WRITTEN HW #1 SOLUTIONS

(1) (10pts) Use induction to prove that

$$\sum_{k=1}^{n}(2k-1) = n^2.$$

(It is possible to prove the above formula without using induction, but for the purposes of this exercise use induction.)

Solution. This is clearly true for $n = 1, 2$. Assuming it is true for n, we will show it is also true for $n + 1$.

$$\sum_{k=1}^{n+1}(2k-1) = \sum_{k=1}^{n}(2k-1) + 2(n+1) - 1 = n^2 + 2n + 1 = (n+1)^2,$$

where we use that $\sum_{k=1}^{n}(2k-1) = n^2$. $\square$

(2) (10pts) The first two terms of the Fibonacci sequence are $F_1 = F_2 = 1$, and all succeeding terms are defined by the recurrence relation

$$F_{n+2} = F_{n+1} + F_n, n \geq 1.$$

A natural question to ask is whether there is an explicit formula for the general term of the Fibonacci sequence.

Consider the polynomial $x^2 - x - 1$. One easily checks that this has roots

$$\rho = \frac{1+\sqrt{5}}{2}, 1 - \rho = \frac{1-\sqrt{5}}{2}.$$

Show that

$$F_n = \frac{\rho^n - (1-\rho)^n}{\sqrt{5}}.$$

(Incidentally, note that $\rho \approx 1.618$ is the golden ratio, a number which appears frequently in classical architechture.)

Solution. We want to show that $F_n = \frac{\rho^n-(1-\rho)^n}{\sqrt{5}}$, where $\rho = \frac{1+\sqrt{5}}{2}, (1-\rho) = \frac{1-\sqrt{5}}{2}$.

For our base case, we need to test two cases: $n = 1, 2$. We do get that $F_1 = 1, F_2 = 1$. Now assume F_{n-1}, F_n are given by the formula suggested; we will show the same holds for F_{n+1}.

$$\begin{aligned}
F_n + F_{n-1} &= \frac{\rho^n - (1-\rho)^n}{\sqrt{5}} + \frac{\rho^{n+1} - (1-\rho)^{n+1}}{\sqrt{5}} \\
&= \frac{\rho^n + \rho^{n-1} - (1-\rho)^n - (1-\rho)^{n-1}}{\sqrt{5}} = \frac{\rho^{n-1}(\rho+1) - (1-\rho)^{n-1}(2-\rho)}{\sqrt{5}} \\
&= \frac{\rho^{n+1} - (1-\rho)^{n+1}}{\sqrt{5}} = F_{n+1},
\end{aligned}$$

where we use the facts that $\rho^2 = \rho + 1$ and $(1-\rho)^2 = 1 - 2\rho + \rho^2 = 2 - \rho$ that follow from $\rho, 1-\rho$ being the roots of the polynomial $x^2 - x - 1$. $\square$

(3) (10pts) The first few rows of Pascal's triangle are

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
1 & & 3 & & 3 & & 1 \\
 & & & \vdots & & &
\end{array}$$

For reasons of notational convenience, we call the top row 'row 0', the next row 'row 1', etc. Each entry in a row is generated by summing the values of the two entries directly to the upper left and upper right of it. For example, the first 3 in the 3rd row is obtained by adding the two numbers immediately above it, which are 1 and 2. The first and last entries of each row are always 1.

Let n, k be integers satisfying $0 \leq k \leq n$. The binomial coefficient $\binom{n}{k}$, sometimes called n *choose* k, is defined by the equation

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

where $n! = n(n-1)(n-2)\dots(2)(1)$, called n *factorial*, is the product of the first n positive integers.

Show that the kth term of the nth row of Pascal's triangle is $\binom{n}{k}$. (Like the numbering of the rows themselves, we start the numbering of the elements in each row with 0, not 1.)

Solution. It is easy to check that the kth term of the nth row of Pascal's triangle is indeed $\binom{n}{k}$. Assume this is true for row n, we will show it is true for row $n+1$. For $k \neq 0, n$, the kth term of the $n+1$st row is equal to the sum of the $k-1$st and kth terms of the nth row. Hence, the kth term of the $n+1$st row is equal to

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\
&= \frac{n!}{(k-1)!(n-k)!}\left(\frac{1}{n-k+1} + \frac{1}{k}\right) = \frac{n!}{(k-1)!(n-k)!}\left(\frac{k+n-k+1}{k(n-k+1)}\right) \\
&= \frac{n!}{(k-1)!(n-k)!}\frac{n+1}{k(n-k+1)} = \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k},
\end{aligned}$$

the expected term. For the cases $k = 0, n$, it is easy to see that $\binom{n}{0} = \binom{n}{n} = 1$, the correct term of Pascal's triangle. □

(4) (5pts) Use the Euclidean algorithm to compute $\gcd(a, b)$ for the following pairs a, b. Make sure to show each Euclidean division you perform.
(a) $a = 255, b = 68$.
(b) $a = 349, b = 17$.
(c) $a = 196, b = 28$.

Solution. We perform the Euclidean algorithm to find the gcd:

(a) $255 = 3(68) + 51$
$68 = 1 \cdot 51 + 17$
$51 = 3 \cdot 17$
$\gcd(255, 68) = 17$.

(b) $349 = 20 \cdot 17 + 9$
$17 = 1 \cdot 9 + 8$
$9 = 1 \cdot 8 + 1$
$\gcd(349, 17) = 1$.

(c) $196 = 7 \cdot 28 + 0$
$\gcd(196, 28) = 28$. □

(5) (5pts) Use the Euclidean algorithm to find a pair of integer solutions (x, y) to

$$31x - 12y = 1.$$

Solution. To find a solution to $31x - 12y = 1$, we start with the Euclidean algorithm on 31 and 12:

$$\begin{aligned} 31 &= 2 \cdot 12 + 7 \\ 12 &= 1 \cdot 7 + 5 \\ 7 &= 1 \cdot 5 + 2 \\ 5 &= 2 \cdot 2 + 1 \end{aligned}$$

Now we start with $1 = 5 - 2 \cdot 2$ and build from there, using the steps from the Euclidean algorithm:

$$\begin{aligned} 1 &= 5 - 2 \cdot 2 \\ &= 5 - 2(7 - 1 \cdot 5) = -2 \cdot 7 + 3 \cdot 5 \\ &= -2 \cdot 7 + 3(12 - 1 \cdot 7) = -5 \cdot 7 + 3 \cdot 12 \\ &= -5(31 - 2 \cdot 12) + 3 \cdot 12 = -5 \cdot 31 + 13 \cdot 12. \end{aligned}$$

Hence $1 = 31 \cdot (-5) - 12 \cdot (-13)$ is a solution to our equation. $\square$

(6) (10pts) Show that the product of any k consecutive positive integers is always divisible by $k!$.

Solution. Let $n+1, n+2, \ldots, n+k$ be any k consecutive positive integers. Therefore $n \geq 0$. Notice that

$$\frac{(n+1)\ldots(n+k)}{k!} = \binom{n+k}{k},$$

and in question 3 of this assignment we shoed that the binomial coefficients are entries of Pascal's triangle. It is clear from the way Pascal's triangle is defined that every entry in it is an integer (being the sum of the two integers above it, which are integers – if you want to make this precise use induction), so the expression above is an integer. In particular, this means $k!$ divides $(n+1)\ldots(n+k)$. $\square$