Math 118. **Combinatorics.** Spring 2013

Problem Set 2. Due on Wednesday, 4/24/2013.

1. Give direct proofs (bijective or using generating functions, or both) of the following statements:

   (a) The number of partitions of n into parts congruent to ± 1 mod 3 equals the number of partitions of n where every part appears at most twice.

   (b) The number of partitions of n into parts congruent to ± 1 mod 6 equals the number of partitions of n into distinct parts congruent to ± 1 mod 3.

2. Let $f(n)$ (respectively, $g(n)$) be the number of partitions $\lambda \vdash n$ into distinct parts, such that the largest part λ_1 is even (respectively, odd). Prove that

$$f(n) - g(n) = \begin{cases} 1, & \text{if } n = k(3k+1)/2 \text{ for some } k \geq 0 \\ -1, & \text{if } n = k(3k-1)/2 \text{ for some } k \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

3. Let $e(n) = \#$ of partitions of n with an even number of even parts,
   $o(n) = \#$ of partitions of n with an odd number of even parts.
   Show that $e(n) - o(n) = \#$ of self-conjugate partitions of n.
   Recall that λ is self-conjugate if $\lambda = \lambda'$.

4. (*) Let $f(n)$ be the number of partitions of $2n$ whose Young diagram can be tiled with n dominoes. For instance, (4,3,3,3,1) is such a partition. Prove that $f(n)$ is equal to the number of ordered pairs (λ, μ) of partitions satisfying $|\lambda| + |\mu| = n$.