① 9/16/07 Barnett

MATH 46 HW 7 ~ SOLUTIONS ~ (2009)

See end for #4

p.243-247 #7

eigvals $\lambda_n = \frac{\pi}{n^2}$ ie, the Fourier sine basis.

efuncs $u_n(x) = \sin nx$

a. $2 \neq$ eigval. so only trivial soln $u \equiv 0$ $\left(c_i = 0 \ \forall i \xRightarrow{\text{completeness}} u \equiv 0\right)$

b. $\frac{\pi}{9} = \lambda_3$ and $f(x) = x(\pi - x) = \frac{8}{\pi}\left[\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \dots\right]$

$$= \sum_{j=1}^{\infty} f_j \phi_j(x)$$

with coeffs $\frac{8}{\pi}, \frac{8}{\pi 3^3}, \frac{8}{\pi 5^3}, \dots$ etc.

this $f_3 \neq 0$ so no solution.

c. $2 \neq$ eigval. so $c_j = \frac{f_j}{\lambda_j - 2} = \frac{\left(\frac{8}{\pi j^3}\right)}{\frac{\pi}{j^2} - 2}$ for j = odd, zero for j even.

coeffs of efunc. expansion of u.

$$u(x) = \sum_{j=1}^{\infty} c_j \phi_j(x) = \frac{8}{\pi} \sum_{\substack{j=1 \\ j \text{ odd}}}^{\infty} \frac{\sin jx}{j^3\left(\frac{\pi}{j^2} - 2\right)} \quad \text{or} \quad \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3\left[\frac{\pi}{(2n-1)^2} - 2\right]}$$

closed-form series solution ... beautiful. (you could also get nonseries ODE soln by taking derivative of $Ku - 2u = f$)

d. $f(x) = \sin 2x$ which is orthog. to $\sin 3x$, so $f_3 = 0$, so c_3 arbitrary.

All other coeffs in eigenfunc expansion given by $c_j = \frac{f_j}{\lambda_j - \pi/9}$ (nonunique soln.)

$f_n = 0$ by orthog, except $f_2 = 1$.

$$\Rightarrow u(x) = \underbrace{c_3}_{\text{arb.}} \sin 3x + \frac{1}{\pi/2^2 - \pi/9} \sin 2x$$

③

$$v(x) = -u_2(x) \int_a^x \frac{u_1 f}{pW} d\xi - u_1(x) \int_x^b \frac{u_2 f}{pW} d\xi$$

p. 250. Where unless stated everything is a func. of ξ inside the integrals.

use Leibniz.

$$v'(x) = -u_2'(x) \int_a^x \frac{u_1 f}{pW} d\xi - \cancel{u_2(x) \frac{u_1(x) f(x)}{p(x) W(x)}} - u_1'(x) \int_x^b \frac{u_2 f}{pW} d\xi + \cancel{u_1(x) \frac{u_2(x) f(x)}{p(x) W(x)}}$$

cancel.

$$-(pv')'(x) = +(p(x) u_2'(x))' \int_a^x \frac{u_1 f}{pW} d\xi + \cancel{p(x)} u_2'(x) \frac{u_1(x) f(x)}{\cancel{p(x)} W(x)} + (p(x) u_1'(x))' \int_x^b \frac{u_2 f}{pW} d\xi - \cancel{p(x)} u_1'(x) \frac{u_2(x) f(x)}{\cancel{p(x)} W(x)}$$

Notice the two non-integral terms in this formula for $(pv')'$

combine to give $\quad \dfrac{u_2'(x)\,u_1(x) - u_1'(x)u_2(x)}{\underbrace{W(x)}_{1}} f(x) = f(x)$

Adding $q(x)\,v(x)$ cancels the other (integral) terms since $\begin{cases} -(p(x)\,u_1'(x))' + q(x)\,u_1(x) = 0 \\ \text{and} \quad -(p(x)\,u_2'(x))' + q(x)\,u_2(x) = 0 \end{cases}$

$\Rightarrow \quad Av = f$ (as functions) $\quad$ because u_1, u_2 are solutions to $Au = 0$.

We must check $v(x)$ satisfies the BCs too! $\quad$ for $x = a$ only 2^{nd} integral survives in $v(x)$ & $v'(x)$

$$B_1 v := \alpha_1 v(a) + \alpha_2 v'(a) = -\alpha_1 u_1(a) \int_a^b \frac{u_2 f}{pW}\,d\xi \; \underbrace{- \alpha_2 u_1'(a) \int_a^b \frac{u_2 f}{pW}\,d\xi}_{\alpha_2 v'(a) \text{ using above } v'(x) \text{ formula}}$$

$$= -\underbrace{\left(\alpha_1 u_1(a) + \alpha_2 u_1'(a)\right)}_{0 \text{ since } B_1 u_1 = 0} \int_a^b \frac{u_2 f}{pW}\,d\xi.$$

Similarly $B_2 v = 0$.

QED: $\quad Lv = f \quad$ so L^{-1} has integral kernel given by $g(x,\xi)$, the Green's Func.

p. 257–258 #1 $\quad u'' + \pi^2 u =: Lu \quad$ Does L have a zero eigenvalue?

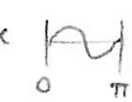
for BCs $u(0) = u(1) = 0$

$Lu = 0$ has soln $\quad A \sin \pi x + \underbrace{B \cos \pi x}_{\text{zero due to BCs}}$

nontrivial $\Rightarrow$ eigenfunction

So has nonunique solution if $(f, \sin \pi x) = 0$, otherwise no soln. (see Thm 4.23 although we didn't do in lecture).

#2 $\quad Lu = u'' + 4u$ on $0 < x < \pi$. $\quad$ L has a $\lambda = 0$ with efunc $\sin 2x$



$\Rightarrow$ no Green's func; only soln. if $(f, \sin 2x) = 0$. (since that's the nontriv. soln to $Lu = 0$)

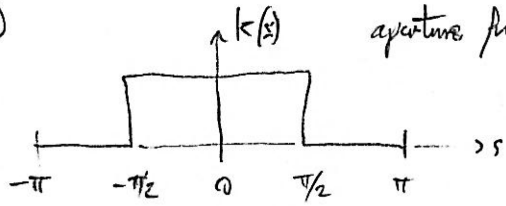
$L = \dfrac{d^2}{dx^2} + \underbrace{4I}_{\text{identity}}$ so has $\begin{cases}\text{eigenvalues} & \lambda_n = -n^2 + 4 \\ \text{efuncs} & \phi_n(x) = \sin nx\end{cases}$, using those of $\dfrac{d^2}{dx^2}$.

So if $u(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$ and $f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x)$ $\quad f_n = \dfrac{(\phi_n, f)}{\|\phi_n\|^2} = \dfrac{2}{\pi} \int_0^\pi (\sin nx) f(x)\,dx$

gives $c_n = \dfrac{f_n}{\lambda_n} \quad \forall n$

So $u(x) = \sum_{\substack{n=1 \\ n \neq 2}}^{\infty} \dfrac{f_n}{-n^2+4} \sin nx + \underbrace{c}_{\text{arbitrary}} \sin 2x$, with f_n given above.

A



aperture func.

Any $\overset{\text{even, } 2\pi\text{-periodic}}{\text{convolution}}$ operator $Ku(x) = \int_{-\pi}^{\pi} k(x-y)\, u(y)\, dy$

has $\{1, \cos nx, \sin nx\}_{n=1,2,\dots}$ as its eigfuncs.

For 1 & $\cos nx$ see Worksheet on deblurring.

Here's proof for $\phi(x) = \sin nx$: $(K\phi)(x) = \int_{-\pi}^{\pi} k(x-y) \sin ny\, dy \overset{s=y-x}{=} \int_{-\pi-x}^{\pi-x} k(-s) \sin n(s+x)\, ds$

$= \cos nx \underbrace{\int_{-\pi}^{\pi} k(s) \sin ns\, ds}_{\text{even}} + \underbrace{\sin nx}_{\phi(x)} \underbrace{\boxed{\int_{-\pi}^{\pi} k(s) \cos ns\, ds}}_{\text{eigenvalue } \lambda_n} =: \pi k_n$ where k_n are Fourier cos coeffs of $k(s)$.

Get Fourier coeffs of k: $k(s) = \frac{k_0}{2} + \sum_{n=1}^{\infty} k_n \cos ns$, $\quad k_n = \frac{1}{\pi}\int_{-\pi}^{\pi} k(s) \cos ns\, ds$, $n \geq 0$.

$\lambda_0 = \pi k_0 = \int_{-\pi}^{\pi} k(s)\, ds = \pi$

$\lambda_n = \pi k_n = \int_{-\pi/2}^{\pi/2} 1 \cdot \cos ns\, ds = \frac{1}{n}\left[\sin ns\right]_{-\pi/2}^{\pi/2} = \begin{cases} 0 & n = 2, 4, \dots \text{ (even)} \\ 2/n & n = 1, 5, 9, \dots \text{ odd} \\ -2/n & n = 3, 7, 11, \dots \text{ odd} \end{cases}$ group as $(-1)^{\frac{n-1}{2}} \frac{2}{n}$

i) Blurring:

$$A_0 = \lambda_0 a_0 = \pi a_0$$
$$\left.\begin{aligned} A_n &= (-1)^{\frac{n-1}{2}} \frac{2}{n} a_n \\ B_n &= (-1)^{\frac{n-1}{2}} \frac{2}{n} b_n \end{aligned}\right\} \quad n = \text{odd},$$

$(A_n = B_n = 0$ for $n > 0$ even$)$.

ii) $n > 0$ even cannot be reconstructed since they've been killed (zero).

$\Rightarrow$ write Fourier series for $f(x) = \frac{A_0}{2\pi} + \sum_{n=1,3,5,\dots}^{\infty} \underbrace{(-1)^{\frac{n-1}{2}} \frac{n}{2}}_{\text{note dividing by } \pi k_n} \left[A_n \cos nx + B_n \sin nx\right]$

iii) 'Blow up' factor of n^{th} coeff when reconstruct is $\frac{n}{2}$

so $\underset{\text{error}}{(0.01)} \frac{n}{2} \leq \underset{\text{max pixel error}}{0.3}$ so $n \leq 60$ in which case $n = 0, 1, 3, 5, \dots, 59$ may be reconstructed. (nonzero n gives 2 coeffs a_n, b_n each) $\Rightarrow$ 61 coeffs total.

#5 General homog. soln $-(K(x)u')' = 0 \xrightarrow{\text{peel off one deriv.}} K(x)\, u'(x) = c$ some const.

$\Rightarrow u'(x) = \frac{c}{K(x)} \xrightarrow{\text{peel off inner deriv.}} u(x) = c \int K(x)^{-1} dx + d$

best we can do since $K(x)$ given only as general func.

Choose u_1, u_2:

$u_1(x) = \int_0^x K(s)^{-1} ds \qquad u_2(x) = \int_x^1 K(s)^{-1} ds \qquad$ sat. $B_1 u_1 = 0$, $B_2 u_2 = 0$.

$\begin{cases} W = u_1 u_2' - u_1' u_2 = -K^{-1}(x) \int_0^x K(s)^{-1} ds - K^{-1}(x) \int_x^1 K(s)^{-1} ds = -K^{-1}(x) \int_0^1 K(s)^{-1} ds \\ p = K(x) \end{cases}$

so $p(\xi)\, W(\xi) = \overset{1}{\cancel{K(\xi) K^{-1}(\xi)}} \int_0^1 K(s)^{-1} ds$ = const. indep. of ξ, nice.

so $g(x, \xi) = \frac{+1}{\int_0^1 K(s)^{-1} ds} \begin{cases} \int_0^x K(s)^{-1} ds \cdot \int_\xi^1 K(s)^{-1} ds & x < \xi \\ \int_x^1 K(s)^{-1} ds \cdot \int_0^\xi K(s)^{-1} ds & x > \xi. \end{cases}$

Soln. $u(x) = \frac{+1}{\int_0^1 k^{-1}(s)ds}\left[\int_0^x u_2(x)\, u_1(\xi)\, d\xi + \int_x^1 u_1(x) u_2(\xi)\, d\xi\right]$ with $u_1(x)$, $u_2(x)$ as given above.

#7 Find an L with eigfuncs $\phi_n = \sin nx$, $n=1,2,\dots$, eigvals $\lambda_n = n^2$: $L = -\frac{d^2}{dx^2}$ on $[0,\pi]$ is such (Dirichlet BCs).

As with Worksheet on this L,

$u_1 = x$
$u_2 = \pi - x$ $\Big\}$ $W = u_1 u_2' - u_1' u_2 = -x - (\pi - x) = -\pi$

$p \equiv 1$, $q \equiv 0$.

split expression for Green's func. (4.46)

So $g(x,\xi) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\,\phi_n(\xi)}{\lambda_n} = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin nx \sin n\xi}{n^2} = \frac{1}{\pi}\begin{cases} x(\pi - \xi) & x<\xi \\ \xi(\pi - x) & x>\xi \end{cases}$

normalized: $\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin nx$.

efunc expansion (p.257)

#8 Recall $\lambda_n = \frac{1}{4} + \pi^2 n^2$, $y_n(x) = x^{-1/2}\sin(n\pi \ln x)$ from #7 p.225.

then $g(x,\xi) = \sum_{n=1}^{\infty} \frac{x^{-1/2}\sin(n\pi \ln x)\; \xi^{-1/2}\sin(n\pi \ln \xi)}{\frac{1}{4} + \pi^2 n^2}$ ← efunc expansion way

and $(L^{-1}f)(x) = \int_1^e g(x,\xi) f(\xi)\, d\xi$

(I forgot normalization).

Alternatively, solve $Au = 0$ ie $-(x^2 u')' = 0$ $\Rightarrow$ $x^2 u' = \text{const} = c$

$\Rightarrow$ $u' = \frac{c}{x^2}$, $u = \frac{-c}{x} + d$

Choose c, d to match each BC in turn:

$u_1(1) = 0$: $c = d$ so $u_1(x) = 1 - \frac{1}{x}$
$u_2(e) = 0$: $\frac{c}{e} = d$ so $u_2(x) = 1 - \frac{e}{x}$ $\Big\}$ $W = u_1 u_2' - u_1' u_2 = \frac{e}{x^2}\left(1 - \frac{1}{x}\right) - \frac{1}{x^2}\left(1 - \frac{e}{x}\right)$

$= \frac{1-e}{x^2}$

$p(\xi) W(\xi) = \xi^2 \frac{(1-e)}{\xi^2} = 1 - e$

So $g(x,\xi) = \frac{1}{e-1}\begin{cases} \left(1 - \frac{1}{x}\right)\left(1 - \frac{e}{\xi}\right) & x<\xi \\ \left(1 - \frac{e}{x}\right)\left(1 - \frac{1}{\xi}\right) & x>\xi. \end{cases}$

#4 a. $k(x,y) = k(x+y) = \frac{k_0}{2} + k_1(\cos x \cos y - \sin x \sin y) + k_2(\cos 2x \cos 2y - \sin 2x \sin 2y) + \cdots$

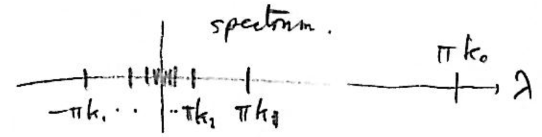
by addition formula.

$\boxed{\text{Note}}$ that $k(x,y) = \cos nx \cos ny$ would give operator which kills all Fourier modes except $\cos ny$, by orthonality on $[-\pi,\pi]$.
Find its eigenvalue? $\lambda \cos nx = \int_{-\pi}^{\pi} \cos nx \cos ny \underbrace{\cos ny}_{u(y)} dy = \cos nx \cdot \frac{2\pi}{2}$ so $\lambda = \pi$.

So by orthogonality, $\{1, \cos nx, \sin nx\}$ are all eigenfunctions. (try each!)

$$\begin{cases} u_0(x) = 1 & \text{has } \lambda = \frac{k_0}{2} \cdot 2\pi = \pi k_0 \\ u_n(x) = \cos nx & \text{has } \lambda = \pi k_1 \\ u_{-n}(x) = \sin nx & \text{has } \lambda = -\pi k_1 \end{cases}$$

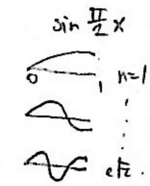
spectrum.



b. $\lambda u(x) \overset{\text{for efunc}}{=} K u(x) = \int_0^1 \min(x,y)\, u(y)\, dy = \int_0^x y\, u(y)\, dy + x \int_x^1 u(y)\, dy$.

$\frac{d}{dx}$ ↳ $\lambda u'(x) = \cancel{x u(x)} + \int_x^1 u(y)\, dy - \cancel{x u(x)}$

$\frac{d}{dx}$ ↳ $\lambda u''(x) = -u(x)$ with BCs $u(0) = 0$, $u'(1) = 0$ from here



so $u_n(x) = \sin[(2n-1)\pi x/2]$ with $\lambda_n = \left(\frac{2}{\pi(2n-1)}\right)^2$ $n = 1, 2, \ldots$ D-N mixed BCs.

c. $\lambda u(x) = K u(x) = (\pi - x)\int_0^x y\, u(y)\, dy + x \int_x^{\pi} (\pi - y)\, u(y)\, dy$ } Leibniz formula.

↳ $\lambda u'(x) = -\int_0^x y\, u(y)\, dy + \cancel{(\pi - x)\, x\, u(x)} + \int_x^{\pi} (\pi - y)\, u(y)\, dy - \cancel{x(\pi - x)\, u(x)}$

↳ $\lambda u''(x) = -\cancel{x u(x)} - (\pi - \cancel{x})\, u(x) = -\pi u(x)$

so $u'' + \frac{\pi}{\lambda} u = 0$ with $u(0) = u(\pi) = 0$ Dirichlet BCs.

so $u_n = \sin nx$ with $\frac{\pi}{\lambda_n} = n^2$ or $\lambda_n = \frac{\pi}{n^2}$ $n = 1, 2, \ldots$

d. Volterra operators have no eigenvalues. (p. 236)