Math 31 Lesson Plan

Day 23: Normal Subgroups

Elizabeth Gillaspy

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Supplies needed:

- Colored chalk

Goals for Myself:

- Watch the time!!

Goals for Students:

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[**Lecture Notes: Write everything in blue, and every equation, on the board. [Square brackets] indicate anticipated student responses. *Italics* are instructions to myself.**]

Collect exams. Return midterms. Introduce Rosa.

Agenda: Today I want to Review characterizations of normal subgroups and have you look at examples of normal subgroups in your groups. Then we'll talk about Quotient groups via modular arithmetic, and look at some Examples of quotient groups.

I know that a lot of people were confused about Theorem 11.7, but I don't think we'll have time to discuss it today. I think it's cool, and I'd like to talk about it in class on Friday; however, it's not a central theorem so we might skip it. We'll see how far we get on Friday.

Questions before we get started?

Who can tell me what the definition of a normal subgroup is?

DEFINITION: Let G be a group. A subgroup $H \leq G$ is *normal* if for every $g \in G$ and every $h \in H$, we have $ghg^{-1} \in H$.

What are some other ways we can say H is normal? (ie, equivalent definitions) *ask for students to provide these characterizations*

alternatively,

1. H is normal iff for all $g \in G$, $gHg^{-1} = H$
2. H is normal iff every $g \in G$, $gH = Hg$ (the left coset of H by g is the same as the right coset of H by g)
3. If $H \leq Z(G)$ then H is normal in G.

4. If G is abelian, then any subgroup H of G is normal.
5. If $[G : H] = 2$ then H is normal.
6. If H is the only subgroup of G of size $|H|$ then H is normal.

I would rather not discuss the proofs of these right now. You're welcome to come ask me in office hours, or I would also encourage you to reread these theorems in the textbook after you've had some R&R to recover from the exam. I would bet that the reading will make a lot more sense after you've slept!

List subgroups on the board and ask students in groups to determine which ones are normal.

Which of these subgroups are normal?

1. $3\mathbb{Z} \leq \mathbb{Z}$
2. $D_3 \times \{0\} \leq D_3 \times \mathbb{Z}_4$
3. $(\mathbb{R}^+, \cdot) \leq (\mathbb{R}^\times, \cdot)$
4. $SL(2, \mathbb{R}) \leq GL(2, \mathbb{R})$
5. $\langle V \rangle \leq D_4$
6. $\langle (12) \rangle \leq S_3$

Discuss as a class

Since $\det(ABA^{-1}) = \det(A)\det(B)\det(A^{-1}) = \det(A)\det(B)\det(A)^{-1}$, and the determinant of a matrix is a real number (so multiplication is commutative), it follows that

$$\det(ABA^{-1}) = \det(B).$$

In other words, if $B \in SL(2, \mathbb{R})$ and $A \in GL(2, \mathbb{R})$, then $ABA^{-1} \in SL(2, \mathbb{R})$. Hence $SL(2, \mathbb{R}) \triangleleft GL(2, \mathbb{R})$.

I want to use the first example, $3\mathbb{Z} \le \mathbb{Z}$, to explain the connection between quotient groups and modular addition.

What are the right cosets of $3\mathbb{Z}$ in $\mathbb{Z}$? $[3\mathbb{Z}, 3\mathbb{Z}+1, 3\mathbb{Z}+2]$ *Draw these on a number line* These are the elements of the group $(\mathbb{Z}/3\mathbb{Z}, +)$. What's $(3\mathbb{Z}+1)+(3\mathbb{Z}+2)$? *Think-pair-share*

$$(3\mathbb{Z}+1)+(3\mathbb{Z}+2) = 3\mathbb{Z}+3 = 3\mathbb{Z}.$$

In general, if $a, b \in \mathbb{Z}$, then $(3\mathbb{Z}+a)+(3\mathbb{Z}+b) = 3\mathbb{Z}+(a+b)$.

What's the Cayley table for $(\mathbb{Z}/3\mathbb{Z}, +)$? Is $(\mathbb{Z}/3\mathbb{Z}, +)$ isomorphic to some familiar group? Is there something special about 3, or would this still be true for $\mathbb{Z}/n\mathbb{Z}$? *Have them figure this out in pairs. While they're working, write two copies of Cayley table for S_3 on board.*

In general, if $H \lhd G$ then we can make the right cosets of H into a group, which we write G/H. The group operation is $Ha * Hb = Hab$.

Example: S_3. We know that $H = \{e, (123), (132)\} \lhd S_3$, but $K = \{e, (12)\}$ is not normal in S_3. If we highlight the cosets of H in S_3, *Do so, labeling them as $H, H(1,2)$* then what's $H * H$? $H * H(12)$? *Draw Cayley table for S_3/H* We see that if we highlight the cosets of H in the Cayley table of S_3, the colored boxes show the same thing as the Cayley table of S_3/H. When we take the quotient group, we basically say "Squish each coset of H down to one element."

However, notice that if we highlight the right cosets of $K = \langle (12) \rangle$ in S_3, we don't get this nice breakdown into boxes. *Highlight boxes and show overlap*

The fact that in this case, we don't get a neat division into boxes, means that we can't make a group out of the cosets! if we zoom out, the color-coded picture doesn't look like the Cayley table of a group.

In math-ese, the operation $Ka * Kb = Kab$ is not a well defined operation on the cosets of K: If we pick different representatives a, a' of the same coset Ka, we can have $Kab \neq Ka'b$.

For example, (13) and (132) represent the same coset of K, and so do (23) and (123). However, $(23)(13) = (123)$ and $(123)(132) = (1)(2)(3) = e$, yet $K(123) \neq Ke$!

Questions about quotient groups?

Please get back into groups of 3 or 4. Please get up, stretch your legs, move around, and find a group with someone that you haven't worked with yet. In your groups, I want you to look at some Examples of quotient groups: In each case, what familiar group is G/H isomorphic to?

1. $H = 0 \times \mathbb{Z}_4, G = S_3 \times \mathbb{Z}_4$
2. $H = \langle 180 \rangle, G = D_4$
3. $H = \langle (2,2) \rangle, G = \mathbb{Z} \oplus \mathbb{Z}$
4. $H = SL(2,\mathbb{R}), G = GL(2,\mathbb{R})$

Decide: Read Section 13 for Friday or Monday?

If extra time: THEOREM: *If $G/Z(G)$ is cyclic, then G is abelian.*

Proof: Suppose $G/Z(G)$ is cyclic, and let $Z(G)g$ be a generator for $G/Z(G)$. Then, let $h, k \in G$ be arbitrary. If either h or k is in $Z(G)$, then $hk = kh$. So suppose that $h, k \notin Z(G)$. Then $h \in Z(G)g^n$ and $k \in Z(G)g^m$ for some $n, m > 1$. That is,

$$h = z_1 g^n; \quad k = z_2 g^m.$$

Now, consider

$$hk = z_1 g^n z_2 g^m = z_1 z_2 g^n g^m = z_1 z_2 g^{m+n},$$

because $z_2 \in Z(G)$ and so it commutes with everything, including g^n. Similarly,

$$kh = z_2 g^m z_1 g^n = z_2 z_1 g^m g^n = z_1 z_2 g^{m+n} = hk.$$

Therefore, G is abelian as claimed. $\square$