2.8: Differentiation Rules

Mathematics 3
Lecture 9
Dartmouth College

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Recall: The Derivative Function

Given a function $y = f(x)$ we derive a new function, called the derivative of f, given by

$$\frac{df}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

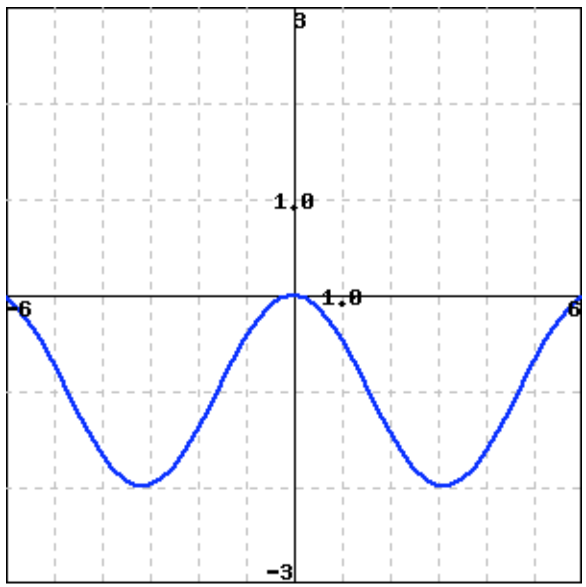
Geometrically, $f'(x) =$ slope of the tangent line to graph of f at the point $(x, f(x))$.

Example 1 (yesterday) Recall the piecewise defined function

$$f(x) = \begin{cases} x, & x \leq 1 \\ 1, & 1 < x < 3 \\ -x + 4, & x \geq 3 \end{cases} \implies f'(x) = \begin{cases} 1 & x < 1 \\ 0 & 1 < x < 3 \\ -1 & x < 3 \end{cases}$$

Example 2

Give of a (rough) sketch of the graph of the derivative function of the following function $y = f(x)$ with graph



Building the Calculus Toolbox

Theorem. *Suppose $y = f(x)$ is a function that has derivative f'. Then,*

$$(cf)' = cf',$$

where c is any constant. Or in Leibniz's notation

$$\frac{d}{dx}(cf(x)) = c \cdot \frac{d}{dx} f(x).$$

Example 3 Find

$$\frac{d}{dx}(7x^2)$$

$$\frac{d^3}{dx^3}(-2x^5)$$

The Addition Rule

Theorem. *If f and g are functions with derivatives f' and g', respectively, then*

$$(f+g)' = f' + g'.$$

In words, the derivative of a sum is the sum of the derivatives.

Example 4 Compute

$$\frac{d}{dx}(3x^2 + 2x - 17)$$

$$\frac{d}{dt}(t - 6\sqrt{t})$$

The Product Rule

Theorem. *If f and g are functions with derivatives f' and g', respectively, then*

$$(fg)' = fg' + gf'.$$

In words, "the derivative of a product is the first factor times the derivative of the second, plus the second factor times the derivative of the first".

WARNING:

$$\frac{d(fg)}{dx} \neq \frac{df}{dx}\frac{dg}{dx}.$$

Example 5

- Find $f'(x)$ in two ways, given $f(x) = (5x+3)(x+2)$.

- If $y = \sqrt{u}(u^2+2)$, find $\frac{dy}{du}$.

The Reciprocal Rule

Theorem. *Suppose f has derivative f'. Then for any x such that $f(x) \neq 0$,*

$$\left(\frac{1}{f}\right)' = -\frac{f(x)'}{f(x)^2}.$$

That is,

$$\frac{d}{dx}\left(\frac{1}{f}\right) = -\frac{\frac{df}{dx}}{(f(x))^2}.$$

Example 6

- Find $D_x f$ given $f(x) = \frac{1}{x^2+1}$.

The Quotient Rule

Theorem. *Suppose* f *and* g *have derivatives* f' *and* g'*, respectively. Then for any* x *such that* $g(x) \neq 0$*,*

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f(x)' - f(x)g(x)'}{g(x)^2}.$$

That is,

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}.$$

In words, "the derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator all divided by the denominator squared".

Example 7

- Find $f'(x)$ given
$$f(x) = \frac{x+1}{x+2}.$$

- Find the slope of the tangent line to
$$y(w) = \frac{1+\sqrt{w}}{w^2+3w+2}.$$

  at the point $(1, \frac{1}{3})$.

Example 8

- For $f(x) = \frac{1}{x} = x^{-1}$, find the derivative three ways, using the power rule, the reciprocal rule, and the quotient rule.

The Chain Rule

Theorem. *Let $(f \circ g)(x) = f(g(x))$ be the function defined from f and g by composition. Assume that g is differentiable at the point x and that f is differentiable at the point $g(x)$. Then the composite function $f \circ g$ is differentiable at the point x, and*

$$(f \circ g)'(x) = [f(g(x))]' = f'(g(x))g'(x)$$

Substitute $u = g(x)$ so that $y = (f \circ g)(x) = f(g(x)) = f(u)$. Using Leibniz's notation:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example 9

- Differentiate

$$f(x) = \sqrt{x^2 + 1}.$$

- Find y' where

$$y = (x^2 + 2)^{10}.$$

Example 10

- Differentiate
$$F(t) = (1 + 3\sqrt{t})^{35}.$$

- Find the tangent line $y = mx + b$ to
$$f(x) = \left(\frac{x+1}{x^2+1}\right)^3.$$
at the point $(0, 1)$.

Differentiability is Stronger than Continuity

Theorem. *If $f'(a)$ exists, then f is continuous at a.*

A function whose derivative exists at every point of an interval is not only continuous, it is smooth, i.e. it has no “sharp corners”.

Have a good weekend!