Math 11, Fall 2007

Lecture 22

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Outline

1. Review and overview
   - Last class
2. Today's material
   - Green's Theorem
3. Group Work
4. Next class

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Line Integrals

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- Conservative vector fields and independence of path
- The Fund. Thm. of Line Integrals
- Conservation of Energy

Outline

One variable: integration by parts

- $$\int_a^b u\ dvb = (uv)|_a^b - \int_a^b v\ du$$
- One interpretation:
    1. Exchange an integral over a region, $[a, b]$, for and integral over its boundary, $\{a, b\}$.
    2. Exchange derivatives for integrals ($dv \to v$ and $u \to du$)

Green's Theorem

Theorem

Let C be a positively oriented, piecewise smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D then

$$\int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\;dA$$

Green's Theorem

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Let C be a **positively oriented**, *piecewise smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D then*

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- $Q_x = P_y$ and link to conservative vector fields
- Exchange the integral over a domain D with an integral over its boundary C
- Exhange integrals and derivatives: $P, Q \rightarrow P_y, Q_x$
- This is a multivariable analogue of integration by parts

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Conservative vector fields

When is a vector field, $\vec{F} = P\,\vec{i} + Q\,\vec{j}$, conservative? Recall: necessary condition is that $P_y = Q_x$. What else? Green's Theorem provides an alternate proof of sufficiency:

- If C is a simple closed path in D and R is the region that C encloses,

$$\oint_F \vec{F} \cdot d\vec{r} = \oint P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dA = 0$$

- Thus the integral is independent of path and so $\vec{F}$ is conservative
- By breaking up any closed curve into simple subcurves, we can prove the general theorem.

Proof of Green's Theorem

We can prove this in the special case of a "simple" region i.e. D is given by

$$D = \{(x, y) | a \leq x \leq b, g_1(x) \leq y \leq g_2(x)$$

or

$$D = \{(x, y) | a \leq y \leq b, h_1(y) \leq x \leq h_2(y)$$

Examples

1. $$\int_C (y + e^{\sqrt{x}})\, dx + (2x + \cos(y^2))\, dy$$
   where C is the boundary of the region enclosed by the parabolae $y = x^2, x = y^2$.
2. $\vec{F}(x, y) =< e^x + x^2 y, e^y - xy^2 >$, C is the cirvle $x^2 + y^2 = 25$ oriented clockwise.

Work for next class

- Reading: 17.5
- f07hw24