Math 9 Fall 2002, 1st Midterm Exam, October 21

1. The direction fields of the four given differential equations are among the five given direction fields. Match the differential equations with their direction fields.

   $$\frac{dy}{dx} = \frac{-x}{y}$$

   $$\frac{dy}{dx} = 1-2xy$$

   $$\frac{dy}{dx} = xy^2$$

   $$\frac{dy}{dx} = x-y$$

2. Solve these two initial value problems.

   a
   $$x\frac{dy}{dx} = x + y, \quad x > 0, \quad y(1) = 10$$

   b
   $$y'' + 2y' - 8y = 0, \quad y(0) = 5, \quad y'(0) = -2$$

3. Find a function $y = f(x)$ whose graph passes through the point $(0, 3)$ and whose tangent line at the point (x, y) has slope $\frac{2x}{y^2}$.

4. Find all the cube roots of $-1 + i$. Write them in the form $re^{i\theta}$ where $0 \leq \theta < 2\pi$.

5. Calculate the radius of convergence for the series

   $$\sum_{n=1}^{\infty} \frac{(x-2)^n}{(3n+1)^5 2^n}.$$

6. a Is the sequence $\left\{\frac{(-1)^n n^2}{n^2+1}\right\}$ convergent? If it is convergent, find its limit. If not, explain clearly why not.

   b Assume that the sequence s_n of the n-th partial sums of the series $\sum_{i=1}^{\infty} a_i$ is given by the formula $s_n = \frac{1}{n^{1/2}}$. Is the series convergent? If it is convergent, find its sum. If not, explain clearly why not.

7. Consider an undamped oscillator consisting of a mass attached to a spring. If an extra mass is added, the oscillator's frequency changes. This observation can be used to "weigh" an astronaut (i.e., to determine the astronaut's mass) in the weightless environment of the space station. A metal cage of mass M is attached to a spring whose spring constant is k (recall that this means that, when the spring is stretched so that the cage is x meters away from its rest position, the restoring force the cage feels is $-kx$, where k is a positive constant). When the cage is empty, the system oscillates with frequency ω_0. With an astronaut of mass m inside the cage, the new system oscillates with frequency ω.

   a Write the differential equation which the displacement function satisfies (with the cage empty).

   b Find the general solution to the equation in part (a). What are the frequencies ω and ω_0 of the two oscillators in terms of the spring constant k and the given masses?

   c Find a formula for the astronaut's mass m in terms of ω_0, ω, and k.

8. Determine whether the series is convergent or not, and explain your reasoning. Specify clearly any convergence criteria you use.

   a $\sum_{n=1}^{\infty} n \sin(\frac{1}{n})$.

   b $\sum_{n=1}^{\infty} \frac{3^n}{4^n+\pi^n}$.

9. Express as a power series the function:

$$f(x) = \frac{1}{(1-x)^3}.$$

   What is the radius of convergence of the series you find?