Math 11
Section 3
Wednesday, November 12, 2008
Sample Solutions

Consider the vector field on $\mathbb{R}^2$

$$\vec{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle.$$

(1.) Compute the line integral of $\vec{F}$ around the unit circle oriented in the counterclockwise direction.

Solution: The easiest way to do this is to parametrize the unit circle by $\vec{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \leq t \leq 2\pi$. Then we have

$$\int_\gamma \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\cos t, \sin t) \cdot \langle -\sin t, \cos t \rangle \, dt =$$

$$\int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt = \int_0^{2\pi} 1 \, dt = 2\pi.$$

(2.) Show that, if we write $\vec{F} = \langle P, Q \rangle$, then almost everywhere[1] we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Where do we not have this?

Solution: Computing the partial derivatives of P and Q, we get

$$\frac{\partial P}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{\partial Q}{\partial x}.$$

The two are equal *except* at the origin, where $x^2 + y^2 = 0$, and neither $\frac{\partial P}{\partial y}$ nor $\frac{\partial Q}{\partial x}$, nor for that matter $\vec{F}$, are defined.

[1] "Almost everywhere" has a precise technical definition, given in more advanced analysis courses. Its use here should make good intuitive sense once you have done the problem.

Notice that this means the region D on which we have $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ is not simply connected. It has a hole in it, at the origin.

(3.) Find a potential function f for $\vec{F}$, that is, a function f such that $\nabla f = \vec{F}$. How do you explain your answer to part (1)?

(Hints: What is the domain of your potential function? What is f in polar coordinates? Why does this make sense?)

Solution: Recalling that the antiderivative of $\dfrac{1}{1+u^2}$ is $\arctan(u)$, we look for a potential function f such that $f_x = P$ and $f_y = Q$:

$$f_y = Q = \frac{x}{x^2+y^2} = \left(-\frac{x}{x^2}\right)\frac{1}{1+\left(\frac{y}{x}\right)^2}$$

$$f = \arctan\left(\frac{y}{x}\right) + C(x)$$

$$f_x = \left(\frac{-y}{x^2}\right)\frac{1}{1+\left(\frac{y}{x}\right)^2} + C'(x) = \frac{-y}{x^2+y^2} + C'(x)$$

$$\frac{-y}{x^2+y^2} + C'(x) = P = \frac{-y}{x^2+y^2}$$

$$C'(x) = 0$$

We can take $C(x) = 0$, and get

$$f(x,y) = \arctan\left(\frac{y}{x}\right).$$

The domain of this function contains all of $\mathbb{R}^2$ except the y-axis. In other words, it does not contain all of the unit circle. This explains how we can have the integral of $\vec{F}$ around the (closed) unit circle unequal to zero; the potential function we have found is not defined on the entire unit circle.

In polar coordinates, we can write

$$f(x,y) = f(r\cos\theta,\, r\sin\theta) = \theta,$$

at least for $x > 0$ and $y > 0$. If we try to define θ as a function of x and y on the entire plane (except the origin, where there are obvious problems), we cannot do so in a continuous manner: Suppose we start at $(1,0)$ and go around the unit circle counterclockwise. The angle θ starts out having a value of 0 at the beginning of the path, and increases to a value of 2π at the end. If θ is supposed to be our potential function for $\vec{F}$, it makes perfect sense that the integral of $\vec{F}$ around the unit circle has value 2π.

Note: If you started with $f_x = P$ instead of $f_y = Q$, you probably came up with the potential function

$$g(x,y) = -\arctan\left(\frac{x}{y}\right).$$

This answer is also correct. Using the fact that $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$, so if $\theta = \tan^{-1}(A)$ then $\theta = \frac{\pi}{2} - \cot^{-1}(A)$, at least for $0 < \theta < \frac{\pi}{2}$, we see:

$$g(x,y) = -\tan^{-1}\left(\frac{x}{y}\right) = -\left(\frac{\pi}{2} - \cot^{-1}\left(\frac{x}{y}\right)\right) = -\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{y}{x}\right)\right) =$$

$$\tan^{-1}\left(\frac{y}{x}\right) + \frac{\pi}{2} = f(x,y) + \frac{\pi}{2}.$$

So our two different potential functions, or antiderivatives, differ by a constant.