A Primer on Power Series

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Suppose that

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (\ddagger)$$

has radius of convergence $R > 0$. Let S be the function defined on $D = N_R(z_0) = \{\, z \in \mathbf{C} : |z - z_0| < R \,\}$ by

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

You should be familar with the following statements and terms.

- The series ($\ddagger$) converges absolutely and uniformly for all z in any closed disk $\{\, z : |z - z_0| \le R_1 \,\}$ contained in D; that is, for all $R_1 < R$.
- S is analytic on D, and ($\ddagger$) is the Taylor series for S about the point z_0. That is,
$$a_n = \frac{S^{(n)}(z_0)}{n!} \quad \text{for } n = 0, 1, 2, \ldots.$$
- The series ($\ddagger$) can be differentiated term-by-term so that for all $z \in D$
$$S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}.$$
- The series ($\ddagger$) can be integrated term-by-term in the following sense. Let C be any contour in D and g any *continuous* function on C^*. Then
$$\int_C g(z) S(z)\, dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n\, dz.$$

Similar statements hold for Laurent series. We haven't proved all of these statements in lecture, but you are welcome to use them. Anyway, suppose we are given two series. First a power series (‡) and a series

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}. \tag{$*$}$$

There is a number $r \geq 0$ (we also allow $r = \infty$) such that $(*)$ converges absolutely if $|z - z_0| > r$ and diverges if $|z - z_0| < r$. If $r < R$, then we can define a function f in the domain $A = \{ z \in \mathbf{C} : r < |z - z_0| < R \}$ by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}. \tag{$**$}$$

Then the following are true.

- The function f is analytic in A and $(*)$ is the Laurent series for f in A. That is, if C is any positively oriented simple closed contour in A that contains z_0 in its interior, then

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \quad \text{for } n = 0, 1, 2, \ldots, \text{ and} \\ b_n &= \frac{1}{2\pi i} \int_C f(z)(z - z_0)^{n-1}\, dz \quad \text{for } n = 1, 2, \ldots. \end{aligned}$$

- The series $(**)$ can be integrated term by term. That is, given a contour C in A and a function g which is continuous on C^*, then

$$\int_C g(z) f(z)\, dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n\, dz + \sum_{n=1}^{\infty} b_n \int_C \frac{g(z)}{(z - z_0)^n}\, dz.$$

- To prove all this we would have needed to prove that both series in $(**)$ converge absolutely and uniformly in any closed annulus of the form

$$A_1 = \{ z : r < r_1 \leq |z - z_0| \leq R_1 < R \}$$

  contained in A.