Barnett
5/21/09

SOLUTIONS

Math 46, Applied Math (Spring 2009): Midterm 2

2 hours, 50 points total, 7 questions, varying numbers of points (also indicated by space)

1. [5 points] Find a 2-term asymptotic expansion for $I(\lambda) = \int_\lambda^\infty t^2 e^{-t^2} dt$ in the large positive parameter $\lambda \to \infty$.

$v = e^{-t^2}$

$$I(\lambda) = \int_\lambda^\infty \overbrace{\frac{-t}{2}}^{u} \cdot \overbrace{-2te^{-t^2}}^{v'} dt \underset{\text{by parts}}{=} \frac{-t}{2}e^{-t^2}\Big|_\lambda^\infty - \int_\lambda^\infty \left(-\tfrac{1}{2}\right)e^{-t^2} dt$$

$$= \cancel{\lim_{t\to\infty}\left(\frac{-t}{2}e^{-t^2}\right)} - \frac{-\lambda}{2}e^{-\lambda^2} - \int_\lambda^\infty \underbrace{\frac{1}{4t}}_{u}\underbrace{(-2te^{-t^2})}_{v'} dt$$

$\to 0$ since exp wins over power.

$$= \frac{\lambda}{2}e^{-\lambda^2} - \frac{1}{4t}e^{-t^2}\Big|_\lambda^\infty + \underbrace{\int_\lambda^\infty \frac{-1}{4t^2}e^{-t^2} dt}_{o\left(\frac{e^{-\lambda^2}}{\lambda}\right)}$$

$$= e^{-\lambda^2}\left[\frac{\lambda}{2} + \frac{1}{4\lambda} + o\left(\frac{1}{\lambda}\right)\right]$$

2-term.

2. [7 points]

(a) Write the first 3 terms (i.e. trivial term plus two more) in the Neumann series for the solution of $u(t) = t + \lambda(Ku)(t)$, where K is a Volterra operator on $[0,b]$ with kernel $k(t,s) = st$ and $\lambda \in \mathbb{R}$ some constant.

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$f(t)$

$$u = f + \lambda K u \quad \Rightarrow \quad (I - \lambda K)u = f$$

$$\Rightarrow \quad u = (I - \lambda K)^{-1} f$$

$$= (I + \lambda K + \lambda^2 K^2 + \cdots) f$$

formally; can be rigorously justified, see book.

$$u(t) = f(t) + \lambda K f(t) + \lambda^2 (K^2 f)(t) + \cdots \quad \text{3-term}$$

$$= t + \lambda \int_0^t k(t,s) f(s)\,ds + \lambda^2 \int_0^t k(t,s) \int_0^s k(s,r) f(r)\,dr\,ds + \cdots$$

$$\int_0^t ts \cdot s\,ds = t\int_0^t s^2\,ds = t \cdot \frac{t^3}{3} = \frac{t^4}{3}$$

$$= t + \lambda \frac{t^4}{3} + \lambda^2 \int_0^t k(t,s) \frac{s^4}{3}\,ds \qquad t\int_0^t s \frac{s^4}{3}\,ds = \frac{t \cdot t^6}{3 \cdot 6}$$

$$= t + \lambda \frac{t^4}{3} + \lambda^2 \frac{t^7}{18} + \cdots$$

2

(b) Use the fact that this series is always uniformly convergent on any bounded interval to prove that K (acting on any bounded interval) has *no* eigenvalues.

eg $[0,b]$

since series uniformly convergent, it must converge to unique solution u.

Taking special case $f \equiv 0$, we have $u = \lambda K u$ has therefore unique solution. Since $u \equiv 0$ is a solution, this is unique, therefore only trivial soln. exist $\forall \lambda \in \mathbb{R}$ in fact, $\mathbb{C}$.

Thus there are no eigenvalues since by definition these are λ values for which u is nontrivial.

3. [6 points] Use an energy argument to show that the eigenvalues of the following Neumann boundary condition problem have definite sign (which?):

4

$$
\begin{aligned}
((1+x)u')' &= \lambda u \qquad \text{for } 0 < x < 1 \\
u'(0) &= u'(1) = 0
\end{aligned}
$$

Mult. ODE by u & integrate over $(0,1)$:

$$\underbrace{\int_0^1 u\,((1+x)u')'\,dx}_{\text{by parts}} = \lambda \underbrace{\int_0^1 u^2\,dx}_{>0}$$

>0 since u nontrivial if an eig.func.

$$\underbrace{\Big[u(1+x)u'\Big]_0^1}_{\text{zero at } x=0,1} - \underbrace{\int_0^1 u'(1+x)u'\,dx}_{\leq 0}$$

$$\text{so} \quad \lambda = \frac{\text{non-positive}}{\text{strictly positive}} \leq 0$$

2

Is $\lambda = 0$ an eigenvalue? If so, what is its eigenspace?

Seek general solu. to ODE with $\lambda = 0$:

$$((1+x)u')' = 0$$

$\int dx$

$$(1+x)u' = c \qquad c \text{ some const.}$$

$$u' = \frac{c}{1+x}$$

$\int dx$

$$u(x) = c\ln|1+x| + d$$

d some other const.

Apply BCs $u'(0) = 0$ gives $\frac{c}{1+0} = 0$ so $c = 0$

$u'(1) = 0$ also forces $c = 0$ but neither constrain d.

$\Rightarrow$ $u(x) = 1$ is an eigenfunction (eigenspace $=$ Span$\{1\}$)

$\lambda = 0$ is an eigenvalue

4. [10 points] Consider the integral operator $Ku(x) := \int_0^\pi x \sin y \, u(y)dy$ acting on functions on $(0, \pi)$.

5 (a) What are all eigenvalue(s) (with multiplicity) and eigenspace(s) of this operator?

K is degenerate Fredholm, with $k(x,y) = \alpha_1(x)\,\beta_1(y)$

$Ku = \lambda u$ $\qquad \begin{cases} \alpha_1(x) = x \\ \beta_1(x) = \sin x \end{cases}$

$\Rightarrow \sum_{j=1}^{n} c_j \alpha_j(x) = \lambda u(x)$

$(\beta_i, \cdot) \hookrightarrow \quad A\vec{c} = \lambda \vec{c}$ with $A = (\beta_i, \alpha_j)_{ij} = [(\sin x, x)]$

$= [\int_0^\pi x \sin x \, dx] = [(-x\cos x)_0^\pi - \underbrace{\int_0^\pi \cos x\, dx}_{0}]$

$= [\pi]$

$\lambda = \pi$ is simple eigval. w/ eigenfunction $\sum_{j=1}^{1} c_j \alpha_j(x) = 1 \cdot x = x$

Also $\lambda = 0$ is ∞-multiplicity eigval. w/ eigenspace all functions orthogonal to $\sin x$
$= \text{Span}\{\sin 2x, \sin 3x, \ldots\}$.

2 (b) Give the general solution to $Ku(x) - 2\pi u(x) = \sin x$, or explain why not possible.

$\sum c_j \alpha_j(x) - 2\pi u(x) = \sin x \qquad (*)$

$(\beta_i, \cdot) \hookrightarrow \quad [\pi] c_1 - 2\pi c_1 = (\sin x, \sin x) \rightarrow \int_0^\pi \sin^2 x \, dx = \pi/2$

so $c_1 = \dfrac{\pi/2}{\pi - 2\pi} = -1/2$ unique.

Sub. back into $(*)$: $\quad u(x) = \frac{1}{2\pi}\left(\sum c_j \alpha_j(x) - \sin x\right) = \frac{1}{2\pi}\left(-\frac{1}{2}x - \sin x\right)$

unique.

2 (c) Give the general solution to $Ku(x) - \pi u(x) = \sin 2x$, or explain why not possible.

notice λ = eigenvalue of K.

$\cancel{[\pi] c_1 - \pi c_1} = (\sin x, \sin 2x) = 0$ $\qquad$ since $F_1 = 0$, it's consistent, but c_1 = arbitrary.

cancels since $\lambda = \lambda_1$

Use $(*)$ again: $\quad u(x) = \frac{1}{\pi}\left(\sum_j c_j \alpha_j(x) - f(x)\right) = \frac{1}{\pi}\left(c_1 x - \sin 2x\right),$

$c_1 \in \mathbb{R}$

(d) Give the general solution to $Ku(x) = \sin x$, or explain why not possible.

This is 1st kind since no λ constant term.

Is RHS f in Range of operator K = Span$\{\kappa_j\}$?

No, since $\kappa_1(x) = x$, so there's no way to reach $\sin x$ by acting K on anything $\Rightarrow$ no solution.

5. [6 points] Consider the integral operator $Ku(x) = \int_1^e k(x,y)u(y)dy$ with kernel

$$k(x,y) = \begin{cases} 1 - \ln y, & x < y \\ 1 - \ln x, & x > y \end{cases}$$

$\leftarrow$ upper } $\leftarrow$ lower } part of y-integral.

split the y-integral:

Convert the eigenvalue problem $Ku = \lambda u$ into a Sturm-Liouville problem on the interval $(1, e)$. Don't forget to find homogeneous boundary conditions [Hint: one will be Dirichlet, one Neumann]

$$(Ku)(x) = \int_1^x (1-\ln x)\, u(y)\,dy + \int_x^e (1-\ln y)\, u(y)\,dy = \lambda u(x)$$

$\frac{d}{dx}$(Leibniz

sign since lower limit

$$\int_1^x \frac{-1}{x} u(y)\,dy + \underbrace{(1-\ln x)u(x) - (1-\ln x)u(x)}_{\text{cancel}} = \lambda u'(x)$$

$\frac{d}{dx}$(Leibniz

$$\int_1^x \frac{1}{x^2} u(y)\,dy - \frac{1}{x}u(x) = \lambda u''(x)$$

you could keep taking $\frac{d}{dx}$, but it would not remove the integral. Instead, recognize this as $\frac{-1}{x} \cdot \lambda u'(x)$ from previous line.

$\Rightarrow$ ODE is $-\frac{1}{x}\lambda u' - \frac{1}{x}u = \lambda u''$

ie $\underbrace{xu'' + u'}_{(xu')'} + \lambda^{-1}u = 0$ in SLP form.

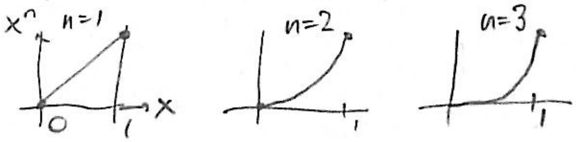
$1 \quad e \quad x$

BCs: 2nd line $\rightarrow$ $\lambda u'(1) = \int_1^1 (\text{something bounded})\,dy = 0$ Neumann left-end

top line $\rightarrow$ $\lambda u(e) = \int_1^e (\cancel{1 - \ln e})\, u(y)\,dy + \int_e^e (\text{something})\,dy = 0$ Dirichlet right-end.

6. [8 points] Short-answer questions

3 (a) Is the sequence of function $\{x, x^2, x^3, \ldots\}$ convergent to 0 in the L^2 sense on the interval $[0,1]$? (prove via a quick calculation)



$f_n(x) = x^n$

L^2 norm squared $\|f_n\|^2 = \int_0^1 f_n^2(x)\,dx = \int_0^1 x^{2n}\,dx = \left.\frac{x^{2n+1}}{2n+1}\right|_0^1 = \frac{1}{2n+1}$

so $\lim_{n\to\infty} \|f_n\| = 0$ Yes.

3 (b) A symmetric Fredholm integral operator K on (a,b) has eigenvalues $\overbrace{1/n^2}^{\lambda_n}$ and normalized eigenfunctions ϕ_n, $n = 1, 2, \ldots$ What condition on f makes the equation $Ku - \frac{1}{4}u = f$ soluble, and what then is the general solution?

by Hilbert-Schmidt theorem.

Use eigenfunction expansion: can since Hilbert-Schmidt thm (complete). (ϕ_n, f) since ϕ_n o.n.b.

get $(\lambda_n - \frac{1}{4})\,c_n = f_n$ for $n = 1, 2, \ldots$ separately

vanishes for $n=2$

Consistency only if $f_2 = 0$ ie $\int_a^b \phi_2(x)\, f(x)\,dx = 0$

(Non-unique) soln. is then $u(x) = \sum_{n\neq 2} \frac{(\phi_n, f)}{1/n^2 - 1/4}\,\phi_n(x) + c\,\phi_2(x)$

c arbitrary.

2 (c) On the interval $(0,\pi)$, the functions $\phi_n(x) = \sin nx$ for $n = 1, 2, \ldots$ form an orthogonal set. If $f = \sum_{n=1}^{\infty} f_n \phi_n$ then use the Cauchy-Schwarz inequality to bound f_1 in terms of $\|f\|$. (BONUS: Show that this either stronger or weaker than Bessel's inequality)

Since orthog, $f_1 = \frac{(\phi_1, f)}{\|\phi_1\|^2}$ $\leftarrow \le \|\phi_1\| \cdot \|f\|$ by Cauchy-Schwarz

$\leftarrow = \int_0^\pi \sin^2 x\,dx = \pi/2.$

so $f_1 \le \frac{\|\phi_1\|}{\|\phi_1\|^2}\|f\| = \frac{1}{\|\phi_1\|}\|f\| = \sqrt{\frac{2}{\pi}}\,\|f\|$

Bonus: Bessel for o.n.b. says $\sum_{n=1}^{N} f_n^2 \le \|f\|^2$ for any N

If instead $\|\phi_n\|^2 = \frac{2}{\pi}$ get $\sum_{n=1}^{N} f_n^2 \le \|f\|^2$, equiv. to above for $N=1$ but otherwise above is weaker than Bessel.

(d) [BONUS] Answer question a) for two other forms of convergence on the same interval

- Uniform convergence to 0 on $[0,1]$? no, since $\sup_{x\in[0,1]} f_n(x) = 1 \ \forall n$
- Pointwise " " " ? no, since $x=1$ is in closed interval, and $f_n(1) = 1 \ \forall n$.

7. [8 points] Now some fun new territory! Consider the inhomogenous SLP

$$-u'' = f$$

where f is some given driving function on $[0, 1]$, with boundary conditions $u(0) = u(1) = 0$ [Hint: apply them wherever possible below]

4 (a) Attack the SLP as you would to convert an IVP into a Volterra integral equation, i.e. integrate from 0 to x, twice, and convert any iterated integrals into single ones. Write your result as $u(x) =$ something involving f and the unknown value $u'(0)$.

$$-u''(s) = f(s)$$

$$\int_0^y ds \quad -u'(y) + \underbrace{u'(0)}_{\text{const.}} = \int_0^y f(s)\,ds$$

$$\int_0^x dy \quad -u(x) + \cancel{u(0)}^{0} + xu'(0) = \int_0^x \int_0^y f(s)\,ds\,dy \quad \text{lemma}$$

$$\int_0^x (x-y) f(y)\,dy$$

$$\text{so} \quad u(x) = x\underbrace{u'(0)}_{\text{unknown.}} - \int_0^x (x-y) f(y)\,dy$$

3 (b) Now find an expression for $u'(0)$ purely in terms of f, as follows: Multiply both sides of the original ODE by $(x - 1)$ then integrate from 0 to 1.

$$-(s-1)\,u''(s) = (s-1) f(s)$$

$$\int_0^1 ds \quad \int_0^1 (1-s)\, u''(s)\,ds = \int_0^1 (s-1) f(s)\,ds.$$

parts.

$$\underbrace{(1-s)u'\Big|_0^1}_{0 - u'(0)} - \underbrace{\int_0^1 (-1)\, u'(s)\,ds}_{u(0)^0 - u(1)^0 = 0}$$

1 (c) Substitute your expression for $u'(0)$ into part a) to get an explicit solution formula for $u(x)$. FUN BONUS: Show that this is actually equivalent to the familiar Greens function solution (from worksheet) to this SLP!

$$u'(0) = \int_0^1 (1-s) f(s)\,ds$$

split up $(0,x) \cup (x,1)$

$$\text{so} \quad u(x) = x \int_0^1 (1-s) f(s)\,ds - \int_0^x (x-s) f(s)\,ds$$

using s instead of y.

Bonus

$$= \int_0^x [\cancel{x} - xs - \cancel{x} + s] f(s)\,ds + \int_x^1 x(1-s) f(s)\,ds$$

$$= \int_0^1 g(x,s) f(s)\,ds \quad \text{with } g(x,s) = \begin{cases} s(1-x), & s<x \\ x(1-s), & s>x \end{cases}$$

cool.

usual Greens func! → does this trick work for general SLP?