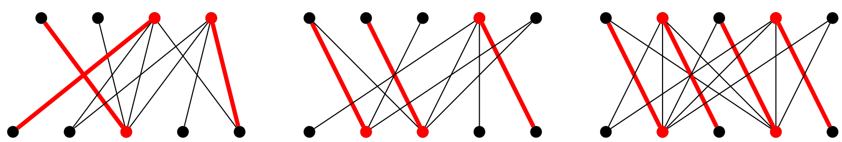
HOMEWORK 7

SOLUTIONS

3.1.1 Find a maximum matching in each graph below. Prove that it is a maximum matching by exhibiting an optimal solution to the dual problem (minimum vertex cover). Explain why this proves that the matching is optimal.

Answer. We saw that there can never be a vertex cover of size smaller than that of a matching. So if we find a vertex cover and a matching of the same size, we have simultaneously minimized and maximized the vertex covering and matching problems. Below I have exhibited vertex covers and matchings of equal zie (3, 3, and 4, respectively), optimizing the respective problems.



□

3.1.3 Let S be the set of vertices saturated by a matching M in a graph G. Prove that some maximum matching also saturates all of S. Must the statement be true for every maximum matching?

Answer: You can convert any non-maximal matching into a maximal matching by adding edges. And you can transform any maximal matching into a maximum matching by iteratively using M-augmenting paths. Both of these operations only add to the set of saturated vertices, never removing them. So if M is a matching, then some maximum matching saturates a set of vertices containing those vertices saturated by M. However, if M is not perfect, then it is not necessarily true for every maximum matching. For example, consider the three maximum matchings of C_3; they each saturate a different set of vertices.

3.1.9 Prove that every maximal matching in a graph G has at least $\alpha'(G)/2$ edges.

Proof. If M is a maximal matching, and S is the set of vertices saturated by M, then $|M| = \frac{1}{2}|S|$. Also, $V(G) - S$ is an independent set. So the smaller M is, the larger of an independent set you get. Recall that

$$\begin{aligned}\alpha(G) &= \text{size of the maximum independent set}\\ \alpha'(G) &= \text{size of the maximum matching}\\ \beta(G) &= \text{size of the minimum vertex cover}\\ \beta'(G) &= \text{size of the minimum edge cover.}\end{aligned}$$

Then

$$2|M| = |S| = n(G) - |V(G) - S| \geq n(G) - \alpha(G) = \beta(G) \geq \alpha'(G)$$

(by Lemma 3.1.21). So $|M| \geq \frac{1}{2}\alpha'(G)$. □

3.1.12 Let C and C' be cycles in a graph G. Prove that $C \triangle C'$ is a single cycle if and only if $C \cap C'$ is a single path.

Sketch. First assume that $C \cap C'$ is a single path. Then the symmetric difference takes the union of the two cycles and subtracts the path that they share, leaving the union of two non-intersecting paths which share endpoints, otherwise known as a cycle.

Now assume that the symmetric difference is a cycle. Then you know that any edges from C (resp. C') which occur consecutively in $C \triangle C'$ also occur consecutively in C (resp. C'). So $C \triangle C'$ is the union of paths alternating from C and C' (a path from C then a path from C' and so on). If it's a union of 2ℓ paths, then the intersection $C \cap C'$ is the union of ℓ paths. But if $C \cap C'$ has more than one component, we will be able to build a cycle of length less than k, a contradiction. □

3.1.18 Two people play a game on a graph G, alternately choosing distinct vertices. Player 1 starts by choosing any vertex. Each subsequent choice must be adjacent to the preceding choice (of the other player). Thus together they follow a path. The last player able to move wins.

Show that the second player has a winning strategy if G has a perfect matching and otherwise the first player has a winning strategy. (Hint: For the second part, the first player should start with a vertex omitted by some maximum matching.)

Proof. If there is a perfect matching, player 2 picks such a matching M. Then player 2 always chooses the vertex in M adjacent to player 1's choice. Since every vertex has exactly one edge in M incident to it. player 2 will win.

If there is no perfect matching, player 1 picks a maximum matching M. Then they start with a vertex unsaturated by M. Player 2 is forced to pick a vertex saturated by M (since M is maximum). There on out, player 1 will pick the vertex adjacent to player 2's choice in M. Player 1 will always be able to move; otherwise there would be an M-augmenting path, which is not possible if M is maximum.

□

3.1.30 Determine the maximum number of edges in a simple bipartite graph that contains no matching with k edges and no star with l edges.

Answer. Suppose G is a X, Y-bigraph. First, if G has no l-stars, then $\Delta(G) \le l - 1$. So

$$\begin{aligned} e(G) &= \sum_{v \in X} d(v) (\le \Delta(G)|X|) \\ &= \sum_{v \in Y} d(v) (\le \Delta(G)|Y|), \end{aligned}$$

implies that $e(G) \le (l-1)\min(|X|, |Y|)$.

Now consider the components of G. Without loss of generality, assume that the smaller partite of every component is in X, and the other partite is in Y. We'll show that $|X| \le k - 1$.

Suppose H is a connected A, B bigraph for which every maximum matching leaves both A and B unsaturated. Now pick a matching for which two unsaturated vertices $a \in A$ and $b \in B$ are as close as possible (i.e. a and b are vertices so that there is no matching in which two unsaturated vertices in opposite parts are closer). Now consider a minimal a, b path. If

□

3.1.42 An algorithm to greedily build a large independent set S iteratively selects a vertex of minimum degree in the remaining graph, adds it to S, and deletes it and its neighbors from the graph. Prove that this algorithm produces an independent set of size at least

$$\sum_{v \in V(G)} \frac{1}{d_G(v) + 1}$$

in a simple graph G.

Proof. Let X be the set of vertices deleted at some step, and let x be the center of that deletion (so $X = \{x\} \cup N_{G'}(x)$ and $d_{G'}(x) \le d_{G'}(v)$ for all $v \in X$, where G' is the subgraph of G before the deletion). Note that $d_{G'}(v) \le d_G(v)$ since G' is a subgraph of G. So

$$\sum_{v \in X} \frac{1}{d_G(v) + 1} \le |X| \frac{1}{d_G(x) + 1} \le |X| \frac{1}{d_{G'}(x) + 1} = |X| \frac{1}{|X|} = 1.$$

Since every vertex in G appears in exactly one deletion,

$$\sum_{v \in V(G)} \frac{1}{d_G(v) + 1} \le \sum_{x \in S} 1 = |S|.$$

□