HOMEWORK 8

SOLUTIONS (SKETCHES)

4.1.1 Give a proof or a counterexample for each statement below.

(a) Every graph with connectivity 4 is 2-connected.

Answer. True. $2 \leq \kappa(G) = 4$.

□

(b) Every 3-connected graph has connectivity 3.

Answer. False. K_5 is 3-connected b/c it is also 4-connected.

□

(c) Every k-connected graph is k-edge-connected.

Answer. True, $\kappa'(G) \geq \kappa(G)$.

□

(d) Every k-edge-connected graph is k-connected.

Answer. False. Consider the bow-tie. □

4.1.7 Obtain a formula for the number of spanning trees of a connected graph in terms of the numbers of spanning trees of its blocks.

Answer. Take the product.

□

4.1.10 Find the smallest 3-regular simple graph having connectivity 1.

Answer. Start with a vertex v that is to be cut. $G-v$ has at least two components, and each component is almost 3-regular (has one or two vertices with degree 2). Consider a component with one vertex of degree 2. Then it has an even number of vertices w degree 3 (degree sum formula). Zero or 2 is not possible; but 4 is. For the comp w/ 2 degree-2 vertices, it is possible to do this w/ 2 deg-3 vertices but not w/ none. THe smallest example has 9 vertices.

□

4.1.14 Let G be a connected graph in which for every edge e, there are cycles C_1 and C_2 containing e whose only common edge is e. Prove that G is 3-edge-connected. Use this to show that the Petersen graph is 3-edge-connected.

Answer. By contradiction.

Suppose removing one edge uv disconnects the graph. That edge belonged to cycles C_1, C_2 whose only common edge was uv. u, v are still connected in the bigger cycle $C_1 \cup C_2 - uv$, thus the whole graph is still connected.

Suppose removing a second edge disconnects the graph. Call that edge $f = ab$. Everything is still in a cycle after we removed uv. If we remove an edge from this cycle, the graph will still be connected. If we remove an edge not in the cycle, then those two vertices a, b are still connected as in the previous case.

□

4.1.20 Let G be a simple n-vertex graph with $n/2 - 1 \leq \delta(G) \leq n - 2$. Prove that G is k-connected for all k with $k \leq 2\delta(G) + 2 - n$. Prove that this is best possible for all $\delta \geq n/2 - 1$ by constructing a simple n-vertex graph with minimum degree δ that is not k-connected for $k = 2\delta + 3 - n$. (Comment: Proposition 1.3.15 is the special case of this when $\delta(G) = (n-1)/2$.)

Proof. Let x, y be any two non-adjacent vertices. Fix k such that $k \leq 2\delta + 2 - n$. Then $\delta \geq (n + k - 2)/2$, and $|N(x)|, |N(y)| \geq \delta \geq (n + k - 2)/2$. Also, $|N(x) \cup N(y)| \leq n - 2$.

$$
\begin{aligned}
|N(x) \cap N(y)| &= |N(x)| + |N(y)| - |N(x) \cup N(y)| \\
&\geq (n + k - 2)/2 + (n + k - 2)/2 - (n - 2) \\
&= k
\end{aligned}
$$

Since x, y were arbitrary, this means that for any pair of vertices, any set of fewer than k vertices cannot disconnect them.

□

4.1.25 $\kappa'(G) = \delta(G)$ ***for diameter 2.*** **Let G be a simple graph with diameter 2, and let $[S, \overline{S}]$ be a minimum edge cut with $|S| \leq |\overline{S}|$.**

(a) Prove that every vertex of S has a neighbor in $\overline{S}$.

Proof. Because of the diameter, only one of S or $\bar{S}$ can contain a vertex which is not adjacent to all vertices in the other set. Suppose S has one such vertex v. Let k be the size of $\bar{S}$. Since $\bar{S}$ has the property that every element of $\bar{S}$ is adjacent to some element of S, the edge cut between them must-have size at least k. Since the edge cut is minimum, $\delta(G) \geq \kappa'(G) \geq k$. But then the degree of v is at least $k = |\bar{S}| \geq |S|$, which is not possible. □

(b) Use part (a) and Corollary 4.1.13 to prove that $\kappa'(G) = \delta(G)$. (Plesnik [1975])

Proof. We know that $|[S, \bar{S}]| \geq |S|$ because every vertex in S has at least one edge connecting it to $\bar{S}$.

Suppose $|[S, \bar{S}]| < \delta$. Then $|S| > \delta$ (Cor 4.1.13). Then we have $|S| > |[S, \bar{S}]|$, a contradiction of what we proved in part (a). Therefore $|[S, \bar{S}]| \geq \delta$, which means they are equivalent. $\kappa' \leq \delta$.

□

4.2.2 Prove that if G is 2-edge-connected and G' is obtained from G by subdividing an edge of G, then G' is 2-edge-connected. Use this to prove that every graph having a closed-ear decomposition is 2-edge-connected. (Comment: This is an alternative proof of sufficiency for Theorem 4.2.10.)

Answer. (Sketch)

If G is 2-edge connected, then every edge (and therefore vertex) is in a cycle. Subdividing an edge keeps this true.

A cycle is 2 edge connected. If we add an edge connecting any two points on the cycle or if we add an edge which is a loop, then the graph is still 2 edge connected. Then we can subdivide the added edge, which is the same as an ear.

□

4.2.8 Prove that a simple graph G is 2-connected if and only if for every ordered triple, (x, y, z), of distinct vertices, G has an $x, z-$path through y.

Proof. (Sketch)

Suppose G is 2-connected. Let (x, y, z) be any ordered triple of vertices. Then let $U = \{x, z\}$. Then by the fan lemma, $\exists$ a y, U fan of 2 paths. These two paths only share y, thus they are disjoint x, y and y, z paths. Concatenate them to create an x, z path through y.

Now suppose G is disconnected or 1-connected. If G is disconnected, the proof is trivial. If G is 1-connected, then removing v disconnects the graph for some v. Consider 2 components C_1, C_2 of $G - v$. Let $x \in V(C_1)$ and $z \in V(C_2)$. Now, consider the ordered triple (v, x, z). We take a v, x path which exists because G is connected, but there is no x, z path which does not go through v. Thus the graph does not have the property above.

$\square$

4.2.22 Suppose that $\kappa(G) = k$ and diam $G = d$. Prove that $n(G) \geq k(d-1)+2$ and $\alpha(G) \geq \lceil (1+d)/2 \rceil$. For each $k \geq 1$ and $d \geq 2$, construct a graph with connectivity k and diameter d for which equality holds in both bounds.

Proof. For $\alpha(G)$. Consider $u, v \in V(G)$ with $d(u, v) = d$. Then the shortest u, v path has d edges and $d+1$ vertices. Every other vertex must not be neighbors, or else a shorter path exists. Depending on if $d+1$ is odd or even, there is at least an independent set of $\lceil (1+2)/2 \rceil$ vertices.

For $n(G)$. Let $d(x, y) = d$ for some $x, y \in V(G)$. We know x, y exists because of the diameter. Then $\exists k$ internally disjoint x, y paths of length $\geq d$. Each of these paths has $d-1$ internal vertices. Thus there are $k(d-1)$ internal vertices, and 2 endpoints, x, y, which means $n(G) \geq k(d-1)+2$.

Given k and d, we will construct a graph such that equalities hold. Create "components" $C_0, ..., C_d$ such that C_0, C_d are single vertices v_0, v_d respectively, and for $0 < i < d$, $C_i = K_k$. Then, if $i = j \pm 1$, connect all the vertices in C_i, C_j.

Clearly this graph G has $n(G) = k(d-1)+1+1$. $d(v_0, v_d) = d$. It is k-connected. If we delete $< k$ vertices, then any internal "component" still has at least one vertex left (which is connected to the top and bottom). If we take one vertex from each C_i with $i \equiv 0 \mod 2$ then we have a vertex cover of size $\lceil (d+1)/2 \rceil$, which bounds α from above.

$\square$