Homework #4

1) Page 102 #10

$$0.01x + y = 0.1$$
$$x + 101y = 11$$

Same as unperturbed problem.

a) Ignore $0.01x$ term! $\Rightarrow y = 0.1$

$$\rightarrow x = 11 - 101y = 11 - 10.1 = 0.9.$$

b) Find True solution

$y = 0.1 - 0.01x$ plug into 2^{nd} eqn.

$$x + 101(0.1 - 0.01x) = 11$$

$$\rightarrow -0.01x = 0.9 \rightarrow x = -90.$$

$$y = 0.1 - 0.01(-90) = 1$$

$\Rightarrow$ The solution from (a) is a poor approximation to the real solution.

c) General exact solution

$$y = 0.1 - \varepsilon x$$

$$(1 - 101\varepsilon)x = 11 - 101(0.1) \rightarrow x = \frac{11 - 0.1(101)}{1 - 101\varepsilon} = \frac{0.9}{1 - 101\varepsilon}$$

$$y = \frac{(1 - 101\varepsilon)0.1 - \varepsilon x}{1 - 101\varepsilon} = \frac{0.1 - 11\varepsilon}{1 - 101\varepsilon}$$

d) Note! everything breaks when $1 - 101\varepsilon = 0$

$$\rightarrow \varepsilon = 1/101 = 0.0099.$$

$\varepsilon = 0.01$ is very close to this value

$$\frac{1}{\text{something very small}} = \text{something large} \Rightarrow \text{BAD!}$$

ii) We already have the leading order term.
now we need 2^{nd} term.

let $x = x_0 + \varepsilon x_1 + \cdots$
$y = y_0 + \varepsilon y_1 + \cdots$

Plug into equations.

$$\varepsilon(x_0 + \varepsilon x_1 + \cdots) + (y_0 + \varepsilon y_1 + \cdots) = 0.1$$
$$x_0 + \varepsilon x_1 + \cdots + 101(y_0 + \varepsilon y_1 + \cdots) = 11$$

ε^1 equation:
$$x_0 + y_1 = 0 \Rightarrow y_1 = -x_0$$
$$x_1 + 101 y_1 = 0 \rightarrow x_1 = -101\, y_1 = +101 x_0$$

→ Two term approximations are

$$x \approx 0.9 + \varepsilon(101(0.9))$$
$$y \approx 0.1 - \varepsilon(0.9)$$

If you were to expand the exact solutions, using geometric series you would get

$$x = 0.9(1 + 101\varepsilon)$$
$$y = (0.1 - 11\varepsilon)(1 + 101\varepsilon)$$
$$= 0.1 + \varepsilon(-11 + (0.1)(101)) = 0.1 - 0.9\varepsilon$$

The same.

2) Page 123 # 10.

$$\varepsilon u'' - a(x)\, u = f(x) \qquad 0 < x < 1$$
$$u(0) = 0 \quad u(1) = \frac{-f(1)}{a(1)}$$

Assume layer near 0.

$\Rightarrow$ outer layer solution satisfies

$$-a(x)u(x) = f(x) \quad \Rightarrow u(x) = \frac{-f(x)}{a(x)}$$

This satisfies the boundary condition.

inner layer let $\cancel{x}w = \frac{x}{\delta} \Rightarrow x = w\delta$.

$$Y(w) = u(x/\delta)$$

Plug in $\frac{\varepsilon}{\delta^2} Y'' - a(w\delta)\, Y = f(w\delta)$

Choose δ st $\frac{\varepsilon}{\delta^2} \sim 1 \Rightarrow \delta = \sqrt{\varepsilon}$

$$\Rightarrow Y'' + a(\sqrt{\varepsilon}\, w) Y = f(\sqrt{\varepsilon}\, w)$$

ε^0 eqn. $Y'' + a(0) Y = f(0)$.

homogeneous solution.

$$Y(w) = C_1 e^{\sqrt{a(0)}\, w} + C_2 e^{-\sqrt{a(0)}\, w}$$

Particular solution
$Y_p(w) = A$ - constant $\Rightarrow$ $-a(0)A = f(0) \Rightarrow A = \frac{-f(0)}{a(0)}$

$$\Rightarrow Y_i(W) = C_1 e^{\sqrt{a(0)}\,W} + C_2 e^{-\sqrt{a(0)}\,W} - \frac{f(0)}{a(0)}$$

$$Y_i(0) = C_1 + C_2 - \frac{f(0)}{a(0)} = 0 \Rightarrow C_2 = \frac{f(0)}{a(0)} - C_1$$

$$\Rightarrow Y_i(W) = \underbrace{C_1 e^{\sqrt{a(0)}\,W}} + \left(\frac{f(0)}{a(0)} - C_1\right) e^{-\sqrt{a(0)}\,W} - \frac{f(0)}{a(0)}$$

This blows up. so set $C_1 = 0$.

$$\Rightarrow Y_i(W) = \frac{f(0)}{a(0)} e^{-\sqrt{a(0)}\,W} - \frac{f(0)}{a(0)}$$

Matching | No unknown constants... I hope they match.

$$\lim_{x \to 0} y_0(x) = \lim_{W \to \infty} Y_i(W)$$

$$-\frac{f(0)}{a(0)} = -\frac{f(0)}{a(0)} \quad \checkmark \text{ Nice.}$$

$\Rightarrow$ The uniform approximation is

$$y_u(x) = y_0(x) + y_i(x) + \frac{f(0)}{a(0)}$$

$$= \frac{-f(x)}{a(x)} + \frac{f(0)}{a(0)}\left[e^{-\sqrt{\frac{a(0)}{\varepsilon}}\,x} - 1\right] + \frac{f(0)}{a(0)}$$

$$= -\frac{f(x)}{a(x)} + \frac{f(0)}{a(0)} e^{-\sqrt{\frac{a(0)}{\varepsilon}}\,x}$$

3) Page 133 #1

$$\varepsilon y' + y = e^{-t} \qquad y(0) = 2. \qquad t \geq 0$$

outer layer. as $t \to \infty$.

$$y = e^{-t}.$$

inner layer let $W = t/\delta$ $Y(W) = y(t/\delta)$

$$\frac{\varepsilon}{\delta} Y' + Y = e^{-W\delta}$$

Scale $\frac{\varepsilon}{\delta} \sim 1 \Rightarrow \delta = \varepsilon.$

Problem becomes

$$Y' + Y = e^{-W\varepsilon}$$

leading order equation. (ε^0, or $\varepsilon = 0$)

$$Y' + Y = 1$$

$$\Rightarrow Y = Ae^{-W} \qquad Y_p = 1.$$

$$\Rightarrow Y = Ae^{-W} + 1$$

$$Y(0) = A + 1 = 2 \Rightarrow A = 1$$

$$Y_i(W) = e^{-W} + 1$$

matching. $\lim_{t \to 0} y_o(t) = \lim_{W \to \infty} Y_i(W)$

$1 = 1$ ✓ good.

uniform approximation is

$y_u(t) = y_o(t) + y_i(t) - 1 = e^{-t} + e^{-t/\varepsilon}$

The residual is

$$r(y_u, \varepsilon) = \varepsilon y_u' + y_u - e^{-t}$$

$$= \varepsilon\left(-e^{-t} + \frac{-1}{\varepsilon} e^{-t/\varepsilon}\right) + e^{-t} + e^{-t/\varepsilon} - e^{-t}$$

$$= -\varepsilon e^{-t} - e^{-t/\varepsilon} + e^{-t} + e^{-t/\varepsilon} - e^{-t}$$

$$= -\varepsilon e^{-t}.$$

Goal: Show uniform convergence. 1st bound 2nd limit.

$$\max_{t \geq 0} \left| -\varepsilon e^{-t} \right| \leq |-\varepsilon|$$

$$\lim_{\varepsilon \to 0} |-\varepsilon| = 0 \quad -$$

$\Rightarrow$ It converges uniformly to 0 as $\varepsilon \to 0$.

4) Page 134 # 3.

$$\begin{cases} \varepsilon y'' + (t+1)^2 y' = 1 & t > 0 \quad 0 < \varepsilon \ll 1 \\ y(0) = 1 \quad \varepsilon y'(0) = 1 \end{cases}$$

outerlayer as $t \to \infty$.

unperturbed problem

$$(t+1)^2 y' = 1 \Rightarrow y' = \frac{1}{(t+1)^2}$$

$$\Rightarrow y = \frac{-1}{t+1} + C$$

innerlayer near $t=0$.

let $w = t/\delta \quad Y(w) = y(t/\delta)$

Plug in.

$$\frac{\varepsilon}{\delta^2} Y'' + \frac{(\delta w+1)^2}{\delta} Y' = 1 \to \frac{\varepsilon}{\delta^2} Y'' + \left(\delta w^2 + 2w + \frac{1}{\delta}\right) Y' = 1$$

$\Rightarrow$ Good!

Scaling. Try $\frac{\varepsilon}{\delta^2} \sim \frac{1}{\delta} \to \delta = \varepsilon \to$ No large terms

So equation becomes

$$\frac{Y''}{\varepsilon} + (\varepsilon w^2 + 2w + 1/\varepsilon) Y' = 1$$

$$\to Y'' + (\varepsilon^2 w^2 + 2\varepsilon w + 1) Y' = \varepsilon$$

leading order equation is

$Y'' + Y' = 0$. let $V = Y' \Rightarrow V' + V = 0$

$$\Rightarrow V = Ae^{-w}$$

$$Y' = Ae^{-w}$$

$$\to Y = Ae^{-w} + B.$$

$$Y(0) = A + B = 1$$

$$Y'(0) = -A = 1 \Rightarrow A = -1 \qquad \Rightarrow B = 2.$$

$$Y_i(w) = 2 - e^{-w}$$

<u>Matching</u>

$$\lim_{t \to 0} y_0(t) = \lim_{w \to \infty} Y_i(w)$$

$$-1 + C = 2 \Rightarrow C = 3$$

So. the uniform approximation is

$$y_u(t) = y_0(t) + y_i(t) - 2.$$

$$= \frac{-1}{t+1} + 3 + (2 - e^{-t/\varepsilon}) - 2 = \frac{-1}{t+1} + 3 - e^{-t/\varepsilon}.$$

5) Page 141 #1 $\quad \lambda$ large

$$y'' - \lambda(1+x^2)^2 y = 0 \qquad y(0) = 0 \quad y'(0) = 1$$

$$\to \frac{1}{\lambda} y'' - (1+x^2)^2 y = 0. \qquad \text{let } \varepsilon^2 = 1/\lambda.$$

This is non-oscillatory WKB Problem. $k(x) = 1+x^2$

$$y_{WKB}(x) = \frac{C_1}{\sqrt{k(x)}} e^{\frac{1}{\varepsilon}\int_0^x k(s)\,ds} + \frac{C_2}{\sqrt{k(x)}} e^{\frac{-1}{\varepsilon}\int_0^x k(s)\,ds}$$

$$\int_0^x k(s)\,dx = \int_0^x 1+s^2\,ds = s + \frac{s^3}{3}\Big|_0^x = x + \frac{x^3}{3}.$$

$$\to y_{WKB}(x) = \frac{C_1}{\sqrt{1+x^2}} e^{1/\varepsilon(x+x^3/3)} + \frac{C_2}{\sqrt{1+x^2}} e^{-1/\varepsilon(x+x^3/3)}$$

$$W_{KB}(0) = C_1 + C_2 = 0 \to C_1 = -C_2$$

$$y'_{WKB}(x) = C_1\left(\frac{1+x^2}{\varepsilon\sqrt{1+x^2}} + \frac{-1}{2}\frac{2x}{(1+x^2)^{3/2}}\right)e^{1/\varepsilon(x+x^3/3)}$$
$$+ C_2\left(\frac{-(1+x^2)}{\varepsilon\sqrt{1+x^2}} - \frac{1}{2}\frac{2x}{(1+y^2)^{3/2}}\right)e^{-1/\varepsilon(x+x^3/3)}$$

$$y'_{WKB}(0) = C_1\left(\frac{1}{\varepsilon}\right) + C_2\left(\frac{-1}{\varepsilon}\right) = 1$$

$$\to C_2\left(\frac{-2}{\varepsilon}\right) = 1 \to C_2 = -\varepsilon/2. \to C_1 = \varepsilon/2$$

Therefore

$$y_{WKB}(x) = \frac{\varepsilon}{2}\frac{1}{\sqrt{1+x^2}}\left(e^{1/\varepsilon(x+x^3/3)} - e^{-1/\varepsilon(x+x^3/3)}\right)$$

b) Page 141 # 2.

$$y'' + \lambda(\pi + x)^4 y = 0 \qquad y(0) = y(\pi) = 0.$$

This is a WKB eigenvalue problem.

Rewrite equation.

$$\frac{1}{\lambda} y'' + (\pi + x)^4 y = 0. \qquad k(s) = (\pi + s)^2$$

$$y_{WKB}(x) = \frac{c_1}{\sqrt{k(x)}} \cos\left(\sqrt{\lambda} \int_0^x k(s)\,ds\right) + \frac{c_2}{\sqrt{k(x)}} \sin\left(\sqrt{\lambda} \int_0^x k(s)\,ds\right)$$

$$y_{WKB}(0) = 0 \Rightarrow c_1 = 0.$$

$$y_{WKB}(\pi) = \frac{c_2}{\sqrt{k(\pi)}} \sin\left(\sqrt{\lambda} \int_0^\pi k(s)\,ds\right) = 0$$

$$\Rightarrow \sqrt{\lambda} \int_0^\pi k(s)\,ds = n\pi. \quad \text{for large } n$$

$$\int_0^\pi k(s)\,ds = \int_0^\pi (\pi^2 + 2\pi s + s^2)\,ds = \pi^2 s + \pi s^2 + \frac{s^3}{3} \Big|_0^\pi$$

$$= \pi^3 + \pi^3 + \frac{\pi^3}{3} = \frac{7\pi^3}{3}$$

$$\rightarrow \sqrt{\lambda} = \frac{3n}{\pi^2(7)} \rightarrow \lambda_n = \frac{9n^2}{49\pi^4}$$

The eigenfunctions are

$$y_n(x) = \frac{1}{\sqrt{k(x)}} \sin\left(\sqrt{\lambda_n} \int_0^x k(s)\,ds\right)$$

$$= \frac{1}{\pi + x} \sin\left(\frac{3n}{7\pi^2}\left(\pi^2 x + \pi x^2 + \frac{x^3}{3}\right)\right)$$

7) Page 141 #3

$$-\frac{1}{x}y'' - \lambda y = 0 \qquad 1 < x < 4$$
$$y(1) = y(4) = 0$$

Rewrite to look like a WKB problem.

$$\frac{1}{\lambda}y'' + xy = 0. \qquad k(x) = \sqrt{x}$$

$$y_{WKB}(x) = \frac{C_1}{\sqrt{k(x)}}\cos\left(\sqrt{\lambda}\int_0^x k(s)\,ds\right) + \frac{C_2}{\sqrt{k(x)}}\sin\left(\sqrt{\lambda}\int_1^x k(s)\,ds\right)$$

$$y_{WKB}(1) = \frac{C_1}{\sqrt{k(1)}} = 0 \rightarrow C_1 = 0.$$

$$y_{WKB}(4) = \frac{C_2}{\sqrt{k(4)}}\sin\left(\sqrt{\lambda}\int_0^4 k(s)\,ds\right) = 0$$

$$\rightarrow \sqrt{\lambda}\int_1^4 k(s)ds = n\pi \qquad n \text{ large.}$$

$$\Rightarrow \sqrt{\lambda}\ \frac{2}{3}s^{3/2}\Big|_1^4 = n\pi \rightarrow \frac{2\sqrt{\lambda}}{3}\left(\overset{8}{4^{3/2}} - 1\right) = \frac{14}{3}\sqrt{\lambda} = n\pi$$

$$\sqrt{\lambda} = \frac{3n\pi}{14} \rightarrow \lambda_n = \frac{9n^2\pi^2}{(14)^2}$$

eigenfunctions are

$$y_n(x) = \frac{1}{\sqrt{k(x)}}\sin\left(\frac{3n\pi}{14}\int_1^x k(s)ds\right)$$

$$= \frac{1}{\sqrt[4]{x}}\sin\left(\frac{3n\pi}{4}\left(\frac{2s^{3/2}}{3}\right)\Big|_x^x\right) =$$

$$= \frac{1}{\sqrt{x}}\sin\left(\frac{n\pi}{2}\left(x^{3/2} - 1\right)\right)$$

8) Page 150 # 11 We need to use integration by parts

$$\int u\,dv = uv - \int v\,du$$

$$Ci(\lambda) = \int_{\lambda}^{\infty} \frac{\cos x}{x}\,dx$$

let $u = \frac{1}{x}$ $v = \sin x$

$du = \frac{-1}{x^2}$ $dv = \cos x\,dx$

$$Ci(\lambda) = \frac{\sin x}{x}\Big|_{\lambda}^{\infty} + \int_{\lambda}^{\infty} \frac{\sin x}{x^2}\,dx$$

$u = \frac{1}{x^2}$ $v = \cos x$

$du = \frac{-2}{x^3}$ $dv = \sin x\,dx$

$$= \frac{\sin x}{x} + \frac{\cos x}{x^2}\Big|_{\lambda}^{\infty} + 2\int_{\lambda}^{\infty} \frac{\cos x}{x^3}\,dx$$

$$= \lim_{x \to \infty} \underbrace{\left(\frac{\sin x}{x} + \frac{\cos x}{x^2}\right)}_{\to 0} - \frac{\sin \lambda}{\lambda} - \frac{\cos \lambda}{\lambda^2} + 2\int_{\lambda}^{\infty} \frac{\cos x}{x^3}\,dx$$

$$= -\left(\frac{\sin \lambda}{\lambda} + \frac{\cos \lambda}{\lambda^2}\right) + 2\int_{\lambda}^{\infty} \frac{\cos x}{x^3}\,dx$$

So the 2 term assymptotic approximation is

$$\boxed{-\left(\frac{\sin \lambda}{\lambda} + \frac{\cos \lambda}{\lambda^2}\right)}$$

9) Page 150 #13.

$$I(\lambda) = \int_0^\infty \frac{e^{-t}}{(t+\lambda)^2}\,dt.$$

We want an asymptotic expansion i.e. each term is smaller than the previous one.

We will get this by integration by parts.

let $u = \frac{1}{(t+\lambda)^2}$ $v = -e^{-t}$

$du = \frac{-2}{(t+\lambda)^3}$ $dv = e^{-t}dt$

$$I(\lambda) = \left.\frac{-e^{-t}}{(t+\lambda)^2}\right|_0^\infty - 2\int_0^\infty \frac{e^{-t}}{(t+\lambda)^3}\,dt$$

$u = \frac{1}{(t+\lambda)^3}$ $v = -e^{-t}$

$du = \frac{-3}{(t+\lambda)^4}$ $dv = e^{-t}dt$

$$I(\lambda) = \left.\frac{-e^{-t}}{(t+\lambda)^2}\right|_0^\infty - 2\left[\left.\frac{-e^{-t}}{(t+\lambda)^3}\right|_0^\infty - 3\int_0^\infty \frac{e^{-t}}{(t+\lambda)^4}\,dt\right]$$

let $u = \frac{1}{(t+\lambda)^4}$ $v = -e^{-t}$

$du = \frac{-4}{(t+\lambda)^5}$ $dv = e^{-t}dt$

$$I(\lambda) = \left.\frac{-e^{-t}}{(t+\lambda)^2}\right|_0^\infty - 2\left[\left.\frac{-e^{-t}}{(t+\lambda)^3}\right|_0^\infty - 3\left(\left.\frac{-e^{-t}}{(t+\lambda)^4}\right|_0^\infty - 4\int_0^\infty \frac{e^{-t}}{(t+\lambda)^5}\,dt\right)\right]$$

$$\Rightarrow I(\lambda) \sim \frac{1}{\lambda^2} - 2\left(\frac{1}{\lambda^3} - 3\left(\frac{1}{\lambda^4}\right)\right) = \frac{1}{\lambda^2}\left(1 - \frac{2}{\lambda} + \frac{3\cdot 2}{\lambda^2} - \cdots\right)$$

The general n^{th} term will be

$$\frac{1}{\lambda^2}(-1)^n \frac{(n+1)!}{\lambda^n}$$