Clifford theory (in ten minutes)

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(See appendix A of [RR] for full details.)

Let A be an algebra, G a group acting on A by automorphisms. THen

$$A \rtimes G = \left\{ \sum_g a_g g \mid a_g \in A \right\}$$

with product $(a_g g)(a_h h) = a_g (a_h)^h gh$. Define $A^G = \{a \in A \mid g(a) = a, \ \forall g \in G\}$. Then

$$A \rtimes G \supset A \rtimes 1 = A \supset A^G.$$

Suppose that the representation theory of A and G are known. Then the game is to find the representation theory of $A \rtimes G$ and A^G.

Let M be a simple $A \rtimes G$ module. Then

$$\mathrm{Res}_A^{A \rtimes G} M = \sum_{\lambda \in \tilde{A}} A^\lambda \otimes L^\lambda,$$

where A^λ is an irrep of A. Let A^λ be a simple submodule in M. Then

$$\sum_{g \in G} g A^\lambda \subset M$$

is an $A \rtimes G$ submodule of M. SInce M is simple, $\sum_{g \in G} g A^\lambda = M$ so

$$\mathrm{Ind}_A^{A \rtimes G}(A^\lambda) = M.$$

It happens that $\mathrm{Ind}_A^{A \rtimes G}(A^\lambda)$ is sometimes isomorphic to $\mathrm{Ind}_A^{A \rtimes G}(A^\mu)$. When is this true? If $g \in G$, then

$$g : A \to A.$$

So $g^* : A\text{modules} \to A\text{modules}$ by $M \mapsto g^*(M)$ (so G acts on the set of A modules, and permutes the simples). Need to check that $gA^\lambda \cong g^*(A^\lambda)$. The *inertia group* of A^λ is

$$H = \{g \in G \mid g^*(A^\lambda) \cong A^\lambda\}$$

So in fact A^λ is an $A \rtimes_\theta H$-module (where the product in $A \rtimes_\theta H$ is $(a_1 h_1)(a_2 h_2) = a_1 h_1(a_2) \theta(h_1, h_2) h_1 h_2$, where $\theta : H \times H \to A$).

Theorem 0.1. *Then $M \cong \mathrm{Ind}_{A \rtimes_\theta H}^{A \rtimes G}(A^\lambda)$ and these are the simple $A \rtimes G$-modules.*

Example 1 Let N be a normal subgroup of G. Then G acts by conjugation on $\mathbb{C}N$ (and $\mathbb{C}G \cong \mathbb{C}N \rtimes_\theta G/N$).

Example 2 $G_{r,1,n}$ is the *imprimitive complex reflection group.* Then

$$G_{r,1,n} = S_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n.$$

where S_n is acting on $(\mathbb{Z}/r\mathbb{Z})^n$ by place permutations. Somtimes this is called the wreath product of S_n and $\mathbb{Z}/r\mathbb{Z}$. Examples: $G_{2,1,n} = WB_n$, signed permutation matrices, and $G_{2,2,n} = WD_n$, signed permutation matrices with even number of signs.

$$1 \to G_{r,p,n} \to G_{r,1,n} \to \mathbb{Z}/p\mathbb{Z} \to 1$$

where $G_{r,p,n}$ is also complex refl groups.

Other examples include $H_r, 1, n$ is the Hecke alg of $G_{r,1,n}$, H_r, p, n is the Hecke alg of $G_{r,p,n}$, and $H_r, p, n = (H_r, 1, n)^{\mathbb{Z}/p\mathbb{Z}}$.

References

[RR] A. Ram and J. Ramagge, *Affine Hecke algebras, cyclotomic Hecke algebras and Clifford theory*, in A tribute to C.S. Seshadri: Perspectives in Geometry and Representation theory, V. Lakshimibai et al eds., Hindustan Book Agency , New Delhi (2003), 428–466, http://www.math.wisc.edu/∼ram/pub/2003Seshadrip428.pdf