Markov Chain Lab

Sociologists have determined that the strongest indicator of a person's income class is the income class of their parents. Let state 1 denote the lower-class, state 2 denote the middle-class, and state 3 represent the upper-class.

Let p_{ij} denote the probability that an individual is in state i given that their parents were in state j.

Suppose sociologists have determined that the probability of someone being lower, middle, or upper class given that their parents were lower class is equal to 0.65, 0.28, and 0.07, respectively. The probability of someone being lower, middle, or upper class given their parents were middle-class is 0.15, 0.67, and 0.18 respectively. And finally, the probability that an individual is lower, middle, or upper-class given that their parents were upper-class is 0.12, 0.36, 0.52, respectively.

1. (a) Draw the transition diagram using the data provided.

   (b) The corresponding transition matrix is:

$$A = \begin{bmatrix} \quad & \quad & \quad \\ \\ \\ \end{bmatrix}$$

   (c) Suppose the current income distribution is $\mathbf{x}_0 = \begin{bmatrix} .21 \\ .68 \\ .11 \end{bmatrix}$. What will be the income distribution in the next generation? In the generation after that? Recall that the population distribution in the next generation is given by $\mathbf{x}_1 = A\mathbf{x}_0$, the following generation has income distribution $\mathbf{x}_2 = A\mathbf{x}_1 = A(A\mathbf{x}_0) = A^2\mathbf{x}_0$, and so on. In general, the income distribution of the $(k+1)^{th}$ generation is $\mathbf{x}_{k+1} = A\mathbf{x}_k = A^{k+1}\mathbf{x}_0$, $k = 0, 1, 2, ...$ Record your answers below:

$$\mathbf{x}_1 = \begin{bmatrix} \quad \\ \\ \\ \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} \quad \\ \\ \\ \end{bmatrix}$$

   (d) The m-file provided includes some code to simulate the change in the income distribution across 20 (future) generations. Run the code by typing the name of the m-file in the MATLAB command window. This Markov Chain model allows us to make predictions about the income distribution for future generations. What do you notice as k increases?

(e) Record the income distribution (to two decimal places) for $k = 20$ below.

$$\mathbf{x}_{20} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

(f) Modify the code by changing the initial conditions and run the code again. Record your choice for the new initial condition and the income distribution (to two decimal places) for $k = 20$ below.

$$\tilde{\mathbf{x}}_0 = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}, \quad \tilde{\mathbf{x}}_{20} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

(g) Compare your answers to (1e) and (1f). What conclusion might you draw from your results?

The following theorem should confirm your hypothesis:

> **Theorem 0.0.1** *If A is an $n \times n$ regular stochastic matrix, then A has a unique steady-state vector $\mathbf{q}$. Furthermore, if $\mathbf{x_0}$ is any initial state and $\mathbf{x}_{k+1} = A\mathbf{x}_k$ for $k = 0, 1, 2, ...$, then the Markov chain $\{\mathbf{x}_k\}$ converges to $\mathbf{q}$ as $k \to \infty$.*

2. The following steps will guide you towards finding the steady-state of our Markov Chain analytically. You may use MATLAB to assist with the computations.

   (a) A vector $\mathbf{q}$ is a steady-state of the Markov chain if it remains unchanged under the action of the matrix A, that is, if $A\mathbf{q} = \mathbf{q}$, where $\mathbf{q}$ is a probability vector. We have that

   $$A\mathbf{q} = \mathbf{q} \text{ iff } A\mathbf{q} - \mathbf{q} = \mathbf{0} \text{ iff } (A - I)\mathbf{q} = \mathbf{0}.$$

   According to Theorem 0.0.1, the equation $(A - I)\mathbf{q} = \mathbf{0}$ along with the condition that the components of $\mathbf{q}$ must be nonnegative and sum to 1 should yield a unique solution. Find the solution to $(A-I)\mathbf{q} = \mathbf{0}$ subject to the additional constraint that $q_1 + q_2 + q_3 = 1$, where q_i is the i^{th} entry of $\mathbf{q}$, and record your answer in the space provided below. This solution is the steady-state of the difference equation.

   $$\mathbf{q} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

   (b) Is your answer in part (2a) consistent with your previous observations?

3. The following questions illustrate the role of eigenvalues and eigenvectors in the Markov chain.

   (a) Recall that λ is an eigenvalue of A iff $det(A - \lambda I) = 0$. However, for this lab, you may determine the eigenvalues of A using the 'eig' function in MATLAB. List the eigenvalues of A below:

   $$\lambda_1 = \underline{\qquad\qquad}$$
   $$\lambda_2 = \underline{\qquad\qquad}$$
   $$\lambda_3 = \underline{\qquad\qquad}$$

   (b) Show that the steady-state vector found in part (2a) is an eigenvector corresponding to one of the above eigenvalues.

(c) Find eigenvectors corresponding to the remaining two eigenvalues (You may use the 'eig' function again for this part, however make sure you know how you would approach this by hand). To find the eigenvectors using 'eig', you will need to use the following syntax:

```
[eigenvec eigenvals] = eig(A);
```

This will return two matrices named *eigenvecs* and *eigenvals*. The diagonal entries of *eigenvals* are the eigenvalues of A; the columns of the matrix *eigenvecs* are eigenvectors corresponding to these 3 eigenvalues.

From yesterday, we have that these vectors form a linearly independent set. List the eigenvectors below, where v_i denotes an eigenvector corresponding to eigenvalue λ_i:

$$\mathbf{v}_1 = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

(d) Because $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of 3 vectors in $\mathbb{R}^3$, it must also be a basis for $\mathbb{R}^3$. Consequently, we may write the initial condition vector $\mathbf{x}_0 \in \mathbb{R}^3$ as a linear combination of these eigenvectors; that is, there exist scalars not all zero such that $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. Use MATLAB to find these scalars c_1, c_2, c_3 and list them below (you may use the 'linsolve' function, or 'rref' in MATLAB to accomplish this):

$$c_1 = \underline{\qquad\qquad}$$
$$c_2 = \underline{\qquad\qquad}$$
$$c_3 = \underline{\qquad\qquad}$$

4. Our goal now is to write the solution to the difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ $(k = 0, 1, 2, ...)$ using the eigenvalues and corresponding eigenvectors of A found above.

   (a) Recall that $\mathbf{x}_{k+1} = A\mathbf{x}_k$ can also be written as $\mathbf{x}_{k+1} = A^{k+1}\mathbf{x}_0$, where $\mathbf{x}_0$ is the initial condition vector. This is equivalent to $\mathbf{x}_k = A^k\mathbf{x}_0$, with $k = 0, 1, 2, ...$ We now know how to write $\mathbf{x}_0$ in terms of the eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Plugging $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ into $\mathbf{x}_k = A^k\mathbf{x}_0$, we obtain (provide justifications in the provided spaces):

$$
\begin{aligned}
\mathbf{x}_k &= A^k(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) \\
&= A^{k-1}c_1(A\mathbf{v}_1) + A^{k-1}c_2(A\mathbf{v}_2) + A^{k-1}c_3(A\mathbf{v}_3) \qquad \underline{\hspace{4cm}} \\
&= A^{k-1}c_1(\lambda_1\mathbf{v}_1) + A^{k-1}c_2(\lambda_2\mathbf{v}_2) + A^{k-1}c_3(\lambda_3\mathbf{v}_3) \qquad \underline{\hspace{4cm}} \\
&= A^{k-2}c_1(\lambda_1 A\mathbf{v}_1) + A^{k-2}c_2(\lambda_2 A\mathbf{v}_2) + A^{k-2}c_3(\lambda_3 A\mathbf{v}_3) \\
&= A^{k-2}c_1(\lambda_1^2\mathbf{v}_1) + A^{k-2}c_2(\lambda_2^2\mathbf{v}_2) + A^{k-2}c_3(\lambda_3^2\mathbf{v}_3) \\
&\;\;\vdots \\
&= c_1\lambda^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + c_3\lambda_3^k\mathbf{v}_3.
\end{aligned}
$$

   That is

$$\mathbf{x}_k = c_1\lambda^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + c_3\lambda_3^k\mathbf{v}_3$$

   for $k = 0, 1, 2, ...$

   Note that the first term $c_1\lambda_1^k\mathbf{v}_1$ is precisely the steady-state vector found in (2a). Explain why $\mathbf{x}_k$ must converge to the steady state as $k \to \infty$.

   (b) Use this construction of the solution $\mathbf{x}_k$ to plot the solution set to the Markov chain and compare your plot to the figure produced by the given code. You should observe that the figures are identical.

   One way of accomplishing this is as follows (Note that indices of x don't match the powers of lambda):

```
[eigenvec eigenvals] = eig(A);

Lambda = diag(eigenvals);

c_i = linsolve(eigenvec,x0);


for k = 1:20
x(k,:) = c_i(1)*Lambda(1)^(k-1)*eigenvec(:,1) +...
         c_i(2)*Lambda(2)^(k-1)*eigenvec(:,2) +...
         c_i(3)*Lambda(3)^(k-1)*eigenvec(:,3);
end

figure
plot(x,'*-')
```