Asymptotic ergodicity of eigenfunctions in a uniformly hyperbolic billiard

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Alex Barnett

barnett@cims.nyu.edu

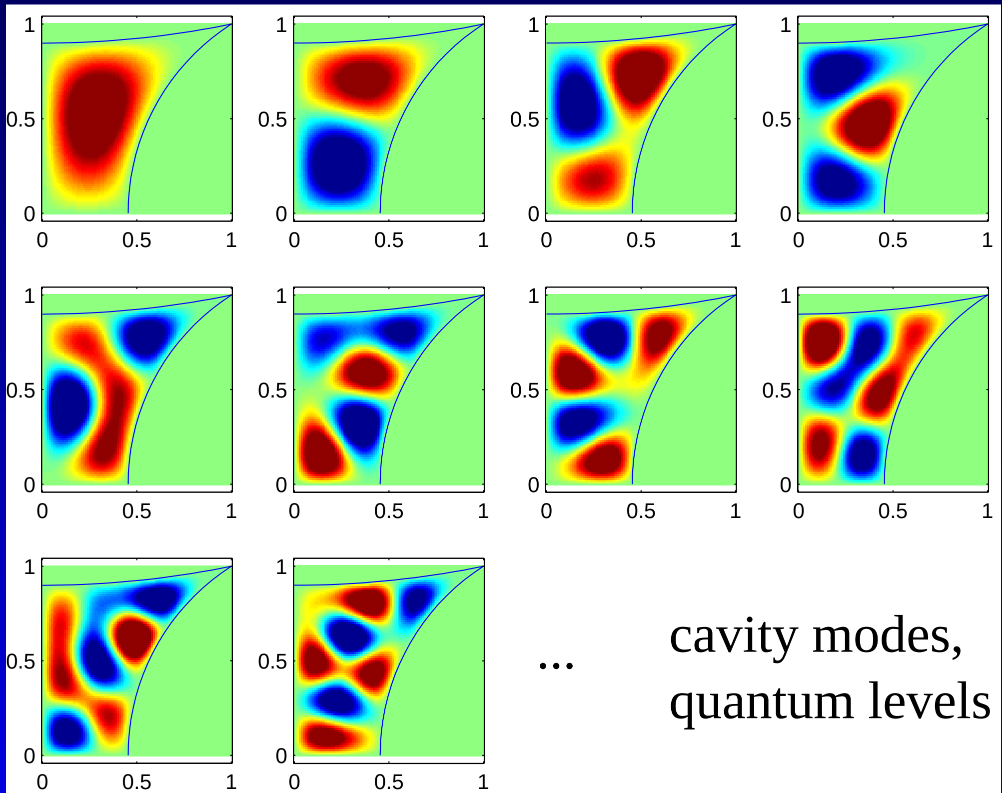
Courant Institute

Thanks to Peter Sarnak, Percy Deift, Eric Heller, Doron Cohen

Dirichlet eigenproblem

Normal modes of elastic membrane (drum)
Eigenfunctions $\phi_n(\mathbf{r})$ of Laplacian in bounded domain $\Omega \subset \mathbb{R}^2$

$$-\Delta\phi_n = E_n\phi_n, \qquad \phi_n|_{\partial\Omega} = 0 \qquad \langle\phi_n, \phi_m\rangle = \delta_{nm}$$



level $n = 1 \cdots \infty$

'energy' eigenvalue E_n

wavenumber $k_n := E_n^{1/2}$
wavelength $\lambda_n := \frac{2\pi}{k_n}$

MOVIE $n \approx 3000$

- If classical dynamics (geodesic flow) in Ω is ergodic, what happens to ϕ_n in semiclassical ($E_n \to \infty$) limit? (87 yr old!)

Classical and quantum ergodicity

Consider $A = A(\mathbf{r})$ classical symbol, indep. of momentum

classical (phase space) average $\overline{A} = \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega A(\mathbf{r}) d\mathbf{r}$

quantum equivalent of A is $\hat{A}$, operator in $L^2(\Omega)$

$$\langle \phi_n, \hat{A}\phi_n \rangle = \int_\Omega A(\mathbf{r}) \underbrace{\phi_n(\mathbf{r})^2 d\mathbf{r}}_{d\mu_{\phi_n} \text{ mass density measure}}$$

diagonal matrix element

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Quantum Ergodicity Theorem (QET):

(Schnirelman '74, Colin de Verdière '85, Zelditch '87, Z-Zworski '96)

For ergodic flow and all well-behaved A,

$$\lim_{E_n \to \infty} \langle \phi_n, \hat{A}\phi_n \rangle - \overline{A} \;=\; 0$$

holds for subset $\{n_j\}$ with density $\lim_{n\to\infty} \#\{n_j < n\}/n = 1$

- Non-ergodic exceptional set of vanishing density can persist

Questions

1. *How* dense is exceptional set?
2. What is *rate* of convergence?

Determine *practical* applicability of QET

Examine numerically for:

- uniformly-hyperbolic (Anosov) Euclidean billiard
- certain choice of A
- up to very high quantum level $n \sim 10^6$

Outline

- Background
- Results: diagonal matrix elements
- Semiclassical theory of Feingold-Peres, results
- Method: computing eigenfunctions at high n

Motivation: Random plane waves

Conj. (Berry '77): random Fourier model of eigenfunctions

$$\phi_n \sim \lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{j=1}^{N} a_j \sin(\mathbf{k}_j \cdot \mathbf{r} + \alpha_j)$$

iid amplitudes $a_j \in \mathbb{R}$
iid phases $\alpha_j \in [0, 2\pi)$

wavevectors $\mathbf{k}_j$: spaced uniformly in direction, $|\mathbf{k}_j| = k_n$

Predicts statistical *variance* of matrix elements

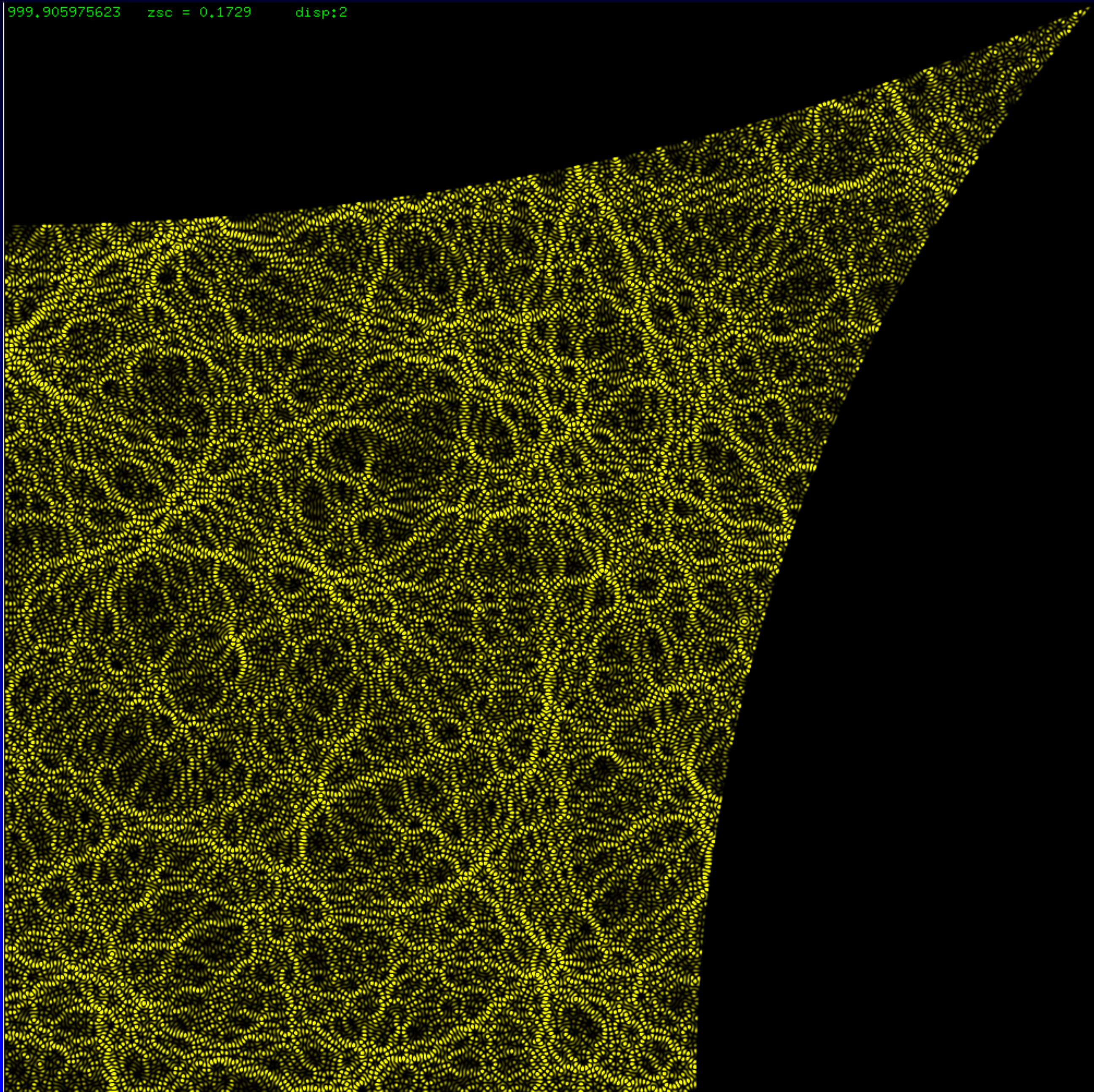
$$\mathrm{Var}[\langle \phi_n, \hat{A}\phi_m \rangle] = \begin{cases} g c_{\mathrm{RW}} E^{-1/2}, & n = m \\ c_{\mathrm{RW}} E^{-1/2}, & n \neq m, \quad |E_m - E_n| = o(E_n^{1/2}) \end{cases}$$

ϕ_n, ϕ_m statistically indep. $\Rightarrow g = 2$ (RMT, time-rev. symm: GOE)

'Coulomb' prefactor $\quad c_{\mathrm{RW}} = \frac{1}{\pi \mathrm{vol}(\Omega)} \iint \frac{A(\mathbf{r}_1)A(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} d\mathbf{r}_1 d\mathbf{r}_2$

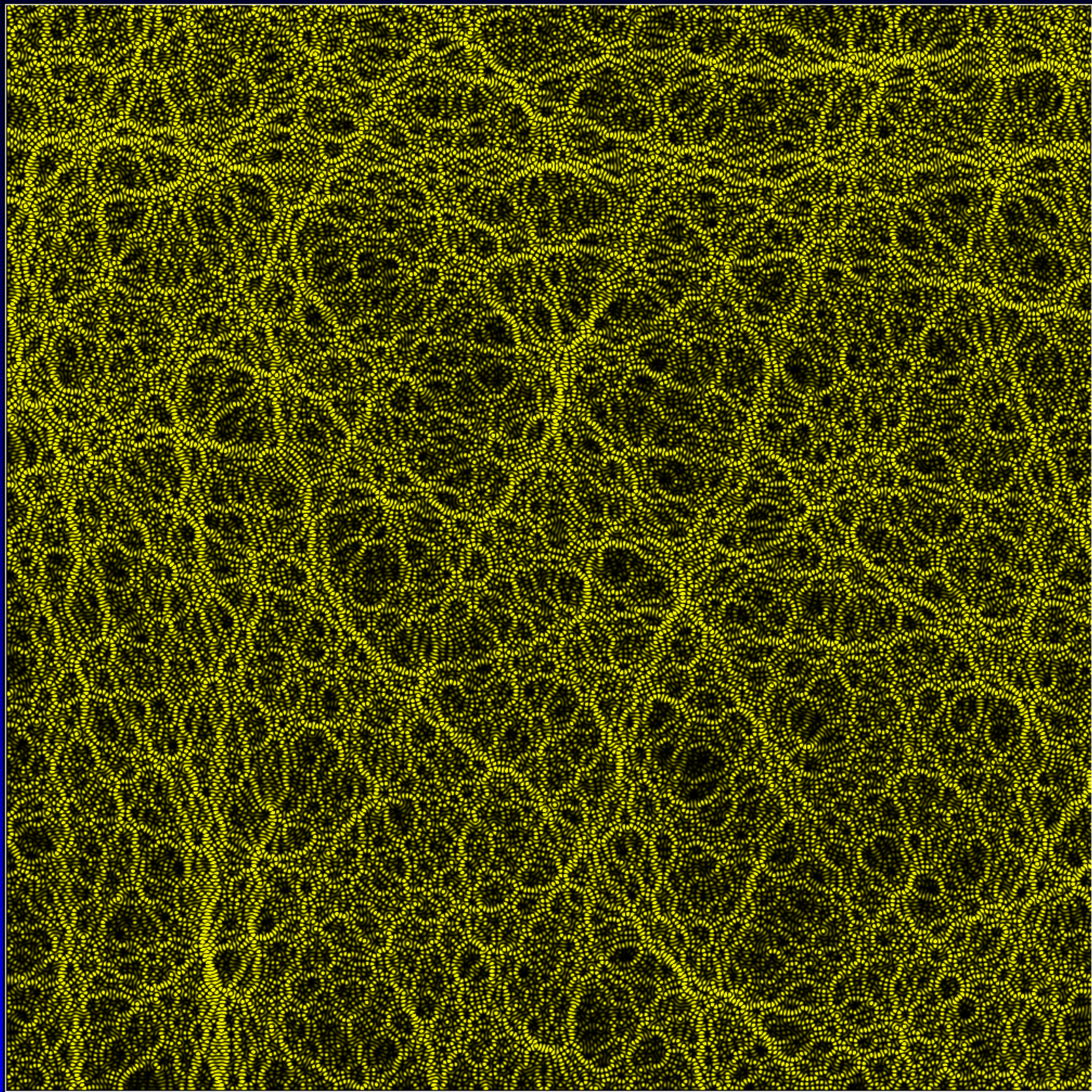
High-energy eigenfunction ϕ_n

$k \approx 10^3$
$E \approx 10^6$
$n \approx 5 \times 10^4$



Random plane waves

stringy ridge structures appear due to $|\mathbf{k}| = \text{const.}$



Theory?

(O'Connor-Heller '86 made a start)

Quantum Unique Ergodicity & Scars

Conj. (Rudnick & Sarnak '94) QUE, stronger form of QET:

For every single level n,

$$\lim_{E_n \to \infty} \langle \phi_n, \hat{A}\phi_n \rangle - \overline{A} \;=\; 0.$$

All ϕ_n *converge (weakly) to a unique measure:* $d\mu_\phi \longrightarrow$ *uniform*

- uniformly hyperbolic manifolds
- *proved* for arithmetic manifolds (Lindenstrauss 2003)

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SCARS: as $E \to \infty$ could have...

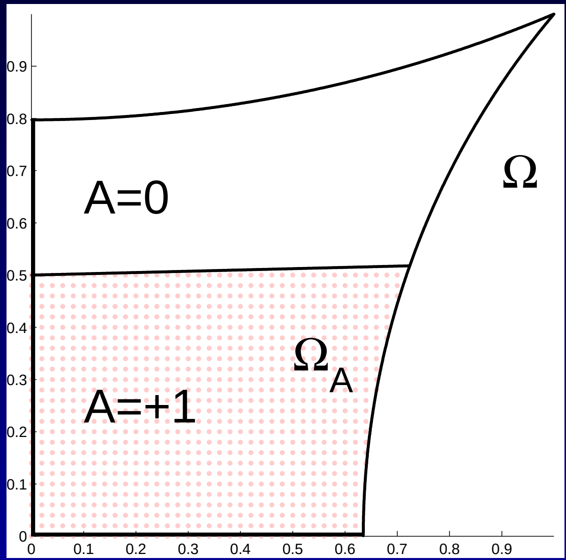
strong: $O(1)$ mass associated with unstable periodic orbit
mathematician's scar

'weak': mass (width) vanishes but *enhancement of* ϕ_n^2 does not
physicist's scar (theory Heller '84, Bogomolny '89, Kaplan '98)

QUE $\Rightarrow$ no *strong* scars persist asymptotically

Numerical tests

Arithmetic manifolds v. special: symmetries, all Lyapunov $\Lambda = 1$



Test generic ‘hard chaotic’ system

- Λ’s differ

Euclidean Sinai-type billiard:
concave walls $\Rightarrow$ Anosov (Sinai ’70)

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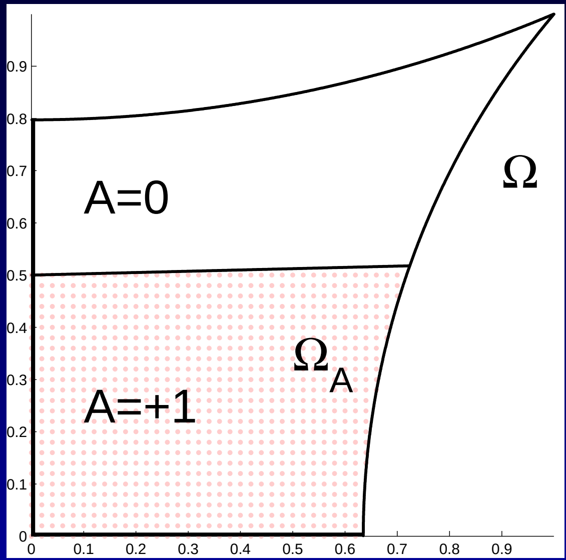


Euclidean Sinai-type billiard:
concave walls $\Rightarrow$ Anosov (Sinai '70)

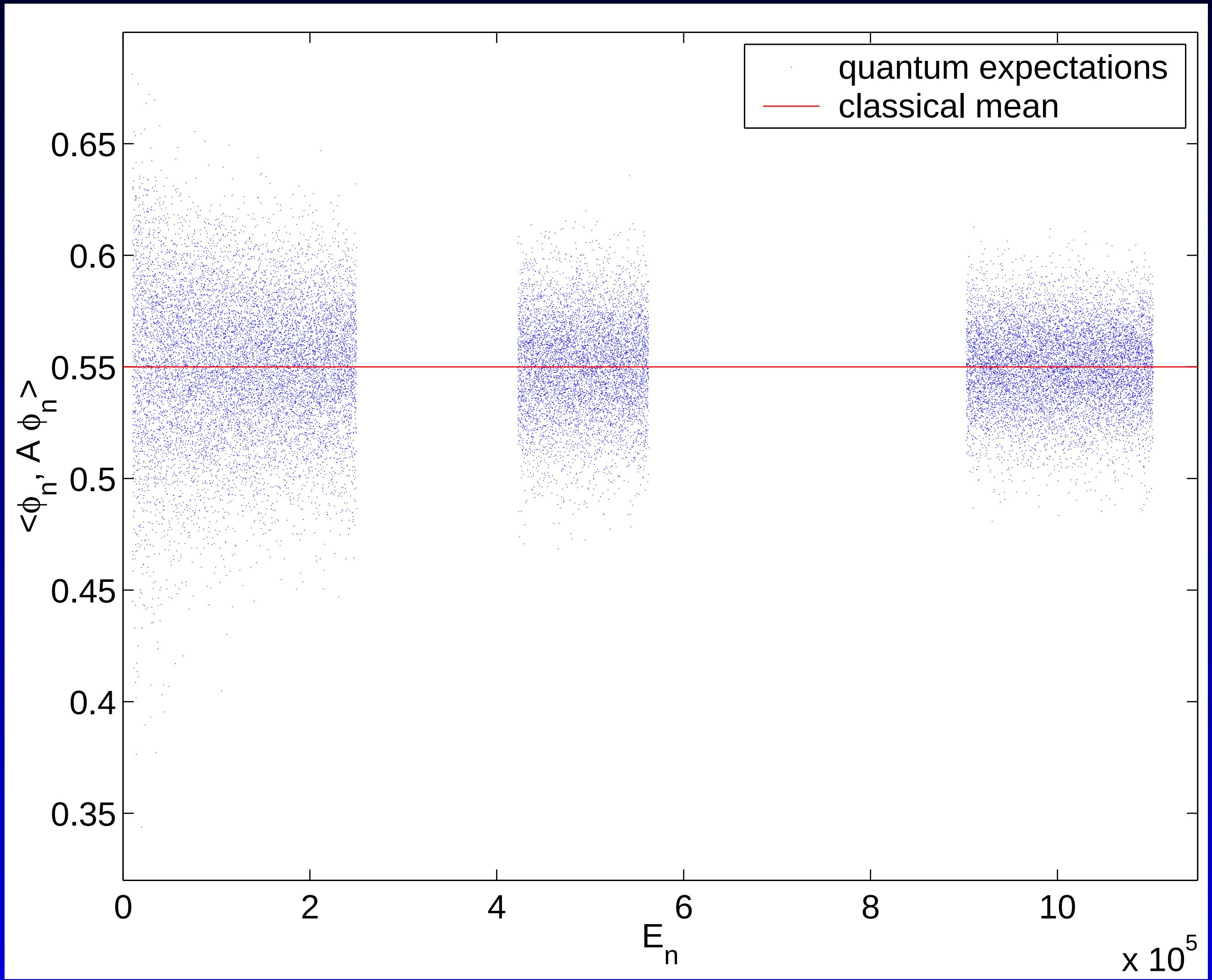
$A(\mathbf{r})$ = piecewise const: quantum calc. *fast*, use only $\partial\Omega_A$ data

- classical $\overline{A} = \mathrm{vol}(\Omega_A)/\mathrm{vol}(\Omega)$
- quantum $\langle \phi_n, \hat{A}\phi_n \rangle$ = mass inside Ω_A

Essential identity: if $-\Delta\phi = E\phi$, have for arbitrary $\phi|_{\partial\Omega_A}$,

$$\langle \phi, \phi \rangle_{\Omega_A} = \frac{1}{2E} \int_{\partial\Omega_A} (E\phi^2 - |\nabla\phi|^2) + 2(\mathbf{n}\cdot\nabla\phi)(\mathbf{r}\cdot\nabla\phi)\, dA$$

Results: Diagonal matrix elements



25000 levels
of first 50000

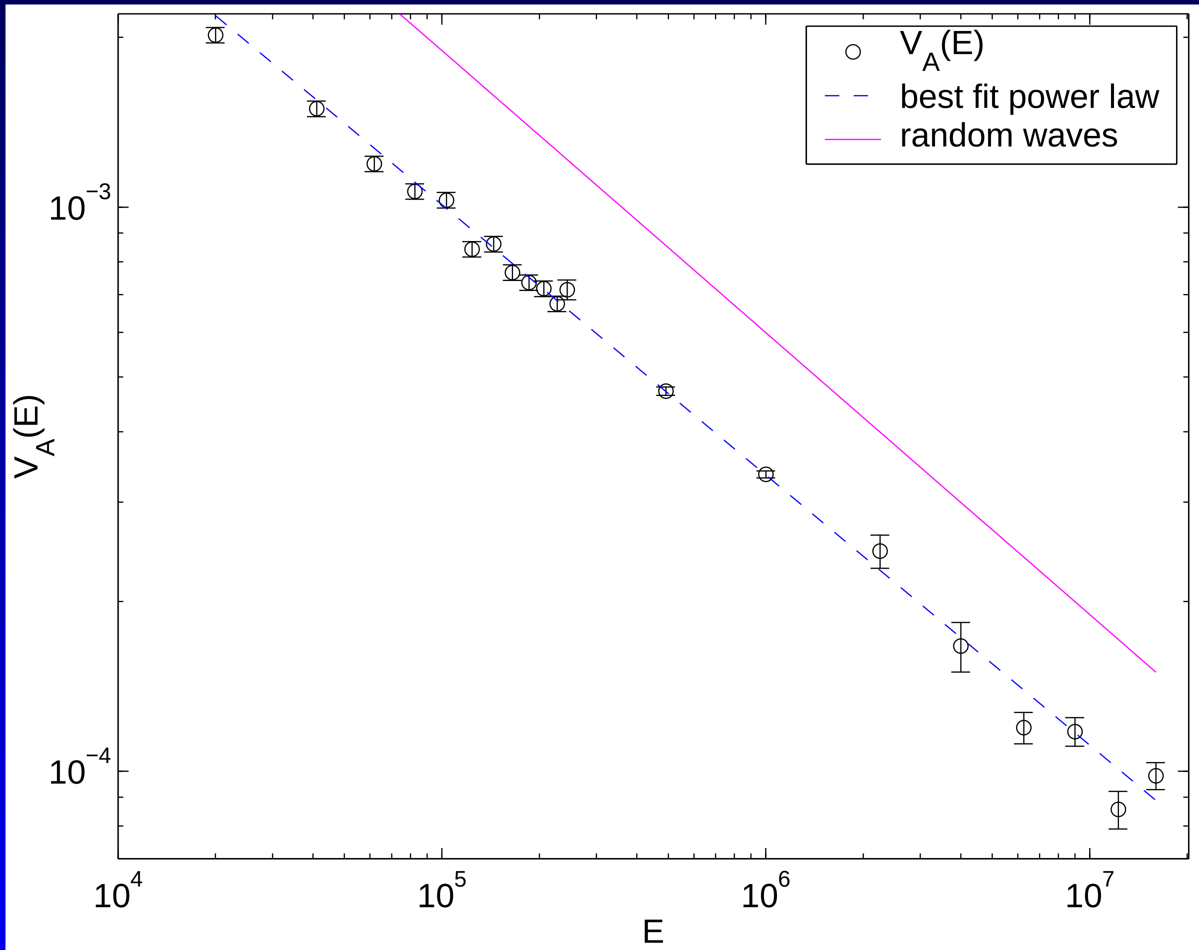
mean
$\langle \phi_n, \hat{A}\phi_n \rangle \to \overline{A}$

- Strong evidence for QUE: exceptional set density $\lesssim 3 \times 10^{-5}$
- Variance slowly decreasing, but how?

Results: Rate of variance decay

Local variance $\quad V_A(E) := \dfrac{1}{m} \displaystyle\sum_{E_n \approx E}^{N \le n < N+m} \left| \langle \phi_n, \hat{A}\phi_n \rangle - \overline{A} \right|^2$

- Hard to measure *e.g.* 1% needs $m \sim 2 \times 10^4$ indep samples!



Results: Power law

Variance model $V_A(E) = aE^{-\gamma}$, find $\gamma = 0.48 \pm 0.01$

- data statistically consistent with model, also with $\gamma = 1/2$

  same γ as $\begin{cases} \text{random waves} \\ \text{scar theory (scar width} \sim E^{-1/4}\text{)} \end{cases}$

- prefactor a: random wave prediction 80% too big (unsurprising)

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Previous studies: (low n, unquantified accuracy, various $\gamma \approx 0.3$-0.5)

- hyperbolic manifolds, level $n < 6000$ (Aurich-Taglieber '98)
- Euclidean, level $n < 6000$, many A choices (Bäcker '98)

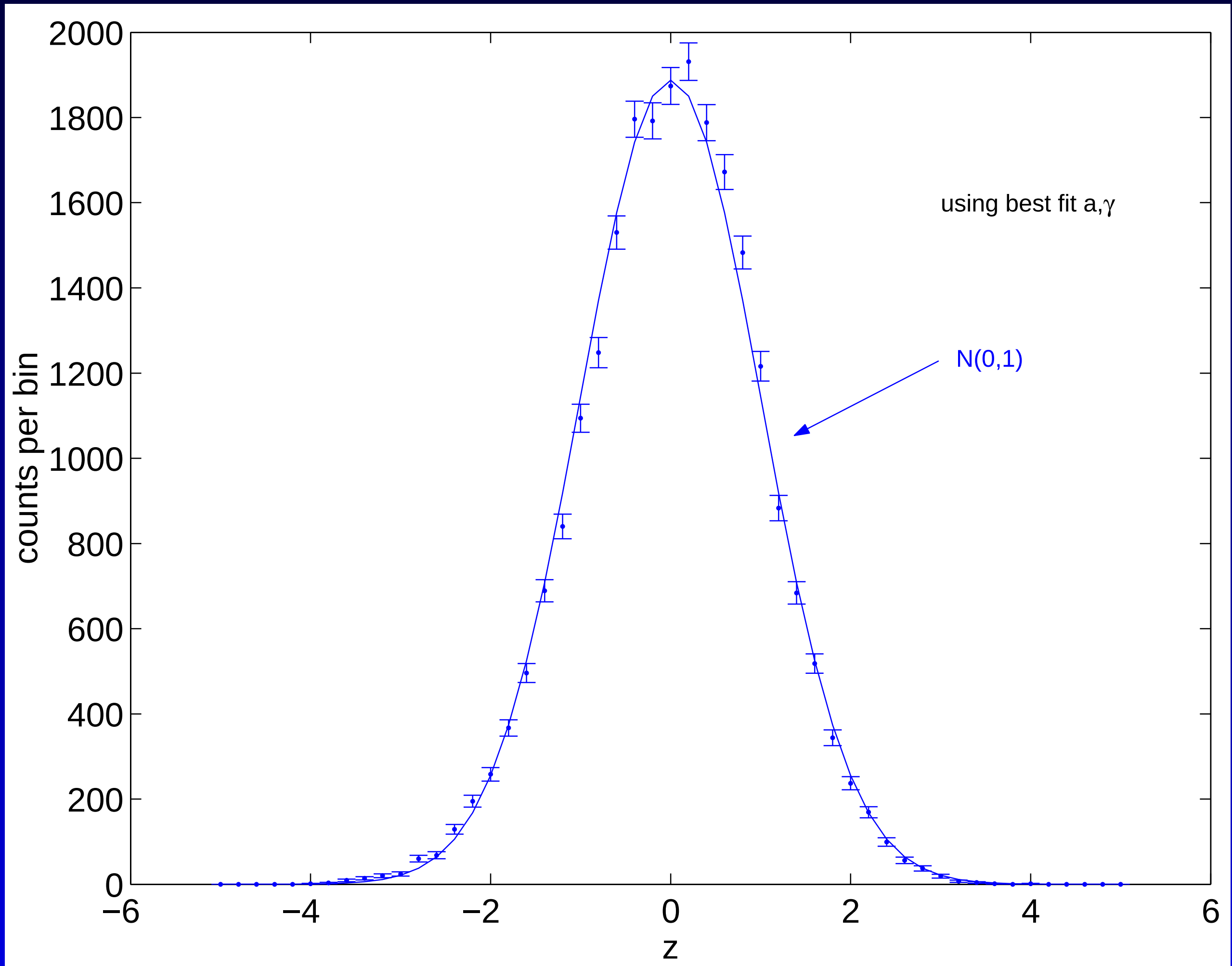
But convergence slow: power law not asymptotic until $n \geq 10^4$

…we go $100\times$ higher!

$n \approx 7 \times 10^5, \quad E \approx 1.6 \times 10^7$

Results: Distribution about the mean

Histogram scaled deviations $(\langle \phi_n, \hat{A}\phi_n \rangle - \overline{A})/\sqrt{V_A(E_n)}$:



- convincing Gaussian (consistent with random wave, RMT)

Theory: Semiclassical variance

(Feingold-Peres ’86)

FP step (i):

Autocorrelation of signal $A(t) := A(\mathbf{r}(t))$ class. trajectory $\mathbf{r}(t)$

$$\langle A(t)A(t+\tau)\rangle_t \overset{\text{ergod}}{=} \overline{A(0)A(\tau)} \underset{\text{QET}}{\overset{\text{almost all } n}{\longleftarrow}} \langle \phi_n, \hat{A}(0)\hat{A}(\tau)\phi_n\rangle$$

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$\downarrow$ Fourier transform $\downarrow$

classical noise power spectrum $\tilde{C}_A(\omega)$

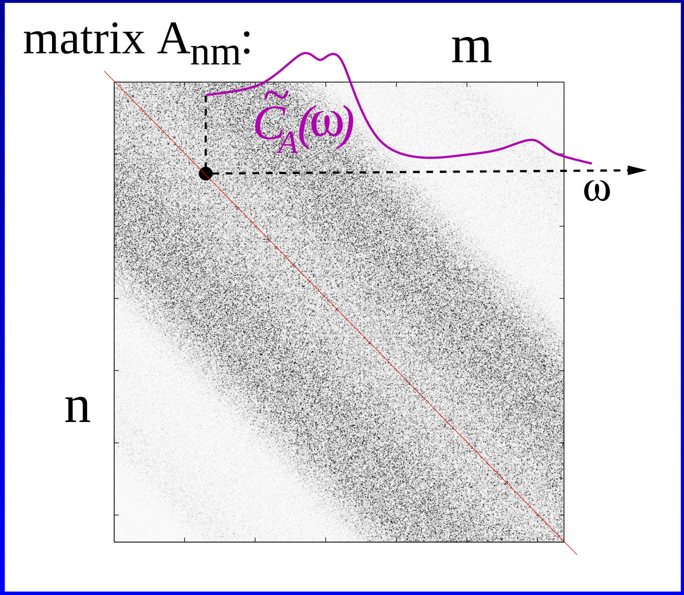
'band profile' of the matrix $A_{nm} := \langle \phi_n, \hat{A}\phi_m\rangle$

Sum rule for local variance, $n \neq m$,

$$\mathrm{Var}[A_{nm}] \to \tilde{C}_A(\omega_{nm})/k_n \mathrm{vol}(\Omega)$$

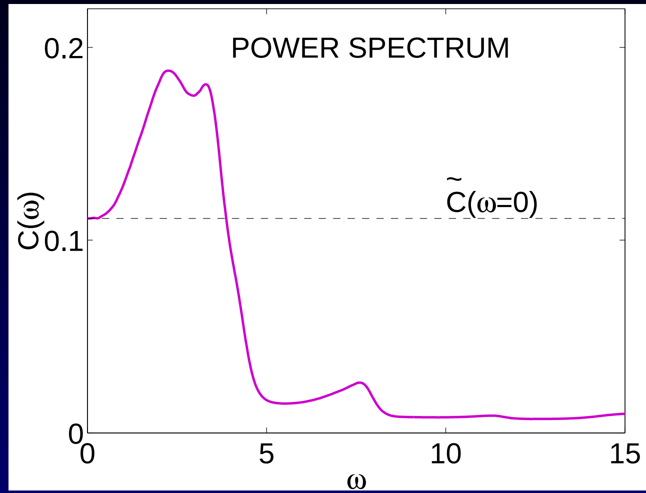
$\omega_{nm} := k_m - k_n$ 'distance' from diag.

(proven, Combescure-Robert '94)



Estimate $\tilde{C}_A(\omega)$ numerically:

Compute power spectrum of $A(t)$ via long trajectories $\boxed{\text{LISTEN to } A(t)}$



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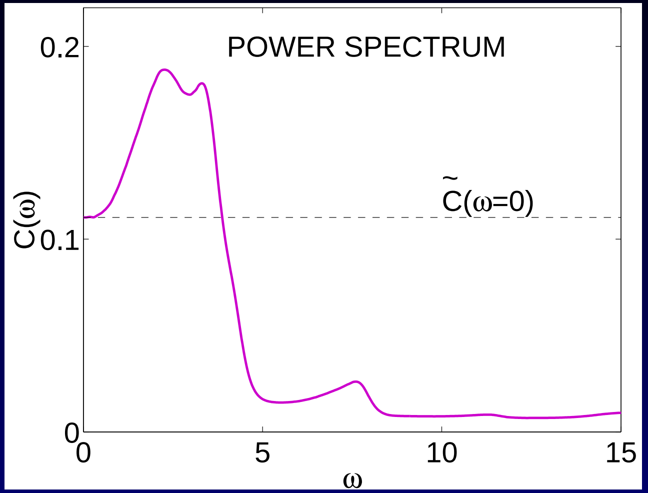


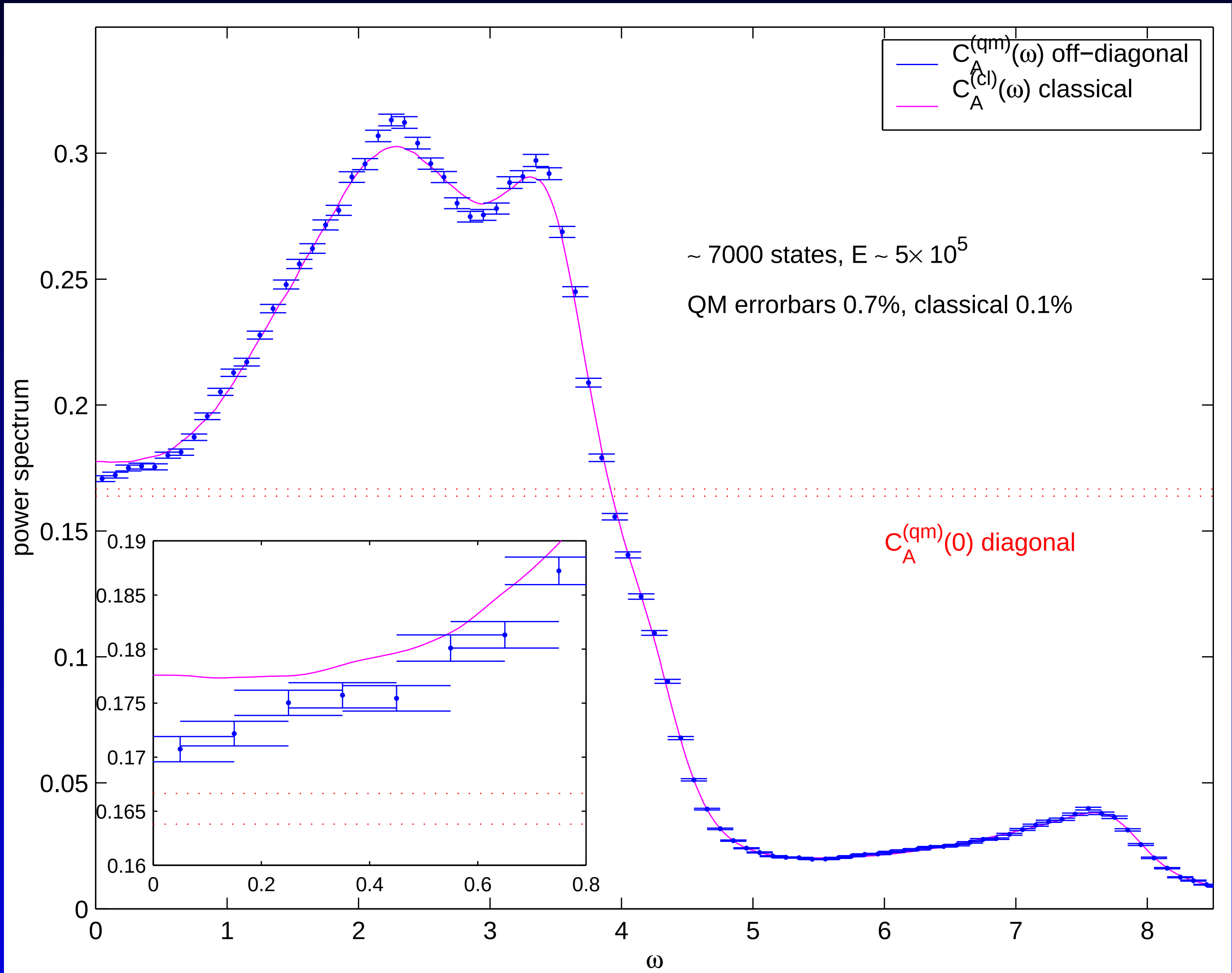
FP step (ii):

Take $\lim_{\omega\to 0}$ then *assume* GOE symm. factor $g = 2$,

$$V_A(E) \;=\; \mathrm{Var}[A_{nn}] \;\to\; \underbrace{\frac{2}{\mathrm{vol}(\Omega)}\tilde{C}_A(\omega=0)}_{\text{prefactor } a_{\mathrm{FP}}} E^{-1/2}, \qquad \leftarrow \;\gamma = 1/2$$

(Eckhardt et al. ’95, bypass GOE symm. argument via periodic orbit sum)

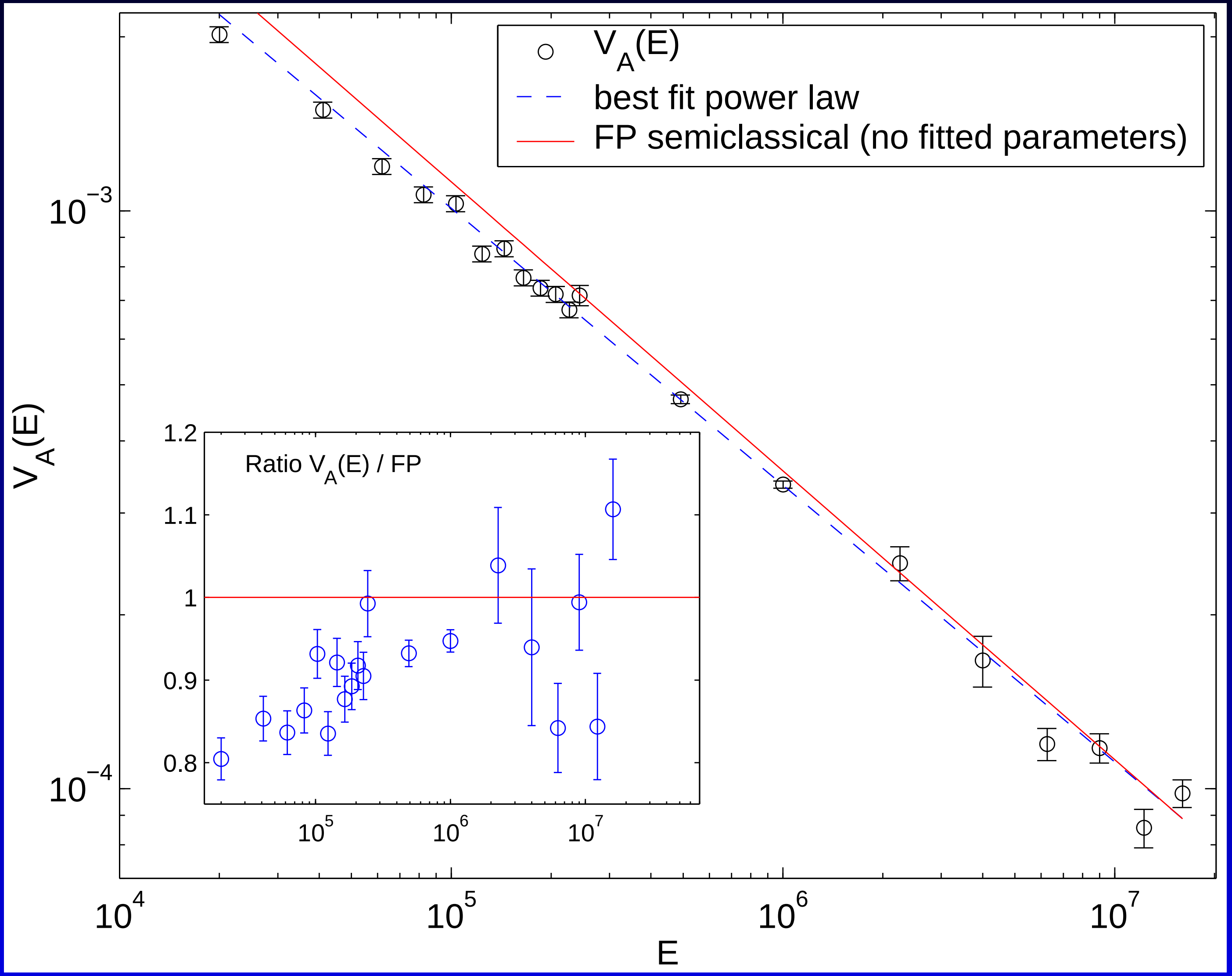
Results: Offdiagonal variance vs. FP step (i)



2.4×10^6 indep. matrix elements

- Most accurate known test of FP for billiards; no fitted params.
- clear 3% error (4σ) near $\omega = 0 \ldots$ convergence is slow!

Results: Diagonal variance vs. FP



- Prefactor: a_{FP} still $7 \pm 1\%$ too big at $n \sim 10^5$
- cf. arithmetic manifolds where $a \neq a_{\text{FP}}$ (Sarnak-Luo '03, proven)

Numerics: scaling method (Vergini-Saraceno '94)

Finds ϕ_j and $k_j^2 = E_j$ efficiently $\sim 10^3$ faster than any other known!

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Relies on 'quasi-orthogonality' of $\partial_n \phi_j$ on $\partial\Omega$:

$$\tilde{Q}_{ij} := \frac{1}{2k_i k_j} \int_{\partial\Omega} \mathbf{r} \cdot \mathbf{n}\, \partial_n \phi_i \partial_n \phi_j \, dA = \delta_{ij} + \text{small error}, \qquad i \approx j$$

- Conj. (V-S '94): error $\sim |E_i - E_j|$.
- (B-Cohen-Heller '00): if Ω erg, FP (i) $\Rightarrow$ error $\sim (E_i - E_j)^2$

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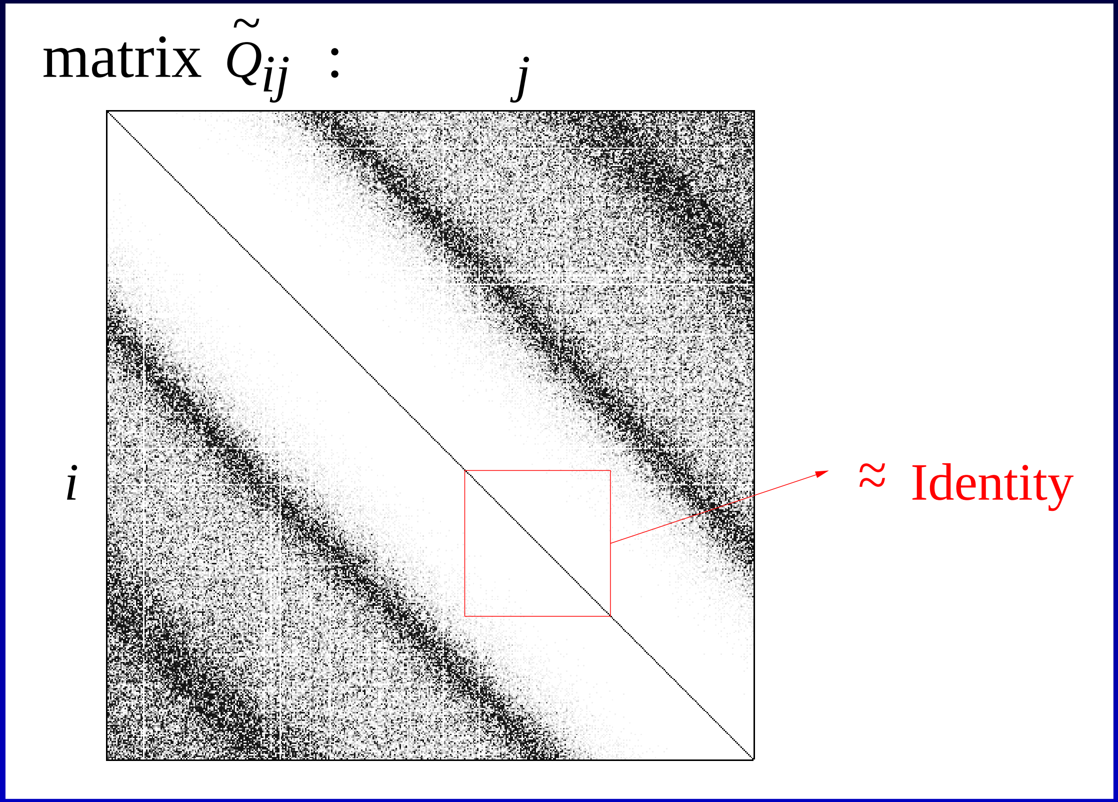
Thm (B '04): *for all* Ω, *regardless of ergodicity,*

$$\tilde{Q}_{ij} = \delta_{ij} + \frac{(E_i - E_j)^2}{8k_i k_j} \underbrace{\langle \phi_i, r^2 \phi_j \rangle}_{\text{bounded by radius of } \Omega}$$

Proof: Divergence Thm, algebra (*e.g.* symbolic 8-by-8 matrix)

Quasi-orthogonality

Example: $\Omega =$ stadium



Corollary: $\tilde{Q}_{ij} \longrightarrow \delta_{ij}$ as $E_i \longrightarrow \infty, \quad |E_i - E_j| = o(E_i^{1/2})$

Scaling method II

Space $\mathcal{H}_\Omega(k) := \{u : -\Delta u = k^2 u \text{ in } \Omega\}$
Let $\omega_j := k - k_j$ (wavenumber offset of level j)

- for $\omega_j = 0$: $\phi_j \in \mathcal{H}_\Omega(k)$
- for $\omega_j < 0$: rescaled (dilated) $\phi_j^k := \phi(k\mathbf{r}/k_j) \in \mathcal{H}_\Omega(k)$

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Let nonneg. bilin. form $f(u,v) := \int_{\partial\Omega} (\mathbf{r}\cdot\mathbf{n})^{-1} uv\, dA$

- f approx diag. in the rescaled eigenbasis $\{\phi_j^k : |\omega_j| < O(1)\}$

  proof, expand in ω: $f(\phi_i^k, \phi_j^k) = \omega_i\omega_j\tilde{Q}_{ij} + O(\omega^2) \approx \omega_i^2\delta_{ij}$
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Method

Simultaneously diagonalize f, g in some basis for $\mathcal{H}_\Omega(k)$ Eigenvectors $\approx \phi_j^k$, eigenvalues $\approx \omega_j$

Notes: Ω must be star-shaped $(\mathbf{r}\cdot\mathbf{n} > 0)$
$u \in \mathcal{H}_\Omega(k)$, $f(u, \phi_j^k) = 0 \;\Rightarrow\; g(u, \phi_j^k) \approx 0$ (in progress…)

Basis set for $\mathcal{H}_\Omega(k)$ **(B '02)**

Plane waves

$\xi_l(\mathbf{r}) = e^{i\mathbf{k}_l\cdot\mathbf{r}}$

- inaccurate *e.g.* $\Omega =$ stadium
- fail as $E \to \infty$ if Ω nonconvex (Gutkin '03, proof)

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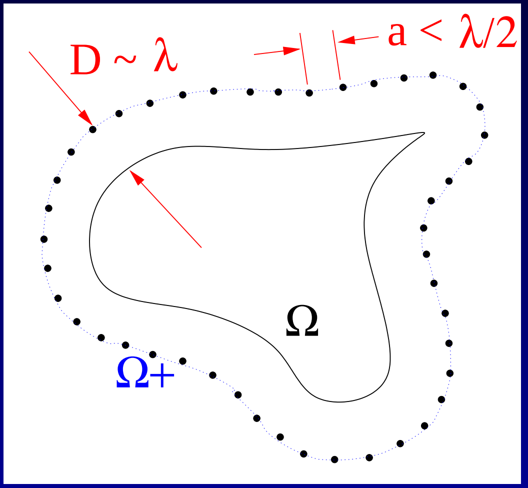

Why?

- $\partial\Omega^+$ layer potential complete in $\mathcal{H}_{\Omega^+}(k)$, for $k \neq$ Dir. eigenvalue of Ω^+
- $\mathcal{H}_{\Omega^+}(k)$ dense $\subset$ $\mathcal{H}_\Omega(k)$ (Lax '56, Runge approx)

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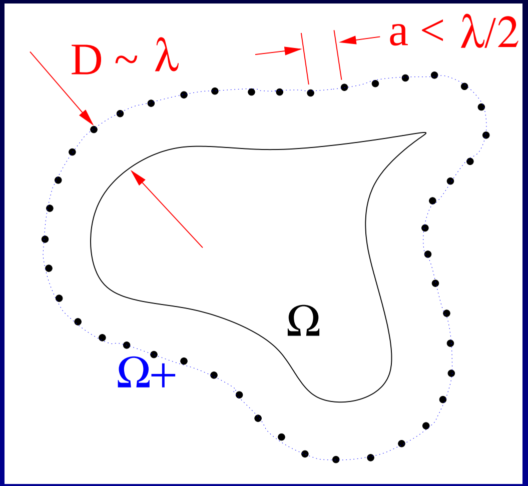

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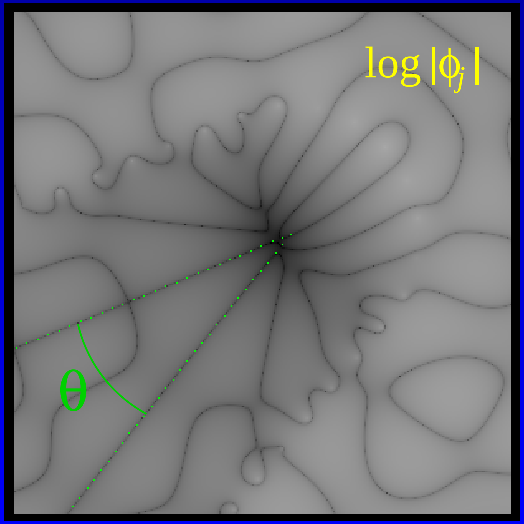
If no analytic continuation of
ϕ_j outside Ω ? (Eckmann-Pillet '95)

e.g. corner, $\theta \neq \dfrac{\pi}{n}$



Empirical: errors small! MOVIE

Numerics: practicalities

- Basis size $N \sim k = E^{1/2} \sim n^{1/2}$ for $d = 2$ *e.g.* $N = 2000$
- Overall effort $O(N^2)$ per level

  $\ldots$cf. boundary integral methods $O(N^3)$ per level
- 10^4 levels at $n \sim 10^5$: a few CPU-days
- Eigenfunction errors $\sim 10^{-5}$, no missing levels

Conclusions

Anosov Euclidean billiard at unprecedented E & number of ϕ_j:

- Strong evidence for QUE
- Matrix elements A_{nm}: variance $\sim E^{-1/2}$
- Feingold-Peres estimate good: diagonal & off-diagonal
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Questions:

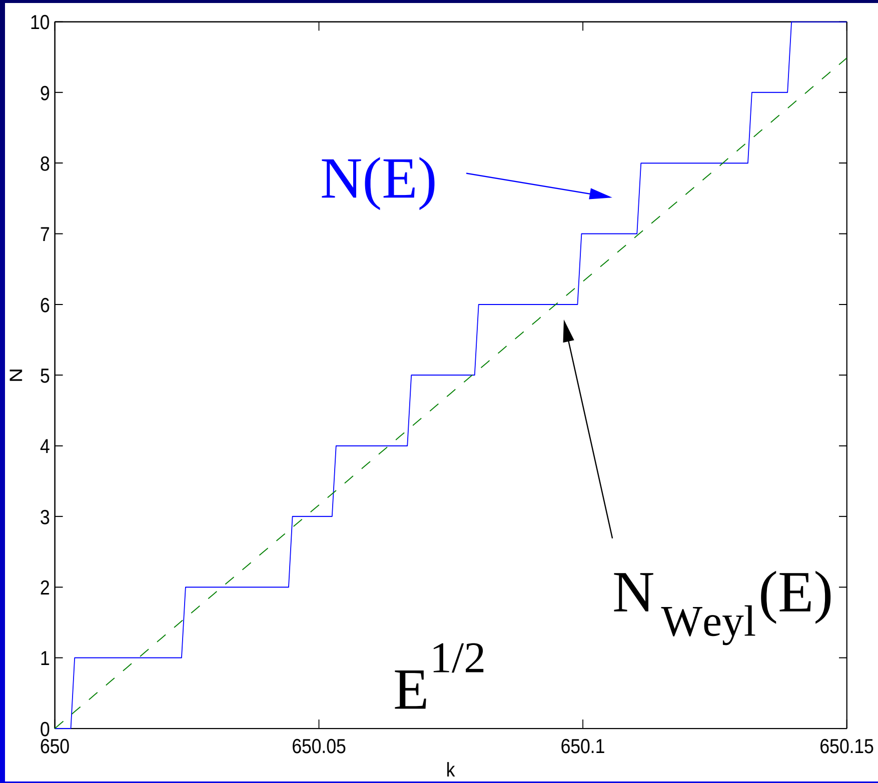
- Do these results depend on choice of A, of domains Ω?
- Efficient numerical method brings math questions:
  - boundary properties of eigenfunctions
  - basis sets for $\mathcal{H}_\Omega(k)$: analysis at corners, re-entrant?

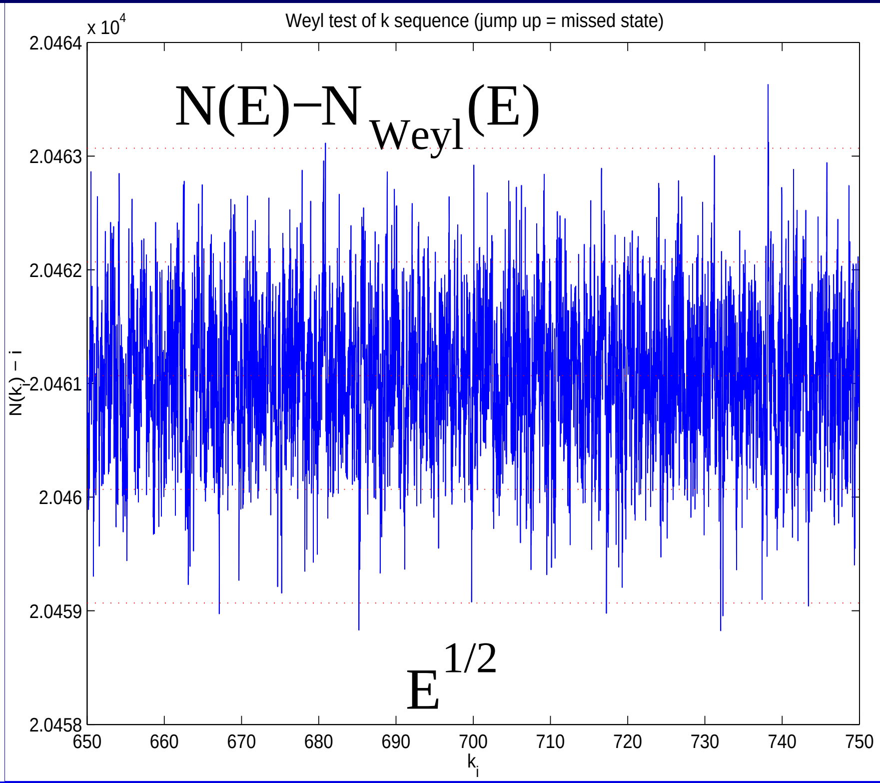
Preprints/software: `http://www.cims.nyu.edu/~barnett`

Missing levels?

Weyl's estimate for $N(E)$, the # eigenvalues $E_j < E$:

$$N_{\text{Weyl}}(E) = \frac{\text{vol}(\Omega)}{4\pi} E - \frac{L}{4\pi}\sqrt{E} + O(1) \cdots$$





- not one level missing in sequence of 6812