Math 23 Fall 2013

Differential Equations

Exam 2

Friday, November 1, 4:00PM - 6:00PM

Your name (please print): SOLUTION ____

Section (circle one): Section 1, Section 2

Instructions: This is a closed book, closed notes exam. The use of calculators is not permitted. The exam consists of **8** problems and this booklet contains **16** pages (including this one). **On problems 3 through 8, you must show your work and justify your assertions to receive full credit**. Justify your answers and simplify your results as much as possible. Also, please clearly mark your final (simplified) answer. The last two pages of this booklet are blank. Good Luck!

The Honor Principle requires that you neither give nor receive any aid on this exam.

FERPA Waiver: By my signature I relinquish my FERPA rights in the following context: My exam may be returned en masse with others present in the classroom. I acknowledge that I understand my score may be visible to others. If I choose not to relinquish my FERPA rights, I understand that I will have to present my student ID at my instructor's office to retrieve my exam.

Signature:____

Math 23 Fall 2013

Your name (please print): ______________________________

Problem	Points	Score
1	10	
2	10	
3	15	
4	10	
5	20	
6	10	
7	10	
8	15	
Total	**100**	

Short Answer Questions - Work will not be graded

1. Determine a lower bound for the radius of convergence ($|x - x_0| < \rho$) of series solution about each given x_0.

① (**5 Points**) $e^x y'' + ty' - 5y = 0, \;\; x_0 = 0$

Answer:$\rho = \infty$

② (**5 Points**) $p'' + \frac{1}{(x-2)(x+2)}p' + \frac{x}{(x+2)(x-5)(x+i)(x-i))}p = 0, \;\; x_0 = 1$

Answer:$\rho = 1$

Short Answer Questions - Work will not be graded

2. Write down the following differential equations into the system of first order differential equations *(DO NOT SOLVE)*.

(a) (**5 Points**)

$$2x'' + 3x' + 5x = 0$$

Answer:

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -\frac{5}{2}y_1 - \frac{3}{2}y_2 \end{aligned}$$

(b) (**5 Points**)

$$z^{(4)} + 10z'' + 2z' + z = t^2e^{-t}$$

Answer:

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_4 \\ y_4' &= -10y_3 - 2y_2 - y_1 + t^2e^{-t} \end{aligned}$$

Show your work

3. A mass weighing 2 lb stretches a spring 6 in. If the mass is pulled down an additional 12 in with an initial velocity 8 ft/s, and if there is no damping, (Note that gravity $g = 32$ ft/s^2 and 1 in = 1/12 ft).

   (a) (**10 Points**) Determine the position u of mass at any time t using ft.
   The mass and spring constant are

$$m = \frac{w}{g} = \frac{2 \text{ lb}}{32 \text{ ft}/s^2} = \frac{1}{16} \text{ lb} \cdot s^2/\text{ft},$$
$$k = \frac{2 \text{ lb}}{1/2 \text{ ft}} = 4 \text{ lb/ft}.$$

   Therefore the spring equation becomes

$$\frac{1}{16}u'' + 4u = 0 \to u'' + 64u = 0$$

   Thus,

$$u(t) = A\cos 8t + B\sin 8t$$

   From the initial condition $u(0) = 1$ ft $u'(0) = 8$ ft/s.

$$u(0) = A = 1$$
$$u'(0) = 8B = 8 \to B = 1$$

   Answer: $u(t) = \cos 8t + \sin 8t$

Show your work

(b) (**5 Points**) Determine the natural frequency ω_0, phase δ, and amplitude R.

$$
\begin{aligned}
u(t) &= \cos 8t + \sin 8t = \sqrt{1+1}\left(\frac{1}{\sqrt{2}}\cos 8t + \frac{1}{\sqrt{2}}\sin 8t\right) = \sqrt{2}\left(\cos\frac{\pi}{4}\cos 8t + \sin\frac{\pi}{4}\sin 8t\right) \\
&= \sqrt{2}\cos\left(8t - \frac{\pi}{4}\right)
\end{aligned}
$$

Answer: $\omega_0 = 8$, $\delta = \frac{\pi}{4}$, $R = \sqrt{2}$

Show your work

4. (**10 Points**) Determine the Taylor series of

$$f(x) = \frac{e^x + e^{-x}}{2}$$

about $x_0 = 0$. Specify at least first 4 terms and n-th term.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$\begin{aligned}
a_0 &= \frac{f(0)}{0!} = \frac{\frac{e^0+e^{-0}}{2}}{0!} = 1 \\
a_1 &= \frac{f'(0)}{1!} = \frac{\frac{e^0-e^{-0}}{2}}{1!} = 0 \\
a_2 &= \frac{f''(0)}{2!} = \frac{\frac{e^0+e^{-0}}{2}}{2!} = \frac{1}{2!} \\
a_3 &= \frac{f^{(3)}(0)}{3!} = \frac{\frac{e^0-e^{-0}}{2}}{3!} = 0 \\
a_4 &= \frac{f^{(4)}(0)}{4!} = \frac{\frac{e^0+e^{-0}}{2}}{4!} = \frac{1}{4!} \\
&\dots
\end{aligned}$$

Answer:

$$f(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots + \frac{1}{(2n)!}x^{2n} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n}$$

Show your work

5. Consider

$$y'' - xy = 0$$

(a) (**10 Points**) Seek power series solution about $x_0 = 1$ by letting $y = \sum_{n=0}^{\infty} a_n (x-1)^n$ (i.e., Find the recurrence relation)

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n (x-1)^n \\ y' &= \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} \end{aligned}$$

Then

$$\begin{aligned} y'' - xy &= \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} - x \sum_{n=0}^{\infty} a_n (x-1)^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - (x-1) \sum_{n=0}^{\infty} a_n (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^{n+1} - \sum_{n=0}^{\infty} a_n (x-1)^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=1}^{\infty} a_{n-1} (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n \\ &= 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=1}^{\infty} a_{n-1} (x-1)^n - a_0 - \sum_{n=1}^{\infty} a_n (x-1)^n \\ &= 2a_2 - a_0 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - a_{n-1} - a_n] (x-1)^n = 0 \end{aligned}$$

Therefore

$$2a_2 - a_0 = 0 \ \rightarrow a_2 = \frac{a_0}{2} \text{ and}$$

$$(n+2)(n+1) a_{n+2} - a_{n-1} - a_n = 0 \rightarrow a_{n+2} = \frac{a_{n-1} + a_n}{(n+2)(n+1)}, n = 1, 2, 3, \cdots .$$

Answer: $a_2 = \frac{a_0}{2}$, $a_{n+2} = \frac{a_{n-1} + a_n}{(n+1)(n+2)}, n = 1, 2, 3, \cdots$

Show your work

(b) (**10 Points**) Find the first three terms in y_1 and y_2 by finding a_n up to a_4.

$$\begin{aligned}
&a_0\\
&a_1\\
&a_2 = \frac{a_0}{2}\\
&a_3 = \frac{a_0 + a_1}{2 \cdot 3} = \frac{a_0}{6} + \frac{a_1}{6} \quad (n = 1)\\
&a_4 = \frac{a_1 + a_2}{3 \cdot 4} = \frac{a_1}{12} + \frac{a_2}{12} = \frac{a_1}{12} + \frac{a_0}{24} \quad (n = 2)
\end{aligned}$$

Therefore

$$\begin{aligned}
&y = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 \cdots\\
&= a_0 + a_1(x-1) + \frac{a_0}{2}(x-1)^2 + (\frac{a_0}{6} + \frac{a_1}{6})(x-1)^3 + (\frac{a_1}{12} + \frac{a_0}{24})(x-1)^4 \cdots\\
&= a_0 + \frac{a_0}{2}(x-1)^2 + \frac{a_0}{6}(x-1)^3 + \frac{a_0}{24}(x-1)^4 + \cdots + a_1(x-1) + \frac{a_1}{6}(x-1)^3 + \frac{a_1}{12}(x-1)^4 \cdots\\
&= a_0(1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4 + \cdots) + a_1((x-1) + \frac{1}{6}(x-1)^3 + \frac{1}{12}(x-1)^4 + \cdots\\
&= a_0 y_1 + a_1 y_2,
\end{aligned}$$

where

$$\begin{aligned}
y_1 &= 1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \cdots\\
y_2 &= (x-1) + \frac{1}{6}(x-1)^3 + \frac{1}{12}(x-1)^4 + \cdots
\end{aligned}$$

Show your work

6. (**10 Points**) Consider an Euler equation

$$t^2y'' - 2ty' + 2y = 0, t > 0.$$

Find a fundamental set of solutions $y_1(t)$ and $y_2(t)$ and show that they form a fundamental set of solutions for $t > 0$.
Let $y = t^r$. Then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. The Euler equation becomes

$$t^2y'' - 2ty' + 2y = t^2r(r-1)t^{r-2} - 2trt^{r-1} + 2t^r = r(r-1)t^r - 2rt^r + 2t^r = 0.$$

r can be found by solving

$$r(r-1) - 2r + 2 = 0 \to r^2 - 3r + 2 = 0 \to r = 1, 2.$$

Therefore $y_1 = t^1$ and $y_2 = t^2$ and Wronskian

$$W = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = t^2 \neq 0 \text{ for } t > 0$$

$\to$ t and t^2 form a fundamental set of solutions.

Answer: $y = c_1t + c_2t^2$

7. (a) (**5 Points**) Find the eigenvalues of

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & -1 \\ -1 & 6 & 3 \end{pmatrix}$$

$$\mathbf{A} - \lambda I = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & 6 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 0 & 0 \\ 1 & 1-\lambda & 1 \\ -1 & 6 & 3-\lambda \end{pmatrix}$$

$$\begin{aligned} \det(\mathbf{A} - \lambda I) &= (2-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ 6 & 3-\lambda \end{vmatrix} - 0\begin{vmatrix} 1 & -1 \\ -1 & 3-\lambda \end{vmatrix} + 0\begin{vmatrix} 1 & 1-\lambda \\ -1 & 6 \end{vmatrix} \\ &= (2-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ 6 & 3-\lambda \end{vmatrix} \\ &= (2-\lambda)((1-\lambda)(3-\lambda)+6) = (2-\lambda)(\lambda^2 - 4\lambda + 9) = 0 \end{aligned}$$

Therefore,

$$2 - \lambda = 0, \quad \lambda^2 - 4\lambda + 9 = 0$$

Finally

$$\lambda = 2, \frac{4 \pm \sqrt{16 - 4(9)}}{2} = \frac{4 \pm i\sqrt{20}}{2} = 2 \pm i\sqrt{5}$$

(b) (**5 Points**) The 3×3 matrix

$$\mathbf{B} = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

has an eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$. Find an eigenvector of $\mathbf{B}$ corresponding to eigenvalue $\lambda_2 = 2$.

$$(A - \lambda I)\vec{x} = \vec{0} \to \begin{pmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus

$$x_1 + x_3 = 0 \to x_3 = -x_1$$

Therefore

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ -x_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ -x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Answer: $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

8. (**5 Points**) (a)(**5 Points**)Find the general solution of

$$\vec{y}\,' = \mathbf{A}\vec{y},$$

where

$$\mathbf{A} = \begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \text{ and } \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

The matrix $\mathbf{A}$ have eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, and eigenvectors

$$\xi_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \xi_2 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \xi_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix},$$

respectively.

$$\vec{y} = c_1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} e^{3t}$$

(b)(**10 Points**) Find the <u>real-valued general solution</u> of

$$\vec{y}(t)' = \mathbf{B}\vec{y}(t),$$

where

$$\mathbf{B} = \begin{pmatrix} 3 & -4 \\ 2 & -1 \end{pmatrix} \text{ and } \vec{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

and the matrix $\mathbf{B}$ has eigenvalues $\lambda_1 = 1 + 2i$ and $\lambda_1 = 1 - 2i$, and eigenvectors

$$\vec{\xi}_1 = \begin{pmatrix} 1+i \\ 1 \end{pmatrix}, \ \vec{\xi}_2 = \begin{pmatrix} 1-i \\ 1 \end{pmatrix},$$

respectively.

Let the real and imaginary part of the eigenvectors be

$$\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Then,

$$\begin{aligned} \vec{y} &= c_1 \begin{pmatrix} 1+i \\ 1 \end{pmatrix} e^{(1+2i)t} + c_2 \begin{pmatrix} 1-i \\ 1 \end{pmatrix} e^{(1-2i)t} \\ &= c_1(\vec{a} + i\vec{b})e^t e^{2it} + c_2(\vec{a} - i\vec{b})e^t e^{-2it} \end{aligned}$$

The first term can be simplified with the Euler formula as

$$\begin{aligned} c_1(\vec{a} + i\vec{b})e^t e^{2it} &= c_1(\vec{a} + i\vec{b})e^t(\cos 2t + i \sin 2t) \\ &= c_1 e^t(\vec{a}\cos 2t - \vec{b}\sin 2t + i(\vec{a}\sin 2t + \vec{b}\cos 2t)). \end{aligned}$$

The second term is the conjugate of the first term

$$c_2(\vec{a} - i\vec{b})e^t e^{-2it} = c_2 e^t(\vec{a}\cos 2t - \vec{b}\sin 2t - i(\vec{a}\sin 2t + \vec{b}\cos 2t)).$$

Therefore

$$\begin{aligned} \vec{y} &= c_1 e^t(\vec{a}\cos 2t - \vec{b}\sin 2t + i(\vec{a}\sin 2t + \vec{b}\cos 2t)) + c_2 e^t(\vec{a}\cos 2t - \vec{b}\sin 2t - i(\vec{a}\sin 2t + \vec{b}\cos 2t)) \\ &= (c_1 + c_2)e^t(\vec{a}\cos 2t - \vec{b}\sin 2t) + (c_1 - c_2)ie^t(\vec{a}\sin 2t + \vec{b}\cos 2t) \end{aligned}$$

Again since $c_1 = \bar{c_2}$, we can let $c_1 + c_2 = C_1$ and $(c_1 - c_2)i = C_2$. Finally,

$$\begin{aligned} \vec{y} &= C_1 e^t(\vec{a}\cos 2t - \vec{b}\sin 2t) + C_2 e^t(\vec{a}\sin 2t + \vec{b}\cos 2t)) \\ &= C_1 e^t(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos 2t - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin 2t) + C_2 e^t(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\sin 2t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos 2t) \\ &= C_1 e^t \begin{pmatrix} \cos 2t - \sin 2t \\ \cos 2t \end{pmatrix} + C_2 e^t \begin{pmatrix} \sin 2t + \cos 2t \\ \sin 2t \end{pmatrix} \end{aligned}$$

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Extra page for scratch work. I will not grade work on this page unless you write on another page “problem continued on page 16”.