Hecke Operators

General outline for **What the heck are Hecke operators**.

- Review of modular forms and a comment on the Fourier series for a modular form
- The definition of Hecke operators
- Hecke's discovery relating Hecke operators and Dirichlet-series.
- Hecke operators and fourier expansions.
- Multiplicative properties of Hecke operators, including commutativity
- Simultaneous eigenforms and their existence
- Conclusion

Recall that a modular form is a function f defined on the complex upper half-plane H so that it is holomorphic everywhere (including ∞) and satisfies the functional equation

$$f(A.z) = f(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).z) = f\left(\frac{az+b}{cz+d}\right) =_{\text{def}} (cz+d)^k f(z)$$

for $A \in \mathrm{SL}_2(\mathbf{Z})$. Last week I asserted that evry modular form had a Fourier expansion. Let me elaborate.

Let $B > 0$ and say H_B is the set of elements in H that have imaginary part above B. The map $z \mapsto e^{2\pi i z} = q_z$ maps H_B onto a punctured disk. Further if we examine H_B/T, essentially a cylinder, then there is a diffeomorphism between the cylinder and the disk.

Recall that a modular form f is periodic since

$$f(T.z) = f\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = f(z+1) = 1^k f(z).$$

Since f is holomorphic on H_B, we get an induced holomorphic map f_∞ on the punctured disk. Since we further require it be holomorphic at the origin, this means that $f_\infty(q)$ be bounded near the origin. Since this function is now analytic on the disk, it has a power series expansion

$$f_\infty(q) = \sum_0^\infty c_n q^n.$$

And, applying the inverse diffeomorphism, we can see that f itself has a power series expansion.

Motivation

Recall the two examples of modular forms I showed you last time, the Eisenstein series and the discriminant. I mentioned that the Eisenstein series had a Fourier expansion of this kind

$$G_k(z) = 2\zeta(k)\left(1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\right)$$

where

- $q = e^{2\pi i z}$,
- the Bernoulli numbers B_k are defined by

$$\frac{x}{x-1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!},$$

- and the function σ_{k-1} is the divisor function of n and is defined by

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}.$$

The discriminanat, it turns out has Fourier expansion

$$\Delta(z) = (2\pi)^{12}\sum_{n=1}^{\infty}\tau(n)e^{2\pi i n z}$$

where τ is Ramanujan's function and is defined as being these coefficients. Both σ and τ are multiplicative arithmetic functions; that is, if $(m, n) = 1$ then

$$\sigma_k(n)\sigma_k(m) = \sigma_k(mn) \qquad \text{and} \qquad \tau(n)\tau(m) = \tau(mn).$$

They also satisfy the more general relations

$$\tau(n)\tau(m) = \sum_{d|(m,n)} d^{11}\tau\left(\frac{mn}{d^2}\right)$$

and

$$\sigma_k(n)\sigma_k(m) = \sum_{d|(m,n)} d^k\sigma_k\left(\frac{mn}{d^2}\right).$$

That two fourier expansions induce two functions with these properties led Hecke to ask what were all the modular forms whose Fourier coefficients satisfied these properties.

Definition of Hecke operators

For a fixed integer k and any $n = 1, 2, 3, ...$ the operator T_n is defined on M_k by

$$(T_n f)(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{nz+bd}{d^2}\right).$$

These operators are called Hecke operators for $\Gamma = \mathrm{SL}_2(\mathbf{Z})$

The special case where $n = p$ is a prime, we get

$$(T_p f)(z) = p^{k-1} f(pz) + \frac{1}{p} \sum_{b(\mathrm{mod} p)} f\left(\frac{z+b}{p}\right)$$

where the sum (mod p) just means that b goes from 0 to $p-1$.

An equivalent definition is the following:

Let $A = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$, so that $A.z = \frac{az+d}{d}$. Then, for $f \in \mathcal{M}_k(\Gamma)$, let

$$(T_n(f))(z) = \left(\tfrac{1}{n}\right) \sum_{\substack{a \geq 1, ad=n \\ b(\mathrm{mod} p)}} a^k f\left(\frac{az+b}{d}\right).$$

Theorem If $f \in \mathcal{M}_k$ and has the Fourier expansion

$$f(z) = \sum_{m \geq 0} a(m) e^{2\pi i m z}$$

then $T_n f$ has the Fourier expansion

$$f(z) = \sum_{m \geq 0} \gamma(m) e^{2\pi i m z}$$

where

$$\gamma(m) = \sum_{d|(n,m)} d^{k-1} a\left(\frac{mn}{d^2}\right).$$

Relating these operators to classical number theory

Hecke, bright guy that he was, discovered a relationship between modular forms with Fourier expansion

$$f(z) = c(0) + \sum_{n \geq 1} c(n) e^{2\pi i n z}$$

and the Dirichlet series

$$\phi(s) = \sum_{n \geq 1} \frac{c(n)}{n^s}$$

formed with the same coefficients. With some basic analysis, we can show that the Dirichlet series converges absolutely for $\sigma = \mathrm{Re}(s) > k + 1$ if $f \in \mathcal{S}_k(\Gamma)$ and if f is a cusp form that it converges for $\sigma > sk$.

The following theorem is mostly a classical analytic number theory result: **Theorem** If the coefficients c_n and c_m satisfy the multiplicative property

$$c_m c_n = \sum_{d|(m,n)} d^{k-1} c\left(\frac{mn}{d^2}\right)$$

then the Dirichlet series will have an Euler product representation of the form

$$\phi(s) = \prod_p \frac{1}{1 - c(p)p^{-s} + p^{k-1}p^{-2s}}.$$

Note that this means that the coefficients of E_k and Δ have these properties and hence the associated Dirichlet series has an Euler product representation. What we'd like is some way to find more such functions. We'd like to do this since getting an Euler product is kind of like factoring. Euler products are sometimes called the analytic version of the fundamental theorem of arithmetic. The Hecke operators, it turns out, help us find an Euler product.

Let $\phi(s)$ be the Dirichlet series associated with the modular form $f \in \mathcal{M}_k(\Gamma)$. Then $\phi(s)$ has an analytic continuations such that

1. if $c_0 = 0$ then $\phi(s)$ is entire
2. if $c_o \neq 0$ then $\phi(s)$ is analytic everywhere except for a pole aat $s = k$ with residue
$$\frac{(-1)^{k/2} c_0 (2\pi)^k}{\Gamma(k)}$$
where Γ is the Gamma function.
3. The function ϕ satisfies the functional equation
$$(2\pi^{-s})\Gamma(s)\phi(s) = (-1)^{k/2}(2\pi)^{s-k}\Gamma(k-s)\phi(k-s).$$

Hecke also proved a kind of converse to this in the sense that if $\phi(s)$ is a Dirichlet series that satisfies this kind of functional equation then, adding some conditions about analyticity and growth, $\phi(s)$ must have come from a modular form.

Properties of $T_n f$

Theorem If $f \in \mathcal{M}_k(\Gamma)$ then $T_n f$ has a Fourier expansion.

Theorem If $f \in \mathcal{M}_k(\Gamma)$ and $V = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, then

$$(T_n f)(z) = (\gamma z + \delta)^k (T_n f)(z).$$

Corollary If $f \in \mathcal{M}_k(\Gamma)$ then $T_n f \in \mathcal{M}_k(\Gamma)$.

Multiplicative properties

Theorem If $(n, m) = 1$, then

$$T_m T_n = T_{mn}.$$

Theorem Any two Hecke operators T_m and T_n defined on $\mathcal{M}_k(\Gamma)$ commute with each other. Moreover,

$$T_m T_n = \sum_{d|(m,n)} d^{k-1} T_{\left(\frac{mn}{d^2}\right)}.$$

Eigenforms

To find out what other forms have these Dirichlet series expansions, we need to find which modular forms give the multiplicative property that we want from Hecke's theorem. It turns out that we need to look at Hecke eigenforms to do this.

Remember if f has Fourier expansion

$$f(z) = \sum_{m \geq 0} c(m) e^{2\pi i z m}$$

then $T_n f$ has expansion

$$(T_n f)(z) = \sum_{m \geq 0} \gamma_n(m) e^{2\pi i z m}$$

where

$$\gamma_n(m) = \sum_{d|(n,m)} d^{k-1} c\left(\frac{mn}{d^2}\right).$$

When $m = 0$ (this means $d|(n, 0) = n$), we get that the constant terms are related by the equation

$$\gamma_n(0) = \sum_{d|n} d^{k-1} c(0) = \sigma_{k-1}(n) c(0).$$

Similarly, when $m = 1$ we get

$$\gamma_n(1) = c(n) \cdot 1 \qquad \text{This means we need } c(m) \text{ to be } 1.$$

The sum on the right of $\gamma_n(m) = \sum_{d|(n,m)} d^{k-1} c\left(\frac{mn}{d^2}\right)$ resembles the multiplicative property of τ and σ. This might suggest that we want those forms f for which $T_n(f)$ has coefficients

$$\gamma_n(m) = c(n)c(m)$$

since this implies

$$c(n)c(m) = \sum_{d|(n,m)} d^{k-1} c\left(\frac{mn}{d^2}\right).$$

The relation $\gamma_n(m) = c(n)c(m)$ can be written as

$$T_n f = c(n) f.$$

A nonzero modular form f satisfying this kind of relation for a nonzero scalar $c(n)$ is an eignenform for T_n and $c(n)$ is the eigenvalue for f.

Examples Any nonzero linear operator on a one-dimensional space has an eigenform. So in particular for $k = 4, 6, 8, 10, 14$ we have that $\mathcal{M}_k(\Gamma)$ is one dimensional. This means that the Eisenstein series for those k are eigenforms.

Simultaneous eigenforms

If f is an eigenform for all T_n then it is called a simultaneous eigenform.

Theorem If $f \in \mathcal{M}_k(\Gamma)$ is a simultaneuous eigenform, then $c(1) \neq 0$.

An eigenform with $c(1) = 1$ is said to be normalized. Hence if $\mathcal{M}_k(\Gamma)$ has a simultaneous eigenform it also has a normalized one.

We also have the characterization for cusp forms that we sought:

Theorem Suppose $f \in \mathcal{S}_k(\Gamma)$. Then f is a normalized simultaneous eigenform iff the coeffecients of $f's$ Fourier expansion satisfy

$$c(n)c(m) = \sum_{d|(n,m)} d^{k-1} c\left(\frac{mn}{d^2}\right)$$

in which case $c(n)$ is an eigenvalue for T_n.

For noncusp forms, it can be shown that the only simultaneous eigenforms are multiples of the Eisenstein series.

The existence of simultaneous eigenforms in $\mathcal{S}_k(\Gamma)$

We know if such things exist, they have these nice properties. The question then becomes "When do they exist?"

Let $\kappa = \dim \mathcal{S}_k(\Gamma)$. Then from what we proved last week we have that

$$\kappa = \begin{cases} \lfloor \frac{k}{12} \rfloor - 1 & \text{if } k \equiv 2(\text{mod})12 \\ \lfloor \frac{k}{12} \rfloor & \text{if } k \not\equiv 2(\text{mod})12 \end{cases}.$$

Let $e(k)$ denote the number of linearly independenty simultaneous eigenforms in $\mathcal{S}_k(\Gamma)$. Hecke showed that $e(k) = 2$ when $\kappa = 2$ and his student Petersson showed that $e(k) = \kappa$ for all k. He did this by introducing an inner product (f, g) on $\mathcal{S}_k(\Gamma)$ defined by

$$\iint_F f(z)\overline{g(z)} v^{k-2} du dv$$

when $z = u + iy$ is in the fundamental domain F. Relative to this inner product the Hecke operators are Hermitian, i.e., they satisfy

$$(T_n f, g) = (f, T_n g)$$

for any two cusp forms. Therefore by standard linear algebra there exist k eigenforms which form an orthonormal bais. They need not *a priori* be simultaneous. But since the T_n commute with other another theorem in linear algebra says they can be simultaneous. Finally, they can easily be normalized. Since the T_n are Hermitian, their eigenvalues are real.

Conclusion

Hecke was looking for the forms that induced a Dirichlet that admitted an Euler product expansion. He showed that if f is a normalized simultaneous eigenform, then it could be written as an Euler product. He also proved a converse: I.e., if a Dirichlet series could be written as an Euler product and satisfied certain analytic conditions, then that series came from a cusp form.

In my next talk, as I generalize the notion of modular forms beyond those for $\Gamma = \mathrm{SL}_2(\mathbf{Z})$, I will have to define Hecke operators in a slightly more general form. They will be defined as a $\mathbf{Q}$-linear combination of double cosets $\Gamma\alpha\Gamma$ where α is an element of some semigroup Δ. These linear combinations can be made into an algebra with a somewhat cumbersome multiplication. We need to go through all this trouble because the definition of Hecke operators as I've given them to you is not readily generalize, whereas the double coset one is.