Your name:

Instructor (please circle): Zajj Daugherty Erik van Erp

Math 11 Fall 2011, Homework 8, due Wed Nov 16
Please show your work. No credit is given for solutions without justification.

(1) *Choose the correct answer. Show relevant work (it will not be graded).*

(a) Let C_1 be the oriented line segment parametrized as $\mathbf{r}(t) = \langle t, 2t, 5t\rangle$, $0 \le t \le 4$. Suppose we have a function $f(x,y,z)$ for which $\int_{C_1} f(x,y,z)ds = 7$, and a vector field $\mathbf{F}(x,y,z)$ for which $\int_{C_1} \mathbf{F}\cdot d\mathbf{s} = 4$.
Now let C_2 be the oriented curve with $\mathbf{r}(t) = \langle t^2, 2t^2, 5t^2\rangle$, $-2 \le t \le 0$. Which one of the following statements is true for the integrals along C_2?

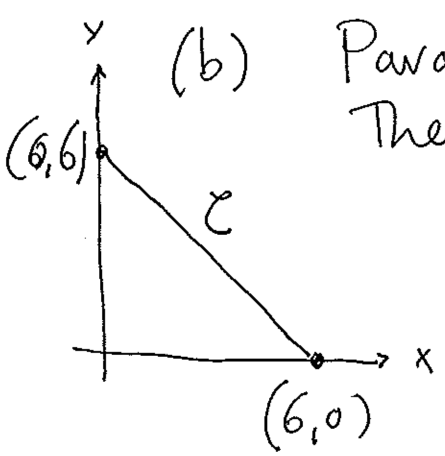
(A) $\int_{C_2} f(x,y,z)ds = 7$ and $\int_{C_2} \mathbf{F}\cdot d\mathbf{s} = 4$.
(B) $\int_{C_2} f(x,y,z)ds = -7$ and $\int_{C_2} \mathbf{F}\cdot d\mathbf{s} = 4$.
(C) $\int_{C_2} f(x,y,z)ds = 7$ and $\int_{C_2} \mathbf{F}\cdot d\mathbf{s} = -4$.
(D) $\int_{C_2} f(x,y,z)ds = -7$ and $\int_{C_2} \mathbf{F}\cdot d\mathbf{s} = -4$.
(E) None of the above.

(b) If C is the straight line from point $(6,0)$ to $(0,6)$, then what is $\int_C x^2 + y^2 ds$?

(A) 72 (B) $72\sqrt{(2)}$ (C) 144 (D) $144\sqrt{2}$ (E) 212 (F) $212\sqrt{2}$

(1)(a) Curve C_1 and C_2 are the same line segment, but C_1 is oriented from $(0,0,0)$ to $(4,8,20)$ while C_2 is oriented from $(4,8,20)$ to $(0,0,0)$. Therefore $\int_C f\,ds$ is the same for both curves (a scalar line integral is independent of orientation) while $\int_C \vec{F}\cdot d\vec{s}$ changes sign.

$\boxed{C}$



(b) Parametrize $\mathcal{C}$ as $\bar{r}(t) = \langle t, 6-t\rangle$, $0 \le t \le 6$
Then $\bar{r}'(t) = \langle 1, -1\rangle$, $\|r'(t)\| = \sqrt{2}$.

$$\int_C x^2+y^2\,ds = \int_0^6 t^2 + (6-t)^2 \cancel{dt}\sqrt{2}\,dt$$
$$= \frac{1}{3}t^3 - \frac{1}{3}(6-t)^3 \Big|_0^6 \sqrt{2}$$
$$= 144\sqrt{2}$$

$\boxed{D}$

(2) Let C be the helix parametrized as $\mathbf{r}(t) = \langle \sin t, t, \cos t \rangle$ for $0 \le t \le \frac{1}{2}\pi$, and let $\mathbf{F}(x,y,z)$ be the vector field

$$\mathbf{F}(x,y,z) = \langle z - y\sin(xy), -x\sin(xy), x \rangle$$

(a) Verify that the vector field $\mathbf{F}$ satisfies the cross-partials test.
(b) Is the cross partials test sufficient in this case to conclude that $\mathbf{F}$ must be conservative? Explain your answer.
(c) Find an explicit potential function $f(x,y,z)$ with $\mathbf{F} = \nabla f$.
(d) Evaluate the vector line integral $\int_C \mathbf{F} \cdot ds$.

(2)(a) Direct calculation shows that

$$\frac{\partial F_2}{\partial x} = -\sin(xy) - xy\cos(xy) = \frac{\partial F_1}{\partial y}$$

$$\frac{\partial F_3}{\partial y} = 0 = \frac{\partial F_2}{\partial z}$$

All cross-partials are equal!

$$\frac{\partial F_1}{\partial z} = 1 = \frac{\partial F_3}{\partial x}$$

(b) The domain of the vector field $\vec{F}(x,y,z)$ is all of $\mathbb{R}^3$. Therefore there are no "holes" and the cross-partials test suffices to conclude that $\vec{F}$ is conservative.

(c) The potential function is

$$f(x,y,z) = xz + \cos(xy)$$

(or $xz + \cos(xy) + C$ in general)

You can find $f(x,y,z)$ by inspection and "guessing". In that case show that the partial derivatives are correct, i.e. $\vec{F} = \nabla f$:

$$\frac{\partial f}{\partial x} = z - y\sin(xy) = F_1$$

$$\frac{\partial f}{\partial y} = -x\sin(xy) = F_2$$

$$\frac{\partial f}{\partial z} = x = F_3$$

Alternatively, you can find $f(x,y,z)$ systematically as follows.

i) $f_x = z - y\sin(xy)$ from F_1.

$f = \int z - y\sin(xy)\,dx = xz + \cos(xy) + g(y,z)$

ii) from i) get $f_y = -x\sin(xy) + g_y$.

From $F_2 = -x\sin(xy)$ get $g_y = 0$.

Then $g(y,z) = h(z)$, i.e., $g(y,z)$ does not depend on y, but only on z.

iii) Now $f = xz + \cos(xy) + h(z)$

$f_z = x + h'(z)$.

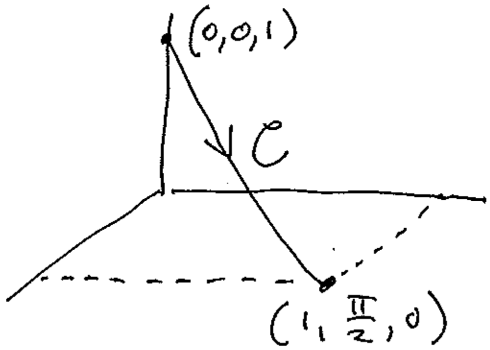
From $F_3 = x$ we see $h'(z) = 0$.

Therefore $h(z) = C$ is a constant and

$$f(x,y,z) = xz + \cos(xy) + C.$$

You can choose any constant C to get a potential function.

(d) The curve C with $\bar{r}(t) = \langle \sin t, t, \cos t \rangle$ starts at $t = 0$, i.e., at $\langle 0, 0, 1 \rangle$ and ends at $t = \frac{1}{2}\pi$, i.e. at point $\langle 1, \frac{\pi}{2}, 0 \rangle$



The shape of the curve is irrelevant because

$$\int_C \vec{F} \cdot d\bar{s} = f(1, \tfrac{\pi}{2}, 0) - f(0,0,1)$$

Use the potential function $f(x, y, z)$ to calculate

$$f(1, \tfrac{\pi}{2}, 0) = \cos \tfrac{\pi}{2} = 0$$

$$f(0,0,1) = \cos 0 = 1$$

$$\int_C \vec{F} \cdot d\bar{s} = 0 - 1 = -1.$$

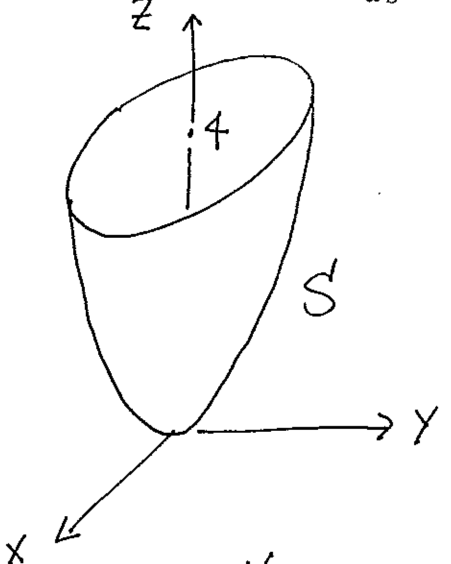
(3) The surface $\mathbf{S}$ is the part of the elliptic paraboloid $z = 2x^2 + 3y^2$ below the plane $z = 4$.

(a) Give a parametrization for the surface S. Call your parameters u and v. Indicate the domain of the parametrization, i.e., the range of parameter values of u, v that corresponds to S.

(b) Find the tangent vectors $\mathbf{T}_u$, $\mathbf{T}_v$ to the grid lines, and the normal vector $\mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v$.

(c) Derive a formula for the length $\|\mathbf{n}\|$.

(d) Set up an iterated integral $\iint \ldots dudv$ (or $dvdu$) that corresponds to the surface integral $\iint_S x + y + z\, dS$. (Do not try to evaluate the integral.)



(a) The standard parametrization for the graph of $z = 2x^2 + 3y^2$ is

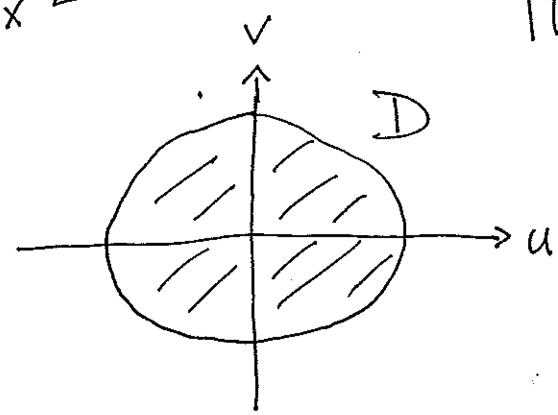
$$x = u$$
$$y = v$$
$$z = 2u^2 + 3v^2$$

$$G(u,v) = (u, v, 2u^2 + 3v^2)$$

The domain D of the parameters is

$$2u^2 + 3v^2 \leqq 4$$

(the interior of the ellipse $2u^2 + 3v^2 = 4$).



(b) $\vec{T}_u = \langle 1, 0, 4u \rangle$, $\vec{T}_v = \langle 0, 1, 6v \rangle$

$$\vec{n} = \vec{T}_u \times \vec{T}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 4u \\ 0 & 1 & 6v \end{vmatrix} = \langle -4u, -6v, 1 \rangle$$

Can use standard formula $\vec{n} = \langle -g_u, -g_v, 1 \rangle$ with $g(u,v) = 2u^2 + 3v^2$.

(c) $\|\vec{n}\| = \sqrt{16u^2 + 36v^2 + 1}$.

(d) • The parameter domain D is described as

$$-\sqrt{2} \leq u \leq \sqrt{2}$$

$$-\sqrt{\frac{4-2u^2}{3}} \leq v \leq \sqrt{\frac{4-2u^2}{3}}$$

• The function $x+y+z$ becomes

$$u + v + 2u^2 + 3v^2$$

• The surface element dS is

$$dS = \|\vec{n}\|\, du\, dv = \sqrt{16u^2 + 36v^2 + 1}\, du\, dv$$

Putting this together:

$$\int_{u=-\sqrt{2}}^{\sqrt{2}} \int_{v=-\sqrt{\frac{4-2u^2}{3}}}^{\sqrt{\frac{4-2u^2}{3}}} (u+v+2u^2+3v^2)\sqrt{16u^2+36v^2+1}\, du\, dv.$$