Math 11, Fall 2007

Lecture 23

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Outline

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1. Review and overview
   - Last class
2. Today's material
   - Curl and Divergence
   - Curl and Divergence: Examples
3. Next class

Integration of a function of two variables

Green's Theorem

Let C be a positively oriented, piecewise smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D then

$$\int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\ dA$$

Example

Find the integral of $\vec{F} = y\,\vec{i} + x\vec{j}$ around a curve simple closed positively oriented curve C enclosing the origin.

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Review of the gradient

- The gradient encodes all the derivative information for a real valued function of more than one variable
- It measures geometric information as well: direction of maximal ascent, normals to surfaces

Differentiation of vector fields

Given a vector fireld $\vec{V} = P\,\vec{i} + Q\vec{j} + R\,\vec{k}$, what is its derivative?

- curl $\vec{V} = \nabla \times \vec{V} =$

$$det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{pmatrix}$$

Here, $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$

Curl

Geometrically, curl measures the circulation of the vector field thought of as a fluid flow. A curl free field is called *irrotational*. The curl also has a connection to conservative vector fields.

Theorem: curl $\nabla f = 0$

Theorem: If $\vec{F}$ is a vector field defined for all points in $\mathbb{R}^3$ whose components has continuous partial derivatives and curl $\vec{F} = 0$, then $\vec{F}$ is conservative.

Divergence

The divergence of a vector field $\vec{F} = P\,\vec{i} + Q\,\vec{j} + R\,\vec{k}$ is

$$\text{div } \vec{F} = \nabla \cdot \vec{F}$$

Divergence

Geometrically, divergence measures how quickly the vector field spreads out (or contracts). A divergence free field is called *incompressible*.

> Theorem: If P, Q, R have continuous second order partial derivatives, then div curl $\vec{F} = 0$

Green's Theorem: new form

Let C be a positively oriented, piecewise smooth, simple closed curve in the plane and let D be the region bounded by C. If $\vec{F} = P\,\vec{i} + Q\,\vec{j}$ and P and Q have continuous partial derivatives on an open region that contains D then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\text{curl } \vec{F}) \cdot \vec{k}\, dA$$

Divergence form

If C is given by the vector equation $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ then

$$\vec{T}(t) = \frac{x'(t)}{|\vec{r}'(t)|}\vec{i} + \frac{y'(t)}{|\vec{r}'(t)|}\vec{j}$$

The outward unit normal to C is

$$\vec{n}(t) = \frac{y'(t)}{|\vec{r}'(t)|}\vec{i} - \frac{x'(t)}{|\vec{r}'(t)|}\vec{j}$$

Then, we can write Green's theorem in divergence form.

$$
\begin{aligned}
\oint_C \vec{F}\cdot\vec{n}\,ds &= \int_a^b (\vec{F}\cdot\vec{n})(t)|\vec{r}'(t)|dt \\
&= \int_a^b \left(\frac{P(x(t),y(t))y'(t) - Q(x(t),y(t))x'(t)}{|\vec{r}'(t)|}\right)|\vec{r}'(t)|\,dt \\
&= \int_a^b (P(x(t),y(t))y'(t) - Q(x(t),y(t))x'(t))\,dt \\
&= \oint_C P\,dy - Q\,dx \\
&= \iint_D (P_x + Q_y)\,dA \\
&= \iint_D \mathrm{div}\vec{F}(x,y)\,dA
\end{aligned}
$$

Outline

Examples

1. Is there a vector field $\vec{G}$ on $\mathbb{R}^3$ so that curl $\vec{G} = xy^2\,\vec{i} + yz^2\,\vec{j} + zx^2\,\vec{k}$?
2. Show that any vector field of the form

$$\vec{F}(x,y,z) = f(y,z)\,\vec{i} + g(x,z)\,\vec{j} + h(z,y)\,\vec{k}$$

is incompressible.
3. Compute the flux integral

$$\oint_C \vec{F} \cdot \vec{n}\, ds$$

where C is the circle of radius 1 union the circle of radius 1/2 and

$$\vec{F} = \frac{y}{\sqrt{x^2+y^2}}\vec{i} - \frac{x}{\sqrt{x^2+y^2}}\vec{j}$$

1. Compute the circulation integral

$$\oint_C \vec{F} \cdot d\vec{r}$$

where C is the circle of radius 1 union the circle of radius 1/2 and

$$\vec{F} = \frac{y}{\sqrt{x^2+y^2}}\vec{i} - \frac{x}{\sqrt{x^2+y^2}}\vec{j}$$

Work for next class

- Reading: 17.6
- f07hw25 (due monday after Thanksgiving)