Homework #9

1) Page 372 # 5

We seek a radically symmetric solotion to $\Delta u = 0$ in $\mathbb{R}^3$.

In spherical coordinates

$$\Delta u = \frac{1}{r^2}\frac{d}{dr}\left(r^2 u_r\right) + \frac{1}{r^2 \sin\phi}\frac{d}{d\phi}\left(\sin\phi \, u_\phi\right)$$

$$+ \frac{1}{r^2\sin^2\phi} u_{\theta\theta}$$

Since radially symmetric u_ϕ & $u_{\theta\theta} = 0$.

$\Rightarrow$ We need to solve

$$\Delta u = \frac{1}{r^2}\frac{d}{dr}\left(r^2 u_r\right) = 0.$$

$$\Rightarrow \frac{d}{dr}\left(r^2 u_r\right) = 0 \rightarrow r^2 u_r = C$$

$$\Rightarrow u_r = \frac{C}{r^2} \rightarrow u(r) = \frac{-C}{r} + D$$

wher C & D are constants

2) Page 372 #6.

$$-\Delta u = \lambda u \qquad x \in \Omega.$$
$$u = 0. \qquad x \in \partial\Omega.$$

Use an energy method.
let u be the solution, then

$$\int_\Omega -\Delta u \; u \, dx = \lambda \int_\Omega u^2 dx$$

$$\int_\Omega \nabla u \cdot \nabla u \, dx - \underbrace{\int_{\partial\Omega} u \frac{\partial u}{\partial n} dA}_{0} = \lambda \int_\Omega u^2 dx.$$

$$\Rightarrow \lambda = \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx} \geq 0 \quad \text{since } u \neq 0.$$

3) Page 396 # 4

$$\mathcal{F}^{-1}(e^{-a|\xi|}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-a|\xi|} e^{-i\xi x} d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{+a\xi} e^{-i\xi x} d\xi + \frac{1}{2\pi}\int_{0}^{\infty} e^{-a\xi - i\xi x} d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{(a-ix)\xi} d\xi + \frac{1}{2\pi}\int_{0}^{\infty} e^{-(a+ix)\xi} d\xi$$

$$= \frac{1}{2\pi}\left[\lim_{\alpha\to-\infty} \frac{1-e^{(a-ix)\alpha}}{a-ix} + \lim_{\beta\to\infty} \frac{e^{-(a+ix)\beta}-1}{-(a+ix)}\right]$$

$$\lim_{\alpha\to-\infty} \left|e^{(a-ix)\alpha}\right| \le \lim_{\alpha\to-\infty} |e^{a\alpha}| = 0.$$

likewise. $\lim_{\beta\to\infty} \left|e^{-(a+ix)\beta}\right| \le \lim_{\beta\to\infty} e^{-a\beta} = 0.$

$$\Rightarrow \mathcal{F}^{-1}(e^{-a|\xi|}) = \frac{1}{2\pi}\left(\frac{1}{a-ix} + \frac{1}{a+ix} = \frac{a+ix+a-ix}{a^2+x^2}\right)$$

$$= \frac{1}{\pi}\left(\frac{a}{a^2+x^2}\right)$$

4) Page 396. #5

b) $\mathcal{F}\left(e^{iax}u\right)(\xi) = \int_{-\infty}^{\infty} e^{i\xi x} e^{iax} u(x)\,dx.$

$$= \int_{-\infty}^{\infty} e^{i(\xi+a)x} u(x)\,dx = \hat{u}(\xi+a)$$

c) $\mathcal{F}\left(u(x+a)\right) = \int_{-\infty}^{\infty} e^{i\xi x} u(x+a)\,dx$ let $s = x+a$ $\Rightarrow x = s-a.$ $dx = ds.$

$$= \int_{-\infty}^{\infty} e^{i\xi(s-a)} u(s)\,ds$$

$$= e^{-i\xi a} \int_{-\infty}^{\infty} e^{i\xi s} u(s)\,ds = e^{-i\xi a}\hat{u}(\xi)$$

5) Page 396 #7

Solve via Fourier Transform

$$\begin{cases} u_t - cu_x - u_{xx} = 0 & x \in \mathbb{R} \quad t > 0. \\ u(x,0) = f(x) & x \in \mathbb{R}. \end{cases}$$

Take the Fourier transform wrt x.

$$\hat{u}_t + ci\xi \hat{u} - (-i\xi)^2 \hat{u} = 0.$$

$$\Rightarrow \hat{u}_t = -(ci\xi + \xi^2)\hat{u}$$

$$\frac{\hat{u}_t}{\hat{u}} = -(ci\xi + \xi^2)$$

$$\ln(\hat{u}) = -(ci\xi + \xi^2)t + C$$

$$\hat{u}(t) = Ce^{-(ci\xi + \xi^2)t}$$

The intial condition in Fourier space is

$$\hat{u}(\xi, 0) = \hat{f}(\xi)$$

$$\Rightarrow C = \hat{f}$$

Now we need to apply the inverse transform to get a solution.

From 5b, we know $\mathcal{F}(e^{iax} u) = \hat{u}(\xi + a)$

$\Rightarrow$ Makes a shift by a.

$\Rightarrow$ The $e^{\pm i(ct)\xi}$ will make a shift in real space of $+ct$

$a = \frac{1}{4t}$

The inverse transform of $e^{-\xi^2 t} = \sqrt{\frac{1}{4\pi t}}\, e^{-x^2/4t}$

Putting everything together, we find

$$u(x,t) = \left(f * \mathcal{F}^{-1}\left(e^{-\xi^2 t}\right)\right)(x+ct)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{(x+ct-y)^2}{4t}}\, dy.$$

6)

$$\mathcal{F}(\delta(x-a)) = \int_{-\infty}^{\infty} e^{i\xi x}\,\delta(x-a)\,dx = e^{i\xi a}$$

Inverse formula

$$\delta(x-a) = \mathcal{F}^{-1}(e^{i\xi a})(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi(x-a)}\,d\xi$$

$$\rightarrow \delta(x+k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi(x+k)}\,d\xi \qquad \text{let } y = -\xi, \quad dy = -d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iy(x+k)}\,dy$$

$$= \frac{1}{2\pi}\mathcal{F}(e^{ikx}).$$

7) Page 396 # 10.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{u}(\xi)|^2 d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\xi)\,\overline{\hat{u}(\xi)}\, d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{i\xi x} u(x)dx\right)\left(\int_{-\infty}^{\infty} e^{-i\xi y}\overline{u(y)}dy\right) d\xi.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x)\,\overline{u(y)} \underbrace{\int_{-\infty}^{\infty} e^{i\xi(x-y)} d\xi}_{\delta(x-y)}\, dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x) \underbrace{\int_{-\infty}^{\infty} \overline{u(y)}\,\delta(x-y)\,dy}_{\overline{u(x)}}\, dx$$

$$= \int_{-\infty}^{\infty} u(x)\overline{u(x)}\, dx = \int_{-\infty}^{\infty} |u(x)|^2 dx.$$

8) Page 397 #11

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} \quad \text{then} \quad u(x) = \frac{1}{1+x^2}$$

By Plancherel relation

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{u}(\xi)|^2 \, d\xi$$

From Problem 4 on Page 396.

$$\mathcal{F}^{-1}(e^{-a|\xi|}) = \frac{2a}{x^2+a^2}$$

We also know that

$$\mathcal{F}^{-1}(\hat{f}(\xi)) = \frac{1}{2\pi}\,\mathcal{F}(\hat{f})(-x)$$

$$a=1$$

$$\Rightarrow \mathcal{F}\left(\frac{1}{1+x^2}\right) = \mathcal{F}^{-1}\left(\frac{1}{1+(-x)^2}\right) = \frac{1}{2}\, e^{-|\xi|}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2}\, dx = \frac{2\pi}{4} \int_{-\infty}^{\infty} e^{-2|\xi|}\, d\xi$$

$$= \pi \int_0^{\infty} e^{-2\xi}\, d\xi$$

$$= \frac{\pi}{-2}\, e^{-2\xi}\Big|_0^{\infty} = \pi/2$$

(9) Page 398 #15

$$\begin{cases} u_{xx} + u_{yy} = 0 & x \in \mathbb{R} \quad y > 0 \\ u_y(x,0) = f(x) \end{cases}$$

Fourier Transform wrt x.

$$\begin{cases} -\xi^2 \hat{u} + \hat{u}_{yy} = 0 \\ \hat{u}_y(\xi,0) = \hat{f}(\xi) \end{cases} \rightarrow \quad \hat{u}_y = C(\xi)\, e^{-|\xi| y}$$

$$\hat{u}_y(\xi,0) = C(\xi) = \hat{f}(\xi)$$

$$\rightarrow \hat{u}_y(\xi, y) = \hat{f}(\xi)\, e^{-|\xi| y}$$

Transform back to real space now.

$$u_y(x,y) = \left(f * \mathcal{F}^{-1}\left(e^{-|\xi| y}\right)\right)(x,y) \qquad \Big| \; a = y \quad \mathcal{F}$$

$$= \left(f * \left(\frac{1}{2\pi} \frac{2y}{y^2+x^2}\right)\right)(x-y)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi}\, y\, \frac{1}{y^2+(x-s)^2}\, f(s)\, ds.$$

now integrate wrt y.

$$\rightarrow u(x,y) = \int_0^y \int_{-\infty}^{\infty} \frac{1}{\pi}\, t\, \frac{1}{t^2+(x-s)^2}\, f(s)\, ds\, dt$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \int_0^y \frac{t}{t^2+(x-s)^2}\, dt\, f(s)\, ds$$

$$= \frac{1}{2\pi} \ln\left(y^2+(x-s)^2\right) f(s)\, ds.$$

10) Bonus.

Page 382 #3b $\quad q<0 \quad p>0$.

$Lu = -\nabla\cdot(p\nabla u) - qu \quad x\in\Omega$

1st. eigenvalues are positive.

consider $Lu=\lambda u$ $\quad(\lambda,u)$ eigenpaire i.e. $u\neq 0$.

Use energy method

$$\int_\Omega uLu\,dx = \lambda\int_\Omega u^2dx$$

$$\int_\Omega -u\nabla\cdot(p\nabla u)\,dx - \int_\Omega q u^2\,dx = \lambda\int_\Omega u^2\,dx$$

$$\int_\Omega -u(\nabla p\cdot\nabla u) - up\underbrace{\nabla\cdot\nabla u}_{\Delta u}\,dx - \int_\Omega q u^2\,dx = \lambda\int_\Omega u^2dx$$

$$= -\int_\Omega u(\nabla p\cdot\nabla u)dx - \Big[\int_\Omega \nabla(u p)\cdot\nabla u\,dx + \underbrace{\int_{\partial\Omega} up\frac{\partial u}{\partial n}dA}_{0 \text{ by BC.}}\Big] - \int_\Omega qu^2dx$$

$$= \underbrace{-\int_\Omega u(\nabla p\cdot\nabla u) + \int_\Omega u(\nabla p\cdot\nabla u)dx}_{=0} + \int_\Omega p\nabla u\cdot\nabla u\,dx - \int_\Omega qu^2dx$$

$$\Rightarrow \underbrace{\int_\Omega p|\nabla u|^2dx}_{\geq 0} - \underbrace{\int_\Omega qu^2dx}_{\geq 0} = \lambda\underbrace{\int_\Omega u^2dx}_{\geq 0}$$

$\rightarrow \lambda \geq 0$

now distinct eigenvalues

let (λ_1, u_1) (λ_2, u_2) be eigenpairs

Then $\int_\Omega u_2 L u_1 \, dx = \lambda_1 \int_\Omega u_1 u_2 \, dx$ | Goal: show $\int_\Omega u_1 u_2 = 0$

$$\lambda_1 \int_\Omega u_1 u_2 \, dx = \int_\Omega -u_2 (\nabla \cdot (p \nabla u_1)) \, dx - \int_\Omega q u_1 u_2 \, dx$$

$$= -\int_\Omega (u_2 (\nabla p \cdot \nabla u_1) + p u_2 \Delta u_1) \, dx - \int_\Omega q u_1 u_2 \, dx$$

$$= -\int_\Omega u_2 (\nabla p \cdot \nabla u_1) + \int_\Omega \nabla (p u_2) \cdot \nabla u_1 - \underbrace{\int_{\partial\Omega} p u_2 \frac{\partial u_1}{\partial n} \, dA}_{=0}$$

$$- \int_\Omega q u_1 u_2 \, dx.$$

$$= -\int_\Omega u (\nabla p \cdot \nabla u_1) \, dx + \int_\Omega u_2 \nabla p \cdot \nabla u_1 + p \nabla u_2 \cdot \nabla u_1 - \int_\Omega q u_1 u_2$$

$$- \int_\Omega q u_1 u_2 \, dx = \int_\Omega u_2 L u_1 \, dx$$

$$= \int_\Omega p \nabla u_2 \cdot \nabla u_1 - \int_\Omega q u_1 u_2$$

$$= \lambda_2 \int_\Omega u_1 u_2 \, dx.$$

$\rightarrow (\lambda_1 - \lambda_2) \int_\Omega u_1 u_2 \, dx = 0$

since $\lambda_1 \neq \lambda_2$. $\int_\Omega u_1 u_2 \, dx = 0.$