HW1 SOLUTIONS MATH 46

① Barnett 9/4/07

p. 7–8 #2

$$[e] = \frac{\text{energy}}{\text{mass}} = \frac{ML^2T^{-2}}{M} = (LT^{-1})^2$$ ← e.g. $KE = \frac{1}{2}mv^2$

$$[D] = LT^{-1}$$

only poss. is $\frac{e}{D^2} = \text{const.}$

#5

$$\begin{array}{c|ccc} & t & x & g \\ \hline L & \left[\begin{matrix} 0 \\ 1 \end{matrix}\right. & \begin{matrix} 1 \\ 0 \end{matrix} & \left.\begin{matrix} 1 \\ -2 \end{matrix}\right] \\ T & & & \end{array}$$

$$\begin{array}{c|ccc} & t & x & g \\ \hline L & 0 & 1 & 1 \\ T & 1 & 0 & -2 \end{array}$$

rank $r=2$, $m=3$ } # dimless is $m-r = 1$.

$$\pi_1 = \frac{gt^2}{x}$$

Law is $x = \frac{gt^2}{2} \Rightarrow 2 = \frac{gt^2}{x} = \pi_1$. ie $\pi_1 = 2$ is the dim'less law. or $f(\pi_1) = 0$ with $f(\pi_1) = \pi_1 - 2$

Can m appear? No since there's nothing to cancel it out when making a dim'less quantity.

Ie

$$\begin{array}{c|cccc} & m & t & x & g \\ \hline M & 1 & & & \\ L & & & 1 & 1 \\ T & & 1 & & -2 \end{array}$$

$$\underbrace{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\vec{x} \in \text{Nul } A$ if $\vec{x}$ gives powers of m, t, x, g in a dimless var.

Consider first row of A: tells you $x_1 = 0$. $\Rightarrow$ no powers of m.

#6

$$\underbrace{x}_{L} = -\underbrace{\tfrac{1}{2}gt^2}_{LT^{-2}T^2 = L} + \underbrace{vt}_{LT^{-1}T = L\ ✓}$$

e.g. divide by x: $$1 = -\underbrace{\frac{gt^2}{2x}}_{\frac{1}{2}y} + \underbrace{\frac{vt}{x}}_{s}$$

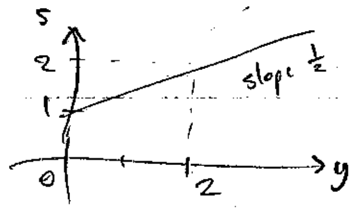
$$\begin{array}{c|cccc} & x & t & v & g \\ \hline L & 1 & & 1 & 1 \\ T & & 1 & -1 & -2 \end{array}$$

x, t: dynamical variables; v, g: params

so law is $1 = -\frac{1}{2}y + s$

or $s = 1 + \frac{1}{2}y$

or $f(s,y) = 0$ with $f(s,y) = -\frac{y}{2} + s - 1$



For all values of v, g, this line contains the formula $x = -\frac{1}{2}gt^2 + vt$.

many other choices, e.g.

$$\begin{cases} y = \dfrac{x}{v^2/g} & \text{rescaled length} \\ s = \dfrac{t}{v/g} & \text{rescaled time.} \end{cases}$$

p. 17–19 #1

$$\begin{array}{c|ccc} & v & \lambda & g \\ \hline L & 1 & 1 & 1 \\ T & -1 & & -2 \end{array}$$

rank $=2$, $m=3$ } $p=1$ dimless quantity $\pi_1 = \frac{v^2}{\lambda g}$

so Pi Thm tells us $f(\pi_1) = 0$ or $\frac{v^2}{\lambda g} = \text{const}$

ie wave speed $= \sqrt{\lambda g}$

have you noticed deep-water waves go faster when they have longer wavelength? It's true.

#4

pressure = $\frac{\text{force}}{\text{area}} = \frac{\text{mass} \times \text{accel}}{L^2} = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}$

~~Sheet1~~

density = $\frac{\text{mass}}{\text{vol.}} = ML^{-3}$

$$\begin{array}{c|c} & \begin{array}{ccccc} p & l & m & t & \rho \end{array} \\ \begin{array}{c} M \\ L \\ T \end{array} & \begin{bmatrix} 1 & & 1 & & 1 \\ -1 & 1 & & & -3 \\ -2 & & & 1 & \end{bmatrix} \end{array}$$

rank appears to be 3, $m=5$, so $m-r=2$ dimless quants.

Check the combinations given in question are dimless: $\pi_1 = \frac{l^3 \rho}{m}$ ← is a density; all dims cancel.

$\pi_2 = \frac{t^6 p^3}{m^2 \rho} \xrightarrow{\text{dims}} T^6 (M^3 L^{-3} T^{-6}) M^{-2} (ML^{-3}) = 1$.

π_1 & π_2 are clearly L.I. (their vectors $\begin{pmatrix} 0 \\ 3 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ for π_1 and $\begin{pmatrix} 3 \\ 0 \\ -2 \\ 6 \\ -1 \end{pmatrix}$ for π_2 are not parallel.)

$\Rightarrow$ Pi Thm says $G(\pi_1, \pi_2) = 0$ is equiv. to original law..

#9

$$\begin{array}{c|c} & \begin{array}{ccccc} Y & C & T & V & r \end{array} \\ \begin{array}{c} M \\ L \\ T \end{array} & \begin{bmatrix} 1 & 1 & & & 1 \\ & -3 & & 3 & -3 \\ -1 & & 1 & & -1 \end{bmatrix} \end{array}$$

rank $r = 3$ (can row reduce), $m = 5$ } $p = 2$ dimless quants.

Now you have freedom choosing π_1 & π_2. We're aiming for something useful when T, r, C are fixed.

$\Rightarrow \pi_1$ = some combo of T, r, c? yes, $\pi_1 = \frac{Tr}{c}$ works dimless.

so any indep. π_2 will do, eg. $\frac{rV}{Y} = \pi_2$

note, doing it the other way round is not useful.

Pi Thm now says $f(\pi_1, \pi_2) = 0$ ie $\pi_2 = G(\pi_1)$

ie $\frac{rV}{Y} = G\left(\frac{Tr}{c}\right)$.

When T, r, C fixed, so is $G\left(\frac{Tr}{c}\right)$. So then $\frac{rV}{Y} = \text{const}$

ie $\boxed{Y = \text{const.}\, rV}$

positively related.

(3)

p. 30-34 #3

dimensions $MT^{-1} = [a]L^2 = [b]L^3$ so $[a] = ML^{-2}T^{-1}$ $[b] = ML^{-3}T^{-1}$

$\boxed{m' = ax^2 - bx^3}$

$m = \rho x^3$ chain rule $\quad m - x - t \quad \frac{dm}{dt} = \frac{dm}{dx}\frac{dx}{dt}$

$\frac{dm}{dt} = 3\rho x^2 x'$

sub in

$3\rho x^2 x' = ax^2 - bx^3$

$\Rightarrow 3\rho x' = a - bx$

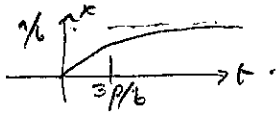
$[\rho] = \left[\frac{m}{x^3}\right] = ML^{-3}$, density.

nondimensionalize (rescale)

$3\rho \frac{x_c}{t_c}\frac{d\bar{x}}{d\bar{t}} = a - bx_c\bar{x}$ $\quad$ choose $t_c = \frac{\rho}{b}$ $\quad x_c = \frac{a}{b}$

so $\cancel{3\rho}\,\cancel{\frac{a}{\rho}}\frac{d\bar{x}}{d\bar{t}} = \cancel{a} - \cancel{a}\bar{x}$ $\quad$ ie $3\frac{d\bar{x}}{d\bar{t}} = 1 - \bar{x}$ $\quad$ no free params!

Solve: $\bar{x}(0) = 0$ $\quad$ gives $\bar{x}(\bar{t}) = 1 - e^{-\frac{1}{3}\bar{t}}$



or $x(t) = x_c\bar{x}(t) = \frac{a}{b}\left(1 - e^{-\frac{b}{3\rho}t}\right)$ $\quad$ exponential growth to size a/b.

#4

ODE $\begin{cases} m x'' = \underbrace{-a x|x'|}_{\text{damping}} \underbrace{- kx}_{\text{restoring}} \end{cases}$ $\quad$ a force MLT^{-2}

ICs $\begin{cases} x(0) = 0 \\ x'(0) = V \end{cases}$

$[a] = \frac{MLT^{-2}}{LLT^{-1}} = ML^{-1}T^{-1}$

$[k] = \frac{MLT^{-2}}{L} = MT^{-2}$

4 params

just params; not x or t. (this is not a 'Pi Thm' matrix)

$$\begin{array}{c|cccc} & a & k & V & m \\ \hline M & 1 & 1 & & 1 \\ L & -1 & & 1 & \\ T & -1 & -2 & -1 & \end{array}$$

rank = 3 so dim Nul A = 1, we know they'll be 1-dim subspace of solns to eqn $A\vec{x} = \vec{b}$.

want the case $\vec{b} = \begin{bmatrix}0\\0\\1\end{bmatrix} = T$, a timescale.

Timescales: $\sqrt{\frac{m}{k}}$ $\quad$ undamped oscillation period

$\sqrt{\frac{m}{Va}}$ $\quad$ time for damping to slow down the initial velocity.

why?

In fact $\sqrt{\frac{m}{k}}\left(\frac{k}{Va}\right)^{\alpha}$ for any α has units of time, but it makes sense to pick $\alpha = 1/2$ to get rid of k.

Want to choose damping timescale if restoring force will be small

$\Rightarrow t_c = \sqrt{\frac{m}{Va}}$ $\quad$ rule for transforming x''

Similarly choose $x_c = \sqrt{\frac{mV}{a}}$ since doesn't involve k.

ODE: $m\boxed{\frac{x_c}{t_c^2}}\frac{d^2\bar{x}}{d\bar{t}^2} = -a\frac{x_c^2}{t_c}\bar{x}\left|\frac{d\bar{x}}{d\bar{t}}\right| - kx_c\bar{x}$

$\Rightarrow \frac{d^2\bar{x}}{d\bar{t}^2} = -\bar{x}\left|\frac{d\bar{x}}{d\bar{t}}\right| - \epsilon\bar{x}$, $\bar{x}(0) = 0$, $\frac{d\bar{x}}{d\bar{t}} = 1$, small dimless param $\epsilon := \frac{k}{aV}$

#10

$$\begin{cases} m x'' = \underbrace{-\gamma\, x'|x'|}_{\text{air resistance}} \; \underbrace{-\, mg}_{\text{weight}} \\ x(0) = 0 \\ x'(0) = V \end{cases}$$

like $(x')^2$ but has the sign of x', what you need for <u>friction</u> (force opp. to vel)

$$\begin{array}{c|c} & \begin{matrix} m & \gamma & g & V \end{matrix} \\ \begin{matrix} M \\ L \\ T \end{matrix} & \begin{bmatrix} 1 & 1 & & \\ & -1 & 1 & 1 \\ & & -2 & -1 \end{bmatrix} \end{array}$$

length scale $x_c = \dfrac{V^2}{g}$ corresponds to height it reaches with no air resistance

timescale $t_c = \dfrac{V}{g}$ time to reach this max height.

neither involve γ, so will be good if γ <u>small</u>.

ODE $\quad m \dfrac{x_c}{t_c^2} \dfrac{d^2\bar{x}}{d\bar{t}^2} = -\gamma \dfrac{x_c^2}{t_c^2} \dfrac{d\bar{x}}{d\bar{t}} \left|\dfrac{d\bar{x}}{d\bar{t}}\right| - mg$

sub. in & cancel, using $x_c/t_c^2 = g$

$$\frac{d^2\bar{x}}{d\bar{t}^2} = -\varepsilon \frac{d\bar{x}}{d\bar{t}}\left|\frac{d\bar{x}}{d\bar{t}}\right| - 1 \quad \text{with } \varepsilon := \frac{\gamma V^2}{mg} \text{ dimless.}$$

IC. $\quad \bar{x}(0) = 0, \quad \dfrac{d\bar{x}}{d\bar{t}}(0) = \dfrac{t_c}{x_c} V = 1$

#11

a) ODE, ICs $\quad \begin{cases} m x'' = -\dfrac{k}{x^2} e^{-t/a} \\ x(0) = L \\ x'(0) = 0 \end{cases}$ (must be dimless: t/a)

b) $$\begin{array}{c|c} & \begin{matrix} L & k & a & m \end{matrix} \\ \begin{matrix} M \\ L \\ T \end{matrix} & \begin{bmatrix} & 1 & & 1 \\ 1 & 3 & & \\ & -2 & 1 & \end{bmatrix} \end{array}$$

note $\beta = \dfrac{k a^2}{m L^3}$ is dimless.

general rescale $\quad \begin{cases} m \dfrac{x_c}{t_c^2} \bar{x}'' = -\dfrac{k}{x_c^2} \bar{x}^{-2} e^{-\frac{t_c}{a}\bar{t}} \\ \bar{x}(0) = L/x_c \\ \bar{x}'(0) = 0 \end{cases}$

$\begin{cases} \text{time scales} & a, \quad \sqrt{\dfrac{L^3 m}{k}}, \quad \text{etc. (there's } a\beta^\alpha, \forall\alpha) \\ \text{length scales} & L, \quad \left(\dfrac{ka^2}{m}\right)^{1/3}, \quad \text{etc.} \quad (L\beta^\alpha, \forall\alpha) \end{cases}$

Possibilities are 4 (2×2 choices for length & time scales which are <u>Lin. Indep.</u>)

i) $x_c = L$, $t_c = a$ gives $\begin{cases} \bar{x}'' = -\beta \bar{x}^{-2} e^{-\bar{t}} \\ \bar{x}(0) = 1 \\ \bar{x}'(0) = 0 \end{cases}$ where prime means $\dfrac{d}{d\bar{t}}$.

ii) $x_c = \dfrac{k^{1/3} a^{2/3}}{m^{1/3}}$, $t_c = \dfrac{L^{3/2} m^{1/2}}{k^{1/2}}$ gives $\begin{cases} \beta \bar{x}'' = -\bar{x}^{-2} e^{-\beta^{-1/2}\bar{t}} \\ \bar{x}(0) = \beta^{-1/3} \\ \bar{x}'(0) = 0 \end{cases}$

which one is appropriate dep. on if $\beta \ll 1$, $\beta \gg 1$, etc.

I won't give other two ones ...

a) int. fac e^{2t} so $(e^{2t}u)' = e^{2t}e^{-t} = e^{t}$

so $e^{2t}u = \int e^{t}dt + c = e^{t} + c$

$u = e^{-t} + ce^{-2t}$.

b) Und. Coeffs but $u'' + 4u = 0$ has solns $\sin 2t, \cos 2t$ so you are driving on-resonance, and will need extra power of t.

$r = \pm 2i$ roots.

Using $s \equiv \sin 2t$, $c \equiv \cos 2t$ and $s' = 2c$ etc.

the other combos $t^2 s$ and tc not needed; this comes with experience.

Try $u = At^2c + Bts$

$u' = 2Atc - 2At^2s + Bs + 2Btc$

$u'' = 2Ac - 4Ats - 4Ats \underbrace{-4At^2c}_{} + 2Bc + 2Bc \underbrace{-4Bts}_{}$ cancels $4u$

so $(2A + 4B)c = 0$ (no $\cos 2t$ in driving)

$-8Ats = \text{driving} = ts$ so $A = -1/8$, $B = -\frac{1}{2}A = \frac{1}{16}$

So $u(t) = -\frac{1}{8}t^2\cos 2t + \frac{1}{16}t\sin 2t$ yuk!

c) Bernoulli with $n=2$ so $w = u^{1-n} = \frac{1}{u}$ so $w' = -\frac{u'}{u^2}$, $u' = -\frac{w'}{w^2}$

so $-\frac{w'}{w^2} - \frac{t}{w} = \frac{t^2}{w^2}$

$\Rightarrow w' + tw = -t^2$ int. fac. $e^{\frac{1}{2}t^2}$ so $(we^{\frac{1}{2}t^2})' = -t^2e^{t^2/2}$

so $w = -e^{-t^2/2}\left[\int t^2 e^{t^2/2}dt + c\right]$ u is inverse of this.

not analytically soluble.

d) Cauchy-Euler $t^2u'' - 3tu' + 4u = 0$ sub. $u = t^m$ $u' = mt^{m-1}$ $u'' = m(m-1)t^{m-2}$

so $t^m\left[m(m-1) - 3m + 4\right] = 0$ ie $m^2 - 4m + 4 = 0$

ie $(m-2)^2 = 0$ so $m = 2$ double root

Look up soln for double root: $u(t) = c_1t^2 + c_2t^2\ln t$.

h) Indep. of t, $G(u,u',u'') = 0$ so use $G(u, v, v\frac{dv}{du}) = 0$ ie replace $\begin{cases} u' \text{ by } v \\ u'' \text{ by } v\frac{dv}{du} \end{cases}$

$uv\frac{dv}{du} - v^3 = 0$ ie $\int u^{-1}du = \int v^{-2}dv \xrightarrow{\text{int.}} \ln u = -v^{-1} + c$

or $\frac{dy}{dt} = v(u) = -(\ln u + c)^{-1}$ Now need to integrate to get $u(t)$

$$\int (\ln u + c)\,du = -\int dt$$

$$u\ln u - u + cu = -t + d \quad \text{ie} \quad u(\ln u - 1 + c) = d - t \qquad c, d \text{ consts.}$$

implicit for $u(t)$, no explicit poss.

#3 a) $mx'' = -\gamma x'|x'| - mg$ same as in #10.

b) $\begin{cases} x'' = -\varepsilon x'|x'| - 1 \\ x(0)=0, \quad x'(0) = 1. \end{cases}$ $\varepsilon := \frac{\gamma V^2}{mg}$ in rescaled units.

ODE is using $v = x'$: $v' = -\varepsilon v^2 - 1$ sep. of var. while going up. (don't worry about going down just yet)

ie $\boxed{\int \frac{dv}{1+\varepsilon v^2}} = -t + c$ recall $\frac{d}{dy}\tan^{-1} y = \frac{1}{1+y^2}$ so sub $y = \sqrt{\varepsilon}\,v$, $dy = \sqrt{\varepsilon}\,dv$.

call I

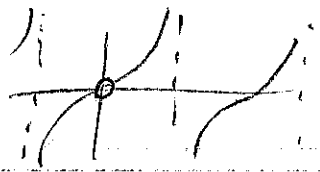
$$I = \varepsilon^{-1/2}\int \frac{dy}{1+y^2} = \frac{1}{\sqrt{\varepsilon}}\tan^{-1}(\sqrt{\varepsilon}\,v)$$

so $\tan^{-1}(\sqrt{\varepsilon}\,v) = \sqrt{\varepsilon}(c - t)$ so $v = \frac{1}{\sqrt{\varepsilon}}\tan\left[\sqrt{\varepsilon}(c-t)\right]$

IC $v(0) = 1$ gives $c = \frac{1}{\sqrt{\varepsilon}}\tan^{-1}\sqrt{\varepsilon}$

c) Max height at $t = \tau$, when $v(\tau) = 0$ ie

Since we are about the zero root of tan func.



then $v = 0 \Rightarrow c - \tau = 0$

ie $\tau = c = \frac{1}{\sqrt{\varepsilon}}\tan^{-1}\sqrt{\varepsilon}$, or real time $= t_c\tau = \frac{V}{g}\frac{\tan^{-1}\sqrt{\varepsilon}}{\sqrt{\varepsilon}}$

Check makes sense when $\varepsilon \to 0$ (vanishing air resistance): $\lim_{x\to 0}\frac{\tan^{-1}x}{x} = 1$

so $\lim_{\varepsilon\to 0}\tau = 1$ giving real time of $\frac{V}{g}$. ✓

this is what expect for no air resistance.