CONSECUTIVE PATTERNS IN PERMUTATIONS

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Abstract. In this paper we study the distribution of the number of occurrences of a permutation σ as a subword among all permutations in $\mathcal{S}_n$. We solve the problem in several cases depending on the shape of σ by obtaining the corresponding bivariate exponential generating functions as solutions of certain linear differential equations with polynomial coefficients. Our method is based on the representation of permutations as increasing binary trees and on symbolic methods.

1. Introduction

Let m, n be two positive integers with $m \leq n$, and let $\pi = (\pi_1, \pi_2, \ldots, \pi_n) \in \mathcal{S}_n$ and $\sigma \in \mathcal{S}_m$ be two permutations. We say that π *contains σ as a subword* if there exist m *consecutive* elements $\pi_{l+1}, \ldots, \pi_{l+m}$ such that $\rho(\pi_{l+1}, \ldots, \pi_{l+m}) = \sigma$, where ρ is the reduction consisting in relabeling the elements with $\{1, \ldots, m\}$ so that they keep the same order relationships they had in π.

For example, if $\sigma = 4132 \in \mathcal{S}_4$, then $\pi = 6725341 \in \mathcal{S}_7$ contains σ as a subword, because $\rho(7253) = 4132$. However, $\pi = 41352 \in \mathcal{S}_5$ does not contain σ as a subword (even though it contains it as a subsequence, that is, in non-consecutive positions); in this case we say that π *avoids* σ. Occurrences of a subword can be overlapped, for instance, 5716243 contains two occurrences of σ, namely 7162 and 6243.

Denote by $A_n(\sigma)$ the set of permutations of $\mathcal{S}_n$ that do not contain σ as a subword, and let $\alpha_n(\sigma) = |A_n(\sigma)|$. If we want to exclude several subwords $\sigma, \tau, \ldots$ we use the corresponding notations $A_n(\sigma, \tau, \ldots)$ and $\alpha_n(\sigma, \tau, \ldots)$. Our main purpose is to compute $\alpha_n(\sigma)$ for a given subword σ. More generally, we are also interested in the distribution of the number of occurrences of σ among all permutations in $\mathcal{S}_n$.

Some well-known counting problems in permutations can be stated in terms of forbidden subwords. For instance, occurrences of 12 correspond to ascents and are counted by Eulerian numbers; up-and-down permutations are those in $A_n(123, 321)$; permutations whose longest increasing run is at most $k-1$ correspond to $A_n(12\cdots k)$; and it is not difficult to see that $\alpha_n(123, 132)$ is precisely the number of involutions. Related results for occurrences of subwords of length three can be found in [2] and, more recently, in [10].

The study of occurrences of *non consecutive* patterns in permutations, initiated in [12], is presently very active. References [1, 3] contain an analysis of the situation where some letters in the pattern are required to be consecutive.

The basis of our work is the use of symbolic methods for specifying combinatorial classes, following the approach described in the books by Flajolet and Sedgewick [8, 11]. The key point is the representation of permutations as binary increasing trees. From this representation, using symbolic methods, we derive differential equations satisfied by the corresponding exponential generating functions. In all cases we have encountered, the differential equations become linear after a suitable substitution. The reader can compare the present paper with [7], where a similar approach is taken for analyzing certain geometric configurations, the difference being that here we deal with exponential GFs that are transcendental instead of being algebraic.

A related approach is taken in [6] for counting occurrences of a given subtree in binary search trees. The main difference in our case is however that patterns corresponding to forbidden subwords may be split into two subtrees in several different ways.

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The organization of the paper is as follows. First, we present some preliminaries on the representation of permutations as increasing trees, on the symbolic method for dealing with exponential generating functions (EGF), and on asymptotic enumeration. In Section 3 we enumerate occurrences of a subword σ in two cases. Firstly, when σ is totally increasing (or decreasing); and secondly when $\sigma = 1, 2, \ldots, a-1, a, \tau, a+1$, and τ is any non empty permutation of the elements $\{a+2, a+3, \ldots, m+2\}$. Notice that the fact that τ and a are arbitrary means that our result covers a very large number of subwords σ. In Section 4 we show how these results specialize in the case of subwords of length three and four. We conclude with some remarks and open problems.

We remark that the results in Section 3 can also be obtained after some derivations from results in Section 2.8 of [9]. However, we believe that the approach taken here helps in understanding the structure of the problem and can be valuable in the solution of further cases of the general problem.

2. Preliminaries

Tree representation of permutations. Following Stanley [13, Chapter 1], we represent a permutation as an increasing binary tree that is, a binary tree in which the labels along any path from the root are increasing. This representation allows us to translate combinatorial properties of permutations (such as avoiding a certain subword) into combinatorial properties of trees, which can be handled more conveniently.

Let $\pi = a_1 a_2 \cdots a_n$ be a word on the alphabet of positive integers with no repeated letters. Define a binary tree $T(\pi)$ as follows. If $\pi = \emptyset$, then $T(\pi) = \emptyset$. Otherwise, let i be the least element of π, so that π can be factored uniquely in the form $\pi = \sigma i \tau$. Now define $T(\pi)$ by induction as the tree with root i, and having $T(\sigma)$ and $T(\tau)$ as left and right subtrees, respectively. This correspondence gives a bijection between $\mathcal{S}_n$ and the set of *increasing binary trees* on n vertices. In particular, we see that the number of such trees is $n!$.

Labeled classes and exponential generating functions. Here we direct the reader to the books by Flajolet and Sedgewick [8] and [11] for a detailed account on combinatorial classes and the symbolic method. Let $\mathcal{A}$ be a class of labeled combinatorial objects and let $|\alpha|$ be the size of an object $\alpha \in \mathcal{A}$. If $\mathcal{A}_n$ denotes the objects in $\mathcal{A}$ of size n and $a_n = |\mathcal{A}_n|$, then the *exponential generating function*, EGF for short, of the class $\mathcal{A}$ is

$$A(z) = \sum_{\alpha \in \mathcal{A}} \frac{z^{|\alpha|}}{|\alpha|!} = \sum_{n \geq 0} a_n \frac{z^n}{n!}.$$

In our context, the size of a permutation is simply its length.

There is a direct correspondence between set-theoretic operations (or "constructions") on combinatorial classes and algebraic operations on EGFs. Table 1 summarizes this correspondence for the operations that are used in the paper. There "union" means union of disjoint copies, "labeled product" is the usual cartesian product enriched with the relabeling operation, and "set" forms sets in the usual sense. Particularly important for us is the construction "boxed product" $\mathcal{A} = \mathcal{B}^{\Box} * \mathcal{C}$, which corresponds to the subset of $\mathcal{B} \star \mathcal{C}$ (the usual labeled product) formed by those pairs in which the smallest label lies in the $\mathcal{B}$ component.

Table 1. The basic combinatorial constructions and their translation into exponential generating functions.

Construction		*Operation on GF*
Union	$\mathcal{A} = \mathcal{B} \cup \mathcal{C}$	$A(z) = B(z) + C(z)$
Labeled product	$\mathcal{A} = \mathcal{B} \star \mathcal{C}$	$A(z) = B(z)C(z)$
Set	$\mathcal{A} = \Pi(\mathcal{B})$	$A(z) = \exp(B(z))$
Boxed product	$\mathcal{A} = \mathcal{B}^{\Box} \star \mathcal{C}$	$A(z) = \int_0^z (\frac{d}{dt} B(t)) \cdot C(t) dt$

Enumerations according to size and an auxiliary parameter χ are described by bivariate generating functions, or BGFs,

$$A(u,z) = \sum_{\alpha \in \mathcal{A}} u^{\chi(\alpha)} \frac{z^{|\alpha|}}{|\alpha|!} = \sum_{n,k \geq 0} A_{n,k} u^k \frac{z^n}{n!},$$

with $A_{n,k}$ the number of objects of size n with χ-parameter equal to k. Throughout the paper the variable z is reserved for marking the length of a permutation, and the variable u is used mostly for marking occurrences of a subword. All derivatives in the paper are taken with respect to z.

Generating functions for increasing trees. We apply the former machinery to increasing trees. The following derivation will be used repeatedly.

Let $\mathcal{I}^0$ be the labeled class of (possibly empty) binary increasing trees. It satisfies the recursive definition

$$\mathcal{I}^0 = \{\epsilon\} + (\{z\}^\Box \star \mathcal{I}^0 \star \mathcal{I}^0),$$

where ϵ is the empty tree, $\{z\}$ represents the tree with one single node, and the box indicates that the root contains the smallest label. So, the equation for the EGF is

$$I^0(z) = 1 + \int_0^z I^0(t)^2 dt,$$

which reduces to ${I^0}'(z) = I^0(z)^2$ with initial condition $I^0(0) = 1$ (derivatives are always with respect to z), admitting the solution $I^0(z) = 1/(1-z)$. Thus, $I^0_n = n!$ as expected.

Equivalent subwords. We say that two subwords $\sigma, \tau \in \mathcal{S}_m$ are *equivalent* if the BGF counting occurrences of σ and τ are the same. We write $\sigma \sim \tau$ to denote equivalence.

There are two simple operations that give equivalent subwords: *reversal*, which transforms $\sigma = \sigma_1 \cdots \sigma_m$ into $\overline{\sigma} = \sigma_m \cdots \sigma_1$, and *complementation*, transforming σ into $\widetilde{\sigma} = (m+1-\sigma_1)\cdots(m+1-\sigma_m)$. The explanation is that a permutation π has as many occurrences of σ as $\overline{\pi}$ has of $\overline{\sigma}$, and as $\widetilde{\pi}$ has of $\widetilde{\sigma}$.

Asymptotic enumeration. Let $A(z)$ be a meromorphic function on a domain of the complex plane including the origin, and let ρ be the unique pole of $A(z)$ such that $|\rho|$ is minimum. Then the following asymptotic estimate holds:

$$[z^n]A(z) \sim \gamma \cdot \rho^{-n},$$

where γ is the residue of A in ρ. See [8, Chapter 4] for a discussion.

3. Main results

This section contains the main results of the paper. We obtain the counting BGF of occurrences of a subword σ in two cases. First we treat the case increasing (or decreasing) subwords.

Theorem 3.1. *Let m be a positive integer, let $\sigma = 1, 2, \ldots, m+1, m+2 \in \mathcal{S}_{m+2}$, and let $P(u,z)$ be the BGF of permutations where u marks the number of occurrences of the subword σ. Then, $P(u,z) = 1/\omega(u,z)$, where ω is the solution of*

$$\omega^{(m+1)} + (1-u)(\omega^{(m)} + \omega^{(m-1)} + \ldots + \omega' + \omega) = 0 \tag{1}$$

with $\omega(0) = 1$, $\omega'(0) = -1$, and $\omega^{(k)}(0) = 0$ for $2 \leq k \leq m$.

Proof. We use the correspondence of permutations as binary increasing trees. We get for $P(u,z)$ a system of $m+1$ first order differential equations, which will be reduced to a differential equation of order $m+1$ with the substitution $P(u,z) = 1/\omega(u,z)$. Let $\mathcal{P}$ be the class of all permutations, let $\mathcal{K}_i$ be the class of permutations not beginning with $1, 2, \ldots, m+2-i$, and let $K_i(u,z)$ be the BGF of $\mathcal{K}_i$ where u marks occurrences of σ. With some abuse of notation, we introduce the parameter u in the equations for classes meaning that it will be placed there when we write the corresponding differential equations for the BGF. With this notation, we can write

$$\mathcal{P} = \{\epsilon\} + \{z\}^\Box \star \mathcal{P} \star [\mathcal{K}_1 + u(\mathcal{P} - \mathcal{K}_1)].$$

This is because occurrences of σ in a permutation (seen as a binary increasing tree) can be separated into occurrences on the left subtree and occurrences on the right subtree, taking into account that if the permutation on the right subtree begins with $1, 2, \ldots, m+1$ (that is, belongs to $\mathcal{P} - \mathcal{K}_1$), then there is an additional occurrence of σ beginning at the root. The corresponding equation for BGFs is, after differentiating,

$$P' = P(K_1 + u(P - K_1)).$$

The next step is to find an equation for $\mathcal{K}_1$. Note that for a permutation not to begin with a $1, 2, \ldots, m+1$, it is not enough that the permutation on the left subtree does not begin with this subword. We have to exclude also the case in which the left subtree is empty and the right subtree begins with a $1, 2, \ldots, m$. This gives us

$$\mathcal{K}_1 = \{\epsilon\} + \{z\}^{\square} \star (\mathcal{K}_1 - \{\epsilon\}) \star [\mathcal{K}_1 + u(\mathcal{P} - \mathcal{K}_1)] + \{z\}^{\square} \star \mathcal{K}_2,$$

which translates to

$$K_1' = (K_1 - 1)(K_1 + u(P - K_1)) + K_2$$

when we differentiate the equation for BGFs. Now it is clear how to find an equation for $\mathcal{K}_2$, in which $\mathcal{K}_3$ will appear, and so on, until we arrive to $\mathcal{K}_m$ (permutations not beginning with $1, 2$), which satisfies

$$\mathcal{K}_m = \{\epsilon\} + \{z\}^{\square} \star (\mathcal{K}_m - \{\epsilon\}) \star [\mathcal{K}_1 + u(\mathcal{P} - \mathcal{K}_1)] + \{z\}.$$

All these equations yield a system of $m+1$ differential equations involving the corresponding BGFs. It is convenient to apply the substitution

$$R = uP + (1-u)K_1.$$

Note that $R' = uP' + (1-u)K_1' = uPR + (1-u)((K_1 - 1)R + K_2) = (uP + (1-u)K_1)R + (1-u)(-R + K_2) = R^2 + (u-1)(R - K_2)$, and we obtain

$$(2) \qquad \begin{cases} P' = PR \\ R' = R^2 + (u-1)(R - K_2) \\ {K_2}' = (K_2 - 1)R + K_3 \\ {K_3}' = (K_3 - 1)R + K_4 \\ \vdots \\ {K_{m-1}}' = (K_{m-1} - 1)R + K_m \\ {K_m}' = (K_m - 1)R + 1 \end{cases} \qquad \text{with } P(0) = R(0) = 1, K_i(0) = 1 \quad \text{for all } i$$

Now we only have to check that setting $P(u,z) = 1/\omega(u,z)$, then ω satisfies (1). The first equation $P' = PR$ gives $R = -\frac{\omega'}{\omega}$. Substituting this into the second one, we get $K_2 = \frac{\omega'' + (1-u)\omega'}{(u-1)\omega}$. By induction on i, we see that

$$K_i = \frac{\omega^{(i)} + (1-u)(\omega^{(i-1)} + \omega^{(i-2)} + \ldots + \omega')}{(u-1)\omega}$$

and $\omega^{(i)}(0) = 0$ for $2 \leq i \leq m$. Finally, (1) is obtained substituting in the last equation of (2) the expressions for K_m and R in terms of ω. $\square$

For $u = 0$ the solution of the differential equation can be expressed as a linear combination of exponentials

$$w = \sum_{j=1}^{m+1} c_j e^{\lambda_j z},$$

where the $\lambda_j = \exp(\frac{2\pi i j}{m+2})$ are the non trivial $(m+2)$th roots of unity, and the indetermined coefficients c_i are the solution of the linear system

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{m+1} \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{m+1}^2 \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_1^m & \lambda_2^m & \cdots & \lambda_{m+1}^m \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \cdots \\ c_{m+1} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \cdots \\ 0 \end{pmatrix}$$

The matrix A of the previous system is easy to invert, since

$$AA^* = (m+2)I - J,$$

where A^* denotes the conjugate transpose of A, and J is the all ones matrix. From this it follows that

$$A^{-1} = \frac{1}{m+2} A^*(I+J),$$

and we can obtain the value of the c_j, thus an explicit expression for $P(0,z)$, the GF of permutations avoiding the subword $\sigma = 1, 2, \cdots, m+2$. For example, for $m = 4$ one gets

$$c_1 = (1+i)/4, \quad c_2 = 1/2, \quad c_3 = (1-i)/4,$$

which agrees with the solution $w = (\cos z - \sin z + e^{-z})/2$ given in the next section.

Our second result is the following.

Theorem 3.2. *Let m, a be positive integers with $a \le m$, let $\sigma = 1, 2, \ldots, a-1, a, \tau, a+1 \in \mathcal{S}_{m+2}$, where τ is any permutation of the elements $\{a+2, a+3, \ldots, m+2\}$, and let $P(u,z)$ be the BGF of permutations where u marks the number of occurrences of the subword σ. Then, $P(u,z) = 1/\omega(u,z)$, where ω is the solution of*

$$\omega^{(a+1)} + (1-u)\frac{z^{m-a+1}}{(m-a+1)!}\omega' = 0 \tag{3}$$

with $\omega(0) = 1$, $\omega'(0) = -1$ and $\omega^{(k)}(0) = 0$ for $2 \le k \le a$. In particular, the distribution does not depend on τ.

Proof. Again we find a system of $a+1$ differential equations for $P(u,z)$ that, after the substitution $P(u,z) = 1/\omega(u,z)$, yield a single differential equation of order $a+1$. Let $\mathcal{P}$ be as before the class of all permutations. For $1 \le i \le m+1$, we denote $\sigma_{>i} = \rho(\sigma_{i+1}, \sigma_{i+2}, \ldots, \sigma_{m+2})$. Note that for $i < a$, $\sigma_{>i}$ has its smallest element in the first position, while for $i \ge a$, the smallest element of $\sigma_{>i}$ is the last one. Now let $\mathcal{K}_i$ be the class of permutations not beginning with any of the following: $\sigma_{>1}, \sigma_{>2}, \ldots, \sigma_{>i}$, and let $K_i(u,z)$ be the BGF of $\mathcal{K}_i$ where u marks occurrences of σ. With this notation, we can write

$$\mathcal{P} = \{\epsilon\} + \{z\}^{\square} \star \mathcal{P} \star [\mathcal{K}_1 + u(\mathcal{P} - \mathcal{K}_1)].$$

The explanation, as in the previous proof, is that in the decomposition of the increasing tree, new occurrences of σ, apart from the ones on the left and right subtrees, can appear beginning in the root, when the permutation on the right subtree begins with $\sigma_{>1}$ (that is, belongs to $\mathcal{P} - \mathcal{K}_1$).

For $\mathcal{K}_1$ we derive the equation

$$\mathcal{K}_1 = \{\epsilon\} + \{z\}^{\square} \star (\mathcal{K}_1 - \{\epsilon\}) \star [\mathcal{K}_1 + u(\mathcal{P} - \mathcal{K}_1)] + \{z\}^{\square} \star [\mathcal{K}_2 + u(\mathcal{P} - \mathcal{K}_1)].$$

Note that the last summand corresponds to the case where the left subtree is empty. Then, the permutation on the right subtree cannot begin with $\sigma_{>2}$, which would produce, together with the root, a permutation beginning with $\sigma_{>1}$. So, after separating on the right subtree the permutations beginning with $\sigma_{>1}$ $(\mathcal{P} - \mathcal{K}_1)$, we are left with those beginning with neither $\sigma_{>1}$ nor $\sigma_{>2}$, that is, $\mathcal{K}_2$.

Analogous expressions can be found for $\mathcal{K}_i$, $1 < i < a$:

$$\mathcal{K}_i = \{\epsilon\} + \{z\}^\square \star (\mathcal{K}_i - \{\epsilon\}) \star [\mathcal{K}_1 + u(\mathcal{P} - \mathcal{K}_1)] + \{z\}^\square \star [\mathcal{K}_{i+1} + u(\mathcal{P} - \mathcal{K}_1)],$$

which yield the equations

$$K_i' = (K_i - 1)(K_1 + u(P - K_1)) + (K_{i+1} + u(P - K_1)).$$

For $\mathcal{K}_a$ (note that $\sigma_{>a} = (\rho(\tau), 1)$) we have

$$\mathcal{K}_a = \{\epsilon\} + \{z\}^\square \star (\mathcal{K}_a - \{\epsilon\} - \{\rho(\tau)\}) \star [\mathcal{K}_1 + u(\mathcal{P} - \mathcal{K}_1)] + \{z\}^\square \star [\mathcal{K}_a + u(\mathcal{P} - \mathcal{K}_1)].$$

The difference now is that in order to avoid a permutation beginning with $\sigma_{>a} = \rho(\tau, a+1)$, the left subtree cannot be $\rho(\tau)$. (Remember that the BGF corresponding to $\{\rho(\tau)\}$ is $z^{m-a+1}/(m-a+1)!$ since it is a permutation of size $m-a+1$ and does not contain the subword σ.) Another difference is that no new variables appear, since when the left subtree is empty there is no danger of beginning with $\sigma_{>a}$, and so there are no additional restrictions for the right subtree.

From all these we get a system of $a+1$ differential equations. After applying the substitutions $R = uP + (1-u)K_1$ as before, and also $S_i = K_{i-1} - K_i$ for $2 \leq i \leq a$, we obtain

$$\left\{ \begin{array}{l} P' = PR \\ R' = R^2 + (u-1)S_2 \\ {S_2}' = S_2R + S_3 \\ {S_3}' = S_3R + S_4 \\ \vdots \\ {S_{a-1}}' = S_{m-1}R + S_a \\ {S_a}' = (S_a + \frac{z^{m-a+1}}{(m-a+1)!})R \end{array} \right. \qquad \text{with } P(0) = R(0) = 1, S_i(0) = 0 \quad \text{for all } i \tag{4}$$

Finally, it only remains to set $P(u,z) = 1/\omega(u,z)$ and to check that ω is the solution of (3). Analogously to the previous proof, it can be shown by induction that $S_i = \frac{\omega^{(i)}}{(1-u)\omega}$ for $2 \leq i \leq a$. Then, (3) follows from the substitution in the bottom equation of the system. $\square$

Note that the fact that $a \leq m$ ensures that τ is not empty. The case in which τ is empty corresponds to the case where σ is the increasing permutation, and has already been treated in the previous theorem.

4. Subwords of length at most four

Occurrences of the two subwords 12 and 21 of length two correspond, respectively, to ascents and descents in permutations, giving rise to the well-known Eulerian numbers [4].

Subwords of length three. Among the 6 permutations of three elements, there are only two different classes regarding its distribution as subwords of permutations since, by reversal and complementation, we have that

$$123 \sim 321$$
$$132 \sim 231 \sim 312 \sim 213$$

By the results in the previous section we get:

Theorem 4.1. *Let $P(u,z)$ and $Q(u,z)$ be the BGFs of permutations where u marks, respectively, the number of occurrences of the subword 123 and 132. Then*

$$Q(u,z) = \frac{1}{1-\int_0^z e^{(u-1)t^2/2}dt}$$

$$P(u,z) = \frac{2e^{\frac{1}{2}(1-u+\sqrt{(u-1)(u+3)})z}\sqrt{(u-1)(u+3)}}{1+u+\sqrt{(u-1)(u+3)}+e^{\sqrt{(u-1)(u+3)}z}(-1-u+\sqrt{(u-1)(u+3)})}$$

$$P(0,z) = \frac{\sqrt{3}}{2}\frac{e^{z/2}}{\cos(\frac{\sqrt{3}}{2}z+\frac{\pi}{6})}$$

Furthermore, the numbers $\alpha_n(123)$ and $\alpha_n(132)$ of permutations avoiding, respectively, the subwords 123 and 132, satisfy

$$\alpha_n(123) \sim \gamma_1 \cdot (\rho_1)^n \cdot n!$$
$$\alpha_n(132) \sim \gamma_2 \cdot (\rho_2)^n \cdot n!$$

where $\rho_1 = \frac{3\sqrt{3}}{2\pi}$, $\gamma_1 = \rho_1 e^{1/(2\rho_1)}$, $(\rho_2)^{-1}$ is the unique positive root of $\int_0^z e^{-t^2/2}dt = 1$, and $\gamma_2 = \rho_2 e^{1/(2\rho_2^2)}$, the approximate values being

$$\rho_1 = 0.8269933, \qquad \gamma_1 = 1.5138274,$$
$$\rho_2 = 0.7839769, \qquad \gamma_2 = 1.7685064.$$

Table 2 indicates the number of permutations of length n with k occurrences of the subwords 123 (top) and 132 (bottom).

Table 2. Occurrences of subwords of length 3 in permutations.

$n\backslash k$	0	1	2	3	4
1	1 1				
2	2 2				
3	5 5	1 1			
4	17 16	6 8	1		
5	70 63	41 54	8 3	1	
6	349 296	274 368	86 56	10	1

The asymptotic estimates are obtained as an application of the result quoted at the end of Section 2. (The computation of ρ_2 has been done numerically using the computer algebra system Maple.) Since $\rho_1 > \rho_2$ in the previous theorem, we see that $\alpha_n(123)$ is asymptotically larger than $\alpha_n(132)$. Looking at Table 2 one is led to conjecture that this is always so. Indeed, we have the following result.

Proposition 4.2. *For every $n \geq 4$, we have*

$$\alpha_n(123) > \alpha_n(132).$$

Proof. Given $\sigma \in \mathcal{S}_m$, let $C_n(\sigma) = \mathcal{S}_n - A_n(\sigma)$ be the permutations of $\mathcal{S}_n$ containing σ. Define a map

$$\Psi : C_n(123) \longrightarrow C_n(132)$$

as follows. If $\pi \in C_n(123)$ contains occurrences of both 123 and 132, then $\Psi(\pi) = \pi$. Otherwise (that is, π contains 123's but not 132's), define $\Psi(\pi)$ as the permutation obtained by traversing π

from left to right and substituting every occurrence of 123 by 132 (transposing the elements in the positions corresponding to 2 and 3).

It only remains to check that Ψ is one to one, and this is because when a 123 is changed to a 132, no new occurrences of 123 appear that did not exist before the substitution. To prove that the inequality is strict for $n \geq 4$, observe that any permutation beginning with 1423 and having no 123 cannot be of the form $\Psi(\pi)$ for any π. $\square$

Subwords of length four. By reversal and complementation, and by the results in the previous section, they fall into seven classes:

I. $1234 \sim 4321$

II. $2413 \sim 3142$

III. $2143 \sim 3412$

IV. $1324 \sim 4231$

V. $1423 \sim 3241 \sim 4132 \sim 2314$

VI. $1342 \sim 2431 \sim 4213 \sim 3124 \sim 1432 \sim 2341 \sim 4123 \sim 3214$

VII. $1243 \sim 3421 \sim 4312 \sim 2134$

The results in the previous section give the BGFs for the distribution of occurrences of subwords in classes I, VI and VII.

Theorem 4.3. *In each of the following cases, let $P(u,z)$ be the BGF of permutations where u marks the number of occurrences of the corresponding subword.*

Case 1342. $P(u,z) = \dfrac{1}{1-\int_0^z e^{(u-1)t^3/6}dt}$.

Case 1234. $P(u,z) = 1/\omega$, *where ω is the solution of*

$$\omega''' + (1-u)(\omega'' + \omega' + \omega) = 0$$

with $\omega(0) = 1$, $\omega'(0) = -1$, $\omega''(0) = 0$. For $u = 0$, the solution is

$$P(0,z) = \frac{2}{\cos z - \sin z + e^{-z}}$$

Case 1243. $P(u,z) = 1/\omega$, *where ω is the solution of*

$$\omega''' + (1-u)z\omega' = 0$$

with $\omega(0) = 1$, $\omega'(0) = -1$, $\omega''(0) = 0$.

Furthermore, the numbers $\alpha_n(1342), \alpha_n(1234)$ and $\alpha_n(1243)$ satisfy

$$\begin{aligned} \alpha_n(1342) &\sim \gamma_1 \cdot (\rho_1)^n \cdot n! \\ \alpha_n(1234) &\sim \gamma_2 \cdot (\rho_2)^n \cdot n! \\ \alpha_n(1243) &\sim \gamma_3 \cdot (\rho_3)^n \cdot n! \end{aligned}$$

where $(\rho_1)^{-1}$ is the smallest postitive solution of $\int_0^z e^{(u-1)t^3/6}dt = 1$, $(\rho_2)^{-1}$ is the smallest positive solution of $\cos z - \sin z + e^{-z} = 0$, and ρ_3 is the solution of a certain equation involving Airy functions. The approximate values are

$$\begin{aligned} \rho_1 &= 0.9546118, & \gamma_1 &= 1.1561986 \\ \rho_2 &= 0.9630055, & \gamma_2 &= 1.1176930 \\ \rho_3 &= 0.9528914, & \gamma_3 &= 1.1696578 \end{aligned}$$

In the last case, the equation $\omega''' + (1-u)z\omega' = 0$, for $u = 0$ and $v = w'$, can be solved in terms of Bessel functions (note that the equation for v is actually a slight variant of the Airy equation). The computations have been performed with the use of the Maple system. Table 3 shows the number of occurrences in each of the seven classes for $n \leq 7$, showing that no two of them have the same distribution. (Entries II, III, IV, V in Table 3 have been computed directly.)

TABLE 3. Occurrences of subwords of length 4 in permutations.

$n\backslash k$		0	1	2	3	4
4	I	23	1			
	II	23	1			
	III	23	1			
	IV	23	1			
	V	23	1			
	VI	23	1			
	VII	23	1			
5	I	111	8	1		
	II	110	10			
	III	110	10			
	IV	110	10			
	V	110	10			
	VI	110	10			
	VII	110	10			
6	I	642	67	10	1	
	II	632	86	2		
	III	631	88	1		
	IV	632	86	2		
	V	631	88	1		
	VI	630	90			
	VII	630	90			
7	I	4326	602	99	12	1
	II	4237	766	37		
	III	4223	794	23		
	IV	4229	782	29		
	V	4218	804	18		
	VI	4210	820	10		
	VII	4204	832	4		

As in the case of length 3, one might expect that again for any two permutations $\sigma, \tau \in \mathcal{S}_4$, if $\alpha_{n_0}(\sigma) > \alpha_{n_0}(\tau)$, then $\alpha_n(\sigma) > \alpha_n(\tau)$ for all $n > n_0$. However, inequality $\alpha_n(1324) \geq \alpha_n(2143)$ holds for $n \leq 11$ but does not hold for $n = 12$. Indeed,

$$\alpha_{11}(1324) = 27959880 > 27954521 = \alpha_{11}(2143),$$

but

$$\alpha_{12}(1324) = 320706444 < 320752991 = \alpha_{12}(2143).$$

This results have been obtained by exhaustive computation, as we do not know the associated EGFs. There is however one relation among classes that we have been able to establish

Proposition 4.4. *For every $n \geq 7$, we have*

$$\alpha_n(1342) > \alpha_n(1243).$$

Proof. As in the proof of Proposition 4.2, let $C_n(\sigma) = \mathcal{S}_n - A_n(\sigma)$ be the set of permutations containing σ. Define a map

$$\Psi : C_n(1342) \longrightarrow C_n(1243)$$

as follows. $\Psi(\pi) = \pi$ if π contains both 1342 and 1243. Otherwise, replace, from left to right, all occurrences of 1342 by 1243. Note that when we replace an occurrence 1342 by 1243, we never create new occurrences of 1342 or 1243. (This is not true in general for other patterns, so that the corresponding Ψ is not a bijection.)

It is clear that $\Psi(C_n(1342) \cap C_n(1243)) \cap \Psi(C_n(1342) \cap A_n(1243)) = \emptyset$, because in the second case there are no 1342 left. Let $\pi \neq \eta \in C_n(1342) \cap A_n(1243)$. Now suppose that $\Psi(\pi) = \Psi(\eta)$. Let i be the smallest index so that $\pi_i \neq \eta_i$. Either π_i or η_i must be moved by Φ.

Now observe that if, say, π_i is changed, it cannot be transposed with any of the preceding elements, so it must be the '3' of a 1342 in π, and thus is interchanged with the '2' in position $i+2$. But now, after replacing this 1342 by 1243, the '3' can no longer be moved, because it is neither the '2' nor the '3' of any other 1342. If η_i is also moved by Ψ, this reasoning implies that $\Psi(\pi)_{i+2} = \pi_i \neq \eta_i = \Psi(\eta)_{i+2}$, which is impossible since $\Psi(\pi) = \Psi(\eta)$. So η_i is not moved by Ψ, but then positions $i-1$, i, $i+1$ and $i+2$ of $\Psi(\pi)$ are $(\pi_{i-1}, \pi_{i+2}, \pi_{i+1}, \pi_i) = (\eta_{i-1}, \eta_i, \eta_{i+1}, \eta_{i+2})$, because this positions of $\Psi(\eta)$ cannot have been moved by Ψ. This is a 1243, which contradicts the fact that $\eta \in A_n(1243)$. □

5. Multiple subwords

Instead of occurrences of a single subword one may consider several subwords. For the case of length three, many of the possible combinations are treated in [10]. For example, the class $A_n(123, 321)$ avoiding 123 *and* 321 is clearly that of *up-and-down* permutations. According to a classical result (see [4]), the corresponding EGF is

$$2\left(\tan z + \frac{1}{\cos z}\right) - 1 - z.$$

The class $A_n(213, 312)$ is that of permutations π having no *valleys*, that is, positions i such that $\pi_{i-1} > \pi_i < \pi_{i+1}$. The BGF where u marks valleys is easily shown to be

$$\frac{\sqrt{1-u}}{\sqrt{1-u} - \tanh(z\sqrt{1-u})}.$$

And the class $A_n(123, 132)$ is equinumerous with the class of involutions, as shown in [3] using the classical correspondence of Foata.

One of the cases not solved in [10] is the simultaneous avoidance of the subwords 123 and 231. We can find a multivariate generating function $P(u, v, z)$ where the coefficient of $u^k v^l z^n/n!$ is the number of elements of $\mathcal{S}_n$ with k occurrences of 123 and l occurrences of 231.

Let $\mathcal{P}$ be as before the class of all permutations, and let $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ be subclasses of $\mathcal{P}$ defined as follows: $\mathcal{K}$ are the permutations not beginning with 12, $\mathcal{L}$ are those not ending with 12, and $\mathcal{M} = \mathcal{K} \cap \mathcal{L}$ are the ones that neither begin nor end with 12. Let $P(u, v, z)$, $K(u, v, z)$, $L(u, v, z)$, $M(u, v, z)$ be respectively the generating functions of these four classes, where u marks occurrences of 123 and v marks occurrences of 231.

We get the following relations for these classes.

$$\begin{aligned}
\mathcal{P} &= \{\epsilon\} + \{z\}^\square \star [\mathcal{L} + v(\mathcal{P} - \mathcal{L})] \star [\mathcal{K} + u(\mathcal{P} - \mathcal{K})] \\
\mathcal{K} &= \{\epsilon\} + \{z\} + \{z\}^\square \star [\mathcal{M} - \{\epsilon\} + v(\mathcal{K} - \mathcal{M})] \star [\mathcal{K} + u(\mathcal{P} - \mathcal{K})] \\
\mathcal{L} &= \{\epsilon\} + \{z\}^\square \star [\mathcal{L} + v(\mathcal{P} - \mathcal{L})] \star [\mathcal{M} - \{z\} + u(\mathcal{L} - \mathcal{M})] \\
\mathcal{M} &= \{\epsilon\} + \{z\} + \{z\}^\square \star [\mathcal{M} - \{\epsilon\} + v(\mathcal{K} - \mathcal{M})] \star [\mathcal{M} - \{z\} + u(\mathcal{L} - \mathcal{M})]
\end{aligned}$$

The idea is the same as in previous proofs. It is based on the fact that a left subtree ending in 12 produces an occurrence of 231 along with the root, and a right subtree beginning with 12 produces a 123 with the root. Therefore, this situations must be marked with v and u respectively. For example, in the third relation, in order for the permutation not to end with 12, the right subtree must be an element of $\mathcal{L} - \{z\}$, and if it belongs to $\mathcal{L} - \mathcal{M}$ (i.e., begins with 12) then it must be marked with a u.

This gives the following system of differential equations for the generating functions (u and v are considered as parameters).

$$\begin{cases} P' = [L + v(P - L)][K + u(P - K)] \\ K' = 1 + [M + v(K - M) - 1][K + u(P - K)] \\ L' = [L + v(P - L)][M + u(L - M) - z] \\ M' = 1 + [M + v(K - M) - 1][M + u(L - M) - z] \end{cases} \quad \text{with } P(0) = K(0) = L(0) = M(0) = 1$$

In particular, if we are interested in the EGF $A(z) = P(0, 0, z)$ whose coefficients are the number of permutations avoiding 123 and 231 simultaneously, we obtain the following result.

Theorem 5.1. *The EGF $A(z)$ of permutations avoiding both 123 and 231 as subwords is the solution the following system of equations, where derivatives are with respect to z:*

$$\begin{cases} A' = CB \\ B' = 1 + (D + z - 1)B \\ C' = CD \\ D' = (D + z - 1)D \end{cases} \quad \textit{with } A(0) = B(0) = C(0) = D(0) = 1$$

An involved explicit form for $A(z)$ can be found in terms of integrals containing the error function, but it seems to us not suitable for obtaining asymptotic results. We can however obtain from the above system as many terms of $A(z)$ as desired, and we get that the corresponding counting sequence begins with

$$1, 2, 4, 11, 39, 161, 784, 4368, 27260, 189540, 1448860, 12076408, 109102564, 1061259548, \ldots$$

6. Concluding remarks

In this paper we have studied the number of permutations $\alpha_n(\sigma)$ avoiding σ as a subword and, more generally, the number of occurrences of σ in permutations of length n. For certain shapes of σ, using bijections between permutations and binary increasing trees, we have been able to find the corresponding bivariate generating functions as inverses of solutions of linear differential equations. The case of length 3 is completely solved, and in the case of length four we have found the solution for three out of seven cases.

Besides extending our results to other subwords not covered by our analysis, there remain some interesting problems. The first one is whether it is true that for every permutation σ there exists some positive constant $\omega < 1$ which depends only on σ,and such that

$$\lim_{n\to\infty} \left(\frac{\alpha_n(\sigma)}{n!}\right)^{1/n} = w.$$

We have seen this is true in the cases analyzed in section 4, and we conjecture that the answer is in the affirmative.

Also, it appears from our computations that the increasing subword $12\cdots k$ is always dominating, in the sense that $\alpha_n(12\cdots k) > \alpha_n(\sigma)$ for any $\sigma \in \mathcal{S}_k$ and n large enough. We conjecture that this is always the case.

Finally, a few comments concerning the distribution of the number of occurrences of a subword. For $\sigma \in \mathcal{S}_m$ and $n \geq m$, let $X_{\sigma,n}$ be the random variable defined on $\mathcal{S}_n$ equal to the number of occurrences of σ. It is easy to see that $E(X_{\sigma,n}) = \frac{n-m+1}{m!}$ and that $Var(X_{\sigma,n}) = c_\sigma n^2$ for some constant c_σ; it follows that the distribution is asymptotically concentrated around the expected value. In fact, using the reasoning in [5], it can be shown that $X_{\sigma,n}$ is asymptotically normal.

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