~ SOLUTIONS ~

Math 46, Applied Math (Spring 2011): Final

3 hours, 80 points total, 10 questions (worth between 5 and 9 points each). Good luck!

1. [9 points] Consider the Dirichlet eigenvalue problem for $y(x)$ in the interval $x \in (1, e)$,

$$-x^2y'' = \lambda y, \qquad y(1) = y(e) = 0\,. \quad \leftarrow \text{Dirichlet BCs.}$$

[3] (a) Prove that any eigenvalues λ have a definite sign (which?) positive.

Energy method: (mult. by y & integrate) $\underbrace{-\int_1^e y x^2 y'' dx}_{\text{fails! since can't kill the } yy' \text{ term...}} = \lambda \int_1^e y^2 dx$

Instead try putting x^2 on other side: [tricky]. $\underbrace{-\int_1^e yy'' dx}_{\text{by parts}} = \lambda \underbrace{\int_1^e \frac{1}{x^2} y^2 dx}_{>0}$. since $\frac{1}{x^2}>0 \ \forall x \in (1,e)$ ($y \equiv 0$ not an eig. func.)

$= \underbrace{\int_1^e y'^2 dx}_{>0} - \underbrace{[yy']_1^e}_{\text{BCs kill this}}$ so $\lambda = \frac{\text{positive}}{\text{positive}} > 0$.

[4] (b) Find WKB approximations to the nth eigenvalue λ_n and corresponding eigenfunction $y_n(x)$.

Put in std form: $\varepsilon^2 y'' + k(x)^2 y = 0$, $\varepsilon^2 = \frac{1}{\lambda}$

$k(x) = \frac{1}{x}$ so $y_{WKB}(x) = \frac{A}{\sqrt{k(x)}} \sin \frac{1}{\varepsilon}\int_1^x k(s)ds + \frac{B}{\sqrt{k(x)}} \cos\frac{1}{\varepsilon}\int_1^x k(s)ds$

easiest is definite integral.

BCs @ $x=1$ mean $y_{WKB}(1) = 0$
so $B = 0$, A = arbitrary, choose 1.

For $y_{WKB}(e) = 0$ need $\frac{1}{\varepsilon_n}\int_1^e k(s)ds = n\pi$

$\int_1^e \frac{1}{s} ds = \ln s\Big|_1^e = 1$.

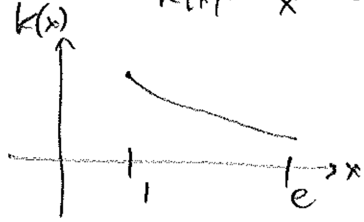
$\Rightarrow \varepsilon_n = \frac{1}{n\pi}$

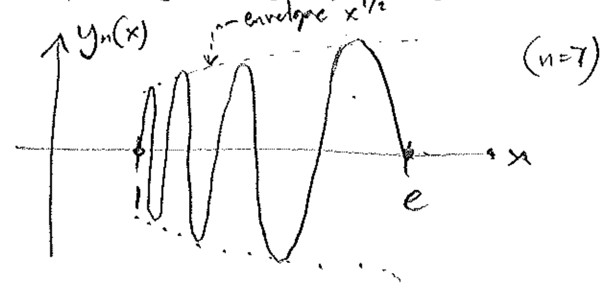
$$\Rightarrow y_n^{WKB}(x) = x^{1/2} \sin\left(n\pi \int_1^x \frac{1}{s} ds\right) = x^{1/2} \sin(n\pi \ln x)$$

$$\lambda_n^{WKB} = \frac{1}{\varepsilon_n^2} \approx n^2\pi^2 \qquad n = 1, 2, \ldots$$

[2] (c) Sketch an eigenfunction of very large eigenvalue magnitude, showing how frequency and amplitude change vs x.

$k(x) = \frac{1}{x}$ controls freq. locally.





[BONUS] In this particular problem, what are the accuracies of λ_n and $y_n(x)$?

It's actually Cauchy-Euler so can solve exactly via $y = x^r$, so $x^2 y'' + \lambda y = 0$ gives $r(r-1) + \lambda = 0$, ie $r = \frac{1}{2}(1 \pm \sqrt{1-4\lambda})$ so $y(x) = x^{1/2} \sin(\sqrt{\frac{1}{4}-\lambda}\,\ln x)$ + cos term

Setting $\sqrt{\frac{1}{4}-\lambda} = n\pi$ gives $\lambda_n = n^2\pi^2 - 1/4$ so error is $+1/4$ for WKB λ's! (relative error $O(n^{-2})$).
Eigfuncs. have zero error (!) · unusual.

2. [6 points] The radius r of the early phase of a nuclear fireball explosion is assumed to depend only on time t, the total energy released e (units ML^2T^{-2}), and the density of the surrounding gas ρ. Following G. I. Taylor in the 1940's, fill a dimensions matrix, and deduce the most specific formula you can for how the radius depends on the other variables.

$$\begin{array}{c|c} & \begin{matrix} r & t & e & \rho \end{matrix} \\ \begin{matrix} M \\ L \\ T \end{matrix} & \begin{bmatrix} & & 1 & 1 \\ 1 & & 2 & -3 \\ & 1 & -2 & \end{bmatrix} \end{array}$$

full rank (since 1st three cols are Lin. indep.), rank = 3.

$\Rightarrow p = \#$ free vars $= \#$cols $-$ rank $= 4 - 3 = 1$

$$\pi_1 = \frac{e t^2}{\rho r^5}$$ by combining cols. to get $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Buckingham Pi Thm tells you. $f(\pi_1) = 0$, or $\pi_1 = c$

$$\Rightarrow r = c\left(\frac{e t^2}{\rho}\right)^{1/5} = c\,\frac{e^{1/5} t^{2/5}}{\rho^{1/5}}$$

c is unknown dim'less parameter.

note this also causes factor ω^2 but it doesn't matter for 2-term approx.

3. [9 points] Consider the perturbed initial-value problem for $y(t)$ on $t > 0$,

$$y'' + y + 4\varepsilon y y'^2 = 0, \qquad 0 < \varepsilon \ll 1, \qquad y(0) = 1, \qquad y'(0) = 0$$

[7] (a) Use the Poincaré-Lindstedt method to give a 2-term approximation. [Hint: rescale $\tau = \omega t$ where ω is perturbed from the value 1. Don't forget to match initial conditions. Partial credit if you can only do regular perturbation; doing that will jog your memory anyway...]

rescale $\tau = \omega t = (1 + \omega_1 \varepsilon + \cdots)t$

so $\frac{d^2}{dt^2}$ replaced by $\omega^2 \frac{d^2}{d\tau^2}$ (just like choosing $t_c = \frac{1}{\omega}$ times)

$\Rightarrow$ in τ variable, (prime $= \frac{d}{d\tau}$)

$$\underbrace{(1 + \omega_1\varepsilon + \cdots)^2}_{\omega^2}(y_0'' + \varepsilon y_1'' + \cdots) + y_0 + \varepsilon y_1 + \cdots = -4\varepsilon\underbrace{(1+\cdots)^2}_{\omega^2}(y_0 + \cdots)(y_0' + \cdots)^2$$

Zeroth-order ($\varepsilon = 0$): $y_0'' + y_0 = 0$ w/ ICs as above, so $y_0(\tau) = \cos\tau$

$O(\varepsilon^1)$: $$2\omega_1 \underbrace{y_0''}_{-\cos\tau} + \underbrace{y_1'' + y_1}_{\text{homog. solns. } \cos\tau,\ \sin\tau} = -4 y_0 y_0'^2 = -4\cos\tau\sin^2\tau$$

$\hookrightarrow$ power-reduction identity $\cos 3\tau - \cos\tau$

So to remove the on-resonance driving $-\cos\tau$ on RHS, choose $\omega_1 = +1/2$

Then, $y_1'' + y_1 = \cos 3\tau$ with ICs for y_1 of $y_1(0) = y_1'(0) = 0$.

Meth. Und. Coeffs: $Y = A\cos 3\tau$, $\Rightarrow Y'' = -9A\cos 3\tau$ so $-9A + A = 1$, $A = -1/8$

Gen. soln. $y_1(\tau) = -\frac{1}{8}\cos 3\tau + c_1 \sin\tau + c_2\cos\tau$

$y_1(0) = 0$ so $c_2 = +1/8$. $y_1'(0) = 0$ so $c_1 = 0$.

Soln. $y_1(\tau) = \frac{1}{8}(\cos\tau - \cos 3\tau)$

Write out 2-term approx: $y_a = y_a(\tau) = \cos\tau + \frac{\varepsilon}{8}(\cos\tau - \cos 3\tau) + \cdots$

where $\tau = (1 + \frac{\varepsilon}{2} + \cdots)t$

sign means oscillator period shorter.

[2] (b) Discuss briefly any differences in the uniformity of the approximation, and the reason, if regular perturbation theory were used instead.

Poincaré–Lindstedt is uniform approx. for $t>0$. (error uniformly bounded).
In contrast, regular perturbation theory would give a secular term of the form $\varepsilon t \sin t$ that is unbounded as $t \to \infty$ for any $\varepsilon > 0$, not a uniform approx.

4. [5 points] By converting into an ODE, find the unique solution $u(t)$ to the integral equation,

$$4\int_0^t (t-s)u(s)ds + u(t) = t, \qquad t > 0.$$

← set $t=0$ to get $0 + u(0) = 0$ so $u(0) = 0$, IC.

$\frac{d}{dt}$ ↳ using Leibniz rule

$$4\cancel{(t-t)}u(t) + 4\int_0^t u(s)ds + u'(t) = 1$$

↖ set $t=0$ to get $4\cdot 0 + u'(0) = 1$ so $u'(0) = 1$, IC.

$\frac{d}{dt}$ ↳

$$4u(t) + u''(t) = 0$$

const-coeff. homog. ODE, 2nd order,

e^{rt} $\quad u'' + 4u = 0$

↳ $r^2 + 4 = 0$ so $e^{\pm 2it}$, or $\{\sin 2t, \cos 2t\}$ are L.I. soln pairs.

IC $u(0) = 0$ means no cos term.

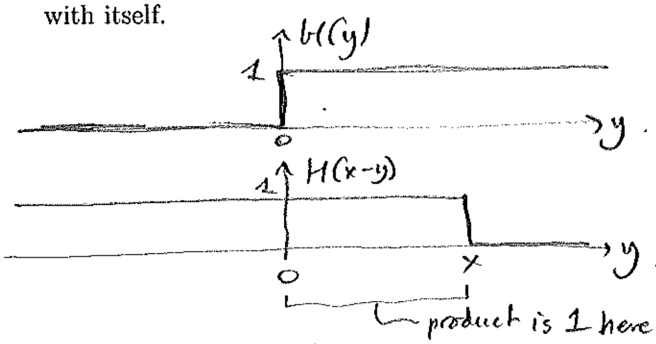
$\Rightarrow$ $u(t) = A\sin 2t$, insert $u'(0) = 1$ to get $A = \frac{1}{2}$.

$\Rightarrow$ $u(t) = \frac{1}{2}\sin 2t$

5. [9 points] Fourier & convolution stuff.

direct computation

[3] (a) Consider the 'step' function $H(x) = 1$ for $x > 0$, zero otherwise. Compute the convolution of H with itself.



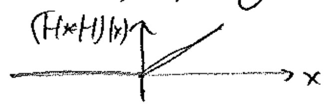
$$(H*H)(x) := \int_{-\infty}^{\infty} H(x-y)\,H(y)\,dy$$

$$= \int_0^{\infty} H(x-y)\,dy$$

$= 0$ if $x<0$ since argument of $H(x-y)$ never >0.

For $x>0$, $H(x-y)$ has positive argument only for $y<x$, giving $\int_0^x 1\,dx = x$.

So, $(H*H)(x) = \begin{cases} x & x>0 \\ 0 & \text{otherwise.} \end{cases}$

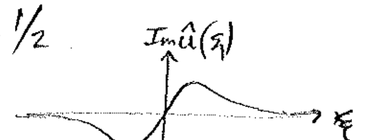


[3] (b) Find the Fourier transform of the function $u(x) = xe^{-x^2/2}$. [Hint: it's a derivative]

If $v(x) = -e^{-x^2/2}$ then $u(x) = \frac{dv}{dx}$.

Taking deriv. corresponds to mult. by $(-i\xi)$ of Fourier transform.

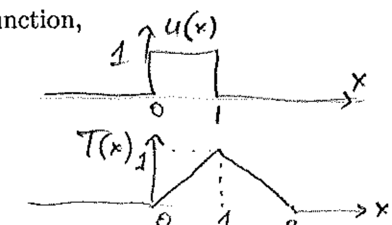
$\hat{v}(\xi) = -\sqrt{2\pi}\,e^{-\xi^2/2}$ using Table w/ $a = 1/2$



$$\Rightarrow \hat{u}(\xi) = -i\xi\,\hat{v}(\xi) = \underbrace{i\sqrt{2\pi}\,\xi\, e^{-\xi^2/2}}_{\text{Note: same functional form as } u \text{ itself!}}$$

[3] (c) Recall that in class you showed that the 'top hat' function $u(x) = 1$ for $0 < x < 1$ and zero otherwise, when convolved with itself, gives the continuous 'triangle hat' function,

$$T(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$



Find the Fourier transform of the function T.

Could be done directly but integrations would be a mess!

If $T = u * u$ in 'real space' (x variable) then by the convolution theorem, $\hat{T} = \hat{u}\hat{u}$ in Fourier space (ξ variable).

Compute,

$$\hat{u}(\xi) = \int_0^1 e^{ix\xi}dx = \frac{1}{i\xi}e^{ix\xi}\Big|_0^1 = \frac{e^{i\xi}-1}{i\xi}$$

is actually prod. of sinc and plane wave $e^{i\xi/2}$

$$\text{so } \hat{T}(\xi) = \hat{u}(\xi)^2 = -\frac{(e^{i\xi}-1)^2}{\xi^2}$$

parabolic PDE is homogeneous. Neumann BC, homogeneous.

6. [8 points] Consider the reaction-diffusion equation in $\Omega \subset \mathbb{R}^3$ with zero-flux boundary condition,

$$u_t = \Delta u - \alpha u, \quad \text{in } \Omega, \quad t > 0, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \qquad u(\mathbf{x}, 0) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega$$

where $\alpha(\mathbf{x})$ is a given spatially-dependent decay rate, and f a given initial distribution. IC.

[6] (a) Consider the simple case $\alpha(\mathbf{x}) = 0$. Prove that there is at most one solution.

Say u_1, u_2 are any two solutions, then let $u = u_1 - u_2$, and let $E(t) := \int_\Omega u(x,t)^2\,dx$, then

$$E'(t) = 2\int_\Omega u u_t\,dx \overset{\text{using PDE}}{=} 2\int_\Omega u\Delta u\,dx \overset{\text{Green's 1st Identity}}{=} -2\int_{\Omega} \underbrace{\vec{\nabla} u \cdot \vec{\nabla} u}_{\|\vec{\nabla} u\|^2 \geq 0}\,dx + 2\int_{\partial\Omega} u \underbrace{\frac{\partial u}{\partial n}}_{\text{zero, BCs.}} \leq 0 \quad \forall t.$$

But $E(0) = 0$ since IC for u is $u_1(x,0) - u_2(x,0) = f(x) - f(x) \equiv 0$.

Also $E(t)$ by construction is $\geq 0 \quad \forall t > 0$.

Combining the three facts gives $E(t) = 0 \quad \forall t \geq 0$, so $u(x,t) \equiv 0 \quad \forall x, \forall t \geq 0$.

$\Rightarrow u_1 \equiv u_2 \quad \forall x \in \Omega, \forall t > 0$ and the solution is unique.

[2] (b) Adapt your proof to general $\alpha(\mathbf{x})$. What condition on $\alpha(\mathbf{x})$ enables your uniqueness proof to still work?

We have instead $\frac{1}{2}E'(t) = \int_\Omega u u_t\,dx \overset{\text{PDE}}{=} \underbrace{\int_\Omega u\Delta u\,dx}_{\leq 0 \text{ by reasoning above.}} - \int_\Omega \alpha(x)\,\underbrace{u^2}_{\geq 0}\,dx$.

so if $\boxed{\alpha(\vec{x}) \geq 0}$ at each $\vec{x} \in \Omega$, we still get $E'(t) \leq 0$ and the rest of the proof is as before.

[in fact, this is the best we can do with energy method, since $\int_\Omega u\Delta u = 0$ is achieved by the nontrivial constant func $u(\vec{x}) = 1$]

$\phi_n(x)$ $\qquad \lambda_n = 1/n^2.$

7. [9 points] K is a symmetric Fredholm integral operator acting on the domain $(0, \pi)$, with a complete set of eigenfunctions $\{\sin nx\}$ and eigenvalues $1/n^2$, labeled by $n = 1, 2, \ldots$.

[2] (a) Find the *general* solution u to the equation $(Ku)(x) - u(x) = \sin x, \quad 0 < x < \pi$, or explain why it has no solution:

$Ku - \lambda u = f \qquad \lambda = 1$, is $= \lambda_1$ an eigenvalue.

$\Rightarrow$ only a solution if $(f, \phi_1) = 0$, but $f(x) = \sin x = \phi_1(x)$

so no solution possible.

[3] (b) Find the *general* solution u to the equation $(Ku)(x) - u(x) = \sin 2x, \quad 0 < x < \pi$, or explain why it has no solution:

as above, $\lambda = \lambda_1$, but now $(f, \phi_1) = (\sin 2x, \sin x)$
$= 0$, so $\exists$ nonunique soln.

Eigenfunction expansion gives $f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x)$ with $f_2 = 1$, $f_n = 0$ for $n \neq 2$.

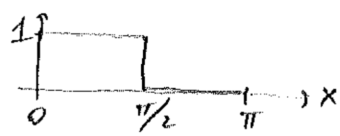
(*) eqn from lectures, $(\lambda_n - \lambda) c_n = f_n \quad \forall n$, so $c_1 =$ anything.

$n=2: \left(\frac{1}{2^2} - \lambda\right) c_2 = f_2 = 1. \quad$ so $c_2 = \frac{1}{1/4 - 1} = -4/3$

$u(x) = c \sin x - \frac{4}{3} \sin 2x, \quad c \in \mathbb{R}.$

[4] (c) Find the *general* solution u to the equation $(Ku)(x) = \begin{cases} 1, & 0 < x \leq \pi/2 \\ 0, & \pi/2 < x < \pi \end{cases}$, or explain why it has no solution:

ϕ_n orthogonal set (by symmetry of K), $\quad f(x) =$



$$f_n = \frac{(f, \phi_n)}{\|\phi_n\|^2} = \frac{\int_0^{\pi/2} \sin nx \, dx}{\int_0^{\pi} \sin^2 nx \, dx} = \frac{2}{\pi} \cdot \frac{-1}{n} \left(\cos nx\right)_0^{\pi/2}$$

$$= \frac{2}{\pi n}\left(1 - \cos\frac{n\pi}{2}\right)$$

this is fine.

not an eigenvalue $\Rightarrow$ unique soln.

$$= \begin{cases} \frac{4}{\pi n}, & n \text{ even} \\ \frac{2}{\pi n}(-1)^{\frac{n-1}{2}}, & n \text{ odd.} \end{cases}$$

Here $\lambda = 0$, so $\lambda_n c_n = f_n$, $c_n = n^2 f_n$

$$u(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} n^2 \frac{1 - \cos\frac{n\pi}{2}}{n} \sin nx = \frac{2}{\pi} \sum_{n=1}^{\infty} n\left(1 - \cos\frac{n\pi}{2}\right) \sin nx$$

Fourier series solution.

[BONUS] Explain whether the solution u to part (c) is in $L^2[(0,\pi)]$:

By Parseval (ϕ_n complete), $\|u\|_2^2 = \frac{2}{\pi}\sum_{n=1}^{\infty}|c_n|^2$ but c_n grow like n.

$\Rightarrow$ sum divergent $\Rightarrow$ soln not in $L^2(0,\pi)$.

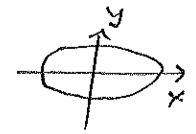
8. [8 points] Electric potential u in an upper half-plane $x \in \mathbb{R}$, $y > 0$, filled with anisotropic medium, satisfies a PDE with a decay condition at infinity,

$$a^2 u_{xx} + u_{yy} = 0, \qquad u(x,0) = f(x),\ x \in \mathbb{R}, \qquad u(x,y) \text{ bounded as } y \to +\infty,$$

with $a > 0$ a given anisotropy constant, and f a given boundary voltage function.

[1] (a) Is the PDE hyperbolic, parabolic, or elliptic?

since $a^2x^2 + y^2 = 1$ is ellipse



Also it's a generalization of Laplace's eqn, which is elliptic.

[7] (b) Use the Fourier transform method to derive the (unique) solution $u(x,y)$. Your answer should be in terms of a and f only. (Don't forget to explain where the $y \to +\infty$ condition enters.)

FT in x, holding y const:

$$-a^2\xi^2\hat{u}(\xi,y) + \hat{u}_{yy}(\xi,y) = 0$$

is a 2nd-order, growth/decay-type ODE in y at each ξ.

Gen. Soln: $\hat{u}(\xi,y) = \hat{A}(\xi)e^{-a\xi y} + \hat{B}(\xi)e^{a\xi y}$, $\hat{A},\hat{B}$ unknown functions.

But $\left.\begin{array}{l}\hat{A}(\xi)=0 \text{ for } \xi<0\\ \hat{B}(\xi)=0 \text{ for } \xi>0\end{array}\right\}$ if u bounded as $y\to+\infty$.

We may gather the remaining halves of $\hat{A},\hat{B}$ as $\hat{u}(\xi,y) = \hat{c}(\xi)e^{-ay|\xi|}$, $\hat{c}$ unknown func.

But at $y=0$, $\hat{u}(\xi,0) = \hat{c}(\xi)\overset{1}{\cancel{e^0}} = \hat{f}(\xi)$ by BCs, so $\hat{c} \equiv \hat{f}$.

$\Rightarrow \hat{u}(\xi,y) = \hat{f}(\xi)\cdot \underbrace{e^{-ay|\xi|}}$

use Table row $e^{-a|x|} \underset{F^{-1}}{\overset{F}{\rightleftarrows}} \frac{2a}{a^2+\xi^2}$

$\int_{-\infty}^{\infty} e^{ix\xi}e^{-a|x|}dx = \frac{2a}{a^2+\xi^2}$

$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}e^{-a|\xi|}d\xi = \frac{1}{\pi}\frac{a}{a^2+x^2}$.

Setting $x \to -x$ we get $\mathcal{F}^{-1}(e^{-a|\xi|}) = \frac{1}{\pi}\frac{a}{a^2+x^2}$

inv FT is $\frac{1}{\pi}\frac{ay}{a^2y^2+x^2}$

the variable

product becomes convolution back in x-space:

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{ay}{a^2y^2+(x-z)^2}f(z)\,dz$$

$\leftarrow$ note a new integration variable (not y!) needed to perform the convolution

note how x replaced by $(x-z)$

note: lin. comb. of Dirichlet & Neumann.

9. [8 points] Consider the Sturm-Liouville problem $-u'' = f(x)$ on the interval $0 < x < 1$, with mixed boundary conditions $\alpha u(0) + u'(0) = 0$, and $u(1) = 0$. Here α is some (Robin) constant.

[6] (a) For fixed $\alpha \neq 1$, compute the Green's function for this SLP.

$A = -\frac{d^2}{dx^2}$ $\leftarrow p \equiv 1$ in SLP std. form.

u_1: want $Au_1 = 0$ & $\alpha u_1(0) + u_1'(0) = 0$, left-hand BC only.

$u_1 = ax + b$ general. so $\left.\begin{array}{l} u_1(0) = b \\ u_1'(0) = a \end{array}\right\} \xrightarrow{BC} \alpha b + a = 0$ ie $a = -\alpha b$

$\Rightarrow u_1(x) = -\alpha b x + b = b(1 - \alpha x)$ can set $b = 1$.

u_2: $u_2 = ax + b$ but $u_2(1) = a + b = 0$ so $a = -b$.

$\Rightarrow u_2(x) = -bx + b = b(1 - x)$ can set $b = 1$.

$W = W(x) = u_1 u_2' - u_2 u_1' = -(1 - \alpha x) + \alpha(1 - x) = \alpha - 1, \ \forall x.$

$$g(x,\xi) = \begin{cases} -\dfrac{u_1(x)\,u_2(\xi)}{p(\xi)\,W(\xi)}, & x < \xi \\ -\dfrac{u_1(\xi)\,u_2(x)}{p(\xi)\,W(\xi)}, & x > \xi \end{cases} = \begin{cases} \dfrac{(1-\alpha x)(1-\xi)}{1-\alpha}, & x < \xi \\ \dfrac{(1-\alpha\xi)(1-x)}{1-\alpha}, & x > \xi \end{cases}$$

Green's func. formula.

[2] (b) Discuss as concretely as you can the solvability for general f, in the case when $\alpha = 1$.

When $\alpha = 1$ $g(x,\xi)$ d.n.e. by above denominator (W) vanishing $\forall x$.

I.e. $Lu = 0$ has non-trivial soln ϕ, what is it? $\phi \equiv u_1 \equiv u_2, \forall x$.

ie $\phi(x) = 1 - x$ spans the nullspace of L.

Either remember SLP has soln only if $(f, \phi) = 0$ ie $\int_0^1 (1-x) f(x)\,dx = 0$.

Or, derive this via: suppose u is soln, then $\underbrace{0 = (u, L\phi)}_{L\phi \equiv 0} \underbrace{=}_{\text{self-adjointness of } L} (Lu, \phi) \underbrace{=}_{SLP} (f, \phi)$.

10. [9 points] Short questions.

[3] (a) Compute the outer solution (with its correct constant) for the perturbed BVP $\varepsilon y'' + (x-2)y' + y = 0$, $y(0) = 1$, $y(1) = 0$, with $0 < \varepsilon \ll 1$.

Where are the boundary layers?

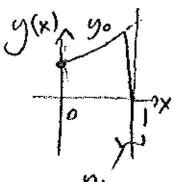
negative at both ends $\Rightarrow$ no BL @ $x=0$, but is bdry layer @ $x=1$.

$\Rightarrow$ Outer soln. must match BC @ $x=0$.

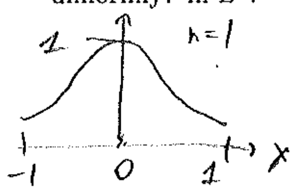
Set $\varepsilon = 0$: $(x-2)y' + y = 0 \Rightarrow \int \frac{dy}{y} = -\int \frac{dx}{x-2} \Rightarrow \ln y = -\ln(x-2) + c$.

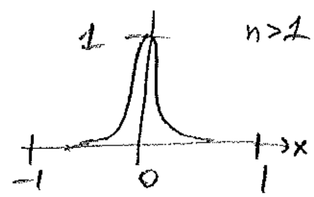
$\Rightarrow y_0(x) = \frac{c}{x-2}$ match $y_0(0) = \frac{c}{-2} = 1$ so $c = -2$

$\Rightarrow y_0(x) = \frac{-2}{x-2} = \frac{1}{1 - x/2}$



[3] (b) Is the sequence $f_n(x) = e^{-nx^2}$, $n = 1, 2, \ldots$, convergent to the zero function on $(-1, 1)$ pointwise? uniformly? in L^2?





, ... sequence gets narrower but has $f_n(0) = 1 \quad \forall n$

$\Rightarrow$ not pointwise.

uniform stronger than pointwise $\Rightarrow$ not uniform, either. (Also: $\max_{-1<x<1} |f_n| = 1, \forall n$)

It is convergent in $L^2(-1,1)$, since $\int_{-1}^{1} e^{-2nx^2} dx \le \int_{-\infty}^{\infty} e^{-2nx^2} dx = \frac{1}{\sqrt{2n}} \underbrace{\int_{-\infty}^{\infty} e^{-y^2} dy}_{\text{some const. } (=\sqrt{\pi})} \to 0$ as $n \to \infty$.

$y = \sqrt{2n}\,x$

[3] (c) Is $10^9(e^x - 1 - x) = O(x^2)$ as $x \to 0$? Prove your answer.

$$\lim_{x\to 0} \frac{10^9(e^x - 1 - x)}{x^2} \overset{\text{l'Hôpital}}{=} \lim_{x\to 0} 10^9 \frac{e^x - 0 - 1}{2x} \overset{\text{l'Hôpital}}{=} 10^9 \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2} 10^9 = \text{const.}$$

(it doesn't matter it's large)

since ratio tends to const, and both funcs continuous,

yes, $10^9(e^x - 1 - x) = O(x^2)$ as $x \to 0$.

$K\phi_n = \lambda_n \phi_n$.

(d) [BONUS] Recall that if K is any symmetric operator with a complete set of eigenfunctions, the kernel of K^{-1} may be written as an eigenfunction expansion. Derive instead an eigenfunction expansion for the kernel of K itself:

this is $g(x,y) = \sum_n \frac{\phi_n(x)\phi_n(y)}{\lambda_n}$.

$u(x) = \sum_n c_n \phi_n(x)$ with $c_n = (\phi_n, u) = \int_a^b \phi_n(y) u(y) dy$.

K acts on each.

So, $(Ku)(x) = \sum_n c_n \lambda_n \phi_n(x) \overset{\text{sub } c_n}{=} \int_a^b \boxed{\sum_n \lambda_n \phi_n(x) \phi_n(y)} u(y) dy$.

must be kernel of K.