Math 123 Homework Assignment #1
Due Friday, April 8th.

Part I:

1. Suppose that X is a normed vector space. Then X is a Banach space (that is, X is complete) if and only if every absolutely convergent series in X is convergent.

2. Let X be a normed vector space and suppose that S and T are bounded linear operators on X. Show that $\|ST\| \le \|S\|\|T\|$.

3. Let X be a locally compact Hausdorff space. Show that $C_0(X)$ is a closed subalgebra of $C^b(X)$.

4. Let A be a unital Banach algebra. Show that $x \mapsto x^{-1}$ is continuous from $G(A)$ to $G(A)$. (Hint: $(a-h)^{-1} - a^{-1} = ((1-a^{-1}h)^{-1} - 1)a^{-1}$.)

ANS: Using the "hint", suppose that $a \in G(A)$ and let $b = a - h$ for some $h \in A$ with

$$\|h\| \le \frac{\|a^{-1}\|^{-1}}{2}. \tag{1}$$

Then standard arguments show that $b \in G(A)$. (Recall that if $\|c\| < 1$, then $1 - c \in G(A)$ and $(1-c)^{-1} = 1 + c + c^2 + \cdots$.) Now we observe that

$$\begin{aligned}
\|b^{-1} - a^{-1}\| &= \|(a-h)^{-1} - a^{-1}\| \\
&= \big\|\big((1-a^{-1}h) - 1\big)a^{-1}\big\| \\
&\le \Big\|\sum_{n=1}^{\infty}(a^{-1}h)^n\Big\|\|a^{-1}\| \\
&\le \sum_{n=1}^{\infty} \|a^{-1}h\|^n\|a^{-1}\| \\
&= \frac{\|a^{-1}h\|\|a^{-1}\|}{1 - \|a^{-1}h\|}
\end{aligned}$$

which, in view of (5) and $\|a^{-1}h\| \le \|a^{-1}\|\|h\|$, is

$$\le 2\|a^{-1}\|^2\|h\|.$$

Since the latter term tends to 0 with h, we're done.

However, Peng Peng Yu came up with a cleaner argument that does not require A to be a Banach algebra. Here it is enough that A be a normed algebra (and therefore with a submultiplicative norm).

Fix $a \in A$ and $\epsilon > 0$. Let $\delta = \frac{1}{2}\min\bigl(\epsilon\|a^{-1}\|^{-2}, \|a^{-1}\|^{-1}\bigr)$. We just need to verify that if $b \in B_\delta(a) \cap G(A)$, then $\|a^{-1} - b^{-1}\| < \epsilon$. However, we first notice that by the "reverse triangle inequality",

$$\begin{aligned} \|b^{-1}\| - \|a^{-1}\| &\leq \|b^{-1} - a^{-1}\| \\ &= \|b^{-1}(a-b)a^{-1}\| \\ &\leq \|b^{-1}\|\|a-b\|\|a^{-1}\| \end{aligned}$$

which, since $\|a - b\| < \frac{1}{2}\|a^{-1}\|^{-1}$, is

$$< \frac{1}{2}\|b^{-1}\|.$$

In particular, this implies that

$$\|b^{-1}\| < 2\|a^{-1}\| \quad \text{if } b \in B_\delta(a) \cap G(A). \tag{2}$$

But then if $b \in B_\delta(a) \cap G(A)$, we have (2) and $\|a - b\| < \frac{\epsilon}{2}\|a^{-1}\|^{-2}$. Therefore

$$\begin{aligned} \|b^{-1} - a^{-1}\| &= \|b^{-1}(a-b)a^{-1}\| \\ &\leq \|b^{-1}\|\|a-b\|\|a^{-1}\| \\ &< \epsilon. \end{aligned}$$

This is what we wanted to show.

Part II:

5. Suppose that X is a compact Hausdorff space. If E is a closed subset of X, define $I(E)$ to be the ideal in $C(X)$ of functions which vanish on E.

(a) Let J be a closed ideal in $C(X)$ and let $E = \{\, x \in X : f(x) = 0 \text{ for all } f \in J \,\}$. Prove that if U is an open neighborhood of E in X, then there is a $f \in J$ such that $f(x) = 1$ for all x in the compact set $X \setminus U$.

(b) Conclude that $J = I(E)$ in part (a), and hence, conclude that *every closed* ideal in $C(X)$ has the form $I(E)$ for some closed subset E of X.

ANS: Fix $x_0 \in X \setminus U$. By definition of E, there is a $f_{x_0} \in J$ with $f_{x_0}(x_0) \neq 0$. Since $|f|^2 = \bar{f}f \in J$ if $f \in J$, we may as well assume that $f_{x_0}(x) \geq 0$ for all $x \in X$, and since J is a subalgebra, we may also assume that $f_{x_0}(x_0) > 1$. Since $X \setminus U$ is compact, there are $x_1, \dots x_n \in X$ so that $f = \sum_k f_{x_k}$ satisfies $f \in J$ and $f(x) > 1$ for all $x \in X \setminus U$. Observe that $g = \min(1, 1/f)$ is in $C(X)$[1]. Since $fg \in J$, we are done with part (a).

Notice that we have proved a bit more than required in part (a): namely there is a $f \in J$ such that $0 \leq f(x) \leq 1$ for all $x \in X$ and $f(x) = 1$ for all $x \notin U$. Thus if h is any function in $I(E)$ and $\epsilon > 0$, then $U = \{\, x \in X : |h(x)| > \epsilon \,\}$ is a neighborhood of E in X. Then we can choose $f \in J$ as above and $\|fh - h\|_\infty < \epsilon$. Thus $h \in \overline{J} = J$. This suffices as we have $J \subseteq I(E)$ by definition. (Notice that if $E = \emptyset$, then we can take $U = \emptyset$ in the above and then $J = C(X)$.)

Remark: Notice that we have established a 1-1 correspondence between the closed subsets E of X and the closed ideals J of $C(X)$: it follows immediately from Urysohn's Lemma[2] that if E is closed and $x \notin E$, then there is a $f \in I(E)$ with $f(x) \neq 0$. Thus $I(E) \neq I(F)$ if E and F are distinct closed sets.

6. Suppose that X is a (non-compact) locally compact Hausdorff space. Let X^+ be the *one-point compactification* of X (also called the Alexandroff compactification: see [Kelly; Theorem 5.21] or [Folland, Proposition 4.36]). Recall that $X^+ = X \cup \{\, \infty \,\}$ with $U \subseteq X^+$ open if and only if either U is an open subset of X or $X^+ \setminus U$ is a *compact* subset of X.

(a) Show that $f \in C(X)$ belongs to $C_0(X)$ if and only if the extension

$$\tilde{f}(\tilde{x}) = \begin{cases} f(\tilde{x}) & \text{if } \tilde{x} \in X, \text{ and} \\ 0 & \text{if } \tilde{x} = \infty. \end{cases}$$

is continuous on X^+.

(b) Conclude that $C_0(X)$ can be identified with the maximal ideal of $C(X^+)$ consisting of functions which 'vanish at ∞.'

ANS: Suppose $\tilde{f}$ is continuous at $x = \infty$, and that $\epsilon > 0$. Then $U = \{\, \tilde{x} \in X^+ : |\tilde{f}(\tilde{x})| < \epsilon \,\}$ is an open neighborhood of ∞ in X^+. But then $X \setminus U$ is compact; but that means $\{\, x \in X : |f(x)| \geq \epsilon \,\}$ is compact. That is, $f \in C_0(X)$ as required.

For the converse, suppose that $f \in C_0(X)$, and that V is open in $\mathbb{C}$. If $0 \notin V$, then $\tilde{f}^{-1}(V) = f^{-1}(V)$ is open in X, and therefore, open in X^+. On the other hand, if $0 \in V$, then there is a $\epsilon > 0$ so that $\{\, z \in \mathbb{C} : |z| < \epsilon \,\} \subseteq V$. Thus, $X^+ \setminus \tilde{f}^{-1}(V) = \{\, x \in X : f(x) \notin V \,\} \bigcap \{\, x \in X : |f(x)| \geq \epsilon \,\}$. Since the first set is closed and the second compact, $X^+ \setminus \tilde{f}^{-1}(V)$ is a compact subset of X, and $\tilde{f}^{-1}(V)$ is a open neighborhood of ∞ in X^+. This proves part (a).

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[1] If $a, b \in C(X)$, then so are $\min(a, b) = (a + b)/2 - |a - b|/2$ and $\max(a, b) = (a + b)/2 + |a - b|/2$. In the above, we can replace f by $\max(f, 1/2)$ without altering g.

[2] For a reference, see Pedersen's *Analysis Now*: Theorems 1.5.6 and 1.6.6 or, more generally, Proposition 1.7.5.

Part (b) is immediate: each $f \in C_0(X)$ has a (unique) extension to a function in $C(X^+)$ and this identifies $C_0(X)$ with the ideal $I\big(\{\infty\}\big)$ in $C(X^+)$. In view of question 5 above, $I\big(\{\infty\}\big)$ is maximal among closed ideals in $C(X^+)$, and, as maximal ideals are automatically closed, maximal among all proper ideals.

7. Use the above to establish the following ideal theorem for $C_0(X)$.

Theorem: Suppose that X is a locally compact Hausdorff space. Then every closed ideal J in $C_0(X)$ is of the form

$$J = \{ f \in C_0(X) : f(x) = 0 \text{ for all } x \in E \}$$

for some closed subset E of X.

ANS: Suppose that J is a closed ideal in $C_0(X)$. Then J is, in view of question 6(b) above, a closed subalgebra of $C(X^+)$. I claim the result will follow once it is observed that J is actually an ideal in $C(X^+)$. In that case, $J = I(E \cup \{\infty\})$, where $E \subseteq X$ is such that $E \cup \{\infty\}$ is closed in X^+. Thus $X^+ \setminus (E \cup \{\infty\}) = X \setminus E$ is open in X, and E is closed in X.

The easy way to verify the claim, is to observe that, in view of the fact that $C_0(X)$ is a maximal ideal in $C(X^+)$, $C(X^+) = \{ f + \lambda : f \in C_0(X) \text{ and } \lambda \in \mathbb{C} \}$. (Here $\lambda \in \mathbb{C}$ is identified with the constant function on X^+.) Then, since J is an algebra, $f(g+\lambda) = fg + \lambda f$ belongs to J whenever f does.

Part III:

8. Assume you remember enough measure theory to show that if $f, g \in L^1\big([0,1]\big)$, then

$$f * g(t) = \int_0^t f(t-s)g(s)\,ds \tag{3}$$

exists for almost all $t \in [0,1]$, and defines an element of $L^1\big([0,1]\big)$. Let A be the algebra consisting of the Banach space $L^1\big([0,1]\big)$ with multiplication defined by (3).

(a) Conclude that A is a commutative Banach algebra: that is, show that $f * g = g * f$, and that $\|f * g\|_1 \le \|f\|_1 \, \|g\|_1$.

(b) Let f_0 be the constant function $f_0(t) = 1$ for all $t \in [0,1]$. Show that

$$f_0^n(t) := f_0 * \cdots * f_0(t) = t^{n-1}/(n-1)!, \tag{4}$$

and hence,

$$\|f_0^n\|_1 = \frac{1}{n!}. \tag{5}$$

(c) Show that (4) implies that f_0 generates A as a Banach algebra: that is, $\operatorname{alg}(f)$ is norm dense. Conclude from (5) that the spectral radius $\rho(f)$ is zero for all $f \in A$.

(d) Conclude that A *has no nonzero complex homomorphisms.*

ANS: First compute that[3]

$$\begin{aligned}\|f * g\|_1 &= \int_0^1 |f * g(t)|\,dt \\ &\leq \int_0^1 \int_0^t |f(t-s)g(s)|\,ds\,dt\end{aligned}$$

which, using Tonelli's Theorem, is

$$\begin{aligned}&= \int_0^1 |g(s)|\Big(\int_s^1 |f(t-s)|\,dt\Big)\,ds \\ &= \int_0^1 |g(s)|\Big(\int_0^{1-s} |f(u)|\,du\Big)\,ds \\ &\leq \|f\|_1\|g\|_1.\end{aligned}$$

To show that $f * g = g * f$ it suffices, in view of the above, to consider continuous functions. Thus, the usual calculus techniques apply. In particular,

$$\begin{aligned}f * g(t) &= \int_0^t f(t-s)g(s)\,ds \\ &= -\int_t^0 f(u)g(t-u)\,du = g * f(t).\end{aligned}$$

This proves (a). However, (b) is a simple induction argument.

Now for (c): the calculation (4) shows that $\operatorname{alg}(f_0)$ contains all polynomials. Since the polynomials are uniformly dense in $C[0,1]$, and the later is dense in L^1, we can conclude that $\operatorname{alg}(f_0)$ is norm dense.

Next, observe that (5) not only implies that $\rho(f_0) = 0$, but that $\rho(f_0^k) = 0$ as well for any positive integer k. However, it is not immediately clear that every element of $\operatorname{alg}(f_0)$ has spectral radius zero. However, there is an easy way to see this. Let $\widetilde{A}$ be the unitalization of A (i.e., $\widetilde{A} := A \oplus \mathbb{C}$), and recall that $a \in A$ has spectral radius zero (a is called *quasi-nilpotent*) if and only if $\tilde{h}(a) = 0$ for all

[3] For a reference for Tonelli's Theorem (the 'uselful' version of Fubini's Theorem), see [*Analysis Now*, Corollary 6.6.8], or much better, see Royden's *Real Analysis*. On the other hand, if you are worried about the calculus style manipulation of limits, consider the integrand

$$F(s,t) = \begin{cases} |f(t-s)g(s)| & \text{if } 0 \leq s \leq t \leq 1, \text{ and} \\ 0 & \text{otherwise.}\end{cases}$$

$\tilde{h} \in \widetilde{\Delta} = \Delta(\widetilde{A})$. Since each $\tilde{h}$ is a continuous algebra homomorphism, $\ker(\tilde{h})$ is a closed ideal in $\widetilde{A}$, and it follows that the collection of quasi-nilpotent elements is actually a *closed ideal* of A given by[4]

$$\operatorname{rad}(A) = \bigcap\nolimits_{\tilde{h}\in\widetilde{\Delta}} \ker(\tilde{h}).$$

Since each f_0^k is in $\operatorname{rad}(A)$, so is the *closed* algebra (in fact, the closed ideal) generated by f_0. Thus, $\operatorname{rad}(A) = A$ in this case, which is what was to be shown.

Of course, (d) is an immediate consequence of (c): if $\rho \in \Delta(A)$, then by defintion there is a $f \in A$ such that $\rho(f) \neq 0$. But then $\rho(f) \geq |h(f)| > 0$, which contradicts the fact that $\operatorname{rad}(A) = A$.

9. Here we want to give an example of a unital commutative Banach algebra A where the Gelfand transform induces and injective isometric map of A onto a proper subalgebra of $C(\Delta)$. For A, we want to take the *disk algebra*. There are a couple of ways that the disk algebra arises in the standard texts, but the most convenient for us is to proceed as follows. Let $D = \{ z \in \mathbb{C} : |z| < 1 \}$ be the open unit disk. We'll naturally write $\overline{D}$ for its closure $\{ z \in \mathbb{C} : |z| \leq 1 \}$, and $\mathbb{T}$ for its boundary. Then A will be the subalgebra of $C(\overline{D})$ consisting of functions which are holomorphic on D. Using Morera's Theorem, it is not hard to see that A is closed in $C(\overline{D})$, and therefore a unital commutative Banach algebra.[5] Notice that for each $z \in \overline{D}$, we obtain $\varphi_z \in \Delta$ by $\varphi_z(f) := f(z)$. We'll get the example we want by showing that $z \mapsto \varphi_z$ is a homeomorphism Ψ of $\overline{D}$ onto Δ. For convenience, let $p_n \in A$ be given by $p_n(z) = z^n$ for $n = 0, 1, 2, \ldots$, and let $\mathcal{P}$ be the subalgebra of polynomials spanned by the p_n.

(a) First observe that Ψ is injective. (Consider p_1.)

(b) If $f \in A$ and $0 < r < 1$, then let $f_r(z) := f(rz)$. Show that $f_r \to f$ in A as $r \to 1$.

(c) Conclude that $\mathcal{P}$ is dense in A. (Hint: show that $f_r \in \overline{\mathcal{P}}$ for all $0 < r < 1$.)

(d) Now show that Ψ is surjective. (Hint: suppose that $h \in \Delta$. Then show that $h = \varphi_z$ where $z = h(p_1)$.)

(e) Show that Ψ is a homeomorphism. (Hint: Ψ is clearly continuous and both $\overline{D}$ and Δ are compact and Hausdorff.)

(f) Observe that if we use the above to identify Δ and $\overline{D}$, then the Gelfand transform is the identify on A, and A is a proper subalgebra of $C(\overline{D})$.

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[4]This result is of interest in its own right. Note that A is always a maximal ideal in $\widetilde{A}$, and so $\operatorname{rad}(A)$ is always contained in A itself.

[5]The maximum modulus principal implies that the map sending $f \in C(\overline{D})$ to its restriction to $\mathbb{T}$ is an isometric isomorphism of A onto a closed subalgebra $A(D)$ in $C(\mathbb{T})$. Of course, our analysis applies equally well to $A(D)$.

10. In this problem, we want to prove an old result to due Wiener about functions with absolutely converent Fourier series using the machinery of Gelfand theory. Recall that if $\varphi \in C(\mathbb{T})$, then the *Fourier coefficients* of φ are given by[6]

$$\check{\varphi}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(e^{it}) e^{-int} \, dt.$$

In some cases — for example if φ has two continuous derivatives — the Fourier coefficients are *absolutely convergent* in the sense that $n \mapsto \check{\varphi}(n)$ defines an element of $\ell^1(\mathbb{Z})$.[7]

We aim to prove the following:

Theorem: (Wiener) *Suppose the $\varphi \in C(\mathbb{T})$ never vanishes and has an absolutely convergent Fourier series. Then $\psi := 1/\varphi$ also has an absolutely convergent Fourier series.*

I suggest the following strategy.

(a) If f and g are in $\ell^1(\mathbb{Z})$, then their *convolution*, $f * g$ is defined by

$$f * g(n) = \sum_{m=-\infty}^{\infty} f(m) g(n-m).$$

Show that $f * g \in \ell^1(\mathbb{Z})$ (so that in particular, $f * g(n)$ is defined for all n), and that convolution makes $\ell^1(\mathbb{Z})$ into a unital, commutative Banach algebra. (Here, $1_{\ell^1(\mathbb{Z})} = \mathbb{1}_{\{0\}}$.[8])

(b) Let $\Delta = \Delta(\ell^1(\mathbb{Z}))$ be the maximal ideal space of $\ell^1(\mathbb{Z})$ equipped with its compact, Hausdorff Gelfand topology. If $z \in \mathbb{T}$, then define $h_z : A \to \mathbb{C}$ by

$$h_z(f) = \sum_{n=-\infty}^{\infty} f(n) z^n.$$

Show that $h_z \in \Delta$.

(c) Let $w = \mathbb{1}_1 \in \ell^1(\mathbb{Z})$. If $h \in \Delta$, then show that $h = h_z$ where $z = h(w)$. (Hint: If $f \in \ell^1(\mathbb{Z})$, then $f = \sum_{n=-\infty}^{\infty} f(n) w^n$ in norm in $\ell^1(\mathbb{Z})$, where for example, $w^2 = w * w = \mathbb{1}_2$ and $w^{-1} = \mathbb{1}_{-1}$.)

[6] I've used $\check{\varphi}$ in place of the traditional $\hat{\varphi}$ to distinguish it from the (other) Gelfand transform to be used in the problem.

[7] Recall that $\ell^1(\mathbb{Z}) = L^1(\mathbb{Z}, \nu)$, where ν is counting measure, is the set of functions $f : \mathbb{Z} \to \mathbb{C}$ such that $\lim_{N\to\infty} \sum_{n=-N}^{n=N} |f(n)| < \infty$.

[8] If S is a subset of X, I use $\mathbb{1}_S$ for the characteristic function of S, which takes the value 1 on S, and 0 otherwise.

(d) Show that $z \mapsto h_z$ is a homeomorphism Φ of $\mathbb{T}$ onto Δ. (Hint: Since both $\mathbb{T}$ and Δ are compact Hausdorff sets, it suffices to see that Φ is a continuous bijection. To show that Φ is continuous, observe that functions of the form $\sum_{n=-N}^{n=N} f(n)w^n$ are dense in $\ell^1(\mathbb{Z})$.)

(e) Since we can identify $\mathbb{T}$ with Δ, if $f \in \ell^1(\mathbb{Z})$, we will view the Gelfand transform of f as a continuous function on $\mathbb{T}$. (So that we write $\hat{f}(z)$ in place of $\hat{f}(h_z)$.) Show that if $\varphi = \hat{f}$ for some $f \in A$, then $\check{\varphi} = f$.

(f) Conclude that the image $\mathfrak{A}$ of $\ell^1(\mathbb{Z})$ in $C(\mathbb{T})$ under the Gelfand transform is exactly the set of φ in $C(\mathbb{T})$ whose Fourier coefficients are absolutely convergent. (That is, $\mathfrak{A}$ is the collection of $\varphi \in C(\mathbb{T})$ such that $n \mapsto \check{\varphi}(n)$ is in $\ell^1(\mathbb{Z})$.)

(g) Now prove Wiener's Theorem as stated above. (Hint: More or less by assumption, $\varphi = \hat{f}$ for some f in $\ell^1(\mathbb{Z})$. Show that f must be invertible in $\ell^1(\mathbb{Z})$ and consider the Gelfand transform of the inverse of f.)

ANS: By Theorem J from lecture, $\sigma(f) = \{ \varphi(z) : z \in \mathbb{T} \}$. Since φ never vanishes, $0 \notin \sigma(f)$ and f is invertible. Let $g = f^{-1}$, and consider $\hat{g}$. Since $g * f = 1_A$, we have $\hat{g}\hat{f} = \hat{g}\varphi = 1_{C(\mathbb{T})}$. It follows that $\hat{g} = \psi$, and hence ψ has an absolutely convergent Fourier series — namely g.