MATH 13 SPRING 2011: MIDTERM # 1 SOLUTIONS

(Note: some of the multiple choice problems do not have choices attached to them in these solutions for the purpose of saving space.)

(1) Calculate the value of

$$\int_0^{\frac{4}{\pi}} \int_0^{\pi} 5(xy + \pi \sin(x))\, dx\, dy.$$

Solution. This problem is a straightforward iterated integral. Evaluating the inner integral yields

$$\int_0^{\frac{4}{\pi}} \left(\frac{5x^2y}{2} - 5\pi \cos x \Big|_{x=0}^{x=\pi} \right) dy = \int_0^{\frac{4}{\pi}} \frac{5\pi^2 y}{2} + 10\pi\, dy = \left(\frac{5\pi^2 y^2}{4} + 10\pi y \right) \Big|_0^{4/\pi} = 60.$$

(2) Using polar coordinates, the integral

$$\iint_D (x + 4y^2)\, dA$$

where D is the top half of the disk of radius two centered at the origin, is equal to

(a):

$$\int_0^{2\pi} \int_0^2 r^2(\cos(\theta) + 4r\sin^2(\theta))\, dr\, d\theta$$

(b):

$$\int_0^{\pi} \int_0^2 r^2(\cos(\theta) + 4r\sin^2(\theta))\, dr\, d\theta$$

(c):

$$\int_0^{2\pi} \int_0^2 r(\cos(\theta) + 4r\sin^2(\theta))\, dr\, d\theta$$

(d):

$$\int_0^{\pi} \int_0^2 r(\cos(\theta) + 4r\sin^2(\theta))\, dr\, d\theta$$

Solution. D is described by polar inequalities $0 \leq r \leq 2, 0 \leq \theta \leq \pi$, so using the polar coordinates change of variable formula with these bounds of integration, the integrand becomes

$$(r\cos\theta + 4(r\sin\theta)^2)r,$$

so the correct integral is choice (b).

(3) Consider the region D in the first quadrant of the xy-plane bounded between the lines $x = 0$, $y = 0$ and $y = -8x + 16$. Which of the following describes this region in polar coordinates?

(a): $\frac{\pi}{2} \leq \theta \leq \pi, 0 \leq r \leq \frac{16}{\sin(\theta)+8\cos(\theta)}$
(b): $0 \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq \frac{16}{\sin(\theta)+8\cos(\theta)}$
(c): $\frac{\pi}{2} \leq \theta \leq \pi, 0 \leq r \leq \frac{8}{\sin(\theta)+16\cos(\theta)}$
(d): $0 \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq \frac{8}{\sin(\theta)+16\cos(\theta)}$

Solution. This region is a triangle with vertices $(0, 0), (0, 16), (2, 0)$, so evidently $0 \leq \theta \leq \pi/2$. To determine inequalities on r, notice that $0 \leq r$ gives a lower bound, and an upper bound is given by the line $y = -8x+16$. Since $y = r\sin\theta, x = r\cos\theta$, this line has equation

$$r \sin\theta = -8r\cos\theta + 16 \Leftrightarrow r = \frac{16}{\sin\theta + 8\cos\theta}.$$

Therefore b is the correct choice.

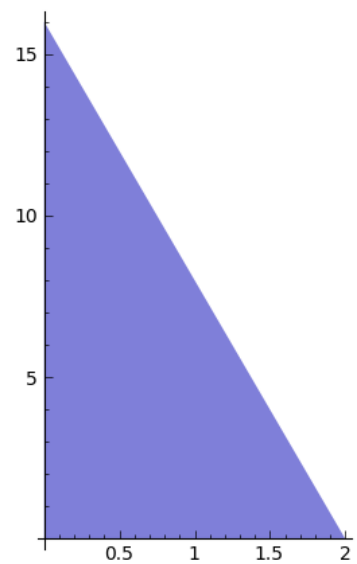

FIGURE 1. Region of integration, Problem 3

(4) Let $\vec{\mathbf{v}} = \langle 3, -1, 2\rangle$ and $\vec{\mathbf{w}} = \langle 1, 2, 5\rangle$. What is $\vec{\mathbf{v}} \times \vec{\mathbf{w}}$?

Solution.

$$\langle 3, -1, 2\rangle \times \langle 1, 2, 5\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 1 & 2 & 5 \end{vmatrix} = -9\mathbf{i} - 13\mathbf{j} + 7\mathbf{k}.$$

(5) By switching the order of integration,

$$\int_0^3 \int_x^3 f(x, y)\, dy\, dx + \int_{-3}^0 \int_x^3 f(x, y)\, dy\, dx$$

is equal to

(a):
$$\int_0^3 \int_{-y}^{y} f(x,y)\,dx\,dy$$
(b):
$$\int_{-3}^3 \int_{-y}^{y} f(x,y)\,dx\,dy$$
(c):
$$\int_{-3}^3 \int_{-3}^{y} f(x,y)\,dx\,dy$$
(d):
$$\int_x^3 \int_0^3 f(x,y)\,dx\,dy + \int_x^3 \int_{-3}^0 f(x,y)\,dx\,dy$$

Solution. Graphing the region of integration gives a triangle with vertices at $(-3,-3),(-3,3),(3,3)$. If we want order of integration $dx\,dy$, this corresponds to inequalities $-3 \le y \le 3, -3 \le x \le y$, so (c) is the correct answer.

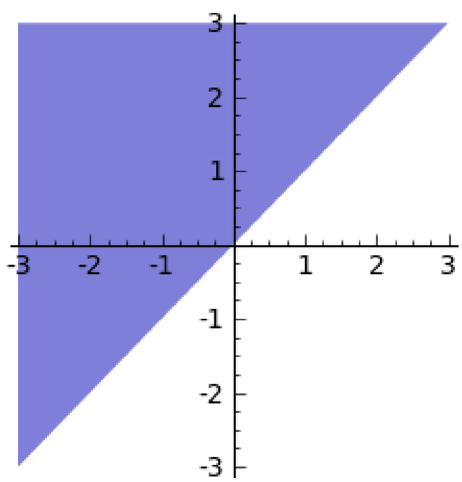

FIGURE 2. Region of integration, problem 5

(6) What is the volume of the region bounded by the surface $z = 1 + y^2$ and the planes $x = 0$, $y = 0$, $z = 0$, and $x + y = 1$?

Solution. The planes $x = 0, y = 0, x+y = 1$ bound a region $0 \le x \le 1, 0 \le y \le 1-x$, and the surfaces $z = 0, z = 1 + y^2$ provide lower and upper bounds on the z coordinates of the region specified in the problem. Therefore, the volume is equal to the double integral

$$\int_0^1 \int_0^{1-x} 1+y^2 \, dy \, dx = \int_0^1 \left(y + \frac{y^3}{3} \right) \Big|_{y=0}^{y=1-x} dx = \int_0^1 (1-x) + \frac{(1-x)^3}{3} \, dx = \frac{-(1-x)^2}{2} - \frac{(1-x)^4}{12} \Big|_0^1 = \frac{7}{12}.$$

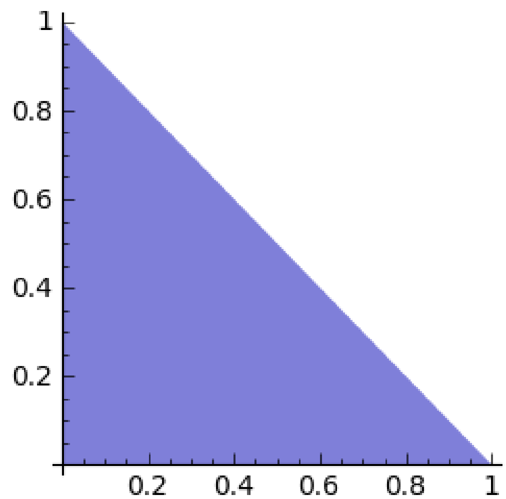

FIGURE 3. Region of integration, Problem 6

(7) The value of the double integral $\iint_R e^{2x^2+y^2} \, dA$, where $R = [0, 1] \times [0, 2]$, lies in the interval
- **(a):** $[-e, 0]$
- **(b):** $[0, 2]$
- **(c):** $[2, 2e^6]$
- **(d):** $[2e^6, +\infty)$

Solution. It is not possible to actually evaluate the double integral in the question in any useful way, but fortunately the question only asks us to estimate the value of the integral. Notice that the integrand $e^{2x^2+y^2}$ takes its minimum value on R at the point $(0, 0)$ and its maximum value at $(1, 2)$. Therefore,

$$\iint_R e^0 \, dA < \iint_R e^{2x^2+y^2} \, dA < \iint_R e^6 \, dA.$$

Since R has area 2, the left and right hand side evaluate to $2, 2e^6$, respectively, so the correct answer is (c).

(8) What is the equation of the tangent plane to the surface $z = x^3y - 3y^2$ at the point $(1, 2, -10)$?

Solution. This is a straightforward application of the equation for the tangent plane to a surface $z = f(x, y)$ at a point (x_0, y_0, z_0):

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

In our situation, $f_x(x, y) = 3x^2y$, $f_y(x, y) = x^3 - 6y$, so $f_x(1, 2) = 6$, $f_y(1, 2) = -11$, and the above equation becomes

$$z + 10 = 6(x - 1) - 11(y - 2) \Leftrightarrow z - 6x + 11y - 6 = 0.$$

(9) Evaluate the integral $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\, dx\, dy$.

Solution. Integrating with respect to the order given in the problem is impossible, because there is no way to find a usable expression for the antiderivative of e^{x^3} with respect to the variable x. Therefore we will switch the order of integration.

The bounds on the integral given in the problem give a region of integration described by inequalities $0 \leq y \leq 1$, $\sqrt{y} \leq x \leq 1$. The graph of this region is the region under the graph $y = x^2$, above the x-axis, over the interval $0 \leq x \leq 1$. (One can see this by squaring both sides of $\sqrt{y} \leq x$, which yields $y \leq x^2$.) Therefore the original integral is equal to

$$\int_0^1 \int_0^{x^2} e^{x^3}\, dy\, dx = \int_0^1 x^2 e^{x^3}\, dx = \frac{e^{x^3}}{3}\Big|_0^1 = \frac{e-1}{3}.$$

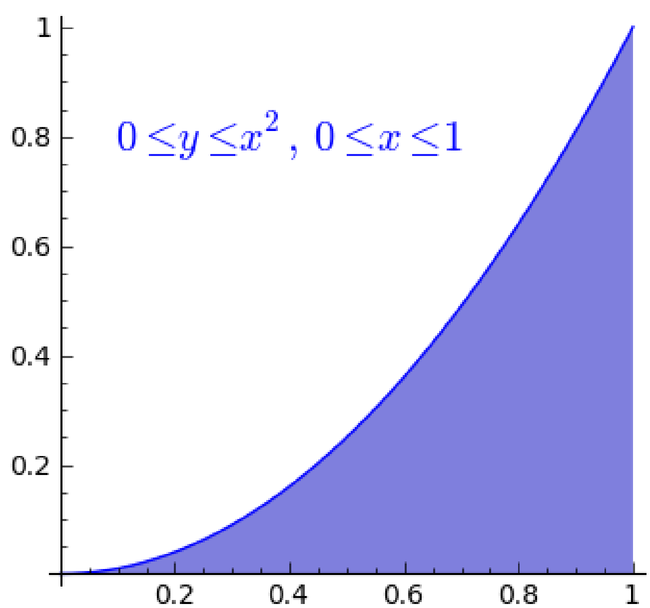

FIGURE 4. Region of integration, Problem 9

(10) Consider the integral $\int_0^1 \int_{-\sqrt{1-x}}^{\sqrt{1-x}} \int_{-\sqrt{1-x-z^2}}^{\sqrt{1-x-z^2}} f(x, y, z)\, dy\, dz\, dx$, where f is a continuous function.

(a) Sketch the projection of the region of integration determined by this integral onto the yz-plane. Please label the boundary curves and relevant intercepts.

Solution. The bounds in the original integral describe a region satisfying

$$0 \leq x \leq 1, -\sqrt{1-x} \leq z \leq \sqrt{1-x}, -\sqrt{1-x-z^2} \leq y \leq \sqrt{1-x-z^2}.$$

We can rewrite the last inequality as

$$0 \leq |y| \leq \sqrt{1-x-z^2} \Leftrightarrow 0 \leq y^2 \leq 1-x-z^2.$$

Notice the middle inequality, by the same reasoning, is equivalent to $z^2 \leq 1-x$, but this is already contained in the previous inequality because $0 \leq y^2 \leq 1-x-z^2 \Leftrightarrow z^2 \leq 1-x$. Therefore our region of integration is described by the pair of inequalities

$$y^2 \leq 1-x-z^2, 0 \leq x \leq 1.$$

At this point you might recognize this as the graph of a circular paraboloid, but this isn't necessary to finish the problem. We can rewrite the first inequality above as $y^2+z^2 \leq 1-x$. When $x=0$, this becomes the inequality $y^2+z^2 \leq 1$, and when $0 \leq x \leq 1$, we have $y^2+z^2 \leq 1-x \leq 1$, so the projection of the region to the yz plane is given by $y^2+z^2 \leq 1$.

(b) Rewrite this integral as an equivalent iterated integral with respect to $dx\,dy\,dz$.

Solution. If you have the answer to the first part of the question, then most of the work in this problem is already done. First, inequalities on x are given by $0 \leq x \leq 1-y^2-z^2$, and the inequalities on y, z are those which describe the disc $y^2+z^2 \leq 1$; namely, $-\sqrt{1-z^2} \leq y \leq \sqrt{1-z^2}, -1 \leq z \leq 1$. Therefore, the answer is

$$\int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{0}^{1-y^2-z^2} f(x,y,z)\, dx\, dy\, dz.$$

(11) Find the volume of the region above the cone $z = \sqrt{x^2+y^2}$ and inside the sphere of radius a centered at $(0,0,0)$. Please simplify your answer as much as possible.

Solution. The region described in the problem looks like an ice-cream cone shape. Algebraically, this region is described by inequalities

$$\sqrt{x^2+y^2} \leq z \leq \sqrt{a^2-x^2-y^2},$$

where the latter inequality comes from the equation for a sphere of radius a centered at the origin. To determine the projection of this region over the xy plane, we want to find all points x, y such that there is some z satisfying the above inequality; ie, the set of points (x,y) such that

$$\sqrt{x^2+y^2} \leq \sqrt{a^2-x^2-y^2}.$$

At this point, the presence of the x^2+y^2 term suggests switching to polar coordinates. In polar coordinates, this inequality becomes $r \leq \sqrt{a^2-r^2}$. Solving for r gives $r^2 \leq a^2/2$, or $r \leq a/\sqrt{2}$. The inequalities on θ are given by $0 \leq \theta \leq 2\pi$.

The volume of the region we want to find is then given by the following double integral, where we use polar coordinates:

$$\int_{0}^{2\pi} \int_{0}^{a/\sqrt{2}} (\sqrt{a^2-r^2} - r) r\, dr\, d\theta.$$

Since there is no dependence on θ in the integrand or the inner bounds of integration, the outer integral will multiply the inner integral by 2π. The inner integral can be done using u-substitution, whose details we omit here:

$$\begin{aligned}\int_0^{2\pi}\int_0^{a/\sqrt{2}}(\sqrt{a^2-r^2}-r)r\,dr\,d\theta &= 2\pi\left((a^2-r^2)^{3/2}\cdot\frac{2}{3}\cdot\frac{-1}{2}-\frac{r^3}{3}\right)\Big|_0^{a/\sqrt{2}}\\ &= 2\pi\left(\frac{-a^3}{6\sqrt{2}}-\frac{a^3}{6\sqrt{2}}+\frac{a^3}{3}\right)\\ &= \frac{\pi a^3}{3}\cdot(2-\sqrt{2}).\end{aligned}$$

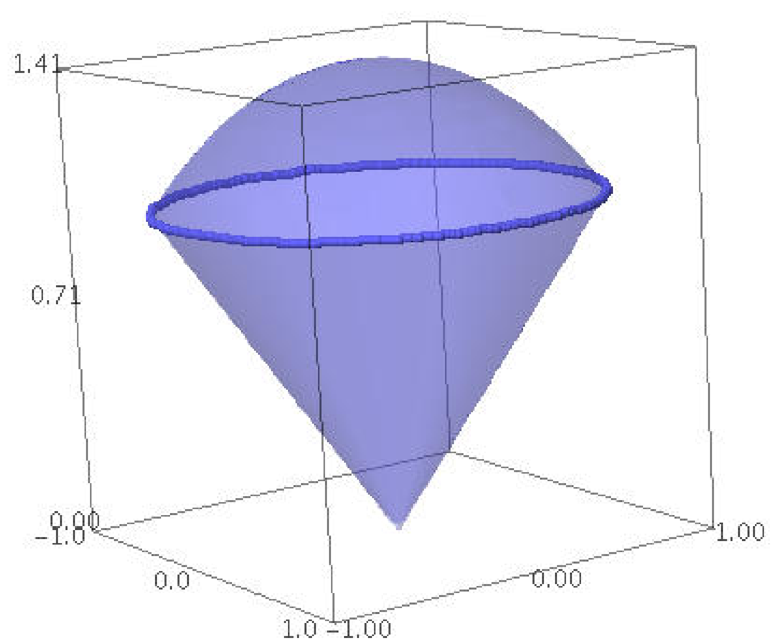

FIGURE 5. Solid described in Problem 11

(12) Find the average value of the function $f(x,y,z)=x+y+2z$ over the tetrahedron in $\mathbb{R}^3$ with vertices $(0,0,0),(1,0,0),(0,1,0)$ and $(0,0,1)$.

Solution. Letting E be the tetrahedron described above, the average value of $f(x,y,z)$ on E is given by

$$\frac{\iiint\limits_E f(x,y,z)\,dV}{\iiint\limits_E 1\,dV}.$$

We need to calculate the integral of the constant function 1 as well as the function $f(x,y,z) = x+y+2z$ over $E$to solve this problem. The tetrahedron has volume $1/6$, since it has a base of area $1/2$ (triangle with vertices at $(0,0),(1,0),(0,1)$) and height 1. Therefore, the integral of the constant function 1 is $1/6$. As for the integral of f, first, we describe E using inequalities. The z coordinate is bounded from below by 0 and from above by the function whose graph describes the plane passing through $(1,0,0),(0,1,0),(0,0,1)$; this evidently is a plane with equation $x+y+z=1$. The projection of E to the xy plane is a triangle with vertices at $(0,0),(1,0),(0,1)$. Therefore, the integral of $f(x,y,z)$ over E is given by

$$\begin{aligned}
\int_0^1\int_0^{1-x}\int_0^{1-x-y} x+y+2z\,dz\,dy\,dx &= \int_0^1\int_0^{1-x}\left((x+y)z+z^2\Big|_{z=0}^{z=1-x-y}\right)dy\,dx\\
&= \int_0^1\int_0^{1-x}(x+y)(1-x-y)+(1-x-y)^2\,dy\,dx\\
&= \int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx\\
&= \int_0^1 (1-x)y-\frac{y^2}{2}\Big|_0^{y=1-x}\,dx\\
&= \int_0^1 \frac{(1-x)^2}{2}\,dx\\
&= \frac{-(1-x)^3}{6}\Big|_0^1 = \frac{1}{6}.
\end{aligned}$$

Therefore the average value is $(1/6)/(1/6) = 1$. Also, notice that the evaluation of the above integral could have been simplified by noting that after the first integration (with respect to the variable z), we have a double integral

$$\int_0^1\int_0^{1-x}(1-x-y)\,dy\,dx,$$

and this integral is actually equal to the volume of the tetrahedron E, since E is the solid below $z=1-x-y$ and over the triangle $0\le y\le 1-x, 0\le x\le 1$ in the xy plane. We already knew the volume of E was $1/6$.

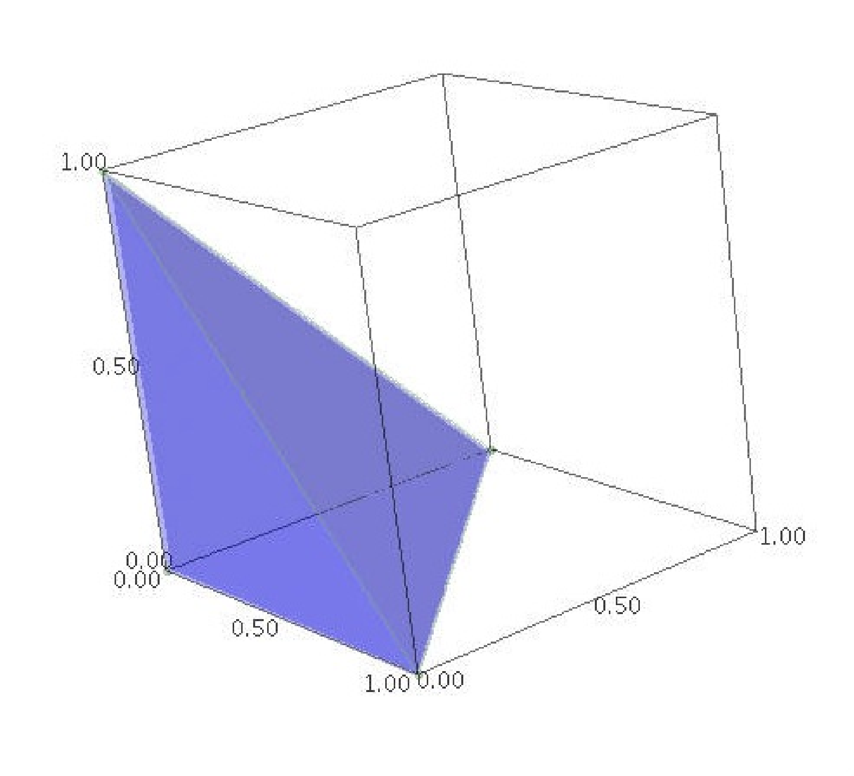

Figure 6. Tetrahedron described in Problem 12