Your name:

Instructor (please circle): Zajj Daugherty Erik van Erp

Math 11 Fall 2011, Homework 7, due Wed Nov 9

Please show your work. No credit is given for solutions without justification.

(1) *Choose the correct answer. Show relevant work (it will not be graded).*

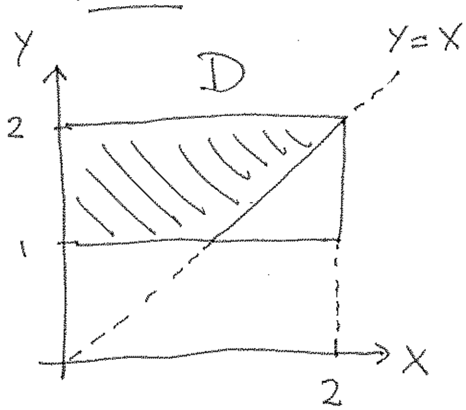
The probability density function $p(x,y)$ for two continuous random variables X, Y is defined as

$$p(x,y) = \begin{cases} \frac{1}{3}xy & \text{if } 0 \le x \le 2 \text{ and } 1 \le y \le 2 \textbf{ (read carefully!)} \\ 0 & \text{otherwise} \end{cases}$$

What is the probability $P(Y \ge X)$?

(A) $\frac{1}{2}$ (B) $\frac{5}{8}$ (C) $\frac{2}{3}$ (D) $\frac{3}{4}$ (E) $\frac{5}{6}$ (F) $\frac{7}{8}$

Picture:



$$P(Y \ge X)$$

$$= \iint_D p(x,y)\,dA$$

$$= \int_{y=1}^{2} \int_{x=0}^{y} \frac{1}{3} xy \,dx\,dy$$

$$\frac{1}{3}\int xy\,dx = \frac{1}{6}x^2 y \Big|_{x=0}^{y} = \frac{1}{6}y^3$$

$$\frac{1}{6}\int y^3\,dy = \frac{1}{24}y^4 \Big|_{y=1}^{2} = \frac{1}{24}(16-1) = \frac{15}{24} = \frac{5}{8}$$

$\boxed{E}$

(2) The solid half-sphere with $x^2+y^2+z^2 \le 9$, and $z \ge 0$ has mass density

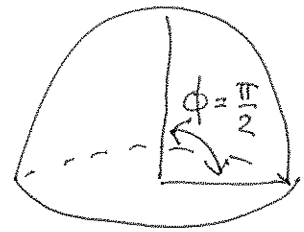
$$f(x,y,z) = \frac{1}{1+(x^2+y^2+z^2)^{3/2}}$$

Calculate the total mass of this object.

Half sphere W: $0 \le \rho \le 3$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \frac{\pi}{2}$



$$\int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{3} \frac{1}{1+\rho^3} \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \cdot \underbrace{\int_0^{\pi/2} \sin\phi \, d\phi}_{1} \cdot \int_0^3 \frac{\rho^2}{1+\rho^3} d\rho$$

$$\int_0^3 \frac{\rho^2}{1+\rho^3} d\rho = \begin{bmatrix} u = 1+\rho^3 \\ du = 3\rho^2 d\rho \end{bmatrix} \int \frac{1}{3u} du = \frac{1}{3} \ln|u|$$

$$= \frac{1}{3} \ln|1+\rho^3| \Big|_{\rho=0}^{3} = \frac{1}{3} \ln 28$$

Solution:

Total mass $= 2\pi \cdot \frac{1}{3} \ln 28 = \frac{2}{3} \pi \ln 28$.

(3) We know that the gravitational field $\mathbf{F}(x,y) = \langle -\frac{x}{r^3}, -\frac{y}{r^3} \rangle$ is conservative with potential function $V = \frac{1}{r}$. Here $r = \sqrt{x^2+y^2}$. Suppose we have a different force field defined by the law

$$\mathbf{F}(x,y) = \langle -\frac{x}{r^n}, -\frac{y}{r^n} \rangle$$

For which values of the exponent n does this vector field satisfy the cross-partial property of a conservative vector field?

$$\vec{F}(x,y) = \left\langle -\frac{x}{(x^2+y^2)^{n/2}}, -\frac{y}{(x^2+y^2)^{n/2}} \right\rangle$$

$$= \langle F_1(x,y), F_2(x,y) \rangle$$

$$\frac{\partial F_1}{\partial y} = -\frac{0 - x \cdot 2y \cdot \frac{n}{2}(x^2+y^2)^{\frac{n}{2}-1}}{(x^2+y^2)^n}$$

$$= \frac{nxy(x^2+y^2)^{\frac{n}{2}-1}}{(x^2+y^2)^n}$$

$\frac{\partial F_2}{\partial x}$ = same formula with $x \leftrightarrow y$ swapped.

$$= \frac{nxy(x^2+y^2)^{\frac{n}{2}-1}}{(x^2+y^2)^n} \quad \text{(no difference)}$$

<u>Conclusion</u>: cross-partials property is satisfied for ANY value of n.