Integratin of Rational Functions by Partial Fractions

September 27, 2006

Problem

- Integrate a rational function

$$f(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials.

- The method of *partial fractions* is to express $f(x)$ by a sum of simpler fractions.

Case 1

- The denominator $Q(x)$ is a product of distinct linear factors

$$Q(x) = (a_1 x + b_1)(a_2 x + b_2) \dots (a_k x + b_k).$$

- Then there exist constants $A_1, A_2, \dots, A_n$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \dots + \frac{A_k}{a_k x + b_k}$$

Example

- Evaluate
$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\mathrm{d}x.$$

- Last time: $2x^3+3x^2-2x = x(2x-1)(x+2)$.

- So
$$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}.$$

- This gives the system of equations:

$$\begin{aligned} 2A + B + 2C &= 1 \\ 3A + 2B - C &= 2 \\ -2A &= -1 \end{aligned}$$

- This gives the system of equations:

$$\begin{aligned} 2A + B + 2C &= 1 \\ 3A + 2B - C &= 2 \\ -2A &= -1 \end{aligned}$$

- Then $A = 1/2$, $B = 1/5$, and $C = -1/10$.

Case 2

- $Q(x)$ is a product of linear factors, some of which are repeated.

- Suppose that the first linear factor $(a_1x + b_1)$ is repeated r times. Then we would use

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \ldots + \frac{A_r}{(a_1x + b_1)^r}$$

- **Example:**

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} \mathrm{d}x$$

Approximate Integration

- There are situations in which is impossible to find the exact value of a definite integral.
- Example

$$\int_0^1 e^{x^2}\mathrm{d}x.$$

- We need to find approximate values of the definite integral.

Midpoint Rule

$$\int_a^b f(x)\mathsf{d}x \simeq M_n = \Delta x[f(\bar{x}_1) + f(\bar{x}_2) + \ldots + f(\bar{x}_n),$$

where

$$\Delta x = \frac{b-a}{n}$$

and

$$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i].$$

Trapezoid Rule

$$\int_a^b f(x)\mathrm{d}x \simeq T_n = \frac{\Delta x}{2}[f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(x_n)],$$

where

$$\Delta x = \frac{b-a}{n},$$

and

$$x_i = a + i\Delta x.$$

Example

- Use the Midpoint Rule and the Trapezoidal Rule with $n = 4$ to approximate the integral

$$\int_1^2 x^2 \mathrm{d}x.$$

Simpson's Rule

$$\int_a^b f(x)\mathrm{d}x \quad \simeq \quad S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)],$$

where n is *even* and

$$\Delta x = \frac{b-a}{n}.$$

Example

- Use Simspon's Rule with $n = 4$ to estimate

$$\int_1^2 x^2 \mathrm{d}x.$$