Numerical study of the asymptotic rate of quantum ergodicity in chaotic billiards

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Thanks to P. Sarnak (NYU), S. Zelditch (JHU),
N. Trefethen (Oxford), T. Betcke (Braunschweig)

Dirichlet eigenproblem

Normal modes of elastic membrane or 'drum' (Helmholtz, Germain, 19^{th}C)
Eigenfunctions ϕ_j of Laplacian $\Delta := \partial_{x_1}^2 + \partial_{x_2}^2$ in bounded domain $\Omega \subset \mathbb{R}^2$

$-\Delta\phi_j = E_j\phi_j$ $\qquad$ $\phi_j|_{\partial\Omega} = 0$ Dirichlet BC $\qquad$ $\langle \phi_i, \phi_j \rangle = \delta_{ij}$

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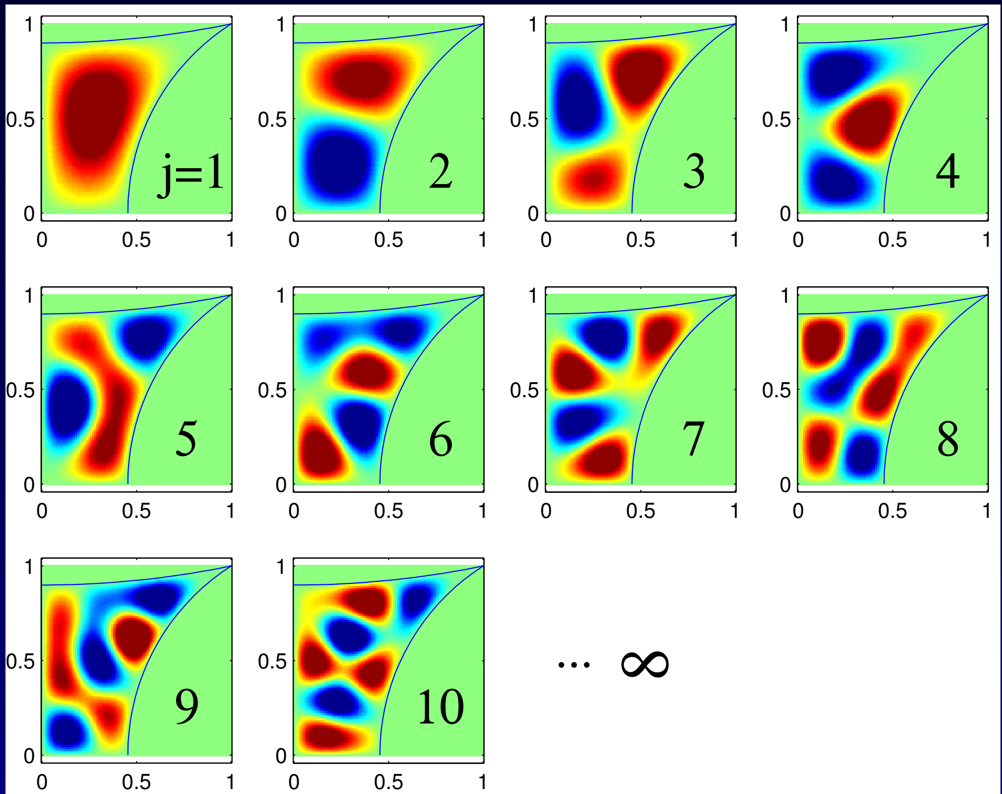

'energy' eigenvalues
$E_1 < E_2 \le E_3 \le \cdots$

wavenumber $k_j = E_j^{1/2}$

wavelength $= 2\pi/k_j$

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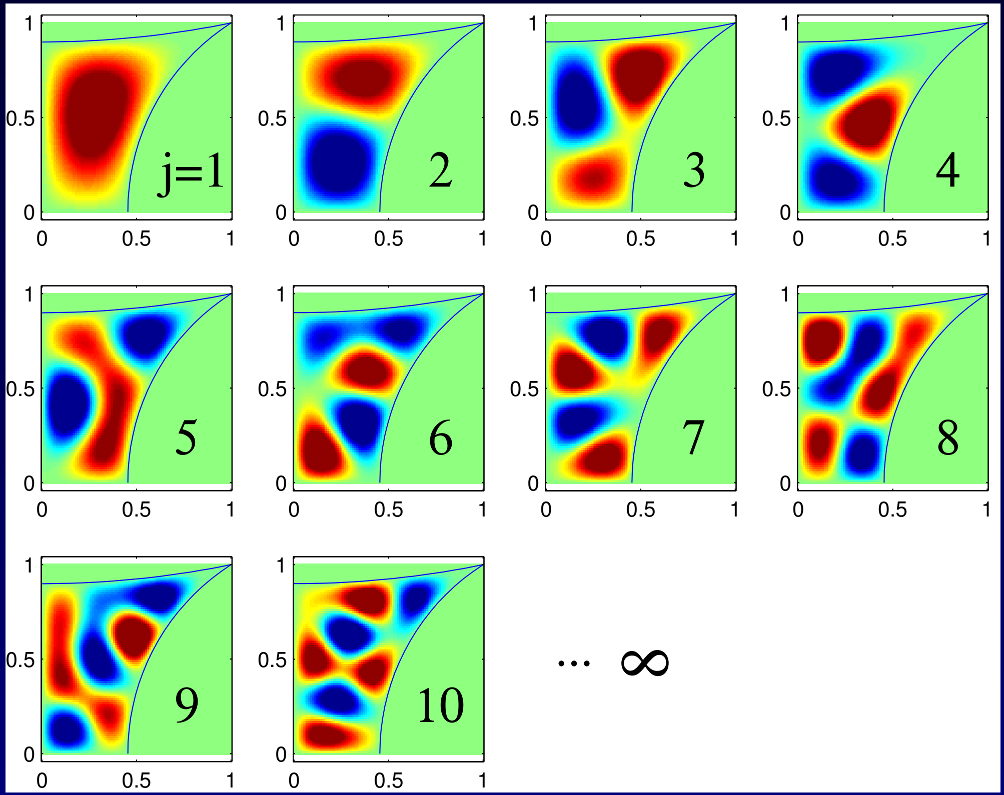

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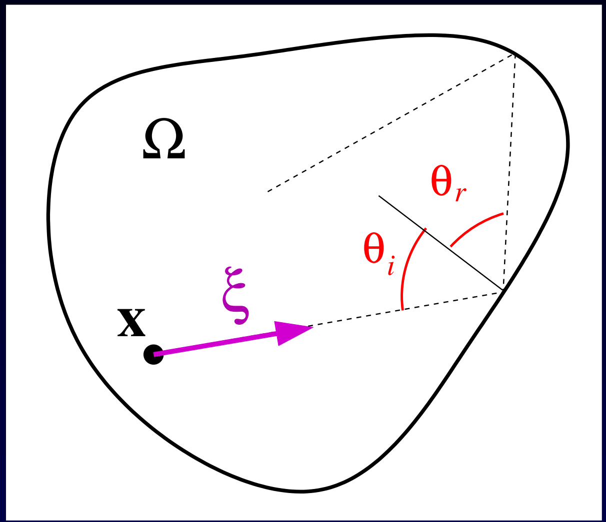
- Time-harmonic solns of wave eqn (acoustics, optics, quantum, etc)
- Asymptotics of ϕ_j as eigenvalue $E_j \to \infty$? Depends on shape…

Outline

- Eigenproblem (quantum)
- Billiard system (classical)
- ‘Quantum chaos’ conjectures and numerical results on
  - exceptional ‘scarred’ eigenmodes
  - rate at which eigenmodes become spatially uniform
- Numerical methods sketch if time

Billiards (the classical analogue)



state $z := (x_1, x_2, \xi_1, \xi_2)$
$\in Z := \Omega \times \mathbb{R}^2$, phase space

free motion (geodesic flow)

trajectory $z_0(t) = G_t z_0$

Hamiltonian dynamical system: $H(x, \xi) = |\xi|^2$ conserved

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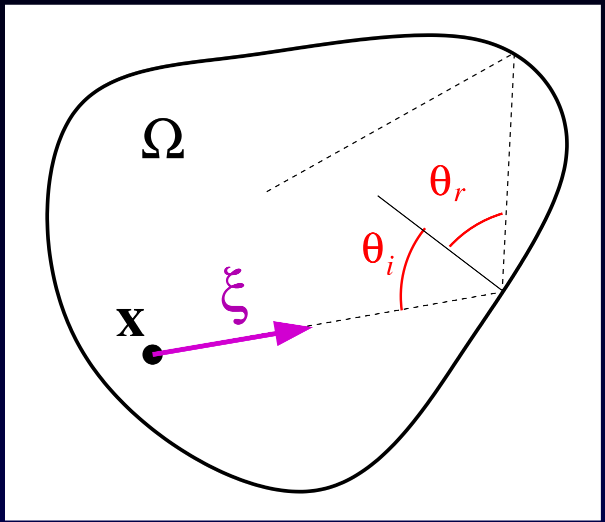

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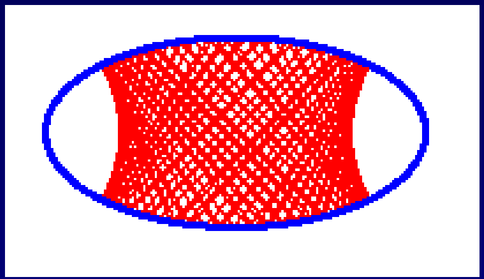
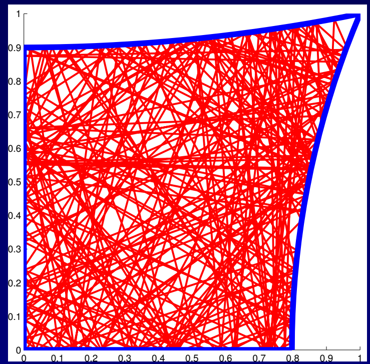
integrable: 

d conserved quantities (d=2)

ergodic: 

only H conserved

Properties of ergodic billiards

Ergodic: $\forall A \in L^1(Z)$, for almost every $z_0 \in Z$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T A(z_0(t))dt \; = \; \frac{1}{\mathrm{vol}(Z)} \int_Z A(z)dz =: \overline{A}$$

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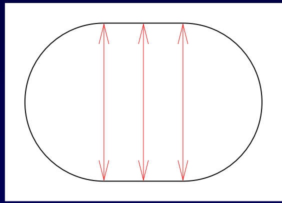
- 'hyperbolic': $|z_1(t) - z_2(t)| \sim ce^{\Lambda t}$ $\quad 0 < \Lambda =$ Lyapunov exponent
- 'Anosov': uniformly hyperbolic (all periodic orbits unstable, isolated)

e.g. Bunimovich stadium ergodic but not Anosov…



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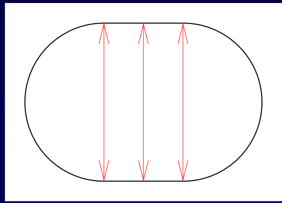

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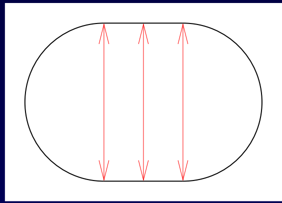

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QUANTUM CHAOS: what are eigenmodes when flow ergodic? (Einstein 1917)

Modes ϕ_j are irregular: $\boxed{\text{VIEW } j \sim 3000\text{: 45 wavelengths across}}$

Quantum Ergodicity Theorem (QET)

For Ω with ergodic flow, and 'well-behaved' $A : \Omega \to \mathbb{R}$,

$$\lim_{E_j \to \infty} \langle \phi_j, A\phi_j \rangle - \overline{A} \; = \; 0 \qquad \forall j \text{ except subseq. of vanishing density}$$

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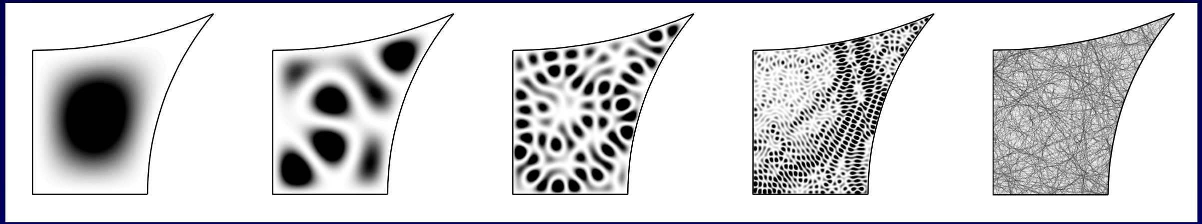

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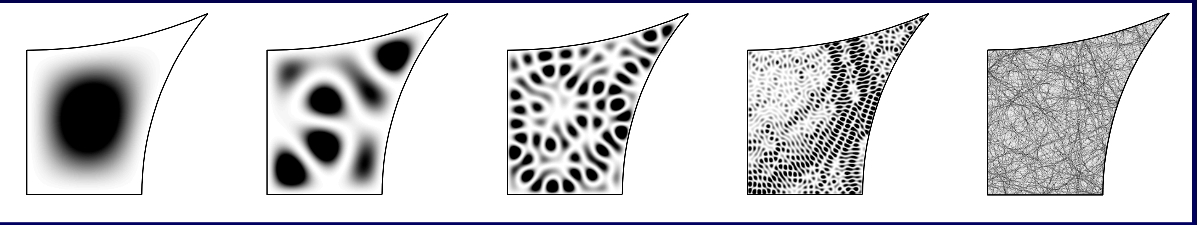

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At what rate is limit reached?
At what rate does the density of excluded subsequence vanish?

Quantum Unique Ergodicity (QUE)

Conj. (Rudnick-Sarnak ’94): There is no excluded subsequence in QET

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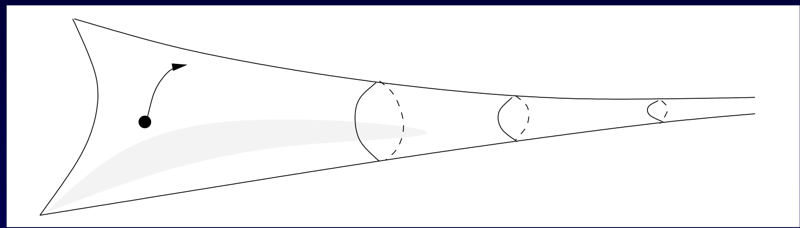

Recent analytic results:

- Proven to hold for *arithmetic* manifold $SL_2(\mathbb{Z}) \setminus \mathbb{H}$ (Lindenstrauss '03)
  ... very special system, symmetries, all $\Lambda = 1$
- Proven *not* to hold for some quantum Arnold cat maps (Faure et al '03)

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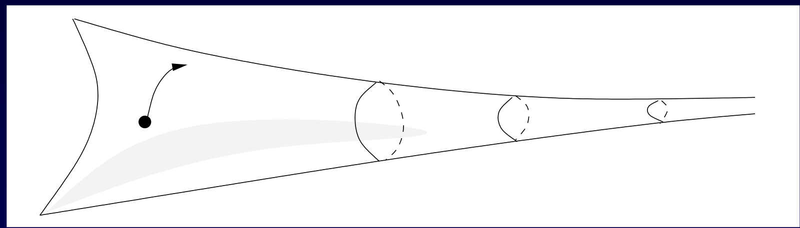

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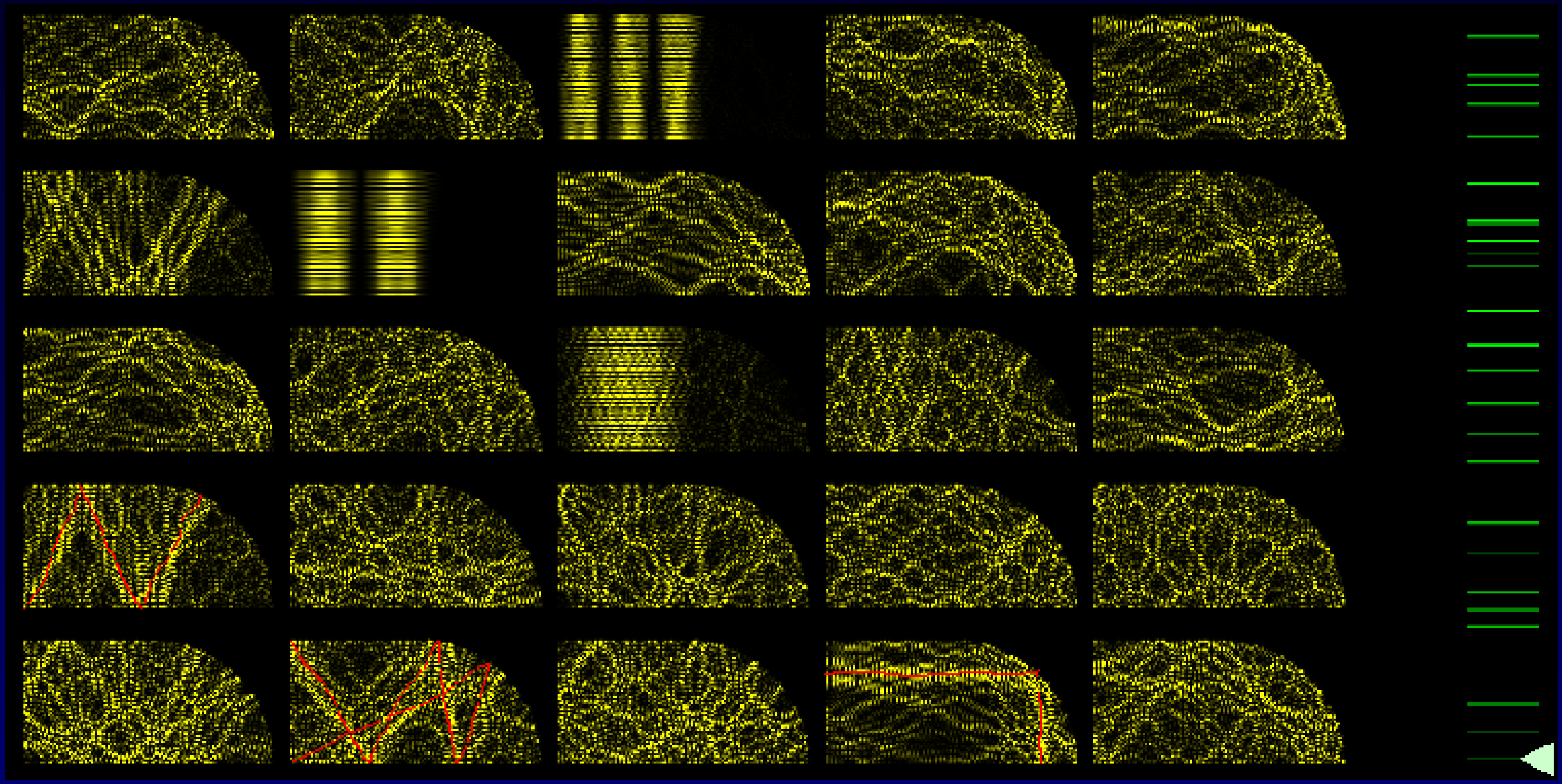
QUE might be a surprise for Euclidean cavity eigenmodes because…

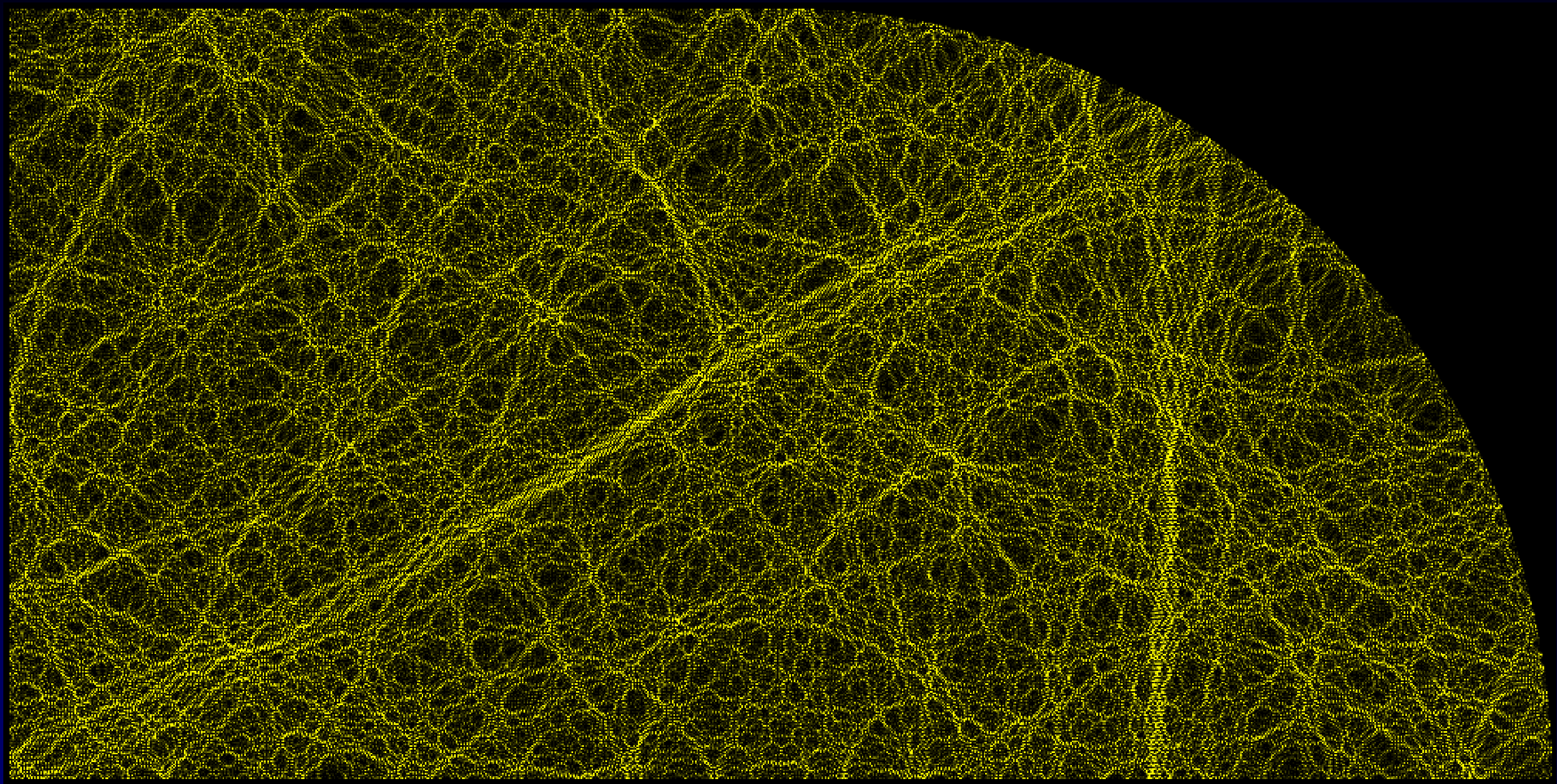
Scars: an exceptional subsequence?

high-j modes $|\phi_j|^2$ often localized on unstable periodic orbits (UPO)
e.g. numerical study of stadium eigenmodes (Heller '84)



- Wavepacket heuristics $\Rightarrow$ transverse width dies as $\sim E^{-1/4}$
- Note also exceptional 'bouncing ball' sequence (since not Anosov)

High-eigenvalue stadium scar



- ‘strong’ scar: $\exists$ sequence $\{\phi_{j_n}\}$ with $O(1)$ mass on UPO
- QUE implies ‘weak’ scarring: UPO mass is $o(1)$

Which occurs? No analytic results in Euclidean billiards…

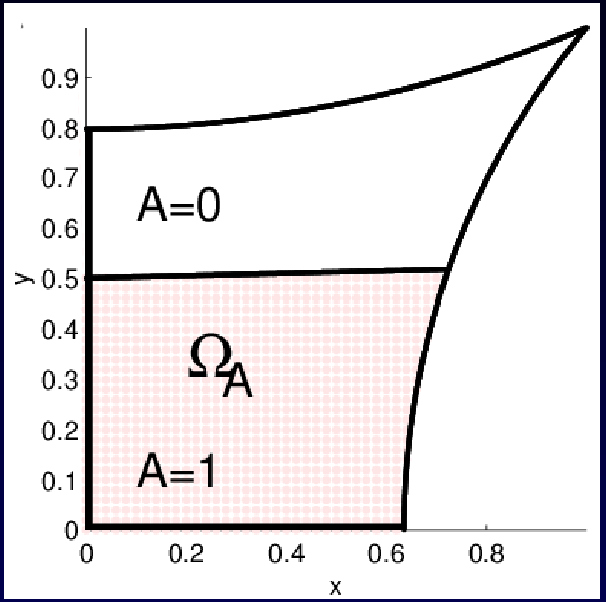
Numerical experiments

(B ’04, to appear CPAM ’06)

dispersing billiard, proven Anosov (Sinai ’70)

desymmetrized, generic Λ exponents

test function $A = \begin{cases} 1 & \text{in } \Omega_A, \\ 0 & \text{otherwise} \end{cases}$



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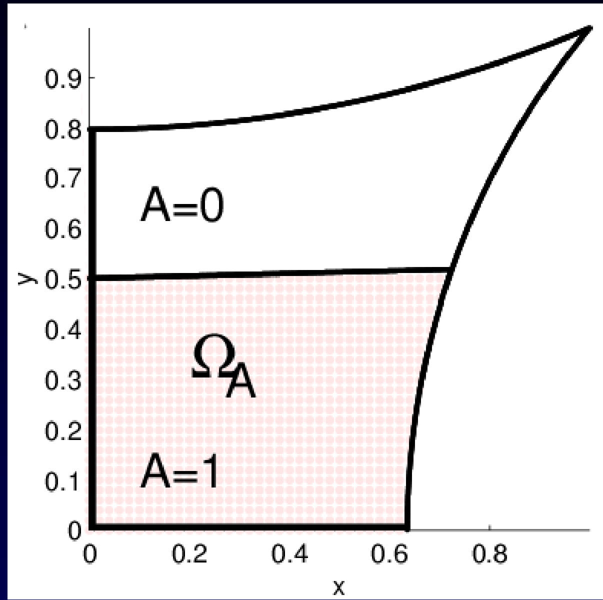


Large-scale study, 30,000 modes in range $j \sim 10^4$ to 10^6, enabled by:

1. Efficient boundary-based numerics for ϕ_j (‘scaling method’)
2. matrix elements $\langle \phi_j, A\phi_j \rangle = \int_{\Omega_A} \phi_j^2 dx$, using integrals on $\partial\Omega_A$

- 100 times higher in j than any previous studies (*e.g.* Bäcker ’98)
- only a few CPU-days total

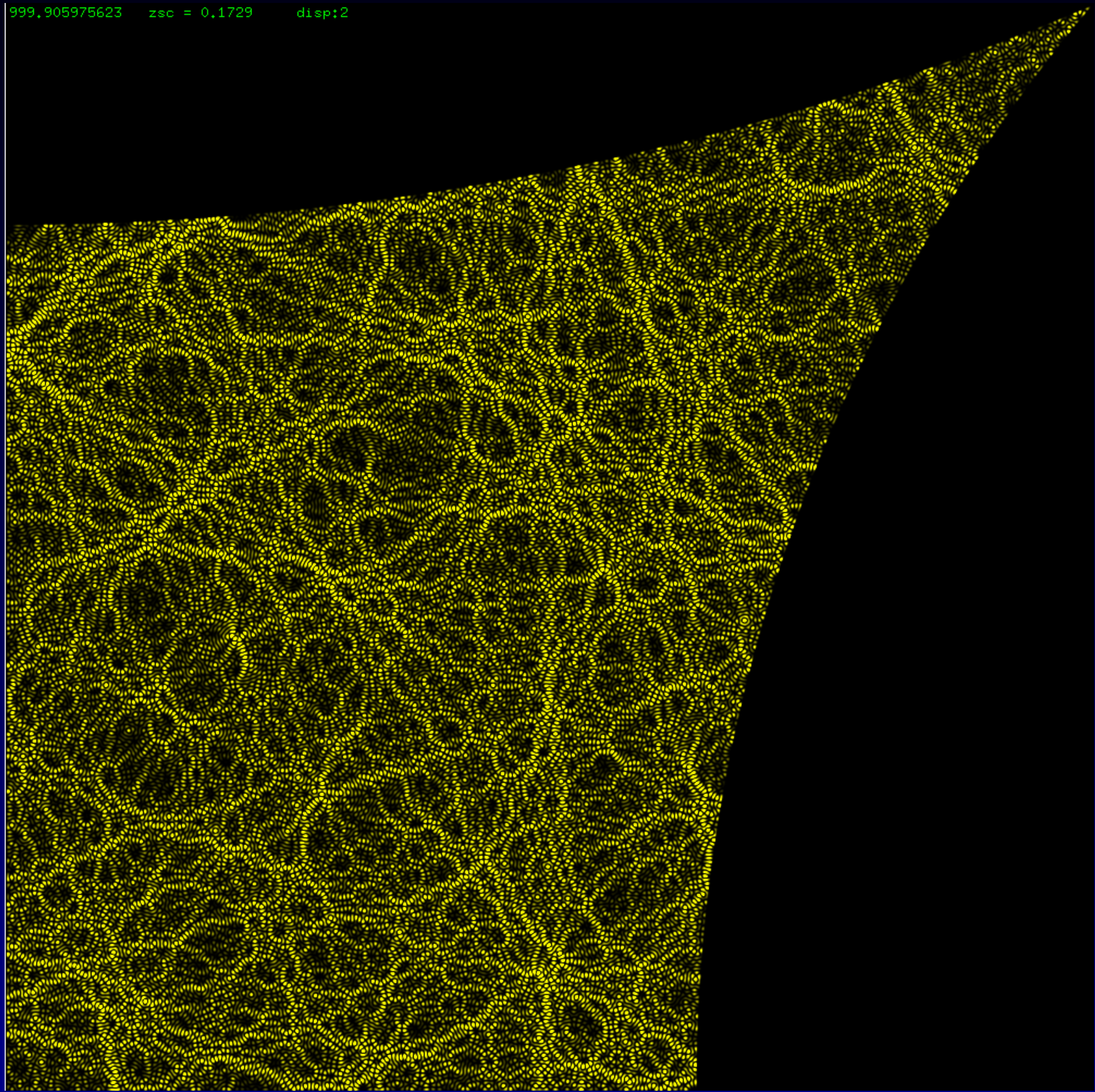
Typical high-frequency mode

225 wavelengths across system

level number $j \approx 5 \times 10^4$

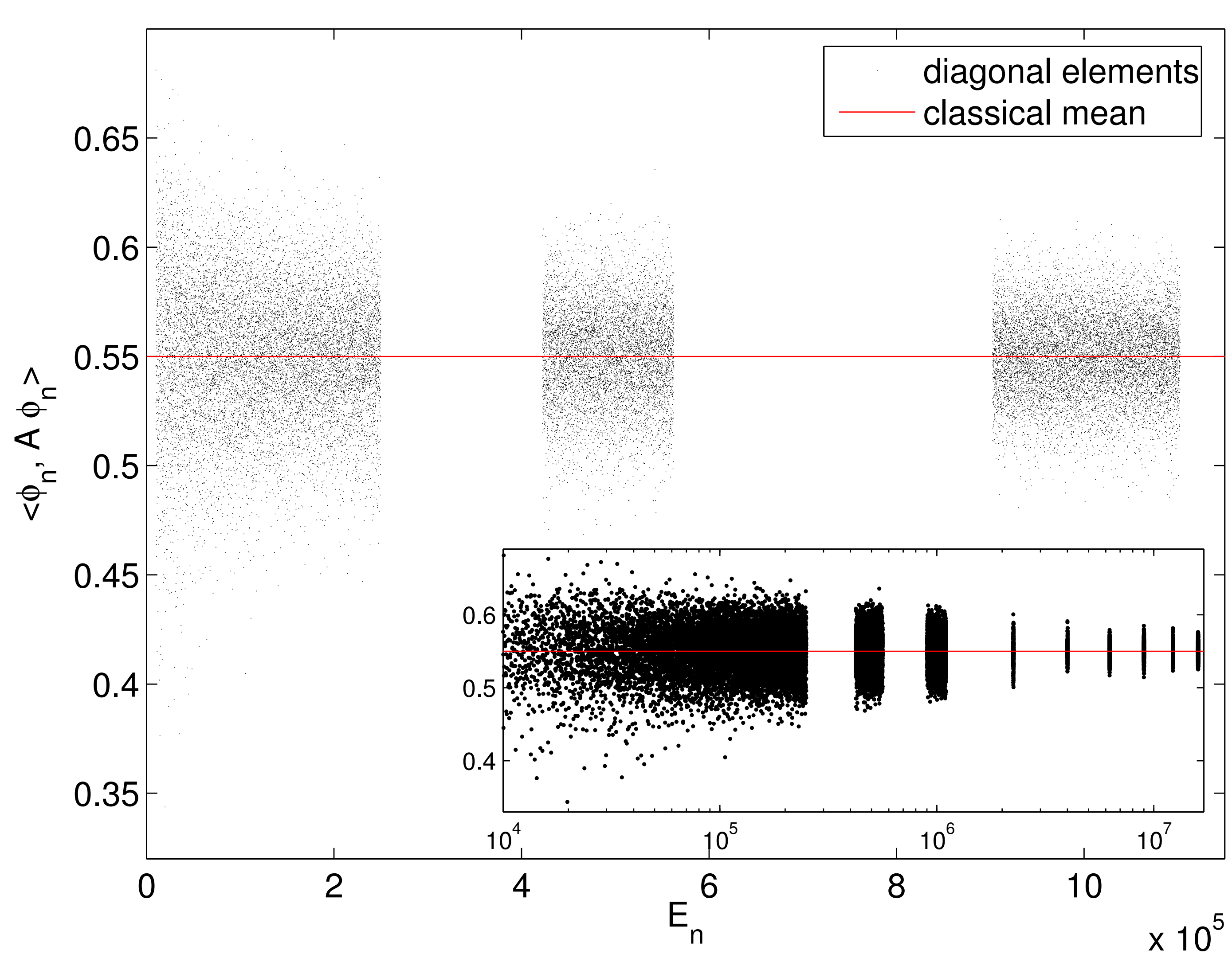
$E_j \approx 10^6$

Note: making this plot is much more expensive than finding $\int_{\Omega_A} \phi_j^2 dx$



Raw diagonal matrix element data

To reach high E, compute ϕ_j only in intervals $E_j \in [E, E + L(E)]$



- strong evidence for QUE (exceptional density $< 3 \times 10^{-5}$)

Rate of quantum ergodicity

energy interval $I_E = [E, E + L(E)]$, width scales as $L(E) = O(E^{1/2})$
mode count in interval $N(I_E) := \#\{j : E_j \in I_E\}$

'quantum variance' $V_A(E) := \dfrac{1}{N(I_E)} \sum_{E_j \in I_E} \left| \langle \phi_j, A\phi_j \rangle - \overline{A} \right|^2$

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Conj. (Feingold-Peres '86): $$V_A(E) \sim \frac{g\tilde{C}_A(0)}{\mathrm{vol}(\Omega)} E^{-1/2}$$

where $g = 2$, as in random matrix theory (RMT), GOE symmetry

and $\tilde{C}_A(\omega) := \int_{-\infty}^{\infty} C_A(\tau) e^{i\omega\tau} d\tau$

Heuristic argument relies on two steps:

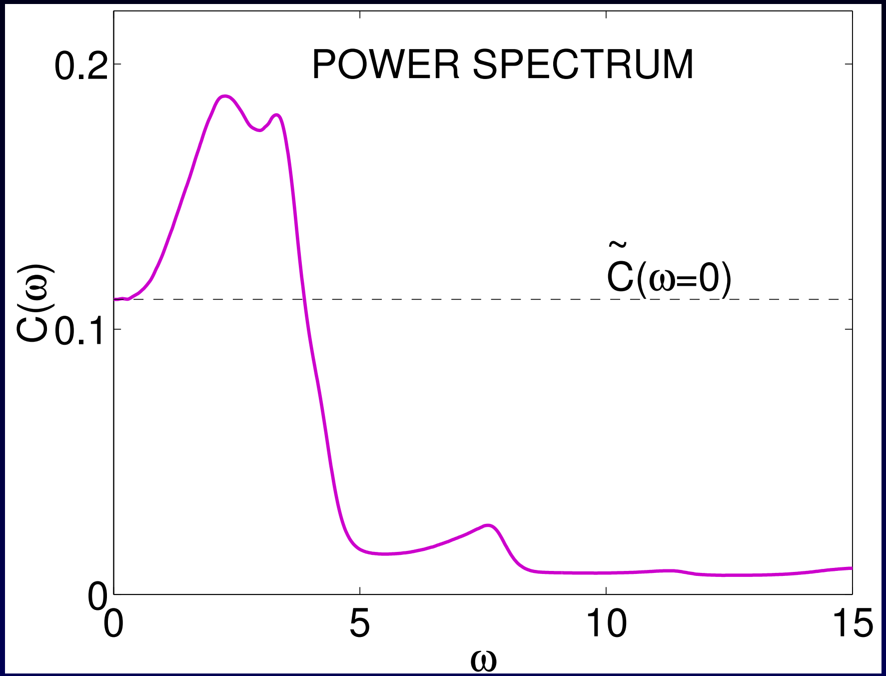
- QET for operator $A(0)A(\tau)$: off-diagonal variance given by $\tilde{C}_A(\omega)$
- assume statistical independence of ϕ_j within I_E (as in RMT)

Classical variance

$$\tilde{C}_A(\omega) := \int_{-\infty}^{\infty} C_A(\tau) e^{i\omega\tau} d\tau$$

FT of autocorrelation = power spectral density of signal $A(z_0(t))$ along ergodic trajectory

LISTEN to it!



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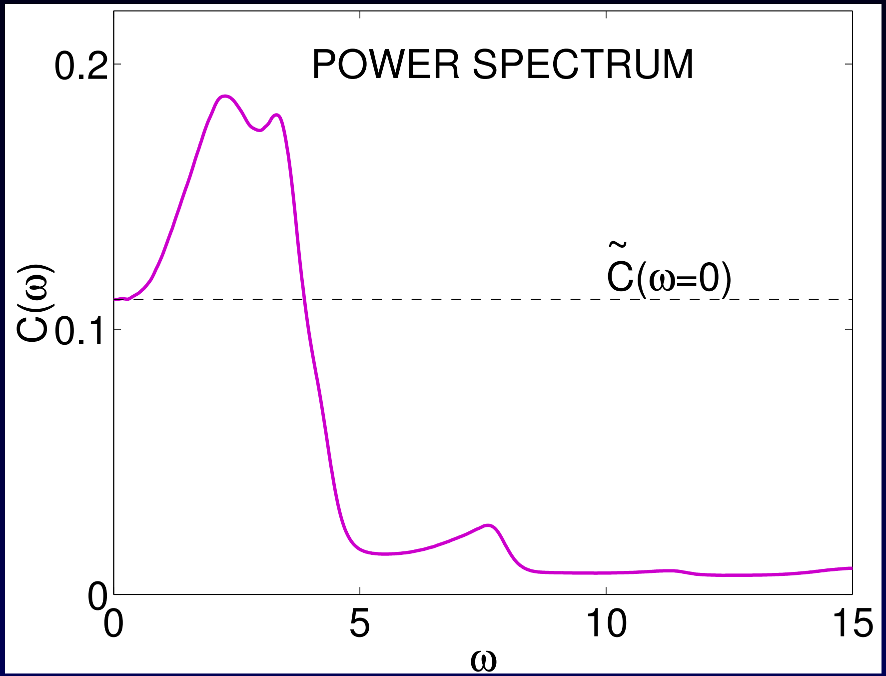


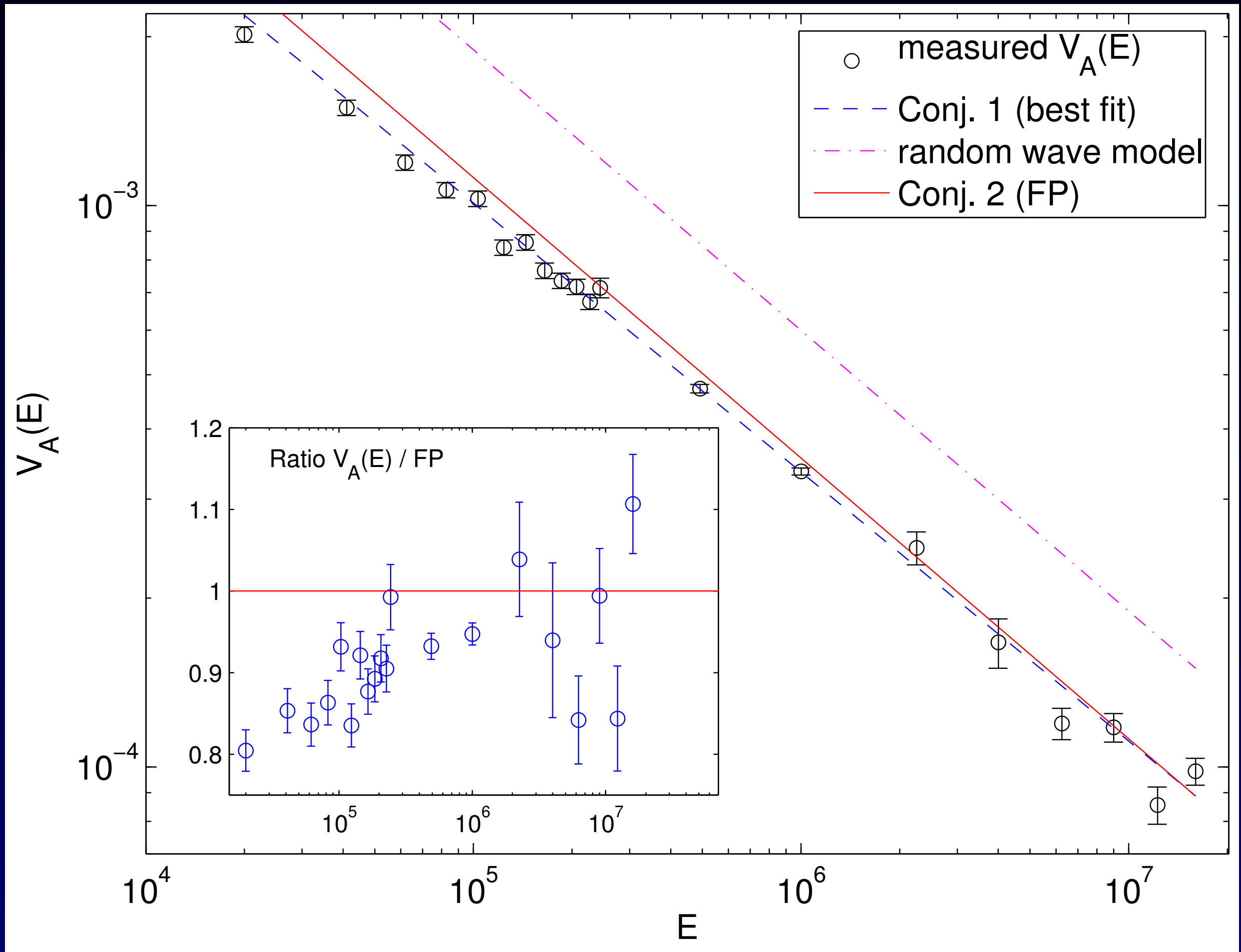
Physically: $\tilde{C}_A(\omega)$ is heating rate due to periodic driving of classical system, at frequency ω, by external field A (Cohen '99, B '00)

FP Conj.: $\omega \to 0$ limit gives prefactor for quantum ergodicity rate

- Note Ω symmetric, A antisymm: conj. fails, $V_A(E) = 0$ (Sarnak '03)

Does it hold in our nonsymmetric case?

Results on ergodicity rate



consistent with power law model $V_A(E) = aE^{-\gamma}$

fit $\gamma = 0.48 \pm 0.01$

$\gamma = 1/2$ agrees with

- FP conjecture
- random-wave model
- scar theory

- large numbers of modes $\longrightarrow$ unprecedented accuracy ($< 1\%$)
- consistent with FP prefactor, convergence slow: 7% off at $j = 10^5$
- asymptotic regime seen for first time (but more data needed!)

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Task: find ϕ_j and E_j such that $(\Delta + E_j)\phi_j = 0$ and $\phi_j|_{\partial\Omega} = 0$

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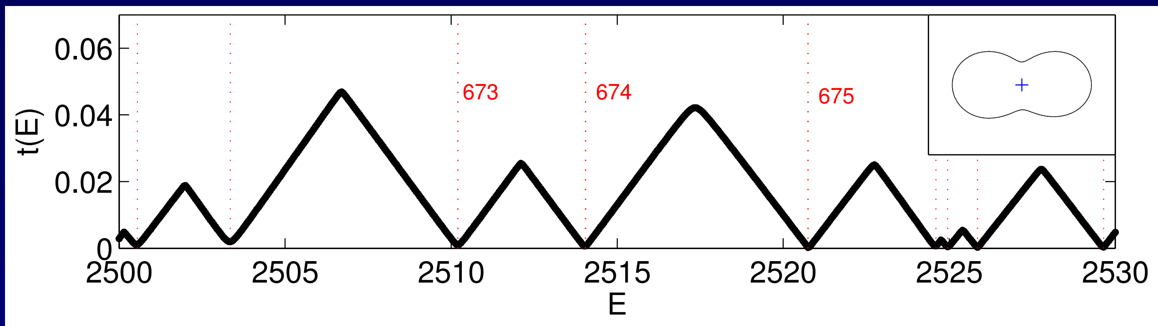
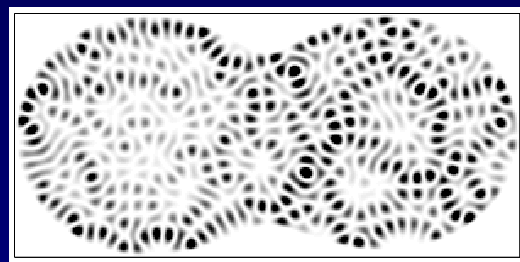
'boundary error' $t(E) := \min_{u \in \text{Span}\{\xi_i\}, u \neq 0} \frac{||u||_{L^2(\partial\Omega)}}{||u||_{L^2(\Omega)}}$, (Rayleigh quotient)



'peanut' cavity, $j \approx 700$



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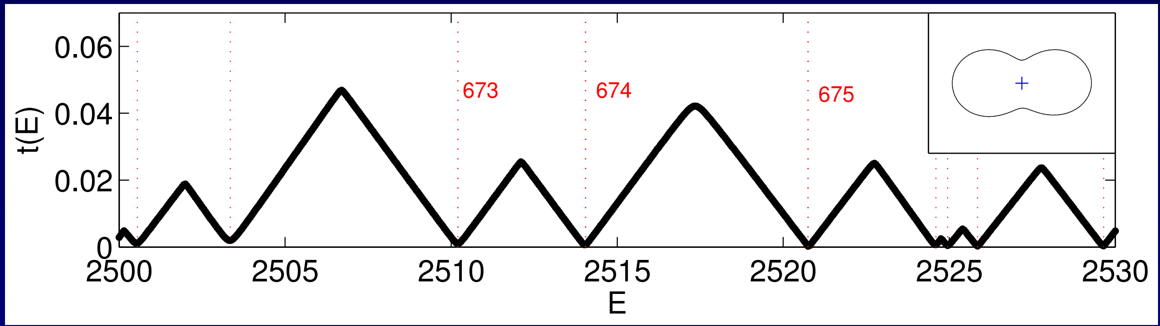

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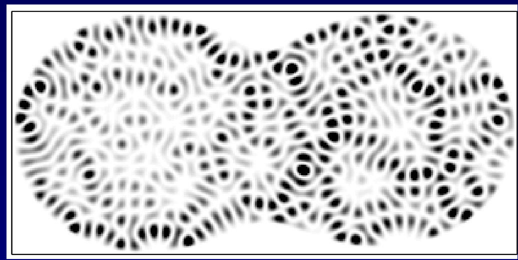


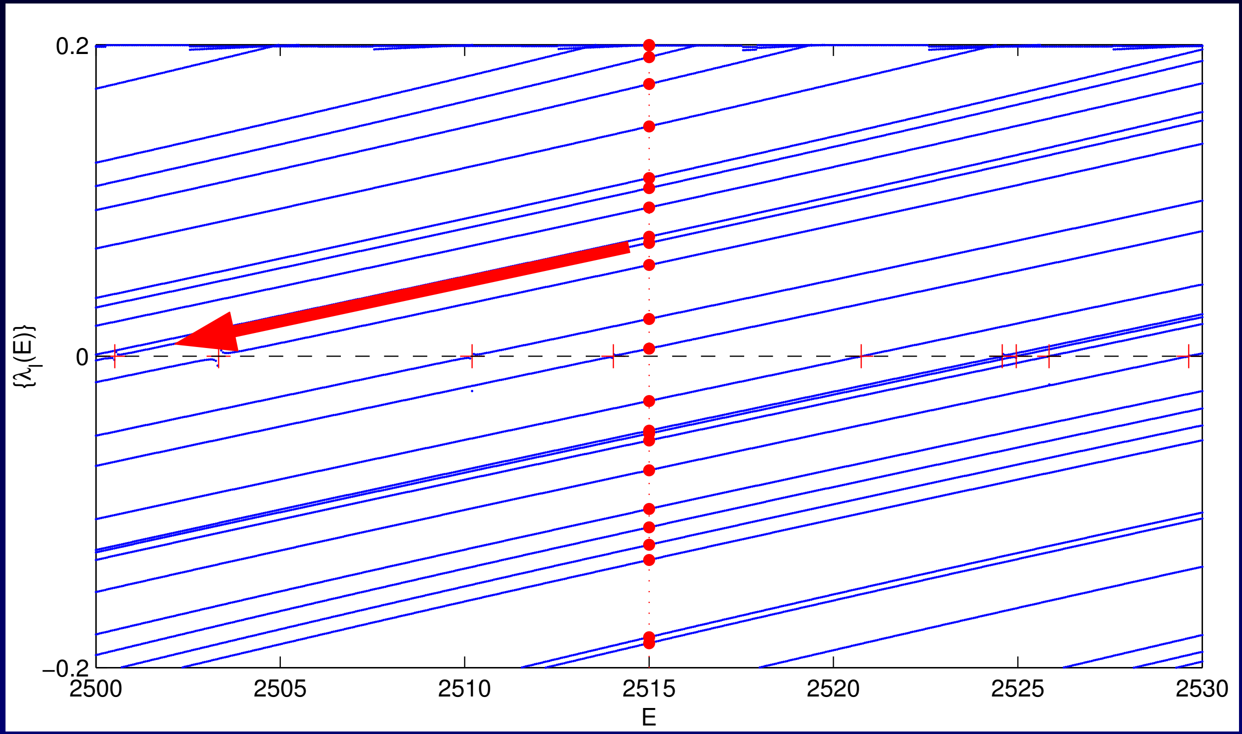
- $t(E)$ is lowest generalized eigenvalue of pair of E-dep. matrices
- finally search in E for minima of $t(E)$

Scaling method

(Vergini '94, B '00, '04)

new Rayleigh quotient $\frac{f(u)}{g(u)}$ where
$$f(u,v) := \int_{\partial\Omega} (\mathbf{x}\cdot\mathbf{n})^{-1} uv,$$
$$g(u,v) := \int_{\partial\Omega} (\mathbf{x}\cdot\mathbf{n})^{-1} (u\mathbf{x}\cdot\nabla v + v\mathbf{x}\cdot\nabla u)$$

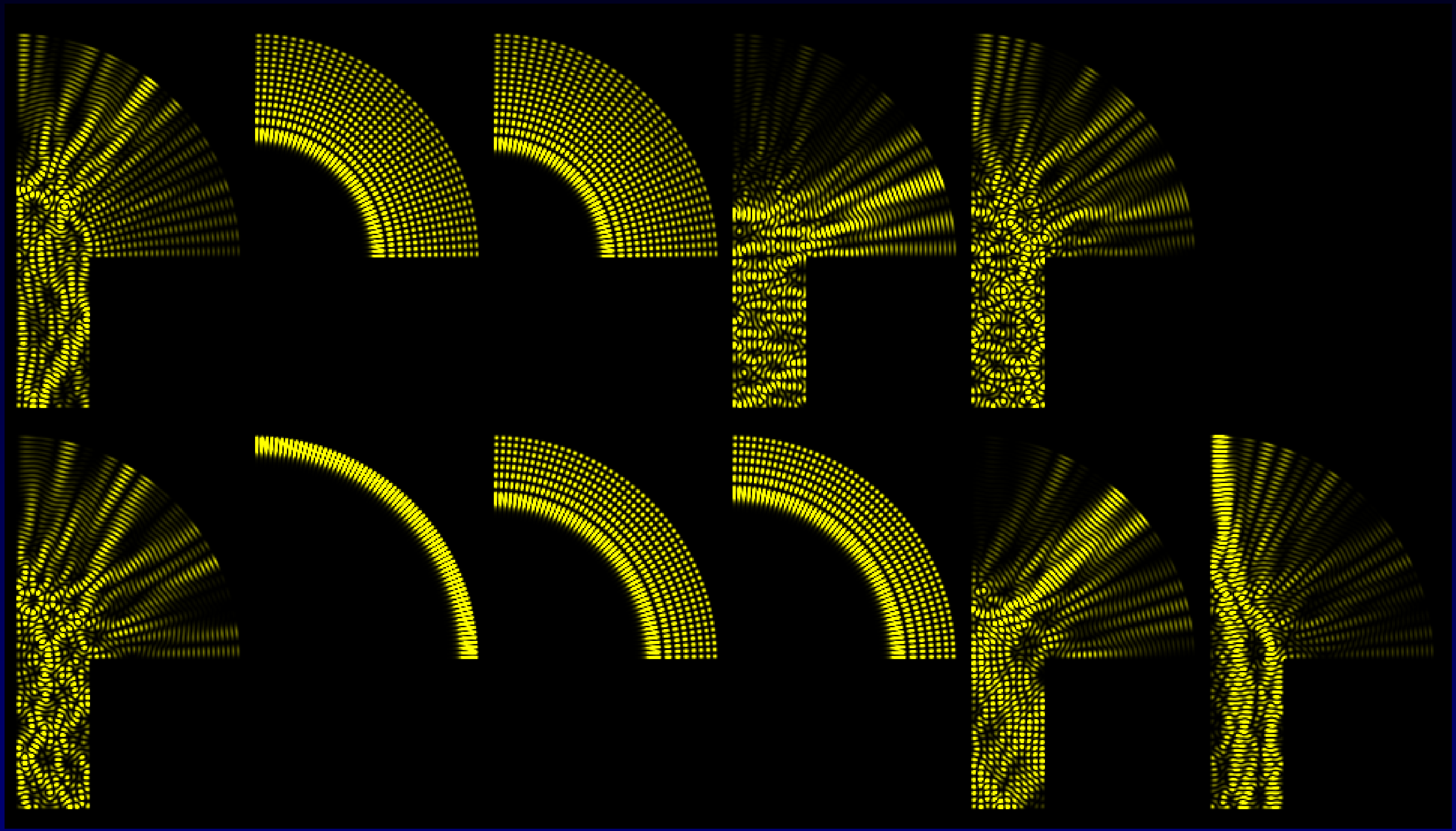
Generalized eigenvalues now approximately linear in $E - E_j$:



- solving $F\mathbf{x} = \hat{\lambda} G\mathbf{x}$ at single E value gives *all* nearest $O(N)$ modes
- no root search, no missing levels, efficiency gain $O(E^{1/2})$
- eigenvectors $\mathbf{x}$ give *dilated* (scaled) approximations to modes ϕ_j

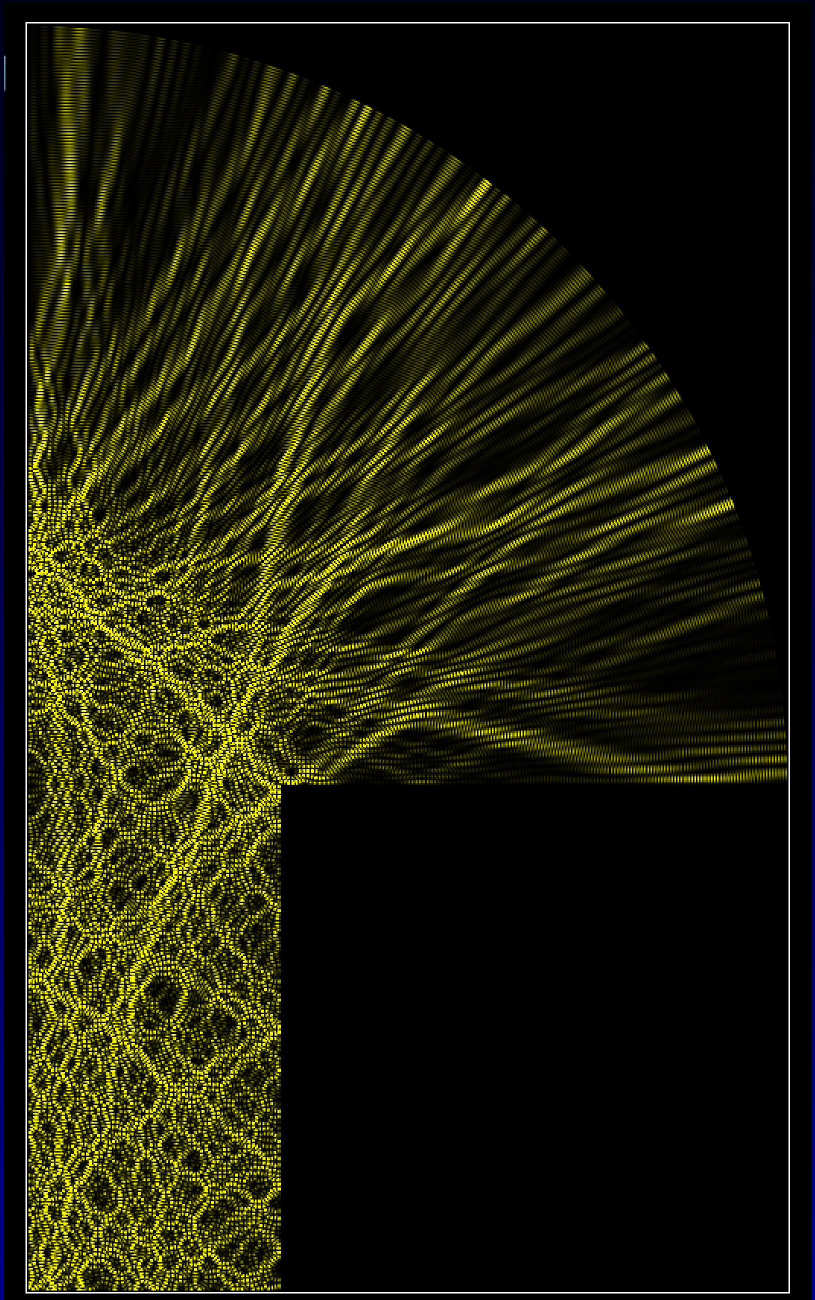
Mushroom billiard (mixed dynamics)

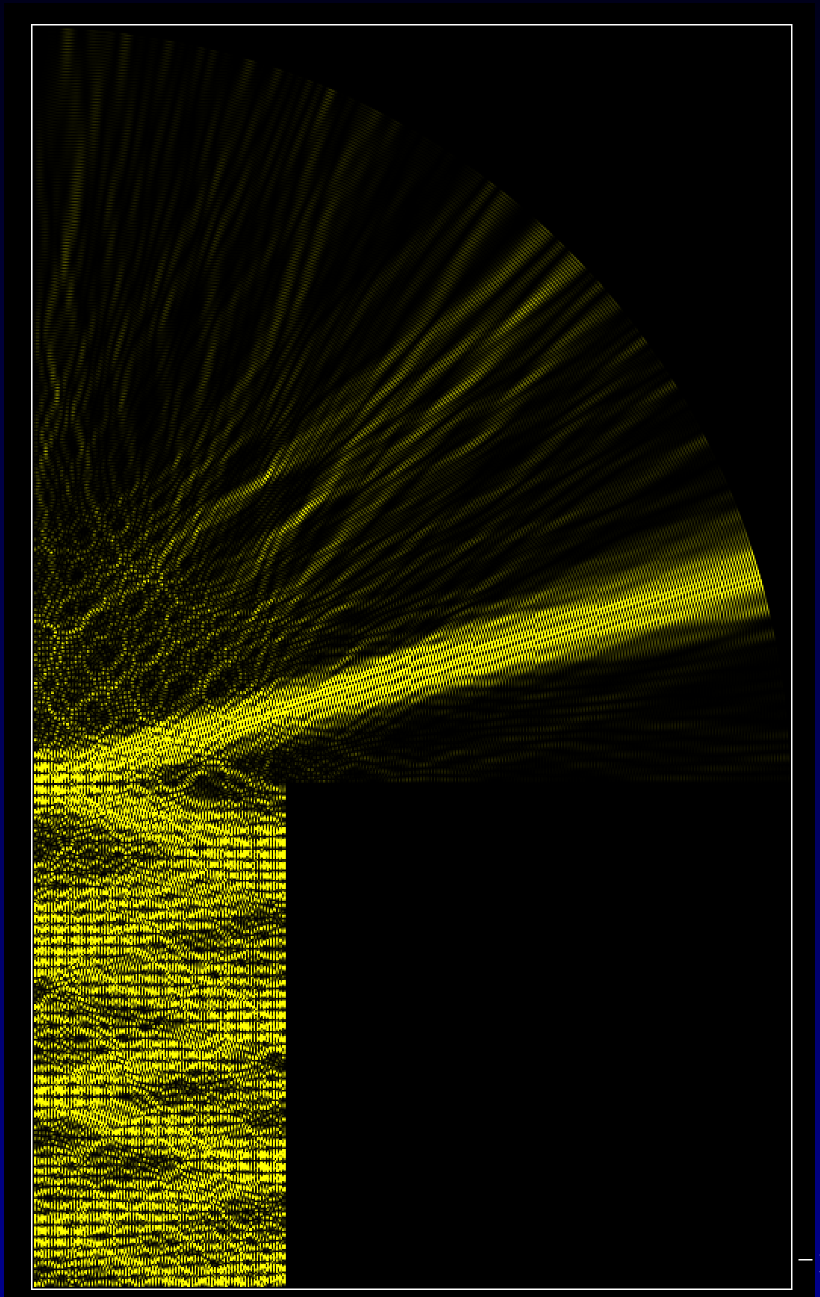
Inspired by AMS Notices March '06, Betcke & I computed. . .



- Due to re-entrant corner, Fourier-Bessel basis choice non-trivial
- We believe first high eigenmodes found for mushroom (last week)

High energy mushroom billiard





Conclusions

Studied nonsymmetric Anosov billiard eigenmodes…

- strong evidence for QUE
- slow asymptotic convergence to Feingold-Peres conjecture

Scaling method and boundary integral identities

- 10^3 faster than any other known method for eigenmodes
- cheaply get large number of high-eigenvalue matrix elements

Ongoing:

- bouncing-ball mode statistics (with A. Hassell)
- basis functions, scaling method numerical analysis (with T. Betcke)

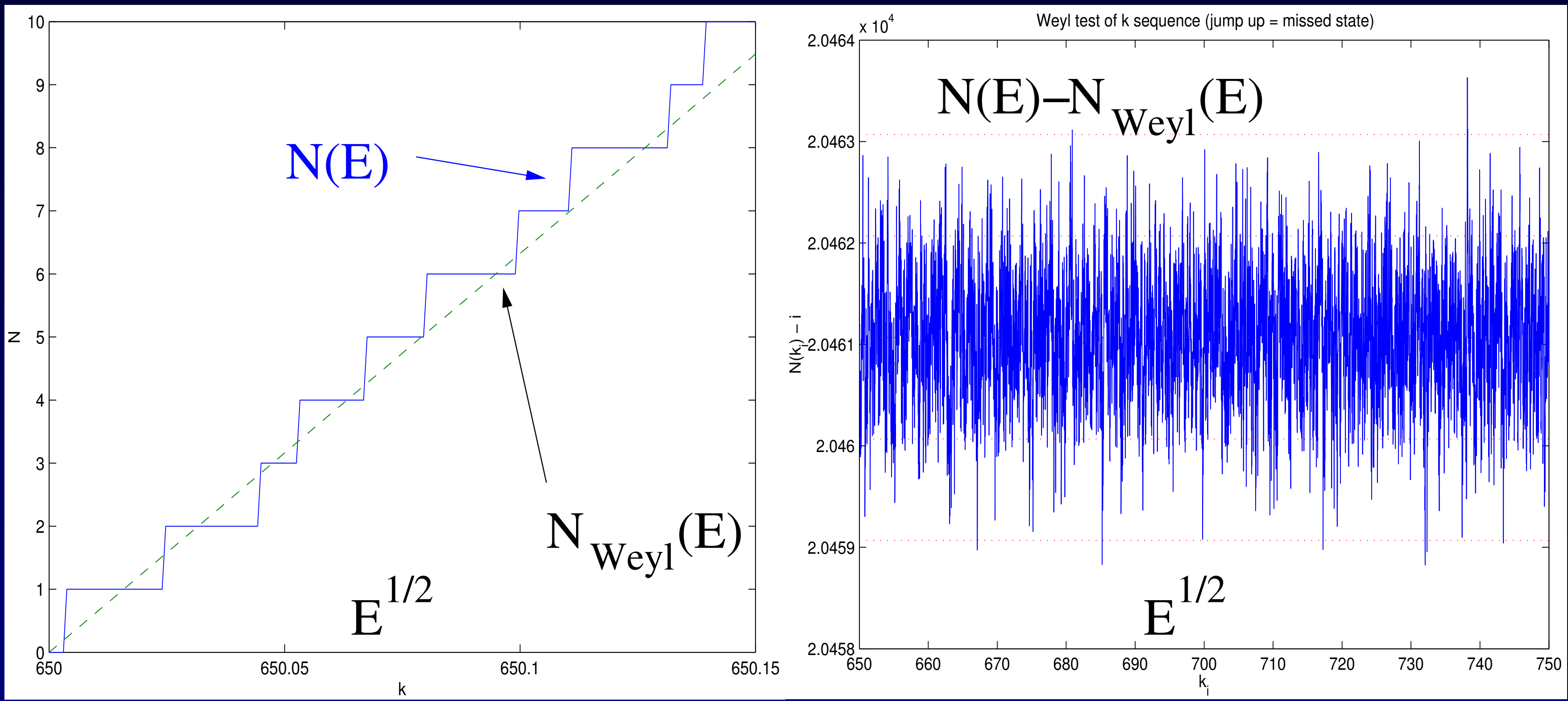
Partial funding by: NSF (DMS-0507614)

Preprints/talks: http://math.dartmouth.edu/∼ahb

Missing levels?

Weyl's estimate for $N(E)$, the # eigenvalues $E_j < E$:

$$N_{\mathrm{Weyl}}(E) = \frac{\mathrm{vol}(\Omega)}{4\pi}E - \frac{L}{4\pi}\sqrt{E} + O(1)\cdots$$



- not one level missing in sequence of 6812