Vector Functions, Space Curves, Derivatives, and Integrals

November 1, 2006

Vector Functions

- A vector function is a function $\mathbf{r}(t)$ whose domain is the set of real numbers and whose range is a set of vectors – in general three-dimensional vectors.

- If $f(t), g(t)$, and $h(t)$ are the components of the vector $\mathbf{r}(t)$, then they are called the **component functions** of $\mathbf{r}$.

- We write

$$\begin{aligned}\mathbf{r}(t) &= \langle f(t), g(t), h(t)\rangle \\ &= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}\end{aligned}$$

Limit of a vector function

- If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \mathbf{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$$

Example

- Find the limit

$$\lim_{t \to \pi/4} \langle \cos t, \sin t, t \rangle$$

Space Curves

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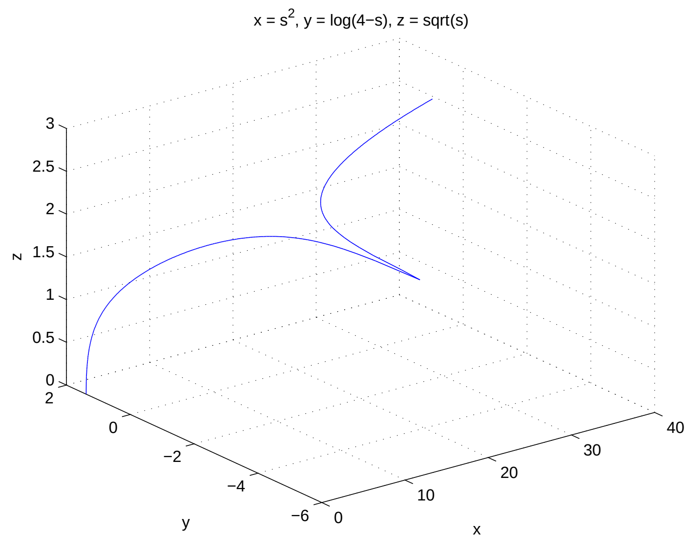

The Helix

- $r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$

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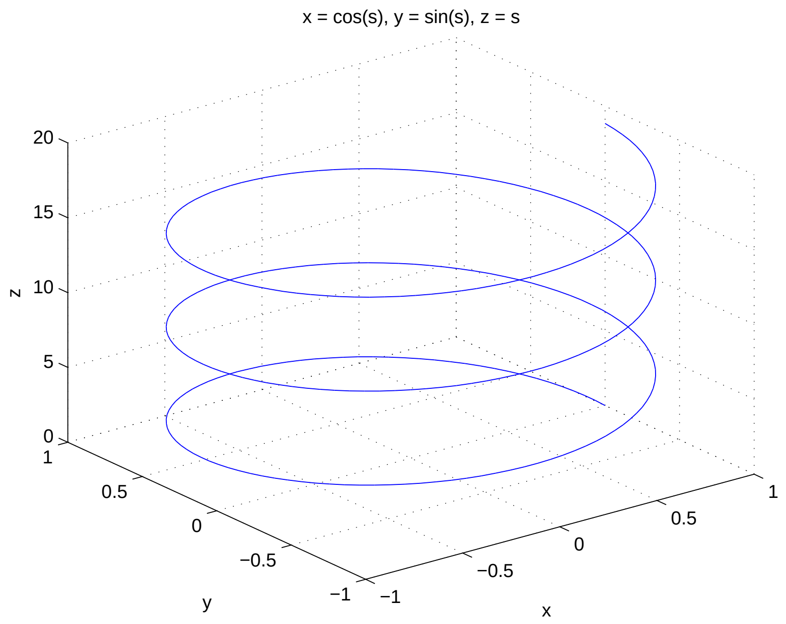

The Toroidal

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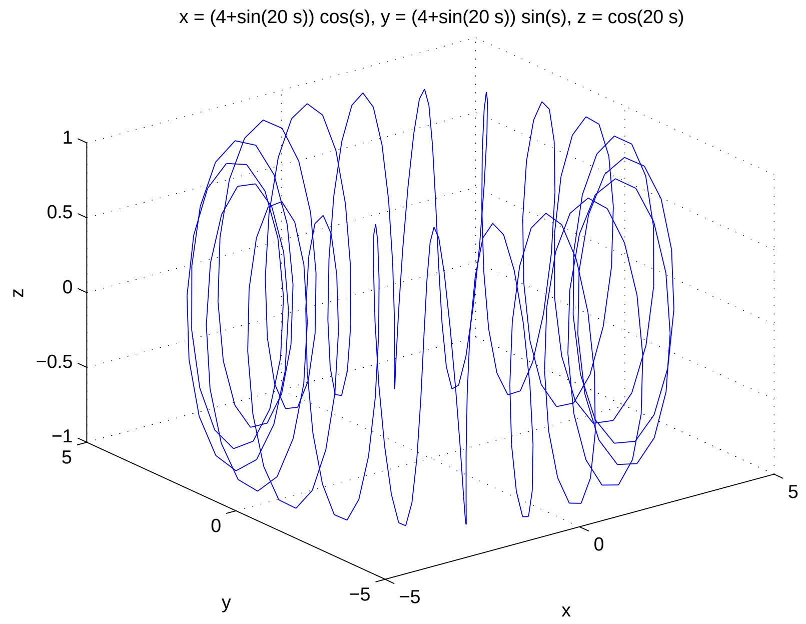

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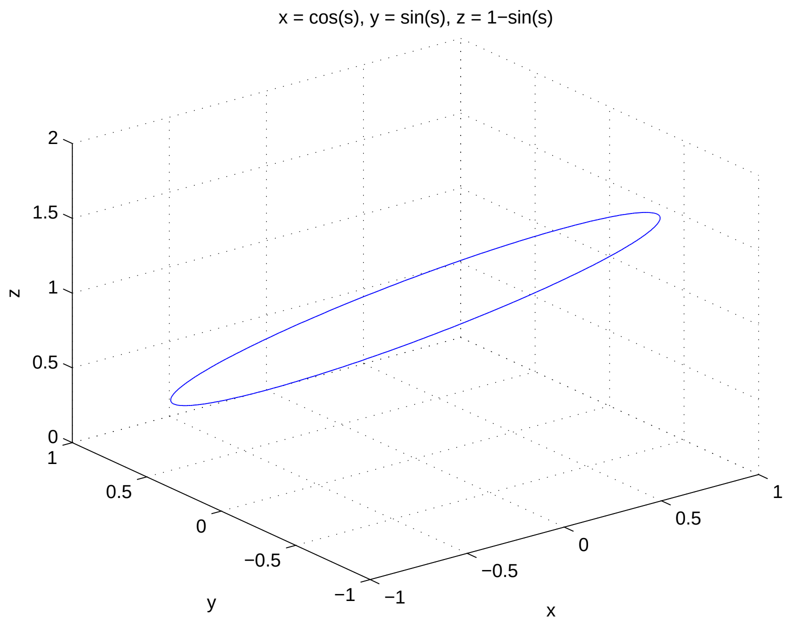

Derivatives of Vector Functions

- The derivative $\mathbf{r}'$ of $\mathbf{r}$ is

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

  if the limit exists.

- The vector $\mathbf{r}'(t)$ is called the **tangent vector**.

- The **unit tangent vector** is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

How to compute $\mathbf{r}'(t)$?

- If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where $f, g,$ and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

Examples

- Find the derivative of $\mathbf{r}(t) = (2t + t^2)\mathbf{i} + e^{-t^2}\mathbf{j} + \sin(2t)\mathbf{k}$.

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- Find parametric equations for the tangent line to the helix with parametric equations

$$x = 2\cos t \; y = \sin t \; z = t$$

at the point $(0, 1, \pi/2)$.

Differentiation Rules

$$\begin{aligned}
[\mathbf{u}(t) + \mathbf{v}(t)]' &= \mathbf{u}'(t) + \mathbf{v}'(t) \\
(c\mathbf{u}(t))' &= c\mathbf{u}'(t) \\
(f(t)\mathbf{u}(t))' &= f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t) \\
(\mathbf{u}(t) \cdot \mathbf{v}(t))' &= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t) \\
(\mathbf{u}(t) \times \mathbf{v}(t))' &= \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t) \\
(\mathbf{u}(f(t)))' &= f'(t)\mathbf{u}'(f(t)).
\end{aligned}$$

Integrals

- The definite integral

$$\int \mathbf{r}(t)\mathrm{d}t = \left(\int_a^b f(t)\mathrm{d}t)\right)\mathbf{i} + \left(\int_a^b g(t)\mathrm{d}t\right)\mathbf{i} + \left(\int_a^b h(t)\mathrm{d}t\right)\mathbf{k}$$

- The Fundamental Theorem of Calculus

$$\int_a^b \mathbf{r}(t)\mathrm{d}t = \mathbf{R}(t)|_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where $\mathbf{R}$ is an antiderivative of $\mathbf{r}$.