Homework Problems

November 2, 2005

Exercise 1. Let $D : \mathbb{P}_4 \to \mathbb{P}_4$ be the linear transformation given by

$$D(p(t)) = (1 - t^2)p''(t) - 2tp'(t) + 20p(t).$$

a. Using coordinates, find bases for the kernel and range of D.

b. Use the result of part (a) to conclude that, up to scalar multiples, there is only one polynomial solution of degree ≤ 4 to the differential equation

$$(1 - t^2)p'' - 2tp' + 20p = 0.$$

What is this solution?

c. Use the result of part (a) to produce a polynomial q of degree at most 4 so that the differential equation

$$(1 - t^2)p'' - 2tp' + 20p = q$$

has *no* polynomial solution of degree ≤ 4.

Recall the following fact from elementary algebra.

Theorem 1. *If $p(t)$ is a polynomial with real coefficients and a is a real number with $p(a) = 0$ then there is a polynomial $q(t)$ with real coefficients so that $p(t) = (t - a)q(t)$.*

In the next exercise we will provide a linear algebraic proof of this fact, at least for polynomials of degree ≤ 3.

Exercise 2. Fix a real number a and consider the linear transformation $T : \mathbb{P}_3 \to \mathbb{R}$ given by

$$T(p) = p(a).$$

a. Using coordinates relative to the bases $\mathcal{B} = \{1, t, t^2, t^3\}$ and $\mathcal{E} = \{1\}$, find bases for the kernel and range of T.

b. Show directly (*without* using Theorem 1 above) that the polynomials in your basis for $\ker T$ are all divisible by $t - a$. Conclude that all the polynomials in $\ker T$ are divisible by $t - a$.

c. Show that part (b) proves Theorem 1 for polynomials of degree ≤ 3?

Exercise 3. [Extra Credit] Apply the technique of Exercise 2 to the linear transformation $T : \mathbb{P}_n \to \mathbb{R}$ given by $T(p) = p(a)$ to prove Theorem 1 in general.