1. (15) Let $\mathbf{F}(x,y) = \langle y\cos(xy) + e^x, x\cos(xy) + 2y\rangle$.

(a) Determine whether or not $\mathbf{F}$ is conservative. If it is, then find a potential function f. If $\mathbf{F}$ is not conservative, then explain why not.

ANS: Since $P_y = Q_x$, $\mathbf{F}$ must be conservative. To find f, use partial integration. We must have $f_x = y\cos(xy) + e^x$. Hence

$$f(x,y) = \sin(xy) + e^x + C(y).$$

Comparing $f_y(x,y) = x\cos(xy)$ with $Q(x,y) = x\cos(xy) + 2y$, we see that $C'(y) = 2y$. Hence $C(y) = y^2 + c$ and $\boxed{f(x,y) = \sin(xy) + e^x + y^2 + c}$.

$f =$ ______________________________

(b) Let $\mathbf{F}$ be the vector field from the previous page and C the upper left-hand quarter of the circle $(x-1)^2 + y^2 = 1$ oriented from $(0,0)$ to $(1,1)$. Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} = I = \int_C (y\cos(xy) + e^x)\,dx + (x\cos(xy) + 2y)\,dy.$$

ANS: By the Fundamental Theorem for Line Integrals,

$$I = f(1,1) - f(0,0) = \sin(1) + e + 1 - 1 = \boxed{\sin(1) + e}.$$

$I =$ ______________________________

2. (10) Let C be the curve consisting of the line segments from $(0,0)$ to $(1,1)$, from $(1,1)$ to $(0,1)$ and from $(0,1)$ back to $(0,0)$. Compute

$$I = \int_C (e^{x^2} + y^2)\,dx + (3xy + \cos(y^3)\,dy.$$

ANS: By Green's Theorem

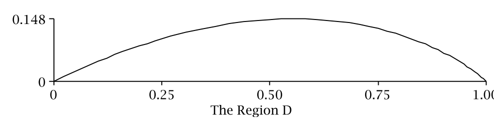
$$\begin{aligned} I = \iint_D (Q_x - P_y)\,dA &= \iint_D (3y - 2y)\,dA = \iint_D y\,dA \\ &= \int_0^1 \int_x^1 y\,dy\,dx \\ &= \frac{1}{2}\int_0^1 (1 - x^2)\,dx \\ &= \frac{1}{2}(1 - \frac{1}{3}) = \boxed{\frac{1}{3}}. \end{aligned}$$

$I = $:______________________________

3. (10) Find the area of the region D sketched below which lies in the first quadrant of the xy-plane and is enclosed by the curve C described by $\vec{r}(t) = \langle 1-t^2, t^2-t^3\rangle$ for $t \in [0,1]$ and the x-axis. (Suggestion: use Green's Theorem.)

ANS: Let C_1 be the *positively oriented* curve consisting of the line segment C_2 from $(0,0)$ to $(1,0)$ followed by C (note that C goes from $(1,0)$ to $(0,0)$). Then by Green's Theorem

$$\begin{aligned}\text{Area} &= \int_{C_1} x\,dy = \int_{C_2} x\,dy + \int_C x\,dy \\ &= 0 + \int_0^1 (1-t^2)(2t-3t^2)\,dt \\ &= (1 - 1 - \frac{1}{2} + \frac{3}{5}) \\ &= \boxed{\tfrac{1}{10}}.\end{aligned}$$



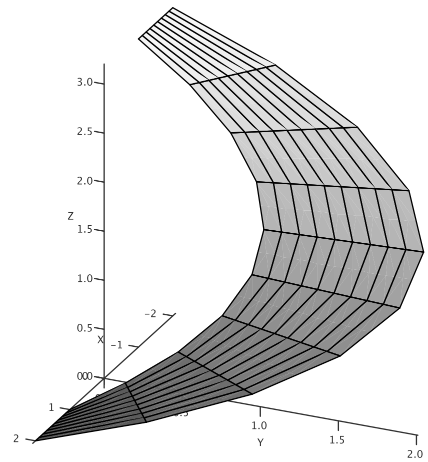
Area = ____________________

4. (10) Let S will be the surface (a helicoid) parameterized by

$$\mathbf{r}(u,v) = \langle u\cos(v), u\sin(v), v\rangle \quad \text{for } u \in [1,2] \text{ and } v \in [0,\pi].$$

A computer sketch of S is given at left.

Express the surface area of S as an iterated integral. ***Do not try to evaluate the integral***.



ANS: Here

$$\begin{aligned}\mathbf{r}_u &= \langle \cos(v), \sin(v), 0\rangle \\ \mathbf{r}_v &= \langle -u\sin(v), u\cos(v), 1\rangle.\end{aligned}$$

Hence

$$\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u\rangle.$$

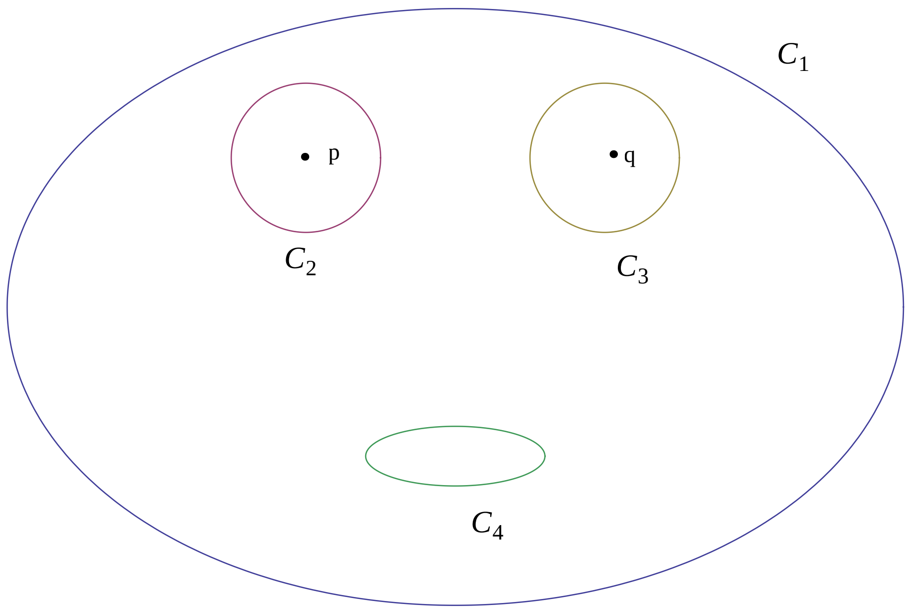
Thus

$$\begin{aligned}A(S) &= \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA = \iint_D \sqrt{1+u^2}\, dA \\ &= \boxed{\int_0^{\pi}\int_1^2 \sqrt{1+u^2}\, du\, dv}.\end{aligned}$$

$A(S) =$ ______________________________

(Your answer should be an iterated integral)

5. (8) Let C_1 be a smooth positively oriented simple closed curve containing the smooth positively oriented simple closed curves C_2, C_3 and C_4 in its interior so that the interiors of C_2, C_3 and C_4 do not overlap as shown below.



Suppose that P and Q have continuous partial derivatives on the region D consisting of all of $\mathbf{R}^2$ with the exception of points p inside C_2 and q inside C_3, and that

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

on all of D. Given that

$$\int_{C_1} P\,dx + Q\,dy = 5 \quad \text{and} \quad \int_{C_2} P\,dx + Q\,dy = 2,$$

then find the values for the following two line integrals:

$$\int_{C_3} P\,dx + Q\,dy = \underline{\hspace{5cm}} \quad \text{and} \quad \int_{C_4} P\,dx + Q\,dy = \underline{\hspace{5cm}}$$

ANS: By Green's Theorem for Multiply connected regions (see pages 1112-3 of the text or your lecture notes),

$$0 = \iint_D (Q_x - P_y)\,dA = \int_{C_1} P\,dx + Q\,dy - \int_{C_2} P\,dx + Q\,dy - \int_{C_3} P\,dx + Q\,dy - \int_{C_4} P\,dx + Q\,dy$$

However, we can apply Green's Theorem again to $\int_{C_4} P\,dx + Q\,dy$ to conclude that its value is 0. Hence the value of $\int_{C_3} P\,dx + Q\,dy$ must be 3.

6. (6) Let $f : \mathbf{R}^3 \to \mathbf{R}$ be a scalar valued function and $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$ a vector field. For each of the following indicate in the blank provided whether the expression below if meaningful (**Y**) or not (**N**).

(a) ____________ $\boldsymbol{\nabla}(\operatorname{div} \mathbf{F})$.

(b) ____________ $\operatorname{div}(\boldsymbol{\nabla} f)$.

(c) ____________ $\boldsymbol{\nabla}(\operatorname{div} f)$.

(d) ____________ $\operatorname{curl}(\operatorname{curl}(\mathbf{F}))$.

(e) ____________ $\operatorname{div}(\operatorname{div}(\mathbf{F}))$.

(f) ____________ $\boldsymbol{\nabla} f \times (\operatorname{div} \mathbf{F})$.

ANS: YYNYNN

7. (8) Let $\mathbf{F}(x, y, z) = \langle xy, x^2 + z^2, x^2 z\rangle$. Compute (a) $\operatorname{curl} \mathbf{F}$ and (b) $\operatorname{div} F$.

(a) $\operatorname{curl} \mathbf{F} =$ ____________________

ANS: $\boxed{\langle -2z, -2xz, x\rangle}$.

(b) $\operatorname{div} F =$ ____________________

ANS: $\boxed{y + x^2}$.

8. (15) Evaluate the following line integrals using the appropriate definition. In each case, C is the curve parameterized by $\mathbf{r}(t) = \langle t, \cos 2t, \sin 2t\rangle$ with $t \in [0, \pi]$.

(a) $I_1 := \displaystyle\int_C x \, ds.$

ANS: $\int_0^\pi t\sqrt{1+4}\, dt = \boxed{\sqrt{5}\frac{\pi^2}{2}}$.

$I_1 =$ ____________________

(b) $I_2 := \displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = \langle y^2 + z^2, -z, y\rangle$.

ANS: $\int_0^\pi \langle 1, -\sin 2t, \cos 2t\rangle \cdot \langle 1, -2\sin 2t, 2\cos 2t\rangle = \int_0^\pi 1 + 2\, dt = \boxed{3\pi}$.

$I_2 =$ ______________________

(c) $I_3 := \displaystyle\int_C y\,dx.$

ANS: $\int_0^\pi \cos(2t)\,dt = \boxed{0}$.

$I_2 =$ ______________________

9. (8) Match each function with the plot of its gradient vector field by placing the appropriate letter in the blank provided.

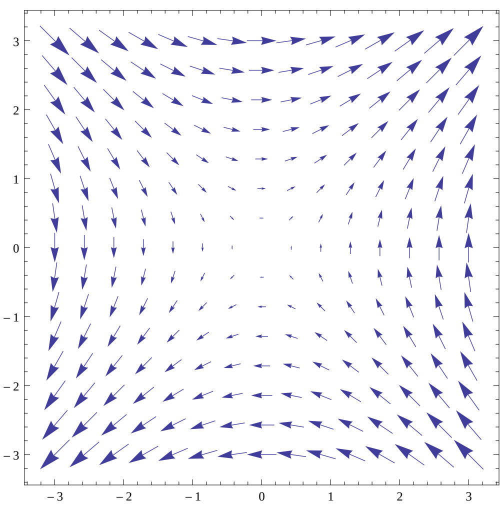
(a) ____________ $f(x, y) = x^2 + y^2$.

(b) ____________ $f(x, y) = x + y$.

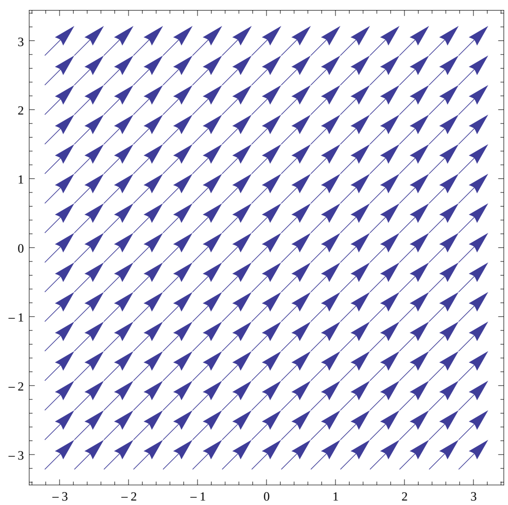
(c) ____________ $f(x, y) = e^x$.

(d) ____________ $f(x, y) = xy$.

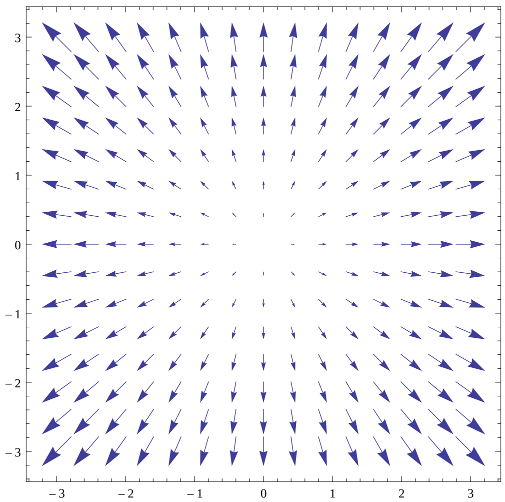
ANS: CBDA



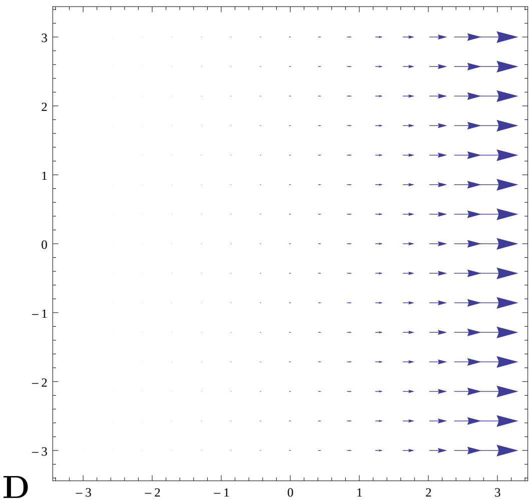
A



B



C



D

10. (10) MULTIPLE CHOICE. Circle the best response. No partial credit will be given on this problem and you do not need to justify your answers.

(a) Which of the following vector fields could be the curl of another vector field $\mathbf{G}$?

A. $\langle x, y, z\rangle$ **B**. $\langle y, z, z\rangle$ **C**. $\langle y^2x, z^2y, x^2z\rangle$ **D**. $\langle z, z, z\rangle$

E. None of These

(b) Let S be the surface parameterized by $\mathbf{r}(u, v) = \langle u^2, v^2, u + 2v\rangle$. Which of the following is the tangent plane to S at the point $(1, 1, 3)$?

A. $x + 2y - 3z + 6 = 0$ **B**. $2x - y + z - 4 = 0$ **C**. $x + y + z - 5 = 0$

D. $x + 2y - 2z + 3 = 0$ **E**. None of These

(c) Let S be the surface formed by the part of the graph of $z = x^2 + y^2$ under the plane $z = 1$. Then the surface area of S is given by which of the following integrals?

A. $\displaystyle\int_0^{2\pi}\int_0^1 r\sqrt{1+4r^2}\,dr\,d\theta$ **B**. $\displaystyle\int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\,dr\,d\theta$

C. $\displaystyle\int_0^{2\pi}\int_{r^2}^1 r\,dr\,d\theta$ **D**. $\displaystyle\int_0^{2\pi}\int_{r^2}^1 r\sqrt{1+4r^2}\,dr\,d\theta$ **E**. None of These

(d) Let C be the curve parameterized by $\mathbf{r}(t) = \langle t^2, t^3\rangle$ for $t \in [0, 1]$. Let $\mathbf{F}(x, y) = \langle y, x\rangle$. Then which of the following is the value of $\displaystyle\int_C \mathbf{F}\cdot\mathbf{T}\,ds$?

A. 1 **B**. 2 **C**. $\sqrt{5}$ **D**. $\sqrt{2}+1$ **E**. None of these

(e) Let C be the smooth positively oriented boundary of a region D. Which of the following is *not* equal to the area of D?

A. $\int_C -y\,dx$ **B**. $\int_C 2xy\,dx + (x^2+x)\,dy$ **C**. $\int_C x\,dy$

D. $\frac{1}{2}\int_C x\,dy - y\,dx$ **E**. All of these give the area

ANS: EDAAE

NAME : ______________________________

SECTION : (circle one) Cho Williams

Math 13

25 February 2013

Hour Exam II

INSTRUCTIONS: This is a closed book, closed notes exam. You are not to provide or receive help from any outside source during the exam except that you may ask either instructor for clarification. You have two hours and you should attempt all 10 problems.

- Wait for signal to begin.
- ***Print*** your name in the space provided and circle your instructor's name.
- Sign the FERPA release below *only if* you wish your exam returned in lecture.
- Calculators or other computing devices are not allowed.
- On problems 1 through 4, you must show your work and justify your assertions to receive full credit. On the "SHORT ANSWER" section, problems 5 – 10, you do not need to show your work and there will be little or no partial credit.
- *Place your final answer in the space provided*!

FERPA RELEASE: Because of privacy concerns, we are not allowed to return your graded exams to the homework boxes without your permission. If you wish us to return your exam via your homework box, please sign on the line indicated below. Otherwise, you will have to pick your exam up in your instructor's office after the exams have been returned in lecture.

SIGN HERE: ______________________________.

Problem	Points	Score
1	15	
2	10	
3	10	
4	10	
5	8	
6	6	
7	8	
8	15	
9	8	
10	10	
Total	100	