Math 74 Midterm Exam

May 22, 2003

Key Theorem: Given a path connected, simply connect topological space X and a properly discontinuous subgroup G of $Homeo(X)$ (the homeomorphisms of X), then $\pi_1(X/G)$ is isomorphic to G.

1. For $n \geq 1$, let $S^{n-1} = \{x \in R^n \mid |x| = 1\}$, $B^n = \{x \in R^n \mid |x| \leq 1\}$, let $a : S^n \to S^n$ be the antipodal map defined by $a(x) = -x$, and let C^n be the *Cross Space* viewed as the identification space formed by identifying the antipodal points of the boundary of B^n, namely the antipodal points of S^{n-1}.

   (a) Prove that the the antipodal mapping generates a properly discontinuous subgroup, G_n, of $Homeo(S^n)$.

   (b) Prove that S^n/G_n is homeomorphic to C^n (Please do this **carefully** using the ideas and results from section 22 of Munkres).

   (c) Assuming $\pi_1(S^n) = id$ for $n \geq 2$, use the Key Theorem stated above to compute $\pi_1(C^n)$ for $n \geq 2$.

2. Let A be the the *loopy topologist's sine curve*. Namely the subspace of R^2 determined by the points $\left(t, sin\left(\frac{1}{t}\right)\right)$ for $t \in \left(0, \frac{1}{2\pi}\right]$ together with the three lines indicated in figure 1.

   (a) Prove A is simply connected.

   (b) Construct a connected space B such that $Homeo(B)$ contains a properly discontinuous subgroup, G, such that G is isomorphic to the integers and such that A is homeomorphic to B/G. (Hint: think about how the real line covers the circle).

   (c) Explain why B's existence does not contradict exercise 8 from section 54 of Munkres.

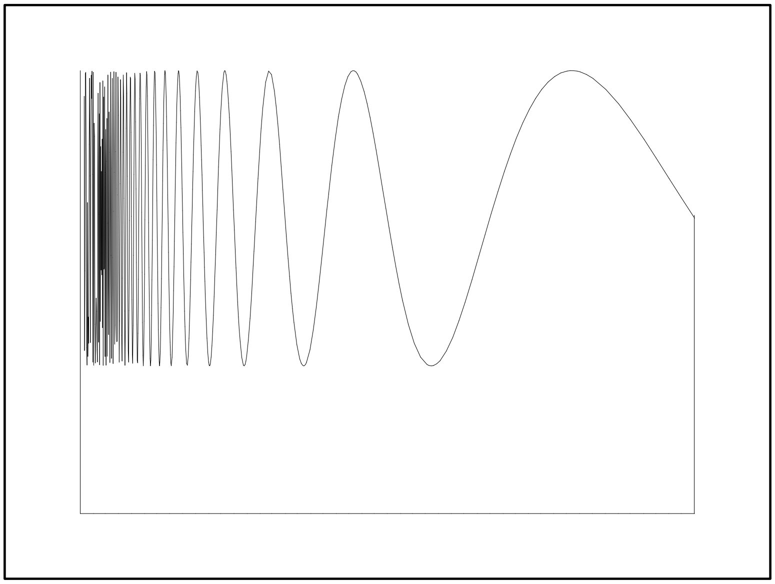

Figure 1: The loopy topologist's sine curve

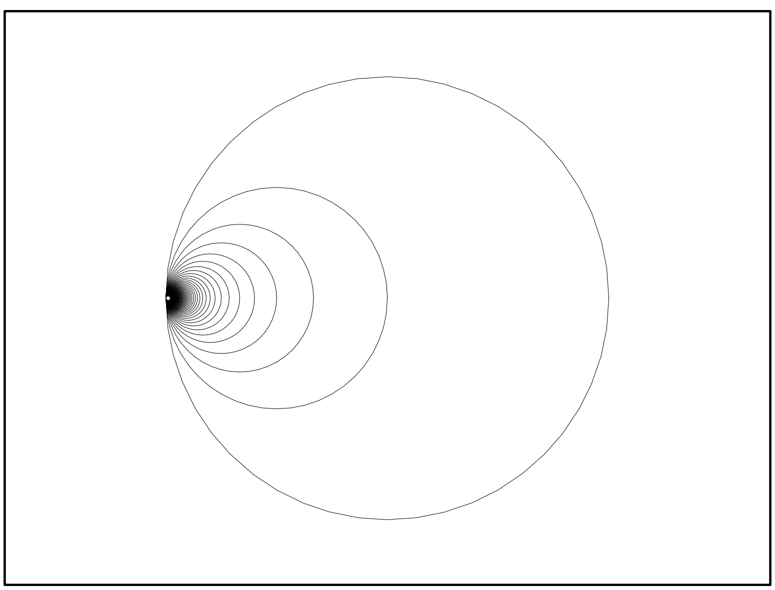

Figure 2: The Hawaiian earrings

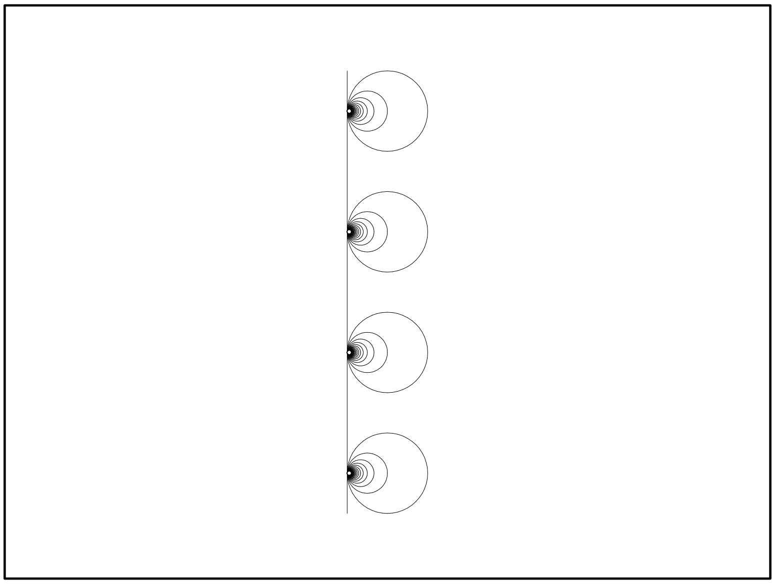

Figure 3: An ∞-fold cover of the Hawaiian earrings

(d) Prove $\pi_1(B/G)$ is not isomorphic to G. Explain why this does not contradict the Key Theorem.

3. Let Z denote the Hawaiian earrings from figure 2; namely, let C_n be the circle of radius $\frac{1}{n}$ centered at $(\frac{1}{n}, 0)$ in R^2, and then let $Z = \bigcup_{i=1}^{\infty} C_n$ viewed as a subspace of R^2.

   (a) Prove Z is compact, path connected and locally path connected.

   (b) For every open neighborhood U of $(0,0)$ in Z, prove that $\pi_1(U,(0,0)) \neq id$.

   (c) Demonstrate that there is a covering map $r : Y \to Z$ where Y is the space described in figure 3.

   (d) Construct a space X and a covering map $q : X \to Y$ such hat $p = r \cdot q : X \to Z$ is **not** a covering map. (Hint: You may restrict your search to q which satisfy $|q^{-1}(y)| = 2$ for every $y \in Y$.)

   (e) Explain why part **??** does not contradict exercise 4 from section 53 in Munkres.