Math 68. Algebraic Combinatorics.

Problem Set 1. Due on Friday, 10/7/2011

1. Prove that
$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$$
for $n \geq m \geq k \geq 0$ by counting certain pairs of sets (A, B) in two ways, and deduce that
$$\sum_{k=0}^{m} \binom{n}{k}\binom{n-k}{m-k} = 2^m \binom{n}{m}.$$

2. Give a combinatorial proof of the equality
$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0,$$
by describing a bijection between subsets of $[n]$ of odd size and subsets of $[n]$ of even size.

3. Let $[n] = \{1, 2, \ldots, n\}$.
   (a) Find the number of k-tuples $(S_1, S_2, \ldots, S_k)$ of subsets of $[n]$ such that $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k$.
   (b) Find the number of k-tuples $(S_1, S_2, \ldots, S_k)$ of subsets of $[n]$ such that $S_1 \cap S_2 \cap \cdots \cap S_k = \emptyset$.

4. A *Delannoy path* is a lattice path in $\mathbb{Z}^2$ from $(0, 0)$ to (m, n) using steps $(1, 0)$ (horizontal), $(0, 1)$ (vertical), and $(1, 1)$ (diagonal). The number of these paths is the Delannoy number $D_{m,n}$. For example, $D_{2,1} = 5$. Prove that
$$D_{m,n} = \sum_{k} \binom{m}{k}\binom{n+k}{m}.$$
Hint: Classify the paths according to the number of diagonal steps.

5. Prove that the number of partitions of n into odd parts equals the number of partitions of n into distinct parts.

6. (a) In how many ways can we choose k points, no two consecutive, from a collection of n points arranged in a line?
   (b) What if the n points are arranged in a circle?

7. A *set partition* π of $[n]$ is a way to subdivide $[n]$ into nonempty blocks. A set partition is called *noncrossing* if it contains no two blocks B and B' such that $i, k \in B$ and $j, l \in B'$ for some $i < j < k < l$. Show that the number of noncrossing partitions equals the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.