m15w06	Sample Midterm	Exam Time: , 6:00 - 8:00
Name:		Student No.:

Instructions:

- Answer ALL questions from Section A
- You may use a handwritten sheet of notes. Calculators are NOT permitted.
- Read all questions carefully
- Unless explicitly told otherwise, you should explain all your answers fully.
- Do NOT seperate the pages of your exam.

Problem	Points	Score
A1	8	
A2	8	
A3	8	
A4	8	
A5	8	
A6	10	
Total	50	

Name:

Section A: Answer ALL questions.

Problem A1: [8 pts] A solid (regular) cone with base radius $2m$ and height $5m$ is situated in space so that its base is centered on the origin and its vertex is at the point $(5,0,0)$. The density of the cone is given by $\rho(x,y,z) = x\ kg/m^3$. What is the mass of the cone?

Solution:

Work in cylindrical coordinates along the x-axis, i.e. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ r\cos\theta \\ r\sin\theta \end{pmatrix}$. The scaling factor is just r and cone is then described by $0 \le x \le 5$, $0 \le \theta \le 2\pi$, $0 \le r \le 2(1 - x/5)$.

$$\begin{aligned} \text{Mass} &= \int_0^5 \int_0^{2\pi} \int_0^{2-2x/5} x r\, dr\, d\theta\, dx \\ &= \pi \int_0^5 x\big(2 - \frac{2x}{5}\big)^2 dx \\ &= -\frac{5\pi}{6}\big(2 - \frac{2x}{5}\big)^3\Big|_0^5 = \frac{20}{3}\pi\ kg \end{aligned}$$

Name:

Problem A2: [8 pts] Recall that the electric potential function for the electric field for a point charge of $+q$ coulombs is given by

$$\frac{q}{4\pi\epsilon_0 r}$$

where r is the distance from the point charge.

Three points charges of $+2$, -1 and $+1$ coulombs are fixed at $(2,0,2)$, $(2,0,0)$ and $(0,0,3)$ respectively. Find the work done by this system in moving a point charge of $+2$ coulombs from $(0,0,0)$ to $(2,0,3)$.

Solution:

The electric potential function for the system is the sum of the electric potential functions for each point charge of the system, i.e.

$$f(x,y,z) = \frac{1}{4\pi e_0}\left(\frac{2}{r_1} - \frac{1}{r_2} + \frac{1}{r_3}\right)$$

where r_1, r_2, r_3 are the respective distances. Now since the electric field is the force per coulomb, the work done is

$$\int_C 2\vec{E} \cdot d\vec{r}$$

where C is any path from $(0,0,0)$ to $(2,0,3)$. But since $\nabla f = -\vec{E}$ we can use the fundamental theorem of line integrals to see

$$\text{Work} = f(0,0,0) - f(2,0,3) = \frac{1}{4\pi e_0}\left[\left(\frac{2}{4} - \frac{1}{\sqrt{8}} + \frac{1}{3}\right) - \left(\frac{2}{1} - \frac{1}{3} + \frac{1}{2}\right)\right] = \frac{1}{4\pi\epsilon_0}\left(-\frac{4}{3} - \frac{1}{\sqrt{8}}\right)$$

Name:

Problem A3: [8 pts]

(**a**) Compute the flux of the vector field $\vec{V}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin(y^2) + x \\ \cos(x^2) - y^3 \end{pmatrix}$ over the unit circle centered at the origin, oriented counterclockwise.

Solution:

Rather than compute directly, use the divergence theorem with $D = \{x^2 + y^2 \leq 1\}$. For then

$$\begin{aligned} \text{Flux over } \partial D &= \int\int_D \text{div } \vec{V} dA = \int\int_D 1 - 3y^2 dA \\ &= \int_0^{2\pi} \int_0^1 r - 3r^3 \sin^2\theta dr d\theta = \int_0^{2\pi} \frac{1}{2} - \frac{3}{4}\sin^2\theta d\theta \\ &= \pi - \frac{3}{4}\pi = \frac{\pi}{4}. \end{aligned}$$

(**b**) Water is pumped into the center of a very shallow pool filling the region $x^2 + y^2 \leq 2$ where x and y are measured in metres. The velocity of the water is given by the vector field $\vec{V} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2}} \end{pmatrix}$ m/s. If the total amount of water in pool is constant, how quickly is the water being pumped in? (We are essentially ignoring the negligible depth, so the units will be m^2/s).

Solution:

The boundary is parametrized by $\vec{r} = \begin{pmatrix} \sqrt{2}\cos t \\ \sqrt{2}\sin t \end{pmatrix}$, so $d\vec{r} = \begin{pmatrix} -\sqrt{2}\sin t \\ \sqrt{2}\cos t \end{pmatrix}$. Thus $\vec{n}ds = \begin{pmatrix} \sqrt{2}\cos t \\ \sqrt{2}\sin t \end{pmatrix}$. The flux over the boundary is then

$$\begin{aligned} \text{Flux} &= \int_0^{2\pi} \begin{pmatrix} \sqrt{2}/2\cos t \\ \sqrt{2}/2\sin t \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2}\cos t \\ \sqrt{2}\sin t \end{pmatrix} dt \\ &= \int_0^{2\pi} 1 dt = 2\pi. \end{aligned}$$

Thus water is leaving the pool at 2π m^2/s, so it must be entering at that rate too.

Name:

Problem A4: [8 pts] (**a**) Is the vector field

$$\vec{F} = \begin{pmatrix} y\sqrt{1+x^2y^2} \\ x\sqrt{1+x^2y^2} \end{pmatrix}$$

conservative everywhere? Justify your answer.

Solution:

$$\text{curl}\vec{F} \cdot \vec{k} = \sqrt{1+x^2y^2} + x^2y^2(1+x^2y^2)^{-1/2} - \sqrt{1+x^2y^2} - x^2y^2(1+x^2y^2)^{-1/2} = 0$$

and $\vec{F}$ and its derivatives are continuous everywhere, so yes it is conservative everywhere.

(**b**) Find $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}$ is the vector field from part (a) and C is the spiral path $\vec{r}(t) = \begin{pmatrix} t^2 \cos t \\ t^2 \sin t \end{pmatrix}$ for $0 \le t \le 2\pi$.

Solution:

Since the vector field is conservative, we know that the line integral is path independent and so we can instead use the path $\vec{r}(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}$ for $0 \le t \le 4\pi^2$. Thus

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{4\pi^2} \begin{pmatrix} 0 \\ t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt = 0.$$

Name:

Problem A5: [8 pts] (**a**) Is the vector field

$$\vec{F} = \begin{pmatrix} e^{(x-y)^2} \\ x - e^{(x-y)^2} \end{pmatrix}$$

conservative everywhere? Justify your answer.

Solution:

$$\text{curl}\vec{F} \cdot k = 1 + 2(x-y)e^{(x-y)^2} - 2(x-y)e^{(x-y)^2} = 1$$

so its not conservative. (But as the curl is very simple, it is a good candidate for use of the curl form of the divergence theorem.)

(**b**) Find $\int_C \vec{F} \cdot d\vec{r}$ where C is the path formed by straight line segments connecting $(0,0)$, $(1,0)$ and $(1,1)$ in that order.

Solution:

The direct computation is very difficult, but if let C_2 be the path parametrized by $\vec{r}(t) = \begin{pmatrix} t \\ t \end{pmatrix}$ $0 \le t \le 1$ and D the triangular region with vertices at $(0,0)$, $(1,0)$ and $(1,1)$, then $\partial D = C - C_2$.

Now

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_{\partial D} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} \\
&= \int\int_D \text{curl } \vec{F} \cdot \vec{k} dA + \int_0^1 \begin{pmatrix} 1 \\ t-1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} dt \\
&= \int\int_D dA + \int_0^1 dt \\
&= \frac{3}{2}.
\end{aligned}$$

Name:

Problem A6: [10 pts] Recall that the gravitational potential for a point mass M is given by

$$P = \frac{GM}{d}$$

where d is the distance from M and G is the universal gravitational constant. Also recall that the kinetic energy of a point mass is given by $\frac{1}{2}mv^2$ where v is the velocity.

(**a**) A unit sphere centered at the origin has uniform density $\rho(x, y, z) = 2\ kg/m^3$. Find the gravitational potential function for the sphere at points along the z-axis with $z > 1$.

Solution:

Use cylindrical coordinates along the z-axis.

$$\begin{aligned}
\text{Potential}(0, 0, a) &= G \int_{-1}^{1} \int_{0}^{2\pi} \int_{0}^{\sqrt{1-z^2}} \frac{2r}{r^2 + a^2} dr d\theta dz \\
&= 2\pi G \int_{-1}^{1} 2\sqrt{1 - z^2 + a^2} - 2adz \\
&= 4\pi G \left(\frac{z}{2}\sqrt{1 - z^2 + a^2} + \frac{a^2 + 1}{2} \sin^{-1} \frac{z}{\sqrt{a^2 + 1}} \right) \Big|_{-1}^{1} - 8\pi a \\
&= 4\pi G(a^2 + 1) \sin^{-1} \frac{1}{\sqrt{a^2 + 1}} - 4\pi G a.
\end{aligned}$$

(**b**) A point mass of $1kg$ is dropped from rest at a point $5m$ above the surface of the sphere in part (a). Using conservation of total energy (kinetic + potential), find the speed of the mass when it is only $3m$ above the surface.

Solution:

By symmetry we can suppose that the mass is dropped at $(0, 0, 5)$ and ends up at $(0, 0, 3)$. Initial kinetic energy is zero, so

$$0 + (4\pi G(26) \sin^{-1} \frac{1}{\sqrt{26}} - 20\pi G = K + 40\pi G \sin^{-1} \frac{1}{\sqrt{10}} - 12\pi G.$$

Thus $K = 4\pi G(26 \sin^{-1} \frac{1}{\sqrt{26}} - 10 \sin^{-1} \frac{1}{\sqrt{10}}) - 8\pi G$. The speed is then

$$v = \sqrt{2K}.$$