Homework #7

1) Page 244 #4

a) Goal: Find eigenvalues & orthonormal eigenfunctions for $(Ku)(x) = \int_{-\pi}^{\pi} k(x+y)u(y)dy$

$k(x) = \frac{k_0}{2} + \sum_{n=1}^{\infty} k_n \cos(nx)$ where k_n is a strictly decreasing sequence.

It seems likely $\{1, \cos(nx)\}$

Lets try $\phi_0 = 1$ first

let $s = x+y \to y = s-x$, $dy = ds$

$$(K\phi_0) = \int_{-\pi}^{\pi} k(x+y)\phi_0(y)\,dy$$

$$= \int_{-\pi}^{\pi} \left(\frac{k_0}{2} + \sum_{n=1}^{\infty}(k_n \cos(n(x+y))\right)\phi_0(y)\,dy$$

$$= \int_{-\pi+x}^{\pi+x} \left(\frac{k_0}{2} + \sum_{n=1}^{\infty} k_n \cos(ns)\right)\underbrace{\phi_0(s-x)}_{1}\,ds.$$

$\downarrow$ since everything is periodic

$$= \int_{-\pi}^{\pi} \left(\frac{k_0}{2} + \sum_{n=1}^{\infty} k_n \cos(ns)\right)ds = \int_{-\pi}^{\pi}\frac{k_0}{2}ds + \sum_{n=1}^{\infty} k_n \underbrace{\int_{-\pi}^{\pi}\cos(ns)ds}_{=0}.$$

$$= \frac{k_0}{2}2\pi = k_0\pi \cdot \frac{\sqrt{2\pi}}{\sqrt{2\pi}} = k_0\pi\sqrt{2\pi}\,\phi_0(x)$$

now $\int_{-\pi}^{\pi} 1\,dx = 2\pi \Rightarrow$ To normalize, $\phi_0(x) = \frac{1}{\sqrt{2\pi}}$

with eigenvalue $\lambda_0 = k_0\pi\sqrt{2\pi}$

Now guess $\phi_n(x) = \cos(nx)$ is eigenfunction

1st normalize

$$\int_{-\pi}^{\pi} \cos^2(nx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} (1-\cos(2nx))\,dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2n}\sin(2nx)\Big|_{-\pi}^{\pi}\right)$$

$$= \frac{2\pi}{2} = \pi$$

$\rightarrow \phi_n(x) = \frac{1}{\sqrt{\pi}}\cos(nx)$ is orthonormal.

Now show it is an eigenfunction & find eigenvalue.

$$(K\phi_m) = \int_{-\pi}^{\pi} k(x+y)\,\phi_n(y)\,dy = \int_{-\pi+x}^{\pi+x} k(s)\,\phi_m(s-x)\,ds$$

$$= \int_{-\pi}^{\pi} k(s)\,\phi_m(s-x)\,ds$$

$$= \int_{-\pi}^{\pi}\left(\frac{k_0}{2} + \sum_{n=1}^{\infty} k_n\cos(ns)\right)\frac{1}{\sqrt{\pi}}\cos(m(s-x))\,ds$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi}\left(\frac{k_0}{2} + \sum_{n=1}^{\infty} k_n(\cos ns)\right)\left[\cos(ms)\cos(mx) + \sin(ms)\sin(mx)\right] ds$$

$$= \frac{1}{\sqrt{\pi}}\cos(mx)\int_{-\pi}^{\pi}\left(\frac{k_0}{2} + \sum_{n=1}^{\infty} k_n\cos(ms)\right)\cos(ms)\,ds \quad = 0 \text{ if } m \neq n.$$

$$+ \frac{\sin(mx)}{\sqrt{\pi}}\underbrace{\int_{-\pi}^{\pi}\left(\frac{k_0}{2}\sin(ms) + \sum_{n=1}^{\infty} k_n\cos(ns)\sin(ms)\right)ds}_{}$$

= 0 since sine is orthogonal to 1, cos(nx)

$$(K\phi_m)(x) = \frac{k_m}{\sqrt{\pi}} \overset{\cos(mx)}{} \int_{-\pi}^{\pi} \cos^2(mx)\, dx = \frac{k_m}{\sqrt{\pi}} \overset{\cos(mx)}{} \frac{1}{2}\int_{-\pi}^{\pi} 1-\cos(2mx)\, dx$$

$$= k_m \sqrt{\pi} \cos(mx) = \underbrace{k_m \sqrt{\pi}}_{\lambda_m} \phi_m.$$

d) $Ku(x) = \int_0^x \sin x \cos(2y)\, u(y)\, dy.$

Text suggest that there are no eigenvalues.

We must show this.

we are trying to solve

$$Ku = \lambda u.$$

$$\Rightarrow u - \frac{1}{\lambda} Ku = \underset{f(x)}{0}$$

By Neumann series. we know the solution is given by

$$u(x) = \sum_{n=0}^{\infty} \left(\frac{1}{\lambda}K\right)^n f(x)$$

In our problem $f=0 \Rightarrow u(x) = 0$ is the only answer. (since solution is unique.)

$\Rightarrow$ only trival solution $\Rightarrow$ no eigenvalues.

2) Page 244 #7

From #4c.

The eigenvalues are $\lambda_n = \frac{\pi}{n^2}$ and the eigenfunctions are $\phi_n = \sin(nx)$

a) $Ku - 2u = 0.$

$\lambda = 2$ is not an eigenvalue of (Ku) thus there is only a trivial soln.

$u = 0$

b) $Ku - \frac{\pi}{9}u = x(\pi - x) = f(x)$

$\lambda_3 = \frac{\pi}{9}$ is an eigen value of K. There is only a solution if $f(x)$ is orthogonal $\sin(3x)$.

Well $f(x) = \frac{8}{\pi}\left[\sin x + \frac{\sin 3x}{3^3} + \dots\right]$

$f_3 \neq 0 \Rightarrow$ not orthogonal.

or you can check via brute force.

$\Rightarrow$ no solution.

c) $Ku - 2u = x(\pi - x)$

$\lambda = 2$ is not an eigenvalue of (Ku)

So our solution $u \in L^2([0,\pi]) \Rightarrow u(x) = \sum_{n=1}^{\infty} u_n \sin(nx)$

Plug into equation.

$$Ku - 2u = \sum_{n=1}^{\infty} (\lambda_n - 2) u_n \sin(nx) = \sum_{n=1}^{\infty} f_n \sin(nx)$$

So $u_n = \frac{f_n}{\lambda_n - 2} = \frac{8}{\pi n^3 \left(\frac{\pi}{n^2} - 2\right)}$ for n odd.

$$= \frac{8}{\pi n(\pi - 2n^2)}$$

Solution is given by

$$u(x) = \sum_{\substack{n=1 \\ n\,odd}}^{\infty} \frac{8}{\pi n(n - 2n^2)} \sin(n x)$$

d) $Ku - \frac{\pi}{9} u = \sin 2x$ (*)

$\lambda_3 = \frac{\pi}{9}$ is an eigenvalue. thus there are infinitely many solutions.

$u_h = C \sin(3x)$ = homogeneous solution.

The particular solution. will be given by

$$u_p(x) = \sum_{\substack{n=1 \\ n\neq 2}}^{\infty} u_n \sin(nx)$$

Plug into (*)

$$\sum_{\substack{n=1 \\ n\neq 2}}^{\infty} \left(\lambda_n - \frac{\pi}{9}\right) u_n \sin(nx) = \sin 2x$$

$\Rightarrow u_n = 0 \;\; \forall n \neq 2$

$$u_2 = \frac{1}{\frac{\pi}{4} - \frac{\pi}{9}} = \frac{36}{5\pi}$$

So the solutions are

$$u(x) = c\sin(3x) + \frac{36}{5\pi}\sin 2x$$

for any constant C.

3) We need the Fourier coefficients of $k(s)$

$k(s)$ is even $\rightarrow$ only need cosine terms.

$$k(s) = \frac{k_0}{2} + \sum_{n=1}^{\infty} k_n \cos(ns)$$

$$k_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} k(s)\,ds = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\,ds = 1$$

$$k_n = \frac{1}{\pi}\int_{-\pi}^{\pi} k(s)\cos(ns)\,ds = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos(ns)\,ds$$

$$= \frac{1}{\pi}\frac{1}{n}\left(\sin(ns)\right)\Big|_{-\pi/2}^{\pi/2} = \frac{1}{\pi}\frac{1}{n}\left[\sin\left(\tfrac{n\pi}{2}\right) - \sin\left(\tfrac{n\pi}{2}\right)\right]$$

only odd terms remain

$$= \frac{1}{\pi}\frac{1}{2m+1}\left[(-1)^{m+1} - (-1)^m\right] = \frac{2(-1)^m}{2m+1}$$

$n = 2m+1 \rightarrow \boxed{m = \frac{n-1}{2}}$

$m = 0, 1, 2, \dots$

This means the eigenvalues of the operator are $\lambda_0 = \pi k_0$, $\lambda_m = 2(-1)^m/(2m+1) = \pi a_m$

$\phi_0 = 1$ $\qquad \phi_m = \cos((2m+1)x)$

$m = 1, 2, \dots, n$

use here

i) We know from worksheet. (or class)

$A_0 = \lambda_0 a_0 = \pi a_0$

$$\left.\begin{aligned} A_n &= \frac{2(-1)^{\frac{(n-1)}{2}}}{n} a_n \\ B_n &= \frac{2(-1)^{\frac{(n-1)}{2}}}{n} b_n \end{aligned}\right\} \text{ For } n \text{ odd.}$$

$A_n = B_n = 0$ $\qquad$ n even

ii)

the blurred image $g(x)$ is given by

$$g(x) = (k * f)(x) = \frac{A_0}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} A_n \cos(nx) + B_n \sin nx)$$

$\to$ only odd terms will be recovered by the deblurring process.

let $\tilde{f}(x)$ = deblurred image. then

$$\tilde{f}(x) = \frac{A_0}{2\pi} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} (-1)^{\frac{n-1}{2}} \frac{n}{2} \left[A_n \cos(nx) + B_n \sin(nx)\right]$$

iii) Blow up factor is $\frac{\text{noise}}{k_n}$

$\to$ we need $\left|\frac{0.01}{k_n}\right| \le \underbrace{0.3}_{\text{max allowed error}}$ $\to$ $\left|\frac{0.01}{2} n\right| \le 0.3$

$\to n \le 60.$

Including zero term we should only take sum of 61 terms.

4) on Page 2.80

$$V(x) = \int_a^b g(x,\xi)\, f(\xi)\, d\xi$$

$$= -\int_a^x \frac{u_1(x)\, u_2(\xi)}{p(\xi)\, W(\xi)} f(\xi)\, d\xi \;-\int_x^b \frac{u_1(\xi)\, u_2(x)}{p(\xi)\, W(\xi)} f(\xi)\, d\xi$$

Goal: Show

$Lv = f$. where $Lu = -(pu')' + qu$

$$v'(x) = -u_1'(x)\int_a^x \frac{u_2(\xi)}{p(\xi)W(\xi)} f(\xi)\, d\xi - u_1(x)\left(\frac{u_2(x)f(x)}{p(x)W(x)}\right) \text{ cancel.}$$

$$+ u_2'(x)\int_b^x \frac{u_1(\xi)}{p(\xi)W(\xi)} f(\xi)\, d\xi + u_2(x)\left(\frac{u_1(x)f(x)}{p(x)W(x)}\right)$$

$$= -u_1'(x)\int_a^x \frac{u_2(\xi)f(\xi)}{p(\xi)W(\xi)} d\xi + u_2'(x)\int_b^x \frac{u_1(\xi)}{p(\xi)W(\xi)} f(\xi)\, d\xi$$

$$\Rightarrow -(pv')' = (p(x)u_1'(x))'\int_a^x \frac{u_2(\xi)f(\xi)}{p(\xi)W(\xi)} d\xi + p(x)u_1'(x)\left(\frac{u_2(x)f(x)}{p(x)W(x)}\right)$$

$$-(p(x)u_2'(x))'\int_b^x \frac{u_1(\xi)f(\xi)}{p(\xi)W(\xi)} d\xi + \cancel{-}\, p(x)u_2'(x)\left(\frac{u_1(x)f(x)}{W(x)p(x)}\right)$$

$$= (p(x)u_1'(x))'\int_a^x \frac{u_2(\xi)f(\xi)}{p(\xi)W(\xi)} d\xi - (p(x)u_2'(x))'\int_b^x \frac{u_1(\xi)f(\xi)}{p(\xi)W(\xi)} d\xi$$

$$+ f(x)\frac{\cancel{(u_2u_1' - u_1u_2')}^{1}}{\cancel{W(x)}}$$

9

now

$$-(pv')' + qv = \left((pu_1')' - qu_1\right)\int_a^{x} \frac{u_2(\xi) f(\xi)}{p(\xi) W(\xi)}\, d\xi$$

$$+ \left(-(pu_2')' + qu_2\right)\int_b^x \frac{u_1(\xi) f(\xi)}{p(\xi) W(\xi)}\, d\xi + f(x)$$

now $(pu_1')' - qu_1 = 0 \to$ since u_1 is a soln to homogeneous problem

and $-(pu_2')' + qu_2 = 0 \to$ same reasoning.

$\therefore$ $Lv = f$ ✓

Boundary conditions.

$$\alpha_1 v(a) + \alpha_2 v'(a) = \alpha_1\left(-\int_a^b \frac{u_1(\xi)u_2(a)f(\xi)}{W(\xi)p(\xi)}\, d\xi\right) + \alpha_2 \left(-u_2'(a)\int_a^b \frac{u_1(\xi) f(\xi)}{W(\xi)p(\xi)}\, d\xi\right)$$

$$= \underbrace{\left(\alpha_1 u_2(a) + \alpha_2 u_2'(a)\right)}_{=0}\int_a^b \frac{u_1(\xi)f(\xi)}{W(\xi)p(\xi)}\, d\xi$$

$$\beta_1 v(b) + \beta_2 v'(b) = \beta_1 \overset{=0}{\left[-u_1(b)\int_a^b \frac{u_2(\xi) f(\xi)}{p(\xi)W(\xi)}\, d\xi\right]} - \beta_2 u_1'(b)\int_a^b \frac{u_2 f}{pW}$$

$$= -\underbrace{\left(\beta_1 u_1(b) + \beta_2 u_2(b)\right)}_{=0}\int_a^b \frac{u_2(\xi) f(\xi)}{p(\xi) W(\xi)}\, d\xi$$ ✓

5) Page 257 #1

Solvability of $u'' + \pi^2 u = f \quad 0<x<1 \quad u(0) = u(1) = 0$

Is zero an eigenvalue of $Lu = u'' + \pi^2 u$?

Homogeneous solution $u(x) = C_1 \cos(\pi x) + C_2 \sin(\pi x)$

$u(0) = C_1 = 0$

$u(1) = C_2 \sin(\pi) = 0$.

$\Rightarrow$ zero is an eigenvalue & the corresponding eigenfunction is $\phi = \sin(\pi x)$

$\Rightarrow$ There exist infinitely many solns if $\langle f, \sin\pi x\rangle = 0$.

ie orthogonal.

Otherwise no solution.

6) Page 257 #2

1^{st} Green's function? $Lu = u'' + 4u \quad u(0) = u(\pi) = 0.$

$\lambda = 0$ is an eigenvalue w/ eigenfunction

$\phi(x) = \sin(2x)$

Now solve. We know $\{ \sin nx \}_{n=1}^{\infty}$ is complete in $L^2([0,\pi])$

This means an function $f(x) \in L^2([0,\pi])$ can be

expressed as

$$f(x) = \sum_{n=1}^{\infty} f_n \sin(nx) \quad \text{where } f_n = \frac{\langle f, \sin nx \rangle}{\langle \sin nx, \sin nx \rangle}$$

We also know

$\circledast \; u'' + 4u = f(x)$ only has a solution

if $\langle f, \sin 2x \rangle = 0 \rightarrow f_2 = 0$,

So we seek a solution $u(x) = \sum_{\substack{n=1 \\ n \neq 2}} u_n \sin(nx)$

Plug into $\circledast$.

$$\sum_{\substack{n=1 \\ n \neq 2}}^{\infty} u_n (-n^2 + 4) \sin nx = \sum_{\substack{n=1 \\ n \neq 2}}^{\infty} f_n \sin(nx)$$

$$u_n (4 - n^2) = f_n \qquad n \neq 2$$

$$\rightarrow u_n = \frac{f_n}{4 - n^2}$$

$\Rightarrow$ all solutions (infinitely many) are given by

$$u(x) = c \sin(2x) + \sum_{\substack{n=1 \\ n \neq 2}}^{\infty} \frac{f_n}{4 - n^2} \sin(nx)$$

If $\langle f, \sin(2x) \rangle = 0$ OTHERWISE NO SOLUTION.

7) Page 257 #5.

$-(ku')' = f \qquad 0 < x < 1 \qquad u(0) = u(1) = 0 \quad k(x) > 0.$

Assume $k \neq 0$. $\forall x \in (0,1)$

1st Solve homogeneous

$(ku')' = 0 \rightarrow \quad k(x)\,u'(x) = C \rightarrow \quad u'(x) = \frac{c}{k(x)}$

$\rightarrow u(x) = c\int_0^x (k(s))^{-1} ds + d$

$u_1(x)$ satisfies Left BC.

$u(0) = d = 0 \rightarrow$ take $u_1 = \int_0^x (k(s))^{-1} ds$

$u_2(x)$ Satisfies Right BC

$\rightarrow u(1) = \int_0^1 (k(s))^{-1} ds + d = 0$

$\Rightarrow$ take $u_2(x) = \int_x^1 (k(s))^{-1} ds$

$$W(x) = \begin{vmatrix} \int_0^x (k(s))^{-1} ds & \int_x^1 (k(s))^{-1} ds \\ (k(x))^{-1} & -(k(x))^{-1} \end{vmatrix}$$

$= -(k(x))^{-1} \int_0^x (k(s))^{-1} - (k(x))^{-1} \int_x^1 (k(s))^{-1} ds$

$= -(k(x))^{-1} \int_0^1 (k(s))^{-1} ds$

$p(x) = k(x)$

$\rightarrow p(\xi)W(\xi) = -\int_0^1 (k(s))^{-1} ds$

So the Green's function is given by

$$g(x,\xi) = \frac{1}{\int_0^1 (k(s))^{-1} ds} \begin{cases} \int_0^x (k(s))^{-1} ds \int_\xi^1 (k(s))^{-1} ds & x < \xi \\ \int_0^\xi (k(s))^{-1} ds \int_x^1 (k(s))^{-1} ds & x > \xi \end{cases}$$

8) Page 258 # 7

$$\sum_{n=1}^{\infty} \frac{\sin(nx)\sin(n\xi)}{n^2} \qquad 0 < x, \xi < \pi.$$

This is the orthogonal expansion (eigen expansion) of the Green's function for the SLP with $\lambda_n = n^2$ as eigenvalues with $\phi_n(x) = \sin(nx)$ as eigen functions.

This corresponds to the equation

$$Lu = \lambda u \quad \text{where } L = -\frac{d^2}{dx^2}$$

with Boundary conditions

$$u(0) = u(\pi) = 0.$$

Since 0 is not an eigenvalue of L, we can construct the Green's function from the Homogeneous Solution

$$u(x) = Ax + B.$$

$u_1(x)$ satisfies Left BC - $u_1(0) = 0 = B$

let $u_1(x) = x$

$u_2(x)$ satisfies Right BC $\Rightarrow u_2(\pi) = A\pi + B = 0$

$\Rightarrow B = -A\pi$

let $A = 1$ $\quad u_2(x) = x - \pi.$

$$W = \begin{vmatrix} x & x-\pi \\ 1 & 1 \end{vmatrix} = x - (x-\pi) = \pi \qquad p(x) = -1$$

$$\Rightarrow g(x,\xi) = \begin{cases} \dfrac{x(\xi-\pi)}{\pi} & x < \xi \\[2ex] \dfrac{\xi(x-\pi)}{\pi} & \xi < x. \end{cases}$$

Therefore

$$\sum_{n=1}^{\infty} \frac{\sin(nx)\sin(n\xi)}{n^2} = \begin{cases} \dfrac{x(\xi-\pi)}{\pi} & x<\xi \\[2ex] \dfrac{\xi(x-\pi)}{\pi} & \xi<x \end{cases}$$

9) Page 257 #8

$$Lu = -(x^2 u')' \quad \text{on} \quad 1 < x < e$$

$$u(1) = u(e) = 0$$

Goal: Find inverse of L

TWO OPTIONS.

1- use eigen expansion to make Green's function
Page 285 #7 gives $\{\lambda_n, \phi_n\}$
Don't foget to normalize
see last page

2- Since $\lambda = 0$ is not an eigenvalue.
We can construct $g(x, \xi)$ via the homogeneous solution.

$$-(x^2 u')' = 0 \Rightarrow (x^2 u')' = 0$$

$$\Rightarrow x^2 u' = C \Rightarrow u' = \frac{C}{x^2}$$

$$u(x) = \frac{C}{x} + D$$

$u_1(x)$ Satisfies left BC.

$$u_1(1) = \frac{C}{1} + D = 0 \Rightarrow C = -D$$

let $D = 1$ $\quad u_1(x) = 1 - \frac{1}{x}$

$u_2(x)$ Satisfies Right BC.

$$u_2(e) = \frac{C}{e} + D = 0 \Rightarrow D = -\frac{C}{e}$$

let $C = -e$

$$u_2(x) = 1 - \frac{e}{x}$$

$$W = \begin{vmatrix} 1-\frac{1}{x} & 1-\frac{e}{x} \\ -\frac{1}{x^2} & \frac{-e}{x^2} \end{vmatrix} = \left(1-\frac{1}{x}\right)\left(\frac{-e}{x^2}\right) + \frac{1}{x^2}\left(1-\frac{e}{x}\right)$$

$$\left.\begin{aligned} &= \frac{-e}{x^2} + \frac{e}{x^3} + \frac{1}{x^2} - \frac{e}{x^3} = \frac{1-e}{x^2} \\ &P(x) = -x^2 \end{aligned}\right\} \quad P(\xi)W(\xi) = (1-e)(-1)$$

$$g(x,\xi) = \begin{cases} -\dfrac{\left(1-\frac{1}{x}\right)\left(1-\frac{e}{\xi}\right)}{1-e} & x < \xi \\[2ex] \dfrac{\left(1-\frac{1}{\xi}\right)\left(1-\frac{1}{x}\right)}{1-e} & \xi < x \end{cases}$$

$$u = L^{-1}f = \int_1^e g(x,\xi) f(\xi)\, d\xi$$

OPTION 1

$$\{\lambda_n, \phi_n(x)\} = \{(n\pi)^2+1 \ , \ x^{-1}\sin(n\pi \ln x)\}$$

$$g(x,\xi) = \sum_{n=1}^{\infty} \frac{x^{-1}\sin(n\pi \ln x) \ \xi^{-1}\sin(n\pi \ln \xi)}{(n\pi)^2+1}$$