Math 12, Fall 2007

Lecture 20

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Outline

1. Recap and overview
   - Last classes
2. Today's material
   - Fundamental Theorem of Line Integrals
3. Next class

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Integration of functions of more than one variable

- Line integrals
- Scalar integrals over curves
- Fundamental theorem of Line Integrals

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Fund. Thm. of Line Integrals

Let C be a smooth curve given by the vector function $\vec{r}(t), a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector field is continuous. Then,

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

Proof:

$$\begin{aligned}\int_C \nabla f \cdot d\vec{r} &= \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t); dt \\ &= \int_a^b (f_x x_t + f_y y_t + f_z z_t)\, dt = \int_z^b \frac{d}{dt} f(\vec{r}(t))\, dt \\ &= f(\vec{r}(b)) - f(\vec{r}(a))\end{aligned}$$

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Consequences

Theorem

$\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C in D

Theorem

Suppose $\vec{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D then $\vec{F}$ is a conservative vector field on D.

Consequences

Definition: A simply connected plane region D is a connected region in the plane with the property that any closed curve in D encloses only points in D.

Theorem

If $\vec{F}(x, y) = P(x, y)\,\vec{i} + q(x, y)\,\vec{j}$ is a vector field defined on an open simply connected region D and $P_y = Q_x$ then $\vec{F}$ is conservative.

Conservation of Energy

Newton's Second Law of motion, $F = ma$, can be rewritten when $\vec{F}$ is a force acting on a particle moving along $\vec{r}(t)$:

$$\vec{F}(\vec{r}(t)) = m\vec{r}''(t)$$

The work done by the force on the object is

$$\begin{aligned} W &= \int_C \vec{F} \cdot d\vec{r} = \int_a^b m\vec{r}''(t) \cdot \vec{r}'(t)\, dt \\ &= \frac{m}{2} \int_a^b \frac{d}{dt}(\vec{r}'(t) \cdot \vec{r}'(t))\, dt \\ &= \frac{m}{2}(|\vec{r}'(b)|^2 - |\vec{r}'(a)|^2) \\ &= K(b) - K(a) \ \ \text{(where } K \text{ is the kinetic energy)} \end{aligned}$$

Conservation of Energy

Suppose now that $\vec{F}$ is conservative and so $\vec{F} = \nabla f$. The potential energy, P, is defined to be $-f$ so $\vec{F} = -\nabla P$ and

$$W = \int_C \vec{F} \cdot d\vec{r} = -\int_C \nabla P \cdot d\vec{r} = P(a) - P(b)$$

Putting this together with $W = K(b) - K(a)$ we have that

$$P(a) + K(a) = P(b) + K(b)$$

This is the law of conservation of energy.

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Work for next class

- Reading: 17.4
- f07hw22