Math46 HW#1

units for energy

1) Page 7 #2

$[e] := \text{energy/mass} = \frac{ML^2T^{-2}}{M} = (LT^{-1})^2$

$[D] = LT^{-1}$

only possible choice is $\frac{e}{D^2} = \text{constant}$.

2) Page 8 #4

What are dimensions

$[t] = T \quad [r] = L \quad [\rho] = M/L^3 \quad [E] = ML^2/T^2$

New variable is P, ambient pressure.

The dimension are force ~~per~~ unit area

$[P] = ML/(T^2L^2) = M/T^2L$

$$\begin{array}{c c} & \begin{array}{ccccc} t & r & \rho & E & P \end{array} \\ \begin{array}{c} T \\ M \\ L \end{array} & \begin{bmatrix} 1 & 0 & 0 & -2 & -2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & -3 & 2 & -1 \end{bmatrix} \end{array}$$

matrix has rank 3. (column 1 2, 3 are linearly independent)

By inspection $\left[\frac{\rho}{E}\right] = \frac{ML^{-3}}{ML^2T^{-2}} = \frac{T^2}{L^5}$

$\Rightarrow \Pi_1 = \frac{r^5\rho}{t^2E}$ is unitless

given that we want to find r as a function of the other variables we need to make the next dimensionless variable not have r.
This means we need some combination of E ρ & P_0
This means we need to find a, b & c st
$\Pi_2 = E^a \rho^b P^c$ is unitless.

$$\Rightarrow \quad 2a - 3b - c = 0$$
and $\quad a + b + c = 0$

add these $\quad 3a - 2b = 0 \quad \Rightarrow \quad a = 2/3\, b$

let $b = 1 \quad a = 2/3$

$c = -a - b = -1 - 2/3 = -5/3$

$$\Rightarrow \Pi_2 = \frac{E^{2/3}\rho}{P^{5/3}}$$

Now the Buckingham Pi theorem says.
$\exists$ a physical law such that
$F(\Pi_1, \Pi_2) = 0 \Rightarrow \exists$ a function g
st $\Pi_1 = g(\Pi_2)$
we can now solve for r.

$$\frac{r^5\rho}{t^2E} = g\left(\frac{E^2\rho}{P^{5/3}}\right) \rightarrow r = \left(\frac{t^2E}{\rho}\, g\left(\frac{E^2\rho}{P^{5/3}}\right)\right)^{1/5}$$

r still varies like $t^{2/5}$.

4) Page 17 #2

$[m] = M \quad [\rho] = M/L^3 \quad [V] = L^3 \quad [s] = L^2$

$$\begin{array}{c|cccc} & m & \rho & V & s \\ \hline M & 1 & 1 & 0 & 0 \\ L & 0 & -3 & 3 & 2 \end{array}$$

The dimension matrix A has size 2x4.
and has rank 2.

The Buckingham Pi Theorem says that there are 2 dimensionless parameters π_1, π_2 st $F(\pi_1, \pi_2) = 0$ for some F.

$\Rightarrow$ $\exists$ a function g st

The dimensionless quantities are

$\pi_1 = \dfrac{mV}{\rho}$ since $\dfrac{m}{\rho}$ is the only way to get rid of M units.

The other dimensionless quantity involves only V & S

$\Rightarrow$ need to pick a & b st $-3a + -2b = 0$

$\Rightarrow$ $a = -2/3\, b$ let $b = 1$

$\Rightarrow$ $\pi_2 = V^{-2/3} S$

$\Rightarrow$ $\exists$ a function g st $\dfrac{mV}{\rho} = g(V^{-2/3} S)$

This would be the physical law the ecologist is looking for.

5) Page 19 #14

Cars are driven at constant speed V.

F = all the forces.

affected by V, C, K,

Goal: Find F in terms of V, C and K.

$$[V] = L/T \quad [C] = L^3/T \quad [K] = m/LT^2$$

$$[F] = mL/T^2$$

Make dimension matrix A

$$\begin{array}{c} \\ M \\ L \\ T \end{array} \begin{array}{c} \begin{array}{cccc} V & C & K & F \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 3 & -1 & 1 \\ -1 & -1 & -2 & -2 \end{bmatrix} \end{array}$$

Goal: Find a vector $\bar{\alpha}$ st

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & -1 \\ -1 & -1 & -2 \end{bmatrix} \bar{\alpha} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$ ← Solve for F

$\rightarrow \alpha_3 = 1$

$\alpha_1 + 3\alpha_2 = 1 + 1 = 2$

$-\alpha_1 - \alpha_2 = -2 + 2 = 0 \rightarrow \alpha_1 = -\alpha_2 = -1$

$\rightarrow -\alpha_2 + 3\alpha_2 = 2 \rightarrow 2\alpha_2 = 2 \rightarrow \alpha_2 = 1$

$\rightarrow F = \beta V^{-1} C^1 K^1$ for a constant β

$\sqrt{\frac{m}{k}}$ is the undamped oscillation period (ie. period of oscillation if $a=0 \Rightarrow$ no damping)

$\sqrt{\frac{m}{aV}}$ is the time for damping to slow down the initial velocity.

Note: $\sqrt{\frac{m}{k}}\left(\frac{k}{Va}\right)^{\alpha}$ for any α has units of T.

but we choose $\alpha = 1/2$. Do you know why?

If the restoring force is small compared to the damping force, we should choose our time scale to be $t_c = \sqrt{\frac{m}{Va}}$

to eliminate "large" numbers.

let $\bar{t} = \frac{t}{t_c}$ & $\bar{x} = \frac{x}{x_c} \Rightarrow x = x_c \bar{x}$

now $\frac{dx}{dt} = \frac{dx}{d\bar{t}}\frac{d\bar{t}}{dt} = \frac{1}{t_c}\frac{dx}{d\bar{t}} \Rightarrow \frac{d(\ )}{dt} = \frac{1}{t_c}\frac{d(\ )}{d\bar{t}}$

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{1}{t_c}\frac{d}{d\bar{t}}\left(\frac{dx}{dt}\right) = \frac{1}{t_c^2}\frac{d^2x}{d\bar{t}}$$

now to switch to $\bar{x}$ units

$$\frac{dx}{d\bar{t}} = \alpha_c \frac{d\bar{x}}{d\bar{t}} \rightarrow \frac{d^2x}{d\bar{t}^2} = x_c \frac{d^2\bar{x}}{d\bar{t}^2}$$

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{x_c}{t_c^2}\frac{d^2\bar{x}}{d\bar{t}^2}$$

Plugging this into the ODE, we find

$$m \frac{x_c}{t_c^2} \frac{d^2\bar{x}}{d\bar{t}^2} = -a x_c \bar{x} \left| \frac{x_c}{t_c} \frac{d\bar{x}}{d\bar{t}} \right| - k x_c \bar{x}$$

$$\frac{d^2\bar{x}}{d\bar{t}^2} = -\frac{a}{m} t_c \bar{x} \left| x_c \frac{d\bar{x}}{d\bar{t}} \right| - \frac{k}{m} t_c^2 \bar{x}$$

$$= \underbrace{-\frac{a}{m}\sqrt{\frac{m}{Va}}\, \bar{x} \left| x_c \frac{d\bar{x}}{d\bar{t}} \right|}_{\text{Choose } x_c \text{ such that this equals } -\bar{x}\left|\frac{d\bar{x}}{d\bar{t}}\right|} - \frac{k}{Va} \bar{x}$$

$$\Rightarrow \boxed{x_c = \sqrt{\frac{mV}{a}}}$$

$$= -\bar{x}\left|\frac{d\bar{x}}{d\bar{t}}\right| - \varepsilon \bar{x}$$ where $\varepsilon = \frac{k}{Va}$ is dimensionless. (and small)

Now for initial conditions

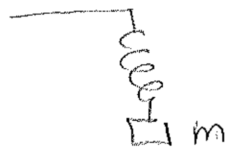
$$x(0) = x_c \bar{x}(0) = 0 \quad \Rightarrow \quad \bar{x}(0) = 0.$$

$$x'(0) = \frac{x_c}{t_c} \bar{x}'(0) = V \quad \Rightarrow \quad \bar{x}'(0) = \frac{V\sqrt{\frac{m}{Va}}}{\sqrt{\frac{mV}{a}}} = 1$$

So the dimensionless problem is

$$\begin{cases} \dfrac{d^2\bar{x}}{d\bar{t}^2} = -\bar{x}\left|\dfrac{d\bar{x}}{d\bar{t}}\right| - \varepsilon \bar{x} \\ \bar{x}'(0) = 1 \\ \bar{x}(0) = 0 \end{cases}$$

7) Page 31 #4



intial velocity V

x = position $\Rightarrow$ $x'(0) = V$

$x(0) = 0$

$$mx'' = -a\,x\,|x'| - k\,x$$

Fundamental Units

$[m] = M \qquad [V] = L/T$

$[V] = L/T^2$

we know $[F] = ML/T^2$ (units of force)

$\Rightarrow \quad [a\,x\,|x'|] = [a]\,L(L/T) = [a]\,L^2\,T^{-1} = MLT^{-2}$

$\Rightarrow [a] = ML^{-1}T^{-1}$

$[kx] = [k]\,L = MLT^{-2} \Rightarrow [k] = MT^{-2}$

The dimension matrix is

$$\begin{array}{c c} & \begin{array}{cccc} m & V & a & k \end{array} \\ \begin{array}{c} M \\ L \\ T \end{array} & \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & -1 & -2 \end{bmatrix} \end{array}$$

No x $\Rightarrow$ not a Pi Thm matrix

rank 3 matrix. $\Rightarrow$ dim(Nullspace) = 1

$\Rightarrow$ There will be a 1-dimensional subspace of solutions to the Equation

$$A\,\bar{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \to \text{only } T \text{ units.}$$

$\Rightarrow$ a timescale.

Possible timescales:

$$t_c = \sqrt{\frac{m}{k}}, \quad t_c = \sqrt{\frac{m}{Va}}$$

7) Page 33 #10. let $x(t)$ denote the position of the ball.
ball of mass m tossed w/ initial velocity V,
$F_{\text{air restistance}} = c(x'|x'|)$

$$m x'' = -mg - c x'|x'|$$
$$x'(0) = V$$
$$x(0) = 0 \quad \leftarrow$$ guess ball was tossed from ground. or tabel origin at pt where tossed.

Fundamental units.

$[m] = M \qquad [V] = L/T$

$[g] = L/T^2$

$[c x'|x'|] = [c]\, L^2 T^{-2} = M L T^{-2} \Rightarrow [c] = M L^{-1}$

Dimension Matrix

$$\begin{array}{c} \\ M \\ L \\ T \end{array} \begin{array}{c} \begin{array}{cccc} m & g & V & c \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -1 & 0 \end{bmatrix} \end{array}$$

$\left[\frac{m}{gc}\right] = M(L^{-1}T^{2})(LM^{-1}) = T^{2}$

Possible time scales

$$t_c = \sqrt{\frac{m}{gc}} \quad \text{or} \quad t_c = \frac{c}{Vm} \quad \text{or} \quad t_c = \frac{V}{g}$$

We want to choose $t_c = V/g$ because it does not depend on c which could be small or large. *

$\Rightarrow \bar{t} = t/t_c \qquad$ let $\bar{x} = \dfrac{x}{x_c} \quad \rightarrow \quad x = \bar{x}\, x_c$

* we do not want to change the dynamics of the problem.

by last problem $\frac{d^2x}{dt^2} = \frac{x_c}{t_c^2}\frac{d^2\bar{x}}{d\bar{t}^2}$, $\frac{dx}{dt} = \frac{x_c}{t_c}\frac{d\bar{x}}{d\bar{t}}$

Plugging this into the equation, we find.

$$\frac{m x_c}{t_c^2}\bar{x}'' = -mg - c\frac{x_c^2}{t_c^2}x'|x'|$$

$$\Rightarrow \bar{x}'' = -\frac{t_c^2}{x_c}g - \frac{c}{m}x_c\, x'|x'|$$

$$= -\frac{V^2}{g^2 x_c}g - \frac{c}{m}x_c\, x'|x'|$$

$$= \underbrace{-\frac{V^2}{g x_c}} - \frac{c}{m}x_c x'|x'|$$

make this term equal to 1. $\Rightarrow x_c = V^2/g$

let $\varepsilon = \frac{cV^2}{mg}$. This is dimensionless

$\Rightarrow$ The ODE becomes

$$\bar{x}'' = -1 - \varepsilon x'|x'|$$

Now initial conditions.

$$x(0) = x_c\bar{x}(0) = 0 \quad \Rightarrow \quad \bar{x}(0) = 0$$

$$x'(0) = \frac{x_c}{t_c}\bar{x}'(0) = V \quad \Rightarrow \quad \bar{x}'(0) = \frac{Vt_c}{x_c}$$

$$= V\left(\frac{V}{g}\right)\left(\frac{g}{V^2}\right) = \underline{1}$$

8) Page 35 #15

IVP for damped pendulum is

$$\frac{d^2\theta}{dt^2} + k\frac{d\theta}{dt} + \frac{g}{l}\sin\theta = 0$$

$\theta(0) = \theta_0$

$\theta'(0) = \omega_0$

Fundamental units.

$[g] = LT^{-2}$ $[l] = L$

$[k] = T^{-1}$ $[\theta_0] = \Theta$ $[\omega_0] = \Theta T^{-1}$

Dimension matrix

$$\begin{array}{c} \\ L \\ T \\ \Theta \end{array}\begin{array}{c} \begin{array}{ccccc} g & l & \theta_0 & \omega_0 & k \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(a) There a 3 time scales

$t_1 = k^{-1}$ $t_2 = \theta_0/\omega_0$ $t_3 = \sqrt{l/g}$

t_3 corresponds to the period of undamped oscillations (we found this in class)

(b) Small amplitude $\Rightarrow$ small period.

$\Rightarrow \sqrt{l/g}$ is small.

It also means the damping coefficient is small.

let: $\bar{t} = t/t_c$ and $\bar{\theta} = \theta/\theta_c$

we know $\dfrac{d^2\theta}{dt^2} = \dfrac{\theta_c}{t_c^2}\dfrac{d^2\bar{\theta}}{d\bar{t}^2}$ & $\dfrac{d\theta}{dt} = \dfrac{\theta_c}{t_c}\dfrac{d\bar{\theta}}{d\bar{t}}$

Plugging this into the differential equation we find.

$$\frac{\theta_c}{t_c^2}\bar{\theta}'' + \frac{k\theta_c}{t_c}\bar{\theta}' + \frac{g}{l}\sin(\theta_c\bar{\theta}) = 0$$

$$\Rightarrow \bar{\theta}'' + k\, t_c\bar{\theta}' + \frac{g}{l}\frac{t_c^2}{\theta_c}\sin(\theta_c\bar{\theta}) = 0$$

We want kt_c small $\Rightarrow$ take $t_c = \sqrt{l/g}$

let. $\varepsilon = k\sqrt{l/g}$

$\Rightarrow$ ODE becomes $\bar{\theta}'' + \varepsilon\bar{\theta}' + 1/\theta_c \sin(\theta_c\bar{\theta}) = 0$

(If $\theta < 1$ then $\sin(\theta_c\bar{\theta}) \approx \bar{\theta}\theta_c$

$\Rightarrow$ $\bar{\theta}'' + \varepsilon\bar{\theta}' + \theta = 0$

take $\theta_c = \theta_0$

Initial conditions

$\theta(0) = \theta_c\bar{\theta}(0) = \theta_0 \rightarrow \bar{\theta}(0) = 1$

$\theta'(0) = \dfrac{\theta_c}{t_c}\bar{\theta}'(0) = \omega_0 \rightarrow \bar{\theta}'(0) = \omega_0 t_c/\theta_c = \dfrac{\omega_0}{\theta_0}\sqrt{\dfrac{l}{g}} = \alpha$

Page 4 #1

a) $u' + 2u = e^{-t}$

$\mu = e^{\int 2dt} = e^{2t+c}$ → leave c behind.

$e^{2t}\mu' + 2e^{2t}u = e^{t}$

$(e^{2t}u)' = e^{t}$

$e^{2t}u = e^{t} + C$

$u(t) = e^{-t} + Ce^{-2t}$

b) $u'' + 4u = t\sin 2t$

1st Find homogeneous solution

① $r^2 + 4 = 0 \Rightarrow r = \pm 2i$

$u(t) = A\sin 2t + B\cos 2t$

2nd Find particular solution.

you can use variation of parameter or undetermined coefficients depending on if you like trig. integrals.

I use undetermined coefficients.

$u_p = t^s(At+B)\cos(2t) + t^s(Ct+D)\sin(2t)$

where S is the # of times $2i$ is a root of ①. ⇒ $S = 1$.

$$u_p = (At^2+Bt)\cos 2t \quad + \quad (Ct^2+Dt)\sin 2t$$

$$u_p' = (At^2+Bt)(-2\sin 2t) + \quad (2At+B)\cos 2t$$
$$+(Ct^2+Dt)(+2\cos 2t) + \quad (2tC+D)\sin 2t$$

$$= [-2(At^2+Bt) + 2tC+D]\sin 2t \quad + [2Ct^2+2Dt+2At+B]\cos 2t$$

$$u_p'' = [-2(At^2+Bt)+2tC+D](2\cos 2t) + [-2(2At+B)+2C]\sin 2t$$
$$+[2Ct^2+2Dt+2At+B](-2\sin 2t) + [4Ct+2D+2A]\cos 2t$$

$$= [-4(At^2+Bt)+4tC+\cancel{2D}+4Ct+2D+2A]\cos 2t$$
$$+[-4Ct^2-4Dt+\cancel{2}-4At-2B] - 4At-2B+2C]$$
$$\sin 2t$$

$$= [-4At^2-4Bt+8tC+4D+2A]\cos 2t \quad +$$
$$[-4Ct^2-4Dt-8At-4B+2C]\sin 2t$$

Plug into ODE

$$u_p'' + 4u_p = t\sin 2t$$

Collect like terms

$t^2\cos 2t: \quad -4A+4A=0$ ✓

$t\cos 2t: \quad -4B+8C+4B=0 \quad \Rightarrow C=0$

$\cos 2t: \quad 4D+2A \quad =0 \quad \Rightarrow D=-\frac{1}{2}A=\frac{1}{16}$

$t^2\sin 2t: \quad -4C+4C=0$ ✓

$t\sin 2t: \quad -8A-4D+4D=1 \quad \Rightarrow A=-1/8$

$\sin 2t: \quad -4B+2C=0 \Rightarrow B=0$

$$\Rightarrow u_p = -\frac{1}{8}t^2\cos 2t + \frac{1}{16}t\sin 2t.$$

$\Rightarrow$ Solution is

$$u(t) = A\sin(2t) + B\cos(2t) + \left(\frac{-1}{8}\right)t^2\cos 2t + \frac{1}{16}t\sin 2t.$$

d) $t^2u'' - 3tu' + 4u = 0$

Guess $u = t^r$ & Plugin

$\Rightarrow r(r-1)t^r - 3rt^r + 4t^r = 0$

$r(r-1) - 3r + 4 = 0$

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0 \Rightarrow r = 2$

$\Rightarrow$ solution is

$u(t) = At^2 + Bt^2 \ln t$

h) $u u'' + (u')^3 = 0$

let $v = u'$

$u'' = \frac{dv}{dt} = \frac{dv}{du}\frac{du}{dt} = \frac{dv}{du} v$

Plug into equation to get a 1st order problem

$uvv' - v^3 = 0$

$\rightarrow v' = \frac{v^3}{uv} = \frac{v^2}{u}$ This is separable

$\int \frac{dv}{v^2} = \int \frac{1}{u} du$

$-v^{-1} = \ln u + c$

$\rightarrow -\frac{1}{v} = \ln u + c$

$\rightarrow v = -\frac{1}{\ln u + c}$

$$\int \ln x \, dx \qquad \begin{array}{ll} u = \ln x & dv = 1\,dx \\ du = \frac{1}{x}\,dx & v = x \end{array}$$

$$= x\ln x - \int dx$$
$$= x\ln x - x$$

$$\frac{du}{dt} = \frac{-1}{\ln u + c}$$ this is separable.

$$\int (\ln u + c)\, du = \int -1\, dt$$

$$u\ln u - u + cu = -t + D$$
$$u\ln u + (c-1)u = -t + D$$