Linear Algebra Practice Final Exam
Final Exam on Sunday, June 2
Bradley 102, 1:00 – 3:00

1 Let

$$A = \begin{pmatrix} -2 & 12 \\ -1 & 5 \end{pmatrix}.$$

Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$. Briefly explain yourself.

Solution

First we need to find the eigenvalues. For this, we look at

$$\det(A - \lambda I) = \det \begin{pmatrix} -2-\lambda & 12 \\ -1 & 5-\lambda \end{pmatrix} = (\lambda - 2)(\lambda - 1),$$

which means that the eigenvalues are $\lambda = 1$ and $\lambda = 2$. Since they are different, we know that there are two linearly independent eigenvectors for A; these will form a basis for $\mathbf{R}^2$, so A is diagonalizable. It remains to find the eigenvectors. Let's do $\lambda = 1$ first. We need to solve $(A - I)\mathbf{x} = \mathbf{0}$, so we row reduce

$$A - I = \begin{pmatrix} -3 & 12 \\ -1 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & -4 \\ 0 & 0 \end{pmatrix}.$$

Now we repeat the process for $\lambda = 2$. Row reduce

$$A - I = \begin{pmatrix} -4 & 12 \\ -1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}.$$

We conclude that we have eigenvectors

$$\mathbf{b}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad \text{for eigenvalue 1} \quad \text{and} \quad \mathbf{b}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{for eigenvalue 2.}$$

The upshot of all of this is that

$$A = \begin{pmatrix} -2 & 12 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}^{-1}.$$

2

(a) Write down a matrix that is not diagonalizable. Explain.

Solution

The matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is not diagonalizable. This is because the only eigenvalue of A is 0, but

$$A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq 0,$$

so the $\mathbf{R}^2$ is not spanned by vectors killed by A.

(b) Explain why the null space of a linear transformation is a subspace.

Solution

Recall that $\mathrm{Null}(T) = \{\mathbf{x} \,|\, T(\mathbf{x}) = \mathbf{0}\}$. So if If $\mathbf{x}$ and $\mathbf{y}$ are both in $\mathrm{Null}(T)$ and $c \in \mathbf{R}$, then

$$T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}) = \mathbf{0}+\mathbf{0} = \mathbf{0} \quad \text{and} \quad T(c\mathbf{x}) = cT(\mathbf{x}) = c\mathbf{0} = \mathbf{0},$$

so $\mathbf{x}+\mathbf{y}$ and $c\mathbf{x}$ are both in $\mathrm{Null}(T)$. This shows that $\mathrm{Null}(T)$ is a subspace.

(c) True or False: If an $n \times n$ matrix A is diagonalizable, then every vector x in $\mathbf{R}^n$ is an eigenvector for A. Explain.

Solution

FALSE! For example, if

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

then

$$A\mathbf{x} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

which is not a multiple of $\mathbf{x}$, so $\mathbf{x}$ is not an eigenvector. But A is obviously a diagonal(izable) matrix!

(d) If the null space of a 7×11 matrix is 5-dimensional, what is the dimension of the column space?

Solution

The dimension of the null space is the number of *non*pivot columns of A and the dimension of the column space is the number of pivot columns of A. So if there are 5 nonpivot columns, there must be 2 pivot columns, which means that the column space is 2-dimensional.

3 Consider the matrix

$$A = \begin{pmatrix} 1 & 3 & 5 & 9 \\ 2 & 4 & 6 & 7 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

(a) Find a basis for Nul(A).

Solution

To find a basis for Nul(A), we simply need to solve the equation $A\mathbf{x} = \mathbf{0}$ by row reduction:

$$\begin{pmatrix} 1 & 3 & 5 & 9 \\ 2 & 4 & 6 & 7 \\ 1 & 2 & 3 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 5 & 9 \\ 0 & -2 & -4 & -11 \\ 0 & -1 & -2 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 5 & 9 \\ 0 & 1 & 2 & 5 \\ 1 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This shows that the vector

$$\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

spans Nul(A), and hence $\{\mathbf{b}\}$ is a basis for Nul(A).

(b) Find a basis for Col(A).

Solution

As we saw in class, the pivot columns of A form a basis for $\mathrm{Col}(A)$. This means that the vectors

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 9 \\ 7 \\ 4 \end{pmatrix}$$

are a basis for $\mathrm{Col}(A)$.

4 Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear transformation with the property that $T(T(\mathbf{x})) = T(\mathbf{x})$ for every vector $\mathbf{x} \in \mathbf{R}^n$ (such a linear transformation is called **idempotent**).

(a) Write V for the image (or range) of T. In other words,

$$V = \{T(\mathbf{x}) \,|\, \mathbf{x} \in \mathbf{R}^n\}.$$

If $\mathbf{v} \in V$, then what is $T(\mathbf{v})$?

Solution

If $\mathbf{v} \in V$, then $\mathbf{v} = T(\mathbf{x})$ for some $\mathbf{x}$. This means that

$$T(\mathbf{v}) = T(T(\mathbf{x})) = T(\mathbf{x}) = \mathbf{v}.$$

In other words, $\mathbf{v}$ is an eigenvector for T with eigenvalue 1.

(b) If $\mathbf{x} \in \mathbf{R}^n$, then what is $T(\mathbf{x} - T(\mathbf{x}))$?

Solution

Again, we just need to calculate

$$T(\mathbf{x} - T(\mathbf{x})) = T(\mathbf{x}) - T(T(\mathbf{x})) = T(\mathbf{x}) - T(\mathbf{x}) = \mathbf{0}.$$

In other words, $\mathbf{v}$ is an eigenvector for T with eigenvalue 0.

(c) Let $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k\}$ be a basis for V. Then we can add some more vectors, $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_l$ to get a basis $\mathcal{B}$ for all of $\mathbf{R}^n$. Show that if you replace $\mathbf{b}_1$ with $\mathbf{a}_1 = \mathbf{b}_1 - T(\mathbf{b}_1)$ then you still have a basis.

Solution

Since the new collection has the same number of vectors as the given basis, we just need to show that the new collection spans $\mathbf{R}^n$. Since $T(\mathbf{b}_1) \in V$, we can write $T(\mathbf{b}_1)$ as a linear combination of the $\mathbf{c}$'s, which shows that

$$\mathbf{b}_1 \in \operatorname{span}\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k, \mathbf{b}_1 - T(\mathbf{b}_1)\}.$$

So whatever the span of the new set of vectors is, it contains all the $\mathbf{c}$'s and all the $\mathbf{b}$'s. This means that the span of the new set of vectors is all of $\mathbf{R}^n$.

(d) In the same way, we can replace each $\mathbf{b}_i$ with $\mathbf{a}_i = \mathbf{b}_i - T(\mathbf{b}_i)$. What is the matrix of T with respect to the basis $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_k, a_1, \mathbf{a}_2, \ldots, \mathbf{a}_l\}$? (This is an *easy* question – not much work is needed!)

Solution

As we saw in part (a), each $\mathbf{c}_i$ is an eigenvector with eigenvalue 1; as we saw in part (b), each $\mathbf{a}_i$ is an eigenvector with eigenvalue 0. This means that T is diagonalizable, and the matrix is

$$D = \begin{pmatrix} 1 & 0 & \cdots & & & & \cdots & 0 \\ 0 & 1 & & & & & & 0 \\ \vdots & & \ddots & & & & & \vdots \\ & & & d_{kk} = 1 & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ \vdots & & & & & & \ddots & \vdots \\ 0 & 0 & \cdots & & & & \cdots & 0 \end{pmatrix}.$$

with 1's down the diagonal until the k^{th} place and 0's afterwards.