The Groupoid

April 17, 2003

Our paths up to homotopy have a very natural algebraic structure. A structure which comes up in enough interesting places that mathematicians have bothered to axiomatize and named it as follows...

Definition: A *groupoid* is a set G, a distinguished subset $G^0 \subseteq G$, two maps $b, e : G \to G^0$, and a composition law

$$\star : G^2 = \{(\gamma_1, \gamma_2) \mid e(\gamma_1) = b(\gamma_2)\} \to G$$

that satisfies the 5 axioms listed below. When discussing groupoids, we say that a given composition *makes sense* provided all the need pairs are in G^2, and we utilize the convention that if a composition is not qualified explicitly by the adjective makes sense, then it is assumed to make sense.

1. $e(\gamma_1 \star \gamma_2) = e(\gamma_2)$ and $b(\gamma_1 \star \gamma_2) = b(\gamma_1)$.
2. For every $p \in G^0$ we have $e(p) = b(p) = p$.
3. $\gamma \star e(\gamma) = b(\gamma) \star \gamma = \gamma$.
4. If $(\gamma_1 \star \gamma_2) \star \gamma_3$ or $\gamma_1 \star (\gamma_2 \star \gamma_3)$ makes sense, then $(\gamma_1 \star \gamma_2) \star \gamma_3 = \gamma_1 \star (\gamma_2 \star \gamma_3)$.
5. For every $\gamma \in G$ there exist γ^{-1} such that $\gamma \star \gamma^{-1} = b(\gamma)$ and $\gamma^{-1} \star \gamma = e(\gamma)$.

One nice thing about groupoids is that they allow to easily construct some natural groups. After completing the next two exercises be sure to remind yourself how to interpret the groups and isomorphisms constructed below in the case where the groupoid is the groupoid of homotopy classes of paths, as introduced in Munkres/lecture.

Exercise 1: Prove that for every $p \in G^0$ that

$$G_p = \{\lambda \in G \mid e(\lambda) = b(\lambda) = p\}$$

is a group when we restrict G's composition to G_p.

Exercise 2: Prove: if there exist $\gamma \in G$ such that $q = \gamma \star p \star \gamma^{-1}$, then G_p and G_q are isomorphic.

Groupoids are very common. For example, a groupoid is born of every equivalence relation!

Exercise 3: Suppose we have an equivalence relation on the elements of X. Recall that you can view an equivalence relation as a subset R of $X \times X$. Let $G = R$, $G^0 = \{(x, x) \mid x \in X\}$, $b((x, y)) = x$, $e((x, y)) = y$, and $(x, y) \star (y, z) = (x, z)$. Confirm that with these choices that we have indeed produced a groupoid from our equivalence relation.