Barnett
6/5/09.

SOLUTIONS

Math 46, Applied Math (Spring 2009): Final

Have fun & good luck!

3 hours, 80 points total, 9 questions worth varying numbers of points

1. [8 points] Find an approximate solution to the following initial-value problem which is uniformly valid on $t > 0$ as $\varepsilon \to 0$, where $0 < \varepsilon \ll 1$ is a perturbation parameter.

$$\varepsilon y'' + 2ty' + ty = 0, \qquad y(0) = 2, \qquad \sqrt{\varepsilon}y'(0) = 1$$

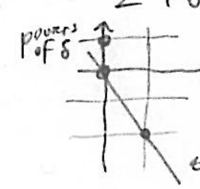
(Be sure to present your answer purely in terms of the variables in the problem, and in a form without any integrals)

singularly-perturbed

This is an initial-layer problem, so we can solve inner layer completely first, then do outer layer.

Rescale time $\tau = \frac{t}{\delta}$ so ODE is $\frac{\varepsilon}{\delta^2}Y'' + 2\underbrace{\tau\delta}_{t}\frac{1}{\delta}Y' + \tau\delta Y = 0$

we need to dominant balance 1st two terms, and find the 3rd term is smaller.



so $\delta\varepsilon^{-2} = 1$ ie $\delta = \sqrt{\varepsilon}$

Sub. for δ: $\cancel{\frac{\varepsilon}{\varepsilon}}Y'' + 2\tau Y' + \cancel{\sqrt{\varepsilon}\tau Y} = 0$ vanish at leading order.

sub. $V = Y'$ then $V' + 2\tau V = 0$ ie $\frac{V}{V'} = -2\tau$

$\Rightarrow V(\tau) = Ae^{-\tau^2}$ $\Rightarrow Y(\tau) = A\int e^{-\tau^2}d\tau + B$
$= C\,\mathrm{erf}(\tau) + B$ (see defn. of erf).

Fit for C, B: $\mathrm{erf}(0) = 0$ so $Y(0) = B = 2$

derivative IC is: $\frac{\sqrt{\varepsilon}}{\delta}Y'(0) = 1$, with $\frac{\sqrt{\varepsilon}}{\delta} = 1$

$\mathrm{erf}'(0) = \frac{2}{\sqrt{\pi}}e^{-0^2} = \frac{2}{\sqrt{\pi}}$ so $Y'(0) = \frac{2}{\sqrt{\pi}}C = 1$ so $C = \frac{\sqrt{\pi}}{2}$

$\Rightarrow$ inner soln is $Y(\tau) = \frac{\sqrt{\pi}}{2}\mathrm{erf}(\tau) + 2$

Common limit $c_m = \lim_{\tau\to\infty} Y(\tau) = \frac{\sqrt{\pi}}{2} + 2$ (since $\mathrm{erf}(+\infty) = 1$)

Outer layer: $2\cancel{t}y_0' + \cancel{t}y_0 = 0$ so $y_0(t) = Ae^{-t/2}$ with $c_m = \lim_{t\to 0} y_0(t) = A$

Unif. approx $y_u(t) = y_0(t) + y_i(t) - c_{m1} = \left(\frac{\sqrt{\pi}}{2}+2\right)\left(e^{-t/2} - 1\right) + \frac{\sqrt{\pi}}{2}\mathrm{erf}\left(\frac{t}{\sqrt{\varepsilon}}\right) + 2$
$= \left(\frac{\sqrt{\pi}}{2}+2\right)e^{-t/2} + \frac{\sqrt{\pi}}{2}\left(\mathrm{erf}\left(\frac{t}{\sqrt{\varepsilon}}\right) - 1\right)$

$k(x)^2$ so $k(x) = 1 + \sin x$

2. [9 points] Consider the Dirichlet eigenvalue problem on $0 < x < \pi$,

$$y'' = \lambda \overbrace{(1 + \sin x)^2}^{k(x)^2} y, \qquad y(0) = y(\pi) = 0$$

(a) Prove that eigenvalues have a definite sign (which?)

Energy method: mult. by y & integrate over interval:

$$\underbrace{\int_0^\pi y y'' \, dx}_{-\int_0^\pi (y')^2 dx \; + \; [yy']_0^\pi} = \lambda \underbrace{\int_0^\pi (1+\sin x)^2 y^2 \, dx}_{>0}$$

>0 since integrand nonnegative and $y \equiv 0$ is not an eigenfunction.

$-\int_0^\pi (y')^2 dx$ is <0 since $y' \equiv 0$ & BCs would have $y \equiv 0$. $[yy']_0^\pi = 0$ by Dirichlet BCs.

so $\lambda = \dfrac{\text{negative}}{\text{positive}}$, $\boxed{\lambda < 0}$

(b) Find WKB approximations to the nth eigenvalue and corresponding eigenfunction.

Standard form $\boxed{\frac{-1}{\lambda}} y'' + k(x)^2 y = 0$, with $\frac{-1}{\lambda} = \varepsilon^2$.

$$y_{WKB}(x) = \frac{A}{\sqrt{k(x)}} \sin \frac{1}{\varepsilon} \int_0^x k(s)\,ds + \frac{B}{\sqrt{k(x)}} \cos \frac{1}{\varepsilon} \int_0^x k(s)\,ds.$$

(lower lim.)

BCs at $x = 0$: have $y_{WKB}(0) = A \cdot 0 + B \cdot 1$ so $B = 0$.

$x = \pi$: need $\frac{1}{\varepsilon} \int_0^\pi k(s)\,ds = n\pi$ so $\sin(\cdot)$ vanishes $(n \in \mathbb{N})$

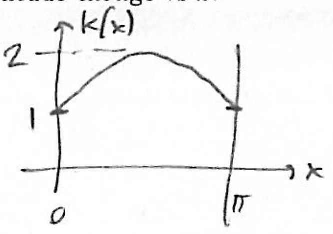
$$\Rightarrow \frac{1}{\varepsilon} = \frac{n\pi}{\int_0^\pi (1 + \sin s)\,ds} = \frac{n\pi}{\pi + 2}$$

$$\Rightarrow \lambda_n = -\frac{1}{\varepsilon_n^2} = -\frac{n^2\pi^2}{(\pi+2)^2}$$

$$y_n(x) = \frac{1}{\sqrt{1 + \sin x}} \sin\left(\frac{n\pi}{\pi+2} \overbrace{(x - \cos x + 1)}^{\int_0^x k(s)ds \text{ definite integral.}}\right)$$

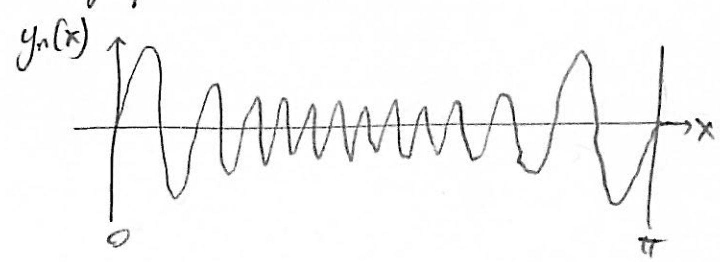
(c) Sketch an eigenfunction with very large eigenvalue magnitude, showing how frequency and amplitude change vs x.

"freq. function":



slow fast slow

so eigenfunction $n \gg 1$



This was a hard question, beyond usual difficulty, needing skill.

3. [9 points] Spread of pollutant concentration $u(\mathbf{x},t)$ in an initially clean body of water $\Omega \subset \mathbb{R}^3$ obeys

$$u_t - \Delta u = f(\mathbf{x}), \quad \mathbf{x} \in \Omega, t > 0, \qquad \alpha u + \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \qquad u(\mathbf{x},0) = 0, \quad \mathbf{x} \in \Omega$$

where f is the pollution source term, and $\alpha > 0$ a boundary absorption constant.

3

(a) Prove that a *steady-state* (time-independent) solution $u(\mathbf{x})$ to the PDE with given boundary conditions is unique. [Hint: set the t-derivative to zero]

Steady state ignores ICs & has $u_t = 0$ $\Rightarrow$ $\begin{cases} -\Delta u = f & \text{in } \Omega \\ \alpha u + \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$

just as with an ODE.

"Poisson Eqn w/ Robin BCs"

Say u_1 & u_2 are solns, then w obeys homog: $\Delta w = 0$ in Ω & same BCs as u.

Mult. by w & integrate:

$$0 \overset{PDE}{=} \int_\Omega w \underbrace{\Delta w}_{=0}\, dx \overset{\text{Green's 1st Identity}}{=} -\int_\Omega |\vec{\nabla} w|^2 dx + \underbrace{\int_{\partial\Omega} w \underbrace{\frac{\partial w}{\partial n}}_{= -\alpha w \text{ by BCs.}} ds}_{= -\int_{\partial\Omega} w^2 ds}$$

Two nonpositive quantities sum to 0 $\Rightarrow$ both are zero.

$\Rightarrow$ $\vec{\nabla} w \equiv \vec{0}$ in Ω, so $w = \text{const}$, but $\int_{\partial\Omega} w^2 ds = 0$ so $w \equiv 0$ in $\Omega \cup \partial\Omega$.

$\Rightarrow$ $u_1 \equiv u_2$ in Ω unique.

4

(b) Prove that the time-dependent solution to the full equations above is unique

Apply same idea: $w = u_1 - u_2$ $\qquad E(t) := \int_\Omega w(x,t)^2 dx$ $\qquad$ w sat. $w_t = \Delta w$ (homog. heat eqn).

$$\text{So } E'(t) = 2\int_\Omega w w_t\, dx \overset{PDE}{=} 2\int_\Omega w \Delta w\, dx = -2\int_\Omega |\vec{\nabla} w|^2 dx - 2\int_{\partial\Omega} w^2 ds$$

reusing above results; both terms nonpositive.

So $E'(t) \le 0$

but $E(t) \geq 0$ by definition, and $E(0) = \int_\Omega 0^2 dx = 0$ for w's homog. IC.

3 facts imply $E(t) = 0$ $\forall t \geq 0$, thus $w \equiv 0$ in Ω, $t > 0$

$\Rightarrow$ $u_1 = u_2$ unique.

[BONUS: $\int_\Omega u\cancel{\Delta u} + \int_\Omega |\nabla u|^2 \overset{G1I}{=} \int_{\partial\Omega} u u_n = -\alpha \int_{\partial\Omega} u^2$

so $\alpha = \frac{\text{pos.}}{\text{neg.}}$ $\quad \alpha < 0$]

[Advanced fact: $u|_{\partial\Omega}$ is eigenfunction of "Dirichlet–Neumann map" for interior Laplace BVP!]

(c) The homogeneous steady-state case of the above is called a Stekloff eigenvalue problem with α as the eigenvalue:

2

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \alpha u + \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

unusual but go with it.

Prove that eigenfunctions from different eigenspaces are *orthogonal on the boundary*. [BONUS: prove α has a definite sign]

unusual in that eigenvalue appears in BC, not in PDE!

u eigfunc : $\Delta u = 0$, $\alpha u + \frac{\partial u}{\partial n} = 0$

v " : $\Delta v = 0$, $\beta v + \frac{\partial v}{\partial n} = 0$

subtract as eqns :

So $\int_\Omega u\cancel{\Delta v} + \int_\Omega \vec{\nabla} u \cdot \vec{\nabla} v \overset{G1I}{=} \int_{\partial\Omega} u v_n = -\beta \int_{\partial\Omega} uv$

$\int_\Omega v\cancel{\Delta u} + \int_\Omega \vec{\nabla} v \cdot \vec{\nabla} u = \int_{\partial\Omega} v u_n = -\alpha \int_{\partial\Omega} vu$

$(\alpha-\beta)\int_{\partial\Omega} uv = 0$

So if $\alpha \neq \beta$, $\int_{\partial\Omega} uv \, ds = 0$, orthog on $\partial\Omega$.

4. [7 points] In 1940 the Russian applied mathematician A. Kolmogorov assumed there was a law for turbulent fluid flow relating the four quantities: l (length), E (energy, units of ML^2T^{-2}), ρ (density, mass per unit volume), and R (dissipation rate, energy per unit time per unit volume). Using this assumption and the Buckingham Pi Theorem, state the simple form the law must have. Show that there is a (famous!) scaling relation $E = \text{const} \cdot l^\alpha$ when other parameters are held constant; give α.

Dimensions matrix:

$$\begin{array}{c|cccc} & l & E & \rho & R \\ \hline M & & 1 & 1 & 1 \\ L & 1 & 2 & -3 & -1 \\ T & & -2 & & -3 \end{array}$$

cancels T powers, leaving $\begin{bmatrix}1\\8\\0\end{bmatrix}$

cancels M, leaving $\begin{bmatrix}0\\11\\0\end{bmatrix}$

Dimensionless quantity $\Pi_1 = \frac{E^3}{R^2} \cdot \frac{1}{\rho} \cdot \frac{1}{l^{11}}$ ← unlikely-sounding but true.

Buck. Pi Thm says $f(\Pi_1) = \text{const}$ ie $\Pi_1 = \text{const}$.

$$\Rightarrow \frac{E^3}{R^2 \rho l^{11}} = c$$

$$\Rightarrow E = c(R^2\rho l^{11})^{1/3} = \text{const} \cdot l^{11/3}$$

when R, ρ fixed.

ie $\alpha = 11/3$

5. [9 points] Bacterial evolution for times $t > 0$ can be modelled by the 1D reaction-diffusion equation in $x \in \mathbb{R}$,

$$u_t = u_{xx} + \alpha u, \qquad u(x,0) = f(x)$$

where α is a breeding/death rate constant.

7 (a) Use the Fourier transform method to write a general solution $u(x,t)$ for $t > 0$ in terms of the initial condition f and α.

$$u_t = u_{xx} + \alpha u \xrightarrow{\mathcal{F} \text{ in } x} \hat{u}_t(\xi,t) = (-i\xi)^2 \hat{u}(\xi,t) + \alpha \hat{u}(\xi,t)$$
$$= (\alpha - \xi^2)\,\hat{u}(\xi,t)$$

Solve as an ODE in t, for ξ fixed: (it's 1st-order linear, const-coeff, simple)

$$\hat{u}(\xi,t) = \hat{c}(\xi) \cdot e^{(\alpha-\xi^2)t}$$

note integration 'const' can still vary w/ ξ.

$$= \underbrace{e^{\alpha t}}_{\text{indep. of } \xi} \cdot \underbrace{\hat{c}(\xi)}\,\underbrace{e^{-\xi^2 t}}$$

(doesn't affect $\mathcal{F}^{-1}$ in ξ).

$\leftarrow$ setting $t=0$, $\hat{f}(\xi) = \hat{c}(\xi)$.

$\Rightarrow$ product in ξ-space $\Rightarrow$ convolution of their inverse FT's in x-space.

Using FT table w/ $\frac{1}{4a} = t$: $\quad e^{-\frac{x^2}{4t}} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} \sqrt{4\pi t}\, e^{-\xi^2 t}$

$$\Rightarrow u(x,t) = e^{\alpha t} \cdot \left(f * \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}\right)(x)$$

$$= e^{\alpha t} \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} f(y)\, dy$$

notice breeding term simply appeared as overall exponential prefactor; the rest is plain old heat eqn on $\mathbb{R}$.

2 (b) Fix $\alpha > 0$, i.e. positive breeding. What range of spatial frequencies ξ in the initial condition lead to exponential *growth* vs t (unstable as opposed to stable behavior)? $\rightarrow$ discussed in §6.6.1 of book, by the way.

looking at Fourier space solution $\hat{u}(\xi,t) = \hat{f}(\xi) e^{(\alpha-\xi^2)t}$

we see Fourier modes $e^{-i\xi x}$ in the solution will grow if $\alpha - \xi^2 > 0$

ie $|\xi|^2 < \alpha$

Ie low spatial frequencies $|\xi| < \sqrt{\alpha}$ grow; higher frequencies decay.

6. [7 points] Solve the following integral equation by converting to an ODE then solving (don't forget the boundary/initial conditions):

it's Volterra

6

$$u(t) + \int_0^t (t-s)u(s)ds = t^2, \qquad t > 0$$

$\frac{d}{dt}$ (use Leibniz's rule for derivatives

$$u'(t) + \cancel{(t-t)u(t)} + \int_0^t 1 \cdot u(s)\,ds = 2t$$

d/dt (

$$u''(t) + u(t) = 2 \qquad \text{ODE}$$

ICs: orig. Volterra w/ $t=0$ gives $u(0) + \int_0^0 (\text{bounded})\,ds = 0$ so $u(0)=0$

1st deriv. eqn w/ $t=0$ " $u'(0) + \int_0^0 \cdots ds = 0$ so $u'(0)=0$

Gen. soln. of ODE: $r^2+1=0$ so $r=\pm i$, osc. w/ freq. 1.

$$u(t) = \underbrace{A\sin t + B\cos t}_{\text{gen. homog. soln}} + 2$$

u_p, a particular soln. (can get via Meth. Und. Coeffs, or just guess it here).

match ICs: $u(0) = B + 2 = 0$

$B=-2$

$u'(0) = A = 0$

so $u(t) = 2(1-\cos t)$ you can check this satisfies the Volterra IE.

Must this solution be unique on each interval $0 < t < T$? If not, characterize the non-uniqueness, or, if so, explain what theorem proves your claim.

1

Yes, soln. is unique on any bounded interval $(0,T)$, by the Thm. for Volterra IEs. It's proof uses uniform convergence of the Neumann series $u = (I-\lambda K)f = f + \lambda Kf + \lambda^2 K^2 f + \cdots$

$\int p(x) \equiv 1$ in SLP form.

7. [10 points] Consider the Sturm-Liouville operator $Au := -u'' - \frac{1}{4}u$ on $[0, \pi]$ with Neumann boundary conditions $u'(0) = u'(\pi) = 0$.

5 (a) Find the set of eigenfunctions and corresponding eigenvalues of A. (If you label by n, be sure to state whether counting starts at $n = 0$ or $n = 1$, etc)

$\lambda < -1/4$ gives decay (no eigvals)

$\lambda = 1/4$ gives linear.

$\lambda > -1/4$ gives oscillation

both possible.

$Au = \lambda u$ so $-u'' - (\frac{1}{4} + \lambda)u = 0$

$\lambda > -1/4$ has gen. soln. $r^2 + \frac{1}{4} + \lambda = 0$ so $r = \pm i\sqrt{1/4 + \lambda}$

$u(x) = A\sin\sqrt{1/4+\lambda}\,x + B\cos\sqrt{1/4+\lambda}\,x$

Neumann BCs : $u'(0) = A\cdot\sqrt{1/4+\lambda}\cos 0 + B\cdot 0$ so $A = 0$.

$u'(\pi) = 0 = -B\sqrt{1/4+\lambda}\,\underbrace{\sin\sqrt{1/4+\lambda}\,\pi}$ zero if $1/4 + \lambda = n^2$

$\Rightarrow \lambda_n = n^2 - 1/4$

$n = 1, 2, \ldots$

But also if $\lambda = -1/4$, $-u'' = 0$ so $u = Ax + B$ & $u(x) \equiv 1$ is Neumann eigfunc.

$\Rightarrow \quad \lambda_n = n^2 - 1/4, \quad u_n(x) = \cos(\sqrt{\frac{1}{4}+\lambda}\,x), \quad n = \underset{\text{covers above case}}{0}, 1, 2, \ldots$

4 (b) Does the equation $Au = f$ with the above boundary conditions have a Green's function? If so, find an expression for it; if not, explain in detail why not.

Green's func. exist if A w/ BCs has no zero eigval, which is true by above calc.

Construct $g(x, \xi)$: $Au = 0$ has soln. gen. $u(x) = a\sin\frac{x}{2} + b\cos\frac{x}{2}$

so $u_1(x) = \cos\frac{x}{2}$ obeys left-hand BC.

$u_2(x) = \sin\frac{x}{2}$ " right-hand BC $(u_2'(\pi) = \cos\frac{\pi}{2} = 0)$

$W = u_1u_2' - u_2u_1' = \frac{1}{2}\cos^2 x/2 - \frac{1}{2}\sin x/2(-\sin x/2) = 1/2 \quad \forall x$

$$g(x, \xi) = \begin{cases} -2\cos\frac{x}{2}\sin\xi/2, & x < \xi \\ -2\sin x/2\cos\xi/2, & x > \xi \end{cases}$$

check $\left.\frac{\partial}{\partial x} g(x,\xi)\right|_{x=0} = 0$ obeys left BC.

1 (c) Use the Green's function, or if not possible, another ODE solution method, to write an explicit formula for the solution $u(x)$ to $Au = f$ with the above boundary conditions, in terms of a general driving $f(x)$.

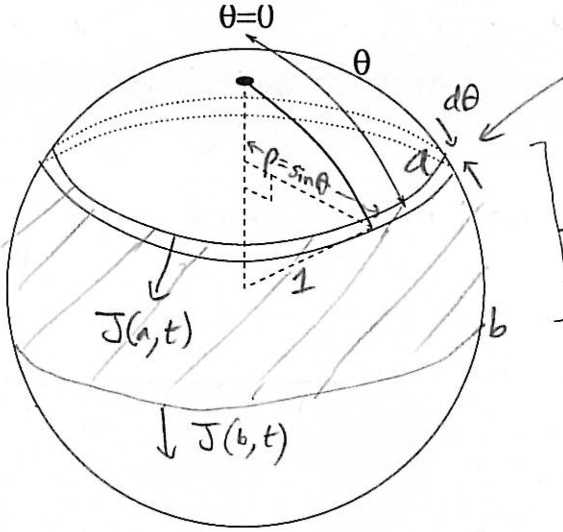
$$u(x) = (L^{-1}f)(x) = \int_0^\pi g(x,\xi)f(\xi)d\xi = -2\int_0^x \sin x/2\cos\xi/2\, f(\xi)d\xi - 2\int_x^\pi \cos x/2\sin\xi/2\, f(\xi)d\xi$$

(d) [BONUS] What is the spectrum of the Green's operator $Gu(x) = \int_0^\pi g(x,\xi)u(\xi)d\xi$, or the solution operator you used above?

$$G = L^{-1} \quad \text{so} \quad \text{eigvals}(G) = \frac{1}{\lambda_n} = \frac{1}{-1/4 + n^2}, \quad n = 0, 1, \ldots$$

8. [7 points] Use the conservation law approach to derive the heat equation on the *surface* of the unit sphere for temperature distributions $u(\theta, t)$ which depend only on polar angle $0 < \theta < \pi$ as shown (and not on longitude), and on time t. As usual you may use Fick's Law that flux is $-k$ times the gradient of u. [Hint: remember you are working on a surface not in a volume. The diagram shows that the radius of the circle at polar angle θ is $\sin\theta$.]

This is analogous to 2d polar case, #5 p.365.



area element $da = \underbrace{2\pi \overbrace{\sin\theta}^{\text{radius}}}_{\text{circle perimeter}} d\theta$

circle perimeter: plays role of $2\pi r$ in the standard planar polar version (#5 p.365)

'mass' in this region

$$= \int_a^b u(\theta,t)\, da = 2\pi \int_a^b u(\theta,t)\sin\theta\, d\theta$$

Conservation Law

$$\frac{d}{dt}\cancel{2\pi}\int_a^b u(\theta,t)\sin\theta\, d\theta = \underbrace{\cancel{2\pi}\sin a\, J(a,t) - \cancel{2\pi}\sin b\, J(b,t)}$$

$$- \int_a^b \left(\sin\theta\, J(\theta,t)\right)_\theta d\theta \qquad \text{by Fund. Thm. of Calc.}$$

true for all a, b so integrands equal:

$$u_t \sin\theta = -\left(\sin\theta\, J\right)_\theta \qquad \text{use } J = -k u_\theta \text{ Fick.}$$

$$\Rightarrow \quad u_t = k\frac{1}{\sin\theta}\left(\sin\theta\, u_\theta\right)_\theta = k\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)$$

since heat eqn is $u_t = k\Delta u$ this tells us $\Delta u = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)$ r^2: (middle term in eqn bottom of p.369)

[BONUS: find the general form of a solution to Laplace's equation on this sphere with the above symmetry] $\Delta u = 0$

integrate up twice: $0 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)$ so $\sin\theta\frac{\partial u}{\partial\theta} = c$

$$\rightarrow \quad u(\theta) = c\int\frac{1}{\sin\theta}d\theta + d = c\ln(\tan\theta/2) + d$$

has ln singularity at $\theta = 0$ but also opposite-sign one at $\theta = \pi$!

9. [14 points] Short-answer questions

(a) Give an example of an interval and an infinite sequence of functions which are orthogonal on this interval but not complete.

2

$[0, \pi]$ $\qquad$ $\{ \sin 2x, \sin 3x, \sin 4x, \cdots \}$

are orthogonal by Fourier series formulae.

However $\int_0^\pi f(x) \sin nx \, dx = 0 \quad \forall n = 2, 3, \cdots$

does not imply $f \equiv 0$ (defn. of completeness). Viz $f(x) = \sin x \not\equiv 0$.

(b) The *variance* of a zero-mean (sorry) probability distribution function $p(x)$ is defined as $\int_{-\infty}^{\infty} x^2 p(x) dx$. Find a formula for the variance as a certain derivative of the Fourier transform of p evaluated at a certain frequency.

3

$$\hat{p}(\xi) := \int_{-\infty}^{\infty} e^{ix\xi} p(x) \, dx$$

so $$\frac{d^2}{d\xi^2} \hat{p}(\xi) = \int_{-\infty}^{\infty} (-ix)^2 e^{ix\xi} p(x) \, dx$$

Choose $\xi = 0$: $$\hat{p}''(0) = - \int_{-\infty}^{\infty} x^2 p(x) \cancel{e^0}^{\,1} dx$$

$$\text{variance} = - \hat{p}''(0)$$

2

(c) Let K be a *self-adjoint* operator with a complete set of orthogonal eigenfunctions. Prove that $Ku - \lambda u = f$ can only be solvable if f is orthogonal to all solutions v of the homogeneous problem $Kv - \lambda v = 0$.

Two ways:

1 Let u be a soln, ie $(K-\lambda)u = f$ then $\underbrace{(v, (K-\lambda)u)}_{\text{self-adjoint so can move } K\ (\&\lambda) \text{ over to otherside}} = (v, f)$

$(\underbrace{(K-\lambda)v}_{=0}, u)$ for any homog. soln. v. $\Rightarrow (v, f) = 0$.

2 $u = \sum_n u_n \phi_n$ & $f = \sum_n f_n \phi_n$ by completeness of eigfuncs ϕ_n (eigvals λ_n)

Expand & equate coeffs: $(\lambda_n - \lambda) u_n = f_n \quad \forall n = 1, 2, \cdots$ $\qquad$ if $\lambda \neq \lambda_n$ then $v \equiv 0$ is only homog. soln.

If $(K-\lambda)v = 0$ has nontriv. soln, then $\lambda = \lambda_m$ for some m (or m's), and v is in that eigenspace.

But $f_m = (f, \phi_m) = 0$ for solvability of u, so $(f, v) = 0$ for any v in eigenspace m.

(d) As $\lambda \to +\infty$, is $e^{-\lambda} = O(\lambda^{-n})$ for each $n = 0, 1, \dots$? (Prove your answer)

means, exists c & $\lambda_0 > 0$ s.t. $\forall \lambda > \lambda_0$, $e^{-\lambda} \le c\lambda^{-n}$

Try taking $\lim_{\lambda\to\infty} \frac{e^{-\lambda}}{\lambda^{-n}} = \lim_{\lambda\to\infty} \frac{\lambda^n}{e^{\lambda}} \xrightarrow{\text{l'Hopital}} \lim_{\lambda\to\infty} \frac{n\lambda^{n-1}}{e^{\lambda}} \xrightarrow{\text{l'H}} \frac{n(n-1)\lambda^{n-2}}{e^{\lambda}} \sim \dots$

3 eventually must get to $= \frac{n!}{e^{\lambda}}$ which has $\lim_{\lambda\to\infty}$ of 0. So in fact $e^{-\lambda} = o(\lambda^{-n})$

which implies big-O as well.

(e) Place the following four terms in the *correct* order to form an asymptotic series as $\varepsilon \to 0$:

2 $f(\varepsilon) \sim \varepsilon^{5/2} + \varepsilon^2 + \varepsilon^{-2} + \varepsilon^2 \ln\varepsilon + \dots$

$$f(\varepsilon) \sim \varepsilon^{-2} + \underbrace{\varepsilon^2 \ln\varepsilon + \varepsilon^2}_{\varepsilon^2 = o(\varepsilon^2\ln\varepsilon) \text{ since } |\ln\varepsilon| \to \infty \text{ as } \varepsilon\to 0.} + \varepsilon^{5/2} + \dots$$

2 (f) A 2π-periodic 1D image f is blurred by a symmetric convolution kernel to give g. Explain when and why it is sometimes impossible to reconstruct f from g.

see deblurring worksheet & HW.

If f has Fourier series coeffs a_n, b_n

$\to$ g " " " $A_n = \pi k_n a_n$, $B_n = \pi k_n b_n$

where k_n are Fourier cos coeffs of $k(x)$ convolution kernel.

If any $k_n = 0$ for some n, then a_n & b_n cannot be determined since $A_n = B_n = 0$. for that n.

[BONUS: Also explain the effect of the smoothness (differentiability) of this kernel on the ability to reconstruct f from a noisy measured data g]

Smoother kernel k $\Rightarrow$ faster decay of $|k_n|$ as $n \to \infty$

$\Rightarrow$ less coeffs $n = 1 \cdots N$ can be included in reconstruction before the formula $\hat{a}_n = \frac{A_n}{\pi k_n}$, $\hat{b}_n = \frac{B_n}{\pi k_n}$ cause too large an amplification of noise. $\Rightarrow$ Less Fourier coeffs in reconstructed image $\Rightarrow$ less resolution (spatial detail)