1. General outline

Variations on a Theme

- Elliptic modular forms – what we saw last week
- Congruence subgroups and their modular forms
- Half integral weight modular forms
- Siegel Modular forms
  - Define $Sp_n(\mathbf{Z})$
  - Show some examples
- General Hecke rings and operators

2. Review of modular forms

A modular form is a function f defined on the complex upper half plane that is holomorphic everywhere (including infinity). Furthermore, it satisfies the functional equation

$$f(A.z) = (cz+d)^k f(z)$$

where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$. We will rewrite this as

$$f|A =_{\mathrm{def}} = (cz+d)^{-k} f(A.z)$$

and for f to be a modular form, we need $f|A = f(z)$.

3. Congruence subgroups and their modular forms

Let $\Gamma = \mathrm{SL}_2(\mathbf{Z})$ and suppose that $\Gamma' \subseteq \Gamma$ is a subgroup of finite index n.

Example 1. *Three common examples of such subgroups are*

- $\Gamma(N) = \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) (modN)\}$
- $\Gamma_0(N) = \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) (modN)\}$
- $\Gamma_1(N) = \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) (modN)\}$

Since Γ' has finite index in Γ, Γ can be written as a disjoint union of n cosets

$$\Gamma = \bigcup_{i=1}^{n} \alpha_i \Gamma'.$$

Proposition 1. *The fundamental domain for Γ' with the above decomposition is*

$$F' = \bigcup_{i=1}^{n} \alpha_i^{-1} F$$

where F is the fundamental domain for Γ. We try to choose the α_i so that the F' is simply connected.

Example 2. *Suppose we wanted to find $F(2)$ the fundamental domain for $\Gamma(2)$. Note that $\Gamma(2)$ is the kernel of $SL_2(\boldsymbol{Z}) \to SL_2(\boldsymbol{Z}/2\boldsymbol{Z}) \backsimeq S_3$. So $[\Gamma : \Gamma(2)] = 6$.*

As coset representatives, pick

$$\alpha_1 = 1 \;\; \alpha_2 = T \;\; \alpha_3 = S \;\; \alpha_4 = TS \;\; \alpha_5 = ST \;\; \alpha_6 = T^{-1}ST.$$

Then, if we apply each of the inverses to F and we get this picture.

Proposition 2. *The following are true:*

- *The only modular forms of weight 0 are constant functions.*

- *Dedekind eta-function is defined to be*
$$\eta(z) = e^{2\pi iz/24}\prod_{i=1}^{\infty}(1-e^{2\pi inz})$$
for $N, k \in \boldsymbol{Z}$ *such that* $k(N+1) = 24$. *A cusp from of weight* k *for* $\Gamma_0(N)$ *is a multiple of*
$$(\eta(z)\eta(Nz))^k.$$
- *Define*
$$G_k^{(a_1,a_2)(mod)N}(z) = \sum_{\substack{(m_1,m_2\in \boldsymbol{Z}^2 \\ (m_1,m_2)\equiv(a_1,a_2)(mod)N}} (m_1z+m_2)^{-k}.$$
Then
  - $G_k^{(a_1,a_2)(mod)N}(z) \in \mathcal{M}_k(\Gamma(N))$
  - $G_k^{(0,a_2)(mod)N}(z) \in \mathcal{M}_k(\Gamma_1(N))$

Note how these two Eisenstein series are pretty natural generalizations of the ones we've already defined. In particular, when $N = 1$ *we get the normalized Eisenstein series from before.*

4. Historical Motivation

The number of ways m can be written as the sum of k squares is equal to the coefficient a_m in th Fourier expansion of
$$\Theta^k = \prod_{j=1}^{k}\sum_{n_j=-\infty}^{\infty} a^{n_j^2} = \sum_{n_1,n_2,\dots n_k=-\infty}^{\infty} q^{\sum n_j^2} = \sum_m a_m q^m.$$
For even k we can construct a modular from of weight k by using
$$\sum_{\vec{n}} q^{\,{}^t\vec{n}A\vec{n}}$$
where A is a symmetric bilenear form.

In particular, one can use the action of the Hecke operators on $\mathcal{M}_2(\Gamma_0(4))$ to derive the formula for the number of ways a number can be written as the sum of four squares:
$$a_n = \begin{cases} 8\sigma_1(n) & \text{for } n \text{ odd} \\ 24\sigma_1(n_0) & \text{for } n = 2^r n_0 \text{ even, } 2 \not| n_0 \end{cases}$$
where $\sigma_1(n) = \sum_{d|n} d$.

Say, instead of wanting to find a way of writing a number as the sum of an even number of squares, we wanted to find a way of writing a number as the sum of an odd number of squares. Clearly our previous approach breaks down since there are no odd weight modular forms. This dilemma leads us to our next definition.

5. Half-integer weight modular forms

For an element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(4N)$ and a complex number z put
$$j(\gamma, z) = \epsilon_d^{-1}\left(\frac{c}{d}\right)(cz+d)^{1/2}$$
where

- we take the branch of $\sqrt{z}$ with argument $(-\pi/2, \pi/2]$,

- $\left(\frac{c}{d}\right)$ is the Kronecker symbol, also know as the *quadratic residue symbol* and is characterized by the following
  (1) $\left(\frac{c}{d}\right) = 0$ if $(c, d) \neq 1$
  (2) If d is an odd prime, $\left(\frac{c}{d}\right)$ it coincides with the usual quadratic residue symbol, i.e., it is one less that number of solutions to $x^2 \equiv c \bmod d$
  (3) If $d > 0$ the map $c \mapsto \left(\frac{c}{d}\right)$ defines a character modulo d
  (4) If $c \neq 0$, the map $d \mapsto \left(\frac{c}{d}\right)$ defines a character modulo a divisor of $4c$
  (5) $\left(\frac{a}{-1}\right) = 1$ or -1 as $a > 0$ or $a < 0$
  (6) $\left(\frac{0}{\pm 1}\right) = 1$.
- $\epsilon_d = \begin{cases} 1 & d \equiv 1 (\mathrm{mod} 4) \\ i & d \equiv -1 (\mathrm{mod} 4) \end{cases}$.

Define

$$\mathfrak{G} = \{[\alpha, \phi(z)]\}$$

where $\alpha \in \mathrm{GL}_2^+(\mathbf{Q})$ and $\phi : H \to \mathbf{C}$ is a holomorphic function such that $\phi(z) = t(\det\alpha)^{-1/2}(cz+d)$ where $|t| = 1$. Note that $(\gamma, j(\gamma, z)) \in \mathfrak{G}$. We often denote $j(\gamma, z)$ by $\gamma *$.

Definition 1. A modular form of half integer weight, level 4N, is a function $f : H \to \mathbf{C}$ such that

- f is holomorphic on H
- f is holomorphic on the cusps of $\Gamma_0(4N)$
- $f|\gamma * = j(\gamma, z)^{-k} f(\gamma.z) = f(z)$ for k odd,

Example 3. *Let* $\Gamma_\infty = \left\{ \pm \left(\begin{smallmatrix} 1 & j \\ 0 & 1 \end{smallmatrix}\right) \right\}$ *Define*

$$E_{k/2} = \sum_{\gamma \in \Gamma_\infty / \Gamma_0(4)} j(\gamma, z)^{-k}.$$

Note this is analogous to the sum

$$E_k = \sum_{(m,n)=1} (mz+n)^{-k}$$

if we let $\gamma = \left(\begin{smallmatrix} a & b \\ m & n \end{smallmatrix}\right)$. *This* $E_{k/2}$ *is a modular form of weight* $k/2$.

6. Siegel Modular Forms

The Siegel half plane, denoted $\mathcal{H}_n$ is the set of all $n \times n$ complex symmetric matrices with positive definite imaginary part.

Definition 2. We define the symplectic group by

$$\Gamma^n = \mathrm{Sp}_n(\mathbf{Z}) = \left\{ M \in \mathrm{GL}_{2n}(\mathbf{Z}) :^t MJM = J \right\}$$

where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n 0 \end{pmatrix}.$$

Notice the similarity between J and S. An equivalent characterization of $\mathrm{Sp}_n(\mathbf{Z})$ is

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^n \iff {}^tAD -^t CB = I_n \text{ and } {}^tAC,^t BD \text{ are symmetric.}$$

Note that $\mathrm{Sp}_1(\mathbf{Z}) = \mathrm{SL}_2(\mathbf{Z})$.

We also define

$$\Gamma_0^n(m) = \left\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^n : C \equiv 0_n (\mathrm{mod}\ m) \right\}.$$

Definition 3. A Siegel modular form of genus n, weight k, and level m is a function F on $\mathcal{H}_n$ so that

- F is holomorphic on $\mathcal{H}_n$
- $F((AZ+B)(CZ+D)^{-1}) = \det D (\det(CZ+D))^k F(Z)$

and so that for $n = 1$ F is also holomorphic at all the cusps.

Remark 1. A quick comment about the dimension of $\mathcal{M}_k(\Gamma^n)$:

$$\dim \mathcal{M}_k(\Gamma^n) \begin{cases} < d_n k^{n(n+1)/2} & \\ = 1 & \text{for } k = 0 \\ = 0 & \text{for } k < 0 \end{cases}$$

where the d_n is a constant that only depends on the genus.

Example 4. *Let A be a symmetric positive definite integral matrix with even diagonal entries and $\det A = 1$. The order of m is necessarily divisible by 8. Then*

$$\Theta_A^n(Z) = \sum_{X \in M_{m,n}} (\boldsymbol{Z}) e^{\pi i^t XAXZ}$$

is a modular form of genus n and weight $m/2$. Moreover, if we don't demand that $\det A = 1$ we can get modular forms of different levels.

7. Generalization of the Hecke operators

Recall that the Hecke operators are defined, in the elliptic case, by

$$T_n f(z) = \frac{1}{n} \sum_{d|n} d^{k-1} \sum_{b \mathrm{mod}\ d} f\left(\frac{nz + bd}{d^2}\right).$$

Via a lot of straightforward linear algebra, this can be rewritten as

$$\frac{1}{n} \sum_A a^k f(A.z)$$

where A runs through a set of representatives of $\Gamma(n) = \{A \in M_n(\mathbf{Z} : \det A = n\}$ of the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ where $0 \le b < d$.

In particular for T_p we get

$$T_p = \frac{1}{p} \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) + \frac{1}{p} \sum_{b \mathrm{mod}\ p} \left(\begin{smallmatrix} 1 & b \\ 0 & p \end{smallmatrix}\right).$$

It turns out that Shimura noticed that the decompostion of $\Gamma \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \Gamma$ into disjoint left cosets was

$$\Gamma \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \Gamma = \Gamma \left(\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) \cup_{b \mathrm{mod}\ p} \left(\begin{smallmatrix} 1 & b \\ 0 & p \end{smallmatrix}\right) \right).$$

This led him to generalize the definition of the Hecke operator as follows. First, let me say a quick word on double cosets.

We say that matrices A and B are equivalent and write $B \sim A$ if there exist unimodular matrices U, V (unimodular matrices are matrices that are invertible and have determinant ± 1) so that $B = UAV$. In this case, for β to be in $\Gamma\alpha\Gamma$ is equivalent to $\beta \sim \alpha$. There's a theorem in the theory of integer matrices that says two matrices are equivalent iff they have the same rank and the same elementary divisors which are the diagonal factors that appear in the Smith normal form of a matrix.

So we assert that for $\alpha \in \mathrm{M}_n(\mathbf{Q})$ we have

$$\Gamma\alpha\Gamma = \bigcup_{i=1}^{\nu} \Gamma\alpha_i \qquad \text{(disjoint)}$$

where ν is the index of $\Gamma\alpha\Gamma$ in Γ. We define the Hecke algebra $R(\Gamma, \Delta)$ to be the $\mathbf{Z}$-module generated by all formal linear sums

$$\sum_k c_k \Gamma\alpha_k\Gamma$$

where $c_k \in \mathbf{Z}$ and $\alpha_k \in \Delta = \mathrm{M}_n^+(\mathbf{Z})$. We define the action of a double coset on a function f in this way: For

$$\Gamma\alpha\Gamma = \bigcup_{i=1}^{\nu} \Gamma\alpha_i \qquad \text{(disjoint)}$$

we define

$$f|\Gamma\alpha\Gamma = f|\bigcup_{i=1}^{\nu} \Gamma\alpha_i \qquad \text{(disjoint)} = \sum_{i=1}^{\nu} f|\alpha_i$$

and extend linearly to arbitrary elements of $R(\Gamma, \Delta)$. Say that

$$\Gamma\alpha\Gamma = \bigcup_{i=1}^{\nu} \Gamma g_i\alpha_i \qquad \text{(disjoint)}$$

was another decomposition of the double coset and $g_i \in \Gamma$. The action I defined above is still well defined since

$$f|\Gamma\alpha\Gamma = f|\bigcup_{i=1}^{\nu} \Gamma g_i\alpha_i = \sum_{i=1}^{\nu} f|g_i\alpha_i = \sum_{i=1}^{\nu} f|\alpha_i$$

since whenever f is a modular form, $f|g = f$ for $g \in \Gamma$.

Finally, we define $T_m(f)$ to be $f| \cup_\alpha \Gamma\alpha\Gamma$ where $\det\alpha = m$.

8. Conclusion

I've shown three natural ways to generalize a modular form and where some of them come from. I've given examples of all three types, spending significant time on Siegel modular forms since that is the one I work with the most. Then I generalize the Hecke operators so that they can used on congruence subgroup modular forms and on Siegel modular forms.