1. (**10 Points**) Consider the matrix $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$. It has eigenvalues $\lambda_1 = -2$ and $\lambda_2 = 5$ with corresponding eigenvectors $\zeta_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\zeta_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$. Please find the solution to the initial value problem

$$\mathbf{x}'(t) = \mathbf{A}\,\mathbf{x}(t), \ \mathbf{x}(0) = \begin{pmatrix} 9 \\ 5 \end{pmatrix}.$$

Don't forget to show all of your work.

2. (**10 Points**) Consider the matrix $\mathbf{B} = \begin{pmatrix} 2 & 4 \\ -1 & 2 \end{pmatrix}$. solve the differential equation

$$\mathbf{x}'(t) = \mathbf{B}\,\mathbf{x}(t),$$

3. (**15 Points**) Consider the initial value problem

$$(x+3)y'' - (x+5)y' = 0,\ y(2) = 1,\ y'(2) = -1.$$

By the existence and uniqueness theorem for second-order linear ODEs, there is a solution $\phi(x)$ to this initial value problem on the interval $-3 < x < +\infty$. Furthermore, since $x_0 = 2$ is an ordinary point, there is an interval I containing x_0 on which $\phi(x)$ is analytic. Let $\sum_{n=0}^{\infty} a_n(x-2)^n$ be the power series expansion of $\phi(x)$ centered at x_0 on I. Find the coefficients a_0, a_1, a_2, a_3 and a_4. Please show all of your work.

In case you need it, here's more space for this problem.

4. (**15 Points**) Find the general solution to the non-homogeneous equation

$$\mathbf{x}'(t) = \mathbf{C}\,\mathbf{x}(t) + \begin{pmatrix} t \\ t+2 \end{pmatrix},$$

where $\mathbf{C}$ is a 2×2-matrix with real entries such that

$$\mathbf{D} = \mathbf{T}^{-1}\mathbf{C}\mathbf{T} = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$$

for $\mathbf{T} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

In case you need it, here's more space for this problem.

5. (**10 Points**)

(a) (**5 Points**) Find the eigenvalues of

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

(b) (**5 Points**) The 3×3 matrix

$$\mathbf{B} = \begin{pmatrix} 3 & \text{-}2 & 0 \\ \text{-}1 & 3 & \text{-}2 \\ 0 & \text{-}1 & 3 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$, and $\lambda_3 = 5$. Find an eigenvector of $\mathbf{B}$ corresponding to the eigenvalue $\lambda_2 = 3$.

6. (**10 Points**) The point x_0 is an ordinary point of the ODE

$$(2x-1)(x^2-1)y'' + xy' + y = 0.$$

Determine a lower bound for the radius of convergence of the series solution to this ODE centered at $x_0 = 0$.

7. (**15 Points**) Consider an Euler equation

$$x^2y'' - 5xy' + 9y = 0, x > 0.$$

(a) (**5 Points**) Find a fundamental set of solutions for this equation on $x > 0$.

(b) (**5 Points**) Compute the Wronskian of the fundamental set of solutions you found in the previous part and verify that it does not vanish on $x > 0$.

(c) (**2 Points**) Write the general solution to this equation.

(d) (**3 Points**) Solve the initial value problem with $y(1) = 2$ and $y'(1) = 1$

8. (**10 Points**) By letting $z = y'$, rewrite the following system of differential equations

$$\begin{aligned} x' &= \frac{1}{2}x - 3y - 3t \\ y'' &= y' + 5x + y - 2\sin(t) \end{aligned}$$

in the form of

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \\ g_3(t) \end{pmatrix},$$

where $\mathbf{A}$ is a 3×3 matrix.