Math 118. **Combinatorics.** Spring 2013

Problem Set 1. Due on Wednesday, 4/10/2013.

1. Recall Sylvester's map $\lambda \mapsto \mu$ defined in class, where λ is a partition into odd parts.
   (a) Prove that the image μ is a partition into distinct parts.
   (b) Prove that Sylvester's map is a bijection between partitions of n into odd parts and partitions of n into distinct parts.
   (c) (Bonus) Prove that the number of different parts in λ equals the number of blocks of consecutive parts in μ. For example, $\lambda = (9,9,7,3,3)$ has three different parts, and $\mu = (9,8,7,4,2,1)$ has three blocks, namely block $9,8,7$, block 4, and block $2,1$.
2. Recall that $p(n)$ denotes the number of partitions of n. Prove that the number of pairs (λ, μ) where $\lambda \vdash n$, $\mu \vdash n+1$, and the Young diagram of μ is obtained from that of λ by adding one square, is equal to $p(0) + p(1) + \cdots + p(n)$.
3. Prove that the number of partitions of n into 4 parts equals the number of partitions of $3n$ into 4 parts of size at most $n-1$.
4. Show that for any partition λ,
$$\sum_i (i-1)\lambda_i = \sum_i \binom{\lambda_i'}{2},$$
where the λ_i' denote the parts of the conjugate partition.
5. Prove the following identities:
$$\prod_{n\geq 1} \frac{1}{1-t^n} = \sum_{k\geq 0} \frac{t^{k^2}}{[(1-t)\cdots(1-t^k)]^2}, \tag{1}$$
$$\prod_{n\geq 1} (1+t^n) = \sum_{k\geq 0} \frac{t^{\binom{k+1}{2}}}{(1-t)(1-t^2)\cdots(1-t^k)}. \tag{2}$$
$$\prod_{n\geq 0} (1+t^{2n+1}) = \sum_{k\geq 0} \frac{t^{k^2}}{(1-t^2)\cdots(1-t^{2k})}, \tag{3}$$
$$\prod_{n\geq 1} \frac{1}{1-qt^n} = \sum_{k\geq 0} \frac{t^{k^2}q^k}{(1-t)\cdots(1-t^k)(1-qt)\cdots(1-qt^k)}, \tag{4}$$
$$\prod_{n\geq 1} (1+qt^n) = \sum_{k\geq 0} \frac{t^{\binom{k+1}{2}}q^k}{(1-t)(1-t^2)\cdots(1-t^k)}. \tag{5}$$

6. Find a bijective proof (using a Franklin-type involution, like in the proof of Euler's Pentagonal Theorem) of Jacobi's identity in the form:

$$\prod_{n=1}^{\infty}(1-x^n y^{n-1})(1-x^{n-1}y^n)(1-x^n y^n) = 1+\sum_{n=1}^{\infty}(-1)^n\left(x^{\frac{n(n+1)}{2}}y^{\frac{n(n-1)}{2}}+x^{\frac{n(n-1)}{2}}y^{\frac{n(n+1)}{2}}\right).$$