Applications of Numbered Undirected Graphs

GARY S. BLOOM, MEMBER, IEEE, AND SOLOMON W. GOLOMB, MEMBER, IEEE

Abstract—Numbered undirected graphs are becoming an increasingly useful family of mathematical models for a broad range of applications. They have found usage in various coding theory problems, including the design of good radar-type codes, synch-set codes and convolutional codes with optimal autocorrelation properties. They facilitate the optimal nonstandard encodings of integers. They have also been applied to determining ambiguities in X-ray crystallographic analysis, to design of a communication network addressing system, to determination of optimal circuit layouts, and to problems in additive number theory. An attempt has been made to systematically present all of these diverse applications in a unifying framework and to indicate the existence of additional applications and to suggest directions for additional research.

I. Introduction

THERE IS an extensive literature dealing with the assignment of numbers to the vertices and edges of directed graphs and a relative paucity of such considerations for undirected graphs. This disparity has resulted from extensive interest in the obvious "real world" applications of numbered directed graphs (e.g., [1]-[4]). Nevertheless, utilitarian models have similarly been formulated using undirected graphs in a variety of less well-known contexts.

In this paper we survey diverse applications of a broad class of assignments of integers to the vertices and edges of graphs. We commence by stating several basic definitions and citing some applications of this unifying family of models.

A *graph* Γ consists of a set of vertices and a set of edges. Every edge must join two distinct vertices, and no more than one edge may join any vertex pair. If a nonnegative integer $\psi(v)$ is assigned to each vertex v, then the vertices of Γ are said to be "numbered." Γ is itself a *numbered graph* if each edge e, is given the value $\psi(e) = |\psi(v_1) - \psi(v_2)|$ where v_1 and v_2 are the endpoints of e. Clearly, in the absence of additional constraints, every graph can be numbered in infinitely many ways. Thus utilization of numbered graph models requires imposition of additional constraints which characterize the problem being investigated.

These necessary constraints arise naturally in studying the wide variety of seemingly unrelated practical applications for which numbered graphs provide underlying mathematical models. Some embodiments of this theory follow.

1) The design of certain important classes of good nonperiodic codes for pulse radar and missile guidance is equivalent to numbering the complete graph in such a way that all the edge numbers are distinct. The node numbers then determine the time positions at which pulses are transmitted. Corresponding radar pulse and missile-guidance code problems have been the subject of investigation for several years (e.g., [5]). Very similar patterns are also used to provide an efficient class of convolutional codes [6].

2) "Nonnatural" methods of encoding the integers from 0 to $b^n - 1$ using n-digit vectors from the b-symbol alphabet have been devised to minimize the seriousness of errors occurring in a single digit. These encodings have been the subject of an extensive literature (e.g., [7]-[10]). The corresponding graph problem involves numbering the nodes of the square lattice grid, b on a side in n dimensions, with the integers from 0 to $b^n - 1$, in a way that optimizes some statistical function (typically the mean or the variance) of the edge numbers.

3) Determination of crystal structures from X-ray diffraction data has long been a concern of crystallographers. The ambiguities inherent in this procedure are now beginning to be understood [11]-[13]. In some cases the same diffraction information may correspond to more than one structure. This problem is mathematically equivalent to determining all numberings of the appropriate graphs which produce a prespecified set of edge numbers.

4) In a small communication network, it might be useful to assign each user terminal a "node number," subject to the constrait that all connecting "edges" (communication links) receive *distinct* numbers. In this way, the numbers of any two communicating terminals automatically specify (by simple subtraction) the link number of the connecting path; and conversely, the path number uniquely specifies the pair of user terminals which it interconnects.

Other applications of numbered graphs have included design of highly accurate optical gauging systems for use on automatic drilling machines [14], design of angular synchronization codes [15], design of optimal component layouts for certain circuit-board geometries [16], and determining configurations of simple resistor networks which can be used to supply any of a specified set of resistance values [17]-[19].

Numbered graph interpretations also apply to other areas of mathematics. Some of the most significant numerical results have resulted from the correspondence between some *ruler problems* in additive number theory and numbered graphs (e.g., [20], [21]).

II. Applications to Coding Theory

Coding theory, to date, has probably inspired more efforts in the area of numbered graphs and used more of its results than any other single field of application. This observation can better appreciate in retrospect, since almost no previous research results have formally been presented in a numbered graph context. To our knowledge, only one investigation exploited numbered graphs in its original formulation. We start our survey with that work.

Manuscript received June 7, 1976; revised September 24, 1976. This research was supported in part by the US Army Research Office under Contract DA-ARO-D-31-124-73-G167.

G. S. Bloom is with the Quantitative Methods Department and the Computer Science Program at California State University, Fullerton, CA 92634.

S. W. Golomb is with the Electrical Engineering and Mathematics Department, University of Southern California, Los Angeles, CA 90007.

A. Optimal Nonstandard Encoding of Integers

In 1954 Kautz [8] examined the question of "minimizing confusion" in the design of codes for digital computers and information processing machines. Although redundancy is designed into codes to detect and in some cases to correct the most likely error patterns in the encoding, transmission, and decoding of messages, some error patterns do pass by undetected. Increasing error correction and detection reliability is generally accomplished at the cost of lowering information transmission rate.

Computers and many other data-handling systems are characterized by very low noise levels and severe reliability requirements. Any error in the message cycle is rare, and multiple errors in a single message are virtually nonexistent. Because of the rarity of these errors, Kautz suggested an alternative to slowing down information transmission rates by adding redundancy. He suggested minimizing confusion resulting from errors by judicious assignment of messages to code words. This idea was based on modeling an n-digit binary code as an n-dimensional cube. Each axis has a 0 or 1 value so that each code word is represented by a corner of the cube. Fig. 1 illustrates the 2^3 code words associated with the 3-dimensional cube. Any single error in communicating an intended message causes reception of a code word at distance 1 from the intended one, i.e., reception of one of the n "neighbors" of the intended message.

Kautz recognized that the best assignment of messages to code words would be dictated by different criteria in different situations. An occasional numerical error might be permissible if it were "small enough," e.g., it is better to receive "15" instead of "27" when actually "19" was the intended message. In other cases it is preferable to make an erroneously received message differ from the intended message as greatly as possible. For example, receiving the string "bxt" is preferable to "bit" when the letter "e" is miscommunicated. The incongruous context indicates an occurrence which might otherwise remain undetected.

Quantification of this technique is straightforward when messages are numbers. When a_i is received instead of a_j, the "confusion index" is defined as $\Delta_{ij} = |a_i - a_j|$. The analysis of how to assign integers to code words for the minimization of functions of Δ_{ij} for applications to space communications was initially proposed by Golomb [23] in 1963 and undertaken by Harper in 1964. Harper [8], [16], [24], [25] solved problems of assigning 2^n numerical messages to 2^n code words for several different criterion functions. He obtained algorithms and analytical results for the following optimization problems on the n-dimensional cube:

1) minimize $\Sigma_{i \neq j} \Delta_{ij}$;
2) maximize $\Sigma_{i \neq j} \Delta_{ij}$;
3) minimize (max Δ_{ij});
4) maximize (min Δ_{ij}).

He also conjectured a solution to;

5) minimize $\Sigma_{i \neq j} \Delta_{ij}^2$.

This solution was independently derived by Crimmins *et al.* [10].

These functions are easily interpreted in the communication context: $\Sigma \Delta_{ij}$ is proportional to the size of the average error which occurs; max Δ_{ij} is the size of the error range; and $\Sigma \Delta_{ij}^2$ indicates the variance in error sizes.

Lindsey [9] generalized some of Harper's work by showing how to optimize $\Sigma \Delta_{ij}$ when the encoding is not binary. Crimmins *et al.* showed that the assignment that minimized the average error size also minimized its variance. Stieglitz and Bernstein [26] indicated that encodings which minimize the mean are not restricted to sequential integers. Bernstein *et al.* [27] considered the minimization of the mean magnitude error for *all* error patterns. They showed that Harper codes are optimal when the channel transmitting the binary encodings has sufficiently small probability of error. For most practical situations this probability bound holds. Numberings minimizing this criterion function on other graphs have also been studied [28]-[30].

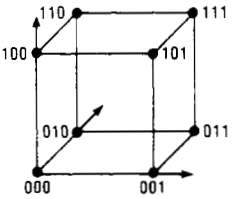

Fig. 1. A 3-digit binary code represented by a 3-dimensional cube.

Properties of numberings which minimize max Δ_{ij} for graphs in general have been investigated as "bandwidth" problems [31]-[36] and summarized by Chvátalová *et al.* [37] and Bloom [12]. Algorithms assigning messages to n-cubes also are applicable for encoding nonnumeric messages. In such cases numeric confusion indices Δ_{ij} must initially be assigned to each (a_i, a_j) pair. When the array (Δ_{ij}) is nonnegative and symmetric, the algorithms cited above apply directly. When (Δ_{ij}) is not symmetric, modification of Harper's results to directed graphs must be made. These extensions have not yet been studied.

B. Binary Codes with Minimum Valued Out-of-Synch Autocorrelation Functions

None of the following three applications was originally formulated in a graph-theoretical context. Nevertheless, they can be viewed as realizations of one incompletely solved problem on a class of numbered graphs.

1) The Graph Model: These coding applications are derived from optimal numberings of complete graphs. The *complete graph on m vertices* K_m consists of m points and $(m^2 - m)/2$ lines joining all pairs of points. Applications of this model require that each edge number be distinct. Such a numbering of a set of edges is *optimal* if it minimizes the largest edge value, which we designate by $G(K_m)$.

An optimally labelled graph with n edges is called *graceful* if its vertices can be numbered with distinct nonnegative integers no larger than n in a manner that assigns to each edge exactly one of the integers from 1 to n. The existence of such numberings (also called *β-valuations*) and their properties have been investigated by Golomb [38], Rosa [39], Kotzig [40], and others and are surveyed in Bloom [12]. Fig. 2 shows a graceful numbering of K_4. However, no complete graph with more than 4 points can be gracefully numbered. Hence, general optimizing numberings for K_m are unknown.

Golomb observed an important equivalence for the coding theory context between a "semi-graceful" numbering which minimizes $G(K_m)$ and a special ruler on which m division marks (including the ends) are placed. The positions of the division marks corresponds to the numbers placed on the m modes of K_m. The edge numbers of K_m thus exactly correspond to the set of measurements which can be made between the marks on what Gardner [41] calls a Golomb Ruler. The

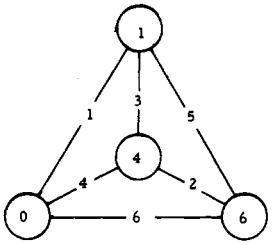

Fig. 2. A graceful numbering of K_4.

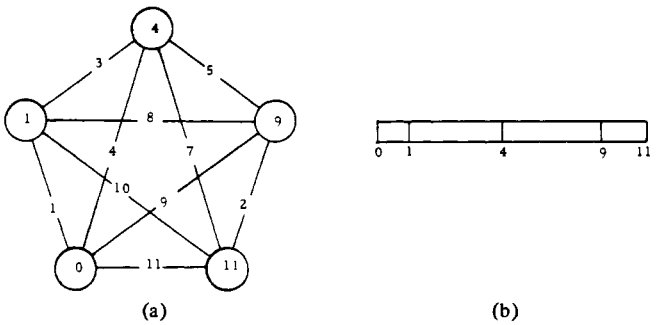

Fig. 3. (a) A semigraceful numbering of K_5. (b) The ruler corresponding to (a).

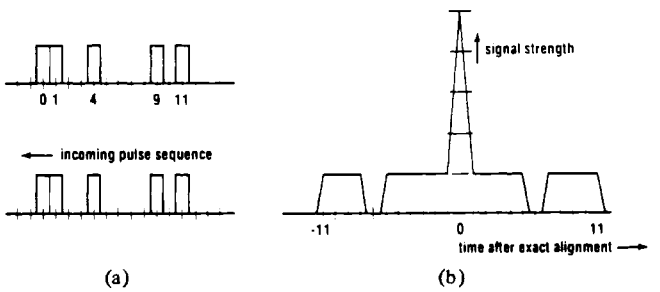

Fig. 4. The correlation of a radar code pulse train with an image of itself. (a) The moment that the returning ruler-spaced pulse train aligns with its template. (b) The autocorrelation function for the returning pulse train.

shortest possible ruler of this type has length $G(K_m)$. Fig. 3 shows a "semigraceful" numbering of K_5 and the corresponding ruler. In the numbering of K_5 no edge is numbered 6 and with the equivalent ruler no measurement of length 6 may be directly made. All optimal rulers have been found for $m \leqslant 11$ marks, [38], [41]-[43], and are summarized in Bloom and Golomb [22]. These rulers have several applications to coding theory. The measurements that they make in the coding context are in time, however, rather than distance.

2) Radar Type Codes: Let us first consider a simple example using the five-mark ruler of Fig. 3. One can generate a radar code from this ruler by transmitting a sequence of five pulses at times corresponding to the marks on the ruler, i.e., 0, 1, 4, 9, and 11. That is, there is a one-unit time interval between the onset of the first and second pulses, a three-unit time interval between the second and third, five units between the third and fourth, and two units between the last two. The time duration between the emission of the signal and its return is determined by correlating all incoming sequences of 11 time units duration with the original sequence. Let each pulse be of one unit duration. Thus, when an incoming string matches the original [see Fig. 4(a)] a signal of strength 5 is generated. For any other line-up of the incoming sequence with the original template there can be at most one incoming pulse. In the absence of noise, then, the unnormalized out-of-synch autocorrelation can attain a maximum magnitude of 1. This is shown in Fig. 4(b).

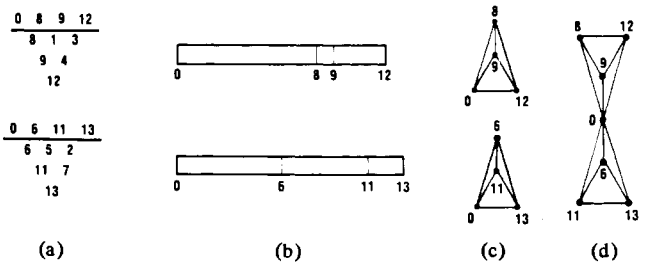

Fig. 5. Representations of a pair of difference triangles used to generate a 2-message, 4-pulse missile code of minimum length. (a) Difference triangles. (b) Rulers. (c) Disconnected graph with 2 components. (d) Connected graph.

A dip in the autocorrelation occurs at ± 6 time units, since there are no pulses which are aligned with a six-unit shift of the pulse sequence out of its synch position. Six, of course, is the only distance of 11 or fewer units that the original ruler could not measure and is the only edge number missing in numbering K_5 (Fig. 3).

Eckler [5] investigated the related coding problem of designing missile guidance codes. In an airborne missile, a receiver passes all incoming signal trains down a delay line. If the line is tapped in several places which correspond to the actual time interval between incoming pulses, then the sum of those pulses will exceed a threshold and initiate some control action.

The command code for such a missile contains two or more different commands. Thus, in terms of instrumentation, the delay line must be tapped by sets of leads corresponding to the delays between pulses for each command. In order to make the code insensitive to random interference pulses (such as electrical storms or jamming efforts), all of the delays between pulses for one command must totally differ from those for every other command. It is also desirable to use the shortest code-word durations possible in order to minimize the delay line and to decrease the time during which interference could occur. Thus Eckler calculated d-1 intervals for the d pulses associated with each of n different commands. In synch, these commands give on reception by the missile, an autocorrelation of height d. Out-of-synch, the maximum autocorrelation is 1, and the noiseless cross-correlation between commands also never exceeds 1.

This problem corresponds to finding a set of n rulers of different lengths each with d-1 marks on it. The marks on these rulers permit measuring each length in only one way. Moreover, the longest of these rulers must be as short as possible. Alternately, the problem corresponds to numbering as gracefully as possible a *disconnected* graph with n components. Each component is a complete graph on d-1 vertices. For this numbering each component of the composite graph has a vertex labeled zero. Still another representation is the connected graph in which the nK_{d-1} components are joined at one point. In Fig. 5 we show these representations for constructing an interference-insensitive, 2-message, 4-pulse missile code with minimum duration sequences.

Although Eckler did not *solve* the problem of optimally designing the codes he specified, he did generate the intervals for many feasible codes of this type. In 1967, some of his results were recalculated and some improved by Robinson and Bernstein in the far different context of convolutional coding. We discuss that problem in the next section.

3) Self-Orthogonal Codes, A Class of Convolutional Codes: In this section we examine the intimate relationship between Golomb Rulers and self-orthogonal codes after a brief discussion of this type of "algebraic coding."

An encoder for convolutional or recurrent codes is unlike an encoder for block codes. For block codes, the encoder partitions a source message into long blocks and then encodes and transmits the entire block across a noisy channel. For convolutional codes, the source messages are partitioned into short blocks whose encoding depends not only on the messages in that block but also upon some fixed number N of previous blocks as well.

Convolutional codes are naturally partitioned into classes by the decoding method they use. In *threshold decoding* a set of "syndrome" digits is calculated from each received code block and its N predecessors to determine errors in each received message. If a majority (the "threshold") of syndrome digits "vote" for an error in a received digit, the received digit is complemented before the block containing the error is decoded. This method is called *direct* decoding. *Feedback decoding* extends the threshold voting method to correction of the syndrome digits themselves. Although syndrome updating improves a decoder's probability of correctly decoding if no decoding errors have previously occurred, it degrades performance when a decoding error does occur; for such an error in a single digit may also cause digits in subsequent blocks to be incorrectly decoded. Such "error propagation" effects cannot occur with direct decoding.

In this paper we are primarily interested in one important aspect of the design of "self-orthogonal" direct codes. More general and more detailed discussions of convolutional codes is beyond the scope of this paper and may be found in the texts by Peterson and Weldon [44], Lin [45], and Berlekamp [46].

In 1967 Robinson and Bernstein [6] devised a systematic method for constructing self-orthogonal codes which depends upon the construction of "difference triangles." Such a triangle is formed from the $\binom{d-1}{2}$ positive differences of an ordered set of d-1 integers $[0, a_1, a_2, \cdots, a_{d-2}]$, where $0 < a_1 < \cdots < a_{d-2}$. An $(n, n\text{-}1)$ code with n bits per block and n-1 information bits can be constructed from n-1 difference triangles in which there are no elements in common. When each of these triangles has $\binom{d-1}{2}$ elements, the minimum distance between code words is d.

The single difference triangles that are used to form codes with $(n, n\text{-}1) = (2, 1)$ can be shown to be equivalent to numbering the vertices of K_{d-1}, the complete graph on d-1 vertices, so that all numbers induced on its edges are distinct. Among the set of graph numberings meeting this requirement, some are much better than others for designing codes.

The transmission rate of (2, 1) self-orthogonal codes is optimized by making the maximum vertex number as small as possible. The design requirements for this code have been previously expressed. Here again, it is necessary to number K_n semigracefully, i.e., to find the shortest ruler for which all $\binom{n}{2}$ measurable distances are distinct.

Robinson and Bernstein found several optimal numberings directly, and, as we noted earlier (Section II-B1), to date all Golomb Rulers for $n \leqslant 11$ marks have been found.

This ruler or graph model also has been applied to the problem of finding the fastest $(n, n\text{-}1)$ codes for $n > 2$. This problem corresponds to finding a set of n-1 rulers of different lengths each with d-1 marks on it. The marks on these rulers must be placed to prevent *any* duplication of measurements among any of the rulers in the set. Moreover, the longest of these rulers must be as short as possible.

This formulation is identical with Eckler's missile-code problem. In this new context Fig. 5 shows representations for constructing a (3, 2) self-orthogonal code with d-1 = 4.

In most cases Robinson and Bernstein's calculations were modestly better than Eckler's. Nevertheless, the optimization of the lengths of these ruler sets remains a largely unstudied open question.

4) Synch Set Codes: In this section we find that most elementary "Synch Set Codes" are another embodiment of Golomb Rulers. The other codes in this class relax the constraint that allows no repetition of measurements made by these rulers.

In 1974 Simmons [15] designed these codes to synchronize the relative angular position of a photodetector on one side of a rotating disk with a stationary, target light source on the other side. If "noise" were no problem in this system, synchronization could be provided by lining up a narrow single slit in an otherwise opaque annular region on the disk with an identically sized, stationary slit in front of the detector. Since the amount of light passed through a narrow slit would produce an insufficiently small electrical signal, "Slit codes" were developed by which identical sets of perforations of the rotation disk and the mask for the detector significantly increased the in-sync signal and minimized passage of out-of-synch light.

Simmons used the average value of the in-synch signal over the maximum out-of-synch noise as his performance criterion for code synchronization. Thus, sync set codes should be designed with many holes ("ones") so that the signal is maximized. However, to simultaneously minimize the number of holes in alignment at out of synch positions, there should be as little repetition as possible among the set of distances separating holes (where each unit of opaque disk is a "zero").

An *$S(k, d)$-synch set* is defined as a set of k distinct, nonnegative integers for which not more than λ pairs have common differences, and for which the maximum element in the set is as small as possible. In context, k is the total number of holes present, and λ is the maximum number of holes that can simultaneously align in an out-of-synch position. A synch set designates positions for the k holes so that the distance from first to last hole is minimized.

When $\lambda = 1$, this another rephrasing of the Golomb Ruler problem. Simmons also discovered many sets for $\lambda > 1$. These sets could be represented either as minimum length rulers that allow measurements to be repeated λ times, or as distinct numberings of K_k which minimize $G(K_k)$, the largest vertex number, and allow up to λ repeats of edge numbers.

The most important unsolved problem in this topic, according to Simmons, is a constructive algorithm that would produce optimal sets given k and λ. Perhaps an equally important goal would be an algorithm to produce optimal sets simply given the overall length of the "code word" allowed. This procedure is equivalent to determining first all synch sets with "small enough" largest element and then evaluating the maximum signal-to-noise (S/N) criterion function to determine the best of these. In this procedure k and λ would be dependent variables, which seem to be not easily predicted by Simmon's data. Another option available to designers of similar codes is the choice of a different criterion function. One might rather choose to maximize: 1) the (signal/largest noise) without averaging the signal strength over the code-word length; or 2) the average of (signal/total noise).

III. Other Applications

In this section we consider an assortment of applications of graph numberings. We believe that these models have great potential to give impetus to a wide variety of research efforts.

A. Ambiguities in X-Ray Crystallography

It sometimes happens that distinct crystal structures will produce identical X-ray diffraction patterns. These inherent ambiguities in the X-ray analysis of crystal structures have been studied by Patterson [47], Garrido [48], and Franklin [11].

As Patterson explained:

> It is well known that the methods of crystal analysis introduce a center of symmetry and that two structures which can be related to one another by a center of symmetry give the same diffraction effects. Any two structures which are congruent in the sense that they can be brought into coincidence with one another by a translation, a rotation, a reflection, or some combination of these operations give the same X-ray pattern. . . . (A) structure analysis is unique if all the structures which can be derived from it are congruent to one another.

Research into these ambiguities is concerned with determination of arrangements of set of points from a knowledge of the vector distances for these points. Extensive results for the case of an infinity of points arranged periodically have been achieved by Patterson and Garrido. The case for ambiguities due to finite arrangements of points has been less thoroughly studied, with results by Piccard [49], Franklin [11], Bloom [50], and Grünbaum [13].

In the infinite case, if two sets of atom positions, say R and S, in some lattice structure A give the same diffraction pattern and are not related by a simple translation or reflection, then R and S are *homometric modulo A*. For instance, a one-dimensional example of such sets was given by Patterson [11, p. 197]

$$R = \{0, 1, 4, 7\}, \quad S = \{0, 4, 5, 7\}, \quad A = 8.$$

To show that these were homometric mod 8, Patterson first showed that R and S were not trivially related, i.e., that $S \neq \pm R + c$ (mod 8) for any c, and more importantly, that the sets of element differences for R and S are congruent. That is

$$D(S) \equiv D(R) \bmod A.$$

Thus, for the example, the sets of all 16 differences are congruent mod 8

$$D(R) \equiv D(S) \equiv \{0, 0, 0, 0, 1, 1, 2, 3, 3, 4, 4, 5, 5, 6, 7, 7\}.$$

Numbered graphs provide a natural representation of these sets. In Fig. 6, for example, the elements of R number the vertices of two K_4 diagrams. In Fig. 6(a), denoted as $D^+(R)$, the positive differences or the usual "absolute value" differences are the edge numbers. In diagram 6(b), the negative differences mod 8 are shown. (Alternatively, the information in Figs. 6(a) and (b) could be presented on a single labeled directed complete multigraph.) Figs. 6(c) and (d) present corresponding representations for $D^+(S)$ and $D^-(S)$. Physically, determining sets of atom positions that are homometric modulo the lattice structure, is equivalent to determining of vertex number sets that result in a prespecified set of edge numbers.

Franklin studied finite sets of points that would give the same diffraction pattern. These he called *strictly homometric*

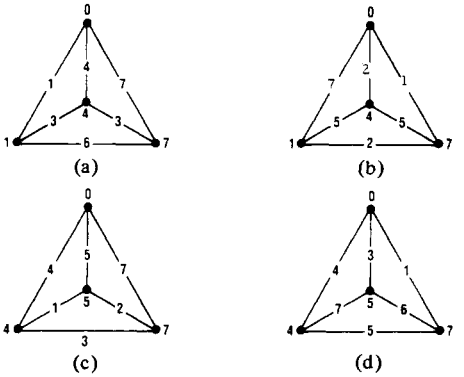

Fig. 6. Two sets of vertex numbers homometric mod 8. The edge numberings show (a) $D^+(R)$, (b) $D^-(R)$, (c) $D^+(S)$, (d) $D^-(S)$.

(or more simply *homometric*) and found that they constitute a subset of infinite point sets that are homometric mod λA, where A is still the lattice matrix and λ is arbitrarily close to 1. In the mathematical statement of this problem, set congruence is replaced by set equality, so that the conditions for sets R and S to be homometric are

$$S \neq \pm R + \underline{c} \text{ for any } \underline{c}, \text{ and } D(S) = D(R).$$

The previous example of Patterson's is not homometric.

Franklin discovered a construction to produce families of homometric sets. Because of the simplicity of restating this problem in numbered graph parameters, we consider an example. Franklin gives

$$R = \{0, 1, 3, 4, 5, 7, 9, 10, 12\}$$
$$S = \{0, 2, 3, 4, 6, 7, 9, 11, 12\}.$$

The set of differences for these (using absolute values only) is

$$D(R) = D(S) = \{1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 6, 6, 6, 7, 7, 7, 8, 8, 9, 9, 9, 10, 11, 12\}.$$

Display of each of these two 9-element homometric sets graphically is done easily with K_9. The vertices are labeled with the elements of the set, and the edges with its difference set. Homometric sets can thus be defined as distinct sets of vertex numbers on complete graphs that generate identical edge number sets. Moreover, every numbered complete graph corresponds to a ruler model [42]. Setting the n marks on a ruler at the distances corresponding to the vertex numbers on the graph generates an isomorphism between edge numbers on the edges of the complete graph and the set of measurements made by the ruler. It is often convenient to consider sets like R and S as rulers with division marks at their element values.

A generalization of these numberings to two, three, or more dimensional graphs is straightforward. Each vertex of the complete graph is labeled with one of the n-dimensional vectors that comprise one set of the homometric pair. The edge vectors are the component-wise differences of the vectors assigned to the vertices defining the edge. A convention to eliminate parity ambiguities is always to take the first nonzero component positive.

An interesting development in the research on homometric sets has recently occurred. Piccard in 1939 presented a theorem which crystallographers immediately accepted for its narrowing of the necessary scope of their investigations into diffraction pattern ambiguities. It was believed that Piccard had proved that if all elements in a difference set are distinct, there is a unique set (except for Euclidean motions) that would gen-

erate it. That is, no pair of homometric sets was believed to exist with a difference set comprised of distinct elements. Franklin's constructions of homometric sets do not violate this condition since his difference sets always include repeated elements.

Recently [50], we determined that this theorem was in error by discovering the following pair of nonredundant homometric sets:

$$R = \{0, 1, 4, 10, 12, 17\} \text{ and } S = \{0, 1, 8, 11, 13, 17\}.$$

The mutual difference set is

$$D(R) = D(S) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17\}.$$

These sets were found by looking among the known Golomb Rulers, which are the minimum length rulers which measure all distinct distances. (To date no other homometric Golomb Ruler pairs have been found.)

This discovery prompted many questions as to whether other such counterexamples exist, whether this is a minimum counterexample, and whether such counterexamples relate to other "special" numberings of the complete graph. It should seem that removal of the nonredundancy constraint in conjunction with the discovery of Franklin's sets reopens the opportunity to understand diffraction pattern ambiguities more completely.

We have generalized the original counterexample [12] to a two-parameter family of nonredundant homometric rulers for which

$$R = \{0, u, u + v, 4u + 2v, 6u + 2v, 8u + 3v\}$$

and

$$S = \{0, u, 5u + v, 5u + 2v, 7u + 2v, 8u + 3v\}.$$

The original counterexample is obtained by setting $u = 1, v = 3$. The set of differences for R is formally identical to the set of differences of S, as expressions in u and v. Not all choices of u and v result in sets of *distinct* differences, but as a concrete example, if v is kept fixed at 3, each positive integer value of u generates a *different* counterexample to "Piccard's Theorem." That is, the differences are all distinct in each set, and for different values of u, the counterexamples differ by more than the Euclidean operations of translation, rotation, dilation and/or contraction. One may allow u and v to be any *complex* numbers (or indeed, elements of any module), and the formal identity between the set of differences of R and the set of differences of S still holds.

A second two-parameter family of nonredundant homometric sets was also found for which

$$R' = \{0, y, y + w, 4y + 2w, 6y + 4w, 8y + 5w\}$$

and

$$S' = \{0, y, 3y + w, 3y + 2w, 7y + 4w, 8y + 5w\}.$$

Again, there is a formal identity between the sets of differences of R' and S'. This set of counterexamples is disjoint from the previous set, as the vector subspaces involved intersect only in the origin. The smallest nonredundant homometric sets of distinct integers from the pair R', S', occurs with $y = 1, w = 2$, giving

$$R' = \{0, 1, 3, 8, 14, 18\} \text{ and } S' = \{0, 1, 5, 7, 15, 18\}.$$

Extensive computer investigation indicates that all counterexamples to "Piccard's Theorem" with six-element sets belong to these two families. We have proved that there are no counterexamples involving sets of fewer than six elements. Moreover, preliminary computer searches have failed to discover any counterexamples involving sets of more than six elements. To date we have inspected all of the approximately 14 000 nonredundant rulers with six elements and lengths between 17 and 29, with seven elements and lengths between 25 and 33, with eight elements and lengths between 34 and 42, with nine elements and lengths between 44 and 53 with ten elements and lengths between 55 and 65.

B. Communication Network Labeling

In a small communication network, it may be desirable to assign each user terminal a "node number," subject to the constraint that all the resulting edges (communication links) receive *distinct* numbers. In this way, the numbers of any two communicating terminals automatically specify (by simple subtraction) the link number of the connecting path; and conversely, the path number uniquely corresponds to the pair of user terminals which it interconnects.

Properties of a potential numbering system for such networks have been explored under the guise of gracefully numbered graphs. That is, the properties of graceful graphs provide design parameters for an appropriate communication network. For example, the maximum number of links in a network with m transmission centers can be shown to be asymptotically limited to not more than $\frac{2}{3}$ of all possible links when m is large [51].

If a graphical model of a particular communications network *cannot* be gracefully numbered, there is, nevertheless, the possibility of using a "semigraceful" numbering in which the constraint requiring the edge numbers to be consecutive integers is relaxed. A "quasi-graceful" numbering which further allows the largest vertex number to exceed the largest edge number might also be used. Quasi-graceful numberings are briefly discussed in the next section, and have been studied by Leech and by Miller as "unrestricted difference sets."

The most important question for utilizing a "graceful addressing and identification system" involve being better able to determine whether an arbitrary model of a communications network is in a graceful configuration. If it is, how should it be numbered? If it isn't, can it be embedded into a graceful structure easily? Or, should it be semigracefully numbered? Moreover, determination needs to be made of growth provisions for any addressing scheme, i.e., of algorithms for numbering an augmented graph in which new nodes and arcs have been added to a gracefully numbered graph.

C. Finite Additive Number Theory and Ruler Problems

In the following paragraphs we restrict our discussion to sketching the relevance of ruler models and complete graph models to several classes of problems posed on finite sets of integers. "Difference-basis" sets form one large subclass of these. The reader who is more interested in the current state of difference-basis problems than is covered here should refer to the survey of results by Miller [21].

Historically, these problems were introduced by an applied problem. In 1945 Brauer [17] studied the problem of placing a minimal number of fixed position contacts on a 30-Ω resistor in a manner that allowed tapping of all integral resistance values 1 Ω through 30 Ω by choosing appropriate pairs of contacts. The general problem of representing the integers $1, 2, \cdots, N$ by taking all differences among the elements of a

minimal *difference-basis set*

$$0 = a_0 < a_1 < \cdots < a_m$$

has been subsequently studied by a series of researchers [18], [21], [52]-[56].

For our purposes we can view a difference basis as a set of mark-positions on a ruler. Thus the set of all differences calculated from a basis is isomorphic to the set of all distances measured by the corresponding ruler.

Moreoever, assigning the positions of the m marks on a ruler, i.e., the difference basis elements, to the vertices of the complete graph K_m causes the $\binom{m}{2}$ edges of K_m to be numbered with the set of $\binom{m}{2}$ measurements made by the (difference-basis) ruler. Consequently, corresponding difference bases, rulers, and complete graphs are all isomorphic and can ordinarily be used interchangeably.

Previously in this paper, we utilized the interchangeable nature of complete graph numberings and ruler models. This was occasioned by discussions of finite number-theoretic problems distinct from those of difference bases. These classes were represented by Golomb Rulers, Sync-Set rulers, and the rulers modeling the sets of Eckler, and of Robinson and Bernstein. Difference-basis sets differ from all of these by requiring their sets of element differences to contain all consecutive integers from 1 up to some prescribed value. These other sets are not in general completely consecutive, but are more usually typified by their nonredundancy, since none (except synch sets) may have any repetition of elements in their sets of element differences. A more extensive comparison of these classes is in Bloom and Golomb [22].

An interesting application of difference basis sets can be made to numbering graphs with m points but some of the $\binom{m}{2}$ possible edges omitted. This procedure can be used to determine the largest graceful graph on m points, that is, the largest graceful subgraph of K_m. Erdös [57], Golomb [58], and Simmons (in [59]) produced successively better results, but Leech (see [60]) and van Rongen [61] recognized that this problem corresponds to one for difference bases. Marks are placed on the ruler at positions of an m-element *restricted difference bases* whose differences include all positive integers up to $n = n(m)$, the length of the ruler. By removing edges whose numbers duplicate those of other edges, one generates graceful numberings for n-edged subsets of K_m. The $n(m)$ edges remaining in the graph are labelled consecutively from 1 to $n(m)$ without duplication.

A related numbering of K_m is produced by labeling them with *unrestricted difference bases* (e.g., [20], [21]). If a ruler is labeled with m elements of an unrestricted difference basis, it measures all consecutive integral distances to a length $N = N(m)$. The overall length of this ruler $L = L(m)$ is not constrained to equal N, so that the parameters N, L, and n (for *restricted* difference bases) are related for any m by

$$n(m) \leqslant N(m) \leqslant L(m).$$

A numbered subgraph of K_m is generated by labeling the vertices of K_m with an unrestricted difference basis and removing all edges with duplicated numbers and all edges numbered beyond the continuous counting sequence. A graph so numbered has $N(m)$ edges numbered consecutively from 1 to $N(m)$. However, it is *quasi-graceful* but not graceful, since the largest vertex number does not equal the largest edge number. Fig. 7 shows a maximal graceful and a maximal quasi-graceful graph

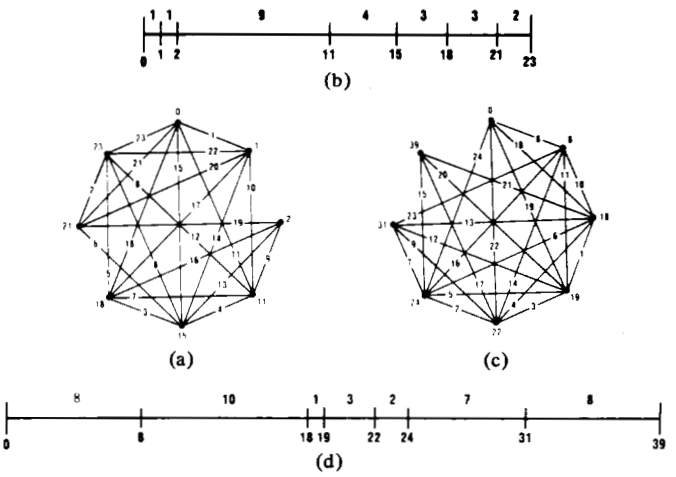

Fig. 7. A comparison of maximal graceful and quasigraceful graphs on 8 vertices. (a) A maximal graceful graph on 8 vertices. 5 chords not used: 0-2, 1-2, 2-23, 11-21, 18-21. (b) The "ruler" generating (a). (c) A maximal quasigraceful graph on 8 vertices. 4 chords not used: 0-31, 0-39, 8-39, 31-39. (d) The "ruler" generating (b).

on 8 vertices and presents their respective defining difference bases as rulers.

The sets most studied in finite additive number theory are probably *cyclic difference sets* [62]-[64]. An m-element basis for such a set may be used to mark a ruler of length $L = L(m)$ which will measure *all* $(m^2 - m)$ nonzero differences on the ruler mod $L = m^2 - m + 1$. The usage of this ruler is somewhat peculiar. Any distance less than the length of the ruler may be measured, including those distances which are broken into two pieces, one measured by a right-end length of the ruler and the other measured by a left-end length. Miller [21] discusses the ruler embodiment of both cyclic-difference sets (also called *perfect-difference cycles*) and a generalization to *general-difference cycles*, as well as the status of both restricted- and unrestricted-difference sets.

In general, relatively little has been done with connecting number theoretical problems to numbered graphs and rulers or in using them as models. We have suggested that graph models and rulers may prove useful. Moreoever, not only should the modeling potential of single rulers be explored, but also one should also examine the utilization of families of rulers. The natural occurrence of the ruler sets in two different coding settings discussed in Section I-B indicates the potential utility of such models.

D. Circuit Layout

Electronic components can be placed on a chassis to minimize either the total amount of wiring between components or the longest wiring correction. Harper [16] formulated this design optimization problem in graph numbering terms and was able to solve some cases using his technique for minimum-confusion code design.

The initial problem he studied was the placement of 16 interconnected components on a linear chassis. The appropriateness of the coding model was striking. The placement of a component determined the number assigned to it: 1 at one end, 16 at the other, with the natural progression between. Since components were equally spaced, the difference of their position numbers indicated the number of units of wire needed to connect them; thus, connecting components at 5 and 11 would require a wire six units long. The problem became the following: Given the graph showing connections be-

tween components, number the components (with position numbers) so that the sum of the differences of integers assigned to connected components is minimized. If the connection

$$\sum_{i<j} \Delta_{ij} = \sum_{\substack{i<j \\ \text{all} \\ \text{connections}}} |j - i|$$

graph had been isomorphic to an n-cube, the results would have been identical to the coding problem discussed in Section I-A. The connection graph here was different enough so that Harper's recursive algorithm to number vertices failed to generate the optimal "nested sets" of solutions that occur for the n-cube. Nevertheless, his algorithm was adequate as a heuristic method and, for this problem, led to the optimal solution.

This physical embodiment of Harper's graph numbering model led him to various generalizations. A layout of the components on a two-dimensional chassis was attempted with a modified heuristic algorithm which gave a result that was close to a provable lower bound. Since it was also considerably better than other reasonable, "intuitively good" numberings, it was considered to be "near optimal." Moreover, the incremental gain in finding the optimal solution would be negligible in relation to the additional effort to do so.

Minimization of the longest wire on the linear chassis received similar analysis and also resulting in an easily implemented, stepwise, suboptimal, heuristic algorithm.

These problems were restated by Harper as integer programming problems. The first of these is a positive semidefinite integer quadratic problem and the second a more general nonlinear integer problem which had no known solutions in 1970 when Harper made this investigation.

IV. Concluding Remarks

In this section we observe the unifying nature of the numbered graph model and reiterate some of the open questions.

A. Graph Numberings as Unifying Models

Graph numberings present a common context for many applied and theoretical problems. This has been illustrated in the previous sections, in which brief sketches of diverse applications are held together by this common thread.

Heretofore there has been little effort to organize and synthesize the scattered results in this area, nor to delineate systematically what is known and unknown. The reward of such efforts is obvious and often immediate. This principle is apparent in the following example which is drawn from this survey.

It was proved that K_n could not be gracefully numbered when $n > 4$. Consideration of the question: "How well *can* we number large graphs on n vertices?" led in two theoretical directions. Each of these has practical applications. The direction taken by numbering K_n "as well as possible" led to relaxing the constraint on the largest allowable vertex number. From these were derived semi-graceful numberings on K_n, which in turn led to the original counterexample of Piccard's Theorem. Thus was discovered a new facet of the nature of possible diffraction pattern ambiguities in crystal structures.

The second direction taken in numbering graphs on n vertices maintained the requirements of graceful numberings. Instead, it was determined that approximately $\frac{1}{3}$ of the edges of K_n needed to be eliminated for the remaining graph to be graceful. Knowledge that $\frac{2}{3}\binom{n}{2}$ is the limit for the number of edges in a graceful graph, in turn, gives design limits for communication networks of this type.

B. Topics for Further Research

We have tried to show that potentially there is a wide variety of applications for numbered graph models, and that relatively few of these have been exploited to date. We will conclude by recapitulating a short list of some specific unresolved questions and potential research projects that we have touched upon. The following list is far from exhaustive, and is intended to be only representative of the many possible areas of fruitful additional research.

1) Extend the tables of optimum code patterns for use as pulse radar signals, synch sets, convolutional codes, etc. Here, a deeper insight into the problem should facilitate progress beyond the point where brute-force computing alone reaches its limit.

2) Explore the related "ruler problems" which have similar communications applications. This includes the problems of finding the shortest rulers with k marks which measure all integer lengths from 1 to n, either i) allowing the same length to be measured in more than one way, or ii) not allowing the same length to be measured in more than one way.

3) Develop a complete theory of the counterexamples to Piccard's Theorem. How, when, and why do they arise? Are there instances of three or more simultaneous homometric sets? What are the implications of these counterexamples, not only for crystallography, but in the application of the same mathematical structures to pulse radar signals, convolutional codes, etc.?

4) Pursue those aspects of the underlying "numbered graphs" which are fundamental mathematical significance. Examples are the problem of finding the number of edges in the largest graceful subgraph of the complete graph, and the "tree conjecture," which asserts that every tree has a graceful numbering.

5) Obtain further results on optimal nonstandard encodings of the integers, and determine the practical situations in which such codes have valid application.

6) Continue the process of finding applications areas which are appropriately modeled as graph-numbering problems. This process has already produced invaluable cross-fertilization, as the most advanced methods developed in one context can be readily applied to another situation arising in a different application.

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