8.3

#6 Evaluate the integral.

$$\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$$

→ Let $x = \sec\theta$

$dx = \sec\theta\tan\theta\,d\theta$

need to change the bounds of integration.

When $x=1$, $\sec\theta = 1 \Rightarrow \theta = 0$

when $x=2$ $\sec\theta = 2 \Rightarrow \theta = \pi/3$

Hence $$\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx = \int_0^{\pi/3} \frac{\sqrt{\sec^2\theta - 1}}{\sec\theta}\sec\theta\tan\theta\,d\theta$$

$$= \int_0^{\pi/3} \frac{\tan\theta}{\sec\theta}\sec\theta\cdot\tan\theta\,d\theta$$

$$= \int_0^{\pi/3} \tan^2\theta\,d\theta$$

$$= \int_0^{\pi/3} \sec^2\theta - 1\,d\theta$$

$$= \tan\theta\Big]_0^{\pi/3} - \theta\Big]_0^{\pi/3}$$

$$= \left(\tan \tfrac{\pi}{3} - \tan 0\right) - \left(\tfrac{\pi}{3} - 0\right)$$

$$= \boxed{\sqrt{3} - \tfrac{\pi}{3}}$$

8.3
#10

$$\int \frac{t^5}{\sqrt{t^2+2}}\, dt$$

Let $t = \sqrt{2}\tan\theta \qquad dt = \sqrt{2}\sec^2\theta\, d\theta$

Hence $\displaystyle\int \frac{t^5}{\sqrt{t^2+2}}\, dt = \int \frac{2^{5/2}\tan^5\theta}{\sqrt{2\tan^2\theta + 2}}\sqrt{2}\sec^2\theta\, d\theta$

$$= 2^{5/2}\int \tan^5\theta \sec\theta\, d\theta$$

$$= 2^{5/2}\int \tan^4\theta \tan\theta \sec\theta\, d\theta$$

$$= 2^{5/2}\int (\sec^2\theta - 1)^2 \tan\theta \sec\theta\, d\theta$$

$$= 2^{5/2}\int (\sec^2\theta - 1)^2 \tan\theta \sec\theta\, d\theta \quad \circledast$$

evaluate it by u-substitution

Let $u = \sec\theta$, $du = \sec\theta\tan\theta\, d\theta$

Hence $\int (\sec^2\theta - 1)^2 \tan\theta \sec\theta\, d\theta$

$$= \int (u^2-1)^2\, du$$

$$= \int (u^4 - 2u^2 + 1)\, du$$

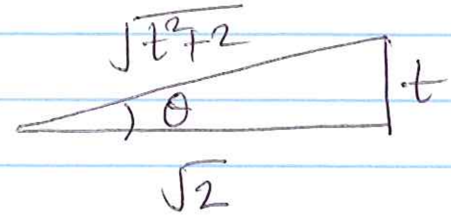
$$= u^5/5 - 2\, u^3/3 + u + C$$

(back to θ) $$= \frac{\sec^5\theta}{5} - \frac{2}{3}\sec^3\theta + \sec\theta + C.$$

From ✱ page 2,

$$\int \frac{t^5}{\sqrt{t^2+2}}\, dt = 2^{5/2}\left(\frac{\sec^5\theta}{5} - \frac{2}{3}\sec^3\theta + \sec\theta\right) + C.$$

From the triangle (Recall $\tan\theta = \frac{t}{\sqrt{2}}$)



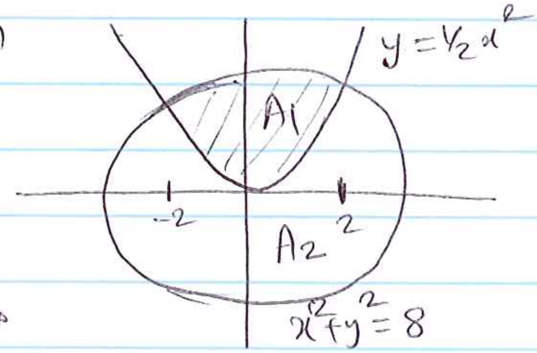
$$\sec\theta = \frac{\sqrt{t^2+2}}{\sqrt{2}}$$

$$\int \frac{t^5}{\sqrt{t^2+2}}\,dt = 2^{5/2}\left(\frac{(t^2+2)^{5/2}}{5(\sqrt{2})^5} - \frac{2}{3}\frac{(t^2+2)^{3/2}}{(\sqrt{2})^3} + \frac{\sqrt{t^2+2}}{\sqrt{2}}\right) + C$$

8.3
#40

Need to find both A_1 & A_2.
First find the points where $y = \frac{x^2}{2}$ intersects the ~~disk~~ circle $x^2+y^2 = 8$.

We will solve the eqn



$$2y + y^2 = 8$$

$$\Rightarrow \quad y = 2 \ \& \ y = -4$$

$y = -4$ does not occur because on the parabola $y = x^2/2$, y is always nonnegative ($y \geq 0$)

Hence $y = 2$. The intersection pts are $(2,2)$ & $(-2,2)$

$y = \sqrt{8-x^2}$ is on top.

$$A_1 = \int_{-2}^{2} \sqrt{8-x^2} - \frac{1}{2}x^2 \, dx$$

$$A_1 = \int_{-2}^{2} \sqrt{8-x^2}\, dx - \frac{1}{2}\int_{-2}^{2} x^2\, dx \quad \text{—} \quad (*)$$

Let's compute $\displaystyle\int_{-2}^{2} \sqrt{8-x^2}\, dx$

Let $x = \sqrt{8}\sin\theta$, $dx = \sqrt{8}\cos\theta\, d\theta$

If $x = -2$, $\sin\theta = -1/\sqrt{2}$

Hence $\theta = -\pi/4$

If $x = 2$ $\sin\theta = 1/\sqrt{2}$, $\theta = \pi/4$

$$\int_{-2}^{2} \sqrt{8-x^2}\, dx = \int_{-\pi/4}^{\pi/4} \sqrt{8-8\sin^2\theta}\,\sqrt{8}\cos\theta\, d\theta$$

$$= 8\int_{-\pi/4}^{\pi/4} \cos^2\theta\, d\theta$$

$$= 8 \int_{-\pi/4}^{\pi/4} \frac{1+\cos 2\theta}{2}\, d\theta$$

$$= 8 \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta \right]_{-\pi/4}^{\pi/4}$$

$$= 4\theta + 2\sin 2\theta \Big]_{-\pi/4}^{\pi/4}$$

$$\cancel{= 4 \quad 2\left[\sin \pi/2 - \right.}$$

$$= \cancel{4} \quad (\pi + 2) - (-\pi - 2)$$

$$= \underline{\underline{2\pi + 4}}$$

From (*) page 5 $\quad A_1 = 2\pi + 4 - \frac{1}{2}\left[\frac{x^3}{3}\right]_{-2}^{2}$

$$= 2\pi + 4 - \frac{1}{6}(16)$$

$$= 2\pi + 4 - 8/3 = \boxed{2\pi + 4/3}$$

A_2 (area of the other part) $=$ area of the disk $- (2\pi + 4/3)$

$$= \pi(\sqrt{8})^2 - 2\pi - 4/3$$

$$= \cancel{40}\ \boxed{6\pi - 4/3}$$

8.8

#8 $$\int_0^\infty \frac{x}{(x^2+2)^2}\,dx$$

First find $\int_0^t \frac{x}{(x^2+2)^2}\,dx$. $(t \geq 0)$

we will evalute this by substituting $u = x^2+2$
$du = 2x\,dx$

$$\int_0^t \frac{x}{(x^2+2)^2}\,dx = \frac{1}{2}\int_2^{t^2+2} \frac{du}{u^2}$$

$$= \frac{1}{2}\left[-\frac{1}{u} \right]_2^{t^2+2}$$

$$= \frac{-1}{2}\left[\frac{1}{t^2+2} - \frac{1}{2} \right]$$

$$\lim_{t\to\infty} \int_0^t \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty} -\frac{1}{2}\left(\frac{1}{t^2+2} - \frac{1}{2}\right)$$

$$= \frac{1}{4}$$

Hence $\int_0^\infty \frac{x}{(x^2+2)^2}\,dx$ is $\boxed{\text{convergent}}$ & is equal to $\boxed{1/4}$

12.1

#24 $a_n = \sqrt{\dfrac{n+1}{9n+1}}$

Sol^n: $\lim\limits_{n\to\infty} \sqrt{\dfrac{n+1}{9n+1}}$

$= \lim\limits_{n\to\infty} \dfrac{\sqrt{1+1/n}}{\sqrt{9+1/n}}$ (dividing numerator & denominator by $\sqrt{n}$)

$= \dfrac{\sqrt{1}}{\sqrt{9}} = \dfrac{1}{3}$

Hence $\{a_n\}_{n=1}^{\infty}$ is cgt & the limit is $\frac{1}{3}$.

#40. $a_n = \dfrac{\sin(2n)}{1+\sqrt{n}}$

$-1 \leq \sin(2n) \leq 1$ for all n.

Hence

$$\frac{-1}{1+\sqrt{n}} \leq \frac{\sin 2n}{1+\sqrt{n}} \leq \frac{1}{1+\sqrt{n}} \quad \text{for all } n.$$

$$\lim_{n\to\infty}\left(-\frac{1}{1+\sqrt{n}}\right) = 0$$

& $$\lim_{n\to\infty} \frac{1}{1+\sqrt{n}} = 0$$

Hence we can apply the squeeze thm.

By squeeze thm

$$\lim_{n\to\infty} \frac{\sin 2n}{1+\sqrt{n}} = 0.$$

Hence $\left\{\frac{\sin 2n}{1+\sqrt{n}}\right\}_{n=1}^{\infty}$ is convergent &

the limit is 0.