MATH 46 HW 6 — SOLUTIONS. 2009. (in a random order). ① Barnett

#24 (p.243-247) $u'' + u = t^2 \qquad u(0) = 1 \qquad u'(0) = 0$

$\int_0^t ds$ replace t by s then integrate

$$u'(t) - \underbrace{u'(0)}_{0 \text{ by IC}} + \int_0^t u(s)ds = \int_0^t s^2 ds = \tfrac{1}{3}t^3$$

$\int_0^t ds$ do the same again

$$u(t) - \underbrace{u(0)}_{1 \text{ by IC}} + \underbrace{\int_0^t\int_0^s u(r)\,dr}_{\text{Lemma: } \int_0^t (t-s)u(s)\,ds} = \tfrac{1}{3}\int_0^t s^3 ds = \tfrac{1}{12}t^4$$

so
$$\int_0^t \underbrace{(t-s)}_{k(t,s) \text{ kernel of } K} u(s)\,ds + u(t) = \underbrace{1 + \tfrac{1}{12}t^4}_{f(t)}$$

(A) see degenerate.pdf (worksheet from X-hr of 5/15/09 & 5/12/11).

irrelevant for HW6:

#6 p.224-226 $\lambda = 0$: $y'' = 0$ so $y = Ax + B$ $\quad$ @$x=0$: $B - aA = 0$; $\quad$ @$x=l$: $Al + B + bA = 0$ subs. in

so $A(l + a + b) = 0 \Rightarrow -l = a + b \iff$ (iff) nontrivial solution ($\lambda=0$ eigval).

#17 $-(x^2y')' = \lambda y$

mult. by y & integrate
$$-\underbrace{\int_1^e y(x^2y')'\,dx}_{\text{by parts}} = \lambda\int_1^e y^2\,dx$$

$$\int_1^e y'x^2y'\,dx - \underbrace{[yx^2y']_1^e}_{\text{vanishes due to BC}} = \int_1^e x^2(y')^2dx > 0$$

so $\lambda > 0$ since ratio of positive quantities.

Efuncs: (expand derivs) $-x^2y'' - 2xy' - \lambda y = 0$ Cauchy–Euler, $\begin{cases} a=1 \\ b=2 \\ c=\lambda \end{cases}$ see p.39 (& type on p.40)

$m^2 + m + \lambda = 0$ so $m = -\tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - \lambda}$

Since BCs are zero Dirichlet, must have $\lambda > 1/4$ (oscillatory case) ... you can check $\lambda \le 1/4$ not eigval. (forces $A=B=0$)

Gen. soln. $y(x) = A x^{-1/2} \sin\Big(\underbrace{\sqrt{\lambda - 1/4}\,\ln x}_{\text{must equal } n\pi \text{ for } x=e}\Big) + \cancel{B} x^{-1/2} \cos\left(\sqrt{\lambda - 1/4}\,\ln x\right)$ since $y(1)=0$

$\Rightarrow \quad \lambda - 1/4 = \pi^2 n^2, \quad \lambda_n = \frac{1}{4} + \pi^2 n^2 \qquad n = 1, 2, \ldots$

efuncs: $y_n(x) = x^{-1/2} \sin(n\pi \ln x)$

#13

$y'' + 2by' + \lambda y = 0 \qquad b>0 \qquad y(0) = y(1) = 0 \qquad 0<x<1$

e^{rt}: $r^2 + 2br + \lambda = 0 \qquad r = -b \pm \sqrt{b^2 - \lambda}$

case $\lambda < b^2$: $Ae^{r_1 x} + Be^{r_2 x}$ with $r_1 \neq r_2$ so $\left.\begin{array}{l} A = -B \\ Ae^{r_1} = -Be^{r_2} \end{array}\right\}$ $A=B=0$ trivial only.

$\lambda = b^2$: $r = -b$ twice, $y = Ae^{-bx} + Bxe^{-bx}$ so $A=0$, $Ae^{-b} + Be^{-b} = 0$ so $B=0$ only trivial.

$\lambda > b^2$: $r = -b \pm i\sqrt{\lambda - b^2}$

$y = e^{-bx}\left(A \sin\sqrt{\lambda - b^2}\,x + \cancel{B \cos\sqrt{\lambda - b^2}\,x}\right)$ for BC @ $x=0$.

eigenvalue condition $\sqrt{\lambda - b^2} = n\pi$ @ $x=1$.

so $\boxed{\begin{array}{l} \lambda_n = n^2\pi^2 + b^2 \\ y_n(x) = e^{-bx}\sin(n\pi x) \end{array}}$

you could also use energy method but it only tells you $\lambda > 0$:

$\underbrace{\int yy''}_{-\int y'^2 + \cancel{[yy']_0^1}^{\,0}} + b\underbrace{\int 2yy'}_{\int (y^2)' = [y^2]_0^1 = 0} + \lambda \int y^2 = 0$

so $\lambda = \dfrac{\int y'^2}{\int y^2} > 0$. (not enough).

p. 243-247 #2

$$\lambda = \begin{array}{ccc} \lambda_1 & \lambda_2 & \lambda_3 \\ 0, & 5, & 5 \end{array} \qquad V = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec v_1 & \vec v_2 & \vec v_3 \\ | & | & | \end{bmatrix} \quad \text{eigvecs}$$

a) orthonormal eigvecs $V = \begin{bmatrix} 2/\sqrt5 & 1/\sqrt5 & 0 \\ -1/\sqrt5 & 2/\sqrt5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ but as I suggested use non-normalized vecs.

b) $\vec f = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} = f_1\vec v_1 + f_2 \vec v_2 + f_3 \vec v_3$ what are coeffs? $\left[\begin{array}{cc|c} 2 & 1 & 1 \\ -1 & 2 & 4 \end{array}\right] \sim \begin{array}{ccc} 1 & -2 & -4 \\ 0 & 5 & 9 \end{array}$

$\lambda = 2$ not an eigval. so $c_i = \dfrac{f_i}{\lambda_i - \lambda} \quad i = 1..3$ $\qquad \sim \begin{array}{ccc} 1 & 0 & -1/5 \\ 0 & 1 & 9/5 \end{array} \begin{array}{l} \leftarrow f_1 \\ \leftarrow f_2 \end{array} \quad (f_3 = 0)$

so $\vec c = \begin{bmatrix} (-1/5)/(-2) \\ (9/5)/3 \\ 0/3 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 3/5 \\ 0 \end{bmatrix}$ ie $\vec x = \frac{1}{5}\begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

or use $f_i = \vec f \cdot \vec v_i / \|\vec v_i\|^2$, $i = 1,2,3$.

c) $A\vec x - 5\vec x = \begin{pmatrix} 1 \\ -1/2 \\ 0 \end{pmatrix}$ $\quad \lambda = 5$ is eigval. but $\vec f$ is orthog to $\lambda = 5$ eigenspace (nonunique)

$\Rightarrow c_1 = \dfrac{f_1}{\lambda_1 - 5} = \dfrac{1/2}{-5} = -\dfrac{1}{10}$, c_2, c_3 arbitrary.

$\vec x = \begin{bmatrix} -1/5 \\ +1/10 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

d) same $\vec{f}$ as b) so $f_2 \neq 0$ and there's no solution. ($0 \cdot c_2 = f_2$ inconsistent.)

#9 $u(t) = e^t \int_0^t e^{-s} u(s)\, ds \quad \xrightarrow[\text{product rule}]{\frac{d}{dt}} \quad u'(t) = \underbrace{e^t \int_0^t e^{-s} u(s)\, du}_{u(t)} + \cancel{e^t}\cancel{e^{-t}} u(t)$

so $\frac{du}{dt} = 2u$ or $u(t) = ce^{2t}$ $\longleftarrow$ but the IC seems to be $u(0) = 0$ giving $c = 0$. I suspect lower limit should have been $-\infty$ (in which case c = arbitrary).

#6 $u(t) = 1 + \int_0^t s \overbrace{\ln\left(\frac{s}{t}\right)}^{(\ln s - \ln t)} u(s)\, ds$

$\frac{d}{dt} \to u'(t) = t \underbrace{\ln\left(\frac{t}{t}\right)}_{0} u(t) + \int_0^t s \overbrace{\left(\frac{-1}{t}\right)}^{\frac{d}{dt}(-\ln t)} u(s)\, ds = -\frac{1}{t} \int_0^t s\, u(s)\, ds \quad (\times)$

$\frac{d}{dt} \to u''(t) = -\frac{1}{t} t\, u(t) + \underbrace{\frac{1}{t^2} \int_0^t s\, u(s)\, ds}$. notice is $\to -\frac{1}{t} u'(t)$

so $tu'' + u' + tu = 0$ don't solve.

Initial conditions: need 2 of them since it's 2nd order. $u(0) = 1$ from original eqn.

From $(\times)$ it's not obvious what $u'(0)$ is: $u'(0) = -\lim_{t \to 0^+} \frac{\int_0^t s\, u(s)\, ds}{t} = \lim_{t\to 0^+} \frac{\frac{1}{2} t^2 u(0)}{t} = 0.$

#8 $(1 - \mu K) u = f$ with driving func. $f(t) = t$

$Ku(t) = \int_0^t (t-s) u(s)\, ds$ Volterra operator

Neumann series $u = (1 + \mu K + \mu^2 K^2 + \dots) f$

2nd term $= \mu K f(t) = \mu \int_0^t (t-s) f(s)\, ds = \mu \int_0^t (t-s)\, s\, ds = \mu \left[\frac{ts^2}{2} - \frac{s^3}{3}\right]_0^t$

$= \frac{\mu t^3}{6}$

3rd term $= \mu^2 K^2 f(t) = \mu^2 K (Kf)(t) = \mu^2 \int_0^t (t-s) \frac{s^3}{6}\, ds = \frac{\mu^2}{6}\left[\frac{ts^4}{4} - \frac{s^5}{5}\right]_0^t$

$= \frac{\mu^2}{6} \cdot \frac{t^5}{20} = \frac{\mu^2 t^5}{120}$

series $u(t) = t + \mu \frac{t^3}{6} + \mu^2 \frac{t^5}{120} + \dots$

$\boxed{4}$

(B) $K(x,y) = \sum_{j=1}^{2} \alpha_j(x)\beta_j(y)$ ← $n=2$ $\alpha_1(x) = 1$, $\alpha_2(x) = -5x^2$ $\beta_1(x) = 1$, $\beta_2(x) = x^2$

see Example 4.15 very similar.

$$A_{ij} = (\beta_i, \alpha_j) = \begin{bmatrix} 1 & -5/3 \\ 1/3 & -1 \end{bmatrix}$$

$(1-\lambda)(-1-\lambda) + 5/9 = 0$
$\Rightarrow \lambda^2 - 4/9 = 0, \quad \lambda = \pm 2/3$

since $(\beta_2, \alpha_1) = \int_0^1 x^2 \cdot 1\, dx = 1/3$

$\lambda = 2/3$ has eigvec $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ so eigfunc of K is $\sum \alpha_j(x) c_j = 5 - 5x^2$ or $\underline{1-x^2}$

$\lambda = -2/3$ " $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ " " $\underline{1-5x^2}$.

Also $\lambda = 0$ is ∞-multiplicity eigenvalue of K with eigenspace all funcs orthog to both 1 & x^2 on $[0,1]$.
1 is not an eigenvalue so $Ku - 1u = f$ is (uniquely) soluble even if f not in the span of the α_j's $=$ Span$\{1, x^2\}$. (Contrast 1st-kind eqn $Ku = f$).

$f_1 = (f, \beta_1) = \int_0^1 x \cdot 1\, dx = 1/2$
$f_2 = (f, \beta_2) = \int_0^1 x \cdot x^2 dx = 1/4$ $\Bigg\}$ $\vec{f} = \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix}$

Solve $A\vec{c} - \vec{c} = \vec{f}$ gives $\begin{bmatrix} 0 & -5/3 \\ 1/3 & -2 \end{bmatrix} \vec{c} = \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & -6 & 3/4 \\ 0 & -5 & 3/2 \end{array}\right]$

$\Rightarrow \left[\begin{array}{cc|c} 1 & 0 & -21/20 \\ 0 & 1 & -3/10 \end{array}\right]$ ← gives $\vec{c}$ solution.

I didn't care about the numbers, just the method

Finally use (*) in lec. notes, so $u(x) = \frac{1}{\lambda}\left(\sum \alpha_j(x) c_j - f(x)\right) = -\frac{21}{20} + \frac{3}{2}x^2 - x$

(13) a. Key is $\int_0^{1/2} u(y)\,dy$ indep. of x $\Rightarrow$ call it c, some const.

then $u(x) = f(x) + \lambda c$

$\int_0^{1/2} dx$ ↳ $c = \int_0^{1/2} f(x)dx + \frac{1}{2} \cdot \lambda c \Rightarrow c(1 - \lambda/2) = \int_0^{1/2} f(x)\,dx$, gives c.

$\Rightarrow u(x) = f(x) + \frac{\lambda}{1-\lambda/2}\int_0^{1/2} f(y)\,dy$ .... soluble if $\lambda \neq 2$ $\forall f$, and if $\lambda = 2$ for f with zero mean $[0, 1/2]$

c. 1-by-1 Fredholm degenerate kernel ($n=1$): $\alpha_1(x) = x$, $\beta_1(x) = x$ so $A = [1/3]$, $\lambda = 1/3$

$A\vec{c} - \vec{c} = \vec{f}$ with $\vec{f} = [f_1] = -\int_0^1 x f(x)\,dx$

so $c_1 = \frac{3}{2}\int_0^1 x f(x)\,dx$ and $u(x) = \frac{3}{2} x \underbrace{\int_0^1 y f(y)\,dy}_{\text{a number}} + f(x)$ using (*). ← since given f.

Always soluble.

#4) c.

$$\lambda u(x) = Ku(x) = (\pi-x)\int_0^x y\,u(y)\,dy + x\int_x^\pi (\pi-y)\,u(y)\,dy$$ } Leibniz formula.

$$\lambda u'(x) = -\int_0^x y\,u(y)\,dy + \cancel{(\pi-x)\,x\,u(x)} + \int_x^\pi (\pi-y)\,u(y)\,dy - \cancel{x(\pi-x)u(x)}$$

$$\lambda u''(x) = -\cancel{x}\,u(x) - (\pi-\cancel{x})\,u(x) = -\pi u(x)$$

by substituting $x=0$ or π into $(Ku)(x)$.

so $u'' + \frac{\pi}{\lambda} u = 0$ with $u(0) = u(\pi) = 0$ Dirichlet BCs.

so $u_n = \sin nx$ with $\frac{\pi}{\lambda_n} = n^2$ or $\lambda_n = \frac{\pi}{n^2}$ $n = 1, 2, \ldots$

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