HOMEWORK 4

SOLUTIONS

2.1.13 Prove that every graph with diameter d has an independent set with at least $\lceil (1+d)/2 \rceil$ vertices.

Proof. Consider a path of minimal length between two vertices u and v with $d(u,v) = d$. This path has $d+1$ vertices, all with the property that if they are not adjacent in the path, then they are not adjacent in G. Then by alternating colors, one can two-color the vertices in the path. The set which includes one or both endpoints (the independent set of a single color) is of size $\lceil d+1)/2 \rceil$. $\square$

2.1.40 Let G be a tree with k leaves. Prove that G is the union of paths $P_1, \ldots, P_{\lceil k/2 \rceil}$ such that $P_i \cap P_j \neq \emptyset$ for all $i \neq j$.

Proof. Consider a collection of $\lceil k/2 \rceil$ paths $\{P_i\}$ that cover the leaves of G. Consider $\sum_i \ell(P_i)$. Suppose that there are two paths P_i (with endpoints u_i and v_i) and P_j (with endpoints u_j and v_j) which do not intersect. Then the paths Q_i from u_i to v_j and Q_j from u_j to v_i together have all the same edges as P_i and P_j, together with the edges on a path which connects P_i to P_j. So $\ell(P_i) + \ell(P_j) < \ell(Q_i) + \ell(Q_j)$. Also, if $\bigcup_{i=1}^{\lceil k/2 \rceil}$ does not cover G, then since it does cover the leaves, $\bigcup_{i=1}^{\lceil k/2 \rceil} P_i$ must be disconnected, and therefore have some pair of paths which are disjoint.

Now take a collection of $\lceil k/2 \rceil$ which maximizes $\sum_i \ell(P_i)$ (such a thing exists because G is finite). Then by the above argument, $\bigcup_{i=1}^{\lceil k/2 \rceil} P_i$ covers G and each of the paths pairwise intersect. $\square$

2.1.44 Prove or disprove: If a simple graph with diameter 2 has a cut-vertex, then its complement has an isolated vertex.

Proof. Consider a vertex u which is not adjacent to every other vertex in $V(G)$ (and therefore is not isolated in $\bar{G}$). Let v be a vertex which is not adjacent to u. We will show that $G - u$ is connected (G is connected because $\operatorname{diam}(G) < \infty$).

Let $a \in V(G) - u$. Then there is a path of length at most 2 from a to v. Since v is not adjacent to u, that path cannot go through u. Therefore that path is an a, v path in $G - u$. By transitivity, this implies that $G - u$ is connected. So u is not cut. $\square$

2.1.47 *Diameter and radius*

(a) **Prove that the distance function $d(u,v)$ on pairs of vertices of a graph satisfies the triangle inequality: $d(u,w) \leq d(u,v) + d(v,w)$.**

Proof. Consider a path P of minimal length from u to v and a path of minimal length Q from v to w. Then there is a path from u to w in $P \cup Q$ of length at most $\ell(P) + \ell(Q)$. This may not be a minimal length path from u to w, but it does bound $d(u, w)$ above by $d(u, v) + d(v, w)$. □

(b) **Use part (a) to prove that** $\operatorname{diam}(G) \leq 2\operatorname{rad}(G)$ **for every graph** G**.**

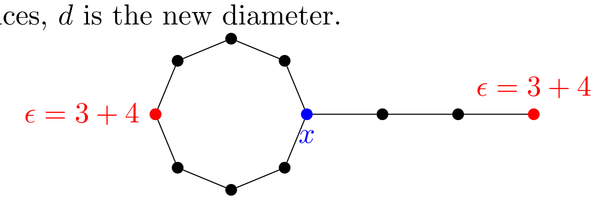
Proof. Consider two anu vertices u and v, and let x be an element of the center (so $\epsilon(x) = \operatorname{rad}(G)$ is minimal). By (a),

$$d(u, v) \leq d(u, x) + d(v, x) \leq \epsilon(x) + \epsilon(x) = 2\operatorname{rad}(G).$$

□

(c) **For all positive integers** r **and** d **that satisfy** $f \leq d \leq 2r$**, construct a simple graph with radius** r **and diameter** d**.** (Hint: build a suitable graph with one cycle.)

Consider the lollypop, a cycle C_{2r} with handle P_{d-r} attached at vertex x, with $r \leq d \leq 2r$. If $d = r$, this is just the cycle, which has radius = diameter = r. By adding a path to x of length less or equal to r, the eccentricity of x remain unchanged (the farthest vertex is still on the other side of the cycle). However, the end of the handle is $d - r$ father away from the antipode of x, and so is $d - r + r = d$ away from x. Since anything closer in on the handle and anything closer in the cycle both have smaller eccentricities than these two vertices, d is the new diameter.

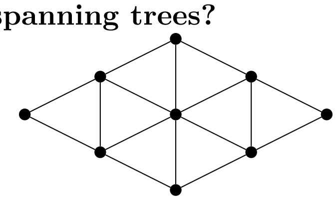


2.1.62 Let G **be a connected graph with** n **vertices. Define a new graph** G' **having one vertex for each spanning tree of** G**, with vertices adjacent in** G' **if and only if the corresponding trees have exactly** $n(G) - 2$ **common edges. Prove that** G' **is connected. Determine the diameter of** G'**.** (an example is given in the book)

Proof. Consider two spanning trees T and T' of G, and say the differ by $2m$ edges. Let e be in T and not in T'. Then there is an edge e' of T' not in T so that $T'' = T' + e - e'$ is also a spanning tree. Since T' differs from T'' by $2m - 2$, we can recursively generate a path in G' from T to T'.

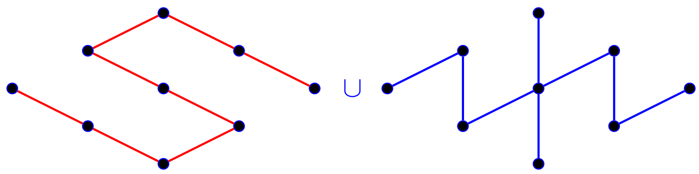
Moreover, this path is length at *most* m, so the diameter of G' cannot be any greater than $e(G) - (n(G) - 1)$ because T' cannot differ from T by more edges than $G - E(T)$. Also, the diameter cannot be any larger than $n(G) - 1$ since $e(T) = e(T') = n(G) - 1$. □

2.1.68 Can the graph below be decomposed into edge-disjoint spanning trees? Into isomorphic edge-disjoint spanning trees?

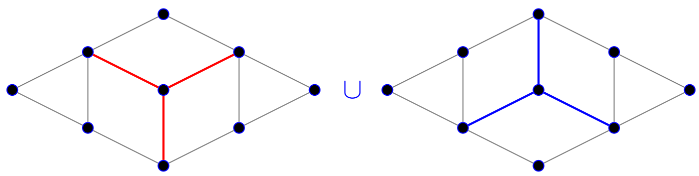


Yes! You can start thinking about it this way: you have nine vertices, so any spanning tree is going to need 8 edges. Since there are 16 edges in G, there is some hope. In fact, you know that if you remove the edges of any spanning subgraph from G, if you can manage to keep the result connected, what remains will be a spanning tree too! Next, since there are two vertices of degree 2, those had better be leaves in both of the disjoint spanning trees.

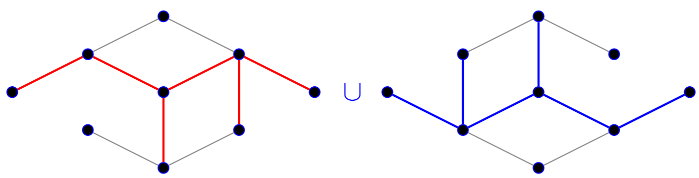
A naive first try, starting at one of those degree-two vertices and drawing a long path that ends at the other degree two vertex works!



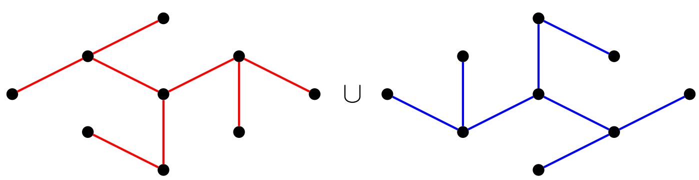
To get two which are isomorphic, we're going to have to be better about exploiting the symmetry of the graph. I started with these two pictures:



Then I tried to keep one the 180° rotation of the other:



Result:



2.2.1 Determine which trees have Prüfer codes that

(a) **contain only one value,**

Answer: The star (there is only one internal vertex).

(b) **contain exactly two values,**

Answer: A tree with two internal vertices u, v (they must be adjacent, and $T \cdot uv$ is a star).

(c) **have distinct values in all positions.**

Answer: The path (every vertex appearing in the list has degree 2, and there are exactly 2 missing).

2.2.8 Count the following sets of trees with the vertex set $[n]$, giving two proofs for each: one using the Prüfer correspondence and one by direct counting arguments.

(a) **trees that have 2 leaves**

Answer: $n!/2$. (1) There are 2 leaves, so these are the lists with $n-2$ distinct values. So pick two values not in the list ($\binom{n}{2}$ ways), and how to rearrange the $n-2$ distinct values in the list ($(n-2)!$ ways. But $\binom{n}{2} * (n-2)! = \frac{1}{2}n(n-1)(n-2)! = \frac{1}{2}n!$. (2) These are paths, so there are $n!$ ways of rearranging the vertices, which double-counts the reflection of the path.

(b) **trees that have $n-2$ leaves**

Answer: $\frac{1}{2}n(n-1)2^{n-2}$. (1) These are lists with exactly two values, since there are 2 non-leaves. Pick a first value, then a second. Then for each element of the list of length $n-2$, decide which of the two values it is. This will double count swapping the first and second chosen values. (2) If there are $n-2$ leaves, then there are 2 internal vertices. Put one on the left and one on the right. Then from the remaining vertices, decide whether they're adjacent to the left vertex or not (if not, they're adjacent to the right). Divide by two, since this double counts exactly the horizontal flip.