Your name:

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Math 11 Fall 2011, Homework 9, due Wed Nov 30

Please show your work. No credit is given for solutions without justification.

(1) *Choose the correct answer. Show relevant work (it will not be graded).*

(a) The vector surface integral $\iint_S \mathbf{F} \cdot d\mathbf{s}$ is zero if

- (A) $\mathbf{F}$ is a radial vector field
- (B) $\mathbf{F}$ is a conservative vector field
- (C) The vector $\mathbf{F}$ is tangent to the surface S at every point of S
- (D) $\mathbf{F} \cdot \mathbf{n} = 0$ at every point of S, where $\mathbf{n}$ is a normal vector for S
- (E) $\mathbf{F} \times \mathbf{n} = \mathbf{0}$ at every point of S, where $\mathbf{n}$ is a normal vector for S
- (F) $\mathbf{F} \times \mathbf{e_n} = \mathbf{0}$ at every point of S, where $\mathbf{e_n}$ is a unit normal vector for S

(b) True or false?

- (i) **True** / **False.** If $\mathcal{C}$ is a simple closed curve in $\mathbb{R}^2$ then $\oint_{\mathcal{C}} y\,dx = \text{Area}(\mathcal{D})$, where $\mathcal{D}$ is the region inside $\mathcal{C}$.
- (ii) **True** / **False.** If $\mathcal{C}$ is a simple closed curve in $\mathbb{R}^2$ then $\oint_{\mathcal{C}} x\,dy = \text{Area}(\mathcal{D})$, where $\mathcal{D}$ is the region inside $\mathcal{C}$.
- (iii) **True** / **False.** The flux of $\nabla \times \mathbf{F}$ through every oriented surface is zero.
- (iv) **True** / **False.** The flux of $\nabla \times \mathbf{F}$ through every closed and oriented surface is zero.
- (v) **True** / **False.** If $\mathbf{F}$ is conservative then $\nabla \times \mathbf{F} = \langle 0, 0, 0 \rangle$.

(i) $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = -1$ because $\vec{F} = \langle y, 0 \rangle$ here.

(ii) $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = 1$ because $\vec{F} = \langle 0, x \rangle$ here

(iii) $\iint_S \text{curl}\, F \cdot dS = \oint_{\partial S} \bar{F} \cdot d\bar{s}$ does <u>not</u> ~~imp~~ have to be zero

(iv) ... but if S is <u>closed</u> it has no boundary, so then $\iint_S \text{curl}\, \bar{F} \cdot d\bar{S} = 0$.

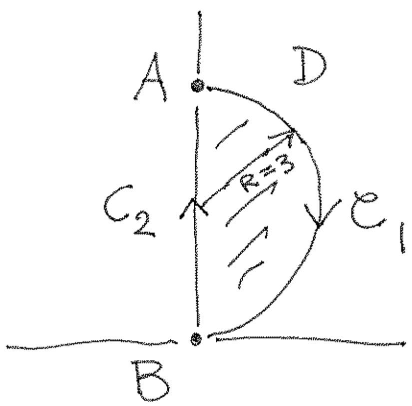
(v) $\vec{F} = \text{grad}\, f$ then $\text{curl}\, \vec{F} = \text{curl}\,(\text{grad}\, f) = \vec{0}$

$$\begin{tikzcd} 1 \arrow[r, "\text{grad}"] \arrow[rr, bend right, "\text{curl}(\text{grad}) = 0"'] & 3 \arrow[r, "\text{curl}"] & 3 \arrow[r, "\text{div}"] & 1 \end{tikzcd}$$

(2) Let C be the semi-circle $x^2 + (y-3)^2 = 9$ in $\mathbb{R}^2$ with $x \geq 0$, oriented from $A = (0,6)$ to $B = (0,0)$. Calculate the line integral

$$\int_C -y\,dx + x\,dy.$$

Hint: Use Green's theorem.



Green's Thm:

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dA = \oint_{\partial D} \bar{F} \cdot d\bar{s}$$

$$= -\oint_{C_1} \bar{F} \cdot d\bar{s} - \oint_{C_2} \bar{F} \cdot d\bar{s}$$

The curves C_1, C_2 are oriented wrongly.

i) We get: $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = 1 - (-1) = 2$.

$\bar{F} = \langle -y, x \rangle$

$$\iint_D 2\, dA = 2 \text{ Area } D = 2 \cdot \tfrac{1}{2} \pi 3^2 = 9\pi.$$

ii) Also: $\int_{C_1} \bar{F} \cdot d\bar{s}$, parametrize C_2 as $\bar{r}(t) = \langle 0, t \rangle$, $0 \leq t \leq 6$

$\bar{r}'(t) = \langle 0, 1 \rangle$

$\bar{F} = \langle -t, 0 \rangle$

$\bar{F} \cdot \bar{r}' = 0$

Then $\int_{C_1} \bar{F} \cdot d\bar{s} = 0$.

Putting it together:

$$9\pi = -0 - \int_{C_2} \bar{F} \cdot d\bar{s} \Rightarrow \int_{C_2} \bar{F} \cdot d\bar{s} = -9\pi$$

(3) Let $\mathbf{F}$ be the vector field in $\mathbb{R}^3$ given by

$$\mathbf{F}(x,y,z) = \langle z\sin(xz),\ x^2+y^2,\ x\sin(xz)\rangle$$

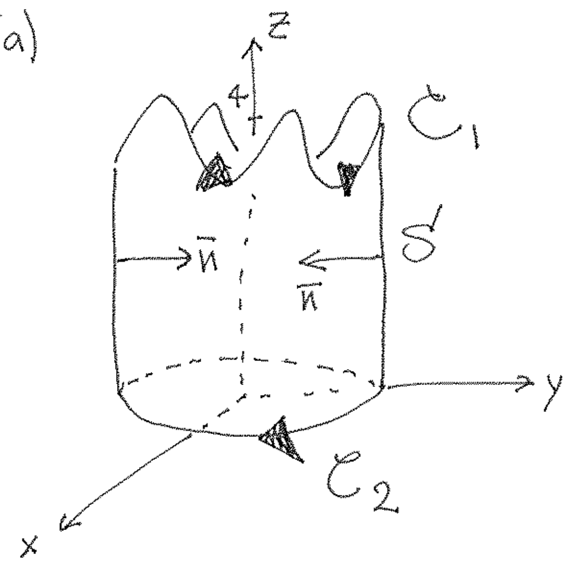
and let $\mathcal{C}$ be the oriented closed curve parametrized as

$$\mathbf{r}(t) = \langle \cos t, \sin t, 4+\sin 4t\rangle \qquad 0 \le t \le 2\pi$$

Let the surface S be the part of the cylinder $x^2+y^2=1$ above the xy plane and below the curve $\mathcal{C}$.

(a) How must S be oriented so that $\mathcal{C}$ is oriented correctly as one of the components of ∂S? Do the normal vectors point inward or outward?
(b) The curve $\mathcal{C}$ is part of the boundary ∂S. Describe the other component of the boundary ∂S, and indicate how it must be oriented.
(c) Use Stokes's Theorem to calculate the line integral $\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s}$.

(a)



C_1 oriented counterclockwise (seen from above)

S' oriented <u>inward</u>

(b) C_2 is the circle in the xy plane with $x^2+y^2=1$, $z=0$.
Oriented <u>clockwise</u>.

(c) $$\iint_S \underbrace{\text{curl}\,\bar F\cdot d\bar S}_{\text{zero}} = \int_{C_1} \bar F\cdot d\bar s + \underbrace{\int_{C_2} \bar F\cdot d\bar s}_{\text{zero}} \Rightarrow \boxed{\int_{C_1} \bar F\cdot d\bar s = 0.}$$

$\text{curl}\,\vec F = \langle 0,0,2x\rangle$.

Since normal vectors of S are <u>horizontal</u>

$\vec n = \langle \cdot, \cdot, 0\rangle$

we get $\text{curl}\,\bar F\cdot\vec n = 0$ and $\iint_S \text{curl}\,\bar F\cdot d\bar S = 0$

Parametrize C_2 as $\bar r(t) = \langle \cos t, \sin t, 0\rangle$, $0\le t\le 2\pi$ (with <u>opposite</u> orientation). Then $\bar r'(t) = \langle -\sin t, \cos t, 0\rangle$.

$\vec F = \langle 0,1,0\rangle$. $\int_{C_2} \bar F\cdot d\bar s = \int_0^{2\pi} \cos t\,dt = 0$.