Disclaimer: This was the Math 8 final exam from Fall 2000. The format of our exam will be as with the midterms, a couple of long problems followed by multiple choice.

1. A radioactive substance has a half-life of h. A sample of the substance is created at time 0. How much time must elapse until only three-fourths of the original amount is present?

2. Find the Taylor series of the function $f(x) = \ln x$ about the point $a = 1$ (i.e. expand $f(x)$ in powers of $(x-1)$). Express your answer in summation notation, so it is clear what the general term of the series is.

3. (a) Solve the initial value problem $xy' + y = 1/x^2$, $y(1) = 0$.

   (b) Determine coefficients a and b such that the functions $y = e^{2t}$ and $y = e^{-4t}$ are solutions to the differential equation $y'' + ay' + by = 0$.

4. Find an equation of the tangent plane to the surface $z^2 = 3x^2 + 6y^2$ at the point $(2, 2, -6)$.

5. Find and classify all critical points of the function $f(x, y) = x^3 + 6x^2 - y^2$. Be sure to justify all your work.

6. A skier is on the side of a mountain whose equation is $z = f(x, y) = x^3 + 6x^2 - y^2$. She is standing at the point $(-4, 1, f(-4, 1))$.

   (a) Suppose that she wishes to start downhill as steeply as possible. In what direction (in the horizontal $xy-$plane) should she point her skis? Make your answer a unit vector.

   (b) Suppose that she is standing at the point $(-4, 1, f(-4, 1))$, and that she wishes to climb uphill at an angle (of elevation) of 45 degrees ($\pi/4$ radians). In what direction (in the horizontal $xy-$plane) should she point her skis to achieve this? Make your answer a unit vector.

7. Consider a curve in three-dimensional space given by $\mathbf{r}(t)$. Suppose that for all times t, the velocity vector $\mathbf{r}'(t)$ is perpendicular to the position vector. Suppose also that $\mathbf{r}(0)$ is a unit vector, so that $\mathbf{r}(0)$ is the position vector of a point on the sphere of radius one centered at the origin. Prove that the entire curve lies on the sphere of radius one centered at the origin. [Hint: Recall that the sphere consists of all points of distance one from the origin, and consider the function $\mathbf{r} \bullet \mathbf{r}$]

8. Consider the surface S whose equation is $x^2 - yz = 1$. Find the point or points on S closest to the origin. (Hint: Use the square of the distance)