Math 747: Lie Algebras

Lecturer:
Arun Ram
Mathematics
University of Wisconsin

Contents

1 The basics (Wed. Sept 5)

A *Lie algebra* is a vector space $\mathfrak{g}$ with a bracket $[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ (the tensor product implies that $[,]$ is bilinear) satisfying

(a) (*skew symmetry*) $[x, y] = -[y, x]$, and

(b) (*Jacobi identity*) $[x, [y, x]] + [y, [z, x]] + [z, [x, y]] = 0$,

for all $x, y, z \in \mathfrak{g}$. Note that a Lie algebra is not an algebra ("Lie" is not an adjective), as algebras A are vector spaces with a product under which A becomes a (associative) ring with identity.

1.1 Homomorphisms

An *algebra homomorphism* is a linear map $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ for which $\varphi(xy) = \varphi(x)\varphi(y)$ (and so $\varphi(\mathbb{1}_{\mathfrak{g}_1}) = \mathbb{1}_{\mathfrak{g}_2}$). A *Lie algebra homomorphism* is also a linear map $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ for which $\varphi([x, y]) = [\varphi(x), \varphi(y)]$.

1.2 Categories, Functors, and the Universal Enveloping Algebra

A *category* is a set of objects together with *morphisms* (functions) between them. Our favorite examples are

Alg = (algebras, algebra homomorphisms)

Lie = (Lie algebras, Lie algebra homomorphisms)

A *functor* is a map between categories

$$F : \mathcal{C}_1 \to \mathcal{C}_2$$

which associates to each object $X \in \mathcal{C}_1$ an object $F(X) \in \mathcal{C}_2$, associates to each morphism $f : X \to Y \in \mathcal{C}_1$ a morphism $F(f) : F(X) \to F(Y) \in \mathcal{C}_2$, and preserves both identity morphisms and composition of morphisms.

There is a functor

$$L : \mathrm{Alg} \to \mathrm{Lie}$$

where if $A \in \mathrm{Alg}$, then the underlying vector spaces of A and $L(A)$ are the same, but the product $a \cdot b \mapsto [a, b] = ab - ba$. There is also a functor

$$U : \mathrm{Lie} \to \mathrm{Alg}$$

where if $\mathfrak{g} \in \mathrm{Lie}$, then the underlying vector space of $U\mathfrak{g}$ is that of the algebra generated by the elements of $\mathfrak{g}$ with the relation $\underbrace{xy - yx}_{\text{as in } U\mathfrak{g}} = \underbrace{[x, y]}_{\text{as in } \mathfrak{g}}$. For example, if $[x, y] = 0$ in $\mathfrak{g}$ then $xy = yx$ in $U\mathfrak{g}$. $U\mathfrak{g}$ is said to be the *universal enveloping algebra* of $\mathfrak{g}$.

So now we have functors

$$L : \mathrm{Alg} \to \mathrm{Lie}$$

and

$$U : \mathrm{Lie} \to \mathrm{Alg}.$$

It may be tempting to speculate that these two functors are inverses of some kind. However, it is easy to see that $U\mathfrak{g}$ is rather large as compared to $\mathfrak{g}$ (often infinite dimensional), whereas $L(A)$ is no that much smaller than A (especially when A is finite dimensional). However, we do have the following theorem:

Theorem 1.1. *The functor U is* left-adjoint *to the functor L, i.e.,*

$$\mathrm{Hom}_{\mathrm{Alg}}(U\mathfrak{g}, A) \cong \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, L(A))$$

as vector spaces.

1.3 Exercises

1. Classify all low-dimensional lie algebras.
2. Prove theorem 1.1.
3. Is there a *right-adjoint* to L? i.e., is there some functor F : Lie *to*Alg for which

$$\mathrm{Hom}_{\mathrm{Lie}}(LA, \mathfrak{g}) \cong \mathrm{Hom}_{\mathrm{Alg}}(A, F\mathfrak{g})?$$

2 Forms and Big Examples (Fri. 9/7)

A *representation* of $\mathfrak{g}$ is a $U\mathfrak{g}$-module. A $U\mathfrak{g}$*-module* is a vector space M with a $U\mathfrak{g}$-action

$$U\mathfrak{g} \otimes M \to M,$$

where

$$(u, m) \mapsto um$$

which is *bilinear*, (i.e., if $c_1, c_2 \in \mathbb{C}$, then

$$(c_1u_1 + c_2u_2)m = c_1u_1m + c_2u_2m$$

$$u(c_1m_1c_2m_2) = c_1um_1 + c_2um_2$$

for $u_1, u_2 \in \mathbb{C}, m_1, m_2 \in \mathbb{C}$) and

$$u_1(u_2m) = (u_1u_2)m.$$

Note: whenever we're using *tensor products*, we're just forcing bilinearity.

2.1 Favorite Examples

Big idea: If A be an algebra, then $L(A)$ is a Lie algebra.

1. For example $M_n(\mathbb{C}) = \{n \times n \text{ matrices}\}$ is a Lie algebra with bracket $[a_1, a_2] = a_1a_2 - a_2a_2$. This is the Lie algebra $\mathfrak{gl}_n$.

   This Lie algebra is associated with, but not the same this as, *general linear group*

$$\mathrm{GL}_n(\mathbb{C}) = \{a \in M_n(\mathbb{C}) | a \text{ is invertible }\}.$$

   Let V be a vector space. Then

$$\mathfrak{gl}(V) = \mathrm{End}(V), \qquad \text{with bracket } [a_1, a_2] = a_1a_2 - a_2a_2.$$

2. Define
$$\mathfrak{sl}(V) = \{a \in \mathfrak{gl}(V) | \mathrm{tr}(a) = 0\}.$$
This is associated to, but is not the same thing as, $SL(V) = \{a \in GL(V) | \det(a) = 1\}$, the *special linear group*.

3. Let $\langle , \rangle : V \times V \to \mathbb{C}$ be bilinear form on V. If $\langle , \rangle$ is *symmetric*, i.e. $\langle u, v\rangle = \langle v, u\rangle$, then
$$\mathfrak{so}(V) = \{a \in \mathfrak{sl}(V) | \langle au, v\rangle + \langle u, av\rangle = 0 \text{ for all } u, v \in V\}.$$
This is related to $\mathrm{SO}(V) = \{a \in \mathrm{SL}(V) | \langle au, av\rangle = \langle u, v\rangle \text{for all } u, v \in V\}$, the *special orthogonal group*.

4. If $\langle , \rangle$ is *skew symmetric*, i.e. $\langle u, v\rangle = -\langle v, u\rangle$, then
$$\mathfrak{sp}(V) = \{a \in \mathfrak{sl}(V) | \langle au, v\rangle + \langle u, av\rangle = 0 \text{ for all } u, v \in V\}.$$
This is related to $\mathrm{Sp}(V) = \{a \in \mathrm{SL}(V) | \langle au, av\rangle = \langle u, v\rangle \text{for all } u, v \in V\}$, the *symplectic group*.

5. Let $\langle , \rangle : V \times V \to \mathbb{C}$ be *Hermitian*, i.e. for vector spaces over $\mathbb{C}$,

   a. $\langle u, c_1 v_1 + c_2 v_2\rangle = \bar{c_1}\langle u, v_1\rangle + \bar{c_2}\langle u, v_2\rangle$, and
   b. $\langle c_1 v_1 + c_2 v_2, u\rangle = c_1\langle v_1, u\rangle + c_2\langle v_2, u\rangle$.

   Then
$$\mathfrak{su}(V) = \{a \in \mathfrak{sl}(V) | \langle au, v\rangle + \overline{\langle u, av\rangle} = 0 \text{ for all } u, v \in V\}.$$
This is related to $\mathrm{SU}(V) = \{a \in \mathrm{SL}(V) | \langle au, av\rangle = \langle u, v\rangle \text{for all } u, v \in V\}$, the *special unitary group*.

Notice that most of these algebras have the same structure, only with different types of forms.

2.2 Exercises

1. Is $\mathfrak{sp}(V) = \{a \in \mathfrak{gl}(V) | \langle au, v\rangle + \langle u, av\rangle = 0 \text{ for all } u, v \in V\}$?

3 More Examples (Mon. 9/10)

3.1 Relating the linear groups of symmetries

From last time, we learned about $\mathfrak{gl}(V) = \mathrm{End}(V)$, $\mathfrak{sl}(V) = \{a \in \mathfrak{gl}(V) \mid \mathrm{tr}(a) = 0\}$, and the three $\mathfrak{sp}(V), \mathfrak{so}(V)$ and $\mathfrak{su}(V)$, all of similar form, but with different forms.

The *orthogonal group* is

$$\mathrm{O}_n(\mathbb{R}) = \{A \in \mathrm{GL}_n(\mathbb{R}) | AA^t = 1\}.$$

The *symplectic group.* is

$$\mathrm{Sp}_{2n}(\mathbb{C}) = \{A \in \mathrm{GL}_{2n}(\mathbb{C}) | AJA^t = J\},$$

where

$$J = \left(\begin{array}{ccc|ccc} & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ \hline -1 & & & & & \\ & \ddots & & & & \\ & & -1 & & & \end{array}\right).$$

$\mathrm{SP}(V)$ is "better" though because is is the group of symmetries of V with a skew symmetric form. For example, our favorite set with fixed distance is the sphere. $\mathrm{SO}(V)$ is the group of symmetries for a true sphere, $\mathrm{SP}(V)$ is the group of symmetries of a "symplectic" sphere (whatever that means), etc.. (A "level set of $\langle , \rangle$ is a subset of $V \times V$ of the form $\{(u, v) \in V \times V | \langle u, v \rangle = c\}$, and $\mathrm{SP}(V)$ preserves this).

What does a skew symmetric form "look like"? Well, what does a vector space look like? First, we choose a basis: V has a basis $\{e_1, \ldots, e_n, e_1^*, \ldots e_n^*\}$. and $\langle , \rangle$ is given by

$$\langle e_i, e_i \rangle = -\langle e_i, e_i \rangle = 0, \quad \langle e_i, e_i^* \rangle, \quad \langle e_i, e_j \rangle = 0, \quad \langle e_i^*, e_j^* \rangle = 0, \quad \langle e_i, e_j^* \rangle = 0, \quad \langle e_i^*, e_j \rangle = 0.$$

Then

$$J = (\langle b_i, b_j \rangle)_{b_i, b_j \in \{e_1, \ldots, e_n, e_1^*, \ldots e_n^*\}}$$

is the matrix of $\langle , \rangle$. So $AJA^t = J$ is a translation of $\langle Au, v \rangle = \langle U, Av \rangle$ by choosing a *very specific* basis. So something we learn by changing to Sp_{2n} by seeing things anew, without really doing anything new. (the message: changing bases is one way to do something that is not new, but still very worthwhile)

Aside: Why use the fields we do? There are lots of GL_n's: $\mathrm{GL}_n(\mathbb{C}), \mathrm{GL}_n(\mathbb{R}), \mathrm{GL}_n(\mathbb{F}_q)$, etc. the ones with $\mathbb{C}, \mathbb{R}$ are "geometric", but so is $\mathbb{F}_q$ in some sense. We change the field as is suits us...

3.2 The free Lie algebra

Motivation: Whenever we learn about a new object, the free *version is usually very interesting.*

The *free Lie algebra* on a set $\{x, y, z\}$ is just the Lie algebra generated by $\{x, y, z\}$, i.e. $\mathfrak{g}$ is the span of $x, y, z, [x, y], [x, z], [y, z], \ldots$.

What else is in there? To get a general idea of what other elements appear, start applying skew symmetry and Jacobi identity relations. For example, $\mathfrak{g}$ being a Lie algebra gives $[y, x] = -[x, y]$, so we don't need to include $[y, x], [z, x]$, etc.. We also have the symbol$[x, [x, y]]$ to include, but $[x, [y, x]] = -[x, [x, y]]$ is superfluous. Moreover $[x, [y, y]] = 0$. Also we could check if the Jacobi identity does anything extra for us: $0 = [x, [x, y]] + [y, [x, x]] + [x, [y, x]] = [x, [x, y]] + 0 - [x, [x, y]]$, so no, not in this case.

The free Lie algebra is *graded* by *degree* (the number of factors bracketed together). For example $\deg([x,[x,y]]) = 3$. So

$$\mathfrak{g} = \bigoplus_{\substack{n\in\mathbb{Z}^+ \\ \dim(\mathfrak{g}_n) \text{ finite}}} \mathfrak{g}_n, \qquad \text{where } \mathfrak{g}_n \text{ is the span of deg } n \text{ brackets}$$

3.3 An aside: two dimensional Lie Algebras and Dick Hain (9/12)

Let

$$\mathfrak{g} = \mathbb{C}\text{-span}\{x,y\}, \qquad \text{with bracket } [x,y] = x.$$

Let $a, b \in \mathfrak{g}$. So $a = a_1x + a_2y$ and $b = b_1x + b_2y$. Then

$$\begin{aligned} [a,b] &= [a_1x + a_2y, b_1x + b_2y] \\ &= a_1[x, b_1x + b_2y] + a_2[y, b_1x + b_2y] \\ &= \dots \\ &= (a_1b_2 - a_2b_1)x. \end{aligned}$$

Dick Hain (Duke U) gave a talk *Modular forms and derivations of free Lie algebras of rank 1* at Sydney U. 10 July 2007 based on the free Lie algebra of rank 2, generated by x and y. The background: Let $V = \text{span}\{x,y\}$. Then

$$\mathrm{SL}_2(\mathbb{C}) = \{A \in \mathrm{GL}_2(\mathbb{C}) \mid ad - bc = 1\}$$

So $\mathrm{SL}_2(\mathbb{C})$ acts on V, and thus $\mathrm{SL}_2(\mathbb{C})$ acts on $\mathfrak{g}$. The submodules of $\mathfrak{g}$ (as SL_2-modules) correspond to modular forms.

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix}[x,y] &= \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}x, \begin{pmatrix} a & b \\ c & d \end{pmatrix}y\right] \\ &= [ax + cy, bx + dy] \\ &= (ad - bc)[x,y] \end{aligned}$$

if

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and } y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

3.4 Exercises

1. What is $\dim(\mathfrak{g}_n)$? [For example, $\dim(\mathfrak{g}_1) = 3$.]

   ((*same as number of irreducible polynomials of degree n over some field??*))

4 Complexification (Wed. 9/12)

How can a Lie algebra with complex coefficients not be a complex Lie algebra? And if it's not, what is its complex counterpart?

Definition. Let $\mathfrak{g}$ be a real Lie algebra with basis $x_1, \ldots, x_n$. So

$$\mathfrak{g} = \mathbb{R}\text{-span}\{x_1, \ldots, x_n\}.$$

The *complexification* of $\mathfrak{g}$ is the complex Lie algebra

$$\mathfrak{g}_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g} = \mathbb{C}\text{-span}\{x_1, \ldots, x_n\}$$

with bracket $[x_i, x_j]$ as before.

4.1 Example: $\mathfrak{su}_1(\mathbb{C})$ is a real Lie algebra

Recall $\mathfrak{su}_1(\mathbb{C}) = \{a \in \mathfrak{sl}_1(\mathbb{C}) | a + \overline{a^t} = 0\}$ with bracket $[a, b] = ab - ba$. So

$$\begin{aligned}
\mathfrak{s}_1(\mathbb{C}) &= \{a \in \mathbb{C} \mid a + \bar{a}^t = 0\} \\
&= \{x + iy \mid a, y \in \mathbb{R}, x + iy + (x - iy) = 0\} \\
&= \{x + iy \mid a, y \in \mathbb{R}, 2x = 0\} \\
&= \mathbb{R}i \\
&= \mathbb{R}\text{-span}\{i\} \text{ with bracket } [i, i] = 0.
\end{aligned}$$

This is a one-dimensional $\mathbb{R}$-vector space, so has *no hope* of being a complex Lie algebra! So $\mathfrak{su}_1(\mathbb{C})$ is a *real* Lie algebra, not a complex Lie algebra.

The standard complexification of $\mathfrak{su}_1(\mathbb{C})$ is the Lie algebra

$$\mathbb{C}\text{-span}\{h\} \qquad \text{with bracket } [h, h] = 0$$

(we use h since, other wise, we would have two i's: the elements are αh where $\alpha \in \mathbb{C}$, so $\alpha = \alpha_1 + i\alpha_2$).

4.2 Example: $\mathfrak{su}_2(\mathbb{C})$ has complexification $\mathfrak{sl}_2(\mathbb{C})$

Recall

$$\begin{aligned}
\mathfrak{sl}_2(\mathbb{C}) &= \{A \in \mathfrak{gl}_2(\mathbb{C}) \mid \operatorname{tr}(A) = 0\} \\
&= \mathbb{C}\text{-span}\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a + d = 0 \right\} \\
&= \mathbb{C}\text{-span}\left\{ x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.
\end{aligned}$$

(sometimes people use $e = x$ and $f = y$)

Then

$$\begin{aligned}
[x,y] &= xy - yx \\
&= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= h.
\end{aligned}$$

The relations

$$[x,y] = h, \quad [h,x] = 2x, \quad \text{and } [h,y] = -2y$$

define $\mathfrak{sl}_2$ (the rest following similarly as above).

Now,

$$\begin{aligned}
\mathfrak{su}_2(\mathbb{C}) &= \left\{ A \in \mathfrak{gl}_2(\mathbb{C}) \;\middle|\; \operatorname{tr}(A) = 0, \quad A + \overline{A^t} = 0 \right\} \\
&= \left\{ \begin{pmatrix} a & b \\ d & c \end{pmatrix} \;\middle|\; a + d = 0,\; a + \bar{a} = 0,\; d + \bar{d} = 0, \quad \text{and } b + \bar{c} = 0 \right\} \\
&= \left\{ \begin{pmatrix} i\alpha & \beta + i\gamma \\ -\beta + i\gamma & -\alpha \end{pmatrix} \;\middle|\; \alpha, \beta, \gamma \in \mathbb{R} \right\}
\end{aligned}$$

is a *real* Lie algebra with dimension 3 (it *must* be real and not complex since it has odd dimension). To calculate its complexificaiton, let

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and } \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These are called the *Pauli matrices* (σ is standard notation from physics), and generate $\mathfrak{su}_2(\mathbb{C})$:

$$\mathfrak{su}_2(\mathbb{C}) = \mathbb{R}\text{-span}\{i\sigma_x, i\sigma_y, i\sigma_z\}.$$

The standard complexification of $\mathfrak{su}_2(\mathbb{C})$ is

$$\begin{aligned}
\mathbb{C}\text{-span}\{i\sigma_x, i\sigma_y, i\sigma_z\} &= \mathbb{C}\text{-span}\{\sigma_x, \sigma_y, \sigma_z\} \\
&= \mathbb{C}\text{-span}\left\{\frac{1}{2}(\sigma_x + i\sigma_y), \frac{1}{2}(\sigma_x - i\sigma_y), \sigma_z\right\} \\
&= \mathbb{C}\text{-span}\{x, y, z\} \\
&= \mathfrak{sl}_2(\mathbb{C}).
\end{aligned}$$

The moral If you are a "hardcore Lie theorist" you *only* know $\mathfrak{sl}_2(\mathbb{C})$. If you are a "hardcore physicist" you *only* know $\mathfrak{su}_2(\mathbb{C})$. If you are a "hardcore number theorist" you *only* know $\mathfrak{sl}_2(\mathbb{R}) = \mathbb{R}\text{-span}\{x, y, h\}$ with bracket the same. The moral of the story is don't be so hardcore.

5 Derivations (Fri. Sept 14 & Mon. Sept 17)

Rob's questions: Why are Lie Algebras a natural object?

Attempt 1: Same reason that going to see "March of the Penguins" was a natural thing to do.

Attempt 2: There are connections to Lie groups. We don't totally understand this connection, but they arise in this way.

Attempt 3: A connection to quantum mechanics comes into play... not that Quantum Mechanics is such a "natural" thing...

Attempt 4: The work of Dick Hain [3.3] (one of the coolest people in the world these days) uses free Lie algebras to study modular forms.

Attempt 5: Today! Derivations...

Definition. Let A be an algebra. A *derivation* of A is a linear map $\mathrm{d} : A \to A$ (linear means I only have to define it on the basis) such that

$$\mathrm{d}(ab) = \mathrm{d}(a)b + a\mathrm{d}(b), \text{ for } a, b \in A.$$

Let Der A be the space of derivations of A.

Theorem 5.1. Der A *is a Lie algebra with bracket*

$$[\mathrm{d}_1, \mathrm{d}_2] = \mathrm{d}_1\mathrm{d}_2 - \mathrm{d}_2\mathrm{d}_1.$$

Proof. Check that the bracket of two d's satisfies the product rule:

$$\begin{aligned}
[\mathrm{d}_1, \mathrm{d}_2](ab) &= (\mathrm{d}_1\mathrm{d}_2 - \mathrm{d}_2\mathrm{d}_1)(ab) \\
&= (\mathrm{d}_1\mathrm{d}_2)(ab) - (\mathrm{d}_2\mathrm{d}_1)(ab) \\
&= \mathrm{d}_1\big(\mathrm{d}_2(a)\cdot b + a\cdot \mathrm{d}_2(b)\big) - \mathrm{d}_2\big(\mathrm{d}_1(a)\cdot b + a\cdot \mathrm{d}_1(b)\big) \\
&= \Big(\mathrm{d}_1\mathrm{d}_2(a)\cdot b + \mathrm{d}_2(a)\cdot \mathrm{d}_1(b) + \mathrm{d}_1(a)\cdot \mathrm{d}_2(b) + a\cdot(\mathrm{d}_1\mathrm{d}_2)(b)\Big) \\
&\quad -\Big((\mathrm{d}_2\mathrm{d}_1)(a)\cdot b + \mathrm{d}_1(a)\cdot \mathrm{d}_2(b) + \mathrm{d}_2(a)\cdot \mathrm{d}_1(b) + a\cdot(\mathrm{d}_2\mathrm{d}_1)(b)\Big) \\
&= (\mathrm{d}_1\mathrm{d}_2)(a)\cdot b + a\cdot(\mathrm{d}_1\mathrm{d}_2)(b) - (\mathrm{d}_2\mathrm{d}_1)(a)\cdot b - a\cdot(\mathrm{d}_2\mathrm{d}_1)(b) \\
&= \big(\mathrm{d}_1\mathrm{d}_2 - \mathrm{d}_2\mathrm{d}_1\big)(a)\cdot b + a\big(\mathrm{d}_1\mathrm{d}_2 - \mathrm{d}_2\mathrm{d}_1\big)(b) \\
&= [\mathrm{d}_1, \mathrm{d}_2](a)\cdot b + a\cdot[\mathrm{d}_1, \mathrm{d}_2](b)
\end{aligned}$$

□

5.1 Example: $A = \mathbb{C}[x]$

Let $d \in$ Der A. Then

$$\begin{aligned}
\mathrm{d}(x^2) &= \mathrm{d}(x)x + x\mathrm{d}(x) = 2x\cdot\mathrm{d}(x). \\
\mathrm{d}(x^3) &= \mathrm{d}(x)x^2 + x\mathrm{d}(x^2) \\
&= \mathrm{d}(x)x^2 + x\cdot 2x\cdot\mathrm{d}(x) \\
&= 3x^2\cdot\mathrm{d}(x). \\
\mathrm{d}(x^k) &= kx^{k-1}\cdot\mathrm{d}(x).
\end{aligned}$$

Since $\mathbb{C}[x]$ has basis $\{1, x, x^2, \dots\}$, any derivation $d \in \text{Der}\ \mathbb{C}[x]$ is determined by your choice of $d(x)$. But

$$d : \mathbb{C}x \to \mathbb{C}[x],$$

so d(x) can be any element of $\mathbb{C}[x]$. So

$$\text{Der}\ \mathbb{C}[x] = \mathbb{C}[x]\text{-span}\left\{\frac{\partial}{\partial x}\right\}, \qquad \text{where } \frac{\partial}{\partial x}x^k = kx^{k-1}.$$

For example

$$\mathrm{d} = (x^2 + 7x + 2)\left(\frac{\partial}{\partial x}\right).$$

Note: $\dim_{\mathbb{C}[x]}\text{Der}\ \mathbb{C}[x] = 1$.

5.2 Example: $A = \mathbb{C}[x, x^{-1}]$

Later

5.3 Example: $A = U\mathfrak{h}$, where $\mathfrak{h}$ is abelian

Let $\mathfrak{h}$ be a vector space with basis $x_1, \dots x_n$ and bracket $[x_i, x_j] = 0$. The enveloping algebra of $\mathfrak{h}$ is

$$U\mathfrak{h} = \mathbb{C}[x_1, \dots, x_n].$$

(one way to think about polynomial rings!)

5.4 Example: the Heisenberg Lie algebra

Let $\mathbb{C}$ be an arbitrary field. Let $\mathfrak{h}$ be a vector space with basis $x_1, \dots, x_n$ and let

$$\mathfrak{h}^* = \text{Hom}(\mathfrak{h}, \mathbb{C})$$

be the *dual* (also a vector space) with basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. Note: we're borrowing notation from calculus: instead of writing x_i^*, we write $\frac{\partial}{\partial x_i}$. It is certainly a dual basis, since $\frac{\partial}{\partial x_i}x_j = \delta_{ij}$. Why our notation makes sense:

$$\begin{aligned}
\left(x_i\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_j}x_i\right)f &= x_1\frac{\partial}{\partial x_j}f - \frac{\partial}{\partial x_j}x_i f \\
&= x_i\frac{\partial}{\partial x_j}f - \left(\delta_{ij}f + x_i\frac{\partial}{\partial x_j}\right)f \\
&= -\delta_{ij}f.
\end{aligned}$$

Definition. The *Heisenberg Lie algebra* $\mathfrak{g}$ is is

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^* \oplus \mathbb{C}c$$

(so has basis $x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, c$) with bracket

$$[c, y] = 0, \qquad [x_i, x_j] = 0, \qquad \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0,$$

$$\left[x_i, \frac{\partial}{\partial x_j}\right] = c\frac{\partial}{\partial x_j}(x_i) = c\delta_{ij}$$

for all $y \in \mathfrak{g}$.

Note that *center* of a Lie algebra $\mathfrak{g}$ is

$$Z(\mathfrak{g}) = \{z \in \mathfrak{g} \mid [z, y] = 0 \text{ for all } y \in \mathfrak{g}\}.$$

So these relations are mostly saying that the Heisenberg Lie algebra is mostly abelian.

The enveloping algebra of $\mathfrak{g}$ is the *Weyl algebra*. It has generators $x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, c$ with relations

$$x_i x_j = x_j x_i, \qquad \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i},$$
$$c x_i = x_i c, \qquad c\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j} c,$$
$$\frac{\partial}{\partial x_i} x_j = x_j \frac{\partial}{\partial x_j} + \delta_{ij} c.$$

Elements of Weyl algebra are linear combinations of

$$c^\ell x_1^{m_1} \cdots x_n^{m_n} \left(\frac{\partial}{\partial x_1}\right)^{\ell_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\ell_n},$$

i.e. polynomial coefficient differential operators.

6 Birds and bees of Lie groups and their algebras (Wed. 9/19)

Definition. Let G be a group, and define

$$\mu : G \times G \to G \qquad \text{via } (g_1, g_2) \mapsto g_1 g_2$$

and

$$\iota : G \to G \qquad \text{via } g \mapsto g^{-1}.$$

1. G is a *topological group* if it is also a topological space and μ and ι are continuous maps.
2. G is a *Lie group* if it is also a manifold and μ and ι are smooth maps.
3. G is a *complex Lie group* if it is also a complex manifold and μ and ι are holomorphic functions.
4. G is a *linear algebraic group* if it is also an affine variety and μ and ι are regular functions.
5. G is a *group scheme* if it is also a scheme and μ and ι are morphisms of schemes.

Remark. About these maps:

1. A *continuous* map is just a morphism of topological spaces .
2. A *smooth* map is just a morphism of manifolds. Really, “smooth” tells me that I can take derivative. So viewing manifolds with respect to their morphisms, what I care about is not that they look locally like $\mathbb{R}^n$, but rather that I know what derivatives look like.

3. A *holomorphic* function is just a morphism of complex manifolds. What's special about the "complex" of complex manifolds? In $\mathbb{C}$, we have complex conjugation, giving it a complex structure. Holomorphic functions additionally preserve complex conjugation.

4. A *regular* function is just a morphisms of varieties, rather, things that look like polynomials.

The above definitions seem to follow a categorical feel! Another way of talking about this idea is via "ringed" objects: A *ringed space* $(X, \mathcal{O}_X)$ is a topological space X with a space of functions $\mathcal{O}_X$ on X. A *tangent vector* to X at an element x is a linear map

$$\xi_x : \mathcal{O}_X \to \mathbb{F}, \qquad \text{satisfying} \qquad \xi_x(f_1 f_2) = \xi_x(f_1) \cdot f_2(x) + f_1(x) \cdot \xi_x(f_2),$$

for $f_1, f_2 \in \mathcal{O}_X$ (so it's almost a derivation). Now, usually in freshman calculus, we make this mistake:

$$\underbrace{\frac{d}{dx}(x^3 + 2x + 1)}_{function} = \underbrace{\lim_{h \to 0} \cdots}_{number},$$

where we treat the right side as a function instead. A *vector field* on X is a derivation of $\mathcal{O}_X$, i.e. a linear map $\mathrm{d} : \mathcal{O}_x \to \mathcal{O}_x$ such that

$$\mathrm{d}(f_1 f_2) = \mathrm{d}(f_1) \cdot f_2(x) + f_1(x) \cdot \mathrm{d}(f_2),$$

for $f_1, f_2 \in \mathcal{O}_X$.

Let $(G, \mathcal{O}_G)$ be a ringed group (totally unstandard phrase, but it seems to be the right thing to call it). Then G acts on $\mathcal{O}_G$ by (left) *regular action*:

$$(L_g f)(h) = f(g^{-1}h) \qquad \text{for } h \in G,$$

and right translation

$$(R_g f)(h) = f(hg) \qquad \text{for } h \in G.$$

Notice, R_g is actually a *left* action:

$$\begin{aligned} (R_g R_{g'} f)(h) &= (R_{g'} f)(hg) \\ &= f(hgg') \\ &= (R_{gg'} f)(h). \end{aligned}$$

A *left invariant vector field* is a derivation

$$\mathrm{d} : \mathcal{O}_G \to \mathcal{O}_G, \quad \text{such that} \quad L_g \mathrm{d} L_{g^{-1}} = \mathrm{d}.$$

(A *right invariant vector field* is a derivation $\mathrm{d} : \mathcal{O}_G \to \mathcal{O}_G$ such that $R_g \mathrm{d} R_{g^{-1}} = \mathrm{d}$.)

Suppose d is left invariant, and let d_g be the tangent vector at $g \in G$, given by

$$\mathrm{d}_g f = (\mathrm{d}f)(g).$$

Notice that if d is left invariant, then d is determined by d_1:

$$\begin{aligned} \mathrm{d}_g f &= (\mathrm{d}f)(g) \\ &= (L_{g^{-1}} \mathrm{d}f)(1) \\ &= (L_{g^{-1}} \mathrm{d} L_g L_{g^{-1}} f)(1) \\ &= (\mathrm{d} L_{g^{-1}} f)(1) \\ &= \mathrm{d}_1(L_{g^{-1}} f). \end{aligned}$$

Now let

$$T_1(G) = \{\text{tangent vectors to } G \text{ at } 1\}.$$

So

$$\{ \text{ left invarient vector fields on } G\} \leftrightarrow T_1(G),$$

and so

$$\{ \text{ left invarient vector fields on } G\} \subset \text{Der } (\mathcal{O}_G).$$

Recall that Der $(\mathcal{O}_G)$ is a Lie algebra under $[\mathrm{d}, D] = \mathrm{d}D - D\mathrm{d}$ for $\mathrm{d}, D \in \text{Der } (\mathcal{O}_X)$. Moreover, if $\mathrm{d}, D \in \text{Der } (\mathcal{O}_G)$ are left-invarient, then $[\mathrm{d}, D]$ is also left invariant. So the set of left invarient vector fields on G is a Lie subalgebra of Der $(\mathcal{O}_G)$. So finally, if $(G, \mathcal{O}_G)$ is a ringed group, then the *Lie algebra of the group* G is $\{$ left invarient vector fields on $G\}$ $(= T_1(G))$. (take a tangent vector to the point 1, and move it around by G; since that movement is a morphism, and get out a whole vector field).

For next time: A *one parameter subgroup* is a morphism $\gamma : \mathbb{R} \to G$. Very fiew of these! Example: find all group homomorphisms $\gamma : \mathbb{R} \to \mathrm{GL}_1(\mathbb{C})$.

7 Birds and bees, part 2 (Fri. 9/21 and Mon. 9.24)

The *Lie algebra* of G is $\mathfrak{g} = T_1(G) = \{\text{tangent vectors to } G \text{ at } 1\}$ with bracket obtained by identifying $T_1(G)$ with the set of left invariant vector fields on G with is a Lie subalgebra of Der $(\mathcal{O}_G)$. So we have a map from Lie groups to Lie algebras. THis is a functor!

Let G is a Lie group. A *one parameter subgroup* of G is a morphism

$$\gamma : \mathbb{R} \to G$$

(our favorite Lie group is $\mathbb{R}$ under the operation *addition*).

Example. What are the one parameter subgroups of $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$:

$$\gamma : \mathbb{C} \to \mathrm{GL}_1(\mathbb{C})?$$

Well, what are the group homomorphisms $\gamma : \mathbb{R} \to \mathrm{GL}_1(\mathbb{R})$, ie.e $\gamma(s + t) = \gamma(s) + \gamma(t)$? So $\gamma(t) = a_0 + a_1 t + a_2 t^2 + \cdots$. We want $\gamma(s + t) = \gamma(s) + \gamma(t)$. The expansion of the left side is

$$\begin{gathered} a_0 \\ +a_1 s + a_1 t \\ +a_2 s^2 + 2a_2 st + a_2 t^2 \\ +a_3 s^3 + 3a_3 s^2 t + 3a_3 st^2 + a_3 t^3 \\ +a_4 s^4 + 4a_4 s^3 t + 6a_4 s^2 t^2 + 4a_3 st^3 + \cdots \end{gathered}$$

The expansion of the right side is

$$\begin{gathered} a_0^2 \\ +a_1 a_0 s + a_1 a_0 t \\ +a_2 a_0 s^2 + a_1^2 st + a_0 a_2 t^2 \end{gathered}$$

$$+a_3a_0s^3 + a_2a_1s^2t + a_2a_1st^2 + a_0a_3t^3$$
$$+a_4a_0s^4 + a_3a_1s^3t + a_2^2s^2t^2 + a_1a_3st^3 + \cdots .$$

So either $0 = a_0 = a_1 = a_2 = \cdots$, or

$$\begin{aligned} a_0 &= 1 \\ a_1 &= a_1 \\ a_2 &= \frac{1}{2}a_1^2 \\ a_3 &= \frac{1}{3}a_2a_1 = \frac{1}{3!}a_1^3 \\ &\vdots \end{aligned}$$

So there is a (almost) unique one-parameter subgroup $\gamma : \mathbb{R} \to \mathrm{GL}(\mathbb{R})$ given by $\gamma(t) = e^{a_1t}$ for each $a_1 \in \mathbb{R}$. Really, the one-parameter subgroup is the function, but we like to actually think about it as the image of γ, which is a subgroup of G (which looks locally like $\mathbb{R}$ near the identity.

Now suppose $\gamma : \mathbb{R} \to G$ is a one parameter subgroup. Define γ_1 as a tangent vector to 1 by

$$(\gamma_1 f) = \lim_{t\to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t - 0}$$

(just some convenient derivation. There are many of them, but this is one that we know a lot about). To Check that this is a tangent vector, we need to check

$$\gamma_1(f_1f_2) = f_1(1)(\gamma_1 f_2) + (\gamma_1 f_1)f_2(1).$$

So now we have a map from one parameter subgroups of G to $T_1(G)$ via

$$\gamma \mapsto \gamma_1.$$

Is this invertible? Baby example:

$$\{\text{tangent vectors at 1 in } \mathrm{GL}_1(\mathbb{C})\} \leftrightarrow \{\text{one parameter subgroup of } \mathrm{GL}_1(\mathbb{C})\}$$

$$a_1 \mapsto e^{a_1t}.$$

So left invariant vector fields on G are tangent vectors at 1 to G are one parameter subgroups of G. We have a functor from Lie groups G to Lie algebras $\mathfrak{g}$.

7.1 Example: $\mathfrak{gl}_n$ maps to GL_n under exponentiation

Recall:

$$\begin{aligned} \mathrm{GL}_n(\mathbb{C}) &= \{g \in M_n(\mathbb{C}) \mid g^{-1} \text{ exists }\}, \text{ and} \\ \mathfrak{gl}_n(\mathbb{C}) &= M_n(\mathbb{C}) \text{ has basis } \{E_{ij} \mid 1 \le i, j \le n\} \end{aligned}$$

where E_{ij} has a 1 in the (i,j) entry and 0 elsewhere. so we have a map from $\mathfrak{gl}_n(\mathbb{C})$ to the one-parameter subgroups of $\mathrm{GL}_n(\mathbb{C})$ via $E_{ij} \mapsto e^{tE_{ij}}$. If $i \neq j$, then

$$
\begin{aligned}
e^{tE_{ij}} &= 1 + tE_{ij} + \frac{t^2}{2!}E_{ij}^2 + \frac{t^3}{3!}E_{ij}^3 + \cdots \\
&= 1 + tE_{ij} + 0 + 0 + \cdots \\
&= \begin{array}{cc} \begin{array}{ccccc} & & i & & \\ & & \downarrow & & \end{array} & \\ \begin{pmatrix} 1 & & & \\ & \ddots & t & \\ & & \ddots & \\ & & & 1 \end{pmatrix} & \begin{array}{c} \\ \leftarrow j \\ \\ \\ \end{array} \end{array} \\
&= x_{ij}.
\end{aligned}
$$

If, on the other hand, $i = j$, we have

$$
\begin{aligned}
e^{tE_{ii}} &= 1 + tE_{ii} + \frac{t^2}{2!}E_{ii}^2 + \frac{t^3}{3!}E_{ii}^3 + \cdots \\
&= 1 + tE_{ii} + \frac{t^2}{2!}E_{ii} + \frac{t^3}{3!}E_{ii} + \cdots \\
&= \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & e^t & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \\
&= h_i(e^t).
\end{aligned}
$$

Let G be a Lie group and let $\mathfrak{g}$ be the Lie algebra. Let $\{x_1, \ldots, x_n\}$ be a basis of $\mathfrak{g}$, and so $e^{tx_1}, \ldots, e^{tx_n}$ are one-parameter subgroups in G. The $x_{ij}(t)$ and $h_i(t)$ are the *elementary matrices* for G.

Theorem 7.1 (Linear algebra theorem 1)**.** *The elementary matrices generate* $\mathrm{GL}_n(\mathbb{C})$.

7.2 Example: $\mathcal{O}_n(\mathbb{C})$

Let $G = \mathcal{O}_n(\mathbb{C}) = \{g \in \mathrm{GL}_n(\mathbb{C}) \mid gg^t = 1\}$. There is a Very Important map,

$$\det : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_1(\mathbb{C})$$

with the Very Important property that it is a homomorphism (and therefore a representation). Really, it is a polynomial function:

$$\det(a_{ij}) = \sum_{w \in S_n} (-1)^{\ell(w)} a_{1w(1)} \cdots a_{nw(n)},$$

so it is any kind of function that we want it to be (holomorphic, continuous, whatever). Here,

$$\det : \mathcal{O}_n(\mathbb{C}) \twoheadrightarrow \{\pm 1\} \quad \text{since } \det(g)^2 = 1$$

for $g \in \mathcal{O}_n(\mathbb{C})$ (certainly, this map is surjective, since if

$$s_1 = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix},$$

then $\det s_1 = -1$).

Topologically, $\mathrm{SO}_n(\mathbb{C})$ and $s_1\mathrm{SO}_n(\mathbb{C})$ are the same, and $\mathcal{O}_n$ is the disjoint union of the two, with the identity sitting inside of $\mathrm{SO}_n(\mathbb{C})$. $\mathrm{Lie}(\mathcal{O}_n) =$ Lie algebra of $\mathcal{O}_n = T_1(\mathcal{O}_n(\mathbb{C})) = T_1(\mathrm{SO}_n(\mathbb{C}))$ (since the identity is in $\mathrm{SO}_n(\mathbb{C})$). So we have a functor from Lie groups to Lie algebras via $G \mapsto T_1(G) = \mathfrak{g}$.

In the last example, theorem [7.1] gave us that the exponentiation map took us the other direction surjectively. However, in this case, exponentiation won't get us into the $s_1\mathrm{SO}_n(\mathbb{C})$ component, so this theorem certainly is not true in general!

8 The Virasoro algebra and the beginning of its motivation (Wed. 9/26)

Example: If $\mathrm{d} \in \mathrm{Der}\,(\mathbb{C}[x])$ the $\mathrm{d}(x^k) = kx^{k-1}\mathrm{d}(x)$, $k \in \mathbb{Z}_{\geq 0}$, where $\mathrm{d}(x) \in \mathbb{C}[x]$. So $\mathrm{Der}\,(\mathbb{C}[x]) = \mathbb{C}[x]$-span$\{\frac{\partial}{\partial x}\}$ and has a $\mathbb{C}$ basis $x^k\frac{\partial}{\partial x}$.

Example: If $\mathrm{d} \in \mathrm{Der}\,(\mathbb{C}[x, x^{-1}])$ the $\mathrm{d}(x^k) = kx^{k-1}\mathrm{d}(x)$, $k \in \mathbb{Z}$, where $\mathrm{d}(x) \in \mathbb{C}[x, x^{-1}]$. So $\mathrm{Der}\,(\mathbb{C}[x, x^{-1}]) = \mathbb{C}[x, x^{-1}]$-span$\{\frac{\partial}{\partial x}\}$ and has a $\mathbb{C}$ basis $x^k\frac{\partial}{\partial x}$.

Step back and thing about the geometry for a moment:

$$\mathbb{C}[x, x^{-1}] = \mathcal{O}_X = \text{functions on } S^1.$$

since $S^1 = \{e^{i\theta} \mid 0 \leq \theta < 2\pi\} = \{z \mid z \in \mathbb{C}, |z| = 1\}$. A functions on S^1 is a functions of z. If $z \mapsto z^{-1}$ is allowed, then your favorite set of functions includes $\mathbb{C}[x, x^{-1}$. (Another way to think of is is functions on $\mathbb{C} \setminus 0$ is the same as functions on the circle).

The *Witt algebra* is the Lie alebra with basis $\ldots, L_{-2}, L_{-1}, L_0, L_1, L_2, \ldots$ and bracket $[l_m, L_n] = (m-n)L_{m+n}$. Do the computation: If $L_m = -x^{m+1}\frac{\partial}{\partial x}$, then

$$\begin{aligned}
[L_m, L_n] &= [-x^{m+1}\frac{\partial}{\partial x}, -x^{n+1}\frac{\partial}{\partial x}] \\
&= (-x^{m+1}\frac{\partial}{\partial x})(-x^{n+1}\frac{\partial}{\partial x}) - (-x^{n+1}\frac{\partial}{\partial x})(-x^{m+1}\frac{\partial}{\partial x}) \\
&= (x^{m+1}\frac{\partial}{\partial x})(x^{n+1}\frac{\partial}{\partial x}) - (x^{n+1}\frac{\partial}{\partial x})(x^{m+1}\frac{\partial}{\partial x}) \\
&= x^{m+1}((n+1)x^n\frac{\partial}{\partial x} + x^{n+1}\frac{\partial^2}{\partial x^2}) - x^{n+1}((m+1)x^m\frac{\partial}{\partial x} + x^{m+1}\frac{\partial^2}{\partial x^2}) \\
&= (n+1)x^{m+n+1}\frac{\partial}{\partial x} - (m+1)x^{n+m+1}\frac{\partial}{\partial x} \\
&= (m-n)(-x^{m+n+1}\frac{\partial}{\partial x}) \\
&= (m-n)L_{m+n}.
\end{aligned}$$

(one could have begun by just defining the Witt algebra, but it is good to understand the geometry hiding in the background).

The *Virasoro algebra* is the Lie algebra with basis

$$\dots, L_{-2}, L_{-1}, L_0, L_1, L_2, \dots, \text{ and } c$$

and bracket

$$[c, y] = 0, \text{ for any } y \in \mathfrak{g},$$
$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{1}{12}(m^3 - m)c.$$

Bam! What's with the $\frac{1}{12}$?? No one really knows. It's a very natural and interesting question, and it comes up in a lot of places, but it's not clear why it keeps popping up.

Where does this come from? We saw c before, in the Heisenberg Lie algebra. It had basis $x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, c$with bracket

$$[c, y] = 0, \qquad [x_i, x_j] = 0, \qquad \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0,$$
$$\left[x_i, \frac{\partial}{\partial x_j}\right] = c\frac{\partial}{\partial x_j}(x_i) = c\delta_{ij}$$

for all $y \in \mathfrak{g}$.

8.1 Semidirect products

Let $\mathfrak{g}$ be a Lie algebra. Let $\mathfrak{d}$ be a Lie algebra actions on $\mathfrak{g}$ by derivations (i.e. $\mathrm{d}([x,y]) = [\mathrm{d}x, y] + [x, \mathrm{d}y]$). The *semidirect product*, $\mathfrak{d} \ltimes \mathfrak{g}$, is $\mathfrak{g} \oplus \mathfrak{d}$ with bracket $[\mathrm{d}, x] = \mathrm{d}(x)$ for $\mathrm{d} \in \mathfrak{d}, x \in \mathfrak{g}$. Let R be a ring and G be a group which acts on R by automorphisms. The *semidirect product* $R \rtimes G$, is the algebra

$$R \rtimes G = \{\sum_{g \in G} r_g g \mid r_g \in R\}$$

with blah given by $(r_1 g_1)(r_2 g_2) = r_1 g_1(r_2) g_1 g_2$. In Math 541, $\mathbb{C}(G \rtimes H) = \mathbb{C}G \rtimes H$. (aut is to grps as der is to lie alg)

8.2 Representations

A *representation* of G is a homomorphism $G \to \mathrm{GL}(M)$. We use M instead of V, since we want to see it as a module: as soon as we have a map from G in to maps on V, we get an *action* of G on V, so V is actually a module. So really, a representation of G "is" a G-module M. A *simple* module is a module with no (stupid) submodules. Many representations (G-modules) have the form

$$M = M_1 \otimes M_2 \otimes \cdots.$$

We'd like the M_i to be simple. The problem is to find the simple modules. THe phiosophy: If I know the simple R-modules and thie simple G-modules, I should be able to build the simple $R \rtimes G$-modules. This is true, but takes work. This work is called *Clifford Theory* (like "row reduction," it is an algorithm).

9 Centralizers and central extensions (Fri. 9/28)

A *representation* is a module. A *module* is a vector space with an action A (a group, and algebra, a Lie algebra, etc). A *simple* module is a module with no submodules except, of course, 0. A module M is *decomposable* if

$$M = M_1 \otimes M_2 \quad \text{as } A\text{-modules.}$$

If $a \in A$ then the matrix of the action of a, a_M on M is *block-diagonal* in the correct basis. We could also hope for get it into Jordan canonical form, though we have to be careful of our context. Often it is impossible to block-diagonalize , but it is often possible to triangularize a_M so it is almost block-diagonal, but stuff all above the diagonal (again, in some good basis of M.) If it is possible for all $a \in A$ simultaniously then M has a submodule M_1. M is *indecomposable* if $M \neq M_1 \oplus M_2$ as$A$-modules.

Let M be an A-module. The *centralizer algebra* of M

$$\operatorname{End}_A(M) = \{\varphi M \to M \mid \varphi a_M = a_M \varphi \text{ for all } a \in A\}$$

which is an algebra under composition.

Lemma 9.1 (Schur)**.** *Let A be a $\mathbb{C}$-algebra.If M is simple, then*

$$\operatorname{End}_A(M) = \mathbb{C}.$$

Proof. Assume M has no submodules (simple and irreducible are the same word). Let $\varphi : M \to M$ ne in $\operatorname{End}_A(M)$. Then $\ker\varphi$ is a submodule of M and $\operatorname{im}\varphi$ is a submodule of M. So, since M is simple, $\ker\varphi = 0$ or $\ker\varphi = M$ and $\operatorname{im}\varphi = 0$ or $\operatorname{im}\varphi = M$. So either $\varphi = 0$ or φ is an isomorphism. Let c be an eigenvalue of φ. Then $c\mathrm{id}_M \in \operatorname{End}_A(M)$, so $\varphi - c\mathrm{id}_M \in \operatorname{End}_A(M)$. The point of this is that by construction $\varphi - c\mathrm{id}_M$ is not invertible (since φ has an eigenvalue in common with $c\mathrm{id}_M$, so $\varphi - c\mathrm{id}_M$ has an eigenvalue of 0). Therefore $\varphi - c\mathrm{id}_M = 0$. So $\varphi = c\mathrm{id}_M$. □

Something which commutes with the action of A acts on a simple module by a constant.

Aside In physics: charge means exactly this constant: there is some simple module and some operator which commutes with the symmetries, so there is some constant called the charge.

Where do we find these things which act like constants?

$$Z(A) = \{z \in A \mid za = az \text{ for all } a \in A\}$$

is the *center* of A.

If you have an operator z that commutes with the A action and it's not in $Z(A)$ then make A bigger so that is *is* in the center.

9.1 Central extensions

Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{c}$ be an abelian Lie algebra (i.e. $[c_1, c_2] = 0$ for $c_1, c_2 \in \mathfrak{c}$). A *central extension* of $\mathfrak{g}$ by $\mathfrak{c}$ is a Lie algebra $junk = \mathfrak{g} \otimes \mathfrak{c}$ such that $\mathfrak{c} \subset Z(junk)$ and if $x, y \in \mathfrak{g}$, then $\tilde{[x, y]} = [x, y] + \varphi(x, y)$ with $\varphi \in \mathfrak{c}$. So to specify a central extension we must specify $\varphi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{c}$. There are conditions on φ if we want $junk$ to be a Lie algebra.

10 Lie algebra cohomology (Mon. 10/1)

First, we need some linear algebra. Let V be a vector space. Any real person thiks about a vector space in terms of its vector space, so pick one: let $b_1, \ldots, b_n$ be a basis. The *tensor algebra* on v is

$$T(V) = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} V^{\otimes k}$$

where $V^{\otimes k}$ has a basis $\{b_{i_1} \otimes \cdots \otimes b_{i_k} \mid 1 \leq i_1, \ldots, i_k \leq n\}$ and product

$$(b_{i_1} \otimes \cdots \otimes b_{i_k}) \cdot (b_{j_1} \otimes \cdots \otimes b_{j_\ell}) = b_{i_1} \otimes \cdots \otimes b_{i_k} \otimes b_{j_1} \otimes \cdots \otimes b_{j_\ell}.$$

Sometimes we write $T^k(V)$ for $V^{\otimes k}$, or $V^{\otimes k} = V \otimes \cdots \otimes V$. The tensor algebra is the free associative algebra generated by $b_1, \ldots, b_n$. If V has basis $b_1, \ldots, b_n$ and W has basis $c_1, \ldots, c_m$, then $V \otimes W$ has basis $\{b_i \otimes c_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$.

Let A be an $n \times n$ matrix, i.e. $A \in \mathrm{End}(V)$, and let B be an $m \times m$ matrix, i.e. $B \in \mathrm{End}(W)$. Denote $A \otimes B$ matrix of the element of $\mathrm{End}(V \otimes W)$ which is given by $(A \otimes B)(v \otimes w) = Av \otimes Bw$ for $v \in V, w \in W$. $A \otimes B$ has entries $(a_{ij} b_{k\ell})$ if $A = (a_{ij})$ and $B = (j_{k\ell})$. If $A \in \mathrm{End}(V)$ then $A^{\otimes k} = A \otimes \cdots \otimes A$ is an elements of $\mathrm{End}(V^{\otimes k})$. So T^k maps vector spaces to vector spaces, and is a functos. The *exterior algebra* is the quotient of $T(V)$ be the relations $v_1 \wedge v_2 = -v_2 \wedge v_1$. Let

$$\bigwedge V = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \bigwedge^k V$$

and $\bigwedge^k V$ has basis $\{b_{i_1} \wedge \cdots \wedge b_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$. (everything is bi-linear). So $\bigwedge^k$ as a map from vector spaces to vectpr spaces is a functor.

The *symmetric algebra* is the quotient of $T(V)$ by the relations $v_1 v_2 = v_2 v_1$ for $v_1, v_2 \in V$. Let

$$S(V) = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} S^k(V).$$

So S^k maps vector spaces to vector spaces, and is a functor:

$$S(V) = \mathbb{C}[b_i, \ldots, b_n].$$

For example, $\bigwedge^n V$ has basis $\{b_{i_1} \wedge \cdots \wedge b_{i_n} \mid 1 \leq i_1 < \cdots < i_n \leq n\}$, so $\bigwedge^n V$ has basis $\{b_1 \wedge \cdots \wedge b_n\}$. Note that if $w \in S_n$ then $b_{w(1)} \wedge \cdots \wedge b_{w(n)} = (-1)^{\ell(w)}$ where

$$S_n \to \{\pm 1\}$$

$$w \mapsto (-1)^{\ell(w)}$$

is the sign homomorphism and $A(b_1 \wedge \cdots \wedge b_n) = Ab_1 \wedge \cdots \wedge Ab_n = \sum_{1 \leq j_i, \ldots j_n \leq n} a_{j_1} 1 b_{j_1} \wedge \cdots \wedge a_{j_n} n b_{j_n} = \sum_{j_i distinct} a_{j_1} 1 \cdots a_{j_n} n (b_{j_1} \wedge \cdots \wedge b_{j_n}) = \sum_{j \in S_n} (-1)^{\ell(w)} a_{j_1} 1 \cdots a_{j_n} n (b_1 \wedge \cdots \wedge b_n) = \det A (b_1 \wedge \cdots \wedge b_n)$. So $\bigwedge^n A = \det(A)$ if $A \in \mathrm{End}(V)$ and $\dim V = n$.

Let $\mathfrak{g}$ be a Lie algebr. Let M be a $\mathfrak{g}$-module, i.e. M is a vector space iwht an action of $\mathfrak{g}$,

$$\mathfrak{g} \otimes M \to M$$

$$(x, m) \mapsto xm$$

such that if $x, y \in \mathfrak{g}$, $m \in M$ then $(xy - yx)m = [x, y]m$, i.e. M is a $U\mathfrak{g}$-module.

The cohomology of $\mathfrak{g}$ wiht coefficients in M is the cohomology of the complex

$$\cdots \to C^i(\mathfrak{g}, M) \xrightarrow{\mathrm{d}} C^{i+1}(\mathfrak{g}, M) \to \cdots$$

where $C^i(\mathfrak{g}, M) = \mathrm{Hom}(\bigwedge^i \mathfrak{g}, M)$ and $\mathrm{d}\phi)(b_{j_i} \wedge \cdots \wedge b_{j_{i+1}} = \sum_{1\le k\le i+1} (-1)^{k+1} b_{j_k} \varphi(b_{j_1} \wedge \cdots \wedge \hat{b_{j_k}} \wedge \cdots \wedge b_{j_{i+1}} + \sum_{1\le k<\ell\le i+1} (-1)^{k+\ell} \varphi([b_{j_k}, b_{j_\ell}] \wedge b_{j_1} \wedge \cdots \wedge \hat{b_{j_k}} \wedge \cdots \wedge b_{j_{i+1}}$ where $\hat{}$ denotes leaving it out.

$$\mathcal{Z}^i(\mathfrak{g}, M) = \ker(\mathrm{d}iC^i(\mathfrak{g}, M) \to C^{i+1}(\mathfrak{g}, M))$$
$$\mathcal{B}^i(\mathfrak{g}, M) = \mathrm{im}\ (\mathrm{d}iC^{i-1}(\mathfrak{g}, M) \to C^i(\mathfrak{g}, M))$$

and

$$H^i(\mathfrak{g}, M) = \frac{\mathcal{Z}^i(\mathfrak{g}, M)}{\mathcal{B}^i(\mathfrak{g}, M)}.$$

Let $\mathfrak{g}$ be a Lie algebra, and let $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ with $[x_1 t^k, x_2 t^\ell] = [x_1, x_2] t^{k+\ell}$ where $x_1, x_2 \in \mathfrak{g}$, $k, \ell \in \mathbb{Z}$.

11 More central extensions (Wed. 10/3)

Recall: let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{c}$ be an abelian Lie algebra. A *central extension* of $\mathfrak{g}$ by $\mathfrak{c}$ is a Lie algebra $\mathfrak{g} \oplus \mathfrak{c}$ (as vector spaces) such that

$$\tilde{[}x, y\tilde{]} = [x, y] + \varphi(x, y)$$

for $x, y, \in \mathfrak{g}$ where $\varphi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{c}$ and $\mathfrak{c} \subset \mathcal{Z}(\mathfrak{g} \oplus \mathfrak{c})$ (i.e. $\tilde{[}c, x\tilde{]} = 0$ for all $c \in \mathfrak{c}$ and $x \in \mathfrak{g} \oplus \mathfrak{c}$).

How do we find the φ? Since $\tilde{[}x, y\tilde{]} = -\tilde{[}y, x\tilde{]}$, we have $\varphi(x, y) = -\varphi(y, x)$. Since $\tilde{[},\tilde{]}$ is bilinear, φ is also bilinear. So $\varphi : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{c}$. Then the Jacobi identity,

$$\tilde{[}x, \tilde{[}y, x\tilde{]}\tilde{]} + \tilde{[}y, \tilde{[}z, x\tilde{]}\tilde{]} + \tilde{[}z, \tilde{[}x, y\tilde{]}\tilde{]} = 0$$

forces

$$\varphi(x, [y, x]) + \varphi(y, [z, x]) + \varphi(z, [x, y]) = 0.$$

Since one "clearly recalls" the formula for d in Lie algebra cohomology, one immediately notices that this relation is more compactly written as

$$\mathrm{d}\varphi = 0$$

where $\mathfrak{g}$ acts on $\mathfrak{c}$ via $x \cdot c = 0$ for $x \in \mathfrak{g}$, $c \in \mathfrak{c}$.

Since $\mathcal{Z}^2(\mathfrak{g}, \mathfrak{c}) = \ker(\mathrm{d} : C^2(\mathfrak{g}, \mathfrak{c}) \to C^3(\mathfrak{g}, \mathfrak{c}))$, this means φ is a 2-cocycle.

Another way to think about the central extension $\mathfrak{e} = \mathfrak{g} \oplus \mathfrak{c}$ is as

$$\mathfrak{e} \overset{e}{\twoheadrightarrow} \mathfrak{g}$$

$$x + c \mapsto x$$

such that $\ker e \subset \mathcal{Z}(\mathfrak{e})$. The process of taking elements of $\mathfrak{g}$ and thinking of them as elements of $\mathfrak{e}$ is the same as choosing a section $s : \mathfrak{g} \to \mathfrak{e}$ of e. THen $\varphi : \mathfrak{g} \times \mathfrak{g} \to \ker(\mathfrak{e})$ defined by

$\tilde{[}s(x), s(y)\tilde{]} = s([x,y]) + \varphi(x,y)$ for $x, y \in \mathfrak{g}$. If $s' : \mathfrak{g} \to \mathfrak{e}$ was a different choice, then the only change must have been the choice of the element of $\mathfrak{c}$ to be added to the element of $\mathfrak{g}$:

$$s'(x) = s(x) + \psi(x), \quad \text{for } x \in \mathfrak{g},$$

with $\psi : \mathfrak{g} \to \mathfrak{c}$ (i.e. $\psi \in C^1(\mathfrak{g}, \mathfrak{c})$). Then $s([x,y]) + \varphi(x,y) = \tilde{[}s'(x), s'(y)\tilde{]} = \tilde{[}s(x) + \psi(x), s(y) + \psi(y)\tilde{]} = s'([x,y]) + \varphi'(x,y) = s([x,y]) + \psi([x,y])\varphi'(x,y)$. So $\varphi(x,y) - \varphi(x,y) = \psi([x,y]) = \mathrm{d}\psi(x,y)$. SO if φ and φ' come from the same $\mathfrak{e}$ then they differ by an element of $B^2(\mathfrak{g}, \mathfrak{c}) = \text{im } * \mathrm{d} : C^1(\mathfrak{g}, \mathfrak{c}) \to C^2(\mathfrak{g}, \mathfrak{c}))$. So central extensions of $\mathfrak{g}$ by $\mathfrak{c}$ are determined by elements of $H^2(\mathfrak{g}, \mathfrak{c}) = Z^2(\mathfrak{g}, \mathfrak{c})/B^2(fg, \mathfrak{c})$.

The *universal central extension* of $\mathfrak{g}$ is a surjective Lie algebra homomorphism $\mathfrak{u} \xrightarrow{u} \mathfrak{g}$ such that $\ker u \subset \mathcal{Z}(\mathfrak{u})$ and if $\mathfrak{e} \xrightarrow{e} \mathfrak{g}$ is another central extension then there is a unique Lie algebra homomorphism $f : \mathfrak{u} \to \mathfrak{e}$ such that $e(f(\mathfrak{u})) = u(\mathfrak{u})$.

First thing to do is think about your favortie examples of universal objects. Examples include tensor products, free groups, products, localizations, quotients of groups, $\mathbb{C}[x_1, x_2, \ldots, x_n]$, fild of fractions.... in general, universal objects just build themselves. SO proving existence is just a matter of building the damn thing by following our noses. Hence, we should be able to get the universal central extension of the Witt algebra to build itself – this will be the Virasoro.

Last thing: If $E \to G$ is a homomorphism of topological groups such that it is a local homeomorphism, then E is a *covering space* or *covering group*. So the central extension is the same as a cover, and a universal central extension is the same as a universal cover.

12 Building an Algebra-to-Topology dictionary (Fri. 10/5)

Last time, in case you hadn't noticed, we covered Hatcher, chapter 3.

A *central extension* of a group G is a surjective homomorphism

$$E \xrightarrow{e} G$$

such that $\ker e \subset \mathcal{Z}(E)$. A *covering group* is a surjective homeomorphism of topological groups $E \to G$ such that it is a local homeomorphism, where *local homeomorphism* means if $x \in G$ there is a neighborhood U of x such that e^{-1} is a discrete disjoint union of neighborhoods isomorphic to U. If $E \to G$ is a covering group, then the kernel of e is a discrete normal subgroup of E, which forces $\ker e \subset \mathcal{Z}(E)$! SO it is becoming important to make an Algebra-to-Topology dictionary:

central extension:	covering map or covering space.
$G = DG$:	G is connected.
universal central extension:	universal cover
centrally closed	simply connected

(where $DG = \langle [g_1, g_2] \mid g_1, g_2 \in G \rangle$, $\quad [g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$)

13 Deriving the central extension of the Witt algebra (Mon. 10/8)

get notes

Recall: The Witt algebra has basis

$$\dots, L_{-2}, L_{-1}, L_0, L_1, L_2, \dots$$

with bracket

$$[L_m, L_n] = (m-n)L_{m+n}.$$

We want $\mathfrak{g} \oplus \mathfrak{c}$ ($\mathfrak{c}$ is abelian) with $\mathfrak{c}$ central in $\mathfrak{g} \oplus \mathfrak{c}$ and

$$[L_m, L_n] = (m-n)L_{m+n} + \varphi(L_m, L_n),$$

where

$$\varphi : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{h}.$$

In order for $\mathfrak{g} \oplus \mathfrak{h}$ to be a Lie algebra,

$$0 = \varphi(L_m, [L_n, L_p]) + \varphi(L_p, [L_m, L_n]) + \varphi(L_n, [L_p, L_m]).$$

We can normalize φ by a coboundary and use

$$\varphi'(L_m, L_n) = \varphi(L_m, L_n) + \psi([L_m, L_n])$$

where $\psi : \mathfrak{g} \to \mathfrak{c}$. So

$$\varphi'(L_m, L_n) = \varphi(L_m, L_n) + (m-n)\psi(L_{m+n}),$$

...

14 Finite dimensional complex semisimple Lie algebras... (Wed. 10/10)

Ram's least favorite phrase in grad school was "Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra..." However, this is an important phrase, so let's unwind it. It means:

1. $\mathfrak{g}$ is a finite dimensional vector space (really, isn't every vector space?)
2. $\mathfrak{g}$ is a vector space over $\mathbb{C}$ (again, aren't they all?)
3. $\mathfrak{g}$ is a Lie algebra (duh - we're *in* a Lie algebras course)
4. $\mathfrak{g}$ is semisimple (ok, maybe there's something here)

What's semisimple? It's a lot like reductive, for one.

Try one: A Lie algebra $\mathfrak{g}$ is *semisimple* if it is a direct sum of simple Lie algebras,

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_\ell$$

as Lie algebras. A *simple* Lie algebra is a Lie algebra with no ideals. An *ideal* of $\mathfrak{g}$ is a subspace $\mathfrak{a}$ such that if $x \in \mathfrak{g}$, $a \in \mathfrak{a}$, then $[x, a] \in \mathfrak{a}$.

Try two: A Lie algebra $\mathfrak{g}$ is *semisimple* if all of the finite dimensional $\mathfrak{g}$-modules are semisimple. A $\mathfrak{g}$-module M is *semisimple* if M is a direct sum of simple $\mathfrak{g}$-modules:

$$M \cong M_1 \oplus \cdots \oplus M_\ell$$

as $\mathfrak{g}$-modules. A $\mathfrak{g}$-module is *simple* if it has no submodule.

Are these two the same? Well, first, $\mathfrak{g}$ is a $\mathfrak{g}$-module, so one is a special case of two. But in some sense, all $\mathfrak{g}$-modules are controlled by $\mathfrak{g}$, i.e. $\mathfrak{g}$ and $\{\mathfrak{g}\text{-modules}\}$ are the same data. The reconstruction theorems explain how to "reconstruct" $\mathfrak{g}$ only from information about the category of $\mathfrak{g}$-modules.

The *dual module* is $M^* = \mathrm{Hom}(M, \mathbb{C})$ with actions $(x\varphi)(m) = \varphi(-xm)$, with $x \in \mathfrak{g}$ and $\varphi \in M^*$. Why the minus? Recall: Let G be a group. Let M be a G-modules. The *dual* of M is the G-module $M^* = \mathrm{Hom}(M, \mathbb{C})$ with actions $(g\varphi)(m) = \varphi(g^{-1}m)$.

If M and N are $\mathfrak{g}$-modules, the tensor product is the $\mathfrak{g}$-module $M \otimes N$ with actions $x(m \otimes n) = xm \otimes n + m \otimes xn$. This is like the thing from group theory where if M and N are G-modules, then $M \otimes N$ is a G-modules with actions $g(m \otimes n) = gm \otimes gn$.

The *trivial* $\mathfrak{g}$*-module* is $\mathbb{C}v$ with actions $xv = 0$. The *trivial* G*-module* is $\mathbb{C}v$ with actions $gv = v$ for $g \in G$. THere are two canonical morphisms

$$M^* \otimes M \to \mathbb{C}$$

$$(\varphi, m) \mapsto \varphi(m)$$

and

$$\mathbb{C} \to M \otimes M^*$$
$$1 \mapsto \sum_{b_i} b_i \otimes b_i^*,$$

where the sum is over a basis $\{b_i\}$ of M and $\{b_i^*\}$ is the dual basis in M^*. (see... many times we do not need an inner product in order to get a dual basis, we just need to calculate $\mathrm{Hom}(M, \mathbb{C})$. Notice that the second canonical map looks a lot like the trace... the basis vectors are col vectors with a 1 in the i^{th} place, and dual basis vector are row vectors with a 1 in the i^{th} place.

homework: With the above definitions, show that these maps are $\mathfrak{g}$- and G-module homomorphisms.

15 Bilinear forms and Hopf algebras (Fri. 10/12)

Last time, we defined (without saying it) a Hopf algebra.

Definition. A *Hopf algebra* is an algebra U with three maps

$$\Delta : U \to U \otimes U,$$

$$\varepsilon : U \to \mathbb{C},$$

$$S : U \to U$$

such that

1. If M and N are U-modules, then $M \otimes N$ with action

$$x(m \otimes n) = \sum_x x_{(1)}m \otimes x_{(2)}n$$

where $\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)}$, is a U-notation. [Note: this notation we're using is called *Sweedler notation*]

2. The vector space $\mathbb{C} = v\mathbb{C}$, with actions $xv_1 = \varepsilon(x)v_1$ is a U-module.

3. If M is a U-module then $M^* = \mathrm{Hom}(M, \mathbb{C})$ with action

$$(x\varphi)(m) = \varphi(S(x)m)$$

is a U-module.

4. The canonical maps

$$\cup : M^* \otimes M \to \mathbb{C}$$

and

$$\cap : \mathbb{C} \to M \otimes M^*$$

are U-module homomorphisms.

15.1 Examples

1. If G is a group and $U = \mathbb{C}G = \mathbb{C}\text{-span}\{g \in G\}$ with

   $\Delta(g) = g \otimes g$ the *coproduct*

   $\varepsilon(g) = 1$ the *counit*

   $S(g) = g^{-1}$ the *antipode*

   is a Hopf algebra

2. If $\mathfrak{g}$ is a Lie algebra, then $U = U\mathfrak{g}$ is a Hopf alegbrea with

   $\Delta(x) = x \otimes 1 + 1 \otimes x,$

   $\varepsilon(x) = 0$

   $S(x) = -x$

   for $x \in \mathfrak{g}$

For a very long time, these were the only two examples.

15.2 More of what forms can do for us

Definition. A *symmetric* bilinear form is

$$\langle , \rangle : M \otimes M \to \mathbb{C}$$

such that

$$\langle x, y \rangle = \langle y, x \rangle$$

for $x, y \in M$.

So notice that
$$M \to M^*$$
via
$$m \mapsto \langle m, \cdot \rangle$$
is a vector space homomorphism.

Let M be a U-module. And *invarient* symmetric bilinear form on M is $\langle , \rangle : M \times M \to \mathbb{C}$ such that $\langle xm_1, ym_2 \rangle = \langle m_1, S(x)m_2 \rangle$ for $x \in U$. If G is a group,
$$\langle gm, gn \rangle = \langle m, n \rangle$$
i.e. $\langle gm, n \rangle = \langle m, g^{-1}n \rangle$. The invarient part means is that
$$M \to M^*$$
via
$$m \mapsto \langle m, \cdot \rangle$$
is a U-module homomorphism. On the level of vector spaces, the symmetric form doesn't know anything about multiplications in U. But an invariant form does!

Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}$ is a $\mathfrak{g}$-module (under the adjoint representation). An *ad-invariant* form is
$$\langle , \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$$
such that
$$\langle y, z \rangle = \langle z, y \rangle$$
$$\langle [x, y], z \rangle = \langle y, [x, z] \rangle.$$
Define
$$\text{rad}\langle , \rangle = \{x \in \mathfrak{g} \mid \langle x, y \rangle \text{ for all } y \in \mathfrak{g}\}$$
i.e., the kernel of
$$\mathfrak{g} \to \mathfrak{g}^*$$
via
$$m \mapsto \langle m, \cdot \rangle.$$

$\langle , \rangle$ is *nondegenerate* if $\text{rad}\langle , \rangle = 0$. Notice that $\text{rad}\langle , \rangle$ is an ideal of $\mathfrak{g}$, i.e. $\text{rad}\langle , \rangle$ is a submodule. So if $\mathfrak{g}$ is simple $\text{rad}\langle , \rangle$ is zero of the whole thing.

16 (Mon. 10/17)

We're still working on what it means for $\mathfrak{g}$ to be a finite dimensional complex semisimple Lie algebra. Again, " finite dimensional complex" means finite dimensional vector space over $\mathbb{C}$, and "Lie algebra" should be obvious. So we're just left with "semisimple". The definition of choice has been that $\mathfrak{g}$ is *semisimple* if all of the $\mathfrak{g}$ modules are semisimple. In particular, $\mathfrak{g}$ is a semisimple $\mathfrak{g}$-module. Reduce to simple $\mathfrak{g}$, i.e. it has no $\mathfrak{g}$-submodules, i.e. it has no stupid ideals.

Let $\langle , \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ be an invariant symmetric bilinear form, i.e.

$$\mathfrak{g} \to \mathfrak{g}^*$$

$$x \mapsto \langle x, \cdot \rangle$$

is a $\mathfrak{g}$-module isomorphism (since $\mathfrak{g}$ is simple – it's always a homomorphism). Also, being that it is simple, the only endomorphisms are constant multiples of the identity. Since $\mathfrak{g}$ carries a symmetric bilinear form, if $\{b_i\}$ is a basis of $\mathfrak{g}$, then there is a dual basis $\{b_i^*\}$ of $\mathfrak{g}$ where $\langle b_i, b_i^* \rangle = \delta_{ij}$.

Homework: Convert section five of T. Halverson and A. Ram's *Partition Algebras* to Lie algebras.

So $\mathfrak{g}$ is semisimple if and only if it has a basis and a dual basis (if it's semisimple, I get a form on each piece, with which I build a good form over the whole thing, and get my dual basis; if it has a basis and dual basis, I can build a good invariant form with these and show that it must be semisimple).

Recall that the *enveloping algebra* of $\mathfrak{g}$ is the algebra generated by $\mathfrak{g}$ with the relation $[x, y] = xy - yx$.

Example. Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C}) = \{n \times n\text{matrices with entries in } \mathbb{C}\}$ with bracket

$$[x, y] = xy - yx.$$

The basis of $\mathfrak{gl}_n$ is $\{E - ij \mid 1 \leq i, j, \leq n\}$ where E_{ij} has a 1 in the (i, j) entry and 0's elsewhere. Let $V = \mathbb{C}$. Then $\mathfrak{gl}_n$ acts on $\mathbb{C}$. V has basis $\{v_1, \ldots, v_n\}$ and

$$xv_i = \sum_{j=1}^{n} x_{ji} v_j$$

for $x \in \mathfrak{gl}_n$. Then define $\langle , \rangle : \mathfrak{gl}_n \times \mathfrak{gl}_n \to \mathbb{C}$ by

$$\langle x, y \rangle = \mathrm{Tr}(x_V y_V)$$

where x_V is the matric of x acting on V. The dual basis with respect to $\langle \rangle$ is $E_{ij}^* = E_{ji}$. $U\mathfrak{gl}_n$ is generated by the symbols E_{ij}. So $E_{ij}E_{ij} \neq 0$ in $U\mathfrak{gl}_n$.

The *Casimir element* is

$$\kappa = \sum_{b_i} b_i b_i^* \in U\mathfrak{g}$$

where the sum is over the basis $\{b_i\}$ and the dual basis $\{b_i^*\}$.

Theorem 16.1. *Let κ be the Cassimir element of $\mathfrak{g}$.*

(a) κ does not depend on the choice of basis.

(b) $\kappa \in \mathcal{Z}(U\mathfrak{g})$.

So κ commutes with any action of $\mathfrak{g}$.

Proof. (a) is a basic result of what it means to change basis

(b) leans hard on our form being ad-invariant

□

We need to get a picture in our heads. A good way to think about this is to think of $\mathfrak{g}$ being a complex semisimple Lie algebra means that is contains lots of $\mathfrak{sl}_2$ subalgebras. What is $\mathfrak{sl}_2$? Recall: $\mathfrak{sl}_2$ is the span of elements x, y, h with bracket $[x, y] = h$, $[h, x] = 2x$, $[h, y] = -2y$.

Suppose M is an $\mathfrak{sl}_2$-module. So h acts on M. So morally, h is a matrix. So h has at least on eigenvalue and eigenvector. Let v be an eigenvector for h, i.e.

$$hv = \lambda v$$

for some $\lambda \in \mathbb{C}$. So

$$\begin{aligned} hxv &= (xh + [hx])v \\ &= (x\lambda + 2x)v \\ &= (\lambda + 2)xv. \end{aligned}$$

So $\mathfrak{sl}_2$ makes more eigenvectors. So

$$\begin{aligned} hx^2v &= (xh + [hx])xv \\ &= (x(\lambda + 2) + 2x)v \\ &= (\lambda + 4)x^2v. \end{aligned}$$

And $hx^3v = (\lambda + 6)x^3v$, and, and, and...

If M is finite dimensional, the $x^kv = 0$ for some $k \in \mathbb{Z}^+$. Let $v^+ = x^{k-1}v$. If M is simple then v^+ generates M. Let $hv^+ = \mu v^+$. Then

$$\begin{aligned} hyv^+ &= (yh + [h, y])v^+ \\ &= (y\mu - 2y)v^+ \\ &= (\mu - 2)yv^+ \end{aligned}$$

and $hy^{@}v^+ = (\mu - 4)y^2v^+$, and ...

So $y^\ell = 0$ for some $\ell \in \mathbb{Z}^+$. So $\{v^+, yv^+, \ldots, y^{\ell-1}v^+\}$ will turn out to be a basis of M.

17 (Fri. 10/19)

Thinking of $\mathfrak{g}$ a semisimple Lie algebra in terms of embedded $\mathfrak{sl}_2$ subalgebras, where $\mathfrak{sl}_2$ is the span of $\{h, x, y\}$ with bracket

$$[x, y] = h, \qquad [h, x] = x, \qquad [h, y] = -2y.$$

Let M be an $\mathfrak{sl}_2$-modules. Let v be an eigenvector for h. So $x^kv = 0$ for some k. Let $v^+ = x^{k-1}v$, where k is minimal. Let μ be such that $hv^+ = \mu v^+$. So

$$\begin{aligned} hyv^+ &= (yh + [h, y])v^+ \\ &= (y\mu - 2y)v^+ \\ &= (\mu - 2)yv^+. \end{aligned}$$

Similarly $hy^2v^+ = (\mu - 4)y^2v^+$. So

$$v^+, yv^+, y^2v^+, \dots, y^{\ell-1}v^+$$

are eignevectors for h in M. Similarly, $xv^+ = 0$, $xyv^+ = \mu v^+$, $xy^2v^+ = (2\mu - 2)yv^+$, $xy^3v^+ = (3\mu - 6)yv^+$, ... So the span of $\{v^+, yv^+, y^2v^+, \dots, y^{\ell-1}v^+\}$ is a submodule of M. If M is simple, then $\{v^+, yv^+, y^2v^+, \dots, y^{\ell-1}v^+\}$ is a basis of M and the $\mathfrak{sl}_2$-action is given by:

- h is a diagonal matrix with μ, $\mu - 2$, $\mu - 4$, ... , $\mu - 2(\ell - 1)$ on the diagonal,
- y has ones on the sub-diagonal and zeros elsewhere, and
- x has the weights μ, $2\mu - 2$, $3\mu - 6$, ... , $(\ell - 1)\mu^{-\binom{\ell-1}{2}2}$ on the super-diagonal.

But $[x, y] = xy - yx$, so if we multiply these matrices together, we get the relation that

$$\mu - 2(\ell - 1) = -(\ell - 1)\mu + (\ell - 1)(\ell - 2)$$

$$\ell\mu = 2(\ell - 1) + (\ell - 1)(\ell - 2) = \ell(\ell - 1)$$

so $\mu = \ell - 1$.

Theorem 17.1. *The irreducible finite dimensional $\mathfrak{sl}_2$ modules are indexed by $\ell \in \mathbb{Z}_{\geq 0}$ with basis $\{v^+, yv^+, y^2v^+, \dots, y^{\ell-1}v^+\}$ and action*

$$hy^jv^+ = (\mu - 2j)y^jv^+, \quad xy^jv^+ = (j\mu - 2\binom{j}{2})y^{j-1}v^+, \quad yy^jv^+ = y^{j+1}v^+.$$

17.1 Jordan decomposition

Let x be a matrix. Then $x = x_s + x_n$ with x_s *semisimple* (as a matrix, i.e. it is diagonalizable), x_n is *nilpotent* (i.e. $y^k = 0$ for some k, or $\mathrm{Tr}(y^k) = 0$ for all k, or all y's eigenvalues are zero), and $x_sx_n = x_nx_s$. SO if $zx = xz$, then $zx_s = x_sz$ and $zx_n = x_nz$.

This decomposition comes from Jordan canonical form, where in an appropriate field, every matrix can be put into a form which looks like eigenvalues on the diagonal and 1's in he super-diagonal.

Remark: A matrix A is a $\mathbb{C}[x]$-module:

$$\mathbb{C}[x] \to \mathrm{End}V$$

$$x \to A$$

So A is semisimple exactly when V is a semisimple $\mathbb{C}[x]$-module.

Theorem 17.2 (Jasobson-Morozov)**.** *If y is a nilpotent matrix then there exist matrices x and h such that*

$$[x, y] = h, \qquad [h, x] = x, \qquad [h, y] = -2y.$$

This choice is relatively unique (with some changes in constants).

Example: Let X be a geometric space. Let c be the class in $H^2(X)$ corresponding to a line bundle $\mathbb{P}^1$ in X. Then multiplications by c is a nilpotent operation on $H^*(X)$ (c pushes up and up and up, and runs out eventually). So $\mathfrak{sl}_2$ acts on the cohomology ring of X.

18 Triangular decomposition (Mon. 10/22)

How we would like to think of complex semisimple Lie algebras.

A *triangular decomposition* of a Lie algebra is

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

as Lie subalgebras, where

$\mathfrak{h}$ is abelian

$\mathfrak{n}^+$ is upper triangular

$\mathfrak{n}^-$ is lower triangular

There is a morphism (*Cartan involution*)

$$\mathfrak{n}^+ \to \mathfrak{h}^-.$$

Example: The algebra

$$\mathfrak{gl}_n = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^-,$$

where $\mathfrak{h}$ is the set of diagonal matrices, $\mathfrak{n}^+$ is the set of upper triangular matrices, and $\mathfrak{n}^-$ is the set of lower triangular matrics. Any decent , self-respecting Lie algebra lives inside ofr $\mathfrak{gl}_n$. In fact, $\mathfrak{g}$ acts on itself by the adjoint representation and so

$$\mathfrak{g} \to \mathfrak{gl}_n$$

when $\dim \mathfrak{g} = n$ via

$$x \mapsto \text{ the matrix of the action of } x \text{ on } \mathfrak{g}.$$

$\mathfrak{g}$ is trying to inherit a triangular decomposition from $\mathfrak{gl}_n$. (From last time, given $x \in \mathfrak{g}$, there exist x_s, x_n, where $x = x_s + x_n$, with x_s *congugate* to something diagonal.)

Now, $U\mathfrak{g}$ gets a decomposition from $\mathfrak{g}$:

$U\mathfrak{g}$ is generated by $\mathfrak{g}$ with
generators $x \in \mathfrak{g}$, and
relations $xy - yx = [x, y]$.

Alternatively, we can construct $U\mathfrak{g}$ with

generators $y \in \mathfrak{n}^-$, $h \in \mathfrak{h}$, and $x \in \mathfrak{n}^+$, and
relations $g_1g_2 - g_2g_1 = [g_1, g_2]$.

The relations here allow me to move y's to the left, x's to the right, leaving h's in the middle. So we get

$$U = U^-U_0U^+ = U^- \otimes U_0 \otimes U^+$$

where

$$U^- = U\mathfrak{n}^-, \qquad U_0 = U\mathfrak{h}, \qquad U^+ = U\mathfrak{n}^+ = \mathbb{C}[h_1, \ldots, h_n],$$

where $\{h_1, \ldots, h_n\}$ is a basis of $\mathfrak{h}$.

18.1 Weights

$\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. So $U\mathfrak{h}$ is a Lie subalgebra of $U\mathfrak{g}$. Let M be a $\mathfrak{g}$-modules. Then M is an $\mathfrak{h}$-module. So M is an $\mathbb{C}[h_1, \dots, h_n]$-module. But what *are* the simple $\mathbb{C}[h_1, \dots, h_n]$-modules?

Simple modules for a commutative algebra are one-dimensional.

Example: $\mathbb{C}[h]$ acts on $\mathbb{C}[h]/ < f >$. M os a $\mathbb{C}[h]$-module determined by the action of h, a matrix with 1's below the diagonal and last row $-(a_0, a_1, \dots, a_{d-1})^T$, with respect to the basis $\{1, h, h^2, \dots, h^{d-1}\}$ if $f = a_0 + a_1 h + a_2 h^2 + \cdots + a_{d-1} h^{d-1}$ (this is *rational canonical form*). Inside of M there is an m such that $hm = \mu m$. So the span of m is a one-dimensional submodule of M, *and is simple.* If M is a $\mathbb{C}[h_1, h_2, \dots, h_n]$-module, then there exists $m \in M$ and $\mu_1, \dots, \mu_n \in \mathbb{C}$ such that $h_i m = \mu_i m$.

So

$$M_\mu = \{m \in M \mid h_i m = \mu_i m\}$$

is a direct sum of simple $\mathbb{C}[h_1, \dots, h_n]$-modules of "type μ".

Theorem 18.1. *The simple $\mathbb{C}[h_1, \dots, h_n]$-modules are indexed by elements of $\mathbb{C}^n$. If $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{C}^n$ then the span of m with actions $h_i m = \mu_i m$ is a simple $\mathbb{C}[h_1, \dots, h_n]$-module.*

Let M be a $\mathfrak{g}$-module. Let $\mu \in \mathbb{C}^n$. Then

$$\mathbb{C}^n \xrightarrow{\sim} \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$$

$$\mu \to \varphi$$

where

$$\varphi : \mathfrak{h} \to \mathbb{C}$$

by

$$h_i \mapsto \mu_i.$$

So write $\mu \in \mathfrak{h}^*$. The *μ-weight space* of M is

$$M_\mu = \{m \in M \mid hm = \mu(h)m \text{ for } h \in \mathfrak{h}\}.$$

This is the piece of M which is all simple $\mathbb{C}[h_1, \dots, h_n]$-modules of type μ on M.

In our example, $\mathbb{C}[h]/ < f >$ for a polynomial f is an indecomposable $\mathbb{C}[h]$-module.

Note: Nobody knows how to classify the indecomposable $\mathbb{C}[h_1, h_2]$-modules.

19 The Cartan subalgebra, weights, and root systems (Wed. 10/24)

Let $\mathfrak{g}$ be a Lie algebra. Embed $\mathfrak{g}$ into $\mathfrak{gl}_n$ for some n (i.e. let $\mathfrak{g}$ act on some module M. Then $\mathrm{End}(M) = \mathfrak{gl}_n$ for some n and

$$\mathfrak{g} \to \mathrm{End}(M) = \mathfrak{gl}_n$$

$$x \mapsto x_M$$

where x_M is the matrix of the action of x on M.

So $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ (upper triangular + diagonal + lower triangular). Call $\mathfrak{h}$ the *Cartan subalgebra*. If $\mathfrak{h}$ has basis $h-1, \ldots h_n$, then $U\mathfrak{h} = \mathbb{C}[h_1, \ldots, h_n]$. If M is a $\mathfrak{g}$-module, then it is a $\mathfrak{h}$-modules. How does M look as an $\mathfrak{h}$-module? The irreducible $U\mathfrak{h}$-modules correspond to elements $\mu \in \mathfrak{h}^*$:

$$\mu : \mathfrak{h} \to \mathbb{C}$$

$$h_i \mapsto \mu_i$$

or $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$ and the corresponding $U\mathfrak{h}$-module is $\mathbb{C}m$ with $h_i m = \mu_i m$ (i.e. $hm = \mu(h)_i m$).

The *μ-weight space* of M is

$$M_\mu = \{m \in M \mid hm = \mu(h)m \forall h \in \mathfrak{h}\}.$$

The *generalized weight space* is

$$M_\mu^{\text{gen}} = \{m \in M \mid \text{if } h \in \mathfrak{h} \text{ then } (h - \mu(h))^k m = 0 \text{ for some } k \in \mathbb{Z}^+\}.$$

A Jordan block looks like μ's on the diagonal and 1's on the super diagonal, so $(h - \mu(h))^k = 0$ for some k. Then

$$M = \bigoplus_{\mu \in \mathfrak{h}^+} M_\mu^{\text{gen}}$$

as $\mathfrak{h}$-modules. If

$$M = \bigoplus_{\mu \in \mathfrak{h}^+} M_\mu$$

then all Jordan blocks of h's are of size 1 and hence h's act semisimple. In fact...

Theorem 19.1. *If $\mathfrak{g}$ is semisimple and M is finite dimensional then*

$$M = \bigoplus_{\mu \in \mathfrak{h}^+} M_\mu.$$

A *weight vector* of weight μ is just an element of M_μ.

19.1 Some people think Lie theory is the same as root systems...

Assume $\mathfrak{g}$ is reductive. Let $\mathfrak{g}$ act on $\mathfrak{g}$ (the adjoint representation). A *root* is a weight of $\mathfrak{g}$.

$$\mathfrak{g} = \bigoplus_{\mu \in \mathfrak{h}^+} \mathfrak{g}_\mu.$$

μ keeps track of the $\mathfrak{h}$-action. $\mathfrak{h}$ acts on $\mathfrak{h}$ by 0 since $\mathfrak{h}$ is abelian (so $[h_1, h_2] = 0$). So $\mathfrak{h} \subset \mathfrak{g}_0$. In fact $\mathfrak{h} = \mathfrak{g}_0$ (since $\mathfrak{h}$ is the maximal abelian subalgebra [we're using reductive here]). So

$$\begin{aligned} \mathfrak{g} &= \mathfrak{g}_0 \oplus \left(\bigoplus_{0 \neq \mu \in \mathfrak{h}^+} \mathfrak{g}_\mu \right) \\ &= \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right) \end{aligned}$$

where

$$R = \{\mu \in \mathfrak{h}^* \mid \mu \neq 0, \mathfrak{g}_\mu \neq 0\}.$$

R is the *root system* of $\mathfrak{g}$.

Let $\langle , \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be the *Killing form*:

$$\langle x, y \rangle = \mathrm{Tr}(\mathrm{ad}_x \mathrm{ad}_y),$$

where ad_x is the matrix of x acting on $\mathfrak{g}$. For us, semisimple = reductive = form is non-degenerate (If $\mathfrak{g}$ is a finite dimensional complex simple Lie algebra, then there is a unique, up to constants, symmetric invariant bilinear form... we'll just always re-normalize to get the Killing form). Let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$. So

$$\begin{aligned} \alpha(h)\langle x, y\rangle &= \langle \alpha(h)x, y\rangle \\ &= \langle [h, x], y\rangle \\ &= -\langle x, [h, y]\rangle \\ &= -\langle x, \beta(h)y\rangle \\ &= -\beta(h)\langle x, y\rangle \end{aligned}$$

for all $h \in \mathfrak{h}$. So if $\alpha \neq -\beta$ (as functions on $\mathfrak{h}$, elements of $\mathfrak{h}^*$), then $\langle x, y\rangle = 0$ and so $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$. So let $R^+ \subset R$ such that

$$R = R^+ \cup R^-$$

where

$$R^- = \{-\alpha \mid \alpha \in R^+\}.$$

$\mathfrak{g}_\alpha$ has nonzero pairin only with $\mathfrak{g}_{-\alpha}$. So

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \right)$$

For $\alpha, \beta \in R^+$, $(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \perp (\mathfrak{g}_\beta \oplus \mathfrak{g}_{-\beta})$. I would like a basis that respects this decomposition and acts nicely with respect to our form. Let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$. Then

$$\begin{aligned} [h, [x, y]] &= -[y, [h, x]] - [x, [y, h]] \\ &= -[y, \alpha(h)x] + [x, [h, y]] \\ &= -\alpha(h)[x, y] + \beta(h)[x, y] \\ &= (\alpha(h) + \beta(h))[x, y] \\ &= (\alpha + \beta)(h)[x, y]. \end{aligned}$$

So if $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$, then $[x, y] \in \mathfrak{g}_{\alpha+\beta}$. So if $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$, then $[x, y] \in \mathfrak{h}$. THe usual choice of R^+ is so that

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^+ = \bigoplus_{\alpha \in R^+} \mathfrak{g}_{-\alpha}$$

and $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ = \left(\bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha\right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha\right)$.

Let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$. We know $[x, y] \in \mathfrak{h}$. So

$$\begin{aligned} \langle h, [x, y]\rangle &= -\langle [x, y], h\rangle \\ &= \langle [h, x], y]\rangle \\ &= \alpha(h)\langle x, y\rangle \end{aligned}$$

Let $H_\alpha \in \mathfrak{h}$ such that $\langle h, H_\alpha\rangle = \alpha(h)$. H_α is well-defined since our form is nondegenerate on $\mathfrak{h}$ and if $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ then $[x, y] = \alpha x, y\rangle H_\alpha$.

Now choose $X_\alpha \in \mathfrak{g}_\alpha$ and $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[X_\alpha.Y_\alpha] = H_\alpha$, $[H_\alpha, X_\alpha] = 2X_\alpha$, and $[H_\alpha, Y_\alpha] = -2Y_\alpha$... hello there Mr. $\mathfrak{sl}_2$!

20 Finding the $\mathfrak{sl}_2$'s (Mon. 10/29)

Let $\mathfrak{g}$ be a complex semisimple (reductive, in characteristic 0) finite dimensional Lie algebra. Then $\mathfrak{g}$ embeds "faithfully" in $\mathfrak{gl}_n$. So

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

We saw on Wednesday that we also have

$$\mathfrak{g} = \left(\bigoplus_{\alpha\in R^+} \mathfrak{g}_{-\alpha}\right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in R^+} \mathfrak{g}_{-\alpha}\right)$$

where $\mathfrak{g}_\mu$ is the μ-weiht space for the action of $\mathfrak{h}$, and $\mathfrak{g}_0 = \mathfrak{h}$. The roots are

$$R = \{\alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq 0, \alpha \neq 0\},$$

and R^+ is a choice of half of R such that $R = R^- \cup R^+$, where

$$R^- = -R^+.$$

We can and will choose $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$, and $H_\alpha \in \mathfrak{h}$ such that $[X_\alpha.Y_\alpha] = H_\alpha$, $[H_\alpha, X_\alpha] = 2X_\alpha$, and $[H_\alpha, Y_\alpha] = -2Y_\alpha$.

What's the point? For each $\alpha \in R^+$, there is a copy of $\mathfrak{sl}_2$ sitting inside of $\mathfrak{g}$.

Example: Let $\mathfrak{g} = \mathfrak{gl}_n$. So $\mathfrak{n}^+$ is the set of upper triangular matrices, and $\mathfrak{h}$ is the set of diagonal matrices. We often write $h_i = E_i i$ as a basis of $\mathfrak{h}$. Our favorite basis of $\mathfrak{gl}_n$ is

$$\{E_{ij} \mid 1 \leq i, j \leq n\}$$

and the action of $\mathfrak{h}$ on $\mathfrak{gl}_n$ is given by

$$\begin{aligned}
[h_i, E_{jk}] &= E_{ii}E_{jk} - E_{jk}E_{ii} \\
&= \delta_{ij}E_{ik} - \delta_{ik}E_{ki} \\
&= (\delta_{ij} - \delta_{ik})E_{kl} \\
&= (\varepsilon_j(h_i) - \varepsilon_k(h_i))E_{jk} \\
&= (\varepsilon_j - \varepsilon_k)(h_i)E_{jk},
\end{aligned}$$

where $\varepsilon_j \in \mathfrak{h}^*$, which sends h_i to δ_{ij}. So the ε_j's form the dual basis.

So $E_{jk} \in \mathfrak{g}_{\varepsilon_j - \varepsilon_k}$. Then $E_{kj} \in \mathfrak{g}_{\varepsilon_k - \varepsilon_j} = \mathfrak{g}_{-(\varepsilon_j - \varepsilon_k)}$. So we can choose $R^+ = \{\varepsilon_j - \varepsilon_k \mid 1 \leq j < k \leq n\}$. Notice that $[E_{jk}, E_{kj}] = E_{jj} - E_{kk} = H_{\varepsilon_j - \varepsilon_k}$, $X_{\varepsilon_j - \varepsilon_k} = E_{jk}$ and $Y_{\varepsilon_j - \varepsilon_k} = E_{kj}$

We are going to exploit the fact that every $\mathfrak{g}$-module is an $\mathfrak{sl}_2$-module (in several ways).

Recall: If M is a simple $\mathfrak{sl}_2$-module, then M can be realized as a 1-dimensional lattice, chopped off after a finite number of dots, labeled $y^k v^+$, and x moves us up and y moves us down.

Now, SL_2 is the group corresponding to the Lie algebra $\mathfrak{sl}_2$. Let

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So let

$$x_\alpha(c) = e^{cX} = 1 + cX = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$$

$$x_{-\alpha}(c) = e^{cY} = 1 + cY = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$$

$$s_\alpha = e^{cX} e^{-c^{-1}Y} e^{cX} = \cdots = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $s_\alpha = s_\alpha(1)$. So

$$s_\alpha = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then conjugation by s_α is an automorphism of SL_2.

$$\begin{aligned} s_\alpha x_\alpha(c) s_\alpha^{-1} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= x_{-\alpha}(-c). \end{aligned}$$

Any finite dimensional $\mathfrak{sl}_2$-module is an SL_2-module. If G is a group and σ is an automorphism of G and M is a G-module, then define a new module $\sigma^*(M)$ whose vector is M and the new action

$$g\dot{m} = \sigma(g)m$$

($\sigma : G \to G$).

In the case of SL_2, us the automorphism σ "conjugation by s_α". In this case $\sigma^*(L(5))$ (our highest-weight lattice) is a 6-dimensional simple SL_2-module. σ "switches X and Y" and $\sigma^*(L(5))$ is just $L(5)$ flipped over. But the lattice is symmetric, so $\sigma^*(L(5)) \cong L(5)$.

Remark: σ is also an automorphism of $\mathfrak{g}$.

$$e^{\mathrm{ad} X_\alpha} e^{-\mathrm{ad} Y_\alpha} e^{\mathrm{ad} X_\alpha} \in G$$

and so

$$\sigma_\alpha : \mathfrak{g} \to \mathfrak{g}.$$

Note that if $y \in \mathfrak{g}$, then $e^{c\mathrm{ad}X} y = e^{cX} y e^{-cX}$.

So the Adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$ corresponds to conjugation action G on $\mathfrak{g}$. SO if $\mathfrak{g}$ is a cssLa, M a $\mathfrak{g}$-module and $\alpha \in R^+$, then σ_α flips all the strings! So a $\mathfrak{g}$-module is *symmetric* in every direction ($\alpha \in R^+$). Recall that the weights of M are $\mu \in \mathfrak{h}^*$ such that $M_\mu \neq 0$. So $M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$ and σ_α flips all weights and is a reflection of order 2.

The *Weyl group* is the group acting on $\mathfrak{h}^*$ generated by the reflections s_α where s_α is the reflection on $\mathfrak{h}^*$ induced by the automorphism σ_α.

Remark: Notice that all weights of finite dimensional M are integra, since they are integra for $\mathfrak{sl}_2$. If $M_\mu \neq 0$, then $\mu(H_\alpha) \in \mathbb{Z}$.

21 $\mathfrak{sl}_3$: the gateway to classification (Wed. 10/31)

Let $\mathfrak{g} = \mathfrak{sl}_3$ be the set of 3×3 matrices with zero traces with bracket

$$[x, y] = xy - yx.$$

We want to decompose fg as

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right),$$

where $\mathfrak{h}$ is the set of traceless diagonal matrices (has dimension 2). Weights μ are elements of $\mathfrak{h}^*$

$$\mu : \mathfrak{h}^* \to \mathbb{C}$$

via

$$h_1 \mapsto \mu(h_1)$$

$$h_2 \mapsto \mu(h_2).$$

Last time we used ε_i such that

$$\varepsilon_i \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = a_i.$$

Note: $\varepsilon_i \in \mathfrak{h}^*$ and $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$.

$$\mathfrak{h}^* = \mathbb{C}\text{-span}\{\mu_1\varepsilon_1 + \mu_2\varepsilon_2 + \mu_3\varepsilon_3 \mid \mu 1 + \mu_2 + \mu_3 = 0\}.$$

Note that E_{12}, E_{13}, E_{23} are in $\mathfrak{sl}_3$ and form a basis of $\mathfrak{n}^+$. If

$$h = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix},$$

then

$$[h, E_{ij}] = (\varepsilon_i - \varepsilon_j)(h) E_{ij}.$$

So $E_{ij} \in \mathfrak{g}_{\varepsilon_i - \varepsilon_j}$. So

$$R^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3\}$$

and be have our desired decomposition.

21.1 drawing h^*

Each $\alpha \in R^+$ gives an embedded $\mathfrak{sl}_2$ inside $\mathfrak{g}$ and a reflection s_α on $\mathfrak{h}^*$.

The picture is in three-space, with axes labeled ε_1, ε_2, and ε_3 with a plane going through $(1,0,0),(0,1,0),(0,0,1)$ shifted back to go through the origin. Off to the side, we drew the hexagonal lattice, with hyperplanes $H_{\varepsilon_1-\varepsilon_2}$, $H_{\varepsilon_1-\varepsilon_3}$, and $H_{\varepsilon_2-\varepsilon_3}$.The copy of $\mathfrak{sl}_2$ corresponding to α has weights in a one-dimensional lattice (in a line-segment), so in our picture, we should have a copy of this lattice in the α direction (perpendicular to H_α). If λ is a weigh in the α directiont, so is a point on the lattice, then

$$\langle \lambda, \alpha^\vee$$

is interpreted as the distance of λ to H_α. So we have a formula for our reflection:

$$s_\alpha = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha.$$

The reflection s_α fixes points on

$$H_\alpha = \{\mu \in \mathfrak{h}^* \mid \langle \mu, \alpha^\vee = 0\}$$

(i.e. all the points on the big lattice in three space which sits on the plane perpendicular to the α direction.

$$s_{\varepsilon_1-\varepsilon_2}(\varepsilon_1 - \varepsilon_2) = -(\varepsilon_1 - \varepsilon_2)$$

and $s_{\varepsilon_1-\varepsilon_2}$ fixes $H_{\varepsilon_1-\varepsilon_2} = \{\mu_1\varepsilon_1 + \mu_2\varepsilon_2 + \mu_3\varepsilon_3 \mid \mu 1 = \mu_2\}$

So back to our big picture. The hexagonal weight lattice gets embedded into the aforementioned plane. Each s_α corresponds to an automorphism σ_α of $\mathfrak{g}$ and induces a flip on weights of $\mathfrak{g}$-modules. If M is a finite dimensional $\mathfrak{g}$-module, then

$$M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu$$

and

$$\dim(M_{s_\alpha \mu}) = \dim(M_\mu).$$

Back on our hexagonal pieces, mark x's on lattice points μ if $M_\mu \neq 0$. The weights of M are in a $\mathbb{Z}$-lattice and are symmetric under the action of the Weyl group, so one point generates a hexagon (if it's not on an H). Now, since we have all these little $\mathfrak{sl}_2$'s, each string of lattice points in every α-direction must be filled in (in any direction, M is the union of strings).

So, in general, finite dimensional $\mathfrak{sl}_3$-modules "look like" hexagons (with two lengths of sides) with strings filled in perpendicular to the H's. Any structure must be symmetric. Woo.

21.2 Philosophy of classification

What possible finite collections of weights could correspond to $\mathfrak{g}$ ($\mathfrak{g}$ is a $\mathfrak{g}$-module)? In order to build a $\mathfrak{g}$, we need a reflection group acting on a lattice (free $\mathbb{Z}$-module, vector space over $\mathbb{Z}$). But I want $\mathfrak{g}$ to be finite dimensional, so W must also be finite, of which there are very few!

22 Classification (Friday. 11/2)

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra.

We can only have one of these if we have a finite reflection group W acting on a lattice P (it's true that every such pair gives us a fdcsLa, but it's not easy to see – this is a big theorem).

Review: For every $\alpha \in R^+$, we get a symmetry (automorphism) σ_α of $\mathfrak{g}$ and a symmetry (reflection) in $\mathfrak{h}^*$.

$$W = \langle s_\alpha \in R^+ \rangle$$

acts on $\mathfrak{h}^*$. The set of weights that actually occue in $\mathfrak{g}$-modules, P, is a vector space over $\mathbb{Z}$ (inside of $\mathfrak{h}^*$).

22.1 Step 1: How many W, P possibilities are there?

Let $\mathfrak{h}^*_\mathbb{Z} = P$ and $\mathfrak{h}^*_\mathbb{C} = \mathfrak{h}^*$. Let $\mathfrak{h}^*_\mathbb{R} = \mathbb{R} \otimes_\mathbb{Z} \mathfrak{h}^*_\mathbb{Z} = \mathbb{R}\text{-span}\{\omega_1, \ldots, \omega_n\}$, where $\omega_1, \ldots, \omega_n$ is a $\mathbb{Z}$-basis of $\mathfrak{h}^*_\mathbb{Z}$.

Example: Rank 1 $\dim \mathfrak{h}^* = 1$. $\mathfrak{h}^*_\mathbb{Z}$ is a line of points, so $\mathfrak{h}^*_\mathbb{R}$ is a line, and $s_\alpha \lambda = \langle \lambda, \alpha^\vee \rangle \alpha$, where $\langle \lambda, \alpha^\vee \rangle$ is the distance from λ to the hyperplane H_α. The set of points which are fixed by the reflection s_α is

$$(\mathfrak{h}^*_\mathbb{R})^{s_\alpha} = \{\lambda \in \mathfrak{h}^*_\mathbb{R} \mid s_\alpha \lambda = \lambda\}.$$

Since a linear transformation s_α fixes 0, and we only have rank 1,

$$W \cong \mathbb{Z}/2\mathbb{Z}.$$

Up to scaling, there is only one choice of (W, P) in dimension 1. Since $W = \langle s_\alpha \mid \alpha \in R^+ \rangle$, we get

$$\mathfrak{g} = \mathfrak{g}_{-\alpha} \oplus \mathfrak{h} \oplus \mathfrak{g},$$

which is three-dimensional... so $\mathfrak{g}$ must be isomorphic to $\mathfrak{sl}_2$.

Example: Rank 1 $\dim \mathfrak{h}^* = 2$. So $\dim \mathfrak{h}^*_\mathbb{R} = 2$. If $W = \langle s_\alpha, s_\beta$, then if the angle between H_α and H_β is not $\pi/2$, then they will generate more reflections. This will turn out to be symmetries of the an n-gon (if W has order $2n$). In fact,

$$W = \langle s_1, s_1 s_2 \mid s_1^2 = 1, (s_1 s_2)^4 = 1 \rangle.$$

Some people play the game with the upper half plane with $\mathrm{SL}_2(\mathbb{Z})$ acting on it. THis has "fundamental regions" and whatnot. We have "fundamental chambers" and what not, which correspond to the cone with walls H_α's.

By thought, we presented by generators s_1, s_2, and relations

$$s_1^2 = 1 = s_2^2, \quad \underbrace{s_1 s_2 s_1 \cdots}_{n\text{factors}} = \underbrace{s_2 s_1 s_2 \cdots}_{n\text{factors}},$$

where $\pi/2 = H_{\alpha_1} angle H_{\alpha_2}$.

Example: Rank big If $\dim \mathfrak{h}^*_\mathbb{R}$ is big, then we'll look for embedded dihedral groups.

23 More classification: building Dynkin diagrams (Mon. 11/5)

Our goal is to classify finite dimensional complex semisimple Lie algebras by classifying pairs (W, P) where $\mathfrak{h}_{\mathbb{Z}}^* = P$ is a $\mathbb{Z}$-lattice, W is a finite subgroup of $\mathrm{GL}(P) \cong \mathrm{GL}_n(\mathbb{Z})$ generated by reflections

$$W = \langle s_\alpha \mid \alpha \in R^+ \rangle$$

where R^+ us an index set for the reflections in W.

23.1 But what *is* a reflection?

We can make P into a vector space over $\mathbb{Q}$:

$$\mathfrak{h}_{\mathbb{Q}}^* = \mathbb{Q} \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}}^*.$$

A *reflections* is a (semisimple) element $s_\alpha \in \mathrm{GL}(\mathfrak{h}_{\mathbb{Q}}^*)$ such that

$$(\mathfrak{h}^*)^\alpha = \{\lambda \in \mathfrak{h}_{\mathbb{Q}}^* \mid s_\alpha \lambda = \lambda\}$$

has codimension 1 in $\mathfrak{h}_{\mathbb{Q}}^*$, i.e. if $\dim(\mathfrak{h}_{\mathbb{Q}}^*) = n$, then the number of eigenvalue 1 is $n-1$. So in some good basis,

$$s_\alpha = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \xi \end{pmatrix} \qquad \text{or } s_\alpha = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}.$$

Then there is a $\alpha^\vee \in \mathfrak{h}_{\mathbb{Q}}$ and $\alpha \in \mathfrak{h}_{\mathbb{Q}}^*$ such that

$$s_\alpha \mu = \mu - \langle \mu, \alpha^\vee \rangle \alpha.$$

Note

$$\begin{aligned} s_\alpha \alpha &= \alpha - \langle \alpha, \alpha^\vee \rangle \alpha \\ &= (1 - \langle \alpha, \alpha^\vee \rangle)\alpha \\ &= \xi \alpha. \end{aligned}$$

So $\langle \alpha, \alpha^\vee \rangle = 1 - \xi$. The pair $\alpha, \alpha^\vee$ is unique up to a constant. If s_α is a reflection in $\mathrm{GL}(\mathfrak{h}_{\mathbb{R}}^*)$ in the ordinary sense, then

$$s_\alpha^2 = 1, \quad \xi = -1, \quad \text{and } \langle \alpha, \alpha^\vee \rangle = 2.$$

23.2 Dealing with the lattice

Last time, we classified the two cases: in rank 1, $W = \mathbb{Z}/2\mathbb{Z}$, and in rank 2, W is a dihedral group. In rank two, we just need that $H_{\alpha_1} \angle H_{\alpha_2} = \pi/n$ (since we need W to be finite). But we forgot! W also has a lattice!

We want $W \in \mathrm{GL}_2(\mathbb{Z})$ for the correct basis. Well, use the basis α_1, α_2 for $\mathfrak{h}_{\mathbb{R}}^*$:

$$s_1 \mu = \mu - \langle \mu, \alpha_1^\vee \rangle \alpha_1,$$

$$s_2 \mu = \mu - \langle \mu, \alpha_2^\vee \rangle \alpha_2,$$

where $\langle \mu, \alpha_1^\vee \rangle$, $\langle \mu, \alpha_2^\vee \rangle \in \mathbb{Z}$. In particular in the basis α_1, α_2

$$s_1 = \begin{pmatrix} -1 & -\langle \alpha_2, \alpha_1^\vee \rangle \\ 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 0 \\ -\langle \alpha_1, \alpha_2^\vee \rangle & -1 \end{pmatrix}.$$

$$s_1 \alpha_2 = \alpha_2 - \langle \alpha_2, \alpha_1^\vee \rangle.$$

Then

$$s_1 s_2 = \begin{pmatrix} -1 + \langle \alpha_1, \alpha_2^\vee \rangle \langle \alpha_2, \alpha_1^\vee \rangle & \langle \alpha_2, \alpha_1^\vee \rangle \\ -\langle \alpha_1, \alpha_2^\vee \rangle & -1 \end{pmatrix}.$$

so $\det(s_1 s_2) = 1)$ and $\operatorname{tr}(s_1 s_2) = -2 + \langle \alpha_1, \alpha_2^\vee \rangle \langle \alpha_2, \alpha_1^\vee \rangle$.

So $\operatorname{char}(s_1 s_2) = t^2 - (-2 + \langle \alpha_1, \alpha_2^\vee \rangle \langle \alpha_2, \alpha_1^\vee \rangle)t + 1$ and eigenvalues of $s_1 s_2$ are

$$\frac{-2 + \langle \alpha_1, \alpha_2^\vee \rangle \langle \alpha_2, \alpha_1^\vee \rangle \pm \sqrt{(-2 + \langle \alpha_1, \alpha_2^\vee \rangle \langle \alpha_2, \alpha_1^\vee \rangle)^2 - 4}}{2}$$

. These ought to be $e^{2\pi i/2}$ and $e^{-2\pi i/2}$ since $(s_1 s_2)^n = 1$. So

$$\frac{-2 + \langle \alpha_1, \alpha_2^\vee \rangle \langle \alpha_2, \alpha_1^\vee \rangle}{2} = \cos(2\pi/n).$$

So $\langle \alpha_1, \alpha_2^\vee \rangle \langle \alpha_2, \alpha_1^\vee \rangle = 2 + 2\cos(2\pi/n) \in \mathbb{Z}$. THe only possibilities are

$n = 1$:

$$\langle \alpha_1, \alpha_2^\vee \rangle \langle \alpha_2, \alpha_1^\vee \rangle = 2 + 2$$

so $s_1 = s_2$.

$n = 2$:

$$\langle \alpha_1, \alpha_2^\vee \rangle \langle \alpha_2, \alpha_1^\vee \rangle = 2 - 2 = 0$$

. ...

In a *Dynkin diagram* (a graph), the number of edges between nodes i and j is $\langle \alpha_1, \alpha_2^\vee \rangle \langle \alpha_2, \alpha_1^\vee \rangle$.

23.3 Classifying the (bigger) general case

$\mathbb{R}^n \cong \mathfrak{h}_{\mathbb{R}}^*$ is a union of closures of connected components C_w of

$$\mathfrak{h}^* \setminus \cup_{\alpha \in R^+} \mathfrak{h}^\alpha$$

(called the *fundamental regions* for action of W). Fix a C_1. Let $H_{\alpha_1}, \ldots, H_{\alpha_n}$ be the walls of C_1. The *simple reflections* are $s_1, \ldots, s_n$. THe *Cartan matrix* of W is

$$\left(\langle \alpha_i, \alpha_j^\vee \right)_{1 \le i, j \le n}.$$

24 Coxeter, Cartan, and Kac-Moody (Fri. 11/9)

Each $\alpha \in R^+$ gives an automorphism σ_α of $\mathfrak{g}$ and a reflection $s_\alpha \in \mathrm{GL}(\mathfrak{h}^*)$ such that

$$M_\mu \tilde{\to} M_{s_\alpha \mu}$$

is a vector space isomorphism when M is a finite dimensional $\mathfrak{g}$-module and μ is a weight (eigenvalue for $\mathfrak{h}$ action). The *Weyl group* is

$$W = \langle s_\alpha \mid \alpha \in R^+ \rangle.$$

We get $P = \mathfrak{h}^*_{\mathbb{Z}}$ a lattice, the weights which appear in finite dimensional $\mathfrak{h}$-modules.

Let $\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{h}^*_{\mathbb{Z}}$ and

$$C = \{\lambda^\vee \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \lambda, \alpha^\vee \rangle > 0 \text{ for all } \alpha \in R^+\}$$

(points on the positive side of $\mathfrak{h}^{s_\alpha} = \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle \lambda, \alpha^\vee \rangle = 0\}$). The chamber C is a fundamental region for the action of W on $\mathfrak{h}^*_{\mathbb{R}}$.

Let $\mathfrak{h}^{s_{\alpha_1}}, \ldots, \mathfrak{h}^{s_{\alpha_n}}$ be the walls of C and $s_1, \ldots, s_2$ be the corresponding reflections.

Theorem 24.1 (Coxeter)**.**

$$W = \langle s_1, \ldots, s_n \mid s_i^2 = 1, \underbrace{s_i s_j s_i \cdots}_{m_{ij} factors} = \underbrace{s_j s_i s_j \cdots}_{m_{ij} factors} \rangle$$

where $\pi/m_{ij} = \mathfrak{h}^{s_{\alpha_i}} \angle \mathfrak{h}^{s_{\alpha_i}}$.

For our calculations, we noticed that the *Cartan matrix*,

$$A = \left(\langle \alpha_1, \alpha_j^\vee \rangle \right)_{1 \le i,j \le n}$$

is a matrix of integers. Recall tha the formula for a reflection is $s_\alpha \lambda = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$, where $\langle \lambda, \alpha^\vee \rangle$ is the "distance from λ to $\mathfrak{h}^{s_\alpha}$.

24.1 Can we go back? Can we build $\mathfrak{g}$ from A?

Let $A = (a_{ij})_{1 \le i,j \le n}$ be any matrix. By rearanging rows and columns, we can assume that $(a_{ij})_{1 \le i,j \le r}$ is non-singular, where r =rank(A). Let ℓ =corank(A), so that $r + \ell = n$. Define $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{d}$ where $\mathfrak{h}'$ has basis $h_1, \ldots, h_n$ and $\mathfrak{d}$ has basis $d_1, \ldots, d_\ell$.

Let $\alpha_1, \ldots, \alpha_n \in \mathfrak{h}^*$ be given by $\alpha_i(h_j) = a_{ij}$ and $\alpha_i(d_j) = \delta_{i,j+r}$. Let $\bar{\mathfrak{h}}' = \mathfrak{h}'/\mathfrak{c}$ where $\mathfrak{c} = \{h \in \mathfrak{h}' \mid \alpha_i(h) = 0 \text{ for } i = 1, \ldots, n\}$. Let $c_1, \ldots, c_\ell$ be a basis for $\mathfrak{c}$ for that $h_1, \ldots, h_r, c_1, \ldots, c_\ell, d_1, \ldots d_\ell$ is another basis of $\mathfrak{h}$. Define $\omega_1, \ldots, \omega_\ell \in \mathfrak{h}^*$ by

$$\omega_i(h_j) = 0, \quad \omega_i(c_j) = \delta_{ij}, \quad , \omega_i(d_j) = 0.$$

(these are the *fundamental weights*!). Let $\mathfrak{a}$ be the Lie algebra generated by $e_1, \ldots e_n, f_1, \ldots f_n$ and $\mathfrak{h}$ with relations

$$[h, h'] = 0, \text{ for } h, h' \in \mathfrak{h},$$

$$[e_i, f_j] = \delta i j h_i, \quad [h, e_i] = \alpha_i(h) e_i, \quad [h, f_i] = -\alpha_i(h) f_i.$$

The *Borcher-Kac-Moody Lie algebra* is $\mathfrak{g} = \mathfrak{a}/\mathfrak{r}$, where $\mathfrak{r}$ is the largest ideal of $\mathfrak{a}$ such that $\mathfrak{r} \cap \mathfrak{h} = 0$. It turns out that this thing is semisimple, though it is rarely finite dimensional. $\mathfrak{g}$, $\mathfrak{a}$, $\mathfrak{r}$ are graded by

$$Q = \sum_{i=1}^{n} \mathbb{Z}\alpha_i$$

by setting $\deg(e_i) = \alpha_i$, $\deg(f_i) = -\alpha_i$, and $\deg(h) = 0$. So, even though $\mathfrak{g}$ isn't finite dimensional, we can still write

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right)$$

where

$$R = \{\alpha \in Q \mid \alpha \neq o, \mathfrak{g}_\alpha \neq 0\}$$

is the set of roots.

Let's just point out that $\mathfrak{n}^+$ is the subalgebra generated by the e's, $\mathfrak{n}^-$ is the subalgebra generated by the f's. So $\mathfrak{n}^+ = \bigoplus_{\alpha \in R^+} \mathfrak{g} + \alpha$ where $R^+ = R \cup Q^+$ and $Q^+ = \sum_i \mathbb{Z}_{\geq 0}\alpha_i$. So

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ = \mathfrak{g}' \rtimes \mathfrak{d}$$

$$\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{n}^- \oplus \mathfrak{h}' \oplus \mathfrak{n}^+$$
$$\mathfrak{c} = \mathcal{Z}(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g}')$$

and

$$\bar{\mathfrak{g}}' = \mathfrak{g}'/\mathfrak{c} = \mathfrak{n}^- \oplus \bar{\mathfrak{h}}' \oplus \mathfrak{n}^+$$

$\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{d}$, $\mathfrak{d}$ acts on $\mathfrak{g}'$ by derivations

$$\bar{\mathfrak{h}}' = \mathfrak{h}'/\mathfrak{c}.$$

25 More on the Cartan matrix and Kac-Moody Lie algebras (Mon. 11/12)

There are a couple of directions we could go from here.

A (generalized) *Cartan matrix* is a matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ such that $a_{ij} \in \mathbb{Z}$, $a_{ii} = 2$ (makes stuff $\mathfrak{sl}_2$'s), $a_{ij} \leq 0$ if $i \neq j$ and $a_{ij} \neq 0$ if and only if $a_{ji} \neq 0$. WHY?? In this case,

$$\mathbb{C}\text{-span}\{e_i, f_i, h_i\} \cong \mathfrak{sl}_2$$

so e_i and f_i act locally nilpotently on $\mathfrak{g}$ (adjoint representation) and

$$\tilde{s}_i = \exp(\mathrm{ad}_{e_i}) \exp(-\mathrm{ad}_{f_i}) \exp(\mathrm{ad}_{e_i}).$$

Here, ad is a code for $\mathfrak{g}$ acting on itself:

$$\mathrm{ad}_x(y) = [x, y].$$

The "exponential map" exp means

$$\exp(S) = 1 + S + S^2/2 + S^3/3! + \cdots.$$

The $\tilde{s_1}, \ldots, \tilde{s_n}$ are automorphisms of $\mathfrak{g}$ and so $\mathfrak{g}$ has lots of symmetry (this is the only kind that we should care about... stuff with lots of symmetry).

Then define $s_i : \mathfrak{h} \to \mathfrak{h}$ and $s_i : \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$s_i h = h - \alpha(h) h_i, \quad \text{ and } s_i \lambda = \lambda(h_i)\alpha_i$$

(by stealing the formula for a reflection). Then indeed, as expected,

$$\tilde{s_i}\mathfrak{g}_\alpha = \mathfrak{g}_{s_i\alpha}, \quad \text{and } \tilde{s_i} h = s_i h$$

(i.e. on weights and on $\mathfrak{h}$, $\tilde{s_i}$ induce reflections).

The *Weyl group* $W \subset \mathrm{GL}(\mathfrak{h}^*)$ is

$$W = \langle s_1, \ldots, s_n \rangle$$

(of course we could have said $W \subset \mathrm{GL}(\mathfrak{h})$, but $\mathfrak{h}$ and $\mathfrak{h}^*$ are dual W-modules, so

$$(w\lambda)(h) = \lambda(w^{-1}h)$$

for $\lambda \in \mathfrak{h}^*$, $w \in W$, and $h \in \mathfrak{h}$.) This W should look kind of small to us... before, we had a reflection for *every* root. We would like to try to get them al back. However, in the non-finite-dimensional case, we *can't* get out an s_α for every α such that $\mathfrak{g}_\alpha \neq 0$! What we do know how to do is get these s_α for the *real roots*:

$$R_{\mathrm{re}} = \bigcup_{i=1}^{n} W\alpha_i.$$

The *imaginary roots* are

$$R_{\mathrm{im}} = R \setminus R_{\mathrm{re}}.$$

Recall:

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right).$$

If $\alpha \in R_{\mathrm{re}}$, then $\alpha = w\alpha_i$ and

$$e_\alpha = \tilde{w} e_i, \quad f_\alpha = \tilde{w} f_i, \quad h_\alpha = \tilde{w} h_i.$$

This gives us some $\mathfrak{sl}_2$'s and a s_α for every real root!

A matrix is *symmetrizable* if there exists a diagonal matrix

$$\mathcal{E} = \begin{pmatrix} \epsilon_1 & & \\ & \ddots & \\ & & \epsilon_n \end{pmatrix}$$

$\epsilon_i \in \mathbb{R}_{\geq 0}$ such that $A\mathcal{E}$ is symmetric. WHY??

Theorem 25.1. $\mathfrak{g}$ *has a nondegenerate symmetric* $\mathfrak{h}$*-invarient bilinear form*

$$\langle , \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}.$$

Do a tiny calculation to see that $\langle , \rangle$ is determined by

$$\langle e_i, f_i \rangle = \epsilon_i, \quad \langle h_i, h \rangle = \alpha_i(h)$$

for $h \in \mathfrak{h}$.

Most papers on this subject begin with the line

> "Let $\mathfrak{g}$ be a symmetizable Kac-Moody Lie algebra..."

Equivalent would be

> "Take some e's and some f's and some h's, use $[e_i, f_i] = h_i$, $[h_i, e_j] = a_{ij} e_j$, suppose there are lots os symmetries W, and a symmetric $\mathfrak{g}$-invarient bilinear form $\langle , \rangle$..."

We said that if A is a Cartan matrix then e_i, f_i act on $\mathfrak{g}$ *locally nilpotently*. More precisely, we must ask for what k is

$$(\mathrm{ad}_{e_i})^k(e_j) = 0?$$

Well, the first such k is $1 - a_{ij}$! (if ij is off of the diagonal, i.e. if I'm not acting on e_i by e_i, so $a_{ij} \leq 0$, and thus $1 - a_{ij}$ is positive... we can find this by experiment) The same is true locally for the f's:

$$\mathrm{ad}(f_i)^{1-a_{ij}}(f_j) = 0.$$

Where is this coming from?? Remember, from Coxeter (Theorem 24.1):

$$W = \langle s_1, \ldots, s_n \mid s_i^2 = 1, \underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{factors}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{factors}} \rangle$$

where $m_{ij} = a_{ij} a_{ji}$. So the fancy version of Serre's theorem goes...

Theorem 25.2 (Serre). *(a) If A is a Cartan matrix then these relations hold in $\mathfrak{g}$ (i.e. $(\mathrm{ad}_{e_i})^{1-a_{ij}}(e_j) \in \mathfrak{r}$, and same with f's).*

(b) If A is symmetrizable then $\mathfrak{r}$ is generated by the relations above (call them (S1) and (S2))

26 Affine Lie algebras (Wed. 11/14)

Let $\mathfrak{g}_0$ be a symmetrizable"?" Kac-Moody Lie algebra with bracket $[,]_0$ and an invariant symmetric bilinear form $\langle , \rangle$ (given to me by the symmetrizableness).

> Note: Most of the literature takes $\mathfrak{g}_0$ to be finite dimensional complex semisimple. But something seems to be happening in the community that indicates that this isn't really what we want. We're not sure if "symmetrizable" is really what we want, but it's worth trying out...

The *loop Lie algebra* is

$$\mathfrak{g}_0[t, t^{-1}] = \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}_0$$

with bracket

$$[t^m x, t^n y]_0 = t^{m+n}[x, y]_0.$$

This has a symmetric bilinear form

$$\langle t^m x, t^n y\rangle = \begin{cases} \langle x, y\rangle_0 & m+n=0, \\ 0 & \text{otherwise} \end{cases}$$

The *universal central extension* of $\mathfrak{g}_0[t,t^{-1}]$ is

$$\mathfrak{g}' = \mathfrak{g}_0[t,t^{-1}] \oplus \mathbb{C}c$$

with bracket

$$[t^m x, t^n y] = t^{m+n}[x,y]_0 + \delta_{m,-n} m\langle x.y\rangle_0 c$$

(c is the "central charge" or something).

Recall Der $(\mathbb{C}[t,t^{-1}])$ is the *Witt Lie algebra*

$$[\mathrm{d}_i, \mathrm{d}_j] = (i-j)\mathrm{d}_{i+j}$$

where $\mathrm{d}_i = t^{i+1}\frac{\mathrm{d}}{\mathrm{d}t}$. Let $\mathrm{d} = t\frac{\mathrm{d}}{\mathrm{d}t}$ and let

$$\mathfrak{g} = \mathfrak{g}_0[t,t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}\mathrm{d}$$

where $[\mathrm{d}, t^m x] = (t\frac{\mathrm{d}}{\mathrm{d}t}t^m)x = mt^m x$, $[\mathrm{d}, c] = 0$, and $[c, t^m x] = 0$. The *affine Lie algebra* is $\mathfrak{g}$ (or $\mathfrak{g}'$ or $\bar{\mathfrak{g}}'$).

With Borchards-Kac-Moody Lie algebra we had

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ = \mathfrak{g}' \rtimes \mathfrak{d}$$

$$\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{n}^- \oplus \mathfrak{h}' \oplus \mathfrak{n}^+$$

$$\mathfrak{c} = \mathcal{Z}(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g}')$$

and

$$\bar{\mathfrak{g}}' = \mathfrak{g}'/\mathfrak{c} = \mathfrak{n}^- \oplus \bar{\mathfrak{h}}' \oplus \mathfrak{n}^+$$

$\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{d}$, $\mathfrak{d}$ acts on $\mathfrak{g}'$ by derivations

$$\bar{\mathfrak{h}}' = \mathfrak{h}'/\mathfrak{c}.$$

Question: When is the affine Lie algebra Kac-Moody?

If $\mathfrak{g}_0$ is finite dimensional complex semisimple then the answer is yes. In this case the two realizations play against eachother. $\mathfrak{g}_0$ has decomplosition as an $\mathfrak{h}_0$-module

$$\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus \mathfrak{h}_0 \oplus \mathfrak{n}_0^+ = \mathfrak{h}_0 \oplus \left(\bigoplus_{\alpha\in R} \mathfrak{g}_\alpha\right)$$

Does the affine Lie algebra have something sumilar?

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathbb{C}c\mathbb{C}d$$

$$\mathfrak{h}' = \mathfrak{h}_0 \oplus \mathbb{C}c$$

$$\bar{\mathfrak{h}}' = \mathfrak{h}_0$$

are possible Cartan subalgebras to use.

27 (Fri. 11/17)

Our story is that we're working on Affine Lie algebras. We start with a $\mathfrak{g}_0$, as symmetrizable K.M. Lie algebra. $\mathfrak{h}_0$ is a Cartan subalgebra, so

$$\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus \mathfrak{h}_0 \oplus \mathfrak{n}_0^+.$$

And we take beacket $[,]_0$ and invariant form $\langle,\rangle_0$. Ler $\bar{\mathfrak{g}}' = \mathfrak{g}_0[t,t^{-1}] = \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}$. We take $\mathrm{d} = t\frac{\mathrm{d}}{\mathrm{d}t}$, so $\mathrm{d}(t^m) = mt^m$.

Note $\mathfrak{g}'$ (the one with the central extension) surjects onto $\bar{\mathfrak{g}}'$, the *loop Lie algebra.* On the other hand, $\mathfrak{g}'$ is contained in $\mathfrak{g} = \mathfrak{g}' \oplus \mathbb{C}d$.

Example Let $\mathfrak{g}_0 = \mathfrak{sl}_2 = \mathrm{span}\{x,y,h\}$. As a K.M. Lie algebra, $\mathfrak{sl}_2$ should have a Cartan matrix [???] and $\mathfrak{g}_0 = \mathbb{C}y \oplus \mathbb{C}h \oplus \mathbb{C}x = \mathfrak{g}_{-\alpha} \oplus \mathfrak{h}_0 \oplus \mathfrak{g}_\alpha$. If $e_1 = x$, $f_1 = y$, then $h_1 = h$. So the cartan matrix is $A = (2)$. The $\mathfrak{sl}_2[t,t^{-1}] = \mathrm{span}\{t^m x, t^m y, t^m h \mid m \in \mathbb{Z}\}$.

$$\bar{\mathfrak{h}}' = \mathfrak{h}_0$$

$$\mathfrak{h}' = \mathfrak{h}_0' \oplus \mathbb{C}c$$

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathbb{C}c \oplus \mathbb{C}\mathrm{d}$$

We want to determine how $\mathfrak{g}$ looks as an $\mathfrak{h}$-module. If $x \in \mathfrak{g}_\alpha$ and $h + zc + w\mathrm{d} \in \mathfrak{h}$, then

$$\begin{aligned}
[h + zc + w\mathrm{d}, t^m x] &= \alpha(h)t^m x + 0 + wmt^m x \\
&= (\alpha(h) + wm)t^m x \\
&= (\alpha + m\delta)(h + zc + w\mathrm{d})t^m x
\end{aligned}$$

where $\delta \in \mathfrak{h}^*$ given by

$$\delta(c) = 0, \quad \delta(\mathrm{d}) = 1$$

and we'll extend α to an element of $\mathfrak{h}^*$ (it started as an element of $\mathfrak{h}_0^*$) by

$$\alpha(c) = 0, \quad \alpha(\mathrm{d}) = 0.$$

Then $t^m x \in \mathfrak{g}_{\alpha+m\delta}$. So

$$\begin{aligned}
\mathfrak{g} &= \mathfrak{g}_0[t,t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}\mathrm{d} \\
&= \mathfrak{h}_0 \oplus \left(\bigoplus_{\substack{\alpha \in R \\ m \in \mathbb{Z}}} \mathfrak{g}_{\alpha+m\delta} \right) \oplus \left(\bigoplus_{m \in \mathbb{Z}} \mathfrak{g}_{m\delta} \right)
\end{aligned}$$

where

$$\mathfrak{g}_{\alpha+m\delta} = \{t^m x \mid x \in \mathfrak{g}_\alpha\}$$

$$\mathfrak{g}_{m\delta} = \{t^m h \mid h \in \mathfrak{h}_0\}.$$

Note: $[\mathrm{d}, t^m h] = mt^m h = m\delta(\mathrm{d})t^m h$.

In $\mathfrak{g} = \mathfrak{sl}_2[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}\mathrm{d}$,

$$\mathfrak{g} = \left(\bigoplus_{m\in\mathbb{Z}} \mathbb{C}t^m x\right) \oplus \left(\bigoplus_{m\in\mathbb{Z}} \mathbb{C}t^m y\right) \oplus \mathbb{C}h \oplus \mathbb{C}c \oplus \mathbb{C}d \oplus \left(\bigoplus_{m\in\mathbb{Z}_{\neq 0}} \mathbb{C}t^m h\right).$$

If $\mathfrak{g}_0$ is finite dimensional complex semisimple then $\mathfrak{g}' = \mathfrak{g}_0[t, t^{-1}] \oplus \mathbb{C}c$. Clearly, this is *no longer finite dimensional*. But it's KM, so it's supposed to have a Cartan... 2×2. THe new Cartan matrix for $\mathfrak{g}'$ should be 2×2... how do we find it?We need to find $e_0, e_1, f_0, f_1, h_0, h_1$. What would we like to use? We would like to keep $e_1 = x$, $f_1 = y$, and $h_1 = h$, as before. THe convention for the rest is $e_0 = ty$, $f_0 = t^{-1}$, and $h_0 = -h + c$. Did we guess right? To check, we want these 0-dealies to be an $\mathfrak{sl}_2$:

$$\begin{aligned} [e_0, f_0] &= [ty, t^{-1}x] \\ &= t^{1-1}[y, x]_0 + 1 \cdot \langle y, x\rangle c \\ &= -h + \mathrm{tr}(yx)c \\ &= -h + c. \end{aligned}$$

Hooray!

To find the Cartan matrix:

$$[h_0, e_0] = \alpha_0(h_0)e_0 = 2e_0, \quad [h_1, e_0] = \alpha_0(h_1)e_0 = -2e_0$$

$$[h_0, e_1] = \alpha_1(h_0)e_1 = -2e_1, \quad [h_1, e_1] = \alpha_1(h_1)e_1 = 2e_1$$

(use the above things we guessed to calculate the brackets) and the Cartan matrix for affine $\mathfrak{sl}_2$ is

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

(can go back and forth with this process).

27.1 Beginning representation theory of KM junk

This means that we want to understand $\mathfrak{g}$-modules. Again, we can use $\mathfrak{h}$ to make weight spaces. But it's highly unlikely that these modules are finite-dimensional, since $\mathfrak{g}$ itself is infinite dimensional (most will be infinite dimensional).

28 Computing the character of $M(\lambda)$ (Wed. 11/28)

Le $\mathfrak{g} = \mathfrak{sl}_2[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$, where if $x, y \in \mathfrak{sl}2$ then $[t^m x, t^n y] = t^{m+n}[x, y]_0 + \delta_{m+n,0}\langle x.y\rangle_0 c$, where $[,]_0$ is the bracket on $\mathfrak{sl}_2$ and $\langle,\rangle_)$ is the invarient form on $\mathfrak{sl}_2$. $c \in \mathcal{Z}(\mathfrak{g})$ and $[d, t^m x] = mt^m x$. $\mathfrak{sl}_2$ has basis x, y, h with $[x, y]_0 = h$, $[h, x] = 2x$, and $[h, y] = -2y$. $\langle x_1, x_2\rangle_0 = \mathrm{tr}(x_1, x_2)$ where $x_1, x_2 \in \mathfrak{sl}_2$ are trace zero matrices.

$\mathfrak{g}$ is a Kac-Moody Lie algebra with generators

$$e_0 = ty, \quad f_0 = t^{-1}x, \quad h_0 = -h + c,$$

$$e_i = x, , \quad f_1 = y, \quad h_1 = h.$$

Note: $[e_0, f_0] = [ty, t^{-1}x] = [y, x]_0 + \langle y, x\rangle_0 c = -h + c$. $\mathfrak{h}$ has basis $\{h, c, d\}$, $\mathfrak{n}^+$ is generated by e_0, e_1 and $\mathfrak{n}^-$ is generated by f_0, f_1.

The adjoint action of $\mathfrak{h}$ on fg gives

$$\begin{aligned} \mathfrak{g} &= \mathfrak{h} \oplus \left(\bigoplus_{\alpha\in R} \mathfrak{g}_\alpha\right) \\ &= \mathfrak{h} \oplus \left(\bigoplus_{m\in\mathbb{Z}} \mathfrak{g}_{\alpha+m\delta}\right) \oplus \left(\bigoplus_{m\in\mathbb{Z}} \mathfrak{g}_{-\alpha+m\delta}\right) \oplus \left(\bigoplus_{m\in\mathbb{Z}_{\neq 0}} \mathfrak{g}_{m\delta}\right) \end{aligned}$$

where $\mathfrak{g}_{\alpha+m\delta} = \text{span}\{t^m x\}$, $\mathfrak{g}_{-\alpha+m\delta} = \text{span}\{t^m y\}$, $\mathfrak{g}_{m\delta} = \text{span}\{t^m h\}$. This is good notation because $\mathfrak{h}^*$ has basis $\omega_1, \omega_0, \delta$ where

$$\omega_1(h) = 1, \quad \omega_1(c) = 0, \quad \omega_1(d) = 0,$$

$$\omega_0(h) = 0, \quad \omega_0(c) = 1, \quad \omega_0(d) = 0,$$

$$\delta(h) = 0, \quad \delta(c) = 0, \quad \delta(d) = 1,$$

and $\alpha = 2\omega_1$.

If $\gamma \in \mathfrak{h}$ and $p \in \mathfrak{g}_{\alpha+m\delta}$ then

$$[\gamma, p] = (\alpha + m\delta)(\gamma)p.$$

What do we mean?

$$[h, t^m x] = t^m[h, x] = 2t^m x = (\alpha + m\delta)(h)t^m x$$

$$[d, t^m x] = mt^m x = (\alpha + m\delta)(d)t^m x.$$

Bracketing e_0, e_1, f_0, f_1 never produces any d's, since $[t^m x_1, t^n, x_2]_0 + \delta_{m+n}\langle x_1, x_2\rangle_0 c$. Some people would like to work instead with $\mathfrak{g}' = \mathfrak{sl}_2[t, t^{-1}] \oplus \mathbb{C}c$. The point is that the elements $t^{-m}x,\ t^{-m}y,\ t^{-m}y\ m \in \mathbb{Z}_{>0}$ and y are in $\mathfrak{n}^-$.

$$f_{\alpha-m\delta} = t^{-m}x$$

$$f_{-\alpha-m\delta} = t^{-m}y$$

$$f_{-m\delta} = t^{-m}h$$

Then $U\mathfrak{n}^-$ is generated by these and has basis monomials in these symbols. In other words,we can take things like

$$\left(f_{-\alpha}\right)^3 \left(f_{-\alpha-2\delta}\right)^5 \left(f_{\alpha-4\delta}\right)^1 \dots$$

A All things aside

A.1 The Kac-Moody Lie algebra. (9/10)

In about 1965, Serre wrote down generators and relations for complex semi-simple Lie algebras $\mathfrak{g}$. The data is a matrix of numbers called the *Cartan matrix*, coming from certain polynomials and their symmetries. The Kac-Moody very deep idea was that "maybe other matrices are fun", i.e., they generalized the Cartan algebra to find another Lie algebra with many of the same nice properties.

A.2 What is a reductive I hear so much about? (9/12)

Something is reductive when it is "almost" semi-simple. A module is *semi-simple* if it decomposes as a direct sum of simple modules:

$$M \cong \bigoplus_i M_i, \qquad M_i \text{ is simple.}$$

The idea is a (group / Lie algebra / algebra / ring) is (reductive / semi-simple) if all of its modules are semi-simple. But what kind of modules do we care about? This is where so many definitions come in relating to the same idea – they all involve different kinds of modules (finite dimensional, integral, topologically smooth, etc.). Alternately, an object is semi-simple if its radical is trivial. But there are also many kinds of radicals.

A.3 Adeles – the study of everything at once. (9/17)

The story: Our favorite field is really $\mathbb{Q}$. The next one that we learned was $\mathbb{R} = \mathbb{Q}_\infty$, which is $\mathbb{Q}$ completed at the infinite *place*. In general, $\mathbb{Q}_p$ is a completion of $\mathbb{Q}$ embedded in $\mathbb{R}$. The next one we liked was $\mathbb{C}[x]$... but that's not a field, so we take $\mathbb{C}(x)$. But even in high school, $\mathbb{C}[x]$ also wasn't enough, as not everything we cared about was a polynomial – for some reason, everyone seemed to be excited about trig, and other such junk! So then we move on to $\mathbb{C}[[x]]$ and $\mathbb{C}((x))$. Besides, our first encounter with $\mathbb{R}$ was to see it as infinite decimal expansions anyway, so really $\mathbb{Q}_\infty = \mathbb{Q}[[\frac{1}{10}]]$. Similarly $\mathbb{Q}_p = [[p]]$ (it's hard to tell the difference between p and p^{-1}). So we have all of this number junk... all of which we study important stuff over. So why study all of these things individually if we could study all of them at once?

So that's what the Adeles are, morally – all of the numbers at once:

$$\text{Adeles} = \prod_p \mathbb{Q}_p,$$

where p can also be ∞. Our corresponding favorite group would then be $\mathrm{GL}_n(\prod_p \mathbb{Q}_p)$. There are, of course, finite conditions to help make things more comfortable... all but a finite number of factors are "trivial", where trivial usually means sitting inside of $\mathbb{Z}_p = \mathbb{Z}[[p]]$. So basically, to understand the Adeles, you just have to get good at doing everything at once.

A.4 Rob wants to know what this "admissible" junk is (9/17)

An *admissible module*, as far as Langlands is concerned, is a module V such that $(V^*)^* \cong V$ (via the canonical homomorphism).

A.5 Tantalizers (9/19)

Tantalizer is short for "tensor power centralizer." So, if G is a group and V is a G-module, then $\mathrm{End}_G(V^{\otimes k}$ is a tantalizer, where

$$\mathrm{End}_G(M) = \{\varphi \in \mathrm{End}(M) \mid \varphi g m = g \varphi m \ \forall m \in M, g \in G\}.$$

A.6 What is this "affine" I keep hearing?

There are many terms in mathematics which are overused, and *affine* is one of those terms. It generally means a "flat" space, but the various affines (affine variety, affine BMW-algebra, etc.) sometimes have very little to do with eachother. "Projective" comes as the counterpart in algebraic geometry, but is not necessarily so in the categorical sense (projective is a categorical term, with Hom's and whatnot).

A.7 Homology vs. Cohomology

Sarah feels that Cohomology wins because it has a ring structure and therefore does more for you. Ram argues that they are the same thing – $V = V^*$ if V is finite dimensional. Even when it's not finite dimensional, the cohomology ring is graded, so can still be matched up with with homology groups in some way... so most of the time they are the same thing. This is the same deal with Atiyah and his K-theory vs. cohomology. Also, same deal with Lie algebras and music... see Jacob Laurie.

B Those other questions

- When you go to a talk, how do you manage to get work done and at the same time know what the talk was about?
- How do you read a paper?
- Suppose I go to a professor to ask him a question, and he starts to talk with me about the topic. Then he refers me to a book and/or a paper on the subject. Does this usually mean that he wants me to read those as opposed to bothering him about it, or is that an invitation to learn more and then come talk to him again, or something else? [ans. Something about absinth...]
- Who do we ask for money? When should we start asking people for money? How do you (personally) get money? Does this change as we progress through our careers? How much money do we ask for? Will you buy me a new computer? Is it appropriate to ask my advisor for a new computer? Lets say I have a friend and their advisor is Professor X and they want a new computer. What advice would you give them with regard to getting a new computer and asking their advisor for money for it?
- How do you prepare seminar? Do you have some advice how to make it more interesting? How do you decide what to say (and how much)? How do you decide what to write on blackboard?
- Why do mathematicians do math? why not philosophy? why not politics or law? Why not something "useful"?

- Who is the greatest mathematician of all time? How about the past 50 years? past 10 year? Who is doing the most exciting math right now? Who is the most influential mathematician of the past 70 years? past 10 year?

- What are the best journals? How much does it matter which journals your papers appear in?

- Why can't mathematicians ever figure out how to split a bill at a restaurant?

- How does one grade quickly?

  In grading, there are two purposes: one is assessment, and one is feedback. So first, one has to put priorities on these goals. If you are grading entirely for feedback, you have to figure out what people are getting wrong and then mark on all of the errors. If it's for assessment, then you have to figure out what deserves points, and mark all of those. Once you've made a decision, then it's important to just go.

  The other thing that comes with grading, or any other tedious job, it's so easy to procrastinate. Something that one can do is to set up little checkpoints. Like make a bunch of small piles of 5-10, and then make it through a pile before you can go do something.

  Now, breaking it up like that can make it so that you are in drastically different moods, compromising fairness. But grading is just not fair. So we have to be honest with ourselves that the system is not ideal. But we do want to do the best for our students. And it's better to do the best for our students to learn than to try to be fair.

- Rob has been feeling very conflicted recently about what the goal is in teaching. What is his job?

  Turn it around. When you were a calculus student, what did you want your teacher or TA to do?

- Job apps:

  Deadlines come Nov 1, Nov 15, Dec 1, Dec 15... Interviews happen at the Joint meetings, beginning of January.

  When do we ask for letter? It takes about 2-4 hours. It takes reading through papers to that they can comment on them. If we average out 3 hours, with thirty letters to do... a month is needed.

  Make sure you get your application pulled at the universities in which you are very interested. Email someone, give a talk, whatnot.