Math 23 Fall 2013

Differential Equations

Exam 1

Friday, October 11, 4:00PM - 6:00PM

Your name (please print): SOLUTION ________________________________

Instructor (circle one): Section 1, Section 2

Instructions: This is a closed book, closed notes exam. The use of calculators is not permitted. The exam consists of **9** problems and this booklet contains **13** pages (including this one). **On problems 4 through 9, you must show your work and justify your assertions to receive full credit**. Justify your answers and simplify your results as much as possible. Also, please clearly mark your final (simplified) answer. The last two pages of this booklet are blank. Good Luck!

The Honor Principle requires that you neither give nor receive any aid on this exam.

FERPA Waiver: By my signature I relinquish my FERPA rights in the following context: My exam may be returned en masse with others present in the classroom. I acknowledge that I understand my score may be visible to others. If I choose not to relinquish my FERPA rights, I understand that I will have to present my student ID at my instructor's office to retrieve my exam.

Signature:________________________________

Math 23 Fall 2013

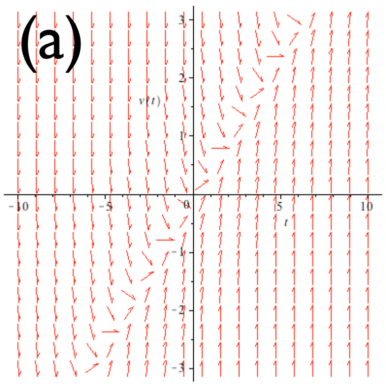
Your name (please print): ______________________________

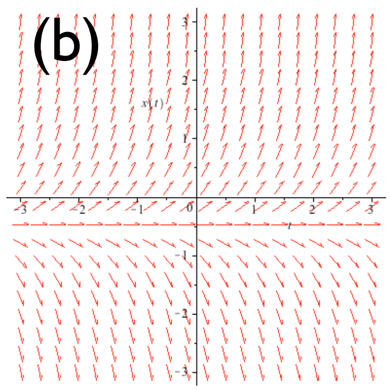
Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10+5	
6	10	
7	15	
8	15	
9	10	
Total	**100+5**	

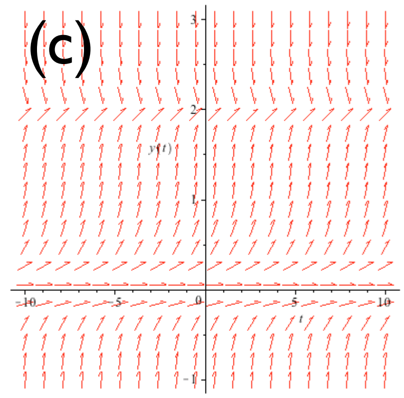
Short Answer Questions - Work will not be graded

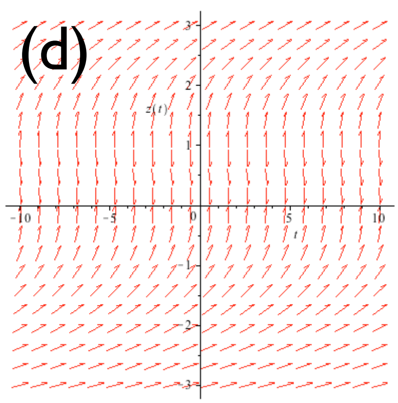
1. (**10 Points**) Match the differential equation with the corresponding direction field.

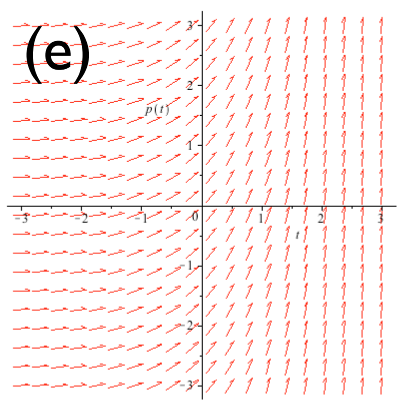
① $y' = y^2(2-y)$ ② $\frac{dv}{dt} = t - 2v$ ③ $\frac{dz}{dt} = \frac{1}{z(z-1)}$ ④ $\frac{dP}{dt} = e^t$ ⑤ $\frac{dx}{dt} = 2x+1$











Answer: ①-(c), ②-(a), ③-(d), ④-(e), ⑤-(b)

Short Answer Questions - Work will not be graded

2. Determine a suitable form for the particular solution y_p if the method of undetermined coefficients is to be used (DO NOT SOLVE).

(a) (**5 Points**)

$$2y'' + 3y' + 5y = 2t^2 \sin t,$$

Answer: $y_p = (A_1 t^2 + A_2 t + A_2) \cos t + (B_1 t^2 + B_2 t + B_2) \sin t$

(b) (**5 Points**)

$$y'' + 2y' + y = t^2 e^{-t},$$

The first guess will be $y_p = (At^2 + Bt + C)e^{-t}$. However, since homogeneous solutions are $y_1 = e^{-t}$ and $y_2 = te^{-t}$, Ce^{-t} and Bte^{-t} in y_p are repeated with both y_1 and y_2. Therefore, y_p must be multiplied by t as $y_p = t(At^2 + Bt + C)e^{-t}$. But again, Cte^{-t} is same as the homogeneous solution y_2. Thus, the next guess will be $y_p = t^2(At^2 + Bt + C)e^{-t}$. Now no terms in y_p are repeated in the homogenous solution.

Answer: $y_p = t^2(At^2 + Bt + C)e^{-t}$

Short Answer Questions - Work will not be graded

3. (**10 Points**) Find the longest interval in which the solution of the initial value problem

$$(t^2 - 1)(t + 4)y'' + ty' + (t - 1)y = t^{10}, y(0) = 1, y'(0) = 0$$

is certain to exist.

Answer: $-1 < t < 1$

Show your work

4. (**10 Points**) A tank originally contains 100 gallons of fresh water. Then water containing 1 lb of salt per gallon is poured into the tank at a rate of 2 gallons per minute, and the mixture is allowed to leave at the same rate. Find the time at which the tank contains 98 lbs of salt. Please justify your answer fully.

$$\frac{dQ}{dt} = 2 - 2\frac{Q}{100} = 2 - \frac{Q}{50} = \frac{100-Q}{50}, \quad Q(0) = 0 \tag{1}$$

By separation

$$\frac{1}{100-Q}dQ = \frac{1}{50}dt \tag{2}$$

Take integral both sides

$$\int \frac{1}{100-Q}dQ = \int \frac{1}{50}dt + C \tag{3}$$

and obtain

$$-\ln|100-Q| = \frac{1}{50}t + C. \tag{4}$$

Then

$$100 - Q = Ce^{-\frac{1}{50}t} \to Q = 100 - Ce^{-\frac{1}{50}t}. \tag{5}$$

From the initial condition $Q(0) = 0$.

$$0 = 100 - C \to C = 100 \tag{6}$$

Therefore,

$$Q(t) = Q = 100 - 100e^{-\frac{1}{50}t}. \tag{7}$$

Let $Q = 98$ and solve for t

$$98 = 100 - 100e^{-\frac{1}{50}t} \to \frac{1}{50} = e^{-\frac{1}{50}t} \to -\frac{1}{50}t = \ln\frac{1}{50} \to t = -50\ln\frac{1}{50} \tag{8}$$

Answer: $t = -50\ln\frac{1}{50}$

Show your work

5. (**10 Points**) Find the general solution of the first-order differential equation

$$\frac{dv}{dt} - \frac{1}{t}v = 2, t > 0.$$

Integrating Factor

$$\mu(t) = e^{\int -\frac{1}{t}dt} = e^{-\ln t} = t^{-1} \tag{9}$$

Multiply both sides

$$t^{-1}\frac{dv}{dt} - t^{-1}\frac{1}{t}v = t^{-1}2 \tag{10}$$

From the product rule

$$(t^{-1}v)' = t^{-1}2 \tag{11}$$

Take integral both sides

$$\int (t^{-1}v)'dt = \int t^{-1}2dt + C \to t^{-1}v = 2\ln t + C \to v = 2t\ln t + Ct \tag{12}$$

Answer: $v = 2t\ln t + Ct$

Extra Credit Problem (**5 Points**) : Determine how the solution behaves as $t \to 0^+$ and justify your answer.

$$\lim_{t\to 0^+} 2t\ln t + Ct = \lim_{t\to 0^+} 2\frac{\ln t}{\frac{1}{t}} + Ct = 0 \text{ (by L'Hospital's rule)} \tag{13}$$

Answer: $v \to 0$ as $t \to 0^+$

Show your work

6. (**10 Points**) Find the explicit solution of the initial value problem

$$(e^x y + 2x) + (2y + e^x)y' = 0, y(0) = 1,$$

$$M = (e^x y + 2x), N = (2y + e^x) \tag{14}$$

$$M_y = e^x, N_x = e^x \rightarrow M_y = N_x \rightarrow \text{exact equation} \tag{15}$$

Therefore

$$\psi_x = (e^x y + 2x) \text{ and } \psi_y = (2y + e^x) \tag{16}$$

Integrate ψ_x with respect to x and obtain

$$\psi = e^x y + x^2 + h(y) \tag{17}$$

Take derivative with respect to y

$$\psi_y = e^x + h'(y) \tag{18}$$

and compare with $\psi_y = (2y + e^x)$. Thus

$$h'(y) = 2y \rightarrow h(x) = y^2 \tag{19}$$

Therefore $\psi = e^x y + x^2 + y^2$ and the solution to the differential equation is

$$y^2 + e^x y + x^2 = C \tag{20}$$

From the initial condition $y(0) = 1$

$$1^2 + e^0 \cdot 1 + 0^2 = C \rightarrow C = 2 \tag{21}$$

$$y^2 + e^x y + x^2 = 2 \rightarrow y^2 + e^x y + x^2 - 2 = 0 \text{ (Quadratic equation)} \tag{22}$$

Therefore

$$y = \frac{-e^x \pm \sqrt{(e^x)^2 - 4(x^2 - 2)}}{2} \tag{23}$$

Check initial condition

$$y(0) = \frac{-e^0 \pm \sqrt{(e^0)^2 - 4(0^2 - 2)}}{2} = \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2} = 1(+) \text{ or } -2(-) \tag{24}$$

Therefore,"+" is the correct choice for the solution

Answer: $y = \frac{-e^x + \sqrt{(e^x)^2 - 4(x^2-2)}}{2}$

Show your work

7. Consider

$$y'' + 3y' + 2y = 10 \sin t.$$

(a) (**5 Points**) Find a fundamental set of solutions of

$$y'' + 3y' + 2y = 0,$$

and show that they form a fundamental set of solutions.

$$y'' + 3y' + 2y = 0 \tag{25}$$

Let $y = e^{rt}$. Then the characteristic equation is

$$r^2 + 3r + 2 = 0 \to (r+2)(r+1) = 0 \to y_1 = e^{-2t} \text{ and } y_2 = e^{-t} \tag{26}$$

Wronskian

$$W = \begin{vmatrix} e^{-2t} & e^{-t} \\ -2e^{-2t} & -e^{-t} \end{vmatrix} = -e^{-3t} + 2e^{-3t} = e^{-3t} \neq 0 \text{ for any } t. \tag{27}$$

Therefore e^{-2t} and e^{-t} form a fundamental set of solutions and the general solution is

$$y_h = c_1 e^{-2t} + c_2 e^{-t} \tag{28}$$

Answer: $y_1 = e^{-2t}$ and $y_2 = e^{-t}$

Show your work

(b) (**7 Points**) Find a particular solution of

$$y'' + 3y' + 2y = 10 \sin t$$

Let

$$y_p = A \cos t + B \sin t \tag{29}$$

Then,

$$y_p' = -A \sin t + B \cos t \tag{30}$$

$$y_p'' = -A \cos t - B \sin t \tag{31}$$

Substitute into the nonhomogenous equation

$$(-A \cos t - B \sin t) + 3(-A \sin t + B \cos t) + 2(A \cos t + B \sin t) = 10 \sin t \tag{32}$$

$$(-A + 3B + 2A) \cos t + (-B - 3A + 2B) \sin t = 10 \sin t \tag{33}$$

$$(3B + A) \cos t + (-3A + B) \sin t = 10 \sin t \tag{34}$$

Therefore

$$A + 3B = 0 \text{ and } B - 3A = 10 \tag{35}$$

Then

$$B = 1 \text{ and } A = -3 \tag{36}$$

Therefore

$$y_p = -3 \cos t + \sin t \tag{37}$$

Answer: $y_p = -3 \cos t + \sin t$

(c) (**3 Points**) Find the general solution of $y'' + 3y' + 2y = 10 \sin t$.

Answer: $y = y_h + y_p = c_1 e^{-2t} + c_2 e^{-t} - 3 \cos t + \sin t$

Show your work

8. (**15 Points**) Suppose a spring follows the differential equation

$$my'' + ky = 0$$

where $y(t)$ is the displacement of the spring, m is the mass of the object attached to the spring, and k is the spring constant. Let $m = 1$ and $k = 4$. Suppose the spring is stretched by 1 when $t = 0$ with an initial velocity $v(0) = -2$. When will the spring come back to the initial location for the first time (*i.e.* $y(t) = 0$)?

$$y'' + 4y = 0, y(0) = 1, y'(0) = 2 \tag{38}$$

Let $y = e^{rt}$

$$r^2 + 4 = 0 \rightarrow r = \pm 2i \tag{39}$$

Therefore

$$y_1 = \cos 2t, y_2 = \sin 2t \rightarrow y = c_1 \cos 2t + c_2 \sin 2t \tag{40}$$

From the initial condition

$$y(0) = c_1 \cos 0 + c_2 \sin 0 = c_1 = 1 \tag{41}$$

$$y'(0) = -2c_1 \sin 0 + 2c_2 \cos 0 = 2c_2 = -2 \rightarrow c_2 = -1 \tag{42}$$

Thus,

$$y = \cos 2t - \sin 2t \tag{43}$$

Let $y = 0$ and find t

$$0 = \cos 2t - \sin 2t \rightarrow \cos 2t = \sin 2t \rightarrow 2t = \frac{\pi}{4} \rightarrow t = \frac{\pi}{8} \tag{44}$$

Answer: $t = \frac{\pi}{8}$

Show your work

9. (**10 Points**) Consider the 2nd-order linear differential equation

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = \frac{1}{t}, t > 0.$$

The functions $y_1 = t^2$ and $y_2 = t$ form a fundamental set of solutions to the associated homogeneous differential equation. The Wronskian of y_1 and y_2 is $-t^2$. Use the variation of parameter ($y = u(t)y_1 + v(t)y_2$) to find the general solution.

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = \frac{1}{t} \tag{45}$$

Let $y_p = u(t)t^2 + v(t)t$ Then

$$u(t) = -\int \frac{y_2(t)g(t)}{W}dt, v(t) = \int \frac{y_1(t)g(t)}{W}dt \tag{46}$$

Therefore

$$u(t) = -\int \frac{y_2(t)g(t)}{W}dt = -\int \frac{t\frac{1}{t}}{-t^2}dt = \int \frac{1}{t^2}dt = -\frac{1}{t} \tag{47}$$

$$v(t) = \int \frac{y_1(t)g(t)}{W}dt = \int \frac{t^2\frac{1}{t}}{-t^2}dt = -\int \frac{1}{t}dt = -\ln t \tag{48}$$

Thus,

$$y_p = -\frac{1}{t}t^2 - (\ln t)t = -t - (\ln t)t \tag{49}$$

Finally, the general solution is

$$y = c_1t^2 + c_2t - t - (\ln t)t \tag{50}$$

Answer: $y = c_1t^2 + c_2t - t - (\ln t)t$ or $y = c_1t^2 + c_2t - (\ln t)t$

Extra page for scratch work. I will not grade work on this page unless you write on another page"problem continued on page 12".

Extra page for scratch work. I will not grade work on this page unless you write on another page "problem continued on page 13".