How Many Borel Sets are There?

Object. *This series of exercises is designed to lead to the conclusion that if $\mathcal{B}_{\mathbf{R}}$ is the σ-algebra of Borel sets in $\mathbf{R}$, then*

$$\operatorname{Card}(\mathcal{B}_{\mathbf{R}}) = \mathfrak{c} := \operatorname{Card}(\mathbf{R}).$$

This is the conclusion of problem 4. As a bonus, we also get some insight into the "structure" of $\mathcal{B}_{\mathbf{R}}$ via problem 2. This just scratches the surface. If you still have an itch after all this, you want to talk to a set theorist. This treatment is based on the discussion surrounding [1, Proposition 1.23] *and* [2, Chap. V §10 #31].

For these problems, you will need to know a bit about well-ordered sets and transfinite induction. I suggest [1, §0.4] where transfinite induction is [1, Proposition 0.15]. Note that by [1, Proposition 0.18], there is an uncountable *well ordered* set Ω such that for all $x \in \Omega$, $I_x := \{\, y \in \Omega : y < x \,\}$ is countable. The elements of Ω are called the *countable ordinals*. We let $1 := \inf \Omega$. If $x \in \Omega$, then $x + 1 := \inf\{\, y \in \Omega : y > x \,\}$ is called the *immediate successor* of x. If there is a $z \in \Omega$ such that $z + 1 = x$, then z is called the *immediate predecessor* of x. If x has no immediate predecessor, then x is called a *limit ordinal*.[1]

1. Show that $\operatorname{Card}(\Omega) \le \mathfrak{c}$. (This follows from [1, Propositions 0.17 and 0.18]. Alternatively, you can use transfinite induction to construct an injective function $f : \Omega \to \mathbf{R}$.)[2]

> **ANS**: Actually, this follows almost immediately from Folland's Proposition 0.17. By the Well Ordering Principle (Theorem 0.3 in Folland), we can assume that $\mathbf{R}$ is well ordered. Then, with this order, $\mathbf{R}$ cannot be isomorphic to an initial segment of Ω because $\mathbf{R}$ is uncountable and every initial segment in Ω is countable. Therefore Ω is either isomorphic to $\mathbf{R}$ or order isomorphic to an initial segment in $\mathbf{R}$. In either case, $\operatorname{Card}(\Omega) \le \operatorname{Card}(\mathbf{R}) := \mathfrak{c}$.

2. If X is a set, let $\mathscr{P}(X)$ be the set of subsets of X — i.e., $\mathscr{P}(X)$ is the *power set* of X. Let $\mathscr{E} \subset \mathscr{P}(X)$. The object of this problem is to give a "concrete" description of the σ-algebra $\mathscr{M}(\mathscr{E})$ generated by $\mathscr{E}$. (Of course, we are aiming at describing the Borel sets in $\mathbf{R}$ which are generated by the collection $\mathscr{E}$ of open intervals.) For convenience, we assume that $\emptyset \in \mathscr{E}$.

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[1] The set of countable ordinals has a rich structure. We let $2 := 1 + 1$, and so on. The set $\{\, n \in \mathbf{N} \,\} \subset \Omega$ is countable, and so has a supremum ω (see [1, Proposition 0.19]). Then there are ordinals $\omega + 1$, $\omega + 2$, ..., 2ω, $2\omega + 1$,..., ω^2, $\omega^2 + 1$, ..., ω^ω, and so on.

[2] The issue of whether or not $\operatorname{Card}(\Omega) = \mathfrak{c}$ is the *continuinum hypothesis*, and so is independent of the usual (ZFC) axioms of set theory.

Let

$$\mathscr{E}^c := \{\, E^c : E \in \mathscr{E} \,\} \quad \text{and} \quad \mathscr{E}_\sigma = \{ \bigcup_{i=1}^{\infty} E_i : E_i \in \mathscr{E} \,\}.$$

(Note, I just mean that $\mathscr{E}_\sigma$ is the set of sets formed from countable unions of elements of $\mathscr{E}$. Since $\emptyset \in \mathscr{E}$, $\mathscr{E} \subset \mathscr{E}_\sigma$.)

We let $\mathscr{F}_1 := \mathscr{E} \cup \mathscr{E}^c$. If $x \in \Omega$, and if x has an immediate predecessor y, then we set

$$\mathscr{F}_x := (\mathscr{F}_y)_\sigma \cup \big((\mathscr{F}_y)_\sigma\big)^c.$$

If x is a limit ordinal, then we set

$$\mathscr{F}_x := \bigcup_{y<x} \mathscr{F}_y.$$

We aim to show that

$$\mathscr{M}(\mathscr{E}) = \bigcup_{x\in\Omega} \mathscr{F}_x \tag{$\dagger$}$$

(a) Observe that $\mathscr{F}_1 \subset \mathscr{M}(\mathscr{E})$.

(b) Show that if $F_y \subset \mathscr{M}(\mathscr{E})$ for all $y < x$, then $F_x \subset \mathscr{M}(\mathscr{E})$. Then use transfinite induction to conclude that $\mathscr{F}_x \subset \mathscr{M}(\mathscr{E})$ for all $x \in \Omega$.

(c) Show that the right-hand side of ($\dagger$) is closed under countable unions.

(d) Conclude that $\bigcup_{x\in\Omega} \mathscr{F}_x$ is a σ-algebra, and that ($\dagger$) holds.

ANS: Since $\mathscr{M}(\mathscr{E})$ is a σ-algebra — and hence is closed under countable unions and complementation — it is clear that $\mathscr{F}_1 \subset \mathscr{M}(\mathscr{E})$. Thus if $A = \{\, x \in \Omega : \mathscr{F}_x \subset \mathscr{M}(\mathscr{E}) \,\}$, we certainly have $1 \in A$. Now suppose that $y \in A$ for all $y < x$. If $x = z + 1$, then because $\mathscr{M}(\mathscr{E})$ is a σ-algebra,

$$F_x = (F_z)_\sigma \cup \big((F_z)\sigma\big)^c \subset \mathscr{M}(\mathscr{E}).$$

But if x is a limit ordinal, then trivially,

$$F_x = \bigcup_{y<x} F_y \subset \mathscr{M}(\mathscr{E}).$$

Then it follows by transfinite induction (Folland, Proposition 0.15) that $A = \Omega$. Therefore $\bigcap_{x\in\Omega} F_x \subset \mathscr{M}(\mathscr{E})$.

Now I claim that $\bigcup_{x\in\Omega} F_x$ is a σ-algebra. Since it clearly contains $\emptyset$ and is closed under complementation, it suffices to see that it is closed under countable unions. So, suppose that $\{E_j\}_{j=1}^\infty \subset \bigcup_{x\in\Omega} F_x$. Say, $E_j \in F_{x_j}$. Since $\{x_j\}$ is countable, there is an $x \in \Omega$ such that $x_j \le x$ for all j by Folland's Proposition 0.19.[3] Then, since Ω has no largest element,

$$\bigcup_{j=1}^\infty E_j \subset (F_x)_\sigma \subset F_{x+1} \subset \bigcup_{x\in\Omega} F_x.$$

This shows that $\bigcup_{x\in\Omega} F_x$ is a σ-algebra containing $\mathscr{E}$. Hence

$$\mathscr{M}(\mathscr{E}) \subset \bigcup_{x\in\Omega} F_x \subset \mathscr{M}(\mathscr{E}).$$

Thus, (†) follows, and this completes the proof.

3. Recall that if A and B are sets, then $\prod_{a\in A} B$ is simply the set of functions from A to B. For reasons that are unclear to me, this set is usually written B^A. Notice that $\prod_{i=1}^\infty B = \prod_{i\in\mathbf{N}} B$ is just the collection of sequences in B. Notice also that $\operatorname{Card}(B^A)$ depends only on $\operatorname{Card}(A)$ and $\operatorname{Card}(B)$.

(a) Check that

$$\prod_{i=1}^\infty \Big(\prod_{j=1}^\infty B\Big) = \prod_{(i,j)\in\mathbf{N}\times\mathbf{N}} B. \tag{$*$}$$

Thus the cardinality of either side of $(*)$ is the same as $\prod_{i=1}^\infty B$.

(b) Use these observations together with the fact that $\operatorname{Card}\big(\prod_{i=1}^\infty \{0,1\}\big) = \mathfrak{c} := \operatorname{Card}(\mathbf{R})$ (which follows from [1, Proposition 0.12]) to show that

$$\operatorname{Card}\Big(\prod_{i=1}^\infty \mathbf{R}\Big) = \mathfrak{c}.$$

(c) Show that if $\operatorname{Card}(\mathscr{E}) = \mathfrak{c}$, then $\operatorname{Card}(\mathscr{E}_\sigma) = \mathfrak{c}$.

ANS: The proof of (a) is immediate from the fact that $\operatorname{Card}(\mathbf{N}\times\mathbf{N}) = \operatorname{Card}(\mathbf{N})$. For (b), just note that

$$\operatorname{Card}\Big(\prod_{j=1}^\infty \mathbf{R}\Big) = \operatorname{Card}\Big(\prod_{j=1}^\infty\Big(\prod_{i=1}^\infty \{0,1\}\Big)\Big),$$

which by part (a) has the same cardinality as $\prod_{i=1}^\infty \{0,1\}$. This proves (b).

[3] This is the property of Ω that is crucial here! Especially notice that $\mathbf{N}$ does *not* have this property. This is why we need countable ordinals to describe $\mathscr{M}(\mathscr{E})$

For (c), we have $\mathscr{E} \subset \mathscr{E}_\sigma$, so $\operatorname{Card}(E) \le \operatorname{Card}(E_\sigma)$. But we have an obvious map of $\prod_{j=1}^\infty \mathscr{E}$ onto $\mathscr{E}_\sigma$. Thus $\operatorname{Card}(\mathscr{E}_\sigma) \le \operatorname{Card}\bigl(\prod_{j=1}^\infty E\bigr) = \operatorname{Card}\bigl(\prod_{j=1}^\infty \mathbf{R}\bigr)$, and the latter is bounded by $\mathfrak{c}$ in view of part (b). This completes the proof.

4. Let $\mathcal{B}_{\mathbf{R}}$ be the σ-algebra of Borel sets in $\mathbf{R}$. In [1, Proposition 0.14(b)], it is shown that if $\operatorname{Card}(A) \le \mathfrak{c}$ and if $\operatorname{Card}(Y_x) \le \mathfrak{c}$ for all $x \in A$, then $\bigcup_{x\in A} Y_x$ has cardinality bounded by $\mathfrak{c}$. By following the given steps, use this observation, as well as problems 2 and 3, to show that

$$\operatorname{Card}(\mathcal{B}_{\mathbf{R}}) = \mathfrak{c}. \tag{‡}$$

(a) Let $\mathscr{E}$ be the collection of open intervals (including the empty set) in $\mathbf{R}$. Then $\operatorname{Card}(\mathscr{E}) = \mathfrak{c}$.

(b) $\mathcal{B}_{\mathbf{R}} = \mathscr{M}(\mathscr{E})$.

(c) Define $\mathscr{F}_x$ as in problem 2. Use transfinite induction and problem 3 to prove that $\operatorname{Card}(F_x) = \mathfrak{c}$ for all $x \in \Omega$.

(d) Use problem 2 to conclude that $\mathscr{M}(\mathscr{E}) = \mathcal{B}_{\mathbf{R}}$ has the cardinality claimed in (‡).[4]

ANS: Parts (a) and (b) are immediate. For c, start by letting $A = \{\, x \in \Omega : \operatorname{Card}(F_x) = \mathfrak{c}\,\}$. It follows from Problem 3(c), that $1 \in A$. Now suppose that $y \in A$ for all $y < x$. If $x = z + 1$, then $F_x \in A$ by Problem 3(c) again. If x is a limit ordinal, then $x \in A$ by the observation the countable union of sets of cardinality $\mathfrak{c}$ has cardinality $\mathfrak{c}$. Thus $A = \Omega$ by transfinite induction.

Now problem 2 implies that $\mathcal{B}_{\mathbf{R}} = \bigcup_{x\in\Omega} F_x$. Since each F_x has cardinality $\mathfrak{c}$ and since Ω has cardinality at most $\mathfrak{c}$, the union has cardinality at most $\mathfrak{c}$ (Folland's Proposition 0.14(b)). This completes the proof.

References

[1] Gerald B. Folland, *Real analysis*, Second, John Wiley & Sons Inc., New York, 1999. Modern techniques and their applications, A Wiley-Interscience Publication.

[2] Anthony W. Knapp, *Basic real analysis*, Cornerstones, Birkhäuser Boston Inc., Boston, MA, 2005. Along with a companion volume *Advanced real analysis*.

[4]It is my ***understanding*** that the classes $\mathscr{F}_x$ are all distinct; that is, $\mathscr{F}_x \subsetneq \mathscr{F}_y$ if $x < y$ in Ω. But I don't have a reference or a proof at hand.