What Makes Mathematics Lessons Easy to Follow, Understand, and Remember?

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As a mathematics teacher at the high-school and college level for many years, I was always intrigued with the question of what makes an effective mathematics teacher. Thus, when I was to decide upon a topic for dissertation research in the domain of mathematics education, I chose to concentrate on a major component of teacher effectiveness: on lesson organization and clarity. A well-organized and clear presentation was defined in that study as a presentation that is easy to follow, understand, and remember. The following is a short description of the method and some findings of that study.

The study was conducted at Stanford University. During the three quarters of the academic year 1978–1979, I visited all 49 sections of a five-course sequence of calculus and analytic geometry taught by 38 teaching assistants. A questionnaire was administered in each class asking students to rate their teacher on lesson organization and clarity. All scores per teacher were averaged. Five teachers who rated very high and four teachers who rated very low were chosen to form two groups for the study.

Two lessons of each of these nine teachers were tape-recorded and transcribed. After analyzing the transcripts, about fifty teaching techniques were identified as contributing to lesson organization and clarity. These teaching techniques were sorted into three groups (not mutually exclusive), reflecting their contribution to making the presentation easy to follow, understand, or remember. Some strategies that belong to more than one group are listed under the group to which they are considered to contribute the most. Table 1 lists the teaching strategies as sorted into the three groups.

Thirty-nine teaching strategies which were related to teacher discourse were chosen to be scored for each transcribed lesson. Analyzing these scores, it was found that on 18 of these techniques, the differences between the two groups of teachers—those rated low and those rated high by their students—were statistically significant. These variables are also recorded in Table 1.

In a second analysis, several single variables that seemed to be closely related were combined into groups. In addition all variables belonging to the same major category were combined. Then, the differences between the two groups of teachers on these combined variables were examined. Results are summarized in Table 2.

Table 1.
What Makes Mathematics Lessons Clear and Organized?

Number if measured in the study	Variable Name	Number if measured in the study	Variable Name
	(1) What Makes a Lesson Easy to Follow?		**(2) What Makes a Lesson Easy to Understand?**
	1.1 Structuring		**2.1 Embedding**
	1.11 Three-division lesson structuring	17	2.11 General review
01*, 02	1.12 General overview, outline, objectives	18*	2.12 "Micro" reviews
03	1.13 Reference to the outline	19	2.13 References
04	1.14 "Micro" outline, overviews, objectives	20	2.14 Interrelations
	1.15 Reference to the "micro" outline	21	2.15 Comparisons/contrasts
05	1.16 Signaling transitions to and from points	22*	2.16 Sorting problems into categories
06	1.17 Emphasizing parts of a procedure	23*	2.17 Breaking down explanations
07	1.18 Clarifying the lesson structure	24*	2.18 Elaborations
08	1.19 Listing past or future topics	25	2.19 Framework for future topics
	1.2 Stimulating Interest		
09*	1.21 General motivation		**2.2 Sequencing**
10**	1.22 "Micro" motivation	26**	2.21 Sequence of topics
	1.23 Dramatization	27**	2.22 Avoiding incoherence
11	1.24 Related anecdotes or historical notes	28**	2.23 Avoiding errors
12	1.25 Questions		
	1.3 Good Oral Presentation		**2.3 Sensitivity to Students**
13**	1.31 Clear speech		
14*	1.32 Simple language		**2.4 Rationalizing**
15**	1.33 Fluency	29*	2.41 Rationalizing
	1.34 Enthusiastic lively presentation	30	2.42 Concluding sentences
	1.35 Humor, jokes		
	1.36 Pauses		**(3) What Makes a Lesson Easy to Remember?**
16**	1.37 Avoiding signals of hesitation/vagueness		
	1.4 Good Visual Presentation		**3.1 What to Remember**
	1.41 Physically dynamic presentation		
	1.42 Supporting gestures		**Emphasizing**
	1.43 Visual devices		3.11 Varying intonation
	1.44 Blackboard technique		3.12 Writing important points on the board
		31	3.13 Principal points
		32**	3.14 Repetitions
		33	3.15 Markers
		34	3.16 Other kinds of emphasis
			Summarizing
		35	3.16 General summary
		36	3.17 "Micro" summaries
			3.2 How to Remember
		37	3.21 Mnemonics
		38*	3.22 Titles
		39*	3.23 Algorithms

*Significant at the .05 level
**Significant at the .01 level

Table 2.

Name of Teaching Strategy	Combined Items as Numbered in Table 1	Significance
Structuring the lesson	1–3, 7–9	*
Good oral presentation	13–15	**
Microreviews and references	18, 19	*
Embedding	17–25	**
Incoherence, errors (neg. score)	27, 28	**
Emphasizing	31–34	**
How to remember	37–39	
Making the lesson easy to follow	1–16	**
Making the lesson easy to understand	17–30	**
Making the lesson easy to remember	31–39	**

*Significant at the .05 level
**Significant at the .01 level

Although not all the variables that were measured yielded differences that were statistically significant, the analysis showed that *the group of high-rated teachers used each of the identified positive teaching techniques more than the group of low-rated teachers, and each of the negative techniques less than the other group of teachers.* For this reason, all the teaching strategies listed in Table 1, not only the ones that yielded significant differences, will be discussed. Teaching strategies that are well known, e.g., providing overviews or reviews, will only be mentioned but less known strategies will be elaborated and illustrated by examples. All examples are excerpts taken from transcriptions of tape-recorded mathematics lessons.

What Makes a Lesson Easy to Follow?

Four major strategies are considered to contribute to making a lesson easy to follow: structuring the presentation, stimulating students' interest, good oral presentation, and good visual presentation.

1. Structuring

Structuring informs students about the content and organization of the material and its place in the structure of the general subject matter. In addition, it provides students who miss parts of the presentation (as a result of wandering thoughts or of concentrating on note taking) with frequent indications of the current development of the lesson, and with what parts of the lesson were missed.

Three-division lesson structuring: *introduction, body, and conclusion*; This concept can be described by the well-known dictum: "Tell students what you're going to tell them and why, tell them, and then tell them what you've just told them and why."

Outlining topics to be studied, listing objectives, general overview; The lesson objectives can be stated, written on the board, or listed in a handout. The last two approaches are preferred because the student can refer to them during the lesson.

Referring to the outline/objectives. The teacher who provides an outline or objectives at the beginning of a lesson refers to each objective in turn. This procedure helps students to identify where the teacher is at that time in reference to the general topic of the lesson.

Using micro-outlines and micro-objectives. The teacher states in advance what is going to be done in the next few minutes or what is expected as a result of doing it. For example: "Next we'll define a sequence of complex numbers. The definition is going to look exactly the same as for a sequence of real numbers, but the meaning is going to be a little different." "First of all, we should make a couple of definitions . . . ." "I'm not going to discuss . . . ."

Signaling transitions to and from main or subordinate points. This procedure alerts the students to the fact that the teacher has completed a point and is moving to the next one. The most commonly used words for this purpose are: "now," "well," "O.K.," "all right," "all right?," "so," "but," "let's." Examples for commonly used combinations of these words and of others are: "But now," "Let's see what happens . . . ," "Well, let's take for example . . . ," "Shall we move on?," "The next thing we want to know . . . ," "On the other hand . . . ."

Emphasizing parts of a procedure. The teacher emphasizes starting/completing of a part of a definition or a proof, e.g., "By this the proof has been completed;" The teacher may also refer to components/steps in a procedure, a definition, or a proof, e.g., "This was the definition . . . ," "So these are our assumptions . . . ," "Then, our conclusion is . . . ."

Listing the topics taught in the past or to be learned next. By this procedure, the student gets the knowledge of how the present topic is placed in the general structure of the general topic.

2. Stimulating Interest in the Topic Taught

Providing general motivation. The teacher provides motivation for the whole lesson or unit of study. Usually the motivation has to do with expected future use, with related applications, or with other factors of relevance to the students.

Providing micro-motivations. The teacher provides motivation for learning a segment of a topic, e.g., a specific theorem or formula. For example: "We will use this formula a lot when . . . ," "and this is going to turn out to be a very important way of representiang a complex function."

Telling related anecdotes or historical notes. Related anecdotes and historical notes make the topic interesting and provide "roots" of the developmental stages of the subject.

3. Good Oral Presentation

Clear speech. This consists of the following factors: loudness—enough to be heard from the back of the classroom; appropriate rate of speech—neither so rapid as to make following difficult nor so slow as to make listening boring; clear articulation —neither mumbling nor speaking in a husky or hoarse manner; variation—in loudness, rate, pitch, etc., to hold attention and give emphasis; fluency—smooth flow of speech.

Simple language. This consists of the use of familiar words, short sentences, conversational but not too informal language, and nonverbosity—avoiding use of unnecessary words.

Enthusiastic, lively presentation, dramatization, humor, jokes. These teaching strategies make the lesson interesting and attractive and alleviate students' tension resulting from trying to follow and understand what is being taught.

Pauses. These provide the students with breathing spaces to digest what was said and with opportunities to ask questions.

Avoiding signals of hesitation. This consists of avoiding the following features:

–Words marking hesitation—oh, eh, umm, ahh, ur.

–Redundant, repeated words or tangles of words—a kind of stammering—such as: "if we look at, at, at the theorem . . . ," "and the, and the, the derivative is . . . ."

–Vague expressions that do not belong to the explanation such as: "you know," "let's say."

–Vague words—using inappropriate words that may cause difficulties in understanding the meaning.

–False starts of sentences, halts in speech, e.g., "It equals to, it's got to be . . . ," "There might be, a function might look . . . ."

4. Good Visual Presentation

The use of both senses—sight and hearing—together, is very beneficial to learning because such use provides two exposures or two chances for student involvement.

Physically dynamic presentation. The teacher moves around, gestures in such a way as to support the explanations, maintains eye-contact with students, but avoids distracting movements or mannerisms.

Blackboard technique. Good blackboard technique consists of the following factors: handwriting which is legible and large enough so that it can be seen from the back of the classroom; good organization of the location of the writing so that the materials have continuity; aesthetically appealing graphs, drawings, and charts; complete and clean erasure of the board; and avoiding blocking the views of the students.

Visual devices: The teacher uses visual aids while teaching. These might be graphs, pictures, or tables drawn on the blackboard, and demonstration aids such as transparencies used on an overhead projector, filmstrips, videotapes, and microcomputers.

What Makes a Lesson Easy to Understand?

In addition to the strategies detailed above, there are four major strategies that facilitate understanding of the material taught. These are: embedding, sequencing, being sensitive to students, and rationalizing.

1. Embedding

Embedding consists of strategies for identifying pre-existing information stored in the learner's memory—information which is relevant to the new material being learned—and of strategies for connecting the new information with the identified related old information.

General review. A review of previously learned material is usually presented in the introduction and serves as a basis for the topic to be taught.

Micro-review. This refers to a review or to mentioning by name of a specific theorem, formula, etc., that is going to be used in presenting the next topic. A few examples of micro-reviews are: "Here you should use the formula $2(a + b) = 2a + 2b \ldots$," "$y$ is an explicit function of x; and when I say explicit, I mean y is a "function of x where x is the independent variable and y is the dependent variable." "We can use here the technique of implicit differentiation."

Interrelations. The teacher indicates the kinds of relationships between the new concepts and the old ones. For example: "The concept of similar triangles is sort of a generalization of congruent triangles," "Rolle's Theorem is a special case of the Mean Value Theorem," "This is a corollary of Newton's Rule."

The strategy of indicating interrelations also includes the identification of categories, that is, the indication of subject area to which a specific topic belongs or may be applied. For example: "This is a theorem of differential calculus," "The Chain Rule is the basic procedure for finding derivatives of composite functions."

Comparisons / contrasts. When solving a problem, proving a theorem, or teaching new material, the teacher compares or contrasts the new material with the old material. For example: "Now, the graph of cosine is very similar to that of the sine, except that it is shifted over a bit." "Unfortunately, the absolute value for complex numbers does not have the same meaning as for real numbers, so we have to go another way to prove this . . . ."

Elaborations. This is the teaching strategy of repeating an idea but not in the same words or at the same difficulty level. The repetition involves enrichment of the previous ideas or highlighting them from a different viewpoint, or presenting them at a higher level of difficulty. Elaborations also include the development of an explanation by small, logical, hierarchically sequential steps, e.g., the Socratic method by which the teacher provides guiding questions and elicits students' answers at each step of the development of the theory. For example: "Why is this function continuous? What kind of a function is it? Polynomial. And what do we know about polynomials? That they are everywhere continuous."

Stimulating imagery. The teacher invokes pre-existing knowledge of graphing principles or of plane and solid figures to promote perception of new concepts. For example: "The Mean Value Theorem is in fact Rolle's Theorem tilted on its side . . . ."

Providing a framework for future topics. Mentioning or discussing a topic before actually teaching it familiarizes the students with the topic and makes the topic more susceptible to learning. For example: "At the time that we'll consider the logarithmic function, we'll come across more problems with the differential percentage error and relative error that we have dealt with today." "We are now going to learn limits in a way which you'll understand very well. In the next calculus course you'll learn it again in a way that you won't understand very well."

2. Sequencing

The topics in a lesson or the ideas within one topic are arranged in an order on a

certain basis. The basis may be logical, hierarchical, topical, spatial, building to climax, inductive, deductive, etc. The teacher should make explicit the basis for the arrangement of ideas. Once this basis is understood, the sequence is easier to retain. The sequencing of topics should proceed from familiar to unfamiliar, from easy to more difficult, from concrete to abstract.

Coherence of topics, appropriate use of transitions. These procedures are important for smoothly connecting what is being taught. The coherent lecture goes in a smooth flow, without skipping intermediate steps or taking-off to irrelevant material. Proper use of transitions is necessary for smooth continuity from one statement to another and for clarification of the relationships between and relative emphasis of the statements.

Avoiding errors. Almost all teachers sometimes make errors in computation, in the use of formulas, in the development of proofs, in inaccurate statements, etc. It is desirable that the errors be promptly acknowledged and corrected and that their number be minimized.

3. Rationalizing Steps

Rationalizing. Teachers provide students with explanations of their actions, decisions, and selections of topics, problems, and strategies for solutions before or while they make specific moves in teaching. Teachers explain what they are doing, why they are doing it, and why they are doing it in this way and not in another way. For example: "We make these assumptions for historical reasons more than anything else." "I'm not going to give an analytic proof because you can find it in your textbook, and it's more important to understand the theorem in terms of the geometry—the graphical meaning."

Concluding statement(s). At the end of a point, the teacher states or explains the meaning/conclusion of what has been done so far. For example: "From all the above, you know that they are similar triangles," "OK, that's the amount of light expressed in terms of the variables," "This is what it means to converge."

4. Sensitivity to the Students

The teacher who is sensitive to the students:

- –is concerned about students' difficulties.
- –knows whether or not the class is understanding what is being said.
- –knows when students are bored or confused.
- –anticipates difficulties and prepares students beforehand.
- –encourages students to speak out during a lecture or discussion.
- –is sensitive to student's desire to ask a question.
- –quickly grasps what a student is asking or telling.
- –is careful and precise in answering questions.
- –relates class topics to students' lives and experiences.
- –teaches at a pace that enables most students to follow the material taught.

What Makes a Lesson Easy to Remember?

Teachers should help the students remember the material taught by identifying what to remember and by providing devices for facilitating retention. In addition to the strategies presented in the previous sections, teachers use for that purpose the following strategies:

1. Identifying What to Remember

From analysis of the transcriptions of the tape-recorded lessons in the present study, it was found that, on the average, the number of words per 50-minute-lesson is about 5000. This enormous amount of information given in such a short time cannot be fully understood and remembered by the students. Therefore, the teacher needs to make a selection of the most important things for the students remember and inform the students about these things by special emphasis or by summaries.

Emphasizing

This is mainly intended to motivate students to pay special attention to what is being said and to help them identify and remember the most important points of the message. Emphasis may also contribute to understanding by helping the students to keep the essential aspect in focus and to reduce the intrusion of the nonessential. The following are different methods that teachers employ to emphasize points:

Intonation: Raising the voice and varying intonation can alert the students to the relative importance of the material presented.

Writing key points on the board: The teacher writes on the board definitions, theorems, or major ideas.

Emphasizing principal points. The teacher identifies and explicitly emphasizes the main points, and the most significant information of the topic. For example: "Now, the important thing to remember is that . . . ," "These are two critical assumptions: . . . ," "Notice that . . . ," "Now, write this down . . . ."

Repetitions. The teacher repeats a sentence or an idea, either in the same words or in different words.

Markers. The teacher uses transitional sentences or phrases to emphasize topics for special qualifications. There are markers of importance, of difficulty, of interest, of usefulness, of beauty, etc. For example: "This is a very important application . . . ," "Pay attention to this critical fact . . . ," "It's easy to do it this way," "This is the beauty of the theorem."

Providing other kinds of emphasis. Sometimes teachers use special words and phrases for emphasis such as: "certainly . . . ," "we are sure that . . . ," "of course we know . . . ," "the problem is that . . . ," "we're interested in . . . ," "notice that . . . ," "we must show . . . ," "I would stress that . . . ," "you should remember . . . ," "I claim that . . . ," "I do want to state . . . ."

Summarizing

General summary. The teacher summarizes the main topics of the lesson. This is usually done at the end of the lesson.

Micro-summaries. The teacher states the rule, the theorem, the result of what was done up to here. Usually, the summaries include previously learned material. For example: "So let's review what we have discussed so far. We have talked about functions, and that every function has a domain, and a range. Now we should add a third thing . . . ."

2. Devising How to Remember

The teacher should provide students with devices for organizing the newly learned material in a structure that can be efficiently stored in memory to facilitate its retrieval.

Mnemonics. The teacher provides mnemonic devices to aid students remember certain concepts, formulas, theorems, and procedures. For example: RST for the three properties of equivalence—Reflective, Symmetric, Transitive; FOIL for the order of product of two binomials—First, Outer, Inner, Last.

Algorithms. Organizing a procedure as an algorithm—breaking it down into components that should be followed in the given order—helps students to store the procedure and retrieve it as one unit.

Categories. When the teacher considers different instances of the same topic, e.g., when providing several applications to some theory, the teacher explains in what way each instance/application differs from the others, and in what instances one should use each as a model for solution. This helps students identify what algorithm to retrieve in each case. For example: "If you have a variable in the numerator, use the Quotient Rule, but, if you have a constant at the top, just take the negative power and take the derivative. Don't use the Quotient Rule for this one. You can use it, but it'll take longer."

Titles. In using titles, the teacher assigns names to procedures and algorithms to serve as a direct access to the procedure storage location in memory (in order to call the procedure from memory). For example: using the Quadratic Formula, combining like terms, solving motion problems.

Summary and Conclusions

From the findings of the study described in this article we may conclude that teachers who are perceived by their students as very clear and organized do not teach the material in its merely simple form, but they say and do a lot of things "around" it, to make their presentation easy to follow, to understand, and to remember. Effective teachers present their lessons in a way that attracts students' attention and that facilitates students' abilities to follow what is being taught. They do this by structuring the material, by stimulating students' interests in what is being taught, and by providing good oral and good visual presentations. In order to help students understand and assimilate the new material, teachers should connect new materials with related old materials stored in the learner's memory, present the new material in a good sequence, explain what they are doing while teaching and why, and adjust their teaching to overcome students' difficulties in learning. Teachers should also help students retain the material taught by identifying what to remember using strategies of emphasizing and summarizing, and by providing devices for organizing the material in a structure that can be efficiently stored in memory and that can be easily retrieved from memory.

Implication for classroom teaching are obvious: When preparing a lesson, the teacher should plan for the various strategies of good teaching detailed above. While teaching, the instructor should get continuous feedback from the students and should be aware of all the techniques that make the presentation easy to follow, understand, and remember.