Math 25, Homework 5, October 27, 2008

1. Show that if a, b, c are positive integers with $b \mid c$, then $a^b - 1 \mid a^c - 1$.

2. Show that if p is an odd prime then $(4^p - 1)/3$ is a composite integer.

3. Show that if $p > 3$ is prime, then $(4^p - 1)/3$ is a base 2 pseudoprime.

4. More generally show that if $b \geq 2$ is an integer and p is an odd prime that does not divide $b^2 - 1$, then $(b^{2p} - 1)/(b^2 - 1)$ is a base b pseudoprime.

5. Prove that 1 is the only odd value of Euler's function.

6. Prove that if n is a positive integer with $n \not\equiv 0 \pmod 4$, then there is some integer $m \neq n$ with $\varphi(m) = \varphi(n)$. (It is unknown if this holds for all positive integers n; it is conjectured that it does.)

7. Compute $\varphi(n)$ for $n = 999$, 1000, and 1001.

8. Rebecca asked if $\varphi(n^2) = n\varphi(n)$ holds for every positive integer n and I replied that it does. Was I right?

9. For a positive integer n, define $\mathrm{rad}(n)$ as the product of the different primes that divide n. For example, $\mathrm{rad}(72) = 6$. Prove that $\varphi(m)/m = \varphi(n)/n$ if and only if $\mathrm{rad}(m) = \mathrm{rad}(n)$.