Midterm 1 solutions

1. MULTIPLE CHOICE: For each of the following questions, please circle the correct answer. You do not need to show any work. (3 points each)

(a) Suppose that a fair, 6-sided die is rolled. Let A be the event that an odd number is obtained. Let B be the event that a 3,4, 5, or 6 is obtained. Then:

(i) $A \subseteq B$ (A is a subset of B) (ii) $A = \overline{B}$ (A is the complement of B)

(iii) A and B are independent (iv) A and B are disjoint

(v) None of the above

(b) Let A and B be two independent events such that $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$. Which of the following must be true?

(i) $\mathbb{P}(A|B) = \mathbb{P}(B)$ (ii) $\mathbb{P}(A \cup B) = \mathbb{P}(A)\mathbb{P}(B)$

(iii) A and B are disjoint (iv) A and B are NOT disjoint

(v) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

(c) There are 100 members in the Roger Federer Fan Club of the Upper Valley: 55 females and 45 males. The RFFCUV chooses 5 different members at random to write obnoxious fan mail. If X is the number of males writing obnoxious fan mail, then:

(i) X has a binomial distribution (ii) X has a geometric distribution

(iii) X has a Poisson distribution (iv) X has a hypergeometric distribution

(v) X isn't a discrete random variable

(d) Suppose that 4/5 of the customers at Morano Gelato love hazelnut-flavored gelato. If we randomly select 10 customers to participate in a survey about gelato flavors, what is the probability that **at least** 9 of them will be fans of hazelnut?

(i) $\binom{5}{4}(9/10) + \binom{5}{4}(9/10)^2$ (ii) $\binom{10}{9}(4/5)^9(1/5)$

(iii) $\binom{10}{9}(4/5)^9(1/5) + \binom{10}{10}(4/5)^{10}$ (iv) $\binom{10}{9}(4/5)^9 + \binom{10}{10}(4/5)^{10}$

(v) None of the above

(e) In the survey described in the previous question, suppose that X is the number of people that had to be surveyed until the first time we discovered a hazelnut lover (i.e. if the first person surveyed didn't like hazelnut but the second person did, $X = 2$). Then:

(i) $\mathbb{P}(X = 2) = (1/5)(4/5)$ (ii) $\mathbb{P}(X = 2) = \binom{10}{2}(4/5)^2(1/5)^8$

(iii) $\mathbb{P}(X \leq 2)$ cannot be determined (iv) $\mathbb{P}(X = 2 \cap X = 1) = \mathbb{P}(X = 1)\mathbb{P}(X = 2)$

(v) None of the above

2. SHORT ANSWER: Answer each of the following questions. You do not need to show any work. (3 points each)

(a) Roger Federer uses 5 alarm systems in his mansion. If someone tries to rob him, each alarm will detect it with probability .99, independently of any other alarm. What is the probability that at least one of Fed's alarms will go off if someone tries to break into his humble home?

Answer: $1-(.01)^5$

(b) There are 37 people in Math 20. Suppose 10 people think the "Honey Badger" video on YouTube is the funniest, 10 prefer the "Thumbs up for Rock N Roll" video, 5 think the best video is "Sleepy Spudgy", and 12 don't ever go on YouTube. What is the probability that if we randomly select 5 different students, none of them have a favorite YouTube video?

Answer: $\dfrac{\binom{12}{5}}{\binom{37}{5}}$

(c) Arthur Weasley has decided to go to Muggle college. Dumbledore tells him that he has a .6 chance of getting in to UVM and a .8 chance of getting into UNH. He also says there is a .1 chance that he will be accepted **only** at UVM. What is the probability that Arthur Weasley gets into at least one of these two colleges?

Answer: $\mathbb{P}(N\cup V) = \mathbb{P}(N)+\mathbb{P}(V)-\mathbb{P}(N\cap V)$

$= .8+.6-.5 = .9$

(d) Suppose a person can have anywhere from 1 to 6 initials, each initial being equally likely. What's the probability that a given person has the initials G.E.L.A.T.O.? (Remember: initials X.Y.Z. and Z.Y.X. are distinct!)

Answer: $\dfrac{1}{26+26^2+26^3+26^4+26^5+26^6}$

(e) A tricky scooper at the gelato shop tells you that, of the 10 flavors available today, if you choose the one she has in mind, you'll get your gelato cup for free. After you make a choice, she points out eight flavors that are NOT the winning ones, leaving just the flavor you originally chose and one other, and asks if you want to switch your choice to the other flavor remaining. What is the probability that you win the free gelato if you choose to switch?

Answer: $9/10$

3. Suppose $\mathbb{P}(\overline{A}) = .3$, $\mathbb{P}(B) = .8$, and $\mathbb{P}(C) = .6$. Suppose furthermore that if C occurs, then A and B both (definitely) occur, but if C does not occur, the chance that A and B both occur is only .2. Find $\mathbb{P}(A \cup B)$.

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

$$= 1 - \mathbb{P}(\bar{A}) + \mathbb{P}(B) - \left[\mathbb{P}(A \cap B \mid C)\,\mathbb{P}(C) + \mathbb{P}(A \cap B \mid \bar{C})\,\mathbb{P}(\bar{C})\right]$$

$$= 1 - .3 + .8 - \left[(1)(.6) + (.2)(.4)\right]$$

4. Novak Djokovic is currently ranked number 1 in the world in men's tennis. Suppose that Roger Federer has a probability of .6 of beating him on hardcourt, probability .9 of beating him on a grass court, and probability .75 of beating him on a clay court. Suppose also that 70% of their games have been on hardcourt, 20% on clay, and 10% on grass. If Fed beats Djokovic in a match, what is the probability that they were playing on a grass court?

By Bayes' rule:

$$\mathbb{P}(\text{grass} \mid \text{Fed wins}) = \frac{\mathbb{P}(\text{Fed wins} \mid \text{grass})\,\mathbb{P}(\text{grass})}{\mathbb{P}(\text{Fed wins})} =$$

$$= \frac{\mathbb{P}(F \mid \text{grass})\,\mathbb{P}(\text{grass})}{\mathbb{P}(F \mid gr)\,\mathbb{P}(gr) + \mathbb{P}(F \mid h)\,\mathbb{P}(h) + \mathbb{P}(F \mid c)\,\mathbb{P}(c)} =$$

$$= \frac{(0.9)(0.1)}{(0.9)(0.1) + (0.6)(0.7) + (0.75)(0.2)}$$

5. (a) Suppose that 150 (indistinguishable) chocolate glazed munchkins are to be distributed among 5 labeled boxes. The manager at Dunkin Donuts wants there to be a fairly even distribution: at least 25 per box. If the munchkins are distributed randomly among the boxes by a very distracted Dunky employee, what is the probability that the distribution will be to the manager's liking?

By using the "stars and bars" approach (the 150 munchkins are "stars" and 4 partitions between boxes are "bars"), $\binom{154}{4}$ ways to put 150 munchkins into 5 boxes.

To get ≥ 25 per box, isolate $25 \times 5 = 125$ of them to go in the boxes, and then randomly distribute 25 into 5 boxes: $\binom{29}{4}$ ways.

So $\mathbb{P}(\text{pleasing manager}) = \dfrac{\binom{29}{4}}{\binom{154}{4}}$

(b) In general, if there are n (indistinguishable) munchkins to be distributed among r labeled boxes, what is the probability that the distracted employee will please his manager if the manager wants to see at least m munchkins per box?

n boxes, n munchkins means n "stars" and $r-1$ "bars": $\binom{n+r-1}{r-1}$ ways to distribute.

To have $\geq m$ in each, isolate $m \times r$, then place $n-mr$ in the boxes randomly: $\binom{n-mr+r-1}{r-1}$ ways.

So $\mathbb{P}(\text{pleasing manager}) = \dfrac{\binom{n-mr+r-1}{r-1}}{\binom{n+r-1}{r-1}}$

6. Suppose that students taking their Defense Against the Dark Arts O.W.L.s – an examination students take at Hogwarts School of Witchcraft and Wizardry– are given 5 questions, and must answer at least 4 correctly in order to pass the exam. Draco Malfoy finds a cheat sheet containing 100 questions, and knows that each of the 5 questions he'll be asked will be chosen randomly from that sheet (each is equally likely).

   (a) If Malfoy comes in to his exam knowing the answer to 85 of the questions, what is the probability that he will pass?

   $\mathbb{P}(\geq 4$ q's chosen from the 85 he knows$)$ =

   choose 4 of the ones he knows

   choose 5 of the ones he knows

   $$\frac{\binom{85}{4}\binom{15}{1} + \binom{85}{5}}{\binom{100}{5}}$$

   choose 1 he doesn't know

   choose 5 questions.

   (b) If the random variable X is "the number of questions the examiner chooses from the list to which Malfoy knows the answers," what distribution does X have? (Be specific – e.g. if the answer were that X has the Dinglehoffer distribution with parameters n, k, and p, you would also have to specify the values of n, k, and p to receive full credit.)

   Hypergeometric with

   $N = 100$ (total #)

   $k = 85$ (total # w/ desired property)

   $m = 5$ (# sampled)

7. (a) Professor Snape claims that he selects students completely at random in his potions class to present their potion. However, Harry Potter seems to get chosen way too often. Suppose there are 40 Potions classes in one term, and 20 students in the class. If Snape is truly choosing randomly, compute the exact probability that Harry will be chosen at least three times during the term to present a potion.

20 students $\Rightarrow$ Probability of being chosen $= 1/20$.

40 classes $\Rightarrow$ 40 "trials", each w/ probability $1/20$ of success (success = Harry is called)

$\mathbb{P}(\#\text{successes} \geq 3) = 1 - \mathbb{P}(\#\text{successes} < 3) =$

$= 1 - \mathbb{P}(\#s = 0) - \mathbb{P}(\#s = 1) - \mathbb{P}(\#s = 2) =$

$= 1 - \left(\frac{19}{20}\right)^{40} - \binom{40}{1}\left(\frac{19}{20}\right)^{39}\left(\frac{1}{20}\right) - \binom{40}{2}\left(\frac{19}{20}\right)^{38}\left(\frac{1}{20}\right)^2.$

(b) Hermione Granger is bummed because she never gets called on anymore in her classes. She interviews a large number of Hogwarts students and alumni and finds that on average, each student is called on four times per term in class. Given that information, approximate the probability that Hermione is selected to show off her skillz no more than twice in all of her classes this term, assuming students are selected randomly.

Can't use binomial because we don't know how many people she asked $\Rightarrow$ Poisson, with $\lambda = 4$.

$X =$ # times Hermione is called.

$\mathbb{P}(X \leq 2) = \mathbb{P}(X=0) + \mathbb{P}(X=1) + \mathbb{P}(X=2) =$

$= e^{-4} + 4e^{-4} + \frac{4^2}{2} e^{-4}$

Bonus Question (3 pts.): If you play roulette using the Labouchere (list) strategy with initial list $\{1, 2, \ldots, n\}$, what are your total winnings (in terms of n) when the list goes down to empty? Give an explanation for why the winnings are the same every time you play with this strategy, despite the fact that roulette is a game of chance!

Recall that the Labouchere strategy is as follows: You start with a list of numbers, and bet the sum of the first and last numbers in the list on red. If you win the bet, you get that amount, and you remove the first and last numbers from your list. If you lose the bet, you lose that amount, and you add the sum of the first and last numbers to the end of your list. If only one number is left in your list, you bet that amount. You keep going until the list becomes empty.

To get an empty list, all #s in the list had to be added to winnings, so winnings $\geq 1+2+3+4+\cdots+n$.

And any #s that weren't originally in the list had to be subtracted from winnings before they were added on, so they don't contribute.

Hence, winnings $= 1+2+\cdots+n = \dfrac{n(n+1)}{2}$.