Workshop 6

More Vector Spaces, Linear Transformations and Subspaces

Instructions:

Get into groups and work on the following exercises. Each group is expected to turn in *one neatly written copy* of their solutions at the end of the class period.

Throughout these exercises $\mathcal{Y}$ and $\mathcal{Z}$ are vector spaces, $S : \mathcal{Y} \to \mathcal{Z}$ is a linear transformation and $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p\}$ is a subset of $\mathcal{Y}$.

Exercise 1. S is called *one-to-one* if, for any given $\mathbf{z} \in \mathcal{Z}$, the equation $S(\mathbf{x}) = \mathbf{z}$ has at most one solution.

a. Show that S is one-to-one if and only if $\ker S = \{\mathbf{0}\}$. [*Hint:* Imitate the proof of the analogous fact for $\mathbb{R}^n$.]

b. Suppose that S is one-to-one. Show that if $S(\mathbf{w}) = S(\mathbf{y})$ then $\mathbf{w} = \mathbf{y}$.

Exercise 2. Show that if $\{S(\mathbf{y}_1), S(\mathbf{y}_2), \ldots, S(\mathbf{y}_p)\}$ is linearly independent then $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p\}$ is also linearly independent. [*Remark:* This is just the abstract version of the last problem on the exam. The proof should be nearly identical.]

Exercise 3. Suppose that S is one-to-one (see Exercise 1). Show that if $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p\}$ is linearly independent then $\{S(\mathbf{y}_1), S(\mathbf{y}_2), \ldots, S(\mathbf{y}_p)\}$ is also linearly independent.

Exercises 2 and 3 together show that if S is one-to-one then $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_p\}$ is linearly independent if and only if $\{S(\mathbf{y}_1), S(\mathbf{y}_2), \ldots, S(\mathbf{y}_p)\}$ is linearly independent.

Exercise 4.* If $\mathcal{V}$ and $\mathcal{W}$ are subspaces of $\mathcal{Y}$, recall that $\mathcal{V} + \mathcal{W} = \{\mathbf{v} + \mathbf{w} \ : \ \mathbf{v} \in \mathcal{V}, \mathbf{w} \in \mathcal{W}\}$ is also a subspace of $\mathcal{Y}$. Suppose that

$$\begin{aligned} \mathcal{V} &= \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}, \\ \mathcal{W} &= \text{Span}\,\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q\}. \end{aligned}$$

Show that $\mathcal{V} + \mathcal{W} = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q\}$.