Workshop Problems 6

Problem 1. Let V and W be vector space, and let $T : V \to W$ be a linear transformation. Given a subspace U of V, let $T(U)$ denote the set of all vectors in W of the form $T(\mathbf{x})$, where $\mathbf{x}$ is in U. Show that $T(U)$ is a subspace of W.

Problem 2. Given $T : V \to W$ as in Problem 1, and given a subspace Z of W, let U be the set of all $\mathbf{x}$ in V so that $T(\mathbf{x})$ is in Z. Show that U is a subspace of V.

Throughout the following exercises, V is a vector space with basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$, and $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is the coordinate map from V to $\mathbb{R}^n$.

Problem 3. Show that the coordinate map is one-to-one. That is, show that if $[\mathbf{u}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{B}}$ for some $\mathbf{u}$ and $\mathbf{v}$ in V then $\mathbf{u} = \mathbf{v}$.

Problem 4. Show that the coordinate map is onto. That is, show that given any $\mathbf{x}$ in $\mathbb{R}^n$ there is a vector $\mathbf{u}$ in V so that $[\mathbf{u}]_{\mathcal{B}} = \mathbf{x}$.

Problem 5. Show that a set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ of vectors in V is linearly independent if and only if the set of coordinate vectors $\{[\mathbf{u}_1]_{\mathcal{B}}, [\mathbf{u}_2]_{\mathcal{B}}, \ldots, [\mathbf{u}_p]_{\mathcal{B}}\}$ is linearly independent in $\mathbb{R}^n$.