MATH 13 SPRING 2011: MIDTERM # 2 SOLUTIONS

(1) $\mathbf{F}$ is defined on $D = \mathbb{R}^2$, C is the closed curve $x^2 + y^2 = 4$, and $\int_C \mathbf{F} \cdot d\mathbf{r} \neq 0$.
(2) $\mathbf{F}$ is defined on $D = \mathbb{R}^2$, C is the closed curve $x^2 + y^2 = 4$, and $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.
(3) $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$ is defined D where D is the interior of the triangle with vertices $(-1, 2), (2, 4), (3, 6)$, and $P_y = Q_x$.

(a): $\mathbf{F}$ is conservative
(b): $\mathbf{F}$ is *not* conservative
(c): There is not enough information

Solution. For the first question, $\mathbf{F}$ cannot be conservative, because conservative is equivalent to saying that the integral of $\mathbf{F}$ across any closed curve is equal to zero. For the second, there is not enough information; we know that the integral around one closed curve is equal to zero, but we do not know anything about any other closed curve. For the third question, because D is an open simply connected set, the fact that $P_y = Q_x$ is sufficient to show that $\mathbf{F}$ is conservative on D.

(4) If $\mathbf{F}$ is a conservative vector field on a region $D \subset \mathbb{R}^2$, then

(a): $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the endpoints of a curve C in D.
(b): $\int_C \mathbf{F} \cdot d\mathbf{r}$ is zero for any closed curve C in D.
(c): $\mathbf{F} = \nabla f$ for some differentiable function f on D.
(d): All of the above.

Solution. The answer is (d); these are the basic properties of a conservative vector field. (For instance, the first two follow from the fundamental theorem of calculus for line integrals.)

(5) Let $\mathbf{F} = \langle e^x, y + e^z, 2x + y \rangle$. Compute $\nabla \times \mathbf{F}$.

Solution. Do a direct calculation:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ e^x & y + e^z & 2x + y \end{vmatrix} = \langle 1 - e^z, -2, 0 \rangle.$$

(6) Let $\mathbf{F}(x, y, z) = \langle x + \sin y, y - \sin z, z \rangle$. Then div $\mathbf{F}$ is equal to

Solution. Do a direct calculation $\nabla \cdot \mathbf{F} = 1 + 1 + 1 = 3$.

Match each of the following functions with the plot of its gradient vector field. Each plot is labelled by the letter located above it.

(7) $f(x, y) = x^2 + y^2$
(8) $f(x, y) = \sin x$
(9) $f(x, y) = e^{x+y}$
(10) $f(x, y) = \frac{1}{xy}$

Solution. (Consult your version of the exam for the sketches.) There are basically two ways to solve this problem: either compute the actual gradients of each of the functions listed above and then match them to the corresponding plots, or use the fact that gradients of $f(x, y)$ are orthogonal to level curves $f(x, y) = C$.

For the former method, for instance, if $f(x, y) = x^2 + y^2$, then $\nabla f = \langle 2x, 2y\rangle$. The only plot which looks like this vector field is (d). One can do the same for the rest of the functions. For the latter method, for instance, one can draw level curves $x^2 + y^2 = C$ for various C, which are circles centered at the origin. The only one of the four pictures which have vectors orthogonal to these circles everywhere is (d).

(11) Find a function $f(x, y, z)$ such that $f(0, 1, 0) = 5$ and

$$\nabla f(x, y, z) = (2xe^y + z^2)\,\mathbf{i} + (x^2e^y + \cos z)\,\mathbf{j} + (2xz - y\sin z)\,\mathbf{k}.$$

Please show your work.

Solution. This type of problem requires a 'partial integration'. For instance, since the first component of ∇f is f_x, we have

$$f_x(x, y, z) = 2xe^y + z^2 \Rightarrow f(x, y, z) = x^2e^y + xz^2 + g_1(y, z),$$

where $g_1(y, z)$ is some yet to be determined function of y, z. Doing the same for f_y, f_z, we get

$$f(x, y, z) = x^2e^y + y\cos z + g_2(x, z) = xz^2 + y\cos z + g_3(x, y).$$

For all three of the different equations for f to be compatible, we must have

$$f(x, y, z) = x^2e^y + y\cos z + xz^2 + C,$$

where C is some constant. To determine C, we use the condition $f(0, 1, 0) = 5$, which yields $1+C = 5$, or $C = 4$. So the final answer is $f(x, y, z) = x^2e^y + y\cos z + xz^2 + 4$.

(12) Let C be the boundary of the square with vertices $(1, 1)$, $(-1, 1)$, $(-1, -1)$ and $(1, -1)$ oriented counterclockwise and let $\mathbf{F} = 2xy\,\mathbf{i} + (3x^2 + \cos(e^y))\,\mathbf{j}$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Solution. Apply Green's Theorem. (If you try to evaluate this line integral directly you get a big mess, which might not be computable). Let D be the interior of C; D is defined by inequalities $0 \le x, y \le 1$, and since $P(x, y) = 2xy, Q(x, y) = 3x^2 + \cos e^y$, Green's Theorem yields

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D Q_x - P_y\,dA = \iint_D 4x\,dA = \int_0^1 \int_0^1 4x\,dx\,dy.$$

This iterated integral is easy to evaluate: the inner integral is $2x^2\Big|_0^1 = 2$, and the outer integral of 2 is still 2.

(13) Let $\mathbf{F} = 4x^3y^3\,\mathbf{i} + 3x^4y^2\,\mathbf{j}$. Find the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve with initial point $(1, 2)$ and terminal point $(-2, 1)$ depicted in the figure below.

Solution. (Consult your exam for the figure. The problem in the exam had a typo, where the $\mathbf{j}$ component was written as $3x^3y^2$. The version above is correct.) The figure depicts a strange looking curve which cannot be easily parameterized, so we will use the fundamental theorem of calculus.

Partial integration on the components of $\mathbf{F}$ yields a potential function $f(x,y) = x^4y^3 + C$, for some constant C. Therefore, the FTC for line integrals yields

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(-2,1) - f(1,2) = 16 + C - (8 + C) = 8.$$

(14) Let E be the region in $\mathbb{R}^3$ defined by the inequalities $1 \leq x^2 + y^2 + z^2 \leq 4$. Find the average value of the function $f(x,y,z) = z^2$ on E.

Solution. The solid E is the region between two spheres of radius 2 and 1, each centered at the origin. To calculate the average value of f, we need to compute both the volume of E as well as the triple integral of z^2 over E.

To compute the volume, we can either evaluate the triple integral of 1 over E, or use geometry. The latter is easier if you remember the formula for the volume of a sphere; a sphere of radius 2 has volume $32\pi/3$, while a sphere of radius 1 has volume $4\pi/3$, so E has volume $28\pi/3$.

To evaluate the triple integral of z^2 over E, we use spherical coordinates. In spherical coordinates, E is described by the inequalities

$$1 \leq \rho \leq 2, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi.$$

Remembering that $z = \rho\cos\phi$ and that the change of variables factor for spherical coordinates in the integrand is $\rho^2 \sin\phi$, we get

$$\iiint_E z^2\, dV = \int_0^{\pi}\int_0^{2\pi}\int_1^2 (\rho\cos\phi)^2\rho^2 \sin\phi \, d\rho\, d\theta\, d\phi = \left(\int_0^{\pi} \cos^2\phi \sin\phi d\phi\right)\left(\int_0^{2\pi} 1\, d\theta\right)\left(\int_1^2 \rho^4\, d\rho\right).$$

(For the last equality, we are using the fact that the integrand is a product of a function purely of ρ, purely of θ, and purely of ϕ.) Each of these integrals can be evaluated using standard methods; for instance, the hardest is probably the first, which can be done using u-substitution:

$$\int_0^{\pi} \cos^2\phi \sin\phi d\phi = \frac{-\cos^3\phi}{3}\Big|_0^{\pi} = \frac{2}{3}, \int_0^{2\pi} 1\, d\theta = 2\pi, \int_1^2 \rho^4 d\rho = \frac{\rho^5}{5}\Big|_1^2 = \frac{31}{5}.$$

Multiplying these together gives $124\pi/15$. We have to divide this by the volume of E to get the final answer:

$$\frac{124\pi}{15} \cdot \frac{1}{V(E)} = \frac{124\pi}{15} \cdot \frac{3}{28\pi} = \frac{31}{35}.$$

(15) Let E be the region in $\mathbb{R}^3$ defined by the inequalities $4 \leq x^2 + y^2 \leq 9, x, y \geq 0, -2 \leq z \leq 3$. Evaluate

$$\iiint_E xyz\, dV.$$

Solution. The region E seems best suited for using cylindrical inequalities (indeed, E is a quarter of a solid cylinder). In terms of cylindrical coordinates (r, θ, z), E is defined by

$$2 \leq r \leq 3, 0 \leq \theta \leq \frac{\pi}{2}, -2 \leq z \leq 3.$$

Remembering that $x = r\cos\theta, y = r\sin\theta$, and that we insert a factor of r in the integrand when switching to cylindrical coordinates, we get

$$\iiint\limits_E xyz\, dV = \int_{-2}^{3}\int_{0}^{\pi/2}\int_{2}^{3} r^3 z\cos\theta\sin\theta\, dr\, d\theta\, dz.$$

Again, we use the fact that the integrand is a product of a function purely of r, purely of θ, and purely of z to write

$$\begin{aligned}\int_{-2}^{3}\int_{0}^{\pi/2}\int_{2}^{3} r^3\cos\theta\sin\theta\, dr\, d\theta\, dz &= \left(\int_{-2}^{3} z\, dz\right)\left(\int_{0}^{\pi/2}\cos\theta\sin\theta\, d\theta\right)\left(\int_{2}^{3} r^3\, dr\right)\\ &= \left(\frac{z^2}{2}\Big|_{-2}^{3}\right)\cdot\left(\frac{\sin^2\theta}{2}\Big|_{0}^{\pi/2}\right)\left(\frac{r^4}{4}\Big|_{2}^{3}\right) = \frac{5}{2}\cdot\frac{1}{2}\cdot\frac{65}{4} = \frac{325}{16}.\end{aligned}$$