The Arrows in Differential Equations Questions

On Friday, we discussed how to use eigenvalues and eigenvectors to solve differential equations. We will first briefly remind ourselves of what we accomplished on Friday.

A *system of differential equations* is:

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \quad & \quad \vdots \qquad \vdots \qquad\qquad \vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned}$$

where the x_i's are variables. On Friday, we thought of these are populations. We also have an *initial condition* which is the set of values $x_1(0) = c_1$, $x_2(0) = c_2$, … $x_n(0) = c_n$.

We can turn a system of differentiable equations into a *matrix differential equation* $\mathbf{x}'$=$A\mathbf{x}$ where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \ A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix}, \text{ and } \mathbf{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}.$$

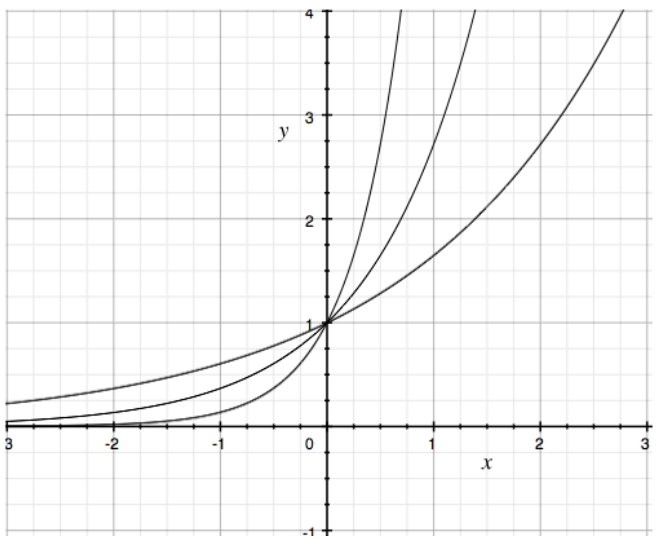
We also have the initial condition $\mathbf{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$, i.e. $\mathbf{x}(0)$ is a vector of constants.

We *decoupled* our matrix differential equation, by assuming that A has n independent eigenvectors. We let $P = \begin{bmatrix} \overline{v}_1 & \overline{v}_2 & \cdots & \overline{v}_n \end{bmatrix}$ and let $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\{\overline{v}_1, \overline{v}_2, \cdots, \overline{v}_n\}$ are the eigenvectors of A and where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A. We know that $A = PDP^{-1}$. Using this information, we made a change of variable using $\mathbf{y}(t) = P^{-1}\mathbf{x}(t)$. Using the change of variable we find that

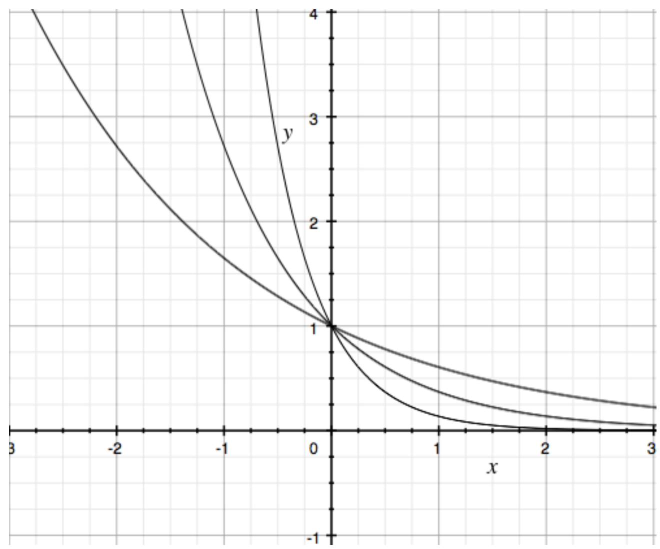
$$\mathbf{y}(t) = \begin{pmatrix} k_1 e^{\lambda_1 t} \\ k_2 e^{\lambda_2 t} \\ \vdots \\ k_n e^{\lambda_n t} \end{pmatrix} \text{ where } \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = \mathbf{y}(0) = P^{-1}\mathbf{x}(0).$$

Then by our change of variable, we find that the general solution of our matrix differential equation is $\mathbf{x}(t) = k_1 e^{\lambda_1 t}\overline{v}_1 + k_2 e^{\lambda_2 t}\overline{v}_2 + \cdots + k_n e^{\lambda_n t}\overline{v}_n$.

The question that we want to answer is **what qualitative information can the eigenvalues of a system of differential equations tell us?**



(a) Graphs of $e^{x/2}$, e^x, and e^{2x}



(b) Graphs of $e^{-x/2}$, e^{-x}, and e^{-2x}

From this point forward, we will concentrate on 2×2 matrices. Since we are working with 2×2 matrices, then $\mathbf{x}(t) = k_1 e^{\lambda_1 t}\overline{v}_1 + k_2 e^{\lambda_2 t}\overline{v}_2$. Thus our discussion must begin with a quick review of the behavior of a function $f(x) = be^{dx}$ where b and d are constants. Below, we see that the behavior of f greatly depends on the sign of d.

Reminding ourselves that eigenvectors give us a new coordinate system in a sense (see the figures on the next page), we begin to make a drawing of possible solutions to our matrix differential equations by drawing the lines that the two eigenvectors lie in. The next question is **what direction do the arrows go in?**

First Eigenvector Line: If $\mathbf{x}(t) = k_1 e^{\lambda_1 t}\overline{v}_1 + k_2 e^{\lambda_2 t}\overline{v}_2$ and if $k_2 = 0$, then we just have that $\mathbf{x}(t) = k_1 e^{\lambda_1 t}\overline{v}_1$. Thus if $\lambda_1 > 0$, then we that $\mathbf{x}(t)$ increases away from 0 as t increases. Thus the arrows along the eigenvalue's line will go away from $(0, 0)$. There is a second case where $\lambda_1 < 0$. In this case, then $\mathbf{x}(t)$ tends to 0 as t increases. So the arrows along the eigenvalue's line will go towards $(0, 0)$.

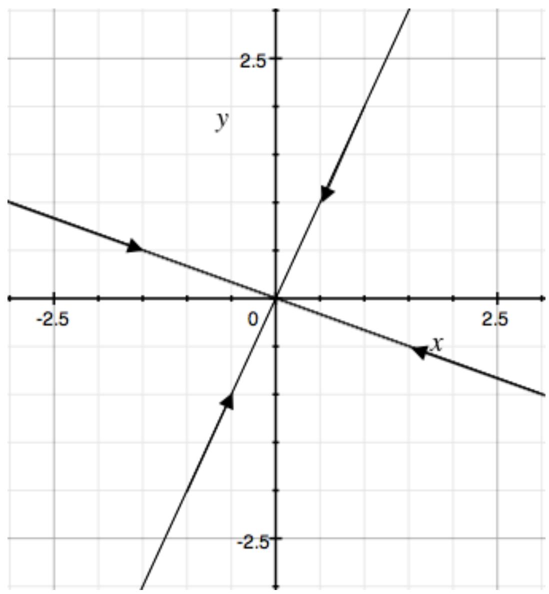
Second Eigenvector Line: Similarly, if $k_1 = 0$, then we just have that $\mathbf{x}(t) = k_2 e^{\lambda_2 t}\overline{v}_2$. Thus if $\lambda_2 > 0$, then we that $\mathbf{x}(t)$ increases away from 0 as t increases. Thus the arrows along the eigenvalue's line will go away from $(0, 0)$. There is a second case where $\lambda_2 < 0$. In this case, then $\mathbf{x}(t)$ tends to 0 as t increases. So the arrows along the eigenvalue's line will go towards $(0, 0)$.

Therefore, we have three cases to consider:

1. Both $\lambda_1, \lambda_2 > 0$
2. Both $\lambda_1, \lambda_2 < 0$
3. $\lambda_1 > 0$ while $\lambda_2 < 0$.

<u>Case 1</u>: $\lambda_1, \lambda_2 > 0$

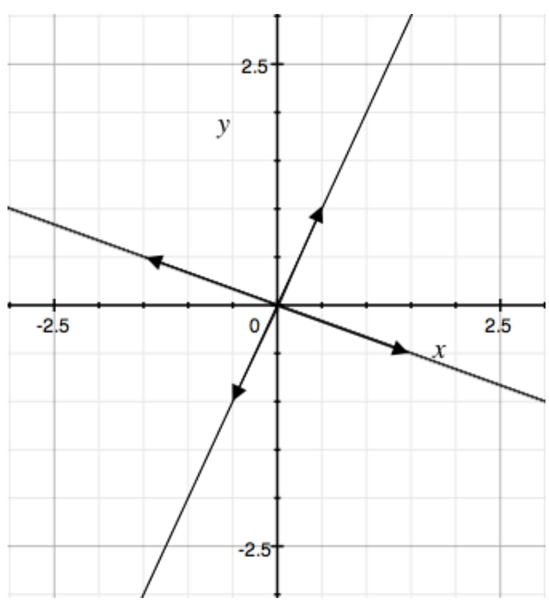
Arrows along the eigenvector lines will all point away from $(0, 0)$. Then we add possible solutions in which both $k_1 \neq 0$ and $k_2 \neq 0$. These added possible solutions, or trajectories, are linear combinations of the solutions along the eigenvector lines and will also have arrows that will point away from $(0, 0)$. In this case, we say that the origin is called an ***attractor*** or a ***sink***.



Eigenvector lines when $(0, 0)$ *is an Attractor*

<u>Case 2</u>: $\lambda_1, \lambda_2 < 0$

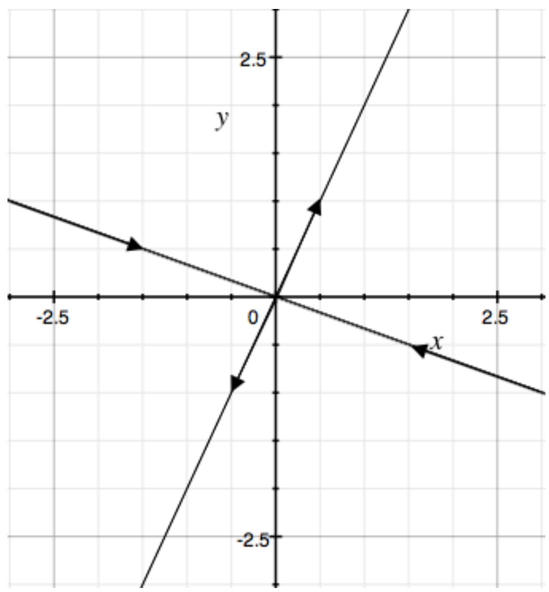
Arrows along the eigenvector lines will all point towards $(0, 0)$. Then we add possible solutions in which both $k_1 \neq 0$ and $k_2 \neq 0$. These added possible solutions, or trajectories, are linear combinations of the solutions along the eigenvector lines and will also have arrows that will point towards $(0, 0)$. In this case, we say that the origin is called an ***repellor*** or a ***source***.



Eigenvector lines when $(0, 0)$ *is a Repellor*

Case 3: $\lambda_1 > 0$ while $\lambda_2 < 0$

Arrows along the eigenvector line associated with the first eigenvalue will both point away from $(0, 0)$. Arrows along the eigenvector line associated with the second eigenvalue will both point towards $(0, 0)$. Then we add possible solutions in which both $k_1 \neq 0$ and $k_2 \neq 0$. These added possible solutions, or trajectories, are linear combinations of the solutions along the eigenvector lines. Some of these trajectories go towards the origin and then move away from the origin. The arrows along those particular trajectories show the paths that the trajectories follow. In this case, we say that the origin is called a ***saddle point***.



Eigenvector lines when $(0, 0)$ *is an Saddle Point*

Therefore eigenvectors and eigenvectors are crucial in giving us an idea of what the possible solutions of a system of differential equations are.

Practice

If you are so inclined, below are some questions to practice arrow drawing on. Some of the questions may look familiar. If you have any questions or comments, please do not hesitate to email me at Katherine [dot] M [dot] Kinnaird [at] Dartmouth [dot] edu.

Each of the below matrices is coefficient matrix A in the matrix differential equations $\mathbf{x}'=A\mathbf{x}$. These matrix differential equations each attempt to model the populations of zoombats and Dartmouth College students in the Hanover area. Please identify if the origin is an attractor, repellor, or a saddle point. Also please draw at least 6 trajectories.

1. $\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$
2. $\begin{pmatrix} 1 & -4 \\ -1 & 1 \end{pmatrix}$
3. $\begin{pmatrix} 2 & 4 \\ -2 & -5 \end{pmatrix}$
4. $\begin{pmatrix} 1/2 & 1/4 \\ 1/9 & 1/2 \end{pmatrix}$
5. $\begin{pmatrix} 1 & 1/2 \\ 1 & 3/2 \end{pmatrix}$
6. $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$
7. $\begin{pmatrix} 1 & 4 \\ -2 & -5 \end{pmatrix}$
8. $\begin{pmatrix} -2 & -1 \\ 3 & 2 \end{pmatrix}$