Discrete Probabilities

Summer 2006

Random Variables and Sample Spaces

- We represent the outcome of the experiment by a capital Roman letter, such as X, called a *random variable*.

- The *sample space* of the experiment is the set of all possible outcomes. If the sample space is either finite or countably infinite, the random variable is said to be *discrete*.

- The elements of a sample space are called outcomes.

- A subset of the sample space is called an *event*.

Distribution Functions

Let X be a random variable which denotes the value of the outcome of a certain experiment, and assume that this experiment has only finitely many possible outcomes. Let Ω be the sample space of the experiment (i.e., the set of all possible values of X, or equivalently, the set of all possible outcomes of the experiment.) A distribution function for X is a real-valued function m whose domain is Ω and which satisfies:

1. $m(\omega) \geq 0$, for all $\omega \in \Omega$, and

2. $\sum_{\omega \in \Omega} m(\omega) = 1$.

For any subset E of Ω, we define the probability of E to be the number $P(E)$ given by

$$P(E) = \sum_{\omega \in E} m(\omega) \ .$$

Examples

Three people, A, B, and C, are running for the same office, and we assume that one and only one of them wins. Suppose that A and B have the same chance of winning, but that C has only 1/2 the chance of A or B. What is the probability to win for each of the three people?

Basic Set Operations

- Then the union of A and B is the set

$$A \cup B = \{x \,|\, x \in A \text{ or } x \in B\} \ .$$

- The intersection of A and B is the set

$$A \cap B = \{x \,|\, x \in A \text{ and } x \in B\} \ .$$

- The difference of A and B is the set

$$A - B = \{x \,|\, x \in A \text{ and } x \notin B\} \ .$$

- The complement of A is the set

$$\tilde{A} = \{x \,|\, x \in \Omega \text{ and } x \notin A\} \ .$$

Properties

The probabilities assigned to events by a distribution function on a sample space Ω satisfy the following properties:

1. $P(E) \geq 0$ for every $E \subset \Omega$.

2. $P(\Omega) = 1$.

3. If $E \subset F \subset \Omega$, then $P(E) \leq P(F)$.

4. If A and B are *disjoint* subsets of Ω, then $P(A \cup B) = P(A) + P(B)$.

5. $P(\tilde{A}) = 1 - P(A)$ for every $A \subset \Omega$.

- For any two events A and B,

$$P(A) = P(A \cap B) + P(A \cap \tilde{B}) \ .$$

- If A and B are subsets of Ω, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \ .$$

Uniform Distribution

The *uniform distribution* on a sample space Ω containing n elements is the function m defined by

$$m(\omega) = \frac{1}{n} \, ,$$

for every $\omega \in \Omega$.

Example

Consider the experiment that consists of rolling a pair of dice. We take as the sample space Ω the set of all ordered pairs (i, j) of integers with $1 \leq i \leq 6$ and $1 \leq j \leq 6$. Thus,

$$\Omega = \{\, (i, j) : 1 \leq i, j \leq 6 \,\} \, .$$

Odds

If $P(E) = p$, the *odds* in favor of the event E occurring are $r : s$ (r to s) where $r/s = p/(1-p)$. If r and s are given, then p can be found by using the equation $p = r/(r+s)$.

Infinite Sample Space

If

$$\Omega = \{\omega_1, \omega_2, \omega_3, \ldots\}$$

is a countably infinite sample space, then a distribution function is defined exactly as before, except that the sum must be *convergent*.

Examples

A coin is tossed until the first time that a head turns up. Let the outcome of the experiment, ω, be the first time that a head turns up. Then the possible outcomes of our experiment are

$$\Omega = \{1, 2, 3, \ldots\}\ .$$

What is the probability that the coin eventually turns up heads.