Math 71 Homework 7

256/7 Define $e: R[x] \to R$ by $e(f(x)) = a_0$, where $f(x) = a_0 + \cdots + a_n x^n$. Show $\ker e = (x)$ and e an epi. $\therefore R[x]/(x) \approx R$. $\therefore$ (x) prime ideal $\Longleftrightarrow$ R an integral domain

256/9 Let g be a cont. function such that $g(\frac{1}{3}) = 0$ and $g(\frac{1}{2}) \neq 0$ and h a cont. function with $h(\frac{1}{3}) \neq 0$ and $h(\frac{1}{2}) = 0$
$gh \in I$ but $g \notin I$, $h \notin I$ and $a \neq 0, b \neq 0$

257/10 Let $a, b \in R$ suppose $ab = 0$ & $\therefore (a+P)(b+P) = P$
$\therefore a + P = P$ or $b + P = P$ Suppose $a \in P$.
$\therefore P$ contains a zero divisor. impossible
Similarly if $b \in P$.

258/19 P prime ideal $\therefore R/P$ finite integral domain $\therefore R/P$ a field
$\therefore P$ max. ideal

[redacted]

258/25 P prime ideal R/P integral domain For $x \in R/P$, $x \neq 0$ $(= P)$, $x^m - x = 0$ for some m. $x(x^{m-1} - 1) = 0$
$\therefore x^{m-1} = 1$ (since R/P integral domain) $\therefore x^{-1} = x^{m-2}$
$\therefore R/P$ field $\therefore P$ max. ideal

258/27 Suppose $a^n = 0$ $(1-ab)(1+ab+\cdots \pm a^{n-1}b^{n-1}) = 1$

278/3 $\min\{N(a) \mid a \in R^*\} = m$ Let $N(a) = m$ Apply the division alg. to 1 and a:

$$1 = qa + r, \quad r = 0 \text{ or } N(r) < N(a)$$

$\therefore r = 0$ $\therefore 1 = qa$ so a is a unit

301/6 a. (2) consists of polynomials with even coefficients

$\therefore (2) = (2\mathbb{Z})[x] \quad \therefore \mathbb{Z}[x]/(2\mathbb{Z})[x] \approx \mathbb{Z}/2\mathbb{Z}[x] = \mathbb{Z}_2[x]$

b. See 256/6

c. Define $e: \mathbb{Z}[x] \longrightarrow \mathbb{Z}\times\mathbb{Z}$ $\quad e(f(x)) = (a_0, a_1)$ where $f(x) = a_0 + a_1x + \cdots + a_nx^n$. Define a mult. in $\mathbb{Z}\times\mathbb{Z}$:

$$(a_0, a_1)(b_0, b_1) = (a_0b_0,\ a_0b_1 + a_1b_0)$$

This mult. gives $\mathbb{Z}\times\mathbb{Z}$ ring structure and e is epi.
Furthermore $\operatorname{Ker} e = (x^2)$

$\therefore \mathbb{Z}[x]/(x^2) \approx \mathbb{Z}\times\mathbb{Z}$ where mult. is defined above

146/6 $\langle(123)\rangle$, $\langle(124)\rangle$, $\langle(134)\rangle$, $\langle(234)\rangle$

146/13 $N_2 = 1$ or 7 $\quad |S_2| = 8$

$N_7 = 1$ or 8 $\quad |S_7| = 7$, $S_7 \approx \mathbb{Z}_7$. Suppose that there are 8 Sylow 7-groups. The intersection of these is $\{1\}$ (why?) and so these subgroups account for $8\cdot 6 + 1 = 49$ elements (counting the identity). If $N_2 > 1$, then there are two Sylow 2-groups which intersect (at best) in a group of order 4. This accounts for ≥ 12 elements (counting the identity).
Impossible $\therefore$ Either $N_2 = 1$ or $N_7 = 1$.

146/15 $351 = 3^3\cdot 13$ $\quad N_3 = 1$ or 13 $\quad N_{13} = 1$ or 27. Can't have $N_3 = 13$ and $N_{13} = 27$ (reason: it takes up too many elements).