Homework #3

P. ...

2.3 Was on last homework sorry. for the error.

A. let $\lambda, \bar{v}$ be an eigenpair for the matrix AB

ie $AB\bar{v} = \lambda\bar{v}$

multiplying both sides by B we se

$BAB\bar{v} = \lambda B\bar{v}$

let $\bar{u} = B\bar{v}$ so $BA\bar{u} = \lambda\bar{u}$

thus λ is an eigenvalue of BA with eigenvector $\bar{u}$.

This proof does not work for $\lambda = 0$ since $AB\bar{v} = 0$ could mean $\bar{v} = 0$. which cannot be an eigenvector.

So we look to the characteristic eqn. under the asumption $\lambda = 0$ is an eigv.

$\det(AB - \lambda I) = 0 \Rightarrow \det(AB) = 0$.

So $0 = \det(AB) = \det(A)\det(B) = \det(B)\det(A)$

$= \det(BA)$.

thus 0 is an eigenvalue of BA. //

T2.7 $b = 0.4$ $f(x,y) = \begin{pmatrix} a - x^2 + 0.4y \\ x \end{pmatrix}$

a)

The fixed pts for the Hénon map are

$$x = \frac{-0.6 \pm \sqrt{0.36 + 4a}}{2}.$$

for $-0.09 < a < .27$, $0 < \sqrt{0.36 + 4a} < 1.2$

This means $x_1 = \frac{-0.6 + \sqrt{0.36 + 4a}}{2}$ is stn.

$x_1 \in (-0.3, 0.3)$

Likewise $x_2 = \frac{-0.6 - \sqrt{0.36 + 4a}}{2}$ is st

$x_2 \in (-0.9, -0.3)$

Now $Df = \begin{bmatrix} -2x & 0.4 \\ 1 & 0 \end{bmatrix}$. fix x & find eigenvalues

$$\begin{vmatrix} -2x - \lambda & 0.4 \\ 1 & -\lambda \end{vmatrix} = -\lambda(-2x - \lambda) - 0.4$$

$$= \lambda^2 + 2\lambda x - 0.4 = 0$$

$$\Rightarrow \lambda = \frac{-2x \pm \sqrt{4x^2 + 1.6}}{2}$$

for x_1, $\lambda_1 = \frac{-2x + \sqrt{4x^2 + 1.6}}{2} \in (0.4, 1)$

$\lambda_2 = \frac{-2x - \sqrt{4x^2 + 1.6}}{2} \in (-1, -0.4)$

$\Rightarrow$ So this fixed pt is stable for X_2 $\lambda_1 \in (1, 2)$

$\lambda_2 \in (-0.4, -0.2)$

Thus X_2 is unstable

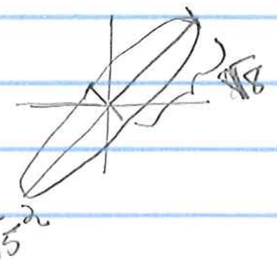
2.8

a) $A = \begin{pmatrix} 2 & 0.5 \\ 2 & -0.5 \end{pmatrix}$

find eigenpairs of AA^T.

$\lambda_1 = 0.5 \quad \bar{V}_1 = \sqrt{2}/2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda_2 = 8.0 \quad \bar{V}_2 = \sqrt{2}/2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



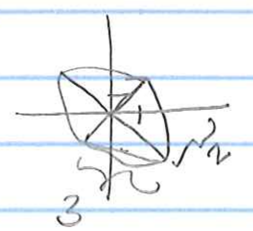
Area of ellipse is $\pi |\det A|$
$= 2\pi$

b) $A = \begin{pmatrix} 2 & 1 \\ -2 & 2 \end{pmatrix}$

eigenpairs of AA^T

$\lambda_1 = 4 \quad \bar{V}_1 = \begin{pmatrix} -0.8944 \\ -.4472 \end{pmatrix}$

$\lambda_2 = 9 \quad \bar{V}_2 = \begin{pmatrix} -0.4472 \\ 0.8944 \end{pmatrix}$



Area $= \pi |\det A| = 6\pi$

T2.8 Find the inverse of the Hénon map.
Does it work if $b=0$.

$f(x,y) = \binom{u}{v}$. swap (x,y) & (u,v) solve for u,v

$$f(u,v) = \begin{pmatrix} a - u^2 + bv \\ u \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\Rightarrow y = u$

$a - y^2 + bv = x \Rightarrow v = b^{-1}(a - y^2 + x)$

$$\Rightarrow f^{-1}\binom{x}{y} = \begin{pmatrix} b^{-1}(a - y^2 + x) \\ y \end{pmatrix}$$

This does not work for $b=0$.

T2.10 $A = \begin{pmatrix} 2/3 & 1 \\ 0 & 2/3 \end{pmatrix}$

eigenpairs of AA^T

$\lambda_1 = 0.111 \quad \bar{v}_1 = \begin{pmatrix} 0.4472 \\ -0.8944 \end{pmatrix} \Rightarrow \sqrt{\lambda_1} = 1/3$

$\lambda_2 = 1.778 \quad \bar{v}_2 = \begin{pmatrix} -0.8944 \\ -0.4472 \end{pmatrix} \Rightarrow \sqrt{\lambda_2} = 4/3$.

So ellipse exits unit circle on 1st application of A.

BUT we know that the origin is a sink since the eigenvalues $(2/3)$ are less than 1

This means that repeated application must cause the ellipse to shrink.

2.9

Find inverse cat map of $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Verify answer by composing w/ cat map.

from Math 22/24. we know $A^{-1} = \frac{1}{\det A}\begin{pmatrix} b & -d \\ -c & a \end{pmatrix}$ otherwise we can derive it.

$A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ solve for u & v

$A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

$A^{-1} \circ A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = A^{-1}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ✓

$A^{-1} \circ A \begin{pmatrix} 0 \\ 1 \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ✓

likewise verify $A \circ A^{-1} e_1 = e_1$, $A \circ A^{-1} e_2 = e_2$

$A^{-1} e_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $A^{-1} e_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

A^{-1} cat map.

Challenge 2

Step 6. $tr(A) = a+d.$

fixed pts $\bar{v}$ are solutions to $(A-I)\bar{v} = 0.$
per Step 5 the number of times this happens is $|\det(A-I)|$

$$\det(A-I) = \begin{vmatrix} a-1 & b \\ c & d-1 \end{vmatrix} = (a-1)(d-1) - bc = \underbrace{ad-bc}_{\det(A)} - \underbrace{(a+d)}_{Tr(A)} + 1$$

$$\Rightarrow |\det(A-I)| = |\det(A) - Tr(A) + 1|$$

Step 8

Goal: Find all fixed pts & period 2 orbits of cat map.
let x & y be rational #s. ie $x = p/q$ $y = r/s$
$p, q, r, s \in \mathbb{Z}$

(x,y) a fixed pt of $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ implies

① $2p/q + r/s = p/q + m$ for $m, n \in \mathbb{Z}$
② $p/q + r/s = r/s + n$

② $\Rightarrow p/q = n$ an integer. Plug into ①.
$r/s = m - n$ an integer.
Only way this is possible in $[0,1)^2$ is if $x = y = 0.$ $\Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the fixed pt.

period 2

$5\,p/q + 3\,r/s = m + p/q \rightarrow 4\,p/q + 3\,r/s = m$ (1)

$3\,p/q + 2\,r/s = n + r/s \Rightarrow 3\,p/q + r/s = n$ (2)

elminate. r/s

$-(4\,p/q + 3\,r/s = m)$

$\underline{9\,p/q + 3\,r/s = 3n}$

$\rightarrow 5\,p/q = (3n-m) = \text{integer. } (\text{mod }1) = 0 (\text{mod }1)$

$\Rightarrow 5\,p/q = 0. \quad p/q \in (0,1)$ & rational.

$\Rightarrow x = 1/5,\ 2/5,\ 3/5,\ 4/5$

by (2) $\quad 3x + y = \text{int}(\text{mod }1) = 0$

$3(1/5) + y = 0 \rightarrow y = 2/5$

$3(2/5) + y = 6/5 + y = 0 \Rightarrow y = 4/5$

$3(3/5) + y = 9/5 + y = 0 \Rightarrow y = 1/5$

$3(4/5) + y = 12/5 + y = 0 \Rightarrow y = 3/5$

fixedpts of A^2 are $(1/5, 2/5), (2/5, 4/5), (3/5, 1/5), (4/5, 3/5)$

NOTE: $(0,0)$ is also a fixed pt.

Now match the orbits.

$A(1/5, 2/5) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1/5 \\ 2/5 \end{pmatrix} = \begin{pmatrix} 2/5 + 2/5 \\ 1/5 + 2/5 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix}$

so, $\{(1/5, 2/5), (4/5, 3/5)\}$ & $\{(2/5, 4/5), (3/5, 1/5)\}$.

are the 2-periodic orbits.

Step 9

# of fixed pts. is $\det|S^n - I|$

$$\begin{vmatrix} F_{2n}-1 & F_{2n-1} \\ F_{2n-1} & F_{2n-2}-1 \end{vmatrix} = \left[(F_{2n}-1)(F_{2n-2}-1) - F_{2n-1}^2\right]$$

For simpler formula use info in step 10.

Step 11

k	# of fixedpts.	# of periodic orbits
1	1	1
2	5	2
3	16	5
4	45	10
5	121	24
6	320	50