Part C: Internal Logic of Topoi

CATEGORIES, TOPOI, AND LOGIC

WINTER 2013

Introduction

For this part of the course, we continue reading through Goldblatt and look into the internal logic of topoi. At this point, we have seen most of the basic tools of topos theory and we are ready to use them to interpret propositional logic inside topoi. The approach that Goldblatt uses is semantic; more syntactic approaches, based on deductive systems and type theories, can be found in Lambek & Scott, for example.

The key to the internal logic of a topos is to understand truth values as elements $1 \to \Omega$ of the subobject classifier, or **truth arrows** as Goldblatt calls them. The definition of the subobject classifier includes the truth arrow $\mathit{true} : 1 \to \Omega$. The characteristic function of $0 \to 1$ is another truth arrow $\mathit{false} : 1 \to \Omega$. This understanding of truth values as truth arrows leads to an understanding of the binary propositional connectives $\wedge, \vee, \supset$ as arrows $\Omega \times \Omega \to \Omega$, which are all obtained as characteristic functions of appropriate subobjects of $\Omega \times \Omega$.

- The conjunction $p \wedge q$ is true when p and q are both true, so $\wedge : \Omega \times \Omega \to \Omega$ is the characteristic function of the subobject
$$\langle \mathit{true}, \mathit{true} \rangle : 1 \to \Omega \times \Omega.$$
- The disjunction $p \vee q$ is true when one or both of p and q is true. The truth of p corresponds to the subobject $\langle \mathit{true}, 1_\Omega \rangle : \Omega \to \Omega \times \Omega$ and the truth of q corresponds to the subobject, so $\vee : \Omega \times \Omega \to \Omega$ is the characteristic function of the *image* of
$$[\langle \mathit{true}, 1_\Omega \rangle, \langle 1_\Omega, \mathit{true} \rangle] : \Omega + \Omega \to \Omega \times \Omega.$$
This is not a monic arrow because of the overlap when p and q are both true, so it is necessary to take the image in order to get an actual subobject of $\Omega \times \Omega$.
- The implication $p \supset q$ is true whenever $p \leq q$, which is the same as saying that $p = p \wedge q$. Thus, $\supset : \Omega \times \Omega \to \Omega$ is the characteristic function of the equalizer of the two parallel arrows
$$\begin{tikzcd} \Omega \times \Omega \arrow[r, shift left, "\wedge"] \arrow[r, shift right, "\mathrm{pr}_1"'] & \Omega. \end{tikzcd}$$
Since equalizers are always monic, this is a well-defined subobject of $\Omega \times \Omega$.

- The negation $\neg p$ is true when p is false, so $\neg : \Omega \to \Omega$ is the characteristic function of $\mathit{false} : 1 \to \Omega$.

This gives us an interpretation of propositional logic in a topos where propositional variables $(p, q, r, \ldots)$ range over $1 \to \Omega$ and complex propositions are evaluated using the above operators on Ω. For example, given truth arrows $p, q, r : 1 \to \Omega$, the proposition $(p \wedge q) \supset r$ evaluates to the composite truth arrow $\supset \circ \langle \wedge \circ \langle p, q \rangle, r \rangle : 1 \to \Omega$. In $\mathbf{Set}$, true and false are the only two truth arrows so we get the usual interpretation of propositional logic where propositions range over these two truth values.

A tautology is a proposition that evaluates to true, no matter how the propositional variables are assigned. These can vary from topos to topos; the propositions that are tautologies in every topos are precisely the tautologies of intuitionistic propositional logic. In a **boolean topos**, such as $\mathbf{Set}$ and $\mathbf{Set}^2$, where $p \vee \neg p$ is a tautology, the tautologies are precisely the tautologies of classical propositional logic. In general, the internal logic of a topos will be an intermediate logic between intuitionistic and classical logic. For example, $(p \supset q) \vee (q \supset p)$ is a tautology in the non-boolean topos $\mathbf{Set}^{\to}$, even though this is not an intuitionistic tautology.

Reading

I recommend reading Goldblatt's chapters 6, 7 and 8 twice. First for a light overview of the concepts and how they are related and then again for the details of these relationships.

- In Chapter 6, Goldblatt reviews classical propositional logic and gradually introduces the ideas necessary to interpret these in topoi and develop the internal (propositional) logic of a topos.
- In Chapter 7, Goldblatt ties in the internal logic with the algebra of subobjects, thus paving the way for the interpretation of predicate logic in topoi.
- In Chapter 8, Goldblatt reviews intuitionistic propositional logic. The main approach is based on Heyting algebras but he does present Kripke models in §8.4.

Exercises

Goldblatt —

Ch. 6 — §6: 1, 2; §7: 1, 2.

Ch. 7 — §2: 1; §3: 1.

Ch. 8 — §3: 1–33 (as needed).

Problems

Here are the assigned problems for this part of the course. Once you have succesfully completed all these problems, we will move on to the next part.

PROBLEM C.1 (The internal logic of the topos $\mathbf{Set}^2$). Recall that the objects in the topos $\mathbf{Set}^2$ are pairs of sets (x_0, x_1) and arrows $(x_0, x_1) \to (y_0, y_1)$ are pairs of functions (f_0, f_1), where $f_0 : x_0 \to y_0$ and $f_1 : x_1 \to y_1$. The terminal object in $\mathbf{Set}^2$ is the pair $(1, 1)$, where 1 is the singleton set $\{0\}$. The subobject classifier in $\mathbf{Set}^2$ is the pair $\Omega = (\{0, 1\}, \{0, 1\})$ together with the arrow $\mathit{true} = (t, t) : (1, 1) \to (\{0, 1\}, \{0, 1\})$ where $t(0) = 1$.

(a) Show that every subobject of (x_0, x_1) is isomorphic to a unique **included subobject** (y_0, y_1) where $y_0 \subseteq x_0$, $y_1 \subseteq x_1$ and the components of $(y_0, y_1) \hookrightarrow (x_0, x_1)$ are inclusions. Describe how to compute the characteristic arrow of such an included subobject.

(b) Find all the truth arrows $1 \to \Omega$ in $\mathbf{Set}^2$ and identify $\mathit{false} : 1 \to \Omega$.

(c) Compute the arrows $\wedge, \vee, \supset : \Omega \times \Omega \to \Omega$ and $\neg : \Omega \to \Omega$ in $\mathbf{Set}^2$.

(d) Verify that $\mathbf{Set}^2$ is a boolean topos.

PROBLEM C.2 (The internal logic of the topos $\mathbf{Set}^{\to}$). Recall that the objects in the topos $\mathbf{Set}^{\to}$ are triples (x_0, x_1, ϕ) where $\phi : x_0 \to x_1$, and arrows $(x_0, x_1, \phi) \to (y_0, y_1, \psi)$ are pairs of functions (f_0, f_1), where $f_0 : x_0 \to y_0$ and $f_1 : x_1 \to y_1$ are such that

$$\begin{tikzcd} x_0 \arrow[r, "\phi"] \arrow[d, "f_0"'] & x_1 \arrow[d, "f_1"] \\ y_0 \arrow[r, "\psi"] & y_1 \end{tikzcd}$$

commutes. The terminal object in $\mathbf{Set}^{\to}$ is the triple $(1, 1, !)$ where 1 is the singleton set $\{0\}$. The subobject classifier in $\mathbf{Set}^{\to}$ is the triple $\Omega = (\{0, \frac{1}{2}, 1\}, \{0, 1\}, \omega)$, where $\omega(0) = 0$ and $\omega(\frac{1}{2}) = \omega(1) = 1$, together with the arrow $\mathit{true} = (t_0, t_1) : (1, 1, !) \to \Omega$ where $t_0(0) = 1$ and $t_1(0) = 1$.

(a) Show that every subobject of an object (x_0, x_1, ϕ) is isomorphic to a unique **included subobject** (y_0, y_1, ψ) where $y_0 \subseteq x_0$, $\phi``(y_0) \subseteq y_1 \subseteq x_1$, ψ is the restriction of ϕ to y_0, and the components of $(y_0, y_1, \psi) \hookrightarrow (x_0, x_1, \phi)$ are the two inclusions $y_0 \hookrightarrow x_0$, $y_1 \hookrightarrow x_1$. Describe how to compute the characteristic arrow of such an included subobject.

(b) Find all truth arrows $1 \to \Omega$ in $\mathbf{Set}^{\to}$ and identify $\mathit{false} : 1 \to \Omega$.

(c) Compute the arrows $\wedge, \vee, \supset : \Omega \times \Omega \to \Omega$ and $\neg : \Omega \to \Omega$ in $\mathbf{Set}^{\to}$.

(d) Verify that $\mathbf{Set}^{\to}$ is a non-boolean topos.

PROBLEM C.3. Topological spaces are a rich source of Heyting algebras and interpretations of intuitionistic logic. This problem takes you through this interpretation. Let X be a topological space and let Θ be its collection of open sets. Recall that a closed set in X is a subset of X whose complement is open.

(a) The **interior** A° of an arbitrary set $A \subseteq X$ is the largest open set contained in A. Dually, the **closure** $\overline{A}$ of an arbitrary set $A \subseteq X$ is the smallest closed set containing A. These are dual because of the relationships

$$X - \overline{A} = (X - A)^\circ, \quad X - A^\circ = \overline{(X - A)}.$$

Show that $A \mapsto A^\circ$ and $A \mapsto \overline{A}$ are well-defined operators and have the following properties:

$$\begin{array}{cc} A^\circ \subseteq A, & \overline{A} \supseteq A; \\ (A^\circ)^\circ = A^\circ, & \overline{(\overline{A})} = \overline{A}; \\ A \cap B^\circ = A^\circ \cap B^\circ, & \overline{A \cup B} = \overline{A} \cup \overline{B}; \\ X^\circ = X, & \overline{\varnothing} = \varnothing. \end{array}$$

The above are the **Kuratowski axioms** for topological interior/closure operators.

Show that any interior operator $A \mapsto A^\circ$ with the above properties defines a topology on X where the open sets consist of all $A \subseteq X$ such that $A^\circ = A$. Therefore, closure operators, interior operators, and topologies are different presentations of the same underlying concepts.

(b) The topological interpretation of intuitionistic propositional logic assigns to every propositional variable p an open subset $[p]$ of X. Contradiction is interpreted by $[\bot] = \varnothing$. Conjunction and disjunction are then interpreted by intersection and union

$$[\alpha \wedge \beta] = [\alpha] \cap [\beta], \quad [\alpha \vee \beta] = [\alpha] \cup [\beta].$$

Implication is interpreted by

$$[\alpha \supset \beta] = ((X - [\alpha]) \cup [\beta])^\circ,$$

which is the largest open set whose intersection with $[\alpha]$ is contained in $[\beta]$. Finally, negation is interpreted by

$$[\neg\alpha] = [\alpha \supset \bot] = (X - [\alpha])^\circ.$$

Thus, every proposition α is interpreted by an open set $[\alpha]$ in X. Verify that this interpretation validates modus ponens:

$$[\alpha] = X \quad \& \quad [\alpha \supset \beta] = X \quad \Longrightarrow \quad [\beta] = X.$$

Then convince yourself that the following nine basic intuitionistic tautologies always evaluate to X:

$$\begin{array}{ccc} \bot \supset p, & p \supset (q \supset p), & (p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r)), \\ p \wedge q \supset p, & p \wedge q \supset q, & p \supset (q \supset p \wedge q), \\ p \supset p \vee q, & q \supset p \vee q, & (p \supset r) \supset ((q \supset r) \supset (p \vee q \supset r)). \end{array}$$

Since modus ponens and these nine axioms form a complete deductive system for intuitionistic logic, it follows from this that all intuitionistic tautologies evaluate to X in this interpretation.

(c) Give topological examples showing that the classical tautologies

$$p \vee \neg p, \quad \neg\neg p \vee \neg p, \quad (p \supset q) \vee (q \supset p)$$

are not intuitionistic tautologies.

PROBLEM C.4. This problem introduces a deep connection between natural deduction proofs and category theory. This connection is further explored in Lambek & Scott, for example. The basic idea is that one can interpret propositional logic as a category where objects are propositions and arrows are proofs. To avoid unnecessary complications, we will restrict our attention to the fragment of propositional logic with only two connectives: conjunction ($\wedge$) and implication ($\supset$).

A natural deduction proof has the form of a tree such as

$$\dfrac{\dfrac{A \wedge (B \wedge C)}{B \wedge C}}{B} \quad \text{and} \quad \dfrac{\dfrac{\overline{A}^{u} \qquad B}{A \wedge B}}{A \supset (A \wedge B)}\,u,$$

where each horizontal line represents the application of an introduction or elimination rule listed below. For the categorical interpretation to work proofs must have only one open hypothesis and must be normalized, i.e., none of the reduction rules listed below can be applied.[1] Thus, objects in the category **Ni** are propositions and arrows are normal proofs with only one open hypothesis.

For example, the natural deduction trees above give arrows

$$(A \wedge (B \wedge C)) \to B, \quad \text{and} \quad B \to (A \supset (A \wedge B)).$$

These two arrows can be composed to get an arrow $(A \wedge (B \wedge C)) \to (A \supset (A \wedge B))$ by superposing the proof trees as follows:

$$\dfrac{\dfrac{\overline{A}^{u} \qquad \dfrac{\dfrac{A \wedge (B \wedge C)}{B \wedge C}}{B}}{A \wedge B}}{B \supset (A \wedge B)}\,u.$$

When necessary, some reductions need to be applied after superposing the trees in order to obtain a normal proof. As you proceed through this problem, you should convince yourself that **Ni** is indeed a well-defined category. You may also assume that natural deductions have the **strong Church–Rosser property**: every natural deduction proof has a unique normal form and any strategy for applying reduction rules will eventually lead to a normal proof.

(a) Conjunction has one introduction rule and two elimination rules:

$$\dfrac{P \qquad Q}{P \wedge Q}(\wedge_I), \quad \dfrac{P \wedge Q}{P}(\wedge_{E\ell}), \quad \dfrac{P \wedge Q}{Q}(\wedge_{Er}).$$

[1] The first requirement of having only one open hypothesis can be relaxed by interpreting in a multicategory, where the domain of an arrow is a tuple of objects rather than just one object. This is not strictly necessary since conjunction can be used to combine multiple hypotheses into one.

These rules cancel each other out and conjunctions shouldn't be used back-to-back in a normal proof. Thus, whenever necessary, we need to apply the reductions:

$$\dfrac{\dfrac{\overset{p\downarrow}{P} \qquad \overset{q\downarrow}{Q}}{P \wedge Q}}{P} \rightsquigarrow \overset{p\downarrow}{P}, \quad \dfrac{\dfrac{\overset{p\downarrow}{P} \qquad \overset{q\downarrow}{Q}}{P \wedge Q}}{Q} \rightsquigarrow \overset{q\downarrow}{Q},$$

and

$$\dfrac{\dfrac{\overset{r\downarrow}{P \wedge Q}}{P} \qquad \dfrac{\overset{r\downarrow}{P \wedge Q}}{Q}}{P \wedge Q} \rightsquigarrow \overset{r\downarrow}{P \wedge Q}.$$

Explain how these introductions, eliminations and reductions make $\wedge$ into a product operator for $\mathbf{Ni}$.

(b) Implication has one introduction rule and one elimination rule:

$$\dfrac{\begin{array}{c}\overline{P}^{\,u} \\ \downarrow \\ Q\end{array}}{P \supset Q}\,u(\supset_I), \quad \dfrac{P \qquad P \supset Q}{Q}\,(\supset_E).$$

These rules cancel each other out and implications shouldn't be used back-to-back in a normal proof. Thus, whenever necessary, we may apply the reductions:

$$\dfrac{\overset{p\downarrow}{P} \qquad \dfrac{\begin{array}{c}\overline{P}^{\,u} \\ q\downarrow \\ Q\end{array}}{P \supset Q}\,u}{Q} \rightsquigarrow \begin{array}{c} p\downarrow \\ P \\ q\downarrow \\ Q \end{array}, \quad \dfrac{\dfrac{\overline{P}^{\,u} \qquad \overset{r\downarrow}{P \supset Q}}{Q}}{P \supset Q}\,u \rightsquigarrow \overset{r\downarrow}{P \supset Q}.$$

Explain how these rules and reductions make $\supset$ into an exponential operator for $\mathbf{Ni}$.