11/15/11
Barnett

SOLUTIONS

Math 53: Chaos!: Midterm 2, FALL 2011

2 hours, 60 points total, 6 questions worth various points (proportional to blank space)

1. [9 points] Complex dynamics. Please show working or some explanation.

1 (a) Is i in the Julia set J(1) ? $c=1$ in $z_{n+1} = z_n^2 + c$ map.

Apply map:

$i \to i^2+1 = 0 \to 1 \to 2 \to 5 \to \cdots \infty$ so, no, i not in Julia set.

once exceeds 2, & $|c|<2$, we proved it goes $\to \infty$.

2 (b) Is 1 in the Mandelbrot set, and why?

Iterate $z_0 = 0$ under $f(z) = z^2+1$: $\to 1 \to 2 \to 5 \to \cdots \infty$ so 0 is in basin of ∞,

so 1 not in Mandelbrot set.

(c) Consider the map $f(z) := z^2 + 1$, for $z \in \mathbb{C}$. Could there exist a periodic sink for this map?

Fatou theorem states that for a polynomial map such as f, each periodic sink must attract a critical point of f.

The only critical point is $f'(z) = 2z = 0$, ie $z=0$, and we found 0 headed to ∞ under the map, so, no, cannot exist any periodic sink.

(d) Could there exist a $z_0 \in \mathbb{C}$ such that $f^n(z_0)$ remains bounded as $n \to \infty$?

No reason why not (such points are in J(1), see BONUS for an example)

ie 1 not in Mandelbrot

(e) Based on your answers above, do you expect $J(1)$ to be connected/disconnected? Have nonzero/zero measure? (circle those that apply; no explanation needed)

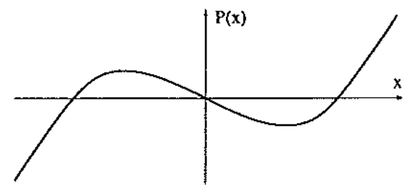
[1] BONUS: Either find an example of a bounded such z_0 from part (d), or prove there cannot exist any.

e.g. a fixed pt of f : solve $f(z)=z$ ie $z^2+1=z$ ie $z=\frac{1\pm\sqrt{-3}}{2}$

$= \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

There are countably infinite (at least), taking fixed pts of f^n.

2. [16 points] Consider 1D motion of a point particle in the potential $P(x) = x^3/3 - x$, which has roughly the following graph:

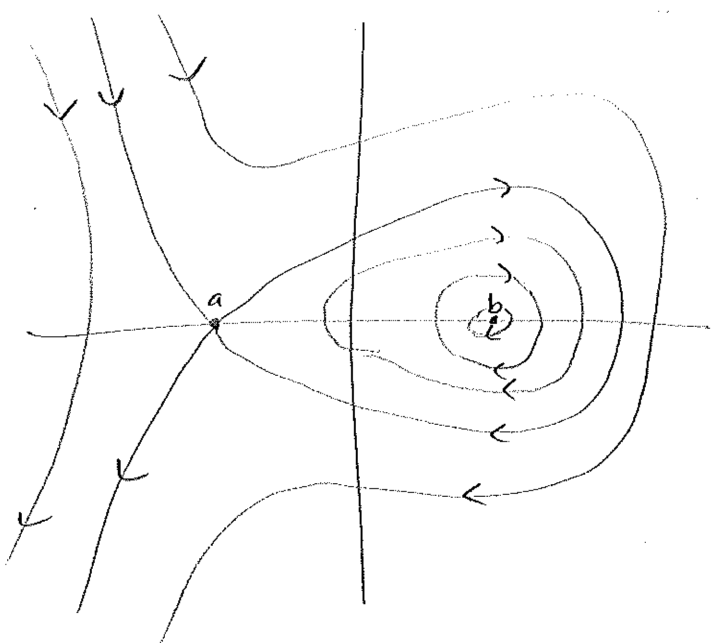


2 (a) Write a system of first-order ODEs for the dynamics in this potential, with no damping.

$$\text{force} = -\frac{dP}{dx} = -x^2+1$$

so $\begin{cases} \dot{x} = y \\ \dot{y} = \text{force} = -x^2+1 \end{cases}$

4 (b) Sketch the phase plane $(x, \dot{x})$ showing several orbits including all the types of motion that can occur:



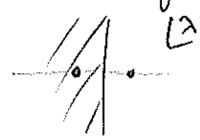
5 (c) Find all equilibria and categorize their stability. Justify your stabilities by giving a rigorous argument in each case. [Hint: use the phase plane]

Jacobean of flow: $\vec{D}\vec{f}(x,y) = \begin{pmatrix} 0 & 1 \\ -2x & 0 \end{pmatrix}$ (2×2 matrix).

force $= 0$ so $-x^2+1 = 0$ ie $x = \pm 1$.

equil. a:

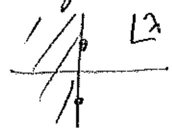
$(-1,0)$: $\vec{D}\vec{f}(a) = \begin{pmatrix} 0 & 1 \\ +2 & 0 \end{pmatrix}$ so eigvals $\lambda^2 - 2 = 0$ ie $\lambda = \pm\sqrt{2}$.



One eigval has positive real part $\Rightarrow$ Unstable.

equil. b:

$(1,0)$: $\vec{D}\vec{f}(b) = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$ eigvals $\lambda^2 + 2 = 0$ ie $\lambda = \pm i\sqrt{2}$



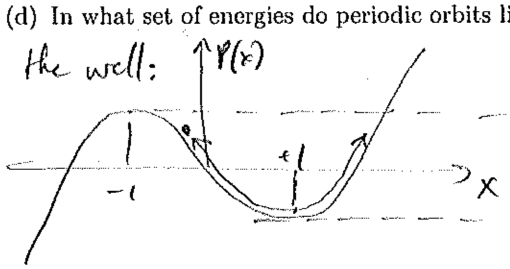
Re parts both zero $\Rightarrow$ nonlinear stability theorem tells us nothing!

However, in any neighborhood $N_\varepsilon(b)$, $\varepsilon>0$, you may find a closed contour of $E(x,\dot{x}) = \frac{\dot{x}^2}{2} + P(x)$ which encloses a neighborhood N_1 of b which never leaves $N_\varepsilon(b)$ $\Rightarrow$ rigorously, b is Stable.

[This is essense of Lyapunov function].

2 (d) In what set of energies do periodic orbits lie? [take care with endpoints]

rolling in the well:



$P(-1) = -\frac{1}{3} + 1 = \frac{2}{3}$

$P(1) = -\frac{2}{3}$

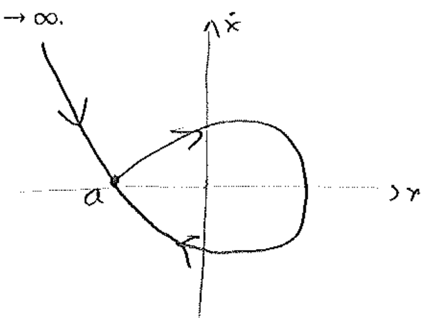
so $-\frac{2}{3} < E < \frac{2}{3}$

not periodic!

note $E = -\frac{2}{3} \Rightarrow$ equilibrium at b

$E = +\frac{2}{3} \Rightarrow$ homoclinic orbit, not periodic.

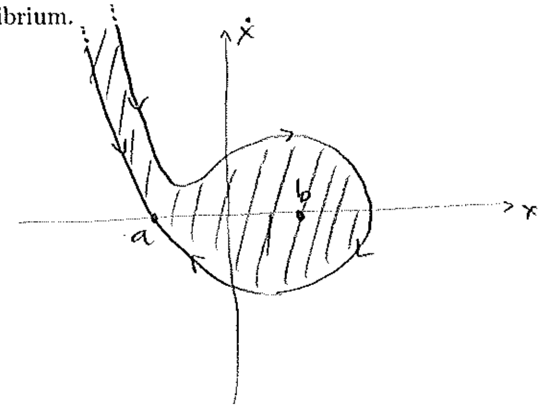
1 (e) Sketch the set of all phase plane points which have the unstable equilibrium as their limit as $t \to \infty$.



Stable manifold of saddle point a

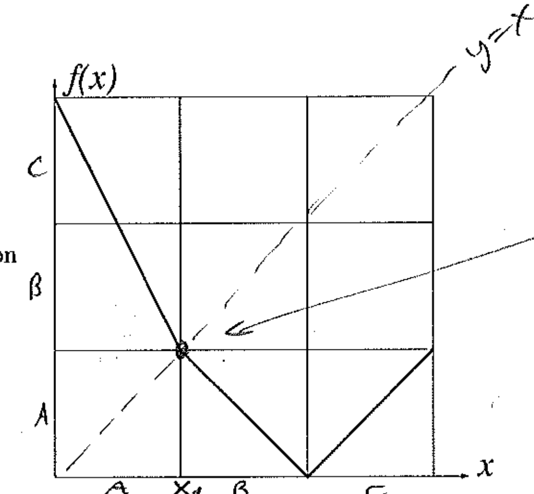
(= also a piece of its unstable manifold!)

2 (f) Imagine a small amount of damping is now added. Sketch on a phase plane the basin of the stable equilibrium.



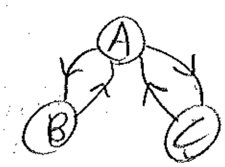
boundary of basin is the stable manifold of saddle a.

3. [10 points] Consider the continuous function f with the following graph:



note fixed point of f, here

2 (a) Draw the transition graph (use three intervals A, B, and C):



2 (b) Prove that a period-2 orbit exists.

since graph has path ABA then $f(ABA) \supset A$ & by fixed pt. theorem, $\exists$ period-2 orbit.

(Note, since $f(A) \cap A$ empty, apart from the point x_1, cannot be a period-1).

2 (c) Can a period-3 orbit exist? Prove your answer.

No, since after 3 iterations, no interval A, B or C has returned to itself (the graph is 'bipartite')

Thus $f^3(A) \cap A = \{\emptyset\}$ so no p-3 possible

$f^3(B) \cap B = \{\emptyset\}$

$f^3(C) \cap C = \{\emptyset\}$ 4

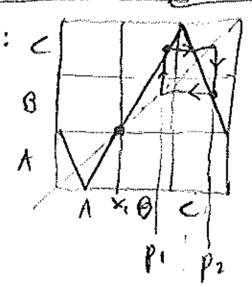
(not strictly true, since $A \cap B$ = single point $\{x_1\}$, but we know this is period-1 already.

4 (d) List all periods that *must* exist, giving a proof of your answer. [Hint: check the obvious before you get fancy]

period	why?
1	$f(x_1) = x_1$ see graph. Note transition graph doesn't see this point since it's on edge of intervals.
2	part (b)
$2n, n \in \mathbb{N}$	$f^{2n}(A) \supset A$ and may construct $\overline{\underbrace{ABACAC\cdots AC}_{2n}}$ which cannot be explained by lower period orbit.

BONUS: Assuming that f is linear in each of the three intervals, prove that if a period-4 orbit exists, it must enter all three intervals. → ie must have $\overline{ABAC}$ rather than $\overline{AB}$ or $\overline{AC}$.

Graph f^2:

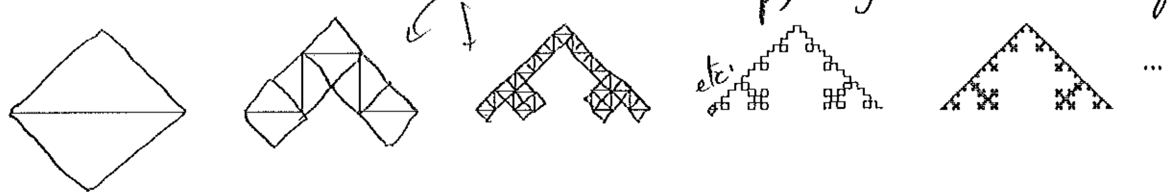


period-4 must involve $f^2(p_2) = p_1$ & $f^2(p_1) = p_2$, $p_2 \neq p_1$. Since the slope of f^2 in C is -2, p_1 & p_2 cannot both be in C, so one must be in B. So orbit is $\overline{CABA}$.

4. [6 points]

4 (a) Find the box-counting dimension of the curve (a subset of $\mathbb{R}^2$) formed by the limiting process sketched below: for each straight line segment remove the middle third and replace it by the other three sides of the square. [Hint: describe your 'boxes'. To avoid colliding boxes you may rotate them to cover without collisions] note boxes do not overlap; they even lie on 45°-rot. grid.

diamond



$\varepsilon = 1$ $\quad$ $\varepsilon = 1/3$ $\quad$ $1/3^2$ $\quad \cdots \quad$ $1/3^n$

$N(\varepsilon) = 1$ $\quad$ $N(\varepsilon) = 5$ $\quad$ 5^2 $\quad \cdots \quad$ 5^n

$$d = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln 1/\varepsilon} = \lim_{n \to \infty} \frac{\ln N(b_n)}{\ln 1/b_n} = \lim_{n\to\infty} \frac{\ln 5^n}{\ln 3^n} = \lim_{n \to \infty} \frac{n \ln 5}{n \ln 3}$$

if $b_n = 1/3^n$ satisfies the conditions $\lim_{n\to\infty} \frac{\ln b_{n+1}}{\ln b_n} = 1$ & $b_n \to 0$, true.

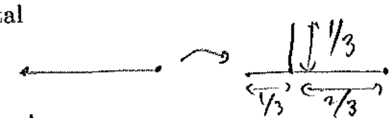
$$= \frac{\ln 5}{\ln 3} \approx 1.465\ldots$$

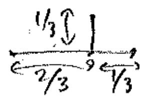
2 (b) Could there be a subset of $\mathbb{R}$ with this same box-counting dimension? Explain.

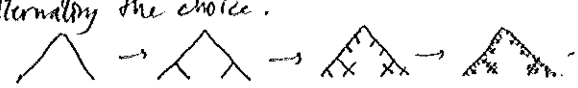
Subsets $S \subset \mathbb{R}$ have a maximum boxdim of 1.

$\Rightarrow$ No, not possible.

[1] BONUS: Describe an alternative construction whose limit gives this same fractal

Start with  then on each line do this: 

or  alternating the choice.



I have no idea why this works, but it does!

OK

5. [8 points] Consider the following map acting on points (x, y) in the unit square $[0, 1]^2$:

$$B(x,y) = \begin{cases} (\frac{x}{4},\ 2y(\text{mod } 1)), & \text{if } y < 1/2, \\ (\frac{x+3}{4},\ 2y(\text{mod } 1)), & \text{otherwise} \end{cases}$$

2 (a) What is the complete set of Lyapunov exponents (for almost all initial conditions) for this map?

$$J = DB(x,y) = \begin{bmatrix} 1/4 & 0 \\ 0 & 2 \end{bmatrix}$$ unless $y = 1/2$, k is constant w.r.t. (x,y).

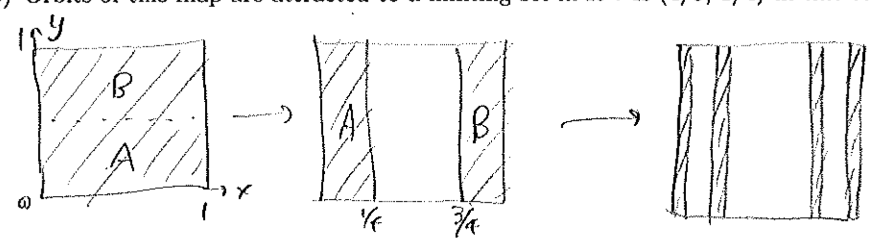
Note J is diagonal.

For $j = 1, 2$, $h_j = \lim_{n\to\infty} \frac{1}{n} \ln \sqrt{\lambda_j(J^n J^{Tn})}$ matrix is $J^{2n} = \begin{bmatrix} 1/4^{2n} & \\ & 2^{2n} \end{bmatrix}$ with eigvals $\lambda = 1/4^{2n}, 2^{2n}$.

so $h_j = \ln \lambda_j(J) = \ln 1/4,\ \ln 2$

$h_1 = \ln 2, \quad h_2 = -2\ln 2$

3 (b) Orbits of this map are attracted to a limiting set in $\mathbb{R}^2$. Is $(4/5, 1/4)$ in this set, and why?



$\cdots K_4 \times [0,1]$ middle-1/2's Cantor set

limiting set.

$x = 4/5$ has quaternary expansion: if only involves 0 & 3 then $x \in K_4$ (defines middle-1/2 Cantor).

Use $4x \pmod 1$ map to extract its expansion: $\frac{4}{5} \to \frac{16}{5} = \frac{1}{5} \to \frac{4}{5} \to \frac{1}{5} \ldots$ etc.
"3" "0" "3" ...

so $x = 0.\overline{30}$ so $x \in K^4$.

Any $y \in [0,1]$ will do, so $(x,y) \in K^4 \times [0,1]$

2 (c) What is the box-counting dimension of this attractor in $\mathbb{R}^2$?

$$\text{box dim}\left(K_4 \times [0,1]\right) = \text{box dim}\,(K_4) + \underbrace{\text{box dim}\,([0,1])}_{1}$$

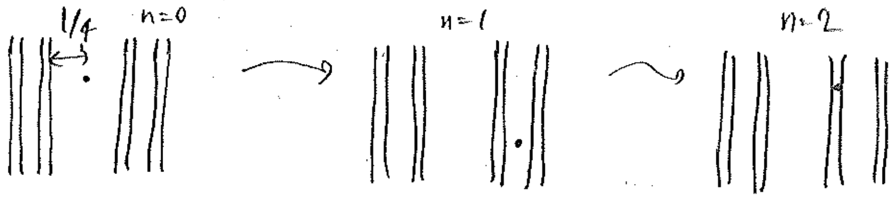
ε	$N(\varepsilon)$
1	1
$1/4$	2
$1/4^2$	2^2
$\vdots$	$\vdots$

so $\dfrac{\ln N(\varepsilon)}{\ln 1/\varepsilon} = \dfrac{n \ln 2}{n \ln 4} = 1/2$

Answer: $1/2 + 1 = 3/2$.

1 (d) Let $(x_0, y_0) \to (x_1, y_1) \to \ldots$ be any orbit with (x_0, y_0) in the unit square. Give a tight upper bound on the distance of (x_n, y_n) from the attractor.

attractor:



each step moves 4x closer.

$$\max_{(x,y)\in[0,1]^2} \text{dist}\left((x,y), \text{attractor}\right) = 1/4$$

$$\text{dist}\left((x_n,y_n), \text{attractor}\right) \le \frac{1}{4^{n+1}}$$

6. [11 points] Random short-answer questions

2 (a) What can you deduce about the stability of $\vec{0}$ for the ODE system $\dot{\mathbf{x}} = A\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^4$ given that the matrix A has eigenvalues -3, -1, 0, and 0?

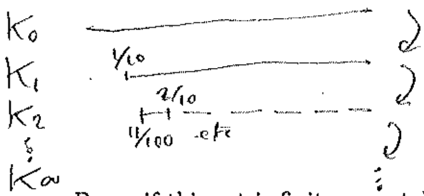
$\lambda = 0$ is degenerate (not simple) so must only have $\operatorname{Re}\lambda_j < 0 \Rightarrow$ AS.

But some $\operatorname{Re}\lambda_j = 0$, so can't deduce stability of $\vec{0}$ (either Stable or Unstable).

2 (b) Give the mathematical definition of an equilibrium point $\mathbf{v}$ of a flow being *stable*.

For each $\varepsilon > 0$, $N_\varepsilon(\vec{v})$ must contain an open set N_1 containing $\vec{v}$, such that all pts in N_1 never leave $N_\varepsilon(\vec{v})$ as $t \to +\infty$.

3 (c) What is the measure of the set of points in $[0,1]$ whose *decimal* expansion never uses the digit "0" ?



construction of Cantor set by removal of $x \in [0,1]$ in which successive digits are "0". Measure multiplied by $9/10$ each time $\Rightarrow$ limit is zero measure.

Prove if this set is finite, countably infinite, or uncountably infinite. (You may use known properties of the set $[0,1]$.)

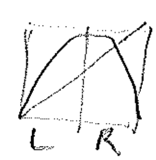
Take any $x \in K_\infty$ above, and map digits "1" through "9" in its decimal expansion to digits "0" through "8". Interpret the result as nonary (base-9). This defines a 1-to-1 map from K_∞ to $[0,1]$, which is uncountably ∞ (by Cantor's diagonal proof).

So K_∞ is uncountable.

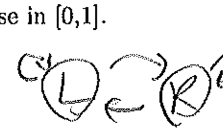
2 (d) What is the Lyapunov exponent of almost all bounded orbits of $G(x) = 4x(1-x)$? Explain why.

$G(x)$ is conjugate to tent map $T(x) := 1 - 2|x - \frac{1}{2}|$ on $[0,1]$. Lyapunov exponent of T is $\ln 2$ since $|T'(x)| = 2$ $\forall x \neq \frac{1}{2}$. Conjugacy preserves Lyapunov exponent. $\Rightarrow$ $\underline{\ln 2}$.

2 (e) Prove that there exists an orbit of $G(x) = 4x(1-x)$ that is dense in $[0,1]$.



G has complete transition graph



$\Rightarrow$ may construct orbit L R LL LR RR RL LLL LLR LRL ... listing all finite-length strings in order. Since length of length-k subinterval $\leq \frac{\pi}{2^{k+1}}$ (it's enough to know upper bound $\to 0$ as $k \to \infty$), orbit passes into every subinterval hence within any $\varepsilon > 0$ of any point in $[0,1]$.