Homework #5

1) Page 150 #12

a) $Ei(\lambda) = \int_\lambda^\infty \frac{e^{-t}}{t}\,dt$ let $u = \frac{1}{t}$, $du = -\frac{1}{t^2}dt$, $v = -e^{-t}$, $dv = e^{-t}dt$

$= \frac{-e^{-t}}{t}\Big|_\lambda^\infty - \int_\lambda^\infty \frac{e^{-t}}{t^2}\,dt$ $u = \frac{1}{t^2}$, $du = -\frac{2}{t^3}$, $v = -e^{-t}$, $dv = e^{-t}dt$

$= \lim_{\alpha\to\infty} \underbrace{\frac{-e^{-\alpha}}{\alpha}}_{0} + \frac{e^{-\lambda}}{\lambda} - \left(\frac{-e^{-t}}{t^2}\Big|_\lambda^\infty - 2\int_\lambda^\infty \frac{e^{-t}}{t^3}\,dt\right)$

$= \frac{e^{-\lambda}}{\lambda} - \left(\lim_{\alpha\to\infty} \underbrace{\frac{-e^{-\alpha}}{\alpha^2}}_{0} + \frac{e^{-\lambda}}{\lambda^2} - 2\int_\lambda^\infty \frac{e^{-t}}{t^3}\,dt\right)$

$= \frac{e^{-\lambda}}{\lambda} - \frac{e^{-\lambda}}{\lambda^2} + \cdots$

b) $\lim_{\lambda\to\infty} |r_n(\lambda)| = \lim_{\lambda\to\infty} n!\left|\int_\lambda^\infty \frac{e^{-t}}{t^{n+1}}\,dt\right|$

$\le \lim_{\lambda\to\infty} n!\, e^{-\lambda} \int_\lambda^\infty t^{-(n+1)}\,dt$

$\le \lim_{\lambda\to\infty} n!\, e^{-\lambda} \frac{t^{-(n+1)+1}}{-(n+1)+1}\Big|_\lambda^\infty$

$= \lim_{\lambda\to\infty} n!\, e^{-\lambda}\left(\lim_{\alpha\to\infty} \underbrace{\frac{\alpha^{-n}}{-n}}_{0} - \frac{\lambda^{-n}}{n}\right)$

$= \lim_{\lambda\to\infty} n!\,\frac{e^{-\lambda}}{n\lambda^n} \to 0.$

c) Goal: Show $r_n(\lambda) = o\left(\frac{(n-1)!\, e^{-\lambda}}{\lambda^n}\right)$ as $\lambda \to \infty$. This will prove that it is an asymptotic expansion because it shows the sequence is decreasing.

You can use the asymptotic expansion of $r_n(\lambda)$, or you can be a bit more clever.

$$|r_n(\lambda)| = n! \left| \int_\lambda^\infty \frac{e^{-t}}{t^{n+1}}\, dt \right| \quad \text{let } t' = t - \lambda \text{ when } t = \lambda \quad t' = 0.$$

$$= n! \int_0^\infty \frac{e^{-t'}\, e^{-\lambda}}{(t'+\lambda)^{n+1}}\, dt'$$

$$= n!\, e^{-\lambda} \int_0^\infty \frac{e^{-t'}}{(t'+\lambda)^{n+1}}\, dt' \quad \text{since } \frac{1}{\lambda + t'} \leq \frac{1}{\lambda} \quad \forall t' \geq 0.$$

$$\leq n!\, \frac{e^{-\lambda}}{\lambda^{n+1}} \underbrace{\int_0^\infty e^{-t}\, dt}_{=1}$$

$$\Rightarrow \lim_{\lambda \to \infty} \left| \frac{r_n(\lambda)}{\frac{(n-1)!\, e^{-\lambda}}{\lambda^n}} \right| \leq \lim_{\lambda \to \infty} \frac{n!\, e^{-\lambda}}{\lambda^{n+1}} \frac{\lambda^n}{(n-1)!\, e^{-\lambda}}$$

$$= \lim_{\lambda \to \infty} \frac{n!}{\lambda} = 0.$$

d) for λ fixed we need to compute $\lim_{n \to \infty} |r_n(\lambda)|$

$$\lim_{n \to \infty} |r_n(\lambda)| \leq \lim_{n \to \infty} n!\, \frac{e^{-\lambda}}{\lambda^n}$$

$$\frac{n!}{\lambda^n} \leq \frac{(n+1)!}{\lambda^{n+1}} \Rightarrow \text{the sequence is increasing.}$$

but there is no bound.

$\Rightarrow$ The sequence diverges.

e) See code.

2) Page 214 #1

1st Verify $\cos\left(\frac{n\pi x}{e}\right)$ are orthogonal on $[0,\ell)$

case $n=0$: $m \neq 0$.

$$\int_0^\ell \cos\left(\frac{m\pi x}{\ell}\right) dx = \frac{\ell}{m\pi} \sin\left(\frac{m\pi x}{e}\right)\Big|_0^e$$

$$= \frac{\ell}{m\pi}\left(\sin(m\pi) - \sin 0\right) = 0.$$

Case $m \neq n \neq 0$.

$$\int_0^e \cos\left(\frac{m\pi x}{e}\right)\cos\left(\frac{n\pi x}{e}\right) dx =$$

$$\int_0^\ell \frac{1}{2}\left[\cos\left(\frac{m+n}{e}\pi x\right) + \cos\left(\frac{m-n}{e}\pi x\right)\right] dx$$

$$= \frac{1}{2}\left[\frac{e}{(m+n)\pi}\sin\left(\frac{m+n}{e}\pi x\right) + \frac{\ell}{(m-n)\pi}\sin\left(\frac{m-n}{\ell}\pi x\right)\right]\Bigg|_0^\ell$$

$$= \frac{1}{2}\left[\frac{\ell}{(m+n)\pi}\left(\underbrace{\sin((m+n)\pi)}_{=0 \text{ since } m+n\in\mathbb{Z}} - \sin 0\right) + \frac{\ell}{(m-n)\pi}\left(\underbrace{\sin((m-n)\pi)}_{=0 \text{ since } m-n\in\mathbb{Z}} - \sin(0)\right)\right]$$

$$= 0.$$

2nd Find C_n.

if $n=0$, simply integrate both sides on $[0,\ell]$

$$\int_0^e f(x)\,dx = \ell C_0 + \sum_{n=1}^{\infty} C_n \underbrace{\int_0^e \cos\left(\frac{n\pi x}{e}\right) dx}_{=0 \text{ by part 1}}$$

$$\to C_0 = \frac{1}{\ell}\int_0^\ell f(x)\,dx$$

if $n \neq n$. multiply by $\cos\left(\frac{m\pi x}{e}\right)$ & integrate over $(0,\ell)$

$$\int_0^\ell f(x)\cos\left(\frac{m\pi x}{e}\right) dx = \sum_{n=0}^{\infty} C_n \underbrace{\int_0^\ell \cos\left(\frac{m\pi x}{e}\right)\cos\left(\frac{n\pi x}{e}\right) dx}_{=0 \text{ if } m\neq n.}$$

if $m=n$.

$$\int_0^l \cos^2\left(\frac{m\pi x}{l}\right) dx = \frac{1}{2}\int_0^l \left(1-\cos\left(\frac{2m\pi x}{l}\right)\right) dx$$

$$= \frac{1}{2}\left(x - \frac{l}{2\pi m} \underbrace{\sin\left(\frac{2\pi m x}{l}\right)}_{=0} \right)\Big|_0^l$$

$$= \frac{l}{2}$$

$$\to C_m = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{m\pi x}{l}\right) dx$$

Now find the fourier series for $f(x) = 1-x$, on $[0,1]$ ie $l=1$

$$C_0 = \int_0^1 1-x\, dx = x - \frac{x^2}{2}\Big|_0^1 = 1 - \frac{1}{2} - 0. = 1/2$$

$$C_n = 2\int_0^1 (1-x)\cos(m\pi x)\, dx$$

$$= 2\left[\int_0^1 \cos(m\pi x)\, dx - \int_0^1 x\cos(m\pi x)\, dx\right]$$

$u = x$, $du = dx$, $v = \frac{1}{m\pi}\sin(m\pi x)$, $dv = \cos m\pi x$

$$= 2\left(\underbrace{\frac{\sin(m\pi x)}{m\pi}\Big|_0^1}_{=0} - \left(\underbrace{\frac{x}{m\pi}\sin(m\pi x)\Big|_0^1}_{=0} - \frac{1}{m\pi}\int_0^1 \sin(m\pi x)\, dx\right)\right)$$

$$= \frac{2}{m\pi}\int_0^1 \sin(m\pi x)\, dx = \frac{-2}{(m\pi)^2}\cos(m\pi x)\Big|_0^1$$

$$= \frac{-2}{(m\pi)^2}\left(\cos(m\pi) - 1\right)$$

$$= \frac{-2}{(m\pi)^2}\left((-1)^m - 1\right) = \begin{cases} 0 & \text{if } m \text{ even} \\ \frac{+4}{(m\pi)^2} & \text{if } m \text{ odd.} \end{cases}$$

$$\to 1-x = \frac{1}{2} + 4\sum_{n=0}^{\infty} \frac{\cos((2m+1)\pi x)}{(2m+1)^2\pi^2}$$

3) Page 214 #3 Proving the Cauchy-Schwarz inequality.

let $q(t) = \langle f+tg, f+tg \rangle$

$$= \langle f,f \rangle + 2t\langle f,g \rangle + t^2 \langle g g \rangle \geq 0$$

$\Rightarrow$ all roots are complex or zero.

$\Rightarrow$ the discriminant

$$4\langle f,g \rangle^2 - 4\langle f,f \rangle \langle g,g \rangle \leq 0$$

$$\Rightarrow |\langle f,g \rangle| \leq \|f\| \|g\|$$

4) Page 215 #5

Legendre polynomials (we are going to derive them :))

on $[-1,1]$

We need to generate P_0, P_1, P_2, P_3

let $P_0(x) = 1$.

$$\langle x,1 \rangle = \int_{-1}^{1} x \, dx = \frac{x^2}{2}\Big|_{-1}^{1} = 0.$$

$$P_1 = x - \frac{\langle x,1 \rangle}{\langle 1,1 \rangle}$$

$$= x$$

$$\langle x^2,x \rangle = \int_{-1}^{1} x^3 dx = \frac{x^4}{4}\Big|_{-1}^{1} = 0.$$

$$\langle x^2,1 \rangle = \int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$$

$$\langle 1,1 \rangle = 2.$$

$$P_2 = x^2 - x\frac{\langle x^2,x \rangle}{\langle x,x \rangle} - 1\frac{\langle x^2,1 \rangle}{\langle 1,1 \rangle}$$

$$= x^2 - 1/3$$

$$P_3(x) = x^3 - x^2 \overbrace{\frac{\langle x^3,x^2 \rangle}{\langle x^2,x^2 \rangle}}^{=0} - x\frac{\langle x^3,x \rangle}{\langle x,x \rangle} - 1 \overbrace{\frac{\langle x^3,1 \rangle}{\langle 1,1 \rangle}}^{=0}$$

$$\int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = 2/3$$

$$= x^3 - \frac{3x}{2}\frac{2}{5} = x^3 - \frac{3x}{5}$$

$$\int_{-1}^{1} x^4 dx = \frac{x^5}{5}\Big|_{-1}^{1}$$

$$c_0 = \frac{\langle P_0, e^x\rangle}{\langle 1, 1\rangle} = \frac{\int_{-1}^{1} 1 e^x\,dx}{\int_{-1}^{1} 1\,dx} = \frac{e^x\big|_{-1}^{1}}{x\big|_{-1}^{1}} = \frac{e - e^{-1}}{2}$$

$$u = x \qquad v = e^x$$
$$du = dx \qquad dv = e^x\,dx$$

$$c_1 = \frac{\langle P_1, e^x\rangle}{\langle x, x\rangle} = \frac{\int_{-1}^{1} x e^x\,dx}{\int_{-1}^{1} x^2\,dx}$$

$$= \frac{x e^x\big|_{-1}^{1} - \int_{-1}^{1} e^x\,dx}{\frac{x^3}{3}\big|_{-1}^{1}} = \frac{(e + e^{-1}) - (e^x\big|_{-1}^{1})}{\frac{1}{3}(1+1)}$$

$$= \frac{3}{2}\left[e + e^{-1} - (e - e^{-1})\right] = 3e^{-1}$$

attached is a plot of the pointwise error

$$e_1(x) = \left|e^x - \left[\frac{e - e^{-1}}{2} + 3e^{-1}x\right]\right|$$

~~and~~ $\cancel{\max_{x \in [-1,1]}\{e_1(x)\}} =$

~~mean square~~ error is ~~the L^2 error.~~

$$\cancel{e_2 = \int_{-1}^{1} \left|e^x - \left[\frac{e - e^{-1}}{2} + 3e^{-1}x\right]\right|^2 dx}$$

$$\cancel{= \int_{-1}^{1} \left(e^{2x} - 2e^x\left(\frac{e - e^{-1}}{2} + 3e^{-1}x\right) + \left(\frac{e - e^{-1}}{2} + 3e^{-1}x\right)^2\right) dx}$$

5) Page 219 # 2.

1st Find Fourier series for $f(x) = x^2$. on $[-\pi, \pi]$
This is an even function over a symmetric interval $\Rightarrow$ $b_n = 0$ $\forall n$. (i.e. no sin terms)
So we only need to compute the a_n.

Thus: $f(x) = x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\left.\frac{x^3}{3}\right|_{-\pi}^{\pi} = \frac{1}{\pi}\frac{2\pi^3}{3} = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx \qquad \begin{array}{ll} u = x^2 & v = \frac{1}{n}\sin(nx) \\ du = 2x & dv = \cos nx\,dx \end{array}$$

$$= \frac{1}{\pi}\left(\underbrace{\left.\frac{x^2}{n}\sin(nx)\right|_{-\pi}^{\pi}}_{=0} - \frac{1}{n}\int_{-\pi}^{\pi} 2x\sin(nx)\,dx\right) \qquad \begin{array}{ll} u = x & v = -\frac{1}{n}\cos nx \\ du = dx & dv = \sin(nx) \end{array}$$

$$= \frac{1}{\pi}\left(\frac{-2}{n}\left[\left.\frac{-x}{n}\cos(nx)\right|_{-\pi}^{\pi} - \underbrace{\frac{1}{n}\int_{-\pi}^{\pi}\cos nx\,dx}_{=0}\right]\right)$$

$$= \frac{2}{\pi n^2}\left[\pi\cos(n\pi) + \pi\cos(n\pi)\right]$$

$$= \frac{4(-1)^n}{n^2}$$

So the Fourier series is

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx).$$

6) Page 225 #3.

Goal: Find eigenvalues & eigenfunctions.

$$\begin{cases} -y'' = \lambda y & 0<x<\ell \\ y(0) = y(\ell) \\ y'(0) = y'(\ell) \end{cases}$$

We know from Math 23 the solution is of the form (when $\lambda \neq 0$)

$$y(x) = C_1 e^{-\sqrt{-\lambda}\,x} + C_2 e^{-\sqrt{-\lambda}\,x}$$

if $\sqrt{-\lambda} \leq 0$ it is impossible to satisfy BC..

if $\lambda > 0$. we get osillatory solution (seems good for periodic).

let $\sqrt{-\lambda} = ik$

$\Rightarrow y(x) = C_1\cos(kx) + C_2\sin(kx)$.

$y(0) = C_1 = C_1\cos(k\ell) + C_2\sin(k\ell) = y(\ell)$

$\Rightarrow k\ell = 2n\pi \Rightarrow k = \frac{2n\pi}{\ell}, \; C_2 = 0$

$y'(x) = -kC_1\sin(kx) + C_2 k\cos(kx)$

$y'(0) = C_2 k = y'(\ell) = -kC_1\sin(k\ell) + C_2 k\cos(k\ell)$

our solution satisfies this.

Thus $\lambda_n = \left(\frac{2n\pi}{\ell}\right)^2 \quad y_n = \cos\left(\frac{2n\pi x}{\ell}\right)$

are the eigenpairs.

if $\lambda = 0 \Rightarrow y = Ax + B$ $\quad y(0) = B = A\ell + B \Rightarrow A = 0$.

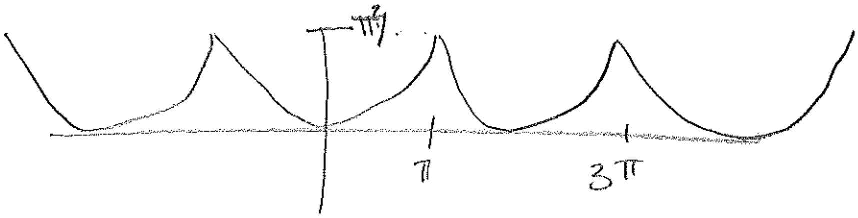
$y' = A$ $\quad \Rightarrow y = \text{constant}$ is an eigen function.

If we take $x=0$. we get.

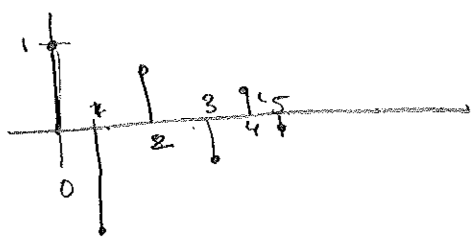
$$-\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \Rightarrow \frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

$$= 1 - \frac{1}{4} + \frac{1}{9} + \cdots$$

Plot of the Fourier series is



Plot of coefficients:



7) Page 225 #4.

$$-y'' = \lambda y \qquad 0<x<1.$$

$$y(0) + y'(0) = 0 \qquad y(1) = 0.$$

If $\lambda = 0$ $\quad y = Ax + B$

$y(1) = A = 0 \rightarrow y(0) = B = 0.$

$\rightarrow \lambda = 0$ is not an eigenvalue

If $\lambda < 0$. $\quad -\lambda = -k^2$

Solutions are

$y(x) = C_1 e^{-kl} + C_2 e^{kl} \rightarrow C_1 = -C_2 e^{2kl}$

$y(l) = C_1 e^{-kl} + C_2 e^{kl} = 0.$

$y(0) + y'(0) = C_1 + C_2 - kC_1 + kC_2 = 0$

$\rightarrow C_2(1+k) + (k-1)C_1 = 0$

$\rightarrow C_2\left[(1+k) + (k-1)e^{2kl}\right] = 0$ $\quad \frac{1+k}{1-k} = e^{2kl}$?Not possible

$\Rightarrow C_2 = 0$ unless

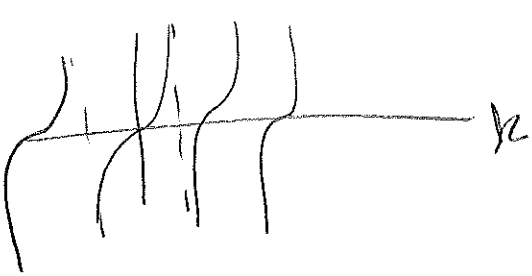
$\rightarrow \lambda < 0$ cannot be an eigenvalue.

If $\lambda > 0$. $\quad \lambda = k^2$

$y = C_1 \cos(kx) + C_2 \sin(kx)$

$y(l) = C_1 \cos(kl) + C_2 \sin(kl) = 0 \Rightarrow$

$y(0) + y'(0) = C_1 + C_2 k = 0 \rightarrow C_1 = -C_2 k \Rightarrow k = -\frac{C_1}{C_2} = \tan k$



8) Page 225 #6

$-y'' = \lambda y \qquad 0 < x < l$

$y(0) - a y'(0) = 0 \qquad y(l) + b y'(l) = 0.$

If $\lambda = 0$: $y'' = 0 \Rightarrow y = Ax + B$

$y(0) - a y'(0) = B - aA = 0$

$\Rightarrow B = aA$

$y(l) + b y'(l) = Al + B + bA = 0$

$\Rightarrow A(l + a + b) = 0$

only way to get a non-trivial solution is if

$l + a + b = 0$ or $l = -(a+b)$

$\Rightarrow \quad -l = a + b \Longleftrightarrow$ nontrivial solution

ie $\lambda = 0$ eigenvalue.

9) a) $\int_{-\pi}^{\pi} \cos^2(nx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}(1-\cos(2nx))dx$

$= \frac{1}{2}\left(x + \frac{1}{2n}\underbrace{\sin(2nx)}_{=0}\Big|_{-\pi}^{\pi}\right) = \pi$

$\left\{\frac{1}{\sqrt{2\pi}}\right\} \cup \left\{\frac{1}{\sqrt{\pi}}\cos(nx), \frac{1}{\sqrt{\pi}}\sin(nx)\right\}_{n=1}^{\infty}$ are orthonormal on $[-\pi,\pi]$

$\int_{-\pi}^{\pi} 1\,dx = 2\pi$

b) $c_n = \langle f_n, f\rangle$

$f(x)=x$ is an odd function over a symmetric interval $\Rightarrow$ sine series

$c_n = \langle \sin nx, f\rangle$

$= \int_{-\pi}^{\pi} x\sin nx\,dx$ $\quad u = x \quad du = dx \quad v = -\frac{1}{n}\cos(nx) \quad dv = \sin nx\,dx$

$= -\frac{x}{n}\cos(nx)\Big|_{-\pi}^{\pi} + \frac{1}{n}\underbrace{\int_{-\pi}^{\pi}\cos nx\,dx}_{=0}$

$= -\frac{\pi}{n}\cos(n\pi) - \frac{\pi}{n}\cos(n\pi) = \frac{2\pi}{n}(-1)^{n+1}$

$f(x) = \sum_{n=1}^{\infty} \frac{2\pi}{n}(-1)^{n+1}\sin(nx)$

c) $0 = \sum_{n=1}^{\infty} \frac{2\pi}{n}(-1)^{n+1}\underbrace{\sin(n\pi)}_{=0}$

d) Parseval's equality says.

$\sum_{n=1}^{\infty} c_n^2 = \|f\|^2 \Rightarrow \sum_{n=1}^{\infty}\frac{4\pi^2}{n^2} = \int_{-\pi}^{\pi} x^2\,dx = \frac{x^3}{3}\Big|_{-\pi}^{\pi} \frac{2\pi^3}{3}$

$\Rightarrow \sum_{n=1}^{\infty} n^{-2} = \frac{\pi}{6}$

11