Math 31 Lesson Plan

Day 19: Sections 9 & 10

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Supplies needed:

- Colored chalk!!
- Quizzes
- Homework
- S_3 and $\mathbb{Z}_8$ tables

Goals for students: Students will:

- Develop their proof-writing skills, both by seeing good proofs modeled and carefully writing up proofs
- Become more comfortable in front of the classroom and writing at the board

[**Lecture Notes: Write everything in blue, and every equation, on the board. [Square brackets] indicate anticipated student responses. *Italics* are instructions to myself.**]

- *Collect homework*
- *Quizzes!*
- *Return homework while taking quiz*
- *Put Cayley tables for S_3 and $\mathbb{Z}_8$ on the board – in an unorthodox order so that the cosets are easy to identify*
- *Introduce Rosa*

Today we're going to talk about Cosets of subgroups. This will allow us to prove Lagrange's Theorem, which allows us to relate the size of a subgroup to the size of a group. Then we can prove that $o(x) \mid |G|$ for any finite group G and any $x \in G$.

Tonight I want you to Read Section 9 and post a comment. You're also welcome to read Section 10, through the first paragraph of page 93, if you'd like. There's no required Blackboard post for Section 10, and we won't be discussing the rest of the section.

Then tomorrow we'll talk about Equivalence Relations, which are discussed in Section 9, and on Wednesday we'll develop a rubric for your presentations, which are starting in 2 weeks – the Tuesday after the midterm is due. 12:40

Are there any questions before we get started?

DEFINITION: Let $H \leq G$. The right cosets of H are the subsets of G of the form

$$Ha = \{ha : h \in H\}$$

for $a \in G$.

Let's look at some examples.

EXAMPLE: Let $H \leq S_3$ be given by $\{e, (123), (132)\}$. Is this a subgroup? Take a few seconds and check. *once they're convinced, calculate the cosets of H.*

$$\begin{aligned} He &= \{e, (123), (132)\} \\ H(123) &= \{e, (123), (132)\} \\ H(132) &= \{e, (123), (132)\} \\ H(12) &= \{(12), (13), (23)\} \\ H(13) &= \{(12), (13), (23)\} \\ H(23) &= \{(12), (13), (23)\} \end{aligned}$$

What do you notice about the cosets in this example? *Class discussion; make sure the following points get mentioned*

- Ha and H always have the same size.
- If $a \in H$ then $Ha = H$.
- If $b \in Ha$ then $Hb = Ha$.
- A coset Ha is not always a subgroup of G.

Let's look at one more example, and then I'd like to have you work in groups to prove (or disprove) some of these observations in general.

EXAMPLE: Find the cosets of $\langle 4 \rangle \leq \mathbb{Z}_8$. *Highlight the different cosets in different colors of chalk* 1:00

$$\begin{aligned} \langle 4 \rangle + 0 = \langle 4 \rangle + 4 &= \langle 4 \rangle \\ \langle 4 \rangle + 1 = \langle 4 \rangle + 5 &= \{1, 5\} \\ \langle 4 \rangle + 2 = \langle 4 \rangle + 6 &= \{2, 6\} \\ \langle 4 \rangle + 3 = \langle 4 \rangle + 7 &= \{3, 7\} \end{aligned}$$

Get into groups of 4 or 5. *Make sure there are 6 groups* Please work on proving the following statements. At the end, I will ask people to present their solutions at the board, so please be prepared!

1. For any $a \in G$, $|Ha| = |H|$.
2. If $a \in H$ then $Ha = H$.

3,4 $Ha \leq G \Leftrightarrow a \in H$.

5,6 $Ha = Hb \Leftrightarrow b \in Ha$.

Once students are in groups, number each group 1-6. I will choose someone from your group to write the proof of the associated statement on the board. *After groups have had time to work on the problems, pick my "victim" from each group and tell them to write on the board when they're ready.* 1:20

Critique proofs as a class to the extent that time allows. At a minimum, ensure that all proofs are correct.

We can use these propositions you just proved to prove Lagrange's Theorem.

THEOREM 10.1 *Let G be a finite group and let $H \leq G$. Then $|H|$ divides $|G|$.*

Proof: We will use the right cosets of H to prove this theorem. First, we observe that if $Ha \cup Hb \neq \emptyset$, then $Ha = Hb$. Why? *Think-pair-share* [If $c \in Ha \cup Hb$, then $c \in Ha$ and $c \in Hb$. But, Proposition 5/6 above tells us that this implies $Hc = Ha$ and $Hc = Hb$, so we must have $Ha = Hb$.]

Now, observe that every element $g \in G$ is in some right coset of H – in particular, $g \in Hg$. Therefore, we can write G as the disjoint union of some right cosets of H:

$$G = Ha_1 \cup Ha_2 \cup \ldots \cup Ha_k$$

for some $k \in \mathbb{Z}^+$, where $Ha_i \cap Ha_j = \emptyset$ unless $i = j$. Why do we know that k is finite? [G is finite] Any other questions? *wait 6 seconds*

Since this union is disjoint, and since $|Ha| = |H|$ for any $a \in G$, it follows that $|G| = k|H|$. Therefore, $|H|$ divides $|G|$ as claimed. $\Box$

A corollary, or consequence, of this theorem is the following:

THEOREM 10.4 Let G be a group of order n and let $x \in G$. Then $o(x)|n$.

I'd like you to take a couple of minutes to think about this, and then I'll prove it at the board – but I want you to tell me how the proof goes! So, figure it out.

If extra time Discuss V_4 via presentation as $(P(\{a, b\}, \Delta)$; Cayley table; lattice of subgroups.

Alternatively: Review for midterm