Math 71 Homework 5 Partial Solutions

111/3 Let S be the set of all $(i\ i{+}1)$. First show any $(1\,b)$ can be expressed in terms of S by induction on b.
$b=2$ obvious. If true for b,

$$(1\ b{+}1) = (b\ b{+}1)(1\,b)(b\ b{+}1).$$

Now $(a\,b) = (1b)(1a)(1b)$. $\therefore$ Any transposition can be written in terms of S

111/4 $(k\ k{+}1) = (1\,2\cdots n)(k{-}1\ k)(n\cdots 2\,1)$

111/5 Let $\sigma = (a_i\,a_j)$ τ contains all the nos. $1, 2, \ldots, p$
For some k, $\tau^k = (a_i\,a_j \cdots)$ Now relabel the a's so $\sigma = (1\,2)$, $\tau^k = (1\,2\,3\cdots p)$. Apply 111/4.

111/11 Let H be a subgroup of S_4 isomorphic to Q_8
$\therefore$ H contains 6 elements of order 4, 1 element of order 2
1 element of order 1. The 6 elements are all the 4-cycles
The element of order 2 is either (ab) or $(ab)(cd)$.
Show this cannot be. (For example, if the element of order 2 is (ab) or $(ab)(cd)$ choose a 4-cycle $(abcd)$ and consider $(abcd)(ab)$ or $(abcd)(ab)(cd)$.)

111/12 Define $\theta: S_{n-2} \to A_n$ by:

$$\alpha \in A_{n-2}, \quad \theta(\alpha) = \alpha$$
$$\alpha \notin A_{n-2}, \quad \theta(\alpha) = \alpha\,(n{-}1\ n)$$

Show θ is a monomorphism.

111/13 Consider $\tau_1\tau_2\cdots\tau_{2k}$ $2k$ ~~no~~ commuting transpositions
Let $(ab)(cd)$ be a pair of adjacent commuting transpositions
Then a, b, c, d are all distinct (check this) and

$$(ab)(cd) = (acbd)^2$$

111/14 An element of order 2 in A_4 has the form $(ab)(cd)$

Let (abc) be the 3-cycle

$$(abc)(ab)(cd) = (acd)$$
$$(ab)(cd)(abc) = (bdc)$$

Now consider all the 3-cycles in the subgroup, their squares, $(ab)(cd)$ and e. Therefore the subgroup generated by (abc) and $(ab)(cd)$ has at least 8 elements. But its order must divide 12. Therefore the subgroup is A_4.

116/1 $\gamma \in G_b \quad \therefore (g^{-1}\gamma g)a = g^{-1}\gamma b = g^{-1}b = a$ so $g^{-1}G_b g \subseteq G_a \quad \therefore G_b \subseteq g G_a g^{-1}$. But $a = g^{-1}b$, so $G_a \subseteq g^{-1}G_b g$

That is, $g G_a g^{-1} \subseteq G_b$

$\operatorname{Ker} \varphi = \bigcap_{b \in A} G_b$

116/2 $G \leq S_A \quad \therefore \varphi : G \to S_A$ is inclusion $\therefore \operatorname{Ker}\varphi = 1$

$\therefore \bigcap_{\sigma \in G} \sigma G_a \sigma^{-1} = 1$

116/3 Suppose $\sigma(a) = a$ some $\sigma \in G$, $a \in A$. If $b \in A$, $\exists \gamma \in G$, $b = \gamma a$. $\therefore \sigma b = (\sigma\gamma)(a) = (\gamma\sigma)(a) = \gamma(a) = b$.

$\therefore \sigma = id$. In particular $G_a = 1$. By transitivity $\mathcal{O}(a) = A \quad \therefore |G/G_a| = |A|$ and $G_a = 1$

$\therefore |A| = |G|$

116/4 For example, take L_i, L_j.

122/8 Let $K = \operatorname{Ker} \pi_H$, $\pi_H : G \to S_n$. By 1st Isomorphism Theorem, G/K isomorphic to a subgroup of $S_n \quad \therefore |G/K| \leq n!$

122/9 First part: by Corollary 5 (p. 120) Second part: $|G| = p^2$

If $a \in G$, then $|a| = 1, p, p^2$. If $\exists a \in G$, $|a| = p^2$ then G is cyclic and so has a normal subgroup of order p (hence index p). If $\exists a \in G$, $|a| = p$, then $\langle a \rangle$ is a subgroup of index p, hence normal

122/11 $\pi(x)(e) = x$, $\pi(x)(x) = x^2$, $\ldots \quad \pi(x)(x^{n-1}) = e$ Thus

$$\pi(x) = (e\ x\ \cdots\ x^{n-1})\ \underbrace{c_2\ \cdots\ c_k}_{\text{disjoint cycles}}$$

$\uparrow$ n-cycle

How is c_2 defined: Take y not one of $e, x, \ldots, x^{n-1}$

$\pi(x)(y) = xy \qquad \pi(x)(xy) = x^2y \quad \cdots \quad \pi(x)(x^{n-1}y) = y$

Show all distinct so have $c_2 = (y\ xy\ \cdots\ x^{n-1}y)$ n-cycle

Continue in this way.