1. General Outline for **Polynomial Hecke Operators**

- Why coset multiplication is onerous
- Spherical map for $\mathrm{GL}_n(\mathbf{Z})$
- Hecke operators for $\mathrm{GL}_n(\mathbf{Z})$
  - Definition and result
  - Image under ω
- Going backwards
  - Define t_k^n
- Why t_k^n
  - They have a nice generating series with Euler product expansion
  - They have nice recursion relations
  - Generate classical Hecke operators.
- What I'm trying to do

2. Review of generalized Hecke operators

Recall that the Hecke operator $T(m)$ is defined as

$$\sum_{\alpha} \Gamma\alpha\Gamma$$

where $\alpha \in M_n^+(\mathbf{Z})$, $\Gamma = \mathrm{GL}_n(\mathbf{Z})$ in this case, and $\det\alpha = m$.

Remember that for a matrix β to be in the double coset $\Gamma\alpha\Gamma$ is implied by saying that α and β have the same Smith Normal form.

3. Coset Multiplication

Last week I called the set of linear combinations

$$\sum c_k \Gamma\alpha_k\Gamma$$

the Hecke algebra relative to Γ and $\Delta = M_n^+(\mathbf{Z})$. For this to be an algebra, there needs to be a multiplication. For $u = \Gamma\alpha\Gamma$ and $v = \Gamma\beta\Gamma$ we define $u \cdot v$ to be

$$u \cdot v = \sum m(u \cdot v; w) w,$$

where the sum is over all $w = \Gamma\xi\Gamma \subset \Gamma\alpha\Gamma\beta\Gamma$. To prove this is well defined is a chore. To carry out a multiplication is close to impossible.

It can be shown, though, that this multiplication has all the desired properties when restricted to the $T(m)$ and to the case where $\Gamma = \mathrm{GL}_2(\mathbf{Z})$. I.e., the multiplication is commutative, $T(m)T(n) = T(mn)$ for relatively prime m, n.

4. Spherical map for $\mathrm{GL}_n(\mathbf{Z})$

Again, let $\Gamma = \mathrm{GL}_n(\mathbf{Z})$ and $G = \mathrm{GL}_n(\mathbf{Z}[p^{-1}])$. Define

$$H_p = H(\Gamma, G) = \left\{ \sum c_k \Gamma\alpha_k\Gamma : \alpha \in G \right\}.$$

Recall that since $[\Gamma : G]\infty$ we see $\Gamma\alpha\Gamma$ has a coset decomposition

$$\bigcup_{i=1}^{\nu} \Gamma\alpha_i.$$

Adrianov proves that every coset in this decomposition has a unique representative

$$\begin{pmatrix} p^{b_1} & * & \cdots & * \\ 0 & p^{b_2} & \cdots & * \\ \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & p^{b_n} \end{pmatrix}$$

where the entries in the column are taken modulo the diagonal entry. He defines a map

$$\omega : \Gamma/G \to \mathbf{Q}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$$

on these representatives by

$$\omega\left(\begin{pmatrix} p^{b_1} & * & \cdots & * \\ 0 & p^{b_2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & p^{b_n} \end{pmatrix}\right) = p^{\sum ib_i} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}.$$

He proves that the restriction of ω to H_p, the Hecke ring, is an ismorphism onto

$$\mathbf{Q}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n}$$

the set of symmetric rational functions.

The full Hecke ring H_p can be thought of as being generated by the n operators

$$\pi_k^n(p) = \Gamma \mathrm{diag}(\underbrace{p, \ldots, p}_{k} \underbrace{1, \ldots, 1}_{n-k})\Gamma$$

for $k = 1 \ldots n$ and by $(\pi_n^n(p))^{-1}$.

The integral subring $\underline{H_p}$ is the ring just generated by the $\pi_k^n(p)$. Under ω, we have $\underline{H_p}$ is ismorphic to the ring of symmetric polynomials.

5. Going backwards

We have this map so let's use it. One application of this map is found in recent work of Rhodes and Shemanske. In this paper, they define families of polynomials $t_k^n(p^\ell)$ that have the following properties

- they have generating series that have Euler product expansions with trivial numerators
$$\sum_{\ell \geq 0} t_k^n(p^\ell) = \prod_{a \leq i_1 < i_2 < \cdots < i_k \leq n} (1 - x_{i_1} x_{i_2} \cdots x_{i_k})^{-1}$$
- the inverse image of t_k^n under ω is a linear combination of honest to goodness Hecke operators
- there are recursion relations among the t_k^n.

The paper demonstrates all these results. Before I talk more about these results, let me say a word on why we care about each.

Generating series are half a step away from Dirichlet series – just let $u = p^{-s}$ for $s \in \mathbf{C}$ and we get a Dirichlet series with Euler product expansion. Then we can ask the standard questions: E.g., does this series have an analytic continuation, what are the fourier coefficients of this expansion, etc.

Knowing that the inverse image of these polynomials are Hecke operators (or linear combinations of them) is inherently interesting. So multiplying these polynomials is nothing more than multiplying these Hecke operators and an onerous task is made easy.

The fact that these polynomial have recursion relations is just saying that the Hecke operators have recursion relations. In the case where $n = 2$, we get the classical recursion relation:

$$T(p^{r+1}) = T(p)T(p^r) - p^{k-1}T(p^{r-1})$$

which is a key step in showing that the Hecke operators in the elliptic case commute.

6. Definition of $t_k^n(p^\ell)$

First, some preliminaries.

- Note that the symmetric group S_n acts naturally on polynomials $\mathbf{Q}[x_1, x_2, \ldots, x_n]$ by
$$\sigma.x_i = x_{\sigma(i)}$$
and extend it appropriately.
- The stabilizer of an element under this action is
$$\mathrm{Stab}(p(x)) = \{\sigma \in S_n : \sigma.p(x) = p(x)\}.$$
The stabilizer is a subgroup of S_n.
- Define the symmetrized polynomials associated to $p(x)$ under S_n as
$$\mathrm{Sym}_n(p(x)) = \sum_{\sigma \in S_n/\mathrm{Stab}(p(x))} \sigma.p$$

Example 1. *Let $p \in \boldsymbol{Q}[x, y, z]$. Let $p = x + y + z$. Since p is itself symmetric, we get that $Stab(p) = S_n$. So then*

$$Sym_n(p) = \sum_{\sigma = id} \sigma(p) = p.$$

In general, then, if we start with p being symmetric, we get that $Sym_n(p) = p$.

Example 2. *Let $p = x + y + z^2 \in \boldsymbol{Q}[x, y, z]$. We can show that*

$$Stab(p) = \{\sigma_1 = id; \sigma_3 = x \mapsto y, y \mapsto x, z \mapsto z\}$$

. Then, we see futher that

$$S_n/Stab(p) = \{\sigma_1; \sigma_2 = (32); \sigma_4 = (123)\}.$$

Then

$$\begin{aligned} Sym_n(p) &= \sigma_1(p) + \sigma_2(p) + \sigma_3(p) \\ &= x + y + z^2 + x + z + y^2 + y + z + x^2 \\ &= 2x + 2y + 2z + x^2 + y^2 + z^2. \end{aligned}$$

So, Sym_n is intuitively the "smallest" way to make a polynomial into a symmetric one and retaining the "shape" of the initial one..

Example 3. *Based on that last piece of intuition, let's find $Sym_n(z_1 \cdots z_k)$. We need a symmetric polynomial that has the same shape (i.e., it has monomials with k indeterminates), and is symmetric. This leaves us with*

$$Sym_n(z_1 \cdots z_k) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} z_{i_1} z_{i_2} \cdots z_{i_k}.$$

- Let $P_n(m)$ denoted the set of partitions of m into n parts, s.t., $\mathbf{i} = (i_1, \ldots, i_n) \in P_n(m)$ iff $m \geq i_1 \geq \cdots \geq i_n \geq 0$ and $\sum i_k = m$. Give $P_n(m)$ the dictionary or lexicographic order.

Definition 1. For indeterminates $z_1, \cdots, z_n$ and $\mathbf{i} \in P_n(m)$, let

$$\begin{aligned} hp(0, \ldots, 0) &= 1 \\ hp(\mathbf{i}) &= \sum_{\substack{\mathbf{j} \leq \mathbf{i} \\ \mathbf{j} \in P_n(m)}} \mathrm{Sym}_n(z_1^{j_1} \cdots z_n^{j_n} \\ h^n(\ell) = hp(\ell, 0, \ldots, 0) &= \sum_{\mathbf{j} \in P_n(m)} \mathrm{Sym}_n(z_1^{j_1} \cdots z_n^{j_n}). \end{aligned}$$

Note that the last sum can be rewritten as

$$\sum_{\substack{\sum j_k = \ell \\ j_k \geq 0}} z_1^{j_1} \cdots z_n^{j_n}.$$

Then, we define

$$t_k^n(p^\ell) = h^{\binom{n}{k}}(\ell)|_{\substack{z_i \mapsto \sigma_i(x_1 \cdots x_k) \\ \sigma_i \in S_n/\mathrm{Stab}(x_1 \cdots x_k)}}.$$

This definition merits some explanation and an example. Note that $|S_n/\mathrm{Stab}(x_1 \cdots x_k)| = \binom{n}{k}$. So, all this definition is saying is that for each σ_i in that set (there are $\binom{n}{k}$ of them), we replace the z_i in the definition of $h^{\binom{n}{k}}(\ell)$ with $\sigma_i(x_1 \cdots x_k)$.

Example 4. *First, note that*

$$h^{\binom{n}{k}}(1) = hp(1, 0, \ldots, 0) = Sym_{\binom{n}{k}}(z_1) = \sum_{i=1}^{\binom{n}{k}} z_i.$$

We start with $h^{\binom{n}{k}}(\ell)$:

$$\begin{aligned} t_1^n(p) &= h^n(1)|_{\substack{z_i \mapsto \sigma_i(x_1) \\ \sigma_i \in S_n/Stab(x_1)}} \\ &= x_1 + x_2 + \cdots + x_n \\ &= s_1(x_1, \ldots, x_n) \end{aligned}$$

since $S_n/Stab(x_1)$ means that we don't count separately the permutations that fix x_1, so the coset reps are $(1x)$.

In general,

$$\begin{aligned} t_k^n(p) &= h^{\binom{n}{k}}(1)\Big|_{\substack{z_i\mapsto\sigma_i(x_1\cdots x_k)\\ \sigma_i\in S_n/Stab(x_1\cdots x_k)}} \\ &= \sum_{1\le i_1<i_2<\cdots<i_k\le n} x_{i_1}x_{i_2}\cdots x_{i_k} \\ &= s_k(x_1,\cdots,x_k) \end{aligned}$$

since in this case $S_n/Stab(x_1\cdots x_k)$ is the set of σ that send at least one $x_i\mapsto x_j$ where $j>i$.

7. Generating Series

The formal generating series induced by these $t_k^n(p^\ell)$ have Euler product factorizations with trivial numerators:

Theorem 1. *For the polynomials $t_k^n(p^\ell)$ defined above, we have*

$$\sum_{\ell\ge 0} t_k^n(p^\ell)u^\ell = \left[\prod_{\sigma\in S_n/Stab(x_1\cdots x_k)} (1-u\sigma(x_1\cdots x_k))\right]^{-1}.$$

Example 5. *For $n=4$, we get*

$$\sum_{\ell\ge 0} t_1^4(p^\ell)u^\ell = [(1-ux_1)(1-ux_2)(1-ux_3)(1-ux_4)]^{-1}$$

$$\sum_{\ell\ge 0} t_2^4(p^\ell)u^\ell = [(1-ux_1x_2)(1-ux_1x_3)(1-ux_1x_4)(1-ux_2x_3)(1-ux_2x_4)(1-ux_3x_4)]^{-1}$$

$$\sum_{\ell\ge 0} t_3^4(p^\ell)u^\ell = [(1-ux_1x_2x_3)(1-ux_1x_2x_4)(1-ux_1x_3x_4)(1-ux_2x_3x_4)]^{-1}$$

$$\sum_{\ell\ge 0} t_1^4(p^\ell)u^\ell = [(1-ux_1x_2x_3x_4)]^{-1}$$

8. Recursion Relations

Since the generating series is writeable as an Euler product expansion, moving the product over, we get

$$\left[\sum_{\ell\ge 0} t_k^n(p^\ell)u^\ell\right]\cdot\left[\prod_{\sigma_i\in S_n\mathrm{Stab}(x_1,\ldots,x_n)} (1-u\sigma(x_1\cdots x_k))\right] = 1.$$

In particular we can show the following proposition:

Proposition 1. *We have these equalities:*

- *For $k=1$ we get*

$$\begin{aligned} t_1^n(p^\ell) &= t_1^n(p^{\ell-1})s_1 - t_2^n(p^{\ell-2})s_2 + \cdots + (-1)^{n-1}t_1^n(p^{\ell-n})s_n \\ &= t_1^n(p^{\ell-1})t_1^n(p) - t_2^n(p^{\ell-2})t_2^n(p) + \cdots + (-1)^{n-1}t_1^n(p^{\ell-n})t_n^n(p) \end{aligned}$$

  since $t_k^n(p)$ is the kth elementary symmetric polynomial s_k.
- *For $k=n$ we get*

$$t_n^n(p^\ell) = [s_n(x_1,\ldots,x_n]^\ell = t_n^n(p)^\ell.$$

In general, using the fact that $\binom{n}{n-k} = \binom{n}{k}$ and the way in which t_k^n is defined in terms of $h^{\binom{n}{k}}$, a general duality can be developed. Using these relations, one can find all the t_k^n by starting with the base case and recursion relations like the ones in the proposition.

9. Recovering Hecke Operators

A useful computational result Rhodes and Shemanske reach is

Theorem 2. *If $\boldsymbol{b} \in P_n(m)$ then*

$$Sym_n(x^{\boldsymbol{b}}) = \sum_{\boldsymbol{a} \leq \boldsymbol{b}} d(a)\omega(\Gamma p^{\boldsymbol{a}}\Gamma$$

for constants $d(a)$. Here $x^{\boldsymbol{b}} = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ and $\Gamma p^{\boldsymbol{a}}\Gamma = \Gamma\, diag(p^{a_1}, p^{a_2}, \ldots, (p^{a_n})\Gamma$.

Using this theorem, one can find the inverse images of the symmetrized polynomials, and using the fact that the $t_k^n(p^\ell)$ are linear combinations of the symmetrized polynomials, one can find the preimages of the $t_k^n(p^\ell)$. These preimages are going to be linear combinations of the classical Hecke operators

$$\pi_k^n(p) = \Gamma \mathrm{diag}(\underbrace{p, \ldots, p}_{k}, \underbrace{1, \ldots, 1}_{n-k})\Gamma.$$

10. Conclusion

The overall purpose of this paper was to find viable candidates with which to pursue the standard questions. In particular, the authors wanted to find polynomials with properties mimicking those of Hecke operators: I.e., they had to satisfy some recursion relations and have generating series with Euler product expansion. With the polynomials in hand, the questions to ask now are whether the generating series with $u = p^{-s}$ has an analytic continuation and if the Fourier coefficients had any arithmetic properties of interest. The stage has been set for those kinds of questions to be asked. Just like the authors of this paper inverted the spherical map for the case when $\Gamma = \mathrm{GL}_n(\mathbf{Z})$ I'm trying to invert it in the case when $\Gamma = \mathrm{Sp}_n(\mathbf{Z})$.