Workshop Problems 3

Problem 1. Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ span $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Suppose that $T(\mathbf{v}_i) = \mathbf{0}$ for $i = 1, 2, \ldots, p$. Show that T is the zero transformation. That is, show that $T(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

Problem 2. Let $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^m \to \mathbb{R}^p$ be linear transformations. Show that the map $\mathbf{x} \mapsto T(S(\mathbf{x}))$ is a linear transformation (from $\mathbb{R}^n$ to $\mathbb{R}^p$). This map is called the *composition of S and T*.

Problem 3. An *affine transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$ has the form $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, with A an $m \times n$ matrix and $\mathbf{b}$ in $\mathbb{R}^m$. Show that an affine transformation T is a linear transformation if and only if $\mathbf{b} = \mathbf{0}$.

Problem 4. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Assume that T is one-to-one. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ be vectors in $\mathbb{R}^n$. Show that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are linearly independent if and only if $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_p)$ are linearly independent.

Caution: The final statement in this problem *may be false* if T is not one-to-one. In other words, not all linear transformations have this property.