Math 24: Introduction to Proofs

Definitions:

(1) The set of natural numbers is $N = \{1, 2, 3, \ldots\}$.
(2) The set of integers is $Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
(3) The set Q of rational numbers consists of the numbers x which can be written in the form $x = \frac{a}{b}$ for some integers a and b with $b \neq 0$.
(4) An integer n is *even* if $n = 2k$ for some integer k.
(5) An integer n is *odd* if $n = 2k + 1$ for some integer k.
(6) Suppose m and n are integers. We say that m *divides* n if $m \neq 0$ and $n = mk$ for some integer k. In this case, we write $m \mid n$.
(7) A natural number $n > 1$ is prime if its only divisors are 1 and n.
(8) A natural number $n > 1$ is composite if it has a divisor that is not equal to 1 nor n.

You may assume the following:

- Basic properties of arithmetic (i.e. the sum and product of two integers is an integer, addition and multiplication are commutative, etc.)
- Every integer is either even or odd
- Every natural number greater than 1 is either prime or composite
- Every rational number can be written as a fraction in lowest terms (i.e. $x = \frac{a}{b}$ where a and b have no common factors)

Prove the following statements:

(1) If two integers are both odd, then their product is odd.

(2) Let n be a natural number. If n^2 is even then n is even.

(3) There do not exist integers m and n such that $14m + 21n = 100$.

(4) Let A and B be any two sets. Then $(A \cup B)' = A' \cap B'$.

(5) Let a and b be non-negative real numbers. If $a^2 \geq b^2$ then $a \geq b$.

(6) Let a, b, and c be natural numbers. If a divides b, b divides c, and c divides a, then $a = b = c$.

(7) If n is a positive multiple of 3, then either n is odd or n is a multiple of 6.

(8) The only even prime number is 2.

(9) Let A, B and C be any three sets. Then $(A \cap B) \cup (A \cap C) = A \cap (B \cup C)$.

Challenge problems:

(1) If $2^n - 1$ is prime then n is prime. (A prime of the form $2n - 1$ is called a Mersenne prime.)

(2) Every four-digit palindrome number is divisible by 11. (A palindrome reads the same backward and forward).

(3) $\sqrt{2}$ is irrational.