Barnett
4/30/09

Math 46, Applied Math (Spring 2009): Midterm 1

~ SOLUTIONS ~

2 hours, 50 points total, 6 questions worth varying number of points

1. [9 points] A gas bubble of mean radius R, containing gas at mean pressure P, in a surrounding fluid of density ρ, can oscillate at frequency ω. This is actually useful in enhancing ultrasound reflection in medical imaging. [Hint: pressure is force per unit area; force is mass times acceleration]

[4] (a) How many (independent) dimensionless quantities are there? Give them. [Hint: a dimensions matrix will help].

$$\begin{array}{c|c} & \begin{array}{cccc} R & \omega & \rho & P \end{array} \\ \begin{array}{c} M \\ L \\ T \end{array} & \begin{bmatrix} & & 1 & 1 \\ 1 & & -3 & -1 \\ & -1 & & -2 \end{bmatrix} \end{array}$$

$$P = \frac{F}{L^2} = \frac{MLT^{-2}}{L^2}$$

$$\#\text{ dimless} = \#\text{ cols} - \text{rank} = 4 - 3 = 1.$$

$$\Pi_1 = \frac{P}{\rho} \cdot \frac{1}{\omega^2 R^2}$$

[2] (b) If a physical law relates the four parameters given in the problem, how must ω scale with R when the other two parameters are fixed?

Buck. Pi Thm says $F(\pi_1) = 0$ ie $\pi_1 = \text{const}$

$$\Rightarrow \frac{P}{\rho \omega^2 R^2} = c \quad \Rightarrow \quad \omega = c\frac{P^{1/2}}{\rho^{1/2} R}$$

ie $\omega \propto \frac{1}{R}$ when others const.

[3] (c) If instead a physical law relates the above four to a fifth parameter v, the velocity of the bubble moving through the fluid, use the Buckingham Pi Theorem to deduce whether the scaling between ω and R you found above *must* still hold when the other three parameters are fixed. (Explain)

Now have 5th column: $\begin{array}{c} v \\ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \end{array}$ & one extra dim'less param $\Pi_2 = \frac{v}{R\omega}$ (for instance).

However, we wish ω & R to appear in only a single π, and they already do in π_1, so let's choose a different π_2:

$$\Pi_2 = \frac{P}{\rho} \cdot \frac{1}{v^2}$$ ← this doesn't involve ω or R.

Then Pi Thm! $\Pi_1 = g(\pi_2)$

so $\frac{P}{\rho \omega^2 R^2} = g\left(\frac{P}{\rho v^2}\right)$ = const if P, ρ, v are.

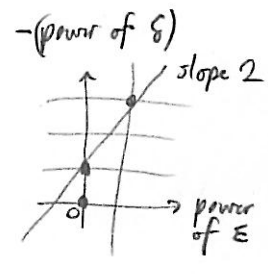
$\Rightarrow \omega \propto \frac{1}{R}$ as before must hold when P, ρ, v held const.

2. [7 points] Consider the algebraic equation

$$\varepsilon x^3 + x + 1 = 0$$

(a) Find leading-order approximations to all solutions valid for small $\varepsilon \ll 1$

$\varepsilon = 0$ gives lin. eqn. $x + 1 = 0 \Rightarrow x_0 = -1$



rescale $y = \frac{x}{\delta}$ gives $\underbrace{\frac{\varepsilon}{\delta^3} y^3 + \frac{1}{\delta} y}_{} + 1 = 0$

dom. balance to give $\delta = \varepsilon^{1/2}$

mult. by $\varepsilon^{1/2}$: $y^3 + y + \cancel{\varepsilon^{1/2}} = 0$ drop to leading order.

so $(y^2+1)y = 0$ ie $y = \pm i,\ 0$ image of root $x_0 = -1$

3 roots are $x = -1 + \cdots,\ -\frac{i}{\sqrt{\varepsilon}} + \cdots,\ +\frac{i}{\sqrt{\varepsilon}} + \cdots$

(b) Find a 2-term approximation to the root which is finite as $\varepsilon \to 0$

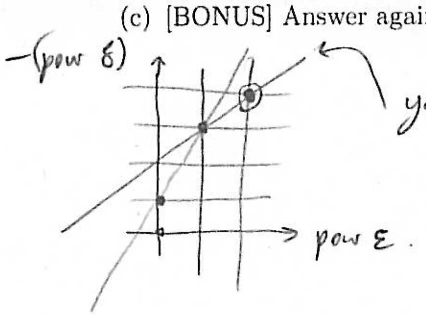
Go back to x variable; do regular perturbation: $x = x_0 + \varepsilon x_1 + \cdots$
$= -1 + \varepsilon x_1 + \cdots$

so $\varepsilon(-1 + \varepsilon x_1 + \cdots)^3 + (-1 + \varepsilon x_1 \cdots) + 1 = 0$

collect ε^1: $-1 + x_1 = 0$ so $x_1 = 1$.

$\Rightarrow x = -1 + \varepsilon + O(\varepsilon^2) \ldots$

(c) [BONUS] Answer again (a) above if the equation is changed to $\varepsilon^2 x^4 + \varepsilon x^3 + x + 1 = 0$



you see now a new possible dominant balancing "slope" arises, and the old one remains.

This is $\delta = \varepsilon$ giving

$$\frac{\varepsilon^2}{\delta^4} y^4 + \frac{\varepsilon}{\delta^3} y^3 + \frac{1}{\delta} y + 1 = 0$$

mult. ε^2: $y^4 + y^3 + \cancel{\varepsilon y} + \cancel{\varepsilon^2} = 0$ ignore at leading order.

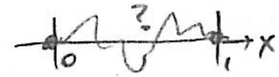
so $y^3(y+1) = 0$, ie $y = 0$ (triple root), -1

images of above 3 roots; new root

Roots are same as above three, with extra fourth root $x = -\frac{1}{\varepsilon} + \cdots$

3. [9 points] Use singular perturbation methods to find a uniform approximate solution to the boundary-value problem

$$\varepsilon y'' - 2y' - e^{y} = 0, \qquad \varepsilon \ll 1, \qquad y(0) = 0, \qquad y(1) = 0$$



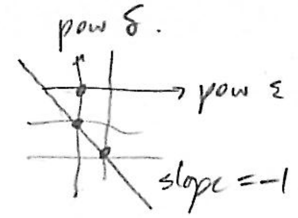
As always, remember to check and explain the location of any boundary layer(s). [Hint: if you can't solve an ODE, express things in terms of its limiting value(s)]

Examine relative signs of $\varepsilon y''$ and $-2y'$ at each end:

@ $x=0$: $\xi = \frac{x}{\delta}$ will give $Y'' - Y' = 0$ unstable $(Ae^{+\xi})$
so no BL possible.
(see Final exam 2007).

@ $x=1$: $\xi = \frac{1-x}{\delta}$ gives correct signs:

Inner layer: substitute rescaling
$$y \to Y$$
$$y' \to -\frac{1}{\delta} Y'$$
$$y'' \to \frac{1}{\delta^2} Y''$$



so $\delta = \varepsilon^{1/(-1)} = \varepsilon$.

so $$\frac{\varepsilon}{\delta^2} Y'' + \frac{2}{\delta} Y' - e^{Y} = 0$$

mult. by ε: $$Y'' + 2Y' - \varepsilon e^{Y} = 0$$
εe^{Y} ignore at leading order.

gen. solve ODE

so $$Y(\xi) = Ae^{-2\xi} + B$$

BC $y(1) = 0$ makes $Y_i(0) = 0$ ie $B = -A$. so $Y_i(\xi) = A(1-e^{-2\xi})$

Outer Layer: (matches BC @ $x=0$)

$$-2y_0' - e^{y_0} = 0 \quad \xrightarrow{\text{gen solve}} \quad \int \frac{dy_0}{e^{y_0}} = -\int \frac{1}{2}\,dx$$

so $$-e^{-y_0} = -\frac{x}{2} + c$$

$\Rightarrow$ $$y_0(x) = -\ln\left(\frac{x}{2} + c\right)$$

BC $y_0(0) = 0$ so $0 = -\ln\left(\frac{0}{2} + c\right)$ so $c = 1$, $y_0(x) = -\ln\left(\frac{x}{2} + 1\right)$

$\Rightarrow$ $C_m = \lim_{x \to 1} y_0(x) = -\ln \frac{3}{2} = \lim_{\xi \to \infty} Y_i(\xi) = A(1 - e^{-\infty})$.

match inner to outer.

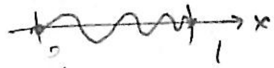
so $A = -\ln \frac{3}{2} = \ln \frac{2}{3}$

$$y_u(x) = y_0(x) + y_i(x) - C_m = -\ln\left(\frac{x}{2} + 1\right) + \left(\ln\frac{2}{3}\right)\left(1 - e^{-\frac{2(1-x)}{\varepsilon}}\right) - \ln\frac{2}{3}$$

$$= \ln\frac{1}{1 + \frac{x}{2}} + \left(\ln\frac{3}{2}\right) e^{-\frac{2(1-x)}{\varepsilon}}$$

4. [10 points] Short answer questions.

[2] (a) Find a WKB approximation to the nth eigenvalue of $\underbrace{\varepsilon^2 y'' + (1+x)^2 y = 0}_{\text{is in std form}}$ with $y(0) = y(1) = 0$ for large n.



$(1+x)^2 = k(x)^2$ so $k(x) = 1+x$

$$y(x) \approx \frac{1}{\sqrt{k(x)}} \sin \frac{1}{\varepsilon} \int_0^x k(s)\,ds$$ ← no cos term since $y(0)=0$.

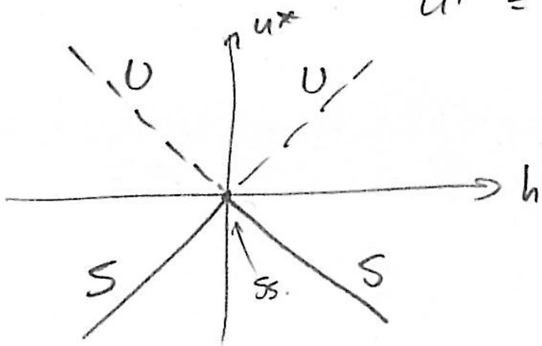
match BC $y(1)=0$ so $\sin(\ldots)=0$ so $n\pi = \frac{1}{\varepsilon_n}\int_0^1 k(s)\,ds = \frac{1}{\varepsilon_n}\left[x+\frac{x^2}{2}\right]_0^1$

so $\varepsilon_n = \dfrac{3}{2\pi n}$.

(3) (b) Sketch a bifurcation diagram showing equilibria and stability for the ODE $du/dt = u^2 - h^2$, as the parameter h varies.

$u' = f(u,h) = u^2 - h^2 = 0$ when equil.

ie $u^* = \pm h$



stability: for any u^* have $f'\big|_{u^*} = \frac{\partial f}{\partial u}\Big|_{u^*} = 2u^*$

so $u^* > 0$ unstable
$u^* < 0$ stable.

(3) (c) Prove or disprove the following claim: $\dfrac{1}{\log \varepsilon} = o(\varepsilon)$ as $\varepsilon \to 0^+$

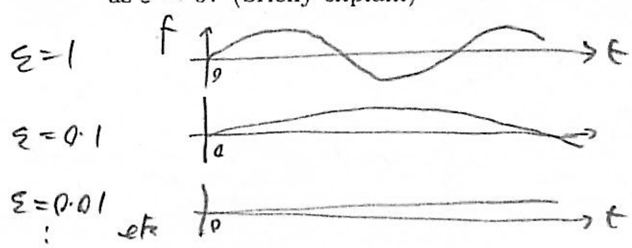
if true, means $\lim_{\varepsilon\to 0} \dfrac{1/\log\varepsilon}{\varepsilon}$ exists and $=0$.

← top & bottom vanish ⇒ l'Hôpital gives $\dfrac{\left(\frac{1}{\log\varepsilon}\right)'}{1} = -\varepsilon^{-1}(\log\varepsilon)^{-2}$ = worse! ...

Instead flip over:

$$= \frac{\varepsilon^{-1}}{\log\varepsilon} \xrightarrow{\text{l'Hôp}} \frac{-\varepsilon^{-2}}{\varepsilon^{-1}} = -\frac{1}{\varepsilon} \to \infty \text{ as } \varepsilon \to 0$$ so <u>No</u>, it's false.

(2) (d) Is $f(t,\varepsilon) = \sin(\varepsilon t)$ pointwise, and/or uniformly, convergent to zero on the interval $t \in (0, +\infty)$, as $\varepsilon \to 0$? (briefly explain)



ptwise yes, since fixing t, $\sin(\varepsilon t) \to 0$ as $\varepsilon \to 0$.

uniform <u>no</u>, since for each $\varepsilon > 0$, $\exists\, t$ large enough that sin reaches its max value 1.

(e) [BONUS] Is it possible for a function $f(t,\varepsilon)$ to be uniformly convergent on some interval of t, but not pointwise convergent, as $\varepsilon \to 0$? (Give an example or explain.)

No, since ptwise values of a func. cannot exceed their max values (which are in turn vanishing if unif. convergent) –

5. [9 points] Consider the perturbed initial-value problem for $y(t)$ on $t > 0$,

$$y'' + y = 4\varepsilon y(1 - y'^2), \qquad 0 < \varepsilon \ll 1, \qquad y(0) = 1, \qquad y'(0) = 0$$

(a) Use the Poincaré-Lindstedt method to give a 2-term uniform approximation. [Hint: set $\tau = \omega t$ where ω is perturbed from the value 1. Don't forget to match initial conditions.]

rescale $\tau = \underbrace{(1 + \omega_1 \varepsilon + \cdots)}_{\text{like } 1/t_c} t.$ so $\frac{d}{dt}$ replaced by $\omega \frac{d}{d\tau}$

ODE becomes, (wrt τ) $(1 + \omega_1\varepsilon + \cdots)^2 y'' + y = 4\varepsilon y(1 - (1+\omega_1\varepsilon + \cdots)^2 y'^2)$

subst. pert. series for $y = y_0 + \varepsilon y_1 + \cdots$

$$(1 + \omega_1\varepsilon + \cdots)^2 (y_0'' + \varepsilon y_1'' \cdots) + y_0 + \varepsilon y_1 = 4\varepsilon(y_0 + \varepsilon y_1 \cdots)(1 - (1 + \cdots)^2 (y_0' + \cdots)^2)$$

$$= 4\varepsilon y_0 (1 - y_0'^2) + O(\varepsilon^2)$$

Zeroth order: $y_0'' + y_0 = 0$ w/ $y_0(0) = 1$, $y_0'(0) = 0$

very std., is $y_0 = \cos\tau$

First order: $2\omega_1 y_0'' + y_1'' + y_1 = 4y_0(1 - y_0'^2)$

ie $y_1'' + y_1 = \text{driving} = -2\omega_1(-\cos\tau) + 4\underbrace{\cos\tau(1 - \sin^2\tau)}_{\cos^3\tau}$

$\cos^3\tau = \frac{1}{4}(3\cos\tau + \cos 3\tau)$ Power-reduction

$$= \underbrace{(3 + 2\omega_1)}_{\text{choose } \omega_1 = -3/2} \cos\tau + \cos 3\tau$$

choose $\omega_1 = -3/2$ to remove secular (on-resonance) term.

This leaves

$y_1'' + y_1 = \cos 3\tau$ use Meth. Und. Coeffs, try $Y = \cos 3\tau$

$\Rightarrow Y'' = -9\cos 3\tau$

$\Rightarrow Y'' + Y = -8\cos 3\tau$, so need $-\frac{1}{8}$ coeff.

Particular soln $y_1 = -\frac{1}{8}\cos 3\tau + c_1\cos\tau + c_2\sin\tau$

Match IC $y_1(0) = y_1'(0) = 0$: $c_1 = \frac{1}{8}$, $c_2 = 0$.

$$\Rightarrow y_u = \cos\tau + \frac{\varepsilon}{8}(\cos\tau - \cos 3\tau) + \cdots \quad \text{with } \tau = \left(1 - \tfrac{3}{2}\varepsilon + \cdots\right)t$$

(b) Has switching on the perturbation $\boxed{\text{increased}}$ or decreased the period of the oscillator?

You must now interpret your ω expansion:

$$\tau = \left(1 - \frac{3}{2}\varepsilon + \cdots\right)t \quad \text{so} \quad \cos\tau \text{ oscillates slower,}$$

longer period than $\cos t$ (unperturbed).

6. [6 points] Radioactivity is modeled by quantum particles leaking through a barrier according to the Schrödinger equation

$$y'' - \frac{\lambda}{x^{3/2}}y = 0, \qquad \text{for } x > 1,$$

It took me a while to come up with a power that gives exp decay for B soln, but also new, and not Cauchy-Euler!

where $\lambda \gg 1$ is a large positive parameter.

(a) Is the ODE oscillatory or growing/decaying? since opposite signs of y'', y terms.

put in std form: $\underbrace{\frac{1}{\lambda}}_{\varepsilon^2} y'' - \underbrace{x^{-3/2}}_{k(x)^2} y = 0$

so $k(x) = x^{-3/4}$

(b) Write down a WKB approximation to the *general* solution

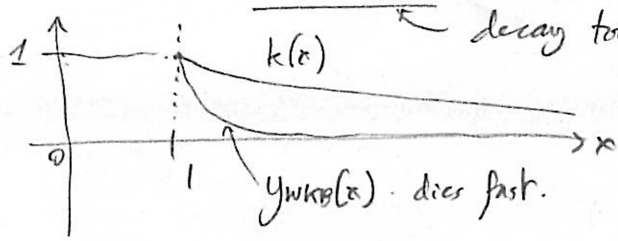
$\boxed{\text{Note: } \frac{1}{\varepsilon} = \sqrt{\lambda}}$ λ large, so ε small.

$$y_{WKB}(x) = \frac{A}{x^{-3/8}} e^{+\sqrt{\lambda}\int x^{-3/4}dx} + \frac{B}{x^{-3/8}} e^{-\sqrt{\lambda}\int x^{-3/4}dx}$$

or, can put as definite integrals, w/ lower limit $x=1$ (left end of interval).

(c) Find a WKB approximation to y in the barrier region $x > 1$, if the initial value is $y(1) = 1$, and a condition $\lim_{x\to+\infty} y(x) = 0$ is imposed.

decay towards $+\infty$.



$$\int x^{-3/4}dx = 4x^{1/4} + c$$

Consider the "$-$" (B) solution:

$$\underbrace{\frac{1}{x^{-3/8}}}_{\text{grows algebraically}} \cdot \underbrace{e^{-\sqrt{\lambda}(4x^{1/4}+c)}}_{\text{decays exponentially} \Rightarrow \text{beats algebraic}}$$

Only the B solution has finite limit (0) as $x \to +\infty$, so $A=0$.

$$\Rightarrow y_{WKB}(x) = \frac{B}{x^{-3/8}} e^{-\sqrt{\lambda}\int_1^x s^{-3/4}ds} = Bx^{3/8} e^{-\underbrace{\sqrt{\lambda}(4x^{1/4}-4)}_{\text{vanishes @ BC } x=1}}$$

$\Rightarrow B \cdot 1^{3/8} = 1$, $B=1$.

$$y_{WKB}(x) = x^{3/8} e^{-4\sqrt{\lambda}(x^{1/4}-1)}$$