1 1.6.14

The diagram gives

$$\begin{gathered} x_1 = 100 + x_2 \\ x_2 + 50 = x_3 \\ x_3 = 120 + x_4 \\ x_4 + 150 = x_5 \\ x_5 = x_6 + 80 \\ x_6 + 100 = x_1 \end{gathered}$$

Hence

$$\begin{gathered} x_1 - x_2 = 100 \\ x_2 - x_3 = -50 \\ x_3 - x_4 = 120 \\ x_4 - x_5 = -150 \\ x_5 - x_6 = 80 \\ x_6 - x_1 = -100 \end{gathered}$$

So to solve this system we need to row reduce the following augmented matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & -1 & 80 \\ -1 & 0 & 0 & 0 & 0 & 1 & -100 \end{bmatrix}$$

$R_5 : R_5 + R_6$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 & -1 & 0 & -150 \\ -1 & 0 & 0 & 0 & 1 & 0 & -20 \\ -1 & 0 & 0 & 0 & 0 & 1 & -100 \end{bmatrix}$$

$R_4 : R_4 + R_5$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ 0 & 0 & 1 & -1 & 0 & 0 & 120 \\ -1 & 0 & 0 & 1 & 0 & 0 & -170 \\ -1 & 0 & 0 & 0 & 1 & 0 & -20 \\ -1 & 0 & 0 & 0 & 0 & 1 & -100 \end{bmatrix}$$

$R_3 : R_3 + R_4$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ 0 & 1 & -1 & 0 & 0 & 0 & -50 \\ -1 & 0 & 1 & 0 & 0 & 0 & -50 \\ -1 & 0 & 0 & 1 & 0 & 0 & -170 \\ -1 & 0 & 0 & 0 & 1 & 0 & -20 \\ -1 & 0 & 0 & 0 & 0 & 1 & -100 \end{bmatrix}$$

$R_2 : R_2 + R_3$

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 100 \\ -1 & 1 & 0 & 0 & 0 & 0 & -100 \\ -1 & 0 & 1 & 0 & 0 & 0 & -50 \\ -1 & 0 & 0 & 1 & 0 & 0 & -170 \\ -1 & 0 & 0 & 0 & 1 & 0 & -20 \\ -1 & 0 & 0 & 0 & 0 & 1 & -100 \end{bmatrix}$$

$R_1 : R_1 + R_2$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & -100 \\ -1 & 0 & 1 & 0 & 0 & 0 & -50 \\ -1 & 0 & 0 & 1 & 0 & 0 & -170 \\ -1 & 0 & 0 & 0 & 1 & 0 & -20 \\ -1 & 0 & 0 & 0 & 0 & 1 & -100 \end{bmatrix}$$

So we have x_1 is free and

$$\begin{aligned} x_1 &= x_1 \\ x_2 &= x_1 - 100 \\ x_3 &= x_1 - 50 \\ x_4 &= x_1 - 170 \\ x_5 &= x_1 - 20 \\ x_6 &= x_1 - 100 \end{aligned}$$

So for $x_1, \ldots, x_6$ to be positive we must have $x_1 \geq 170$ which implies $x_6 \geq 70$ so the smallest possible value for x_6 is 70.

2 2.1.8

If BC has 3 rows then by the definition of matrix multiplication B has 3 rows.

3 2.1.12

Let $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ Now $AB = \begin{bmatrix} 3b_{11} - 6b_{21} & 3b_{12} - 6b_{22} \\ -b_{11} + 2b_{21} & -b_{12} + 2b_{22} \end{bmatrix}$

So if $AB = \mathbf{0}$ we must have

$$\begin{array}{cc} 3b_{11} - 6b_{21} = 0 & 3b_{12} - 6b_{22} = 0 \\ -b_{11} + 2b_{21} = 0 & -b_{12} + 2b_{22} = 0 \end{array}$$

So finding the solution amounts to solving two systems of simultaneous equations.

$$\begin{array}{cc} 2b_{21} = b_{11} & 2b_{22} = b_{12} \\ 2b_{21} = b_{11} & 2b_{22} = b_{12} \end{array}$$

So choose $b_{21} = b_{22} = 1$ so $b_{11} = b_{12} = 2$then

$$AB = \begin{bmatrix} 3(2) - 6(1) & 3(2) - 6(1) \\ -2 + 2(1) & -2 + 2(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$

4 2.1.24

Let $\mathbf{b} \in \mathbb{R}^m$. If $AD = I_m$ we must have $AD\mathbf{b} = I_m\mathbf{b}$ giving $AD\mathbf{b} = \mathbf{b}$. Now $D\mathbf{b}$ is a vector, so if we take $\mathbf{x} = D\mathbf{b}$ by above we have $A\mathbf{x} = \mathbf{b}$. Hence for every $\mathbf{b} \in \mathbb{R}^m$ $A\mathbf{x} = \mathbf{b}$ has a solution. This implies that the columns of A span $\mathbb{R}^m$, hence A has at least m columns. But $AD = I_m$ implies A has exactly m rows which implies A has at least as many columns as rows.

5 2.1.24

Let $A_1, \ldots, A_n$ be the columns of A since the columns of A span $\mathbb{R}^3$ there exist $b_{11}, \ldots, b_{n1} \in \mathbb{R}$ such that

$$A_1 b_{11} + \cdots + A_n b_{n1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Similarly there exist $b_{12}, \ldots, b_{n2} \in \mathbb{R}$ and $b_{13}, \ldots, b_{n3} \in \mathbb{R}$ such that

$$A_1 b_{12} + \cdots + A_n b_{n2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and

$$A_1 b_{13} + \cdots + A_n b_{n3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Let

$$D = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & b_{n3} \end{bmatrix}$$

Then by construction $AD = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ as desired.

6 2.2.8

If A is invertible and $AD = I$ we must have $A^{-1}AD = A^{-1}I$ giving $ID = A^{-1}$ hence $D = A^{-1}$ as desired.

7 2.2.20

a) if $(A - AX)^{-1} = X^{-1}B$ then

$$X(A - AX)^{-1} = XX^{-1}B$$

giving

$$X(A - AX)^{-1} = IB$$

hence

$$X(A - AX)^{-1} = B$$

now X and $(A - AX)^{-1}$ are invertible, hence B is the product of invertible matrices and is therefore invertible.

b)

$$(A - AX)^{-1} = X^{-1}B$$

multiplying on the left by X yields

$$X(A - AX)^{-1} = IB$$

multiplying on the right by $(A - AX)$ yields

$$X = B(A - AX)$$

distributing the B gives

$$X = BA - BAX$$

$$X + BAX = BA$$

$$(I + BA)X = BA$$

$(I + BA) = BAX^{-1}$ so $I + BA$ is the product of invertible matrices and therefore invertible. Hence $X = (I + BA)^{-1}BA$