Math 74 Final: Second Version

Due by noon on June 4th.

1. Here is a presentation of the alternating group

$$A_5 = < a, b | a^2, b^3, (ab)^5 >$$

   which is the first non-Abelian simple group. Construct a topological space with fundamental group A_5, and **prove** it has this fundamental group.

2. Let f be continuous map from S^1 to S^1 such that $f(-x) = -f(x)$ for every $x \in S^1$. (See the Borsuk Ulam sheet for hints.)
   (a) Let z^2 be the covering map from example 3 on page 338 of Munkres. Prove there exist a continuous map $\tilde{f}$ such that $\tilde{f}z^2 = z^2 f$.
   (b) Prove $deg(\tilde{f}) = deg(f)$ (See the Borsuk Ulam sheet for the definition of $deg(f)$).
   (c) Using an isomorphism between $\pi_1(S^1, y)$ and $\mathbf{Z}$, prove that an element $[\gamma] \in \pi_1(S^1, y)$ corresponds to an odd integer if and only if the lifting of γ to S^1 with respect to the covering map z^2 has distinct end points.
   (d) Prove $deg(\tilde{f})$ is odd and finish the proof of the Theorem 2 from the Borsuk-Ulam theorem handout. (Hint: notice that $[\tilde{f}(z^2(\lambda))] = [z^2(f(\lambda))]$, where λ is a path that connects anti-podal points in S^1.)

3. See the rotation sheet for notation and hints.
   (a) Let $\phi_l(m)$ be the homeomorphism determined by $\phi_l(m)(z_1, z_2) = (e^{i2\pi\frac{m}{k}} z_1, e^{i2\pi\frac{ml}{k}} z_2)$. Prove that the mapping of $\phi_l : \frac{\mathbf{Z}}{k\mathbf{Z}} \to Homeo(S^3)$, sending $[m]$ to $\phi_l(m)$, is an injective homomorphism when $l \in \mathbf{Z}$ is relatively prime to k.
   (b) Prove that $\phi_l(\frac{\mathbf{Z}}{k\mathbf{Z}})$ a properly discontinuous subgroup of $Homeo(S^3)$.
   (c) Let $L(k, l) = S^3/\phi_l(\frac{\mathbf{Z}}{k\mathbf{Z}})$, and prove that if $L(k, l)$ is homeomorphic to $L(k_1, l_1)$ then $k = k_1$.

4. See the rotation sheet for notation and hints.
   (a) Prove that **UH** forms a group under the $*$. (Hint: $q^{-1} = \bar{q}$.)
   (b) Let $E^3 = \{\vec{v} = xi + yj + zk \mid x, y, z \in \mathbf{R}$ where we view as $\{i, j, k\}$ as an orthonormal basis. Prove that the map $\Psi : \mathbf{UH} \to Homeo(E^3)$ determined by $\psi(q)(\vec{v}) = q\vec{v}q^{-1}$ has its image contained inside $SO(3)$ and is a continuous homomorphism.
   (c) Prove Ψ is the two fold covering map that identifies the anti-podal the points of S^3 (Hint: $q(\theta, u) = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\vec{v}$ will satisfy that $\Psi\left(q(\theta, u)\right)$ is the right-handed rotation by θ about the axis u.)
   (d) Use the previous problem to prove that $SO(3)$ is homeomorphic to C^3 (Hint: you may use problem 1 from the first exam.)