Math 3: Fall 2008

Exam 1 Solutions

1. The limit

$$\lim_{x \to 2} \frac{|x-2|}{x-2}$$

is equal to

(a) ∞

(b) 0

(c) 1

(d) -1

(e) the limit does not exist.

Answer: (e) The left-side limit is -1 but the right-side limit is 1.

2. The equation $3^x - 3x^2 + 1 = 0$ has

(a) No real solutions.

(b) Only one solution, which is in the interval $[-1, 0]$.

(c) Only one solution, which is in the interval $[0, 1]$.

(d) Only one solution, which is in the interval $[1, 2]$.

(e) Two or more solutions.

Answer: (e) Notice that $f(x) = 3^x - 3x^2 + 1$ is continuous so we use the Intermediate Value Theorem. We have $f(-1) = -5/3 < 0$ and $f(0) = 2 > 0$ so there is a root in the interval $(-1, 0)$. We also have $f(1) = 1 > 0$ and $f(2) = -2 < 0$, so there is a root in the interval $(1, 2)$.

3. Let $f(x) = e^{x/3}$ and let $g(x) = \ln x - \ln 2$. The value of $(f \circ g)(16) = f(g(16))$ is

(a) 3

(b) 2

(c) e

(d) 14/3

(e) None of the above.

Answer: (b) $f(g(16)) = f(\ln 16 - \ln 2) = f(\ln(\frac{16}{2})) = f(\ln 8) = e^{(\ln 8)/3} = (e^{\ln 8})^{1/3} = 8^{1/3} = 2$.

4. Assume that f is differentiable at x_0. The tangent line at the point $(x_0, f(x_0))$ is

(a) The line that intersects the graph of f only at the point $(x_0, f(x_0))$.

(b) The line passing through $(x_0, f(x_0))$ whose slope is the limit as h approaches 0 of the slope of the line passing through $(x_0, f(x_0))$ and $(x_0+h, f(x_0+h))$.

(c) The line passing through $(x_0, f(x_0))$ and $(x_0, f'(x_0))$.

(d) The line with equation $y = f'(x)$.

(e) The line with slope $\frac{\sin x_0}{\cos x_0}$.

Answer: (b)

5. For what real values of a is the function $f(x) = a^x$ increasing?

(a) Only for $a = e$.

(b) For every $a > 0$.

(c) For every $a \neq 0$.

(d) For every $a > 1$.

(e) For every real value of a.

Answer: (d)

6. The domain of the function $f(x) = \sqrt{3 - \sqrt{x-2}}$ is

(a) $[2, 3]$

(b) $[0, 3]$

(c) $[2, 11]$

(d) $[4, 9]$

(e) None of the above.

Answer: (c) The domain of the function $g(x) = \sqrt{x-2}$ is $[2, \infty)$ because we need $x - 2 \geq 0$. To plug the output y of $\sqrt{x-2}$ into $\sqrt{3-y}$, we would need $y \leq 3$. Thus, given $x \geq 2$, we also need $\sqrt{x-2} \leq 3$, which means $x - 2 \leq 9$, or $x \leq 11$.

7. The function $f(x) = 3 - \sin(x^2)$ is

(a) even.

(b) odd.

(c) both even and odd.

(d) neither even nor odd.

Answer: (a) $f(-x) = 3 - \sin((-x)^2) = 3 - \sin(x^2) = f(x)$ for all x, so f is even. The only function which is both even and odd is the zero function, so f is not odd.

8. The limit

$$\lim_{x\to\infty} \frac{x+2}{\sqrt{4x^2+1}}$$

is

(a) 0

(b) 1/4

(c) 1/2

(d) 1

(e) does not exist

Answer: (c) Dividing top and bottom by x, we have

$$\frac{x+2}{\sqrt{4x^2+1}} = \frac{1+\frac{2}{x}}{\sqrt{4+\frac{1}{x^2}}}$$

for all $x \neq 0$. The limit of the numerator is $1+0=1$, and the limit of denominator is $\sqrt{4+0} = 2$.

9. The range of the function $f(x) = 3 + \cos(2x)$ is

(a) $[2, 4]$

(b) $[-1, 1]$

(c) $[-2, 2]$

(d) $(-\infty, \infty)$

(e) None of the above.

Answer: (a) The range of the function $g(x) = \cos x$ is $[-1, 1]$, hence the range of $h(x) = \cos(2x)$ is also $[-1, 1]$ (the graph of h simply contracts the graph of g by a factor of 2). Now f shifts everything up 3, so the range of f is $[2, 4]$.

10. Suppose that an object moves along the x-axis in such a way that its position at time t (in seconds) is $x = t^4 + t$ meters to the right of the origin. The average velocity of the particle over the interval $[1, 2]$ is

(a) 5 meters/second

(b) 16 meters/second

(c) 19 meters/second

(d) 33 meters/second

(e) None of the above.

Answer: (b) At time $t = 2$, we have $x = 2^4 + 2 = 16 + 2 = 18$. At time $t = 1$, we have $x = 1^4 + 1 = 2$. Thus, the average velocity of the particle over the interval $[1, 2]$ is

$$\frac{18 - 2}{2 - 1} = \frac{16}{1} = 16 \text{ meters/second}$$

NON-MULTIPLE CHOICE. PLEASE SHOW ALL YOUR WORK.

1. Let

$$f(x) = \begin{cases} 4x+1 & x \le 1 \\ 2x^2 + kx & x > 1 \end{cases}$$

(a) (5 pts) Find the value of the constant k such that f is continuous at $x = 1$.

Answer: Notice that $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 4x + 1 = 4 \cdot 1 + 1 = 5$ and that $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x^2 + kx = 2 \cdot 1^2 + k \cdot 1 = 2 + k$. For f to be continuous at $x = 1$, we need these two limits to be equal, so we need $5 = 2 + k$ and hence $k = 3$. We then have $\lim_{x\to 1} f(x) = 5 = f(1)$.

(b) (5 pts) For the value of k that you found in part (a), is f differentiable at $x = 1$? Explain your reasoning (no credit will be given without justification).

Answer: No. The derivative of the function $4x + 1$ is 4, so we know that slope at the point $x = 1$ looking only from the left is 4. The derivative of the function $2x^2 + 3x$ is $4x + 3$, so we know that the slope at the point $x = 1$ looking only from the right is $4 \cdot 1 + 3 = 7$. Thus,

$$\lim_{h\to 0^-} \frac{f(1+h) - f(h)}{h} = 4 \neq 7 = \lim_{h\to 0^+} \frac{f(1+h) - f(h)}{h}$$

so f is not differentiable at $x = 1$.

(c) (5 pts) Is there any value of k for which f is differentiable at $x = 1$?

Answer: No. If a function is differentiable at a point, then it must be continuous at that point. We know from part (a) that the only value of k making f is continuous at $x = 1$ is $k = 3$, and we saw in part (b) that for this value f fails to be differentiable at $x = 1$. Therefore, there is no value of k such that f is differentiable at $x = 1$.

2. Let

$$f(x) = \frac{3x-4}{\sqrt{x^2-3}}.$$

(a) (5 pts) Find its derivative $f'(x)$.

Answer: We have

$$f(x) = \frac{3x-4}{(x^2-3)^{1/2}}$$

Therefore,

$$\begin{aligned} f'(x) &= \frac{(x^2-3)^{1/2}\cdot 3 - (3x-4)\cdot \frac{1}{2}(x^2-3)^{-1/2}\cdot 2x}{((x^2-3)^{1/2})^2} \\ &= \frac{3\sqrt{x^2-3} - (3x^2-4x)(x^2-3)^{-1/2}}{x^2-3} \\ &= \frac{3(x^2-3) - (3x^2-4x)}{(x^2-3)^{3/2}} \\ &= \frac{3x^2-9-3x^2+4x}{(x^2-3)^{3/2}} \\ &= \frac{4x-9}{(x^2-3)^{3/2}} \end{aligned}$$

(b) (5 pts) Find an equation of the tangent line to $f(x)$ at $x=2$.

Answer: The slope of the tangent line to $f(x)$ at $x=2$ is

$$f'(2) = \frac{4\cdot 2-9}{(4-3)^{3/2}} = \frac{-1}{1} = -1$$

A point on the line is given by $(2, f(2))$. Since

$$f(2) = \frac{3\cdot 2-4}{\sqrt{4-3}} = \frac{2}{1} = 2$$

that point is $(2,2)$. Thus, an equation for the tangent line to $f(x)$ at $x=2$ is $y-2=(-1)(x-2)$, or $y-2=-x+2$, or more simply $y=-x+4$.

3. Let

$$f(x) = \frac{x-1}{x^2+2x-3}$$

Evaluate each of the following limits. Explain your answers!

(a) (5 pts) $\lim_{x\to 0} f(x)$

Answer: Since f is continuous at every point of its domain, and 0 is in its domain, we have

$$\lim_{x\to 0} f(x) = f(0) = \frac{0-1}{0+0-3} = \frac{-1}{-3} = \frac{1}{3}$$

(b) (5 pts) $\lim_{x\to 1} f(x)$

Answer: Notice that $x^2+2x-3 = (x-1)(x+3)$, so for all $x \neq 1$ we have

$$f(x) = \frac{x-1}{x^2+2x-3} = \frac{x-1}{(x-1)(x+3)} = \frac{1}{x+3}$$

Hence, using the fact that the function $\frac{1}{x+3}$ is continuous at every point of its domain, we have

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{1}{x+3} = \frac{1}{1+3} = \frac{1}{4}$$

(c) (5 pts) $\lim_{x\to\infty} f(x)$

Answer: We have

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{x-1}{x^2+2x-3} \\ &= \lim_{x\to\infty} \frac{\frac{1}{x}-\frac{1}{x^2}}{1+\frac{2}{x}-\frac{3}{x^2}} \\ &= \frac{0-0}{1+0-0} \\ &= 0\end{aligned}$$