Math 71 Homework 3 Partial Solutions

48/2a What is $(13)(12)$?

48/5 Suppose $|H| = n-1$. $\therefore \exists\, x_0 \notin H$. Then for some $x \in H$, $x_0 x \in H$ (Consider left mult. $L_{x_0}: H \to G$)
But $x_0 x \in H$ and $x \in H$ implies $x_0 \in H$ contradiction

49/13 Suppose $H \neq \{0\}$ $\therefore \exists\, a/b \in H$. Can assume $b > 0$
$\therefore a = b\left(\frac{a}{b}\right) = \frac{a}{b} + \cdots + \frac{a}{b} \in H$ $\therefore \exists$ positive integer $k \in H$ $\therefore 1 = \frac{1}{k} + \cdots + \frac{1}{k} \in H$ $\therefore \mathbb{Z} \subseteq H$
$\therefore \frac{r}{s} = r\left(\frac{1}{s}\right) \in H$.

60/11 Is $\{e, r^4, s, sr^4\}$ a proper subgroup?

60/13a $\mathbb{Z} \times \mathbb{Z}_2$ has an element of order 2

60/18 Define $\phi: \mathbb{Z}_n \to H$ by $\phi(x^a) = h^a$. Show that ϕ is a unique homo. Note we are not assuming that $|h| = n$.

66/8 $(12) = (1234)(1243)^3(1234)$

66/11 Every element of S_4 has order 1, 2, 3, 4. Find an element $A \in SL(2, \mathbb{Z}_3)$ such that $A^n \neq I$ for $n = 1, 2, 3$ or 4.
(Try $\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$).

65/13 Every positive rational can be written
$\frac{p_1 p_2 \cdots p_s}{q_1 q_2 \cdots q_t}$ ~~where the p's and q's are primes (not necessarily distinct~~

65/14c (c) Let $H \leq G$ be f.g. with generating set $\left\{\frac{a_1}{b_1}, \ldots, \frac{a_\ell}{b_\ell}\right\}$
Every element $x \in H$ can be written
$$x = n_1 \frac{a_1}{b_1} + \cdots\ n_\ell \frac{a_\ell}{b_\ell} \quad \text{(for integers } n_1, \ldots, n_\ell)$$
$$= N/b_1 \cdots b_\ell = N/k \quad \text{for some integer } N$$
$\therefore x = N\left(\frac{1}{k}\right) \in \left\langle \frac{1}{k} \right\rangle$ $\therefore H \leq \left\langle \frac{1}{k} \right\rangle$
Hence H is cyclic (subgroup of a cyclic group).

(d) $\mathbb{Q}$ is not cyclic

65/19 (a) $\forall a$ and $k \in \mathbb{Z}$, $k \neq 0$, $a = kx$ for some x

Given $p/q \in \mathbb{Q}$ and $k \neq 0$

$$p/q = k(p/kq)$$

(b) Let G be a finite abelian group. Then $\exists$ positive integer N such that $Na = 0$ $\forall a \in G$ (Proof: every $a \in G$ has finite order n_a. Let $N = \prod n_a$) Now assume that G is divisible and let $a \neq 0 \in G$

$\therefore$ $a = Na'$ some $a' \in G$.

$\therefore$ $a = 0$. Contradiction.