Your name:

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Math 11 Fall 2011, Homework 4, due Wed Oct 19
Please show your work. No credit is given for solutions without justification.

(1) Do the following calculations.

(a) The mixed fourth derivative f_{stst} for $f(s,t) = \frac{1+t}{\sqrt{1+s+s^2}}$.

(b) The direction $\mathbf{u}$ (expressed as a unit vector) in which the function $h(x,y,z) = \ln(x^2 + xy + z^2)$ has its maximal rate-of-change at the point $(x,y,z) = (2,2,1)$.

(c) The maximal value of the directional derivative $D_{\mathbf{u}}f$ of the function $f(x,y) = \cos(xy)$ at the point $(x,y) = (\pi, 2)$.

(d) The directional derivative $D_{\mathbf{u}}g$ of the function $g(x,y) = x^2y^{-4}$ at point $P = (2,1)$ in the direction of a unit vector $\mathbf{u}$ that makes an angle $\theta = \pi/3$ with the direction of the gradient vector ∇f.

(a) Clairaut's theorem: $f_{stst} = f_{ttss}$.
Start with the $\partial/\partial t$ derivatives:

$$f_t = \frac{1}{\sqrt{1+s+s^2}}, \quad f_{tt} = 0, \quad \cdots, \quad \boxed{f_{ttss} = 0}$$

(b) max rate-of-change is in direction of ∇h.

$$\nabla h = \left\langle \frac{2x+y}{x^2+xy+z^2}, \frac{x}{\sqrt{x^2+xy+z^2}}, \frac{2z}{\sqrt{x^2+xy+z^2}} \right\rangle$$

This vector has the same direction as

$$\langle 2x+y, x, 2z \rangle.$$

At $(2,2,1)$ this is $\langle 6,2,2 \rangle$. Therefore

$$\bar{u} = \frac{1}{\sqrt{44}} \langle 6,2,2 \rangle = \frac{1}{\sqrt{11}} \langle 3,1,1 \rangle$$

(c) The maximal value of $D_{\bar{u}}f$ is $\|\nabla f\|$.

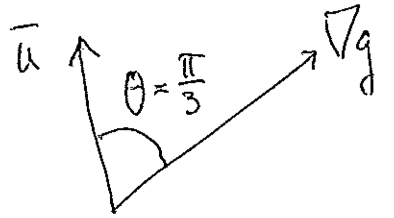
$$\nabla f = \langle -y\sin(xy),\ -x\sin(xy)\rangle$$

$$\nabla f_{(\pi,2)} = \langle -2\sin 2\pi,\ -\pi\sin 2\pi\rangle = \langle 0,0\rangle.$$

So $D_{\bar{u}}f = 0$ for any direction $\bar{u}$.

The maximum value is $\boxed{0.}$

(d) We use the formula $D_{\bar{u}}g = \nabla g \cdot \bar{u} = \|\nabla g\|\cos\theta$.



$$\nabla g = \langle 2xy^{-4},\ -4x^2y^{-5}\rangle$$

$$\nabla g_{(2,1)} = \langle 4, -16\rangle = 4\langle 1,-4\rangle$$

$$\|\nabla g_{(2,1)}\| = 4\sqrt{1+16} = 4\sqrt{17}$$

$$\cos\theta = \cos\frac{\pi}{3} = 1/2$$

$$\boxed{D_{\bar{u}}g = 2\sqrt{17}}$$

(2) Consider the function $f(x,y) = x^2y^{-4}$.

(a) Derive an equation for the tangent plane to the graph $z = f(x,y)$ at the point where $(x,y) = (1,1)$.

(b) Using the methods of Chapter 13, calculate the cosine of the angle ψ between the tangent plane and the xy-plane. The angle ψ is called the *angle of inclination* of the tangent plane. (See section 15.5, p. 827.)

(c) Based on the result of (b), calculate the "slope" of the tangent plane, which equals $\tan\psi$. You can use the trigonometric identities $\cos^2\psi + \sin^2\psi = 1$ and $\tan\psi = \sin\psi/\cos\psi$.

(d) Calculate the magnitude $\|\nabla f\|$ of the gradient vector at the point $(1,1)$. How does this value compare to the slope of the tangent plane?

(a) Tangent plane : $z = L(x,y)$

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Here $(a,b) = (1,1)$ and $f(1,1) = 1$, while

$$f_x = 2xy^{-4}, \quad f_x(1,1) = 2$$

$$f_y = -4x^2y^{-5}, \quad f_y(1,1) = -4$$

Tangent plane:

$$z = 1 + 2(x-1) - 4(y-1)$$

or $\quad z = 2x - 4y + 3.$

(b) Normal vector to plane $2x - 4y - z = -3$ is

$$\bar{n}_1 = \langle 2, -4, -1 \rangle$$

Normal vector to xy-plane is $\bar{n}_2 = \langle 0,0,1 \rangle$.

$$\cos\psi = \frac{\bar{n}_1 \cdot \bar{n}_2}{\|n_1\| \, \|n_2\|} = \frac{-1}{\sqrt{4+16+1}} = -\frac{1}{\sqrt{21}}.$$

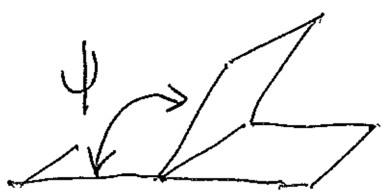
(c) $\sin^2\psi = 1 - \cos^2\psi = 1 - \frac{1}{21} = \frac{20}{21}$

$$\sin\psi = \sqrt{\frac{20}{21}} = \frac{\sqrt{20}}{\sqrt{21}}$$

$$\tan\psi = \frac{\sin\psi}{\cos\psi} = \frac{\sqrt{20}/\sqrt{21}}{-1/\sqrt{21}} = -\sqrt{20}.$$

REMARK: the sign of this slope is irrelevant. The minus sign indicates that we calculated the obtuse angle of inclination.

We usually take ψ acute.



(d) $\nabla f = \langle f_x, f_y \rangle = \langle 2xy^{-4}, -4x^2y^{-5} \rangle$

$\nabla f_{(1,1)} = \langle 2, -4 \rangle$ ← from (a)

$\|\nabla f_{(1,1)}\| = \sqrt{4+16} = \sqrt{20}$

This is the same answer as $\tan\psi$, except for the sign. In general

$\|\nabla f\| = \tan\psi$ (if ψ is acute).

(3) (a) Sketch a contour diagram of $f(x,y) = \sqrt[3]{x/y}$ with contour interval $m = 1/3$.

(b) Without doing any computations, sketch a number of gradient vectors (ten or more) at various points in your diagram. Make sure the vectors that you draw point in the right directions. The length of individual vectors in your picture does not have to be accurate, but they should have roughly the correct lengths *relative to each other.*

(c) Find an expression for the linear approximation $L(x,y)$ of $f(x,y) = \sqrt[3]{x/y}$ at the point $(a,b) = (8,27)$.

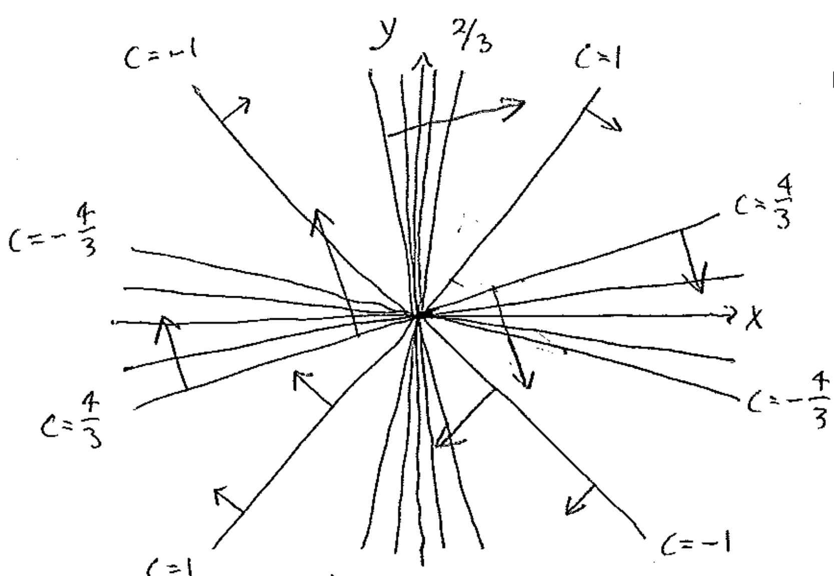
(d) Estimate the cube root $\sqrt[3]{7/29}$ without the help of a calculator.

(a) $\sqrt[3]{X/Y} = C$, $X/Y = C^3$, $X = C^3 Y$. (but $Y \neq 0$)

For $C = 0, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1, \pm\frac{4}{3}$ we find

$C^3 = 0, \pm\frac{1}{27}, \pm\frac{4}{27}, \pm 1, \pm\frac{64}{27}$

So the straight lines $X = C^3 Y$ will look like this:



contour diagram
with gradient vectors

(b). ∇f is perpendicular to the contour lines.

- Points towards <u>increasing</u> c-values.
- Is <u>longer</u> if contours get <u>closer</u> together

(c) $f(x,y) = x^{\frac{1}{3}} y^{-\frac{1}{3}}$ $\qquad$ $f(8,27) = 2 \cdot 3^{-1} = \frac{2}{3}$

$f_x = \frac{1}{3} x^{-\frac{2}{3}} y^{-\frac{1}{3}}$ $\qquad$ $f_x(8,27) = \frac{1}{3} \cdot 2^{-2} \cdot 3^{-1} = \frac{1}{36}$

$f_y = -\frac{1}{3} x^{\frac{1}{3}} y^{-\frac{4}{3}}$ $\qquad$ $f_y(8,27) = -\frac{1}{3} \cdot 2 \cdot 3^{-4} = -\frac{2}{3^5}$

Linear approximation of $f(x,y)$ near $(8,27)$:

$$L(x,y) = f(8,27) + f_x(8,27)(x-8) + f_y(8,27)(y-27)$$

$$= \frac{2}{3} + \frac{1}{36}(x-8) - \frac{2}{3^5}(y-27).$$

(d) To approximate $f(7,29) = \sqrt[3]{7/29}$ use the formula from (c):

$$L(7,29) = \frac{2}{3} - \frac{1}{36} - \frac{2}{3^5} \cdot 2$$

$$= \frac{2}{3} - \frac{1}{36} - \frac{4}{3^5}.$$

This is acceptable as a final answer (without use of calculators). It can be further simplified to

$$L(7,29) = \frac{605}{972}$$

 A calculator would tell you that

$$\frac{605}{972} = 0.622427\cdots$$

while $\sqrt[3]{7/29} = 0.622634\cdots$