Math 68. Algebraic Combinatorics.

Problem Set 3. Due on Friday, 11/4/2011.

1. Prove that
$$\frac{1}{1-z} = \prod_{j\geq 0}(1+z^{2^j}).$$

2. For fixed k, give the exponential generating function for the number of surjective maps from $[n]$ onto $[k]$.

3. (a) Let b_n denote the number of (labeled) rooted trees on the vertex set $[n]$ whose leaves are colored either red of blue. Find an equation satisfied by the exponential generating function
$$B(z) = \sum_{n\geq 0} b_n \frac{z^n}{n!} = 2z + 4\frac{z^2}{2!} + 24\frac{z^3}{3!} + \dots$$
   (b) Use the Lagrange inversion formula to deduce that
$$b_n = \sum_{k=0}^{n} \binom{n}{k} k^{n-1}.$$
   (c) * Give a direct combinatorial proof of (b).

4. Let $M(n)$ be the set of all subsets of $[n]$, with the ordering $A \leq B$ if the elements of A are $a_1 > a_2 > \cdots > a_j$ and the elements of B are $b_1 > b_2 > \cdots > b_k$, where $j \leq k$ and $a_i \leq b_i$ for $1 \leq i \leq j$. (The empty set $\emptyset$ is the bottom element of $M(n)$.)
   (a) Draw the Hasse diagrams (with vertices labeled by the subsets they represent) of $M(1)$, $M(2)$, $M(3)$, and $M(4)$.
   (b) Show that $M(n)$ is graded of rank $\binom{n+1}{2}$. What is $\mathrm{rank}(\{a_1, \dots, a_k\})$?
   (c) Define the *rank-generating function* of a graded poset P to be
$$F(P,q) := \sum_{x\in P} q^{\mathrm{rank}(x)}.$$
   Show that the rank-generating function of $M(n)$ is given by
$$F(M(n),q) = (1+q)(1+q^2)\cdots(1+q^n).$$

5. Let q be a prime power, and let V be an n-dimensional vector space over $\mathbb{F}_q$. Let $B_n(q)$ denote the poset of all subspaces of V, ordered by inclusion. It's easy to see that $B_n(q)$ is graded of rank n, the rank of a subspace of V being its dimension.

(a) Show that the number of elements of $B_n(q)$ of rank k is given by the q-binomial coefficient

$$\left[\begin{array}{c} n \\ k \end{array}\right]_q = \frac{(q^n-1)(q^{n-1}-1)\dots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\dots(q-1)}.$$

(One way to do this is to count in two ways the number of k-tuples $(v_1,\dots,v_k)$ of linearly independent elements from $\mathbb{F}_q^n$: (1) first choose v_1, then v_2, etc., and (2) first choose the subspace W spanned by $v_1,\dots,v_k$, and then choose v_1, v_2, etc.)

(b) Show that $B_n(q)$ is rank-symmetric. (You can use (a).)

(c) Show that every element $x \in B_n(q)_k$ covers $[k]_q = 1+q+\cdots+q^{k-1}$ elements and is covered by $[n-k]_q = 1+q+\cdots+q^{n-k-1}$ elements.

(d) Define operators $U_i : \mathbb{R}B_n(q)_i \to \mathbb{R}B_n(q)_{i+1}$ and $D_i : \mathbb{R}B_n(q)_i \to \mathbb{R}B_n(q)_{i-1}$ by

$$U_i(x) = \sum_{\substack{y\in B_n(q)_{i+1} \\ y>x}} y, \qquad D_i(x) = \sum_{\substack{z\in B_n(q)_{i-1} \\ z<x}} z.$$

Show that $D_{i+1}U_i - U_{i-1}D_i = ([n-i]_q - [i]_q)I_i$.

(e) Deduce that $B_n(q)$ is rank-unimodal and Sperner.

6. * Let h_n be the number of ways to choose a permutation π of $[n]$ and a subset S of $[n]$ such that if $i \in S$, then $\pi(i) \notin S$. Find an expression for the exponential generating function $\sum_{n\geq 0} h_n \frac{z^n}{n!}$.