Math 71 - Homework 4 - Partial Solutions

85/1 $E \lhd H \Rightarrow \varphi^{-1}(E) \lhd G$:
Let $x \in G$, $y \in \varphi^{-1}(E)$ Show $xyx^{-1} \in \varphi^{-1}(E)$
$\varphi(xyx^{-1}) = \varphi(x)\varphi(y)(\varphi(x))^{-1} \in E$ since $E \lhd H$ and
$\varphi(y) \in E$ $\therefore xyx^{-1} \in \varphi^{-1}(E)$

85/5 $|gN| = n \Leftrightarrow n$ smallest positive integer such that $(gN)^n =$
N. $(gN)^n = N \Leftrightarrow g^n N = N \Leftrightarrow g^n \in N$
Example (one of many): $\langle \bar{3} \rangle \in \mathbb{Z}_{12}$ Consider
$\mathbb{Z}_{12}/\langle \bar{3} \rangle$ $|\bar{2}| = 6$ (in $\mathbb{Z}_{12}$) $|\bar{2} + \langle \bar{3} \rangle| = 3$ in
$\mathbb{Z}_{12}/\langle \bar{3} \rangle$.

85/9 $\operatorname{Im} \varphi = \mathbb{R}^{>0}$, $\operatorname{Ker} \varphi = S^1$ (unit circle in $\mathbb{C}^*$)
$r > 0$ positive real Fiber of r circle of radius $\sqrt{r}$

86/10 Write $\varphi(a + 8\mathbb{Z}) = a + 4\mathbb{Z}$. Show φ well-defined:
$a + 8\mathbb{Z} = b + 8\mathbb{Z}$ $\therefore 8 \mid a-b$ $\therefore 4 \mid a-b$ $\therefore$
$a + 4\mathbb{Z} = b + 4\mathbb{Z}$ $\therefore \varphi$ well-defined. $\operatorname{Ker} \varphi = \{8\mathbb{Z}, 4 + 8\mathbb{Z}\}$
fiber over $1 + 4\mathbb{Z}$ is $\{1 + 8\mathbb{Z}, 5 + 8\mathbb{Z}\}$.

88/32 $Q_8 / \langle i \rangle$ has order 2 $\therefore$ Isomorphic to $\mathbb{Z}_2$ (same for
j, k) $Q_8/\{\pm 1\}$ has order 4 $\therefore$ Isomorphic to $\mathbb{Z}_4$ or
$\mathbb{Z}_2 \times \mathbb{Z}_2$. Show it has no element of order 4.

88/34 Let r^i be the rotations in D_{2n} and let sr^i be the reflections
Show $r^i \langle r^k \rangle r^{-i} \subseteq \langle r^k \rangle$ and $sr^i \langle r^k \rangle r^{-i} s \subseteq \langle r^k \rangle$.
Now let ρ^i be the rotations in D_{2k} and $s\rho^i$ the reflections
in D_{2k}. Define $\phi(r^1) = \rho^1$, $\phi(r^2) = \rho^2$, ...,
$\phi(r^{k-1}) = \rho^{k-1}$, $\phi(r^k) = \rho^0$, ... and similarly
$\phi(sr^1) = \sigma\rho^1$. $\phi(sr^2) = \sigma\rho^2$, ...
Now apply the first isomorphism to ϕ.

89/35 Consider $\det : GL(n, \mathbb{F}) \longrightarrow \mathbb{F}^*$ and apply the first isomorphism theorem.

89/38 Define $\phi : A \times A \to A$ by $\phi(a,b) = a-b$

95/5 (a) Define $\theta : H \to gHg^{-1}$ by $\theta(h) = ghg^{-1}$. Show θ is an isomorphism.

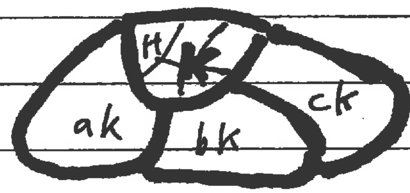
(b) H and gHg^{-1} are subgroups of order n. Since there is exactly one subgroup of order n, $H = gHg^{-1}$

95/8 Let $|H| = m$, $|K| = l$. $\exists A, B$ $Am + Bl = 1$

Let $x \in H \cap K$. Then $x = x^1 = (x^m)^A (x^n)^B = 1$

96/11 Here is a picture – you provide the proof.

$H \leq K \leq G$



G

Here are the cosets of K in G (K, aK, bK, cK) and the cosets of H in K (unlabeled but call them H, $\alpha H, \beta H, \gamma H$).

Consider the function $L_a : K \to aK$ left mult. by a. It is a bijection. The cosets of H which are in aK are $L_a(H)$, $L_a(\alpha H), L_a(\beta H), L_a(\gamma H) = aH, a\alpha H, a\beta H, a\gamma H$.

Similarly for bK and cK. Nuff said.

96/15 $G = S_n$ Fix i, $1 \leq i \leq n$. Then G_i are all permutations in S_n which fix i. Note S_{n-1} is isomorphic to G_n, all permutation in S_n which fix n. Therefore it suffices to show $G_i \cong G_n$

Given $\sigma \in G_i$, then

$$\sigma = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & n \\ \sigma(1) & \cdots & \sigma(i-1) & i & \sigma(i+1) & & \sigma(n) \end{pmatrix}$$

assign to σ a perm. $\rho \in G_n$ as follows: From the above array delete $\begin{smallmatrix} i \\ i \end{smallmatrix}$, leave the integers $1, \ldots, i-1$ unchanged, replace the integers $i+1, \ldots, n$ by $i, \ldots, n-1$ and add the column $\begin{smallmatrix} n \\ n \end{smallmatrix}$ on the right. Then $\rho \in G_n$. This defines a function $G_i \to G_n$.