Homework #1

T1.3 Goal: Find x st $\left|\frac{3x^2 - x^3}{2}\right| > |x|$

$$|3x - x^3| > 2|x|$$

Break in 2 case

$3x - x^3 \leq -2x$ or $3x - x^3 \geq 2x$

$x^3 \geq 5x$ $\quad$ $x^3 \leq x$

$x(x^2 - 5) \geq 0$ $\quad$ $x(x^2-1) \leq 0$

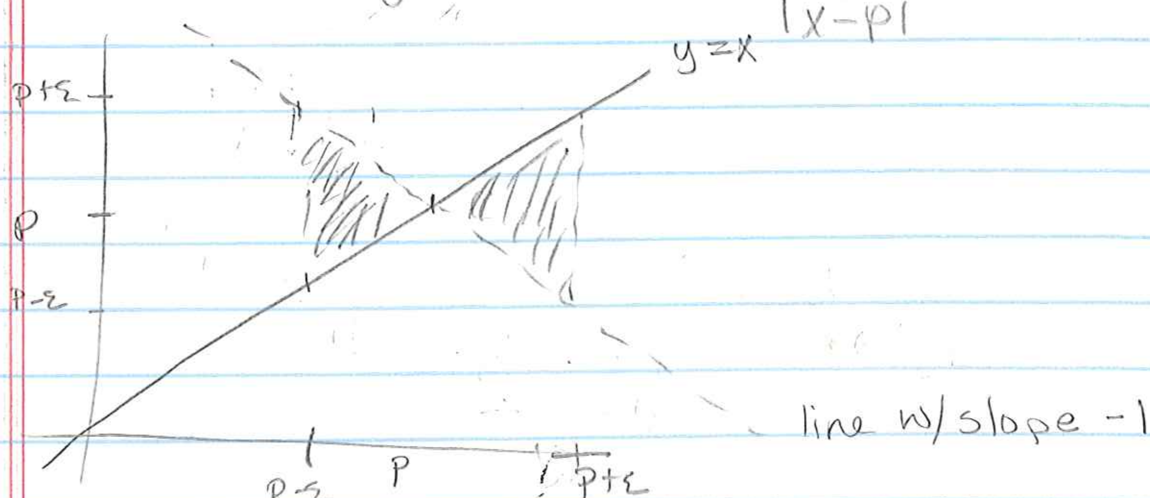
$x \geq \pm\sqrt{5}$ $\quad$ $x \leq \pm 1$

So x satisfies $|x| > \sqrt{5}$ & $|x| < 1$

T1.4 p is a fixed pt of f. ie $f(p) = p$

Given $\varepsilon > 0$, Goal: Find a geometric condition. Under which all pts $x \in N_\varepsilon(p)$ are in the basin of p.

ε is the largest value st $\frac{|f(x) - p|}{|x-p|} < 1$



f must go through the shaded region.

T1.5 $f(x) = 2x^2 - 5x$ on $\mathbb{R}$ has fixed pts at $x=0, x=3$

Goal: find fixed pts of $f^2(x)$

$$f^2(x) = 8x^4 - 40x^3 + 40x + 25x$$

We know $x=0$ & $x=3$ are fixed pts. We can use this info to find the 2-periodic pts.

So we need to find the roots of $\frac{f^2(x) - x}{x}$

Use synthetic division

$$\begin{array}{r|rrrr} 3 & 8 & -40 & 40 & 24 \\ & & 24 & -48 & -24 \\ \hline & 8 & -16 & -8 & 0 \end{array}$$

$8(x^2 - 2x - 1) = 0 \qquad x = \frac{2 \pm \sqrt{4 - 4(-1)}}{2} = 1 \pm \sqrt{2}$

T1.8 $G(x) = 4x(1-x)$

Goal: Prove that for each $k \in \mathbb{Z}^+$, there is an orbit of period-k.

Since G has 2 roots, $G^k(x)$ has 2^k roots.
Some of the lower order functions will have fixed pts that carry on to G^k.
So an upper bound on the # of lower-p orbit

$$M = \underset{p=\phi}{2} + \underset{p=1}{2^2} + \cdots + \underset{p=k-1}{2^{k-1}}$$

we know that the 1^{st} 2 will be fixed pts for all of them so we can subtract them out (ie not repeatedly count)

# of roots not in previous levels

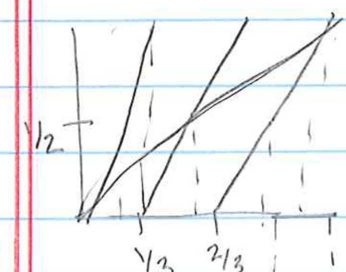
$$\Rightarrow 2^k - (2 + 2^2 + \cdots + 2^{k-1}) + 2k > k$$

$\Rightarrow$ must be at least 1 k-periodic orbit.

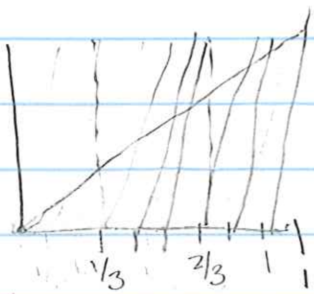
8T1.11 Construct a periodic table for
$f(x) = 3x \pmod 1$

Period k	# of f^k fixed pts	# of fixed pts lower orbits	orbits of period k
1	2	0	2
2	8	2	3.
3	26	2	$24/3 = 8$
4	80	$2+3\cdot 2 = 8$	9
k	$3^k - 1$.	# of fixed pts orbits multiples of k.	difference k.

f



f^2



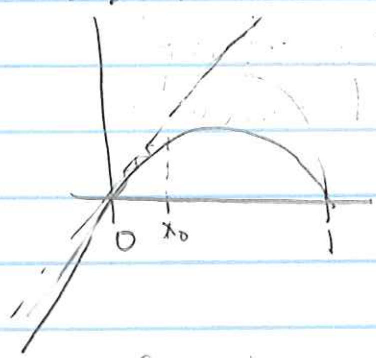
1.1 $\ell(x) = ax+b$ a, b constants
for what values of a & b does $\ell(x)$ have an attracting fixed pt.

The fixed pt is attracting if $|\ell'(p)| < 1$

$\ell'(x) = a$ NOTE: Does not depend on fixed pt.

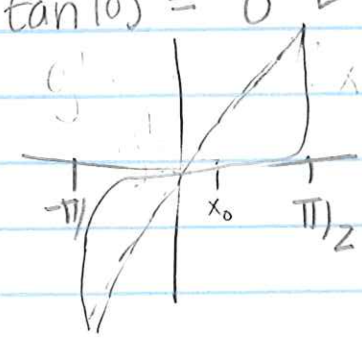
So attracting if $|a| < 1$ $\forall b$
repelling if $|a| > 1$ $\forall b$

1.2 (a) $f(x) = x - x^2$
(i) $f(0) = 0$ $\therefore$ 0 is a fixed pt
(ii) Draw a cobweb plot.



$x > 0 \rightarrow$ attracting (stable)
$x < 0 \rightarrow$ repelling (unstable.

Plot w/ computer. (b) $g(x) = \tan x$ $-\pi/2 < x < \pi/2$
(i) $\tan(0) = 0$ ✓ fixed pt.
(ii)



c) Need a function h st $h'(0) = 1$
& $x = 0$ is an attracting fixed pt

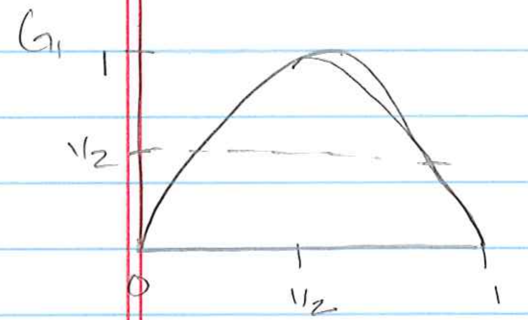
$h(x) = \arctan x$
(There are others)

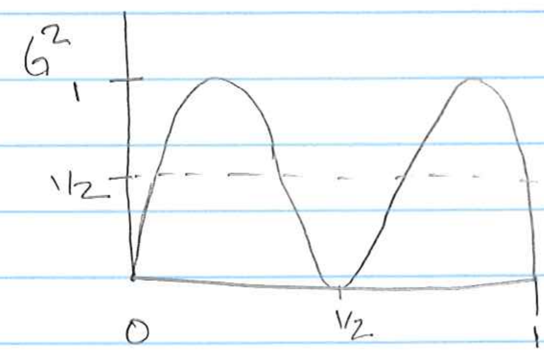
d) Need h st $h'(0) = 1$ & $x = 0$ is a
repelling fixed pt.
$h(x) = \tan(x)$

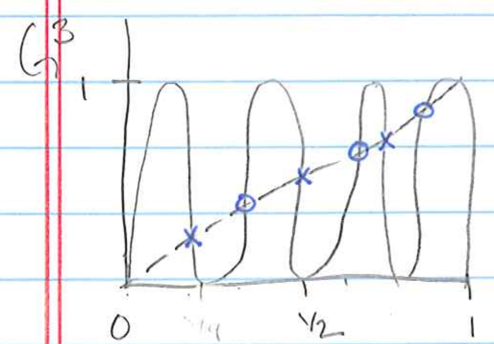
1.4 $x_1, \dots, x_8$ are the eight fixed pts of $G^3(x)$ where $G(x) = 4x(1-x)$.

$x_1 = 0$.

(a) for which i is $x_i = 3/4$







$i = 6$.

(b)

We know there are 2 groups of orbit 3.

o → $\frac{d}{dx}(G^3)(p_j) > 0$ where $\{p_j\}$ is the set of

x → $\frac{d}{dx}(G^3)(p_j) < 0$. fixed pts.

So The 1st orbit consist of the fixedpts st $\frac{d}{dx}(G^3)(p_j) < 0$.

" 2nd " " st $\frac{d}{dx}(G^3)(p_j) > 0$.

1.9 $x_{n+1} = \dfrac{x_n + 2}{x_n + 1}$

a) let $x_0 \geq 0$. Goal: Compute $L = \lim_{n \to \infty} x_{n+1}$

Consider the function $f(x) = \dfrac{x+2}{x+1}$

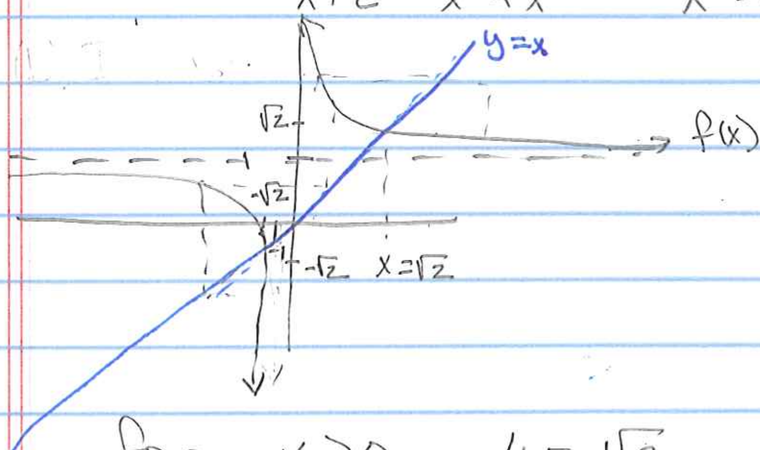
note: $f(x) > 1$

What are the fixed pts of $f(x)$?

i.e. find x s.t. $f(x) = \dfrac{x+2}{x+1} = x$

$\Rightarrow x + 2 = x^2 + x \Rightarrow x = \pm\sqrt{2}$

(not drawn to scale)



for $x_0 \geq 0$. $L = \sqrt{2}$.

(b) for $x_0 < 0$.

if $x_0 = -1$ $f(x)$ is undefined $\rightarrow$ no limit.

if $x_0 = -\sqrt{2}$ stay there $\rightarrow$ fixed source.

for all other $x_0 < 0$. $L = \sqrt{2}$.

1.15 Goal: prove any orbit $\{x_0, x_1, \dots\}$ of $G(x) = 4x(1-x)$ is given by

$$x_n = 1/2 - 1/2 \cos(2^n \arccos(1-2x_0))$$

Proof (Work backwards)

let x_0 be the first fixed pt. in orbit $\ni$ $\cos\theta_0 = 1-2x_0$

$\Rightarrow \theta_n = 2^n \theta_0$ (Below explains where this comes from)

note: $\cos\theta_n = 1-2x_n$

$$\Rightarrow x_n = \frac{1}{2} - \frac{1}{2}\cos\theta_n$$

$$= \frac{1}{2} - \frac{1}{2}\cos(1-2^n \arccos(1-2x_0))$$

How is this connected to the iteration?

If $\cos\theta_n = 1-2x_n$

then $x_{n+1} = 4x_n(1-x_n)$

$$= 4\left(\tfrac{1}{2}(1-\cos\theta_n) - \tfrac{1}{4}(1-\cos\theta_n)^2\right)$$

$$= 2-2\cos\theta_n - (1-2\cos\theta_n + \cos^2\theta_n)$$

$\frac{1}{2} - \frac{1}{2}\cos\theta_{n+1} = 1 - \cos^2\theta_n$

$$\Rightarrow \cos\theta_{n+1} = (2-2\cos^2\theta_n - 1)(-1)$$

$$= 2\cos^2\theta_n - 1$$

$$= \cos(2\theta_n)$$

by half angle formula.

$\rightarrow \theta_{n+1} = 2\theta_n$

MORAL OF STORY: Logistic Map is a reparametrization of the map $\theta \rightarrow 2\theta \pmod{2\pi}$ in $[0, 2\pi)$.

P.31 The behavior of the iterations is the same independent of x_0.

for $\varepsilon = 0.001$ after 1 step the difference is already larger than $1/2$.

for $\varepsilon = 1e-15$, it takes ~ 50 steps for the error to accumulate to $1/2$.