Math 74: Rotation Madness

Let $\mathbf{E}^n$ denote $\mathbf{R}^n$ with a choice of Euclidean inner product and an orientation. Let $SO(n)$ be the subgroup of the invertible linear transformations of $\mathbf{E}^n$ that preserve the Euclidean inner product and the orientation. Notice by choosing an orthonormal basis of $\mathbf{E}^n$ we can view $SO(n)$ as subset of the of the n by n matrices, $M_{n\times n}(\mathbf{R})$. In fact

Theorem 1 *Fixing an orthonormal pairs we have*

$$SO(n) = \{A \in M_{n\times n}(\mathbf{R}) \mid AA^{tr} = I, det(A) = 1\}.$$

Proof:$A \in SO(n)$ will will sends a positively oriented orthonormal basis to another positively oriented orthonormal basis and in particular $A^{tr}A = I$ is necessary since this is precisely this statement. Conversely any such matrix that satisfies $AA^{tr} = I$ will preserve the inner product since $Av \cdot Aw = A^{tr}Av \cdot w = v \cdot w$.

Recall a linear transformation A preserves orientation if and only if $det(A) > 0$. Since $det(AA^{tr}) = det(A)^2 = det(I) = 1$ we see that $det(A) = \pm 1$, and the orientation preserving condition forces the $det(A) = 1$.

q.e.d.

Example 1: $SO(2)$ For any $A \in SO(2)$, since A preserves the notion of an oriented basis, we have that B is determined by what B does to single vector (by the right-hand rule). Viewing A in an orthonormal basis $\{e_, e_2\}$ we see that the fact that $A(e_1) = a_{11}e_1 + a_{21}e_2$ and since the norm is preserved $a_{11}^2 + a_{12}^2 = 1$ and hence $a_{11} = \cos(\theta)$ and $a_{11} = \sin(\theta)$ for some $\theta \in [0, 2\pi)$. By the right-hand rule, $A(e_2) = -\sin(\theta)e_1 + \cos(\theta)e_2$ and as a matrix

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Notice, in complete generality, we can view $SO(n)$ as subspace of the space of n by n matrices, $M_{n\times n}(\mathbf{R})$ which is topologically $\mathbf{R}^{n^2}$. Notice in its subspace topology, multiplication and inversion are continuous operations (in fact via the nice rational functions learned in linear algebra). Utilizing this topology, $SO(2)$ is in fact a very familiar topological space. Namely by the continuity of $\sin$ and $\cos$ the mapping of $S^1 = \mathbf{R}/\mathbf{2}\pi\mathbf{Z}$ to the above matrices is bijective and continuous. Hence, since S^1 is compact and and any subspace of $\mathbf{R}^{n^2}$ is Hausdorff, this mapping is a homeomorphism. Hence $SO(2)$ is homeomorphic to S^1 and in a particularly natural way.

Example 2: $SO(3)$ Notice, since we are in $\mathbf{E}^3$, if we fix a point $u \in S^2$, then the notion of a right-handed rotation about this axis makes sense via the right-hand rule. We can characterize the elements of $SO(3)$ as follows.

Theorem 2 *Every $A \in SO(3)$ can described by fixing some point $u = (x, y, z) \in S^2$ and performing a right-handed rotation by some angle θ about u.*

Proof: First observe we can explicitly write down the matrix that performs a right-handed rotation by an angle of θ about $u = (x, y, z) \in S^2$ via $R(u, \theta) =$

$$\left[\begin{array}{ccc} 1 + (1-\cos(\theta))(x^2-1) & -zsin(\theta) + (1-\cos(\theta))xy & y\sin(\theta) + (1-\cos(\theta))xz \\ z\sin(\theta) + (1-\cos(\theta))xy & 1 + (1-\cos(\theta))(y^2-1) & -x\sin(\theta) + (1-\cos(\theta))yz \\ -y\sin(\theta) + (1-\cos(\theta))xz & x\sin(\theta) + (1-\cos(\theta))yz & 1 + (1-\cos(\theta))(z^2-1) \end{array}\right].$$

By inspection $R(u, \theta) \in SO(3)$.

Given $A \in SO(3)$, we will first prove A must fix a non-zero vector. Recall $\det(A^{tr}) = \det(A)$ hence

$$\det((I - A)A^{tr}) = \det(I - A)\det(A^{tr}) = \det(I - A).$$

By definition

$$(I - A)A^{tr} = (A^{tr} - I) = -(I - A^{tr}),$$

and by taking the determinant we have

$$\det((I - A)A^{tr}) = -\det(I - A^{tr}) = -\det(I - A),$$

and hence $det(I - A) = 0$. This equivalent to $I - A$ having a nontrivial kernel, which implies there exist $v \neq 0$ such that $v - Av = 0$, as needed.

Take any w in our fixed vector's, v's, orthogonal compliment. Note $Av = v$ hence Aw is still orthogonal to to v. As such we may utilize a rotation, $R(v, \theta)$, about v to send Aw to w. Notice $R(v, \theta)A = B \in SO(3)$. For any $B \in SO(3)$ since B preserves the notion of an oriented basis, we have that B is determined by what B does to any pair of orthogonal vectors (by the right-hand rule). By construction $R(v, \theta)A$ preserves two vector hence, from this observation, $R(v, \theta)A = I$. In other words $A = R(v, \theta)^{-1} = R(v, -\theta)$, as needed. **q.e.d.**

Once again in its subspace topology $SO(3)$ is a familiar space.

Theorem 3 *$SO(3)$ is homeomorphic to C^3.*

Proof: First we will construct a map Ψ from B^3 to $SO(3)$. Let

$$\Psi(x) = \begin{cases} R(x/|x|, \pi d(0, u)) & x \neq 0 \\ I & x = 0 \end{cases}.$$

By looking at the formula for $R(u, \theta)$ from the proof of theorem 2, we find that Ψ is continuous. Now, since rotating by π clockwise around an axis is the same as rotating by π counter-clockwise around this same axis, Ψ is constant on C^3 equivalence classes. Hence (by theorem 2.22 of Munkres) we have a bijective continuous map from C^3 to $SO(3)$. The map is a homeomorphism since C^3 is compact and $SO(3)$ is Hausdorff. **q.e.d.**

Recall from the first problem of our first exam told us that

$$S^3 = \{(w, x, y, z) \in \mathbf{R}^4 \mid w^2 + x^2 + y^2 + z^2 = 1, \}$$

naturally covers C^3 (notice the use of (w, x, y, z) as coordinates of R^4). In problem three of the final we will explore S^3 and end up with a much better understanding of this relationship between S^3 and $SO(3)$.

To do so it is convenient to observe that S^3 has some natural algebraic structures on it. First, we may identify R^4 with $\mathbf{C}^2$ via the map sending (w, x, y, z) to $(w + ix, y + iz)$. Hence $S^3 = \{(z_1, z_2) \in \mathbf{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$. Recall if $z = x + iy$ then $\bar{z} = x - iy$, and $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$.

We may also identify R^4 with $\mathbf{H}$, the quaternions, via the map sending (w, x, y, z) to $w + xi + yj + zk$. Recall the quaternions are the vector space with basis $\{1, i, j, k\}$, formed into a ring (in fact a division ring), via the multiplication rule, $*$, which satisfies

$$i * i = -1, j * j = -1, k * k = -1$$

and

$$i * j = -j * i = k, j * k = -k * j = i, k * i = -i * k = j$$

on the basis elements and is extended by linearity to all of $\mathbf{H}$. As with the complex numbers, we have that if $q = w + xi + yj + zk$ then we let $\bar{q} = w - xi - yj - zk$, and $|q| = \sqrt{q\bar{q}} = \sqrt{w^2 + x^2 + y^2 + z^2}$. Notice, by definition, S^3 is naturally equivalent to $\mathbf{UH} = \{q \mid |q| = 1\}$, the set of unit quaternions.