Trigonometric Substitution

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- *The inverse substituion:*

$$\int f(x)\mathrm{d}x = \int f(g(t))g'(t)\mathrm{d}t \quad \text{if } x = g(t)$$

- For $\sqrt{a^2 - x^2}$ use the substitution $x = a\sin\theta$, $-\pi/2 \le \theta \le \pi/2$ and the identity $1 - \sin^2\theta = \cos^2\theta$.

- **Example:** $\int x^3\sqrt{9 - x^2}\mathrm{d}x$.

- For $\sqrt{a^2 + x^2}$ use the substitution $x = a \tan\theta$, $-\pi/2 < \theta < \pi/2$ and the identity $1 + \tan^2\theta = \sec^2\theta$.

- **Example:**

$$\int \frac{\mathrm{d}x}{\sqrt{4 + x^2}}.$$

$$\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}} \mathrm{d}x$$

- For $\sqrt{x^2 - a^2}$ use the substitution $x = a\sec\theta$, $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$ and the identity $\sec^2\theta - 1 = \tan^2\theta$.

- **Example:**

$$\int \frac{\mathrm{d}t}{\sqrt{t^2 - 6t + 5}}$$

Integration of Rational Functions by Partial Fractions

- **Problem:**Integrate a rational function

$$f(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials.

- The method of *partial fractions* is to express $f(x)$ by a sum of simpler fractions.

- If $deg(P) < deg(Q)$, then it is possible to express f as such a sum.
- If $deg(P) \geq deg(Q)$ we must take the preliminary step of dividing Q into P by long division:

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

where S and R are also polynomials.

Examples

- Find

$$\int \frac{x^3 + x}{x - 1} \mathrm{d}x$$

Case 1

- The denominator $Q(x)$ is a product of distinct linear factors

$$Q(x) = (a_1 x + b_1)(a_2 x + b_2) \dots (a_k x + b_k).$$

- Then there exist constants $A_1, A_2, \dots, A_n$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \dots + \frac{A_k}{a_k x + b_k}$$

Example

- Evaluate

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} \mathrm{d}x.$$

Case 2

- $Q(x)$ is a product of linear factors, some of which are repeated.

- Suppose that the first linear factor $(a_1 x + b_1)$ is repeated r times. Then we would use

$$\frac{A_1}{a_1 x + b_1} + \frac{A_2}{(a_1 x + b_1)^2} + \ldots + \frac{A_r}{(a_1 x + b_1)^r}$$

- **Example:**

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} \mathrm{d}x$$

Case 3

- $Q(x)$ contains irreducible quadratic factors, none of which is repeated

- Then the expression for $R(x)/Q(x)$ will have a term of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

- Use $\int \frac{\mathrm{d}x}{x^2+a^2} = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + C$.

Example:

$$\int \frac{2x^2 - x + 4}{x^3 + 4x} \mathsf{d}x$$