Note

Four-Dimensional Polytopes: Alicia Boole Stott's Algorithm

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Between 1850 and 1852, the Swiss mathematician Ludwig Schläfli developed a theory of geometry in n-dimensions. In *Theorie der vielfachen Kontinuität* ([4]), he defined the n-dimensional sphere, introduced the concept of four-dimensional polytopes, which he called polychemes, and proved that there are exactly six regular polytopes in four dimensions but only three in dimensions higher than four. Unfortunately, his work was not accepted for publication, and only fragments were published some years later. The entire manuscript did not appear until 1901. Thus, mathematicians in the second half of the century were unaware of Schläfli's discoveries. Between 1880 and 1900 the six regular polytopes were independently rediscovered by, among many others, Stringham in 1880 [5], Gosset in 1900 [3] and Boole Stott in 1900 [1]. The work of Stringham and Gosset is well known today, but that of Boole Stott, an amateur mathematician, has remained almost unnoticed.

In this note we present her original algorithmic approach, together with her original drawings and models. We emphasize that, as a woman born in the mid-nineteenth century, Boole Stott never received any formal mathematical training. Her discoveries came from an extraordinary capacity to visualize the fourth dimension. Rigorous mathematical proofs can therefore not be expected in her work, but instead we find a watershed of surprising and original ideas.

Alicia Boole was born near Cork (Ireland) in 1860, the third daughter of the famous logician George Boole. He died when Alicia was four years old, and her mother became an innovative educator. The amateur mathematician Howard Hinton, a frequent guest in their home, was deeply interested in the fourth dimension. He taught the children to visualize four-dimensional shapes with small cubes; this may (or may not) have inspired Alicia's later research. Whatever the inspiriation, Alicia Boole Stott (she married in 1890) rediscovered the six four-dimensional polytopes by computing their three-dimensional sections. In 1895 she was introduced to the Dutch geometer P. H. Schoute. They collaborated for more than 20 years, combining Schoute's analytical methods with her unusual visualization ability; in 1914, after Schoute's death (1913), the University of Groningen awarded Boole Stott an honorary doctorate. After that, she was isolated from the mathematical community until about 1930, when her nephew, G. I. Taylor, introduced her to H. S. M. Coxeter. Despite the nearly 50 year difference in their ages, Boole Stott and Coxeter collaborated productively until her death in 1940. (For more details, see [2].)

To clarify her approach to four-dimensional polytopes, we first apply Boole Scott's method to the five regular polyhedra (Figure 1).

Constructing the parallel two-dimensional sections of any polyhedron (i.e., the sections parallel to one of its faces) is quite elementary. To compute, for example, the sections of the cube, we intersect the plane containing a given face of the cube with the cube itself. This intersection is, of course, the face of the cube; that is, the parallel section is a square. Translating the plane towards the center of the cube, we see that all parallel sections are isometric squares. Similarly, parallel sections of the tetrahedron are decreasing triangles, triangles and hexagons for the octahedron, pentagons and decagons for the dodecahedron and triangles, hexagons and dodecagons for the icosahedron.

Diagonal sections of a regular polyhedron P are sections $H \cap P$, where H is a plane perpendicular to the segment OV joining the center of the polyhedron with a vertex.

We can visualize a regular solid by unfolding it to a planar net. Roughly speaking, this means "cutting" certain edges of the polyhedron and mapping it to a two-dimensional space. The well-known net for the cube is shown in Figure 2.

Note that to recover the three-dimensional cube from the unfolded version, one must identify certain edges. This allows us to describe the parallel sections of the cube in a very easy way. Namely, one parallel section could be one of the squares in Figure 2, for example, the middle square (call it MS). In order to obtain the other sections (which will be parallel to the square MS after folding the net) one just needs to move the four edges of MS in the unfolded cube parallel towards the remaining squares. In each case, one obtains a square isometric to the square MS (after necessary identification of end points of the edges).

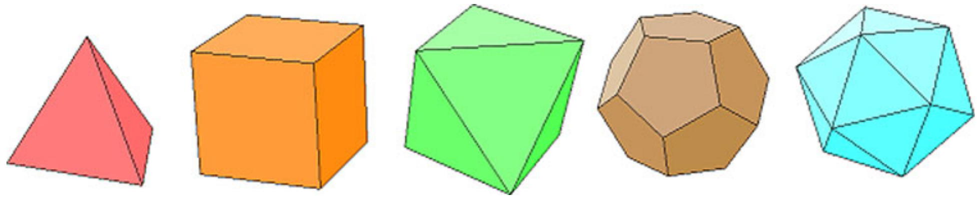

Figure 1. The five regular polyhedra: The tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron.

Boole Stott's Sections of Polytopes

In "On certain series of sections of the regular four-dimensional hypersolids" [1], Boole Stott describes an original method for obtaining the three-dimensional sections of the regular polytopes.

A *polytope* in 4-dimensional space is a subset of the four-dimensional Euclidean space bounded by polyhedra such that every face of each polyhedron is also a face of exactly one other polyhedron. For any polytope, we define a *flag* (P, L, V, C) to be the figure consisting of a vertex P, an edge L containing P, a face V containing L, and a cell C containing V. The polytope is said to be *regular* if there is an isometry mapping any flag to any other, in such a way that

$$(\sigma(P), \sigma(L), \sigma(V), \sigma(C)) = (P', L', V', C').$$

The number of vertices, edges, faces and cells of the six regular polytopes are listed in Figure 3.

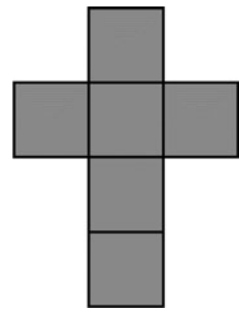

Figure 2. An unfolded cube.

Polytope	v	e	f	c	cell
Hypertetrahedron or 5-cell	5	10	10	5	tetrahedron
Hypercube or 8-cell	16	32	24	8	cube
Hyperoctahedron or 16-cell	8	24	32	16	tetrahedron
24-cell	24	96	96	24	octahedron
120-cell	600	1200	720	120	dodecahedron
600-cell	120	720	1200	600	tetrahedron

Figure 3. Polytopes in four dimensions.

Boole Stott begins with an intuitive uniqueness proof, roughly as follows. Let P be a regular polytope whose cells are cubes. Let V be one of the vertices of P, and consider the diagonal section of P by an affine subspace K, close enough to V so that K intersects all the edges coming from V. The corresponding section must be a regular polyhedron bounded by equilateral triangles, i.e., the tetrahedron (bounded by 4 triangles), the octahedron (bounded by 8 triangles) or the icosahedron (bounded by 20 triangles). Therefore the polytope can only have 4, 8, or 20 cubes meeting at each vertex. Considering the possible angles in 4 dimensions, Boole Stott shows that P must have 4 cubes at a vertex (8 and 20 are too many), which gives the 8-cell (also called a hypercube). She finds the remaining five polytopes in a similar manner.

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Next, she studies three-dimensional parallel sections of these polytopes. Let H be an affine three-dimensional subspace perpendicular to the line OC, where O is the center of a given polytope P, and C the center of one of its cells. The parallel section is $H \cap P$.

Although she treats only parallel sections of polytopes in [1], Boole Stott also made models of diagonal sections $K \cap P$. Here K is an affine three-dimensional subspace perpendicular to the segment OV, where again O is the center of P and V is one of its vertices.

Boole Stott uses the unfolding of a four-dimensional body in a three-dimensional space, analogous to our discussion above. This unfolding operation can be described as "cutting" some of the two-dimensional edges between the three-dimensional faces and mapping the polytope to the third dimension. For example, the unfolded hypercube is the famous tesseract (Figure 4). Note that some two-dimensional faces (i.e., squares) must be identified to

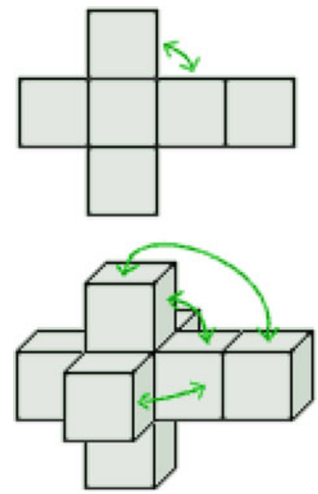

Figure 4. Unfolded hypercube.

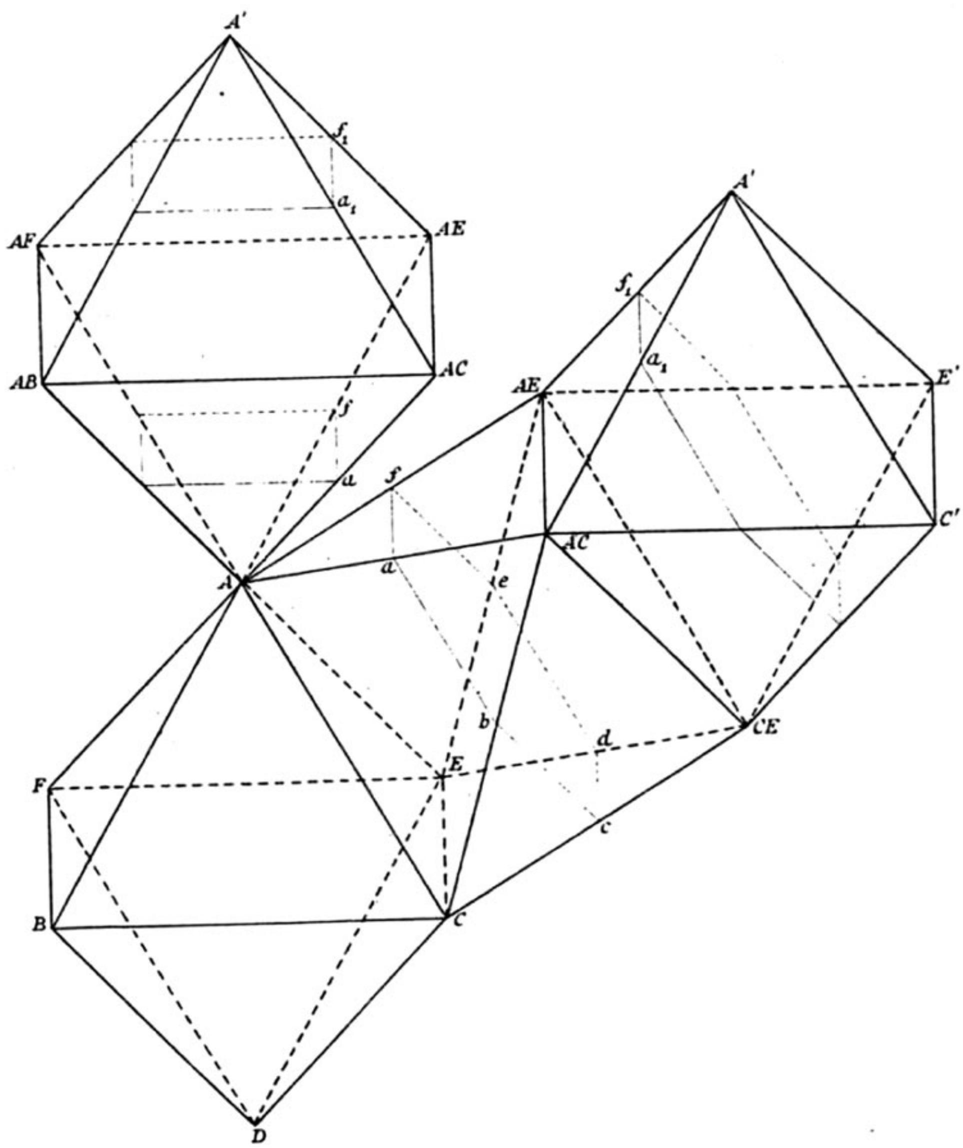

Figure 5. Four octahedra of the 24-cell [1].

recover the hypercube (this identification, of course, is only possible in four dimensions).

We have formalized in the following algorithm Boole Stott's method for computing these three-dimensional parallel sections. A similar algorithm can be described for the case of diagonal sections.

Sketch of the Algorithm

Let P be a four-dimensional regular polytope.

- Step 1: Unfold the polytope P into the three-dimensional space.

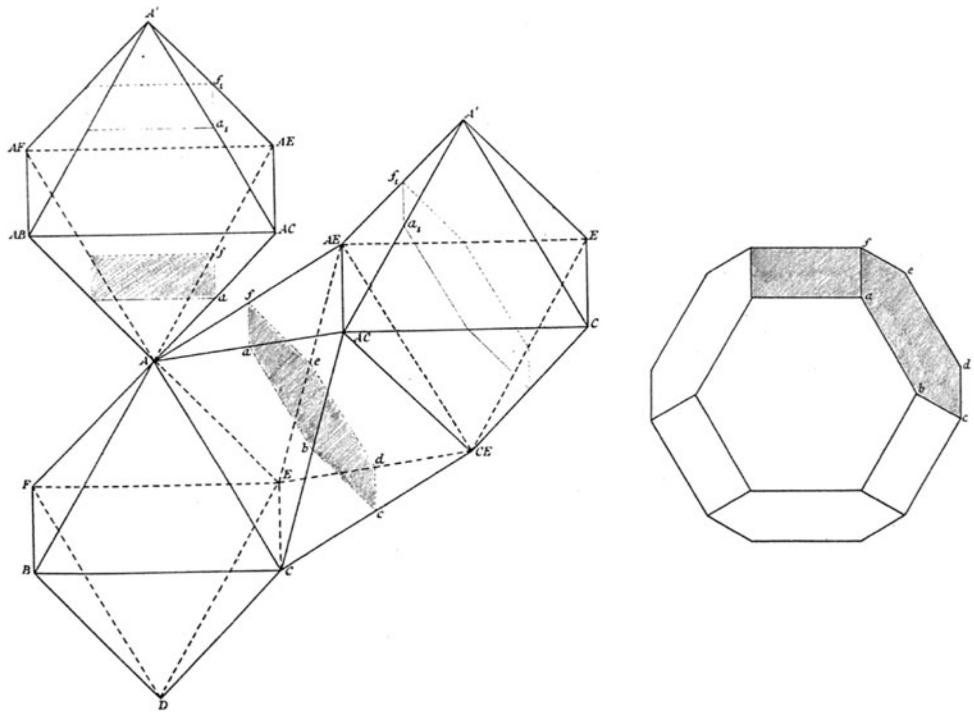

Figure 6. Section $H_2 \cap P$ of the 24-cell [1].

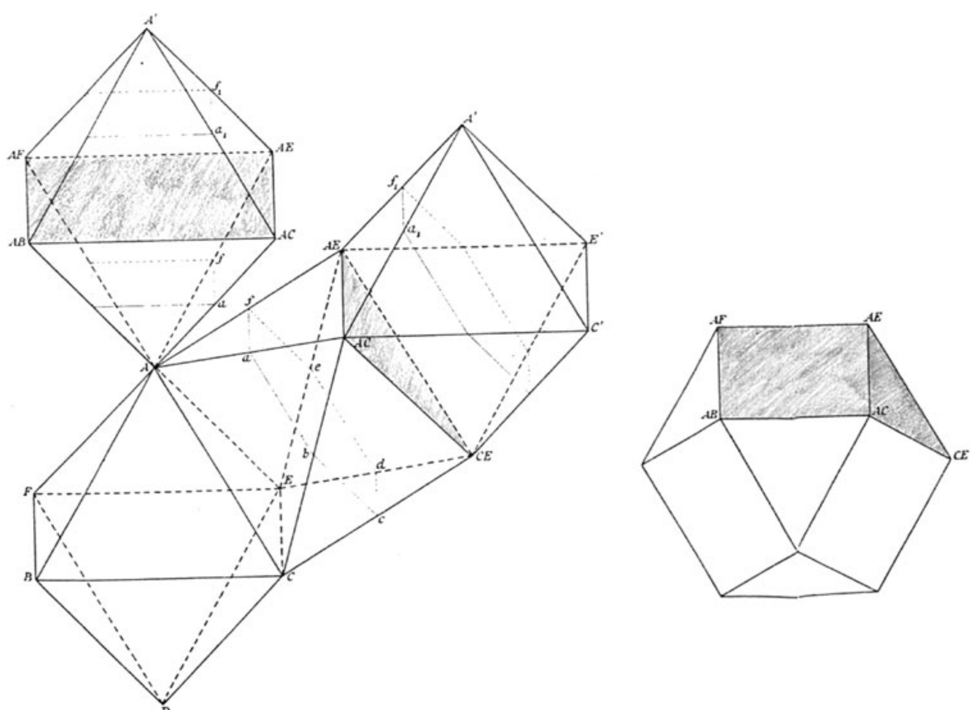

Figure 7. Section $H_3 \cap P$ of the 24-cell [1].

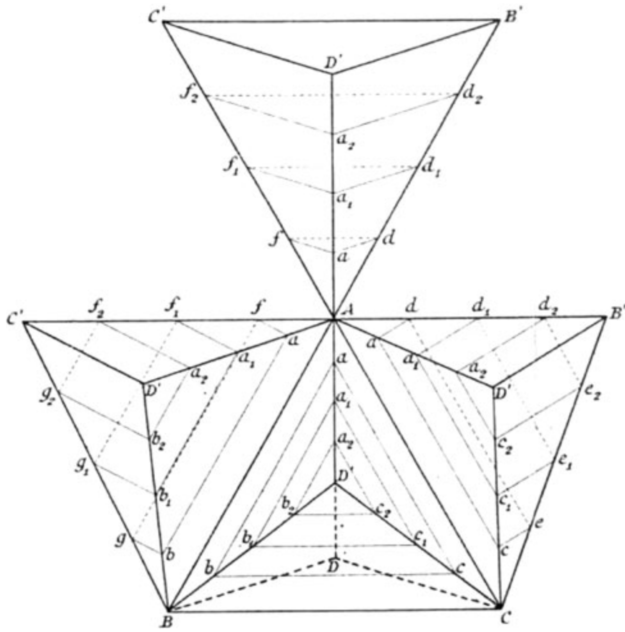

Figure 8. Five tetrahedra of the 16-cell [1].

Figure 11. Drawings and models of parallel sections of the 600-cell. (Courtesy of the University Museum of Groningen.)

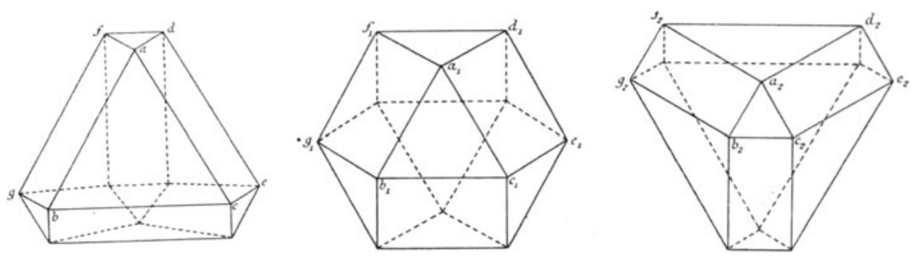

Figure 9. Second, third and fourth sections of the 16-cell [1].

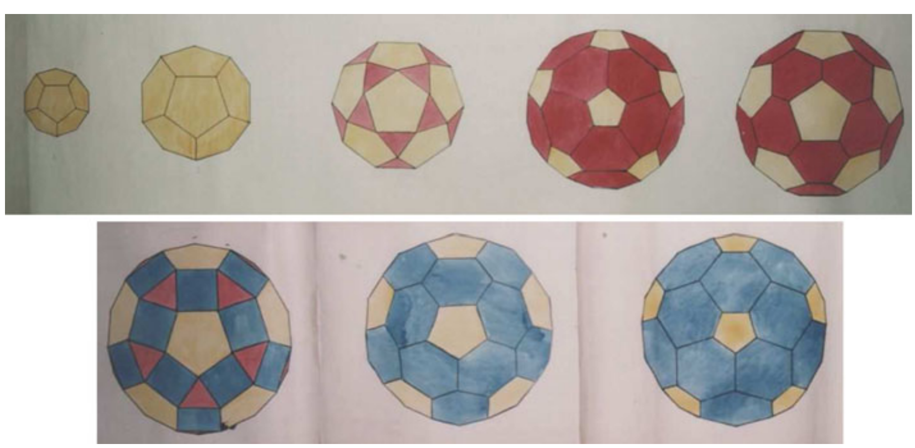

Figure 10. Sections of the 120-cell. (Courtesy of the University Museum of Groningen.)

- Step 2: Let Γ be the graph whose nodes are the vertices and the midpoints of the edges of the unfolded *P*. Two nodes are connected if one is the midpoint of an edge and the other a vertex contained in that edge.
- Step 3: Fix a cell *C* of the polytope *P* on the unfolded figure.
- Step 4: The first three-dimensional sections S_1 of *P* will be *C*. (Note that the cell *C*, in the folded polytope *P*, is contained in a three-dimensional subspace H_1. The section S_1 is therefore $H_1 \cap P = C$.)

- Step 5: Let $\mathcal{V}_2$ be the elements of Γ at distance 1 of C; these are just the midpoints of the edges of $P \setminus C$ meeting C. (Note that in this case, $\mathcal{V}_2$, in the folded polytope P, is contained in a hyperplane H_2 parallel to H_1. These points will be the vertices of a polyhedron S_2 that will be the second section of P.)
  One can easily compute the faces of S_2: for any cell D of the unfolded polytope that intersects $\mathcal{V}_2$, the polygon, segment or point given by the convex hull of $D \cap \mathcal{V}_2$ will be a face, edge or vertex of S_2, respectively. The natural folding of P gives the identification of every face of the polyhedron S_2.
- Step 6: Let $\mathcal{V}_3$ be the elements of Γ at distance 1 of S_2 that are not contained in S_1. (Note that $\mathcal{V}_3$ is contained, in the folded polytope, in a hyperplane H_3 parallel to C. $S_3 = P \cap H_3$ is the third section of P.)
  We compute the faces of S_3 as above. The natural folding of P gives the identification of every face of the polyhedron S_3.
- Step 7: Repeat step 6 until $\mathcal{V}_i = \emptyset$.

For a non-trivial example, let P be the 24-cell. (This polytope is the only one without an analogue in three dimensions). Its cells are octahedra, with 6 meeting at each vertex. Step 1 of the algorithm gives the unfolding of P. In Boole Stott's representation (see Figure 5) only 4 octahedra are drawn. Note that the figure is again an unfolding. The two A' should be identified and similarly, for the vertices AE and AC.

Fix an octahedron cell of P (step 3). Let H_1 be the three-dimensional space containing the octahedron $ABCDEF$. The first section $H_1 \cap P$ is clearly the octahedron $ABCDEF$ itself (step 4).

Let H_2 be the space parallel to H_1 and passing through the point a (the mid-point between A and AC). The second section $H_2 \cap P$ is a three-dimensional solid whose faces are either parallel to the faces of the octahedron $ABCDEF$ or to the rectangle $BCEF$. In Figure 6 two of these faces are shaded. Since the drawing of the octahedra meeting at A is not complete (3 octahedra are missing), we only see part of the final section. The remaining part can be deduced by symmetry (step 5).

Following step 6, let H_3 be the space parallel to H_1 and passing through the vertex AC. The section $H_3 \cap P$ contains a rectangle $ABACAEAF$ parallel to the rectangle $BCEF$ and a triangle $AEACCE$ parallel to the face ACE (the shaded faces of Figure 7).

By symmetry, the fourth section passing through a_1 (the mid-point between AC and A') is isomorphic to the second section (step 7). Again by symmetry, the last section through A' is an octahedron (step 8).

Exercise: Use the algorithm to compute the three-dimensional sections of the 5-cell and the 8-cell.

We conclude with Boole Stott's drawings for the parallel sections of the 120-cell, and her cardboard models for the diagonal sections of the 600-cell, which you can see on display at the University of Groningen (Figures 8–11).

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