advantage to switch your choice of doors?

Question: What is the conditional probability that you win if you switch, given that you have chosen door 1 and that Monty has chosen door 3.



Problem

Assume that E and F are two events with positive probabilities. Show that if $P(E|F) = P(E)$, then $P(F|E) = P(F)$.

In this case, we say that the events E and F are independent.

It is also equivalent to the fact that $P(E \cap F) = P(E)P(F)$

Problem

A die is rolled twice. What is the probability that the sum of the faces is greater than 7, given that

1. the first outcome was a 4?
2. the first outcome was greater than 3?
3. the first outcome was a 1?
4. the first outcome was less than 5?