Littlewood-Richardson rule

Let μ, ν, λ be partitions such that $\mu, \nu \subset \lambda$. A tableau T of shape λ/μ of weight λ is a filling of μ/λ with ν_1 1's, ν_2 2's, $\dots$, such that the row filling are weakly increasing and the column fillings are strictly decreasing. We can derive a word $w(T) = a_1 a_2 \cdots a_N$, where $N = |\lambda/\mu|$, by reading off the numbers in T rights to left, top to bottom. For example, if

$$\mu = \ydiagram{1,1}, \quad \nu = \ydiagram{4,2,2}, \quad \lambda = \ydiagram{5,3,2},$$

$$T = \begin{ytableau} \none & 1 & 1 & 1 & 2 \\ \none & 2 & 3 \\ 1 & 3 \end{ytableau},$$

has word $w(T) = 2\ 1\ 1\ 1\ 3\ 2\ 3\ 1$.

A word $w = a_1 a_2 \cdots a_N$ is a *lattice permutation* if for each $1 \leq r \leq N$ and $2 \leq i \leq n$, there are no more occurrences of i than that of $i-1$. So $2\ 1\ 1\ 1\ 3\ 2\ 3\ 1$ is not a lattice permutation, but $1\ 2\ 1\ 1\ 3\ 2\ 3\ 1$ is. In the above example, the only tableau of shape λ/μ of weight ν which is also generates a lattice permutation is the one generating the word $1\ 1\ 1\ 1\ 2\ 2\ 3\ 3$.

Theorem 1 (Littlewood-Richardson rule)**.** *Let s_λ be the Schur function indexed by the partition λ. Then*

$$s_\mu s_\lambda = \sum_{\substack{\mu, \nu \subset \lambda \\ |\lambda| = |\mu| + |\nu|}} c^\lambda_{\mu,\nu} s_\lambda$$

where $c^\lambda_{\mu\nu}$ is the number of tableau of shape λ/μ of weight ν which generates a lattice permutation.

Example.

$$\ydiagram{2,1} \cdot \ydiagram{2,1} = \begin{array}{lll} \,0 \cdot (2,1,1,1,1) & +\,1 \cdot (2,2,1,1) & +\,1 \cdot (3,1,1,1) \\ +\,1 \cdot (2,2,2) & +\,2 \cdot (3,2,1) & +\,1 \cdot (3,3) \\ +\,1 \cdot (4,1,1) & +\,1 \cdot (4,2) & +\,0 \cdot (5,1) \end{array}$$

The interesting factor in here $2 \cdot (3,2,1)$, represented by

$$\begin{ytableau} \none & \none & 1 \\ \none & 1 \\ 2 \end{ytableau} \quad \text{and} \quad \begin{ytableau} \none & \none & 1 \\ \none & 2 \\ 1 \end{ytableau}.$$

Incidentally, $c^\lambda_{\mu\nu}$ is also the number of ways to build λ from μ by laying down boxes from the first row of ν into rows 1 through $\ell(\mu)+1$, then from the second row of ν into rows f_1+1 through $\ell(\mu)+2$ (where f_1 is the first row with boxes from ν_1 in it), then from the third row of ν into rows f_2+1 through $\ell(\mu)+3$, and so on, so that at each step you have a partition.

Special Cases

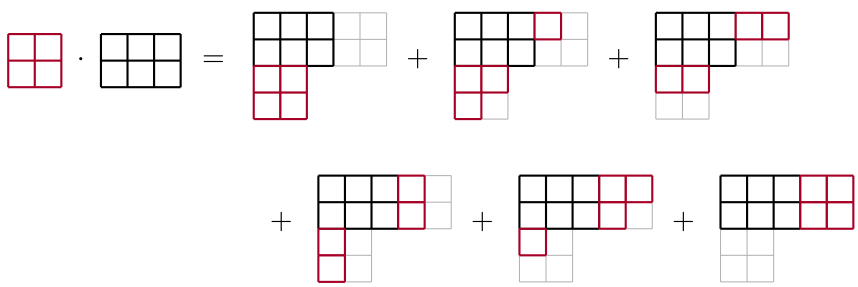
Horizontal strips: If $\nu = (n)$, then each $c^{\lambda}_{\mu\nu}$ is at most 1, since there is only one way to fill in a bunch of boxes with 1's. Moreover, λ/μ must be a *horizonal strip* (a skew shape where every column has at more one box), since if there were more than one box in any column, we would have to fill with only weakly decreasing numbers.

Vertical strips: If $\nu = (1^n)$, then, again, each $c^{\lambda}_{\mu\nu}$ is at most 1, and is nonzero exactly when λ/μ is a length n *vertical strip* (a skew shape where every row has at more one box).

Rectangles: Let $m \leq n$ and s, t be non-negative integers. Then each $c^{\lambda}_{(s^m)(t^n)}$ is 1 if $\lambda \in \mathcal{T}(s^m, t^n)$, and is zero otherwise, where $\mathcal{T}(s^m, t^n)$ is the set of partitions λ with length $\leq m + n$ such that

$$\begin{aligned}
&\lambda_{m+1} = \lambda_{m+2} = \cdots = \lambda_n = t, \\
&\lambda_m \geq \max(s, t), \\
&\lambda_i + \lambda_{m+n-i+1} = s + t, \quad i = 1, \ldots, m.
\end{aligned}$$

In other words, $\mathcal{T}(s^m, t^n)$ is the set of partitions made of stacking (s^m) on top of (t^n), carving a corner out of (t^n), rotating it 180° and gluing it to the end of (s^m). For example,



References

[Mac] I. G. Macdonald *Symmetric functions and Hall polynomials*, 2nd ed., Oxford University Press, 1995.