Part B: Introducing Topoi

CATEGORIES, TOPOI, AND LOGIC

WINTER 2013

Introduction

For this part of the course, we continue reading through Goldblatt and begin to look at logical aspects of categories. In classical set theory, for every predicate ϕ of elements of a set A, we can form the subset $P = \{x \in A : \phi(x)\}$ which consists of all elements of A for which ϕ holds. Since $x \in P$ is itself a predicate of elements of A, this establishes an equivalence between predicates and subsets. In order to replicate this process in a categorical setting, we must first introduce the idea of a **subobject** and their basic properties. The main difficulty with this process is that objects have no elements *per se* so the formation of subobjects is not nearly as straightforward as in set theory.

To facilitate the handling of subobjects, a topos is required to have a **subobject classifier**, which is usually denoted Ω. The elements of Ω are the **truth values** of the topos. In classical set theory (and thus in the topos $\mathbf{Set}$), there are only two truth values — $true$, $\top$ or 1, and $false$, $\bot$ or 0 — which means that Ω is just the two-element set $\{true, false\}$, $\{\top, \bot\}$ or $2 = \{0, 1\}$. We will soon see that in an elementary topos, Ω can be any Heyting algebra and hence the internal logic of a topos is naturally intuitionistic rather than classical. Even when the internal logic of an elementary topos is classical, Ω can be any Boolean algebra so even that case is more general than classical set theory.

There are three standard topoi that are used to illustrate the variety of internal logic for topoi.

- The topos $\mathbf{Set}$ has for subobject two-element Boolean algebra $\{0, 1\}$, which means that the internal logic of $\mathbf{Set}$ is the usual interpretation of classical logic.

- The topos $\mathbf{Set}^2$ has for subobject classifier four-element Boolean algebra $\{0, 1\}^2$, which means that the internal logic of $\mathbf{Set}^2$ is also classical but the interpretation is not the standard one.

- The topos $\mathbf{Set}^{\rightarrow}$ has for subobject classifier the three-element Łukasiewicz algebra $\left\{0, \frac{1}{2}, 1\right\}$, which means that the internal logic $\mathbf{Set}^{\rightarrow}$ is an interpretation of intuitionistic logic.

In Chapter 4, Goldblatt introduces elementary topoi and some fundamental examples. He then continues in Chapter 5 to derive their basic properties, such as the quintessential **epi-mono factorization** which says that every arrow in a topos factors as an epic followed by a monic in an essentially unique way. This and our earlier work on limits and colimits will take care of the basic internal structure of topoi. In the next part of the course, we will use these tools to dig deeper into the internal logic of topoi.

Reading

I recommend the following path:

(1) Read Goldblatt's Chapter 4. The more general examples in §4.5 and §4.6 are dense but they are important and will come back over and over as fundamental examples throughout the course. Goldblatt's introduction to these examples is too brief to do them justice.

(2) Read Goldblatt's Chapter 5. Work carefully through the first three sections as mastering the tools presented there is key to understanding topoi.

Note that what Goldblatt presents as the "topos of sheaves over I" in §4.5 is one of many equivalent ways to look at sheaves. Goldblatt will present sheaves in another manner in Chapter 14; the latter is the most prevalent presentation of sheaves in the literature. Subtle changes in presentation like this can lead to confusion.

Exercises

Goldblatt —

Ch. 4 — §1: 1, 2, 3; §5: 1, 2; §6: 1, 2, 3; §7: 1, 2, 3, 4; §8: 1, 2.

Ch. 5 — §1: 1; §2: 1, 2, 3; §4: 1, 2, 3; §5: 1, 2, 3.

Problems

Here are the assigned problems for this part of the course. Once you have succesfully completed all these problems, we will move on to the next part. Problems B.1, B.2, B.3 are intended to introduce you to some of the essential topological aspects of topos theory. Make sure to discuss these problems in order to get familiar with the necessary tools from topology.

PROBLEM B.1. Many examples of topoi are based on topological spaces. In this problem, you are asked to show that the category $\mathbf{Top}$ of topological spaces is complete and cocomplete. Many topoi are based on $\mathbf{Top}$, but $\mathbf{Top}$ itself is not a topos since it does not have exponentials nor a subobject classifier.

A **topological space** is a set X together with a collection Θ of subsets of X which are called **open sets** and such that:

i. $\varnothing$ and X are open;

ii. the intersection of any finite number of open sets is open; and

iii. the union of any (finite or infinite) number of open sets is open.

Thus the same base set can X have many topologies. The finest topology on X is the **discrete topology** wherein all subsets of X are open. The coarsest topology on X is the **indiscrete topology** which only has two open sets, $\varnothing$ and X. In general, a topology Θ on X is **finer/coarser** than a topology Θ' on X if Θ has more/fewer open sets than Θ'.

The objects in the category **Top** are topological spaces and the arrows are the **continuous maps**: $f : Y \to X$ is continuous if $f^{-1}(U)$ is an open set in Y for every open set U in X. Note that if Θ and Θ' are two topologies on the set X, then Θ is finer than Θ' if and only if the identity map is continuous from X with Θ to X with Θ'. So a bijective continuous map is not necessarily an isomorphism in **Top** since the inverse map may fail to be continuous. A bijection $f : Y \to X$ such that it and its inverse $f^{-1} : X \to Y$ are continuous is called a **homeomorphism**; these are the isomorphisms in **Top**.

(a) A **topological base** on a set X is a collection of subsets of X called **basic open sets** such that:

- every $x \in X$ is an element of some basic open set, and
- if U, V are basic open sets and $x \in U \cap V$ then there is a basic open set W such that $x \in W \subseteq U \cap V$.

Show that the collection of all subsets of X that are unions of basic open sets defines a topology on X. Show that to verify that $f : Y \to X$ is continuous it is enough to check that $f^{-1}(U)$ is open in Y for every basic open set U.

(b) The **topological sum** $X + Y$ of two topological spaces X and Y is the disjoint union of X and Y with the topology determined by the base consisting of all open subsets of X and open subsets of Y (considered as subsets of the disjoint union). Show that this defines a coproduct in **Top**.

(c) The **topological product** $X \times Y$ of two topological spaces X and Y is the cartesian product of X and Y with the topology determined by the base consisting of all sets $U \times V$ where U is open in X and V is open in Y. Show that this defines a product in **Top**.

(d) A **subspace** of X is a subset X' of X together with the topology consisting of all sets $U \cap X'$ where U is open in X. Verify that this **subspace topology** is indeed a topology on X' and use this idea to show that **Top** has equalizers.

(e) If $\sim$ is an equivalence relation on X then the finest topology on $X/{\sim}$ that makes the quotient map continuous is defined by

$$W \subseteq X/{\sim} \text{ is open} \iff \bigcup W = \{x \in X : [x]_\sim \in W\} \text{ is open in } X.$$

Verify that this **quotient topology** is indeed a topology on $X/{\sim}$ and use this idea to show that **Top** has coequalizers.

PROBLEM B.2. In §4.5, Goldblatt briefly introduces the topos $\mathbf{Top}(I)$. The objects of $\mathbf{Top}(I)$ are pairs (X, p) where $p : X \to I$ is a **local homeomorphism**: for every point $x \in X$ there are an open set U of X containing x and an open set J of I containing $f(x)$ such that the restriction of p to U is a homeomorphism onto J. The arrows of $\mathbf{Top}(I)$ are commutative triangles

$$\begin{tikzcd} X \arrow[dr, bend left, "p"] \arrow[d, "f"'] & \\ Y \arrow[r, bend right, "q"'] & I. \end{tikzcd}$$

Objects in $\mathbf{Top}(I)$ are also called **display spaces** and local homeomorphisms are also called **display maps**.[1]

Goldblatt gives a detailed description of the subobject classifier in $\mathbf{Top}(I)$. In this problem, you will show that $\mathbf{Top}(I)$ is complete and cocomplete. Through this, you will discover the basic tricks to work with local homeomorphisms.

(a) Given local homeomorphisms $p : X \to I$ and $\ell : L \to I$, show that every map $f : X \to L$ such that $\ell \circ f = p$ is also a local homeomorphism. Use this to show that the comma category $\mathbf{Top}(I) \downarrow (L, \ell)$ is isomorphic to $\mathbf{Top}(L)$.

(b) Given local homeomorphisms $p : X \to I$ and $q : Y \to I$, show that the topological sum $X + Y$ comes equipped with a local homeomorphism $[p, q] : X + Y \to I$ and that this defines a coproduct in $\mathbf{Top}(I)$.

(c) Given local homeomorphisms $p : X \to I$ and $q : Y \to I$, show that the diagonal $r : X \times_I Y \to I$ of the pullback square

$$\begin{tikzcd} X \times_I Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "q"] \\ X \arrow[r, "p"'] & I \end{tikzcd}$$

in $\mathbf{Top}$ is a local homeomorphism and hence that this defines a product in $\mathbf{Top}(I)$.

(d) Given the isomorphism from part (a), conclude from part (c) (applied to $\mathbf{Top}(L)$) that $\mathbf{Top}(I)$ has pullbacks.

(e) Suppose that $p : X \to I$ is a local homeomorphism and suppose $\sim$ is an equivalence relation on the set X such that $x \sim x' \implies p(x) = p(x')$. Show that there is a unique map $\tilde{p} : X/{\sim} \to I$ such that

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[d] & I \\ X/{\sim} \arrow[ur, "\tilde{p}"'] & \end{tikzcd}$$

commutes and that this map is a local homeomorphism. Conclude that $\mathbf{Top}(I)$ has coequalizers.

PROBLEM B.3. The **Sierpiński space** is the topological space Σ with two points $\{0, 1\}$ and the collection of open sets $\Theta = \{\varnothing, \{0\}, \{0, 1\}\}$. The **Sierpiński topos** $\mathbf{Top}(\Sigma)$ consists of pairs (X, p) where X is a topological space and $p : X \to \Sigma$ is a local homeomorphism. This means that for each point $x \in X$ one of the following two cases holds:

[1] The terms "etale space" and "etale map" are also used in this context, so the topos $\mathbf{Top}(I)$ is sometimes denoted $\mathbf{Etale}(I)$ or $\mathbf{Et}(I)$ (e.g., Mac Lane & Moerdijk, §II.6). Unfortunately, this usage originates from a poor translation of French terminology and ultimately to a confusion between two unrelated but similar looking French terms — *étalé* (meaning spread or displayed) and *étale* (meaning slack water, the time between low tide and high tide where water is still) — which both have a technical meaning. It is best to reserve "etale" for the other meaning and use "display" for this one.

Ad 0. $p(x) = 0$ and $\{x\}$ is open in X; thus p restricts to a homeomorphism from the open set $\{x\}$ in X onto the open set $\{0\}$ in Σ.

Ad 1. $p(x) = 1$ and $\{x\}$ is not open in X but there is a point $y \in X$ such that $p(y) = 0$ and $\{x, y\}$ is open in X; thus p restricts to a homeomorphism from the open set $\{x, y\}$ in X onto the open set $\{0, 1\}$ in Σ.

(a) Show that the point $y \in p^{-1}(0)$ associated to $x \in p^{-1}(1)$ in case 1 is unique. So the correspondence $x \mapsto y$ is a function $p^{-1}(1) \to p^{-1}(0)$.

(b) Show that the Sierpiński topos $\mathbf{Top}(\Sigma)$ is equivalent to the topos of functions $\mathbf{Set}^{\rightarrow}$. Specifically, show that there are functors $F : \mathbf{Top}(\Sigma) \to \mathbf{Set}^{\rightarrow}$ and $G : \mathbf{Set}^{\rightarrow} \to \mathbf{Top}(\Sigma)$ and natural isomorphisms $I \cong F \circ G$ and $I \cong G \circ F$ (where I denotes the identity functor on the appropriate category).

(c) Describe the subobject classifier for $\mathbf{Top}(\Sigma)$ using the method in §4.5 of Goldblatt. Then describe the relationship between that subobject classifier and that for $\mathbf{Set}^{\rightarrow}$ described in §4.4 of Goldblatt.

(d) Let Σ_n denote the topological space with n points $\{0, 1, \ldots, n-1\}$ and open sets

$$\Theta = \{\varnothing, \{0\}, \{0, 1\}, \{0, 1, 2\}, \ldots, \{0, 1, \ldots, n-1\}\}.$$

(So the Sierpiński space is Σ_2.) Find a category equivalent of the topos $\mathbf{Top}(\Sigma_n)$ similar to the one for $\mathbf{Top}(\Sigma)$ in part (b).

(e) Repeat part (c) for $\mathbf{Top}(\Sigma_n)$ and the other category you found in part (d)

PROBLEM B.4. Consider the two-element group C_2 with elements ± 1 under multiplication. A C_2-set can be viewed as a set X equipped with an involution $x \mapsto -x$; an equivariant map between C_2-sets X and Y is a map $f : X \to Y$ such that $f(-x) = -f(x)$ for all $x \in X$. The topos C_2-$\mathbf{Set}$ consists of C_2-sets with equivariant maps (Goldblatt §4.6).

(a) Show that the monics in C_2-$\mathbf{Set}$ are precisely the injective equivariant maps.

(b) Show that the epics in C_2-$\mathbf{Set}$ are precisely the surjective equivariant maps.

(c) Show that every equivariant bijection is an isomorphism in C_2-$\mathbf{Set}$.

(d) A **retraction** (or **left inverse**) of an arrow $f : X \to Y$ is an arrow $r : Y \to X$ such that $r \circ f = 1_X$. Note that a map with a retraction must be monic (in any category); such an arrow is sometimes called **split monic**. Show that a monic $f : X \to Y$ in C_2-$\mathbf{Set}$ has a retraction if and only if $X \neq \varnothing$ or $X = Y = \varnothing$.

(e) A **section** (or **right inverse**) for $f : X \to Y$ is a map $s : Y \to X$ such that $f \circ s = 1_Y$. Note that a map with a section must be epic (in any category); such maps are often called **split epic**. Show that there is an epic map in C_2-$\mathbf{Set}$ that does not have a section. Characterize the splic epics in C_2-$\mathbf{Set}$.