

# Combinatorics of affine Hecke algebras of type C.

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(joint with Arun Ram)

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$$T_0 \quad T_1 \quad T_2 \quad \cdots \quad T_{k-2} \quad T_{k-1} \quad T_k$$

Pictorially, the generators of  $\mathcal{B}_k$  are identified with the diagrams

$$T_k = \text{Diagram of } T_k \text{ (horizontal line with circles)} , \quad T_0 = \text{Diagram of } T_0 \text{ (vertical line segment)} ,$$

and

$$T_i = \text{Diagram of } T_i \text{ (vertical strands with labels i, i+1)} \quad \text{for } i = 1, \dots, k-1.$$

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$$\begin{array}{ccccccc} T_0 & T_1 & T_2 & & T_{k-2} & T_{k-1} & T_k \\ \textcircled{=} & \textcircled{=} & \textcircled{=} & \cdots & \textcircled{=} & \textcircled{=} & \textcircled{=} \end{array}$$

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$$T_i T_{i+1} T_i = \text{Diagram showing } T_i T_{i+1} T_i = \text{Diagram showing } T_{i+1} T_i T_{i+1}$$

The diagram illustrates the relation  $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ . It consists of two parts separated by an equals sign. Each part shows three strands. In the first part, the strands are labeled  $i$ ,  $i+1$ , and  $i$  from left to right. The strand  $i$  passes over strand  $i+1$ , which then passes over strand  $i$ . In the second part, the strands are labeled  $i+1$ ,  $i$ , and  $i+1$  from left to right. The strand  $i+1$  passes over strand  $i$ , which then passes over strand  $i+1$ .

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(similar picture for  $T_k T_{k-1} T_k T_{k-1} = T_{k-1} T_k T_{k-1} T_k$ )

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**Two (isomorphic) quotients, two perspectives:**

- Fix  $t, t_0, t_k \in \mathbb{C}^\times$ . The affine Hecke algebras of type C  $H_k$  is the quotient of  $\mathbb{C}\mathcal{B}_k$  by

$$(*) \quad 0 = (T_0 - t_0)(T_0 - t_0^{-1}) = (T_k - t_k)(T_k - t_k^{-1}) = (T_i - t^{1/2})(T_i + t^{-1/2})$$

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for  $i = 1, \dots, k-1$ .

2. Let  $A, B, C$  be finite dim'l  $U_q\mathfrak{g}$ -modules. Then  $\mathbb{C}\mathcal{B}_k$  acts on

$$B \otimes \underbrace{C \otimes \cdots \otimes C}_{k \text{ factors}} \otimes A$$

Under good (to be defined) conditions, this action factors through the quotient (\*).

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**Goal today:**

Tell you 3 descriptions of “calibrated” irreducible reps of  $H_k$ .

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$W$  is a group of signed permutations generated by transpositions  $s_0, s_1, \dots, s_{k-1}$  with relations

$$\begin{array}{ccccccccc} & s_0 & & s_1 & & s_2 & & & s_{k-2} & s_{k-1} \\ & \bullet & = & \bullet & - & \bullet & - & \cdots & \bullet & - & \bullet \end{array}$$

$$s_j^2 = 1, \quad s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0 \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

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$W$  acts on the subscripts of the  $Y_i$ 's with  $Y_{-i} = Y_i^{-1}$ .

## Central characters

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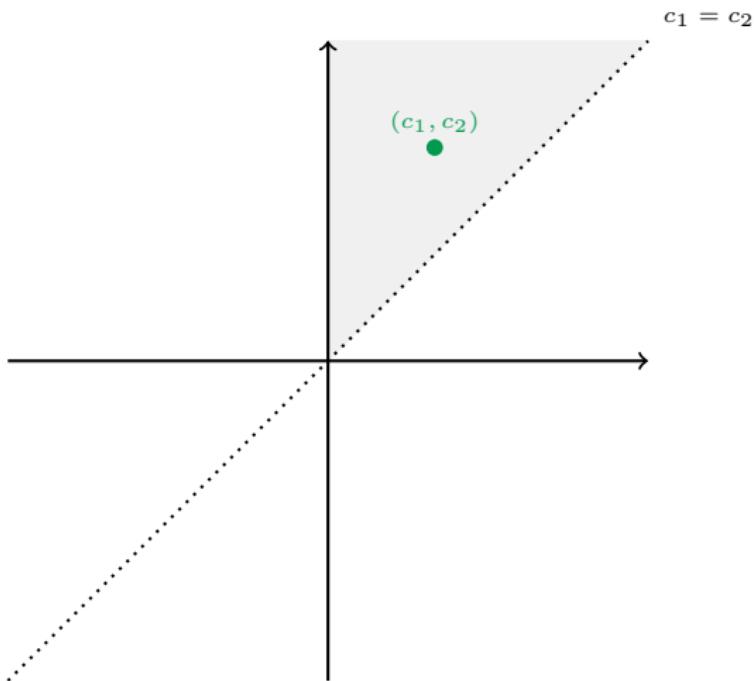
Description 1: Central characters are indexed by points in  $k$  dimensions.

## Central characters as points

Restrict to real points.

Fav equivalence class reps:  $0 \leq c_1 \leq \cdots \leq c_k$ .

When  $k = 2$ :

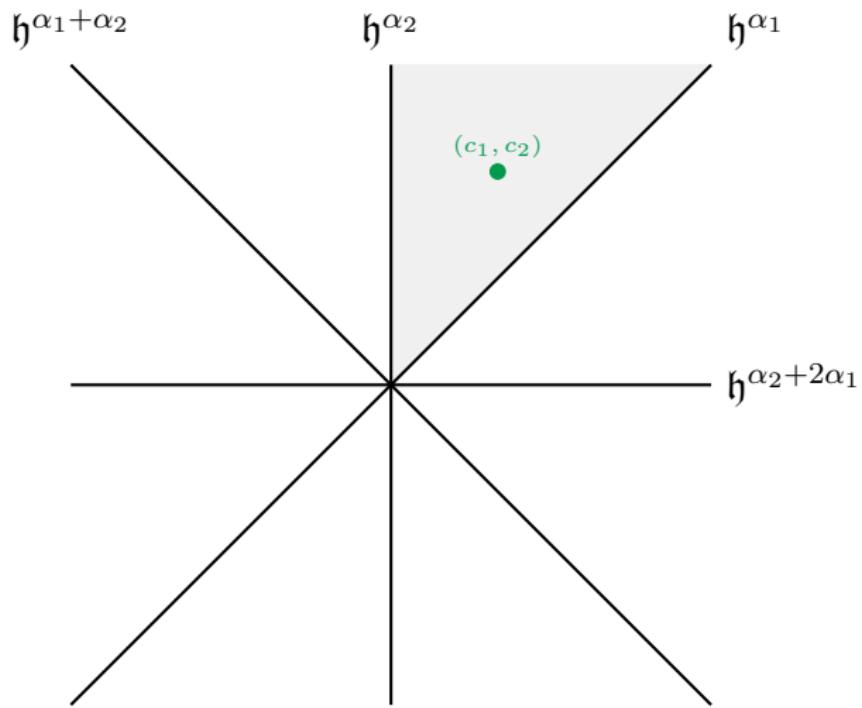


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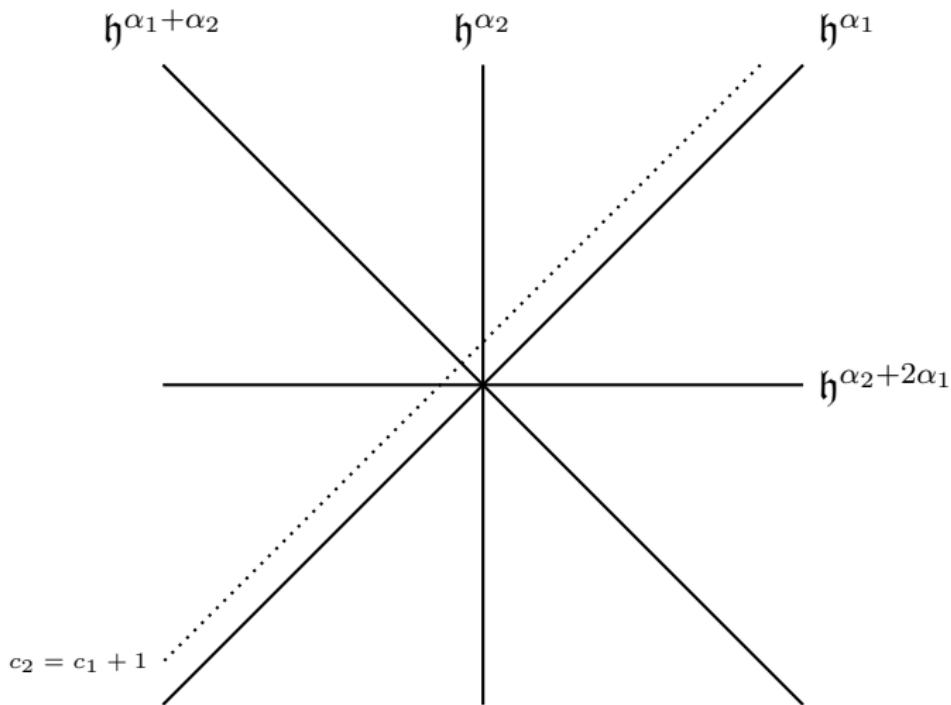


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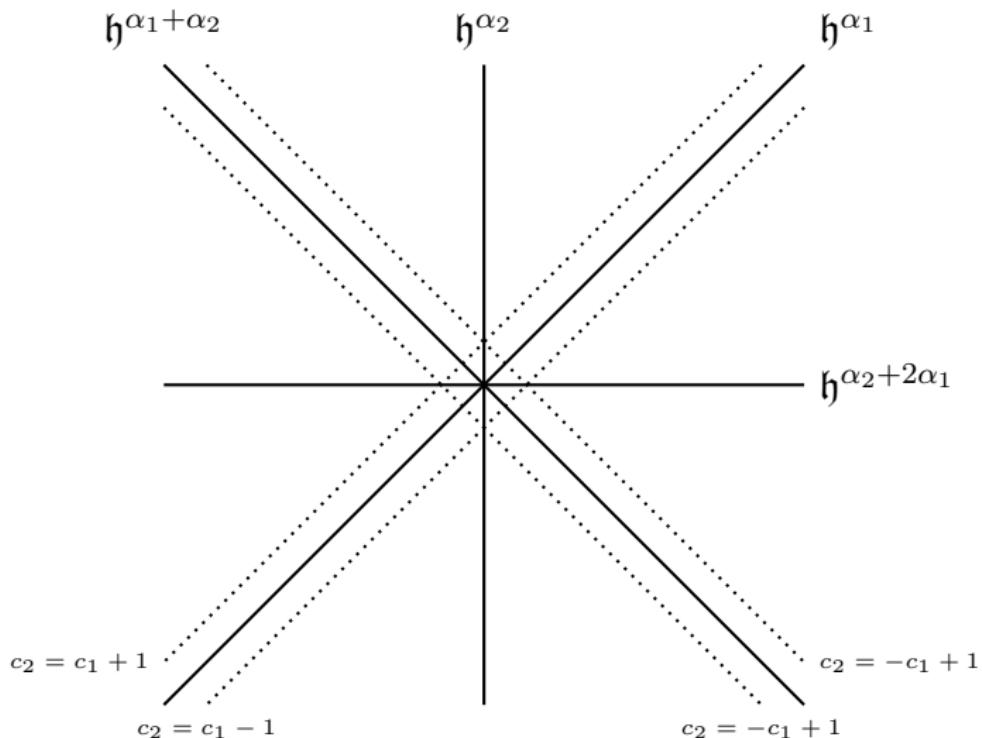


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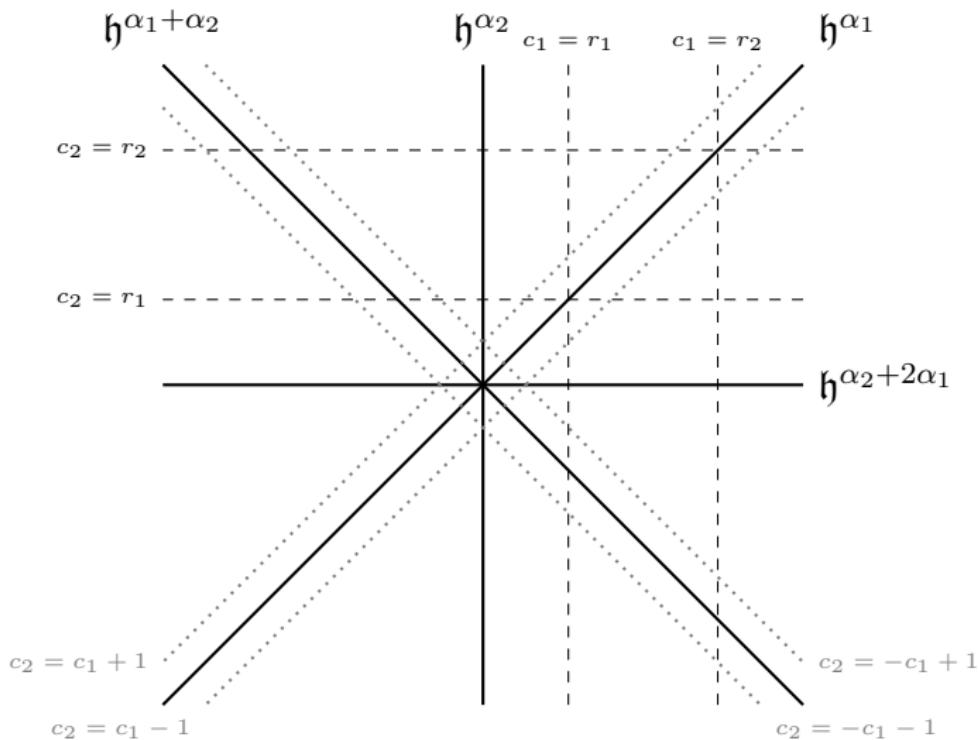


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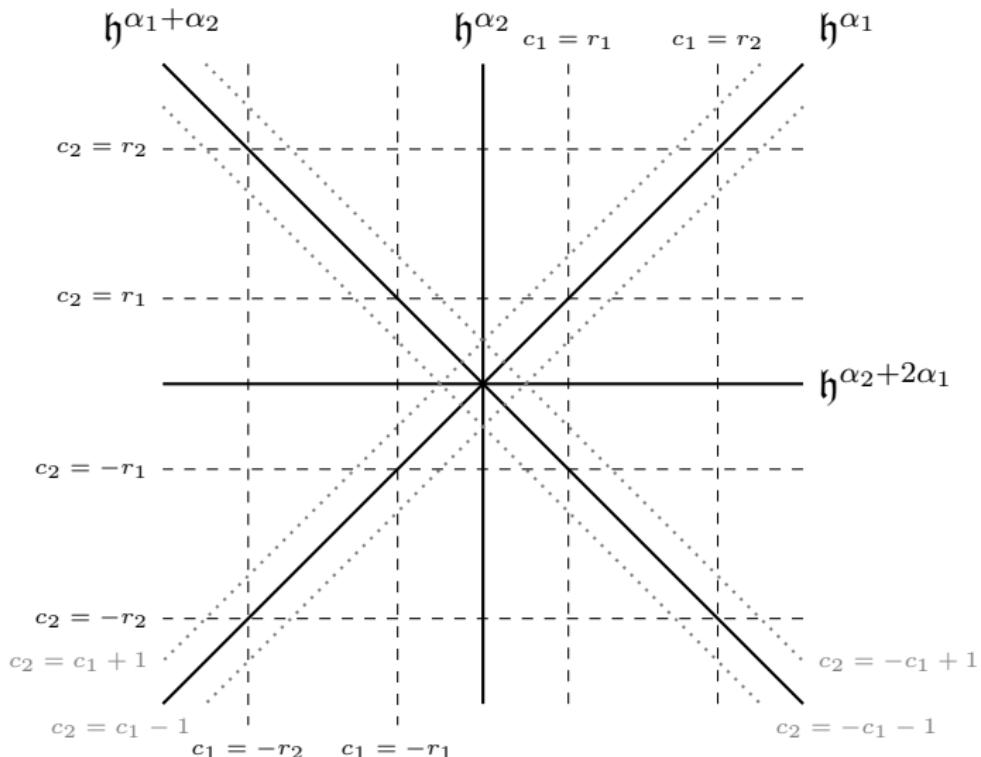
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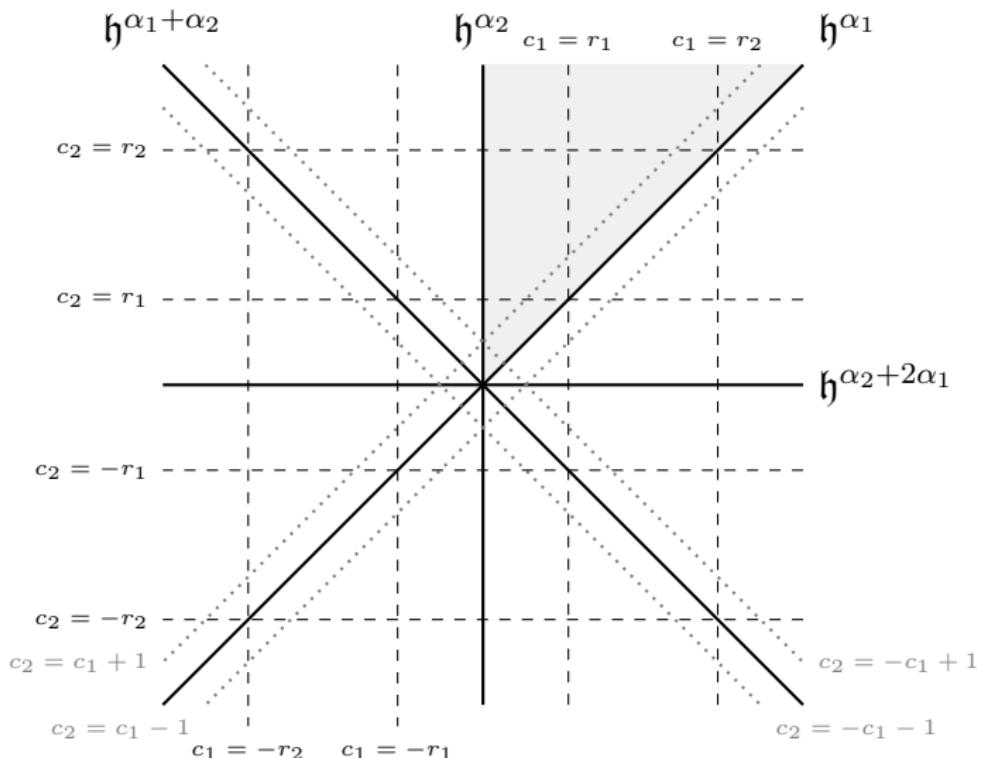
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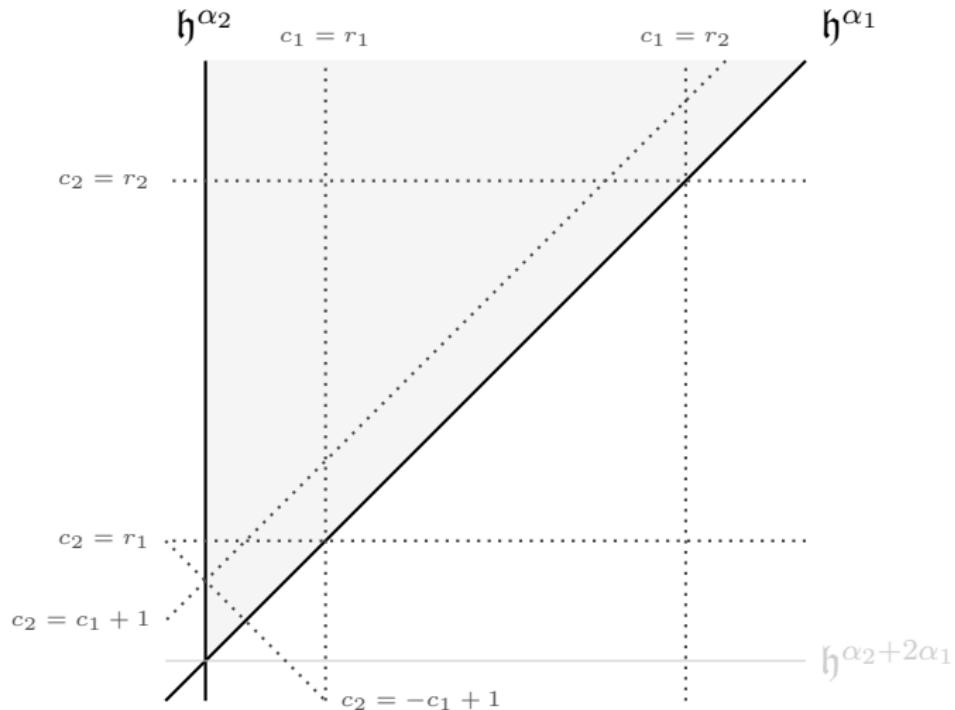
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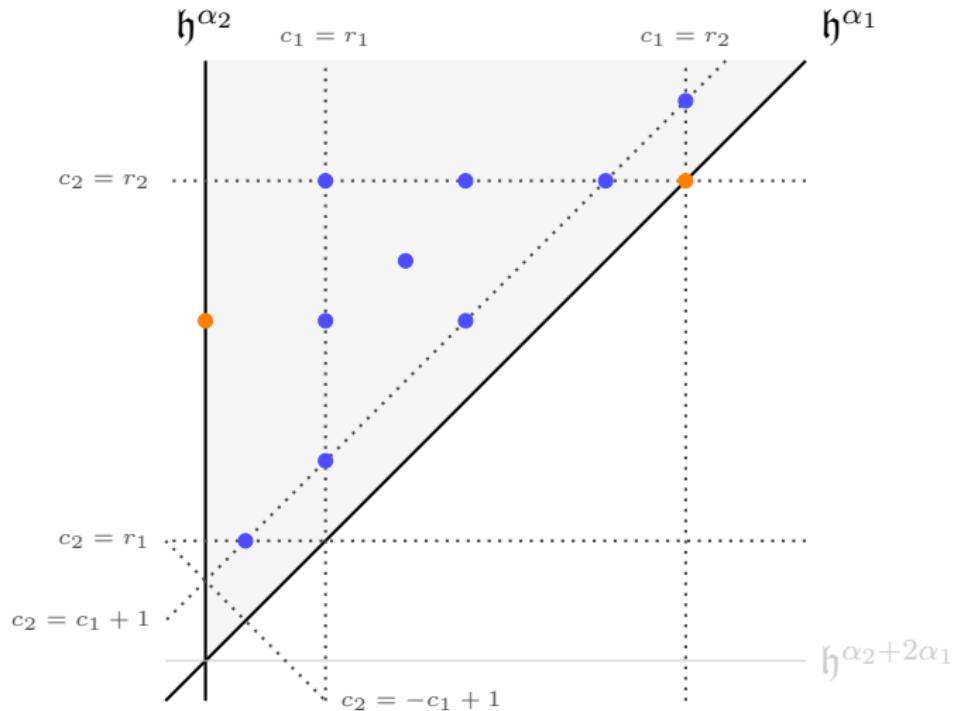
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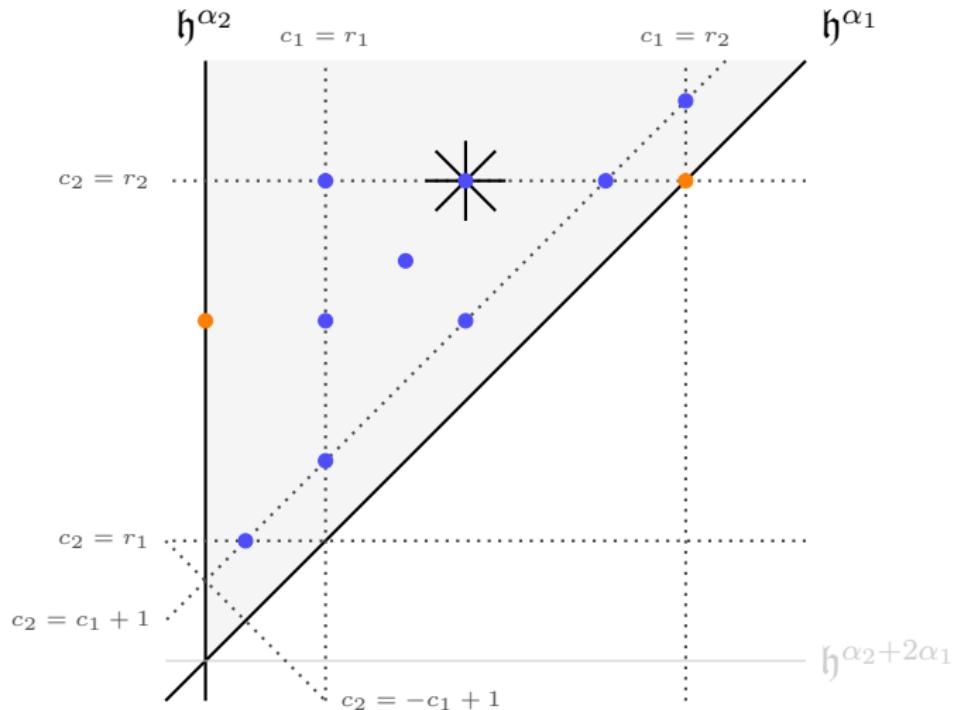
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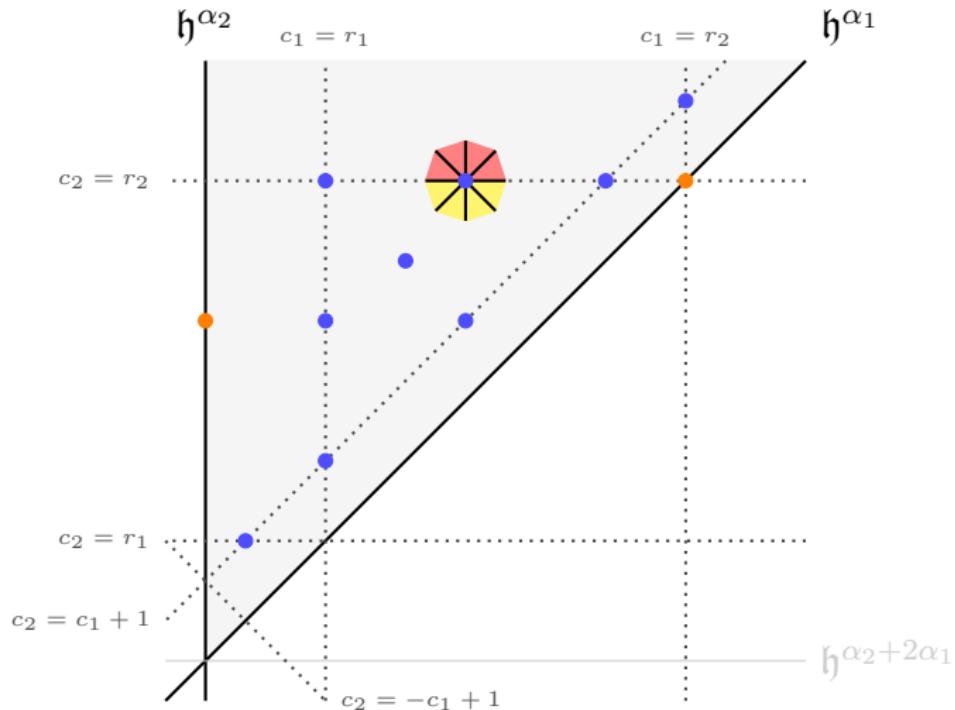
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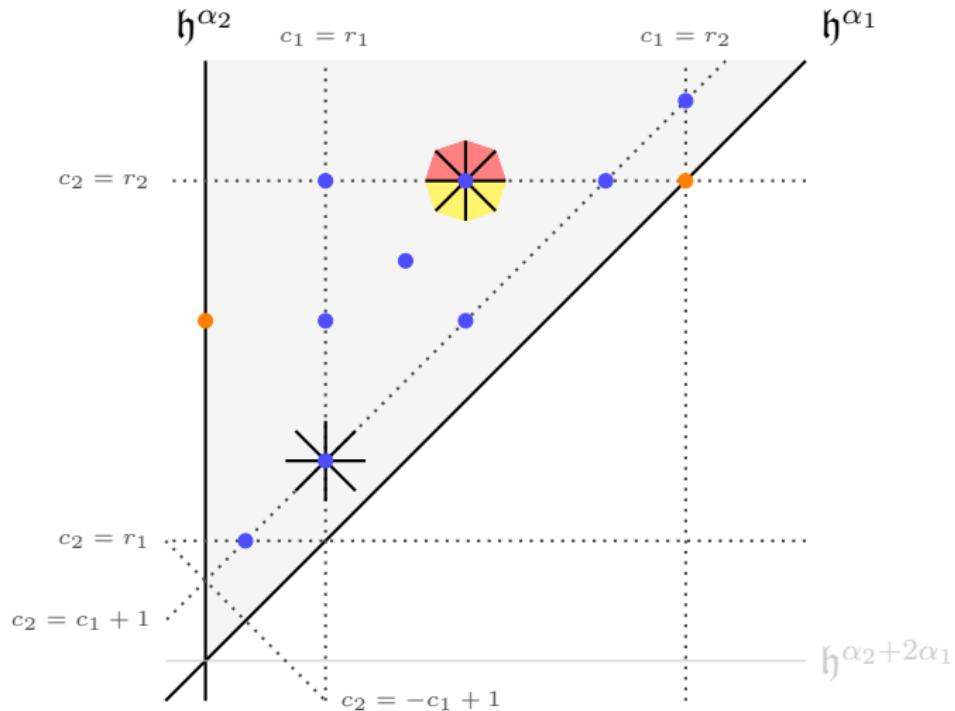
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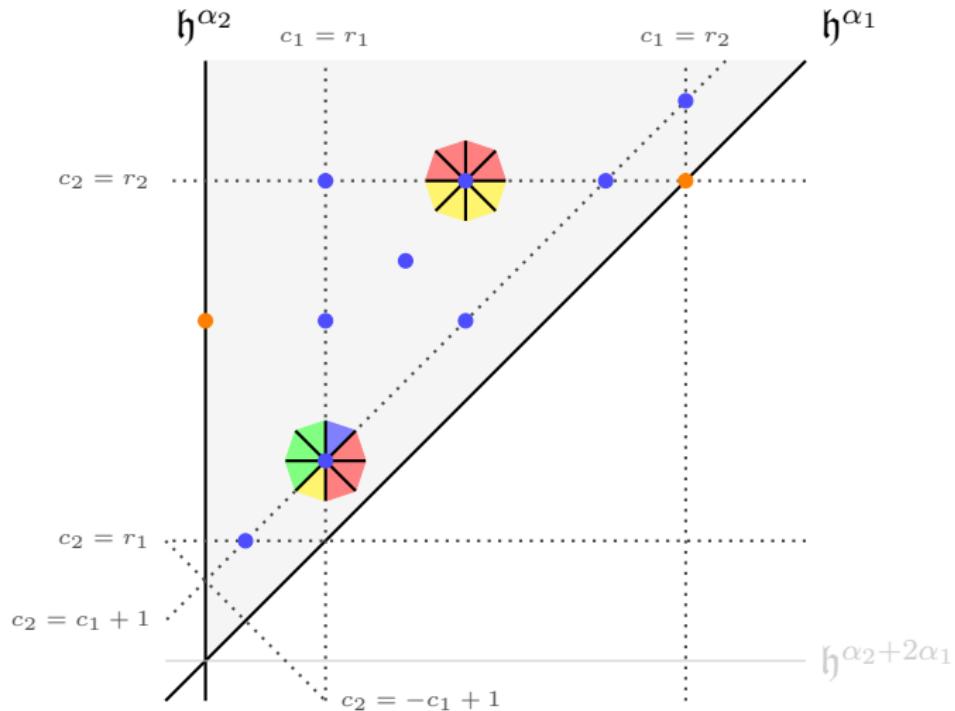
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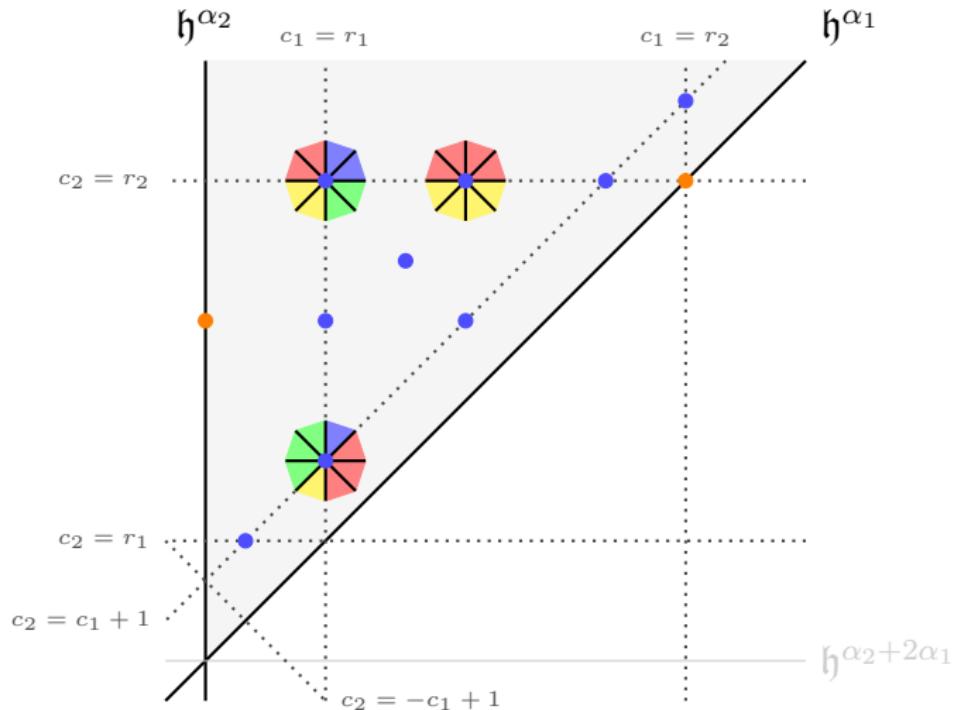
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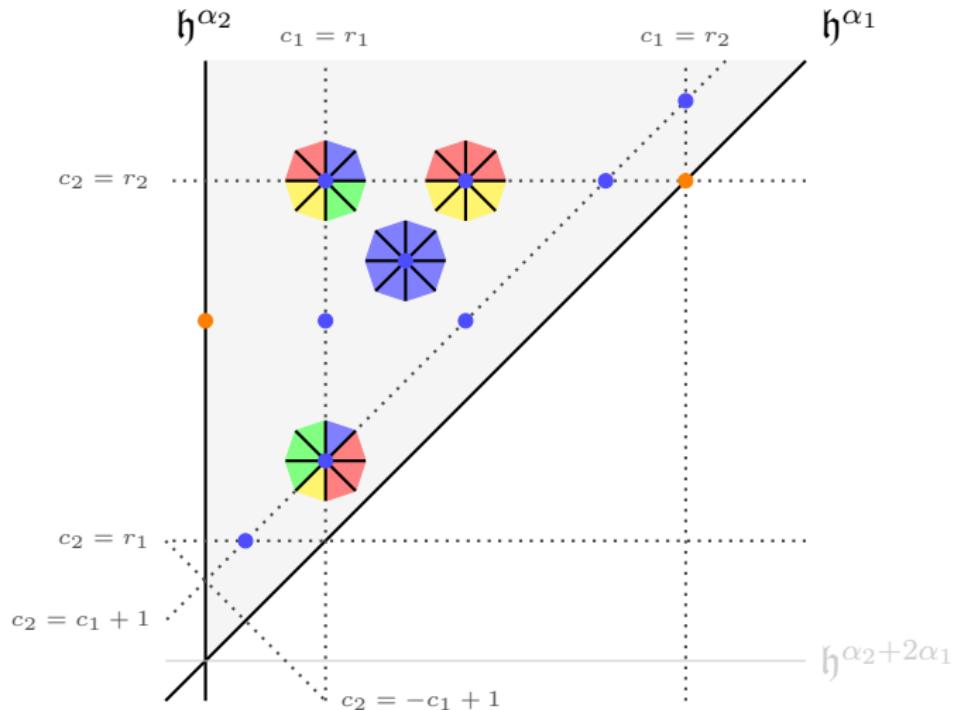
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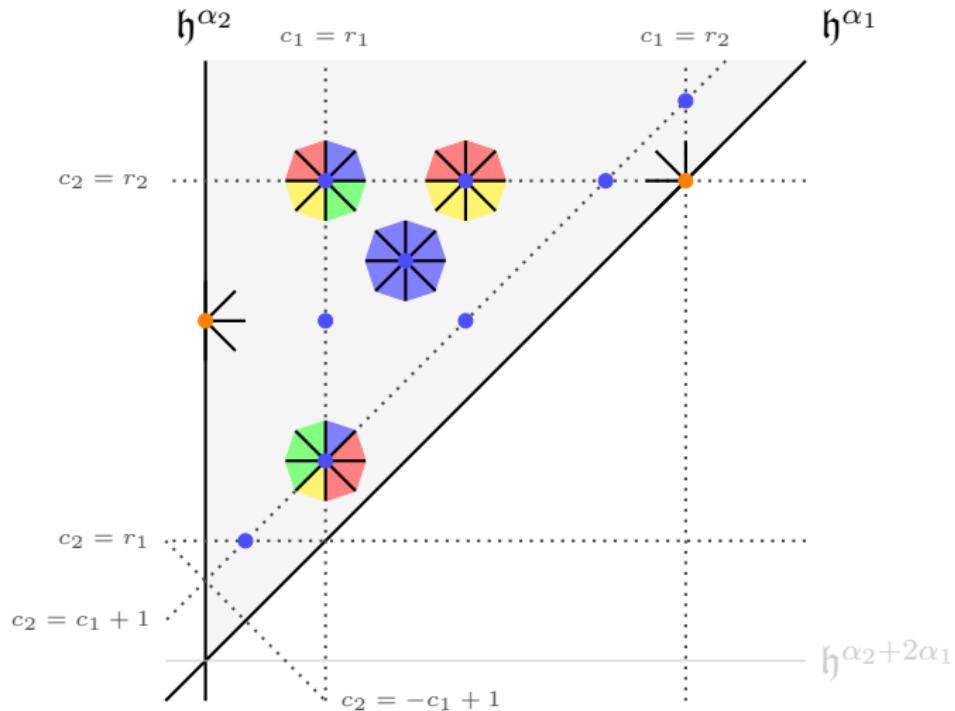
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Irreps are indexed by skew local regions around points.

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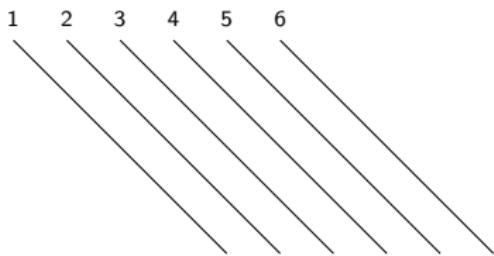
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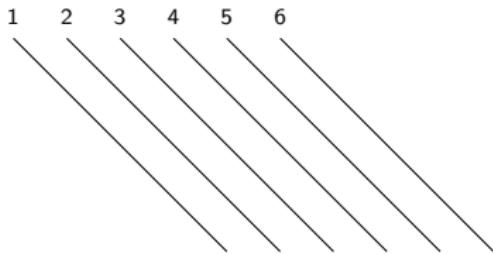
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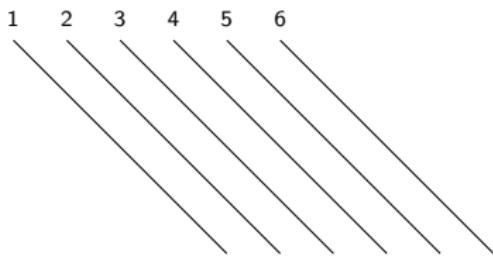
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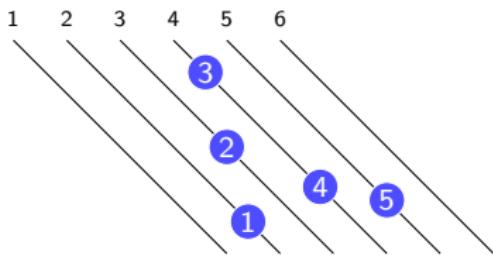
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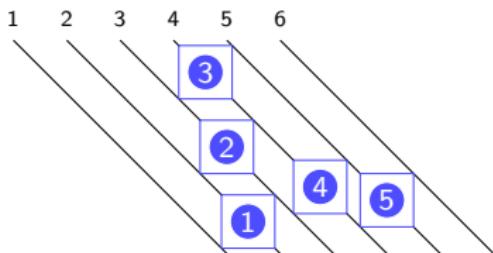
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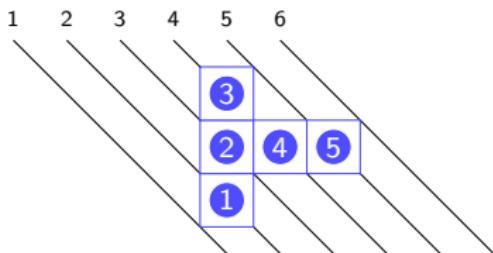
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Start with diagonal lines labeled by  $\mathbb{Z}$ . Restrict to points in  $(\mathbb{Z} + \beta)^k$ . A central character  $c$  gives a list of diagonal placements.

For example:

$$c = (2, 3, 4, 4, 5)$$



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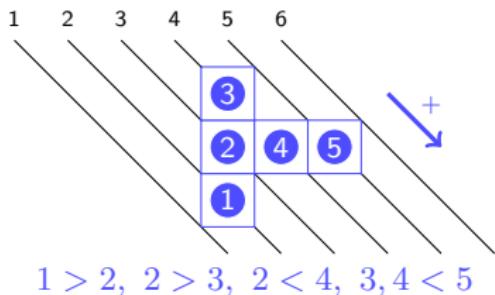
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$$1 > 2, 2 > 3, 2 < 4, 3, 4 < 5$$

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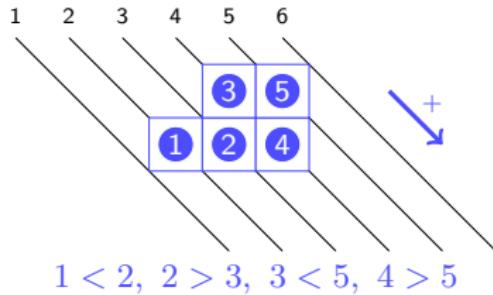
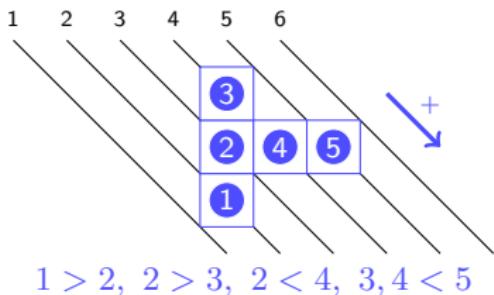
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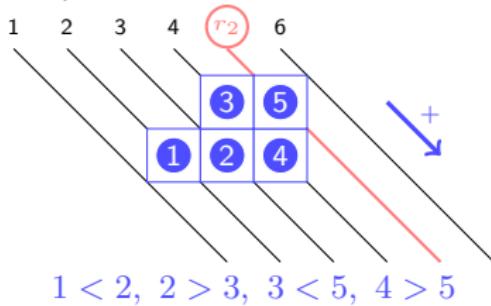
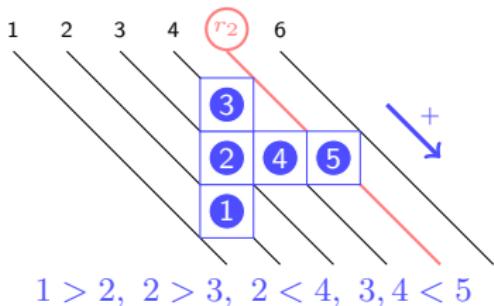
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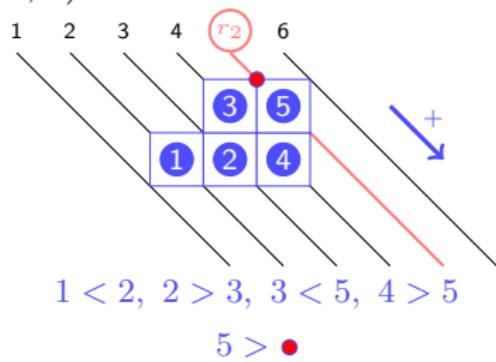
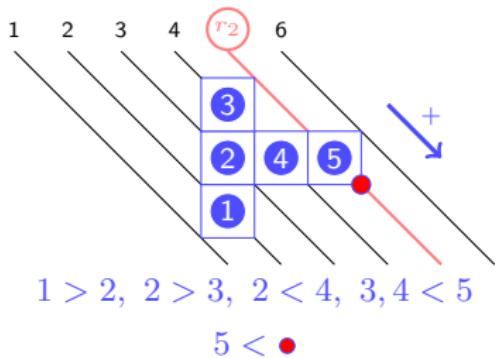
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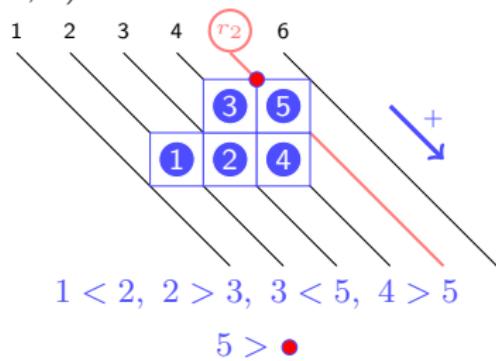
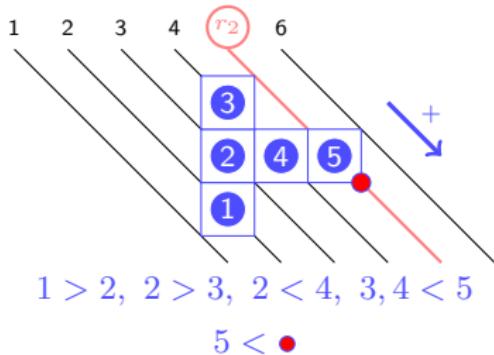
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Basis indexed by standard fillings with  $\{\pm 1, \dots, \pm k\}$  with restrictions:

(1) Exactly one of  $i$  or  $-i$  appears.

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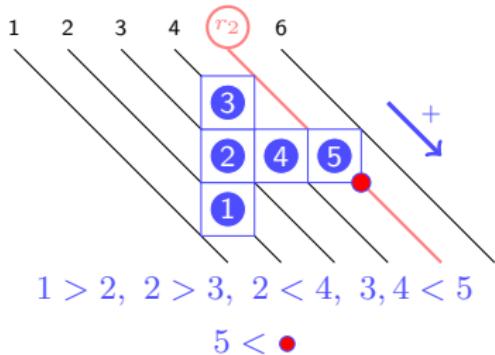
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For example:

$$c = (2, 3, 4, 4, 5)$$



-5			
-4	-2	-1	●
-3			
-5			
-4	-3	-2	●
-3			
-5			
-4	-3	-2	●
1			
5			

ok!

bad!

ok!

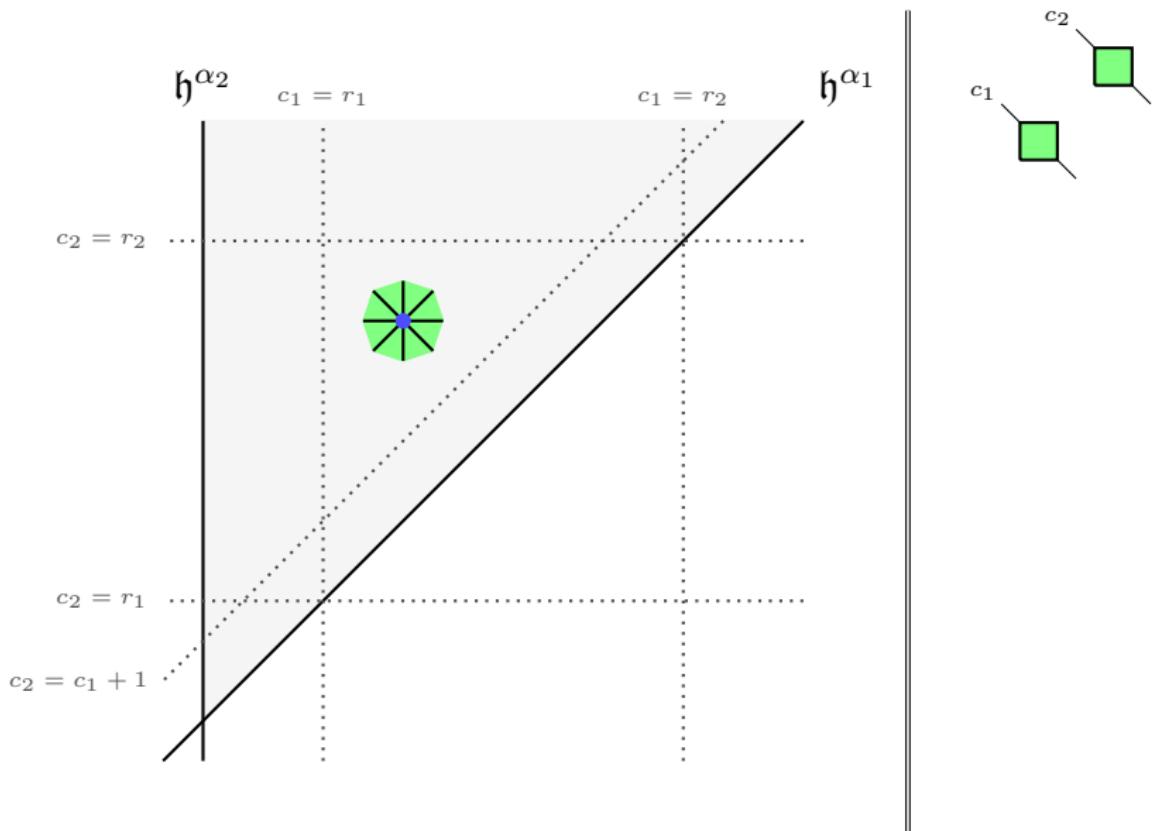
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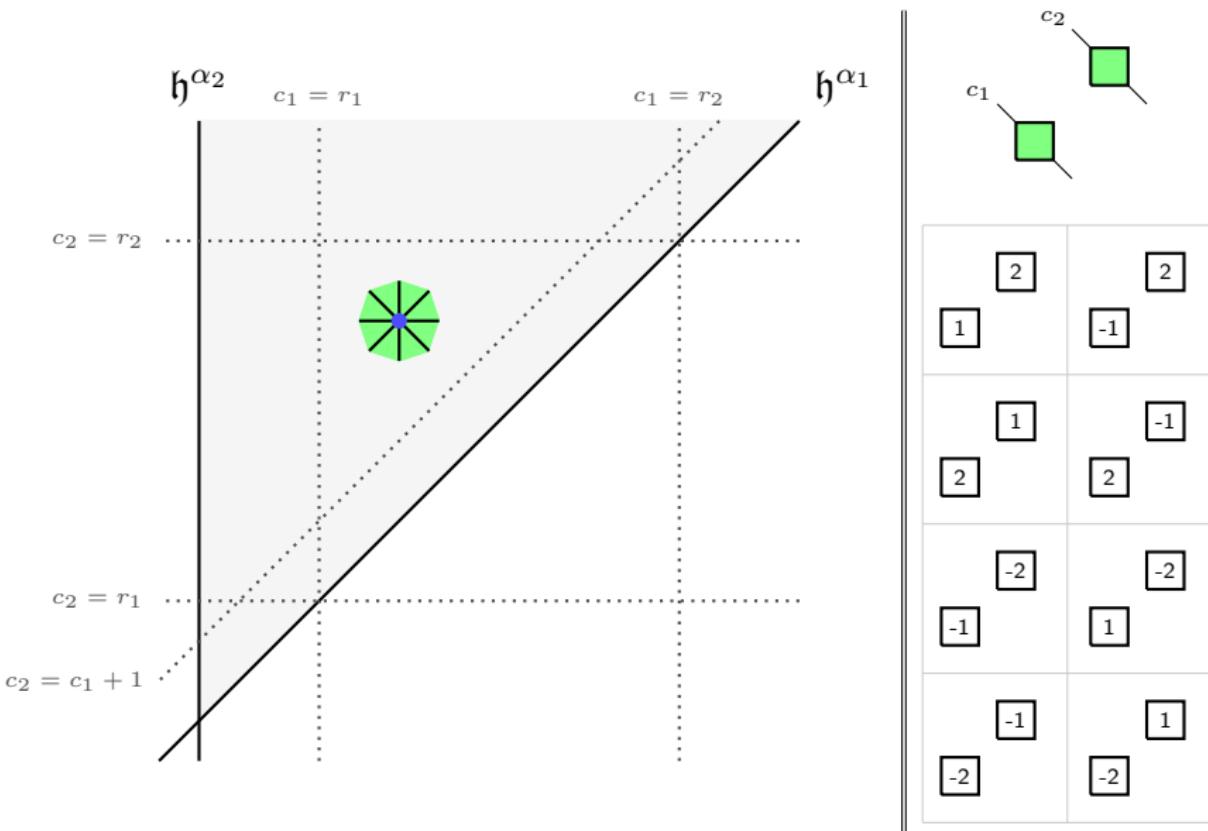
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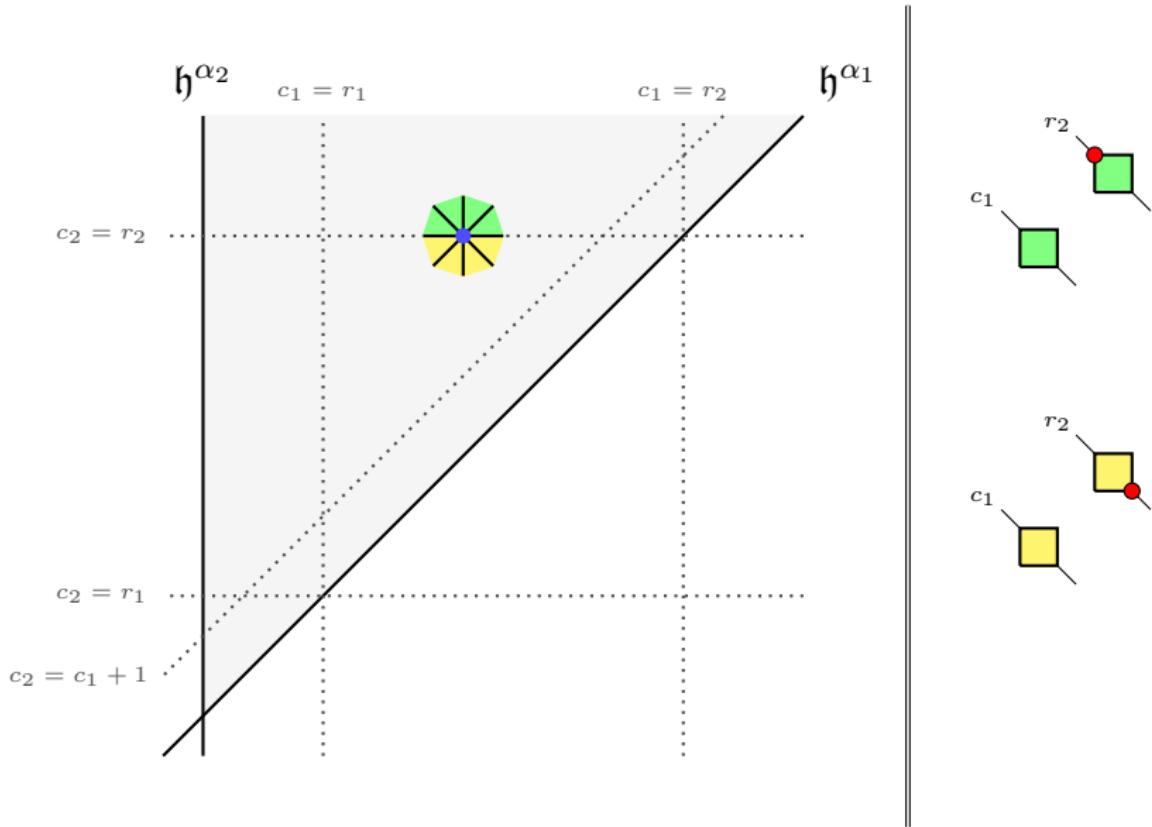
## Points versus box arrangements



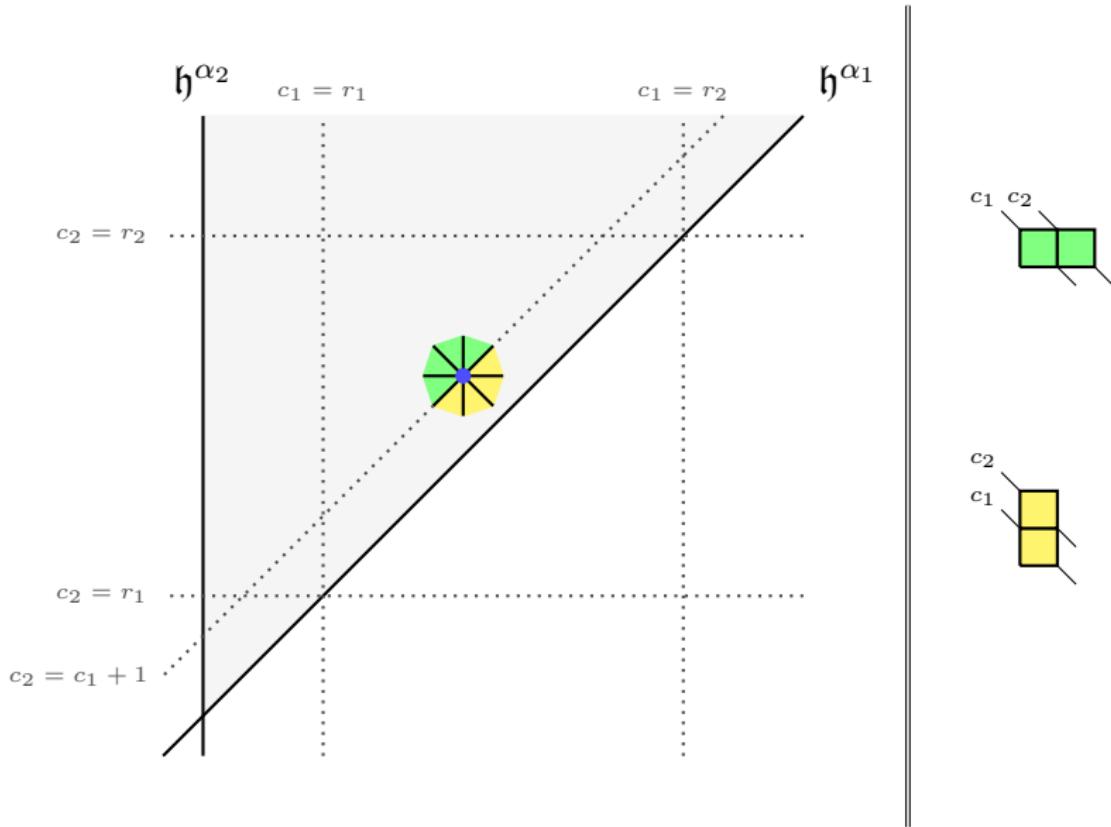
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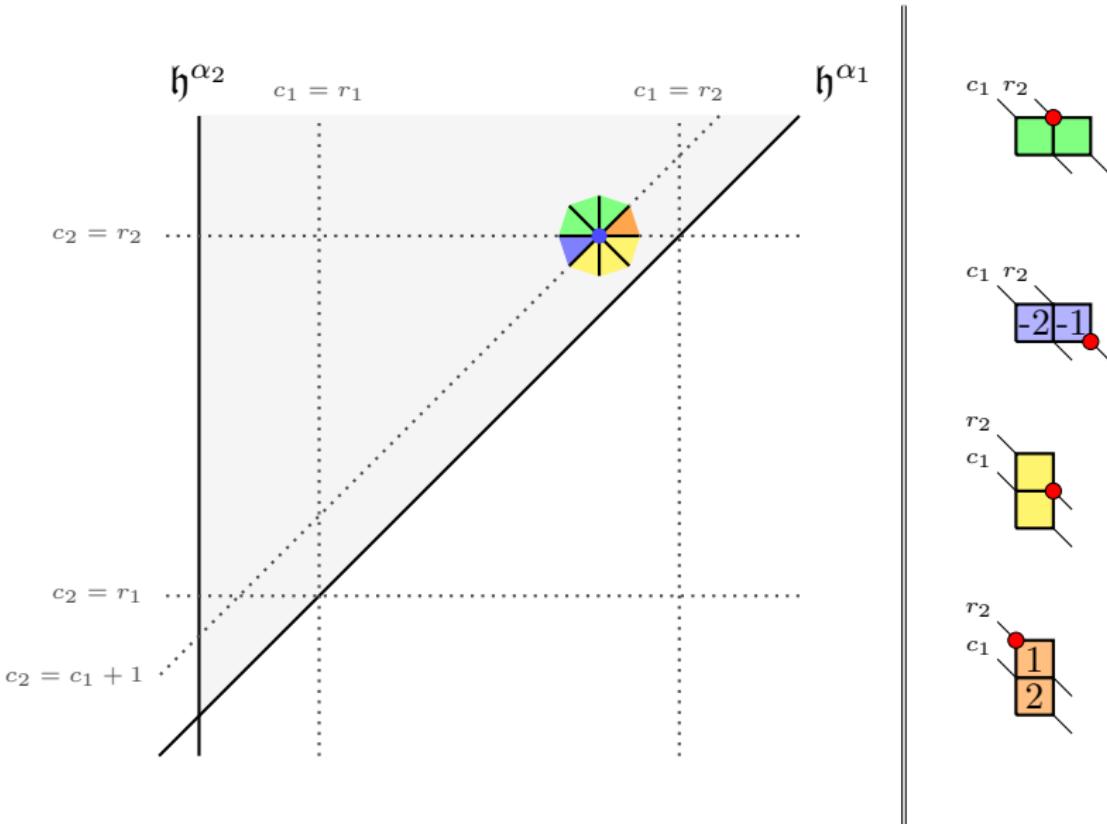
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Basis is indexed by chambers in each region.

**Description 2:** Marked box arrangements.  
Basis indexed by good fillings.

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**Description 3:** Partitions.  
Representation arise in Schur-Weyl duality with certain  $U_q\mathfrak{gl}_n$  reps.

## Centralizer properties

Let  $U = U_q \mathfrak{gl}_n$  be the quantum group for  $\mathfrak{gl}_n(\mathbb{C})$ . We're interested in certain finite dimensional simple  $U$ -modules  $L(\lambda)$  indexed by partitions:

$$\lambda = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array}$$

(drawn as a collection of boxes piled up and to the left)

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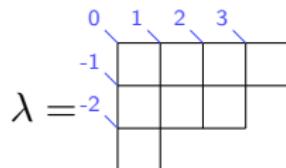
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In particular, rectangular partitions:

$$(a^c) = c \begin{array}{|c|c|c|} \hline & & a \\ \hline & & \\ \hline \end{array}$$

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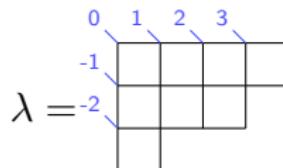
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The content of a box is its diagonal number.

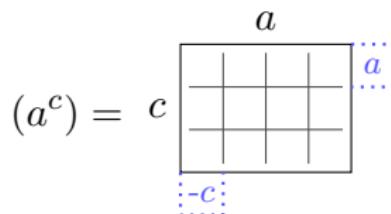
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The content of a box is its diagonal number.

The eigenvalues of  $T_0$  and  $T_k$  are controlled by the contents of addable boxes to  $(a^c)$  and  $(b^d)$ .

## Centralizer properties

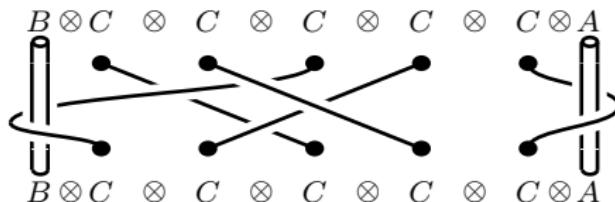
Theorem (D.-Ram)

1. Let  $U = U_q\mathfrak{g}$ , and let  $A$ ,  $B$ , and  $C$  be finite dim'l  $U$ -modules. The two-boundary braid group  $\mathcal{B}_k$  acts on  $B \otimes (C)^{\otimes k} \otimes A$  (via  $R$ -matrices) and this action commutes with that of  $U$ .

# Centralizer properties

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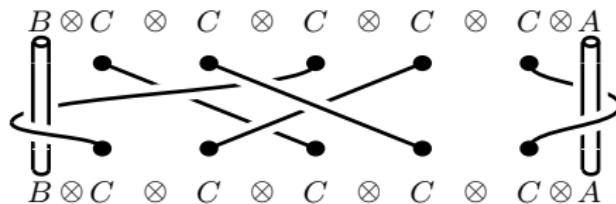
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$R$ -matrices:  $U$  has an associated invertible element  $R = \sum_{\mathcal{R}} R_1 \otimes R_2$  of  $U \otimes U$  that gives us a map

$$\check{R}_{MN}: M \otimes N \longrightarrow N \otimes M$$

This map acts on a component  $L(\lambda)$  of  $L(\mu) \otimes L(\square)$  by  $q^{2c(\lambda/\mu)}$ .

## Centralizer properties

### Theorem (D.-Ram)

2. If  $\mathfrak{g} = \mathfrak{gl}_n$ ,  $A = L((a^c))$ ,  $B = L((b^d))$ , and  $C = L(\square)$ , then the action in 1. factors through the quotient by

$$0 = (T_0 - t_0)(T_0 - t_0^{-1}) = (T_k - t_k)(T_k - t_k^{-1}) = (T_i - t^{1/2})(T_i + t^{-1/2})$$

where  $t = q^2$ ,  $t_0 = t^{\frac{1}{2}(b+d)}$ , and  $t_k = t^{\frac{1}{2}(a+c)}$ .

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$$T_0 : \begin{array}{c} B \otimes C \\ \text{---} \\ \text{---} \\ B \otimes C \end{array} \cdots$$

$$T_k : \cdots \begin{array}{c} C \otimes A \\ \text{---} \\ \text{---} \\ C \otimes A \end{array}$$

$$T_i : \begin{array}{c} C \otimes C \\ \text{---} \\ \text{---} \\ C \otimes C \end{array}$$

$$d \begin{array}{|c|c|c|} \hline & b & \\ \hline & \boxed{-d} & \\ \hline \end{array} \quad \boxed{b}$$

$$c \begin{array}{|c|c|c|} \hline & a & \\ \hline & \boxed{-c} & \\ \hline \end{array} \quad \boxed{a}$$

$$\begin{array}{|c|} \hline \square \\ \hline \end{array} \quad \boxed{1} \quad \boxed{-1}$$

# Exploring our new favorite tensor space, $A \otimes B \otimes C^{\otimes k}$

Move the right pole to the left:

$$\begin{array}{c} B \otimes C \otimes C \otimes C \otimes C \otimes C \otimes A \\ \text{---} \\ | \quad | \quad | \quad | \quad | \quad | \quad | \\ B \otimes C \otimes C \otimes C \otimes C \otimes C \otimes A \end{array} = \begin{array}{c} A \otimes B \otimes C \otimes C \otimes C \otimes C \otimes C \\ \text{---} \\ | \quad | \quad | \quad | \quad | \quad | \quad | \\ A \otimes B \otimes C \otimes C \otimes C \otimes C \otimes C \end{array}$$

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 \text{---} \\
 | \quad | \quad | \quad | \quad | \quad | \quad | \\
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 \end{array}
 =
 \begin{array}{c}
 A \otimes B \otimes C \otimes C \otimes C \otimes C \otimes C \\
 \text{---} \\
 | \quad | \quad | \quad | \quad | \quad | \quad | \\
 \text{---} \\
 A \otimes B \otimes C \otimes C \otimes C \otimes C \otimes C
 \end{array}$$

New favorite generators:

$$T_0 = \text{---} \quad T_i = \begin{array}{c} i \quad i+1 \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \quad \text{and} \quad Y_1 = \text{---} .$$

$$\text{Let } Y_2 = T_1 Y_1 T_1 = \text{---} .$$

## Exploring our new favorite tensor space, $A \otimes B \otimes C^{\otimes k}$

Products of rectangles:

$$L((a^c)) \otimes L((b^d)) = \bigoplus_{\lambda \in \Lambda} L(\lambda) \quad (\text{multiplicity one!})$$

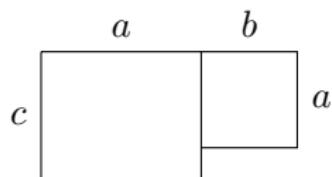
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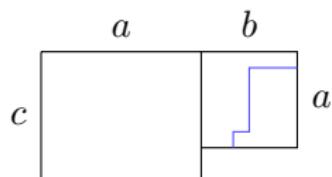


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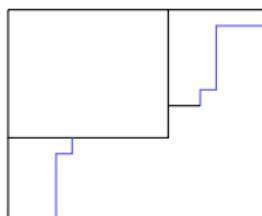


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$$(a^c) \otimes \begin{array}{|c|c|}\hline & c \\ \hline c & \end{array} = \begin{array}{|c|c|c|}\hline & & c \\ \hline & c & \end{array} \oplus \begin{array}{|c|c|c|}\hline & & c \\ \hline & c & \end{array} \oplus \begin{array}{|c|c|c|}\hline & & c \\ \hline & c & \end{array} \oplus \dots$$

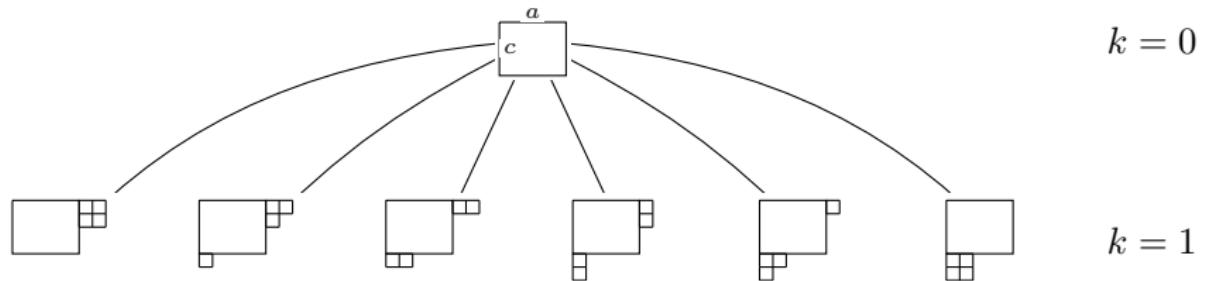
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Exploring our new favorite tensor space,  $A \otimes B \otimes C^{\otimes k}$

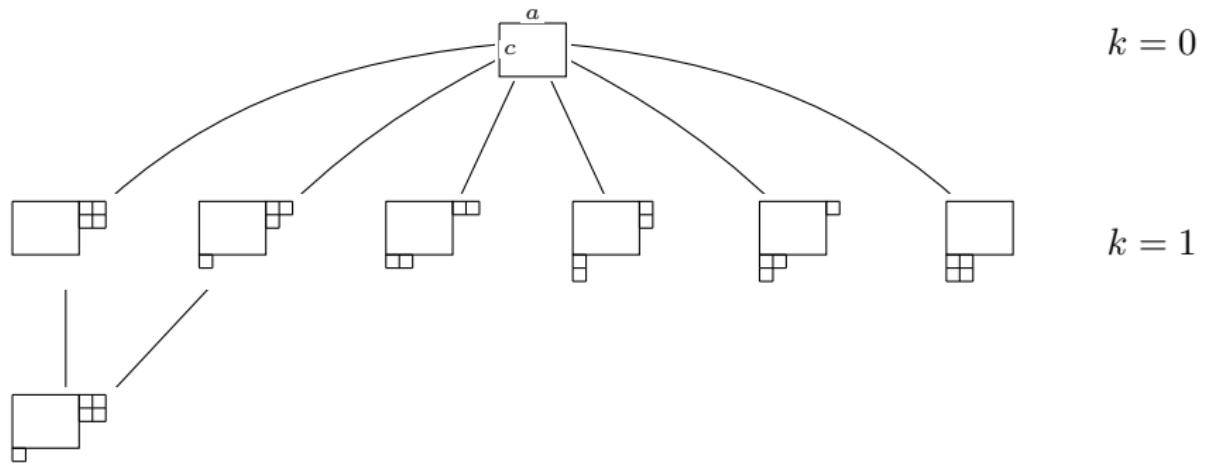


$k = 0$

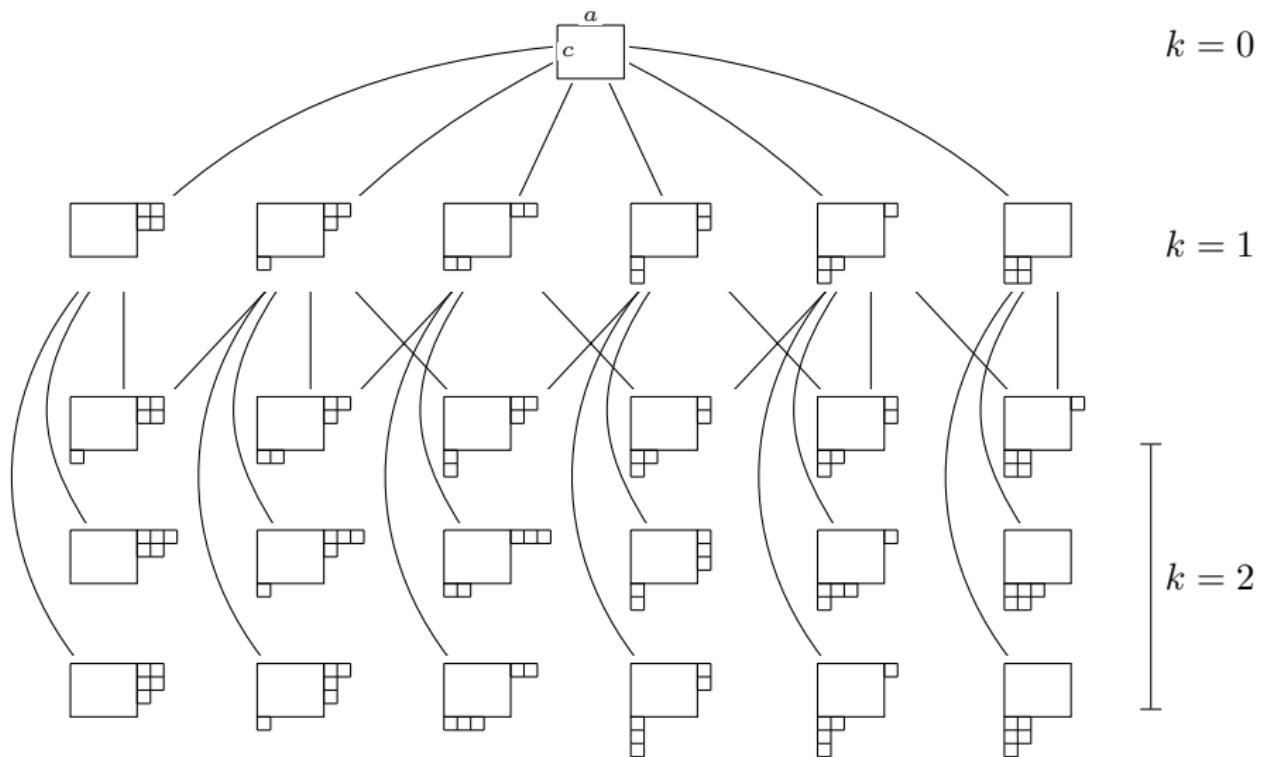
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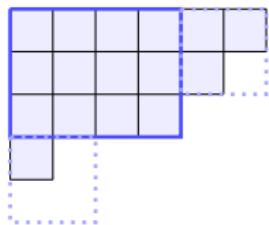
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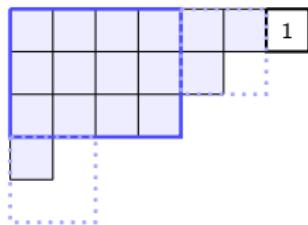
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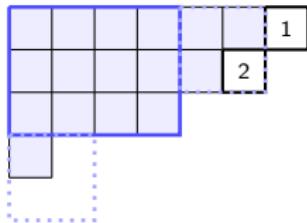
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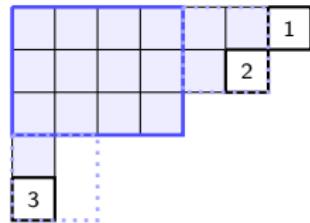
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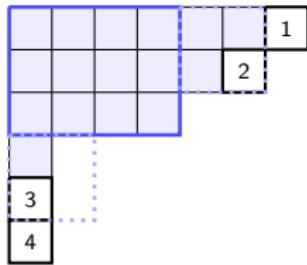
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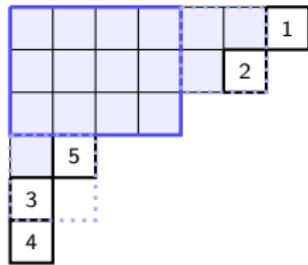
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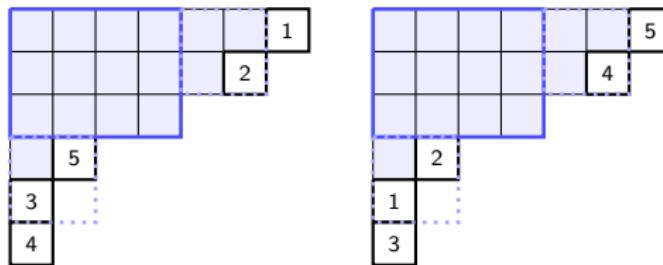


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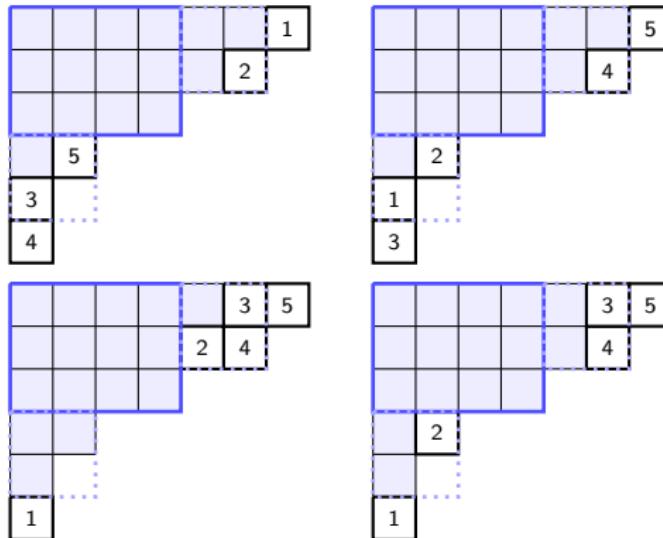
(\*)  $H_k$  representations in tensor space are labeled by certain partitions  $\lambda$ .

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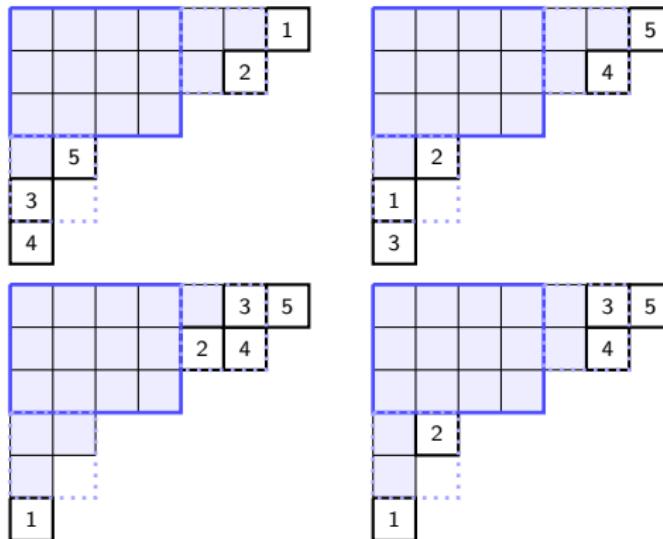
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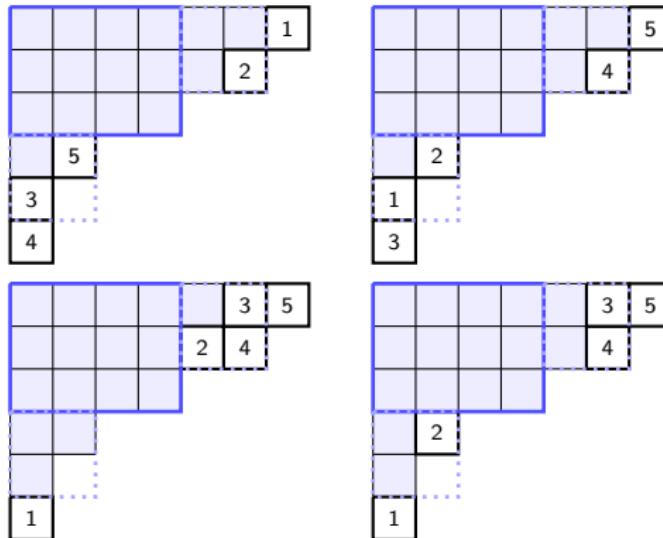
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$$L \left( \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \right) \otimes L \left( \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \right) \otimes L \left( \begin{array}{|c|} \hline \end{array} \right)$$



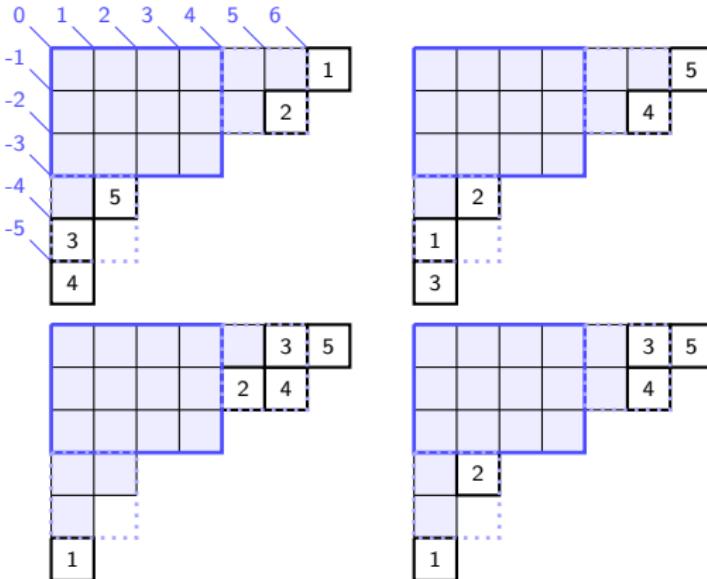
- (\*)  $H_k$  representations in tensor space are labeled by certain partitions  $\lambda$ .
- (\*) Basis labeled by tableaux from some partition  $\mu$  in  $(a^c) \otimes (b^d)$  to  $\lambda$ .

$$L \left( \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \right) \otimes L \left( \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \right) \otimes L \left( \begin{array}{|c|} \hline \end{array} \right)$$



- (\*)  $H_k$  representations in tensor space are labeled by certain partitions  $\lambda$ .
- (\*) Basis labeled by tableaux from some partition  $\mu$  in  $(a^c) \otimes (b^d)$  to  $\lambda$ .
- (\*) Calibrated

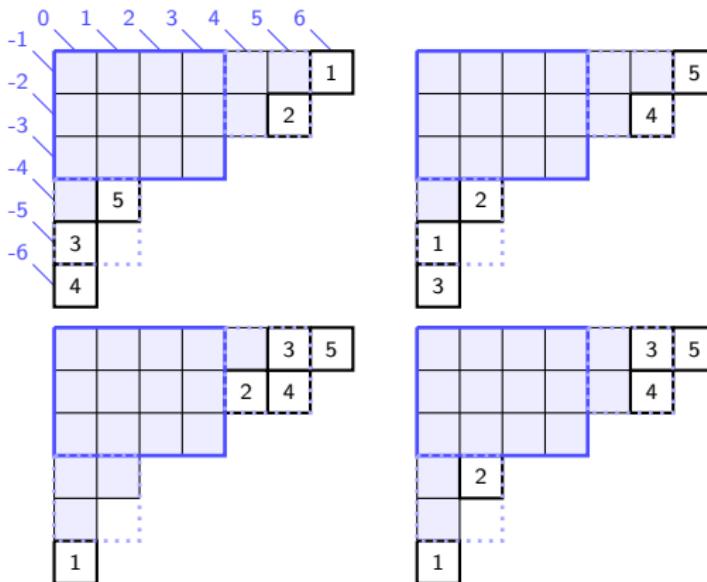
$$L \left( \begin{array}{|c|c|c|c|c|c|} \hline & & & & & \\ \hline \end{array} \right) \otimes L \left( \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \right) \otimes L \left( \begin{array}{|c|} \hline \end{array} \right)$$



- (\*)  $H_k$  representations in tensor space are labeled by certain partitions  $\lambda$ .
- (\*) Basis labeled by tableaux from some partition  $\mu$  in  $(a^c) \otimes (b^d)$  to  $\lambda$ .
- (\*) Calibrated

$$L \left( \begin{array}{|c|c|c|c|} \hline & & & \\ \hline \end{array} \right) \otimes L \left( \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \right) \otimes L \left( \square \right)$$

Shift by  $\frac{1}{2}(a - c + b - d)$

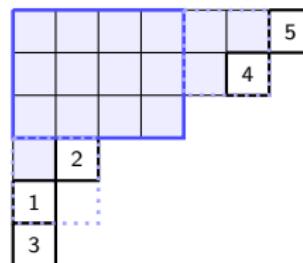
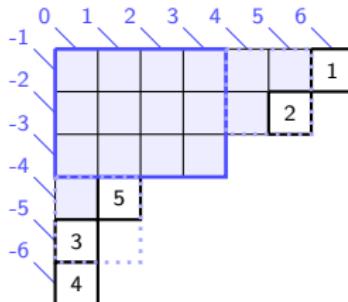


- (\*)  $H_k$  representations in tensor space are labeled by certain partitions  $\lambda$ .
- (\*) Basis labeled by tableaux from some partition  $\mu$  in  $(a^c) \otimes (b^d)$  to  $\lambda$ .
- (\*) Calibrated

$$L\left(\begin{array}{|c|c|c|c|c|c|c|}\hline & \square & \square & \square & \square & \square & \square \\ \hline \end{array}\right) \otimes L\left(\begin{array}{|c|c|}\hline \square & \square \\ \hline \end{array}\right) \otimes L\left(\square\right) \otimes L\left(\square\right) \otimes L\left(\square\right) \otimes L\left(\square\right) \otimes L\left(\square\right)$$

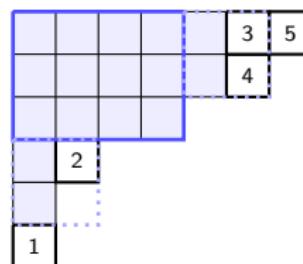
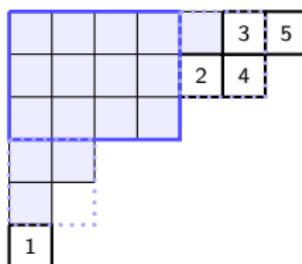
Shift by  $\frac{1}{2}(a - c + b - d)$

$$\begin{aligned} Y_1 &\mapsto t^{5.5} \\ Y_2 &\mapsto t^{3.5} \\ Y_3 &\mapsto t^{-4.5} \\ Y_4 &\mapsto t^{-5.5} \\ Y_5 &\mapsto t^{-2.5} \end{aligned}$$



$$\begin{aligned} Y_1 &\mapsto t^{5.5} \\ Y_2 &\mapsto t^{3.5} \\ Y_3 &\mapsto t^{-4.5} \\ Y_4 &\mapsto t^{-5.5} \\ Y_5 &\mapsto t^{-2.5} \end{aligned}$$

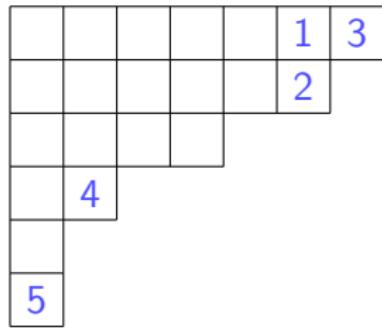
$$\begin{aligned} Y_1 &\mapsto t^{-5.5} \\ Y_2 &\mapsto t^{2.5} \\ Y_3 &\mapsto t^{4.5} \\ Y_4 &\mapsto t^{3.5} \\ Y_5 &\mapsto t^{5.5} \end{aligned}$$



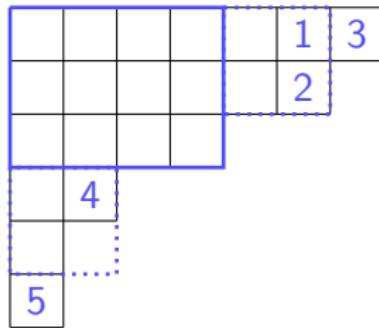
$$\begin{aligned} Y_1 &\mapsto t^{5.5} \\ Y_2 &\mapsto t^{3.5} \\ Y_3 &\mapsto t^{-4.5} \\ Y_4 &\mapsto t^{-5.5} \\ Y_5 &\mapsto t^{-2.5} \end{aligned}$$

- (\*)  $H_k$  representations in tensor space are labeled by certain partitions  $\lambda$ .
- (\*) Basis labeled by tableaux from some partition  $\mu$  in  $(a^c) \otimes (b^d)$  to  $\lambda$ .
- (\*) Calibrated:  $Y_i$  acts by  $t$  to the shifted content of box <sub>$i$</sub> .

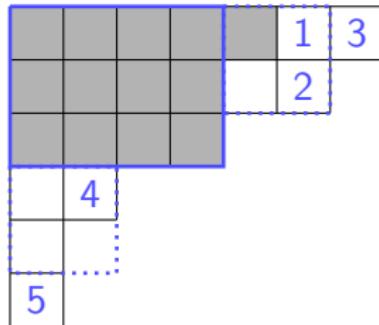
From  $\{\text{partitions in tensor space}\}$  to  $\{\text{box arrangements}\}$



From {partitions in tensor space} to {box arrangements}

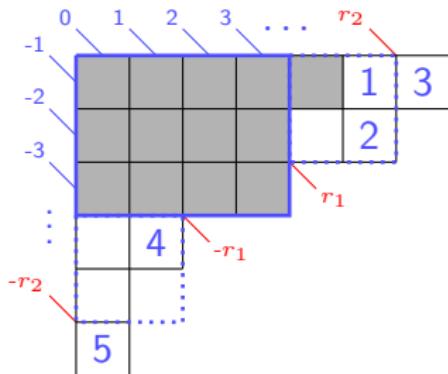


From  $\{\text{partitions in tensor space}\}$  to  $\{\text{box arrangements}\}$



■ = boxes that must appear in the partition at level 0.

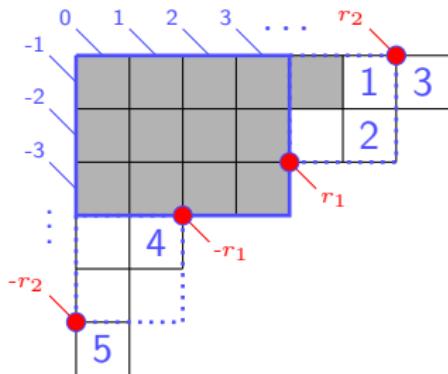
From  $\{\text{partitions in tensor space}\}$  to  $\{\text{box arrangements}\}$



■ = boxes that must appear in the partition at level 0.

$$\gamma(Y_1) = t^{4.5}, \gamma(Y_2) = t^{3.5}, \gamma(Y_3) = t^{r_2}, \gamma(Y_4) = t^{-2.5}, \gamma(Y_5) = t^{-r_2}.$$

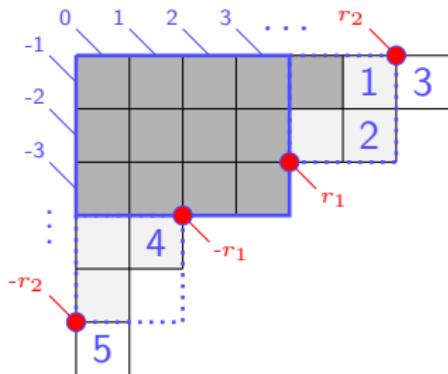
From  $\{\text{partitions in tensor space}\}$  to  $\{\text{box arrangements}\}$



■ = boxes that must appear in the partition at level 0.

$$\gamma(Y_1) = t^{4.5}, \gamma(Y_2) = t^{3.5}, \gamma(Y_3) = t^{r_2}, \gamma(Y_4) = t^{-2.5}, \gamma(Y_5) = t^{-r_2}.$$

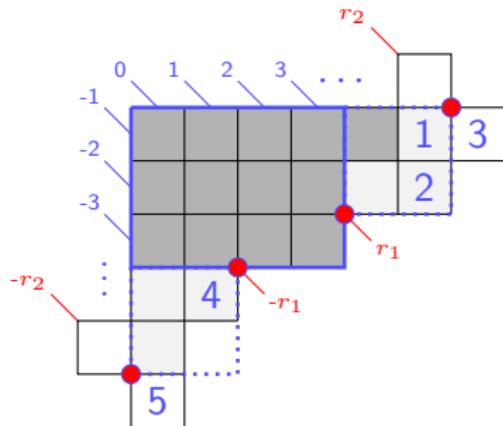
From  $\{\text{partitions in tensor space}\}$  to  $\{\text{box arrangements}\}$



■ = boxes that must appear in the partition at level 0.

$$\gamma(Y_1) = t^{4.5}, \gamma(Y_2) = t^{3.5}, \gamma(Y_3) = t^{r_2}, \gamma(Y_4) = t^{-2.5}, \gamma(Y_5) = t^{-r_2}.$$

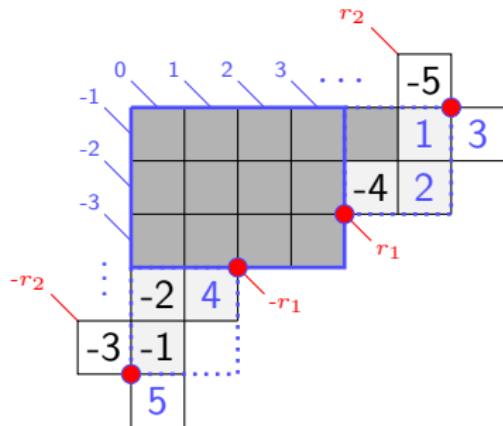
From  $\{\text{partitions in tensor space}\}$  to  $\{\text{box arrangements}\}$



■ = boxes that must appear in the partition at level 0.

$$\gamma(Y_1) = t^{4.5}, \gamma(Y_2) = t^{3.5}, \gamma(Y_3) = t^{r_2}, \gamma(Y_4) = t^{-2.5}, \gamma(Y_5) = t^{-r_2}.$$

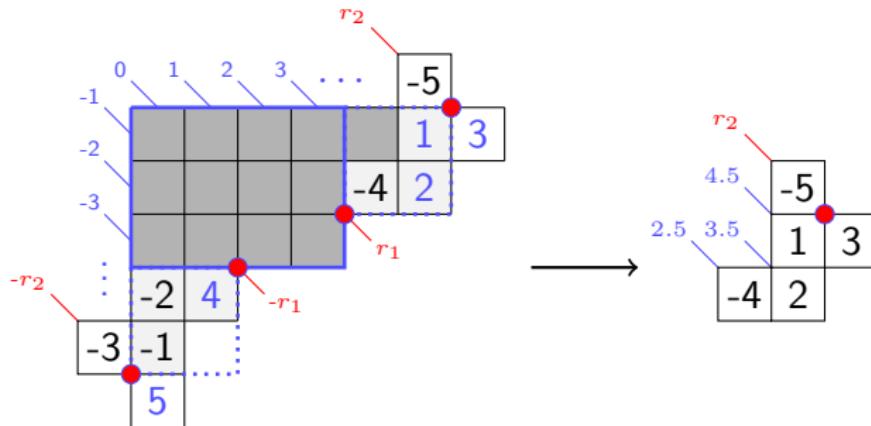
From  $\{\text{partitions in tensor space}\}$  to  $\{\text{box arrangements}\}$



■ = boxes that must appear in the partition at level 0.

$$\gamma(Y_1) = t^{4.5}, \gamma(Y_2) = t^{3.5}, \gamma(Y_3) = t^{r_2}, \gamma(Y_4) = t^{-2.5}, \gamma(Y_5) = t^{-r_2}.$$

From  $\{\text{partitions in tensor space}\}$  to  $\{\text{box arrangements}\}$



$\blacksquare = \text{boxes that must appear in the partition at level 0.}$

$$\gamma(Y_1) = t^{4.5}, \gamma(Y_2) = t^{3.5}, \gamma(Y_3) = t^{r_2}, \gamma(Y_4) = t^{-2.5}, \gamma(Y_5) = t^{-r_2}.$$

versus

$$\gamma(Y_1) = t^{4.5}, \gamma(Y_2) = t^{3.5}, \gamma(Y_3) = t^{r_2}, \gamma(Y_4^{-1}) = t^{2.5}, \gamma(Y_5^{-1}) = t^{r_2}.$$

# Thanks!

$\Phi(c) = \{ \xi_i | \xi_i c \in J \} \cup \{ \xi_i | c - \xi_i \in J \}$   
 $= \{-81, -80, -77, -78, -79, 80, 81, 82, 83, -c_1 - c_2\}$   
 $\frac{1}{2}(a-c+b-d) = \frac{1}{2}(8-6+6-5) = \frac{3}{2} = 0$   
 $\frac{1}{2}(a+c+b+d) = \frac{1}{2}(8+6+6+5) = 12$   
 central character =  $(-\frac{1}{2}\xi_1 - 24\xi_2, -14\xi_3)$   
 $Z(c) = \{ \xi_i | c_i = 0 \} \cup \{ \xi_i | c_i = \xi_j \}$   
 $R = \{ \xi_i | c_i = 0 \} \cup \{ \xi_i | c_i = -k_1, -k_2, -k_3 \} \cup \{ \xi_i | c_i = k_1, k_2, k_3 \}$   
 "standard tableau" =  $\begin{pmatrix} -81 & -80 & -77 & -78 & -79 & -81 \\ 82 & 83 & 80 & 81 & 82 & 83 \\ 81 & 82 & 83 & 80 & 81 & 82 \\ 80 & 81 & 82 & 83 & 80 & 81 \\ 81 & 82 & 83 & 80 & 81 & 82 \\ 82 & 83 & 80 & 81 & 82 & 83 \end{pmatrix} = w$   
 solid & dashed regions must be on pos. side of solid hypers  
 pos. side is defined by which side each dashed region is on  
 dashed hypers local region is on neg. side of

$\text{standard tableau} = \{ w(c), J \} \cup R(w) \cap Z(c) = \emptyset$   
 $R(w) \cap Z(c) = \emptyset$   
 pos. side solid dashed

<img alt="A diagram showing a grid of numbers from 1 to 84. A red bracket on the left indicates c=6, and a blue bracket at the top indicates a=8. A green bracket on the right indicates b=6. A yellow bracket at the bottom indicates d=6. There are several yellow boxes containing numbers like 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84. Some numbers are crossed out with a green line. A red arrow points to the number 18. A blue arrow points to the number 22. A green arrow points to the number 26. A yellow arrow points to the number 30. A red arrow points to the number 34. A blue arrow points to the number 38. A green arrow points to the number 42. A yellow arrow points to the number 46. A red arrow points to the number 50. A blue arrow points to the number 54. A green arrow points to the number 58. A yellow arrow points to the number 62. A red arrow points to the number 66. A blue arrow points to the number 70. A green arrow points to the number 74. A yellow arrow points to the number 78. A red arrow points to the number 82. A blue arrow points to the number 86. A green arrow points to the number 90. A yellow arrow points to the number 94. A red arrow points to the number 98. A blue arrow points to the number 102. A green arrow points to the number 106. A yellow arrow points to the number 110. A red arrow points to the number 114. A blue arrow points to the number 118. A green arrow points to the number 122. A yellow arrow points to the number 126. A red arrow points to the number 130. A blue arrow points to the number 134. A green arrow points to the number 138. A yellow arrow points to the number 142. A red arrow points to the number 146. A blue arrow points to the number 150. A green arrow points to the number 154. A yellow arrow points to the number 158. A red arrow points to the number 162. A blue arrow points to the number 166. A green arrow points to the number 170. A yellow arrow points to the number 174. A red arrow points to the number 178. A blue arrow points to the number 182. A green arrow points to the number 186. A yellow arrow points to the number 190. A red arrow points to the number 194. A blue arrow points to the number 198. A green arrow points to the number 202. A yellow arrow points to the number 206. A red arrow points to the number 210. A blue arrow points to the number 214. A green arrow points to the number 218. A yellow arrow points to the number 222. A red arrow points to the number 226. A blue arrow points to the number 230. A green arrow points to the number 234. A yellow arrow points to the number 238. A red arrow points to the number 242. A blue arrow points to the number 246. A green arrow points to the number 250. A yellow arrow points to the number 254. A red arrow points to the number 258. A blue arrow points to the number 262. A green arrow points to the number 266. A yellow arrow points to the number 270. A red arrow points to the number 274. A blue arrow points to the number 278. A green arrow points to the number 282. A yellow arrow points to the number 286. A red arrow points to the number 290. A blue arrow points to the number 294. A green arrow points to the number 298. A yellow arrow points to the number 302. A red arrow points to the number 306. A blue arrow points to the number 310. A green arrow points to the number 314. A yellow arrow points to the number 318. A red arrow points to the number 322. A blue arrow points to the number 326. A green arrow points to the number 330. A yellow arrow points to the number 334. A red arrow points to the number 342. A blue arrow points to the number 346. A green arrow points to the number 350. A yellow arrow points to the number 354. A red arrow points to the number 362. A blue arrow points to the number 366. A green arrow points to the number 370. A yellow arrow points to the number 374. A red arrow points to the number 382. A blue arrow points to the number 386. A green arrow points to the number 390. A yellow arrow points to the number 394. A red arrow points to the number 402. A blue arrow points to the number 406. A green arrow points to the number 410. A yellow arrow points to the number 414. A red arrow points to the number 422. A blue arrow points to the number 426. A green arrow points to the number 430. A yellow arrow points to the number 434. A red arrow points to the number 442. A blue arrow points to the number 446. A green arrow points to the number 450. A yellow arrow points to the number 454. A red arrow points to the number 462. A blue arrow points to the number 466. A green arrow points to the number 470. A yellow arrow points to the number 474. A red arrow points to the number 482. A blue arrow points to the number 486. A green arrow points to the number 490. A yellow arrow points to the number 494. A red arrow points to the number 502. Ablue arrow points to the number 506. Agreen arrow points to the number 510. Ayellow arrow points to the number 514. Ared arrow points to the number 522. Ablue arrow points to the number 526. Agreen arrow points to the number 530. Ayellow arrow points to the number 534. Ared arrow points to the number 542. Ablue arrow points to the number 546. Agreen arrow points to the number 550. Ayellow arrow points to the number 554. Ared arrow points to the number 562. Ablue arrow points to the number 566. Agreen arrow points to the number 570. Ayellow arrow points to the number 574. Ared arrow points to the number 582. Ablue arrow points to the number 586. Agreen arrow points to the number 590. Ayellow arrow points to the number 594. Ared arrow points to the number 602. Ablue arrow points to the number 606. Agreen arrow points to the number 610. Ayellow arrow points to the number 614. Ared arrow points to the number 622. Ablue arrow points to the number 626. Agreen arrow points to the number 630. Ayellow arrow points to the number 634. Ared arrow points to the number 642. Ablue arrow points to the number 646. Agreen arrow points to the number 650. Ayellow arrow points to the number 654. Ared arrow points to the number 662. Ablue arrow points to the number 666. Agreen arrow points to the number 670. Ayellow arrow points to the number 674. Ared arrow points to the number 682. Ablue arrow points to the number 686. Agreen arrow points to the number 690. Ayellow arrow points to the number 694. Ared arrow points to the number 702. Ablue arrow points to the number 706. Agreen arrow points to the number 710. Ayellow arrow points to the number 714. Ared arrow points to the number 722. Ablue arrow points to the number 726. Agreen arrow points to the number 730. Ayellow arrow points to the number 734. Ared arrow points to the number 742. Ablue arrow points to the number 746. Agreen arrow points to the number 750. Ayellow arrow points to the number 754. Ared arrow points to the number 762. Ablue arrow points to the number 766. Agreen arrow points to the number 770. Ayellow arrow points to the number 774. Ared arrow points to the number 782. Ablue arrow points to the number 786. Agreen arrow points to the number 790. Ayellow arrow points to the number 794. Ared arrow points to the number 802. Ablue arrow points to the number 806. Agreen arrow points to the number 810. Ayellow arrow points to the number 814. Ared arrow points to the number 822. Ablue arrow points to the number 826. Agreen arrow points to the number 830. Ayellow arrow points to the number 834. Ared arrow points to the number 842. Ablue arrow points to the number 846. Agreen arrow points to the number 850. Ayellow arrow points to the number 854.</p>