Math 8 Midterm 1 Solutions

October 13, 2011

(1) Determine whether the following series converges. If any method from class allows you to do so, give the value of the sum.

$$\sum_{k=2}^{\infty} 3^{k+4} 2^{-2k-2}$$

Solution:

$$\begin{aligned}
\sum_{k=2}^{\infty} 3^{k+4} 2^{-2k-2} &= \sum_{k=2}^{\infty} \frac{3^{k+4}}{2^{2k+2}} \\
&= \frac{3^4}{2^2} \sum_{k=2}^{\infty} \frac{3^k}{2^{2k}} \\
&= \frac{3^5}{4^2} \sum_{k=2}^{\infty} \left(\frac{3}{4}\right)^k \frac{4}{3} \\
&= \frac{3^5}{4^2} \sum_{k=2}^{\infty} \left(\frac{3}{4}\right)^{k-1} \\
&= \frac{3^5}{4^2} \left(\sum_{k=2}^{\infty} \left(\frac{3}{4}\right)^{k-1} + 1 - 1 \right) \\
&= \frac{3^5}{4^2} \left(\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^{k-1} - 1 \right) \\
&= \frac{3^5}{4^2} \left(\frac{1}{1 - \frac{3}{4}} - 1 \right) \\
&= \frac{3^5}{4^2}(4-1) = \frac{3^6}{4^2}
\end{aligned}$$

(2) Evaluate the following integral.

$$\int_0^1 \sqrt{3 - x^2 - 2x}\, dx$$

Solution: First, we must complete the square.

$$\sqrt{3 - x^2 - 2x} = \sqrt{3 - (x^2 + 2x)} = \sqrt{3 - (x^2 + 2x + 1) + 1} = \sqrt{4 - (x+1)^2} = 2\sqrt{1 - \left(\frac{x+1}{2}\right)^2}$$

Notice this is of a similar form as $\sqrt{1 - \sin^2\theta}$. So we should do a trig. sub. Let $\frac{x+1}{2} = \sin\theta$. Then $x = 2\sin\theta - 1$ and $dx = 2\cos\theta d\theta$.

$$\begin{aligned}
\int_0^1 \sqrt{3 - x^2 - 2x}\, dx &= \int_{\pi/6}^{\pi/2} 4\sqrt{1 - \sin^2\theta}\cos\theta d\theta \\
&= \int_{\pi/6}^{\pi/2} 4\cos^2\theta d\theta \\
&= \int_{\pi/6}^{\pi/2} 4(\frac{1}{2}(1 + \cos(2\theta))d\theta \\
&= 2(\theta + \frac{1}{2}\sin(2\theta))|_{\pi/6}^{\pi/2} \\
&= 2\left(\pi/2 + \frac{1}{2}\sin(\pi) - (\pi/6 + \frac{1}{2}\sin(\pi/3))\right) \\
&= 2\pi/3 + \sqrt{3}/2
\end{aligned}$$

(3) Determine whether the following series converges. If any method from class allows you to do so, give the value of the sum.

$$\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$$

Solution: It is not possible to the comparison test so we must look an alternative technique. We should use the integral test.

$$\begin{aligned}
\int_2^{\infty} \frac{\ln x}{x^2} dx &= \lim_{b\to\infty} \int_2^{b} \frac{\ln x}{x^2} dx \\
&= \lim_{b\to\infty} \frac{-\ln x}{x}\Big|_2^b + \int_2^b \frac{1}{x^2} dx \quad \text{(integration by parts)} \\
&= \lim_{b\to\infty} \frac{-\ln x}{x} - \frac{1}{x}\Big|_2^b \\
&= \lim_{b\to\infty} \frac{-\ln b}{b} - \frac{1}{b}\Big| - \frac{-\ln 2}{2} - \frac{1}{2} \\
&= \frac{\ln 2}{2} - \frac{1}{2}
\end{aligned}$$

Since the integral exist, the series converges. We do not have a way to determine what it converges to.

(4) Determine whether the following series converges. If any method from class allows you to do so, give the value of the sum.

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$$

Solution: Since we want to know the value of the series, if it exist, we should do partial fractions to determine if it is a telescoping series. Solving

$$\frac{2n+1}{n^2(n+1)^2} = \frac{A}{n} + \frac{B}{n^2} + \frac{C}{n+1} + \frac{D}{(n+1)^2},$$

we find $A = C = 0$, $B = 1$, and $D = -1$. Looking at partial sums, we find

$$S_1 = 1 - \frac{1}{2}$$

$$S_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{9}$$

$$S_N = 1 - \frac{1}{(N+1)^2}$$

$$S = \lim_{N\to\infty} S_N = 1$$

Thus the series converges and its sum is 1.

(5) Evaluate the following integral.

$$\int_2^3 \frac{2}{x^2\sqrt{x^2-4}}\,dx$$

Solution:

$$\int_2^3 \frac{2}{x^2\sqrt{x^2-4}}\,dx = \int_2^3 \frac{1}{x^2\sqrt{(x/2)^2-1}}\,dx$$

Let $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta$. Substituting this into the integral, we get the following.

$$\begin{aligned}
\int \frac{2\sec\theta\tan\theta}{4\sec^2\theta\sqrt{\sec^2\theta-1}}\,d\theta &= \int \frac{\tan\theta}{2\sec\theta\tan\theta}\,d\theta \\
&= \int \frac{1}{2\sec\theta}\,d\theta \\
&= \int \frac{1}{2}\cos\theta\,d\theta \\
&= \frac{1}{2}\sin\theta
\end{aligned}$$

Since the limits of integration did not transfer nicely, we must transform back to x. Doing this we find

$$\sin\theta = \frac{\sqrt{x^2-4}}{x}$$

Evaluating at the limits, we find

$$\int_2^3 \frac{2}{x^2\sqrt{x^2-4}}\,dx = \frac{\sqrt{5}}{6}$$

(6) For what values of c does the following series converge? Is the convergence absolute or conditional?

$$\sum_{n=1}^{\infty} \frac{nc^{n+1}}{(n-1)!}$$

Solution: We use ratio test.

$$\lim_{n\to\infty} = \left| \frac{(n+1)c^{n+2}}{n!} \frac{(n-1)!}{nc^{n+1}} \right| = \lim_{n\to\infty} = \left| \frac{(n+1)c}{n^2} \right| = 0 < 1$$

So the series converges for all c.

(7) Determine whether the following series converges. If any method from class allows you to do so, give the value of the sum.

$$\sum_{n=4}^{\infty}\left(\frac{1}{\sqrt{n^2+n}-\sqrt{n^2+1}}\right)^n$$

Solution: We shall use the the n^{th} root test.

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+n}-\sqrt{n^2+1}}\right)^{n/n} &= \lim_{n\to\infty}\frac{1}{\sqrt{n^2+n}-\sqrt{n^2+1}} \\
&= \lim_{n\to\infty}\frac{1}{\sqrt{n^2+n}-\sqrt{n^2+1}}\left(\frac{\sqrt{n^2+n}+\sqrt{n^2+1}}{\sqrt{n^2+n}+\sqrt{n^2+1}}\right) \\
&= \lim_{n\to\infty}\frac{\sqrt{n^2+n}+\sqrt{n^2+1}}{n+1} \\
&= \lim_{n\to\infty}\sqrt{\frac{n^2+n}{(n+1)^2}}+\sqrt{\frac{n^2+1}{(n+1)^2}} \\
&= 2
\end{aligned}$$

(8) Does the following series converge? If so, does it converge conditionally or absolutely?

$$\sum_{n=3}^{\infty}(-1)^{n-1}\frac{\sqrt[3]{2n^6-3n^3+1}}{\sqrt{4n^7+n^3-12}}$$

Solution: First, we check if the series converges absolutely, this means we need to check the convergence of

$$\sum_{n=3}^{\infty}\frac{\sqrt[3]{2n^6-3n^3+1}}{\sqrt{4n^7+n^3-12}}.$$

Lets use comparison test with the series $\sum_{n=3}^{i} nftyb_n$ where $b_n = \frac{\sqrt[3]{n^6}}{\sqrt{n^7}}$.

$$\begin{aligned}\lim_{n\to\infty}\frac{\sqrt[3]{2n^6-3n^3+1}}{\sqrt{4n^7+n^3-12}}\frac{\sqrt[3]{n^6}}{\sqrt{n^7}} &= \lim_{n\to\infty}\sqrt[3]{\frac{2n^6-3n^3+1}{n^6}}\sqrt{\frac{n^7}{4n^7+n^3-12}}\\ &= \frac{\sqrt[3]{2}}{2}\end{aligned}$$

This means we can compare. Now, $\sum_{n=3}^{i} nftyb_n$ converges by p-test. Thus

$$\sum_{n=3}^{\infty}(-1)^{n-1}\frac{\sqrt[3]{2n^6-3n^3+1}}{\sqrt{4n^7+n^3-12}}$$

converges absolutely.