Ergodicity of quantum eigenfunctions in classically chaotic systems

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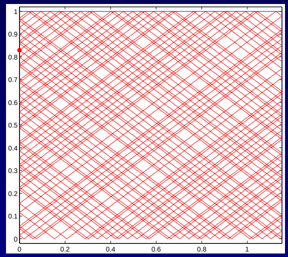
work in collaboration with Peter Sarnak, CIMS/Princeton

Classical billiards

Point particle in 2D domain Ω, elastic reflections off boundary Γ.
Position $\mathbf{r} \equiv (x, y)$. Phase space $(\mathbf{r}, \theta)$. Energy is conserved.

- Type of motion depends on billiard table shape:

Regular:  has other conserved quantities (*e.g.* θ)

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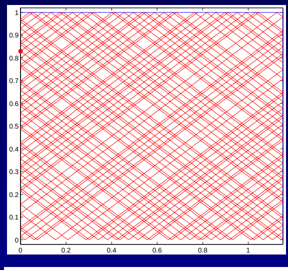
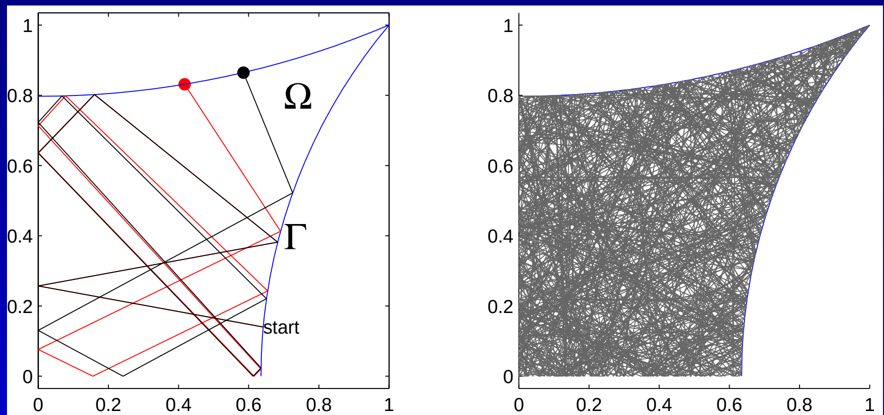
Regular:  has other conserved quantities (*e.g.* θ)

Chaotic: 

ergodic: nearly every trajectory covers phase space

- Hyperbolicity: exponential divergence of nearby trajectories

$$|\mathbf{r}_1(t) - \mathbf{r}_2(t)| \sim e^{\Lambda t}, \qquad \Lambda = \text{Lyapunov}$$

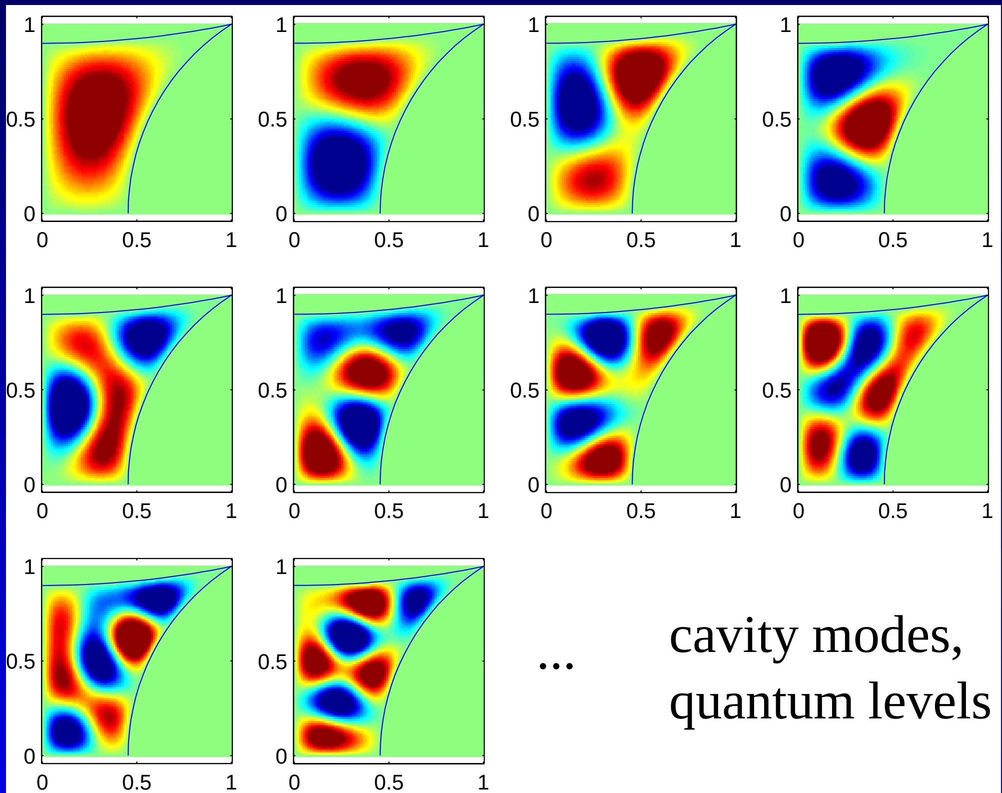
- Also: *Anosov property* (all $\Lambda > 0$), *mixing* (phase space flow)

‘Quantum’ billiards

‘quantum’ just means ‘wave’

Membrane (drum) problem: eigenfunctions $\phi_n(\mathbf{r})$ of laplacian

$$-\Delta\phi_n = E_n\phi_n, \qquad \phi_n(\mathbf{r}\in\Gamma) = 0 \qquad \int_\Omega \phi_n^2 d\mathbf{r} = 1$$



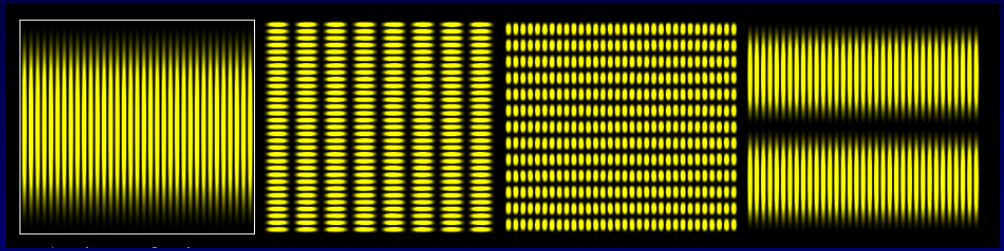
‘energy’ eigenvalue E
wavenumber k

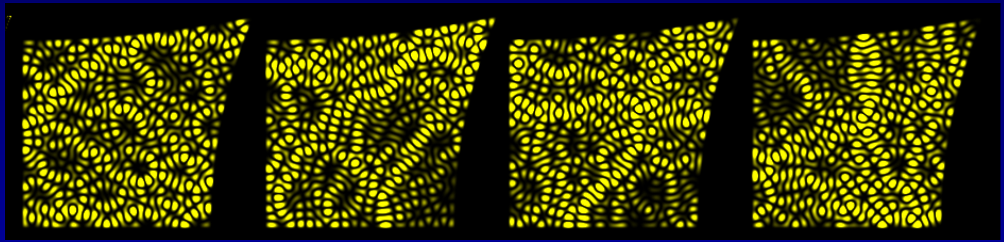
$$k \equiv \sqrt{E} \equiv \frac{2\pi}{\lambda}$$

‘quantized’ equivalent of classical billiards (momentum$\rightarrow i\nabla$)

Quantum chaos

What happens at higher E? Depends on classical dynamics:

Regular:  ϕ_n separable

Chaotic:  ϕ_n disordered

MOVIE

'low' energy: $n \sim 700, \quad E \sim 10^4, \quad k \sim 100, \quad 15\lambda$ across

1970's to present day, field of QUANTUM CHAOS:

- eigenvalues (spacings, correlations, RMT…)
- eigenfunctions (ergodicity, correlations, matrix els…)
- dynamics (scattering, resonances, dissipation, electron physics, q. chemistry…)

Classical and quantum averages

Choose 'test function' $A(\mathbf{r})$:

classical (phase space) average $\bar{A} \equiv \dfrac{1}{\mathrm{vol}(\Omega)} \displaystyle\int_{\Omega} A(\mathbf{r}) d\mathbf{r}$

quantum version is $\hat{A} =$ operator in linear space of ϕ_n's

Expectation (average) $\langle \phi_n, \hat{A}\phi_n \rangle \equiv \displaystyle\int_{\Omega} A(\mathbf{r}) \underbrace{\phi_n(\mathbf{r})^2 d\mathbf{r}}_{d\mu_{\phi_n} \text{ density measure}}$

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Quantum ergodicity:

$$\boxed{\langle \phi_n, \hat{A}\phi_n \rangle - \bar{A} \;\longrightarrow\; 0 \qquad \text{as} \quad E_n \to \infty}$$

- Does this happen? For all states n? At what *rate*?
- We test numerically for certain A, up to very high $n \sim 10^6$.

If true for all A $\Rightarrow$ equidistribution in space, $d\mu_{\phi_n} \to$ uniform

Outline

- Motivation: random waves, scars
- Ergodicity theorems, conjectures
- Numerical test results
- Rate of equidistribution: semiclassical estimate
- Sketch of numerical techniques which make this possible
- Conclusion

Motivation: Random plane waves

Conjecture (Berry '77): statistical model of eigenfunctions

$$\phi_n \sim \frac{1}{\sqrt{N}} \sum_{j=1}^{N} a_j \sin(\mathbf{k}_j \cdot \mathbf{r} + \alpha_j)$$

iid amplitudes $a_j \in \mathbb{R}$
iid phases $\alpha_j \in [0, 2\pi[$

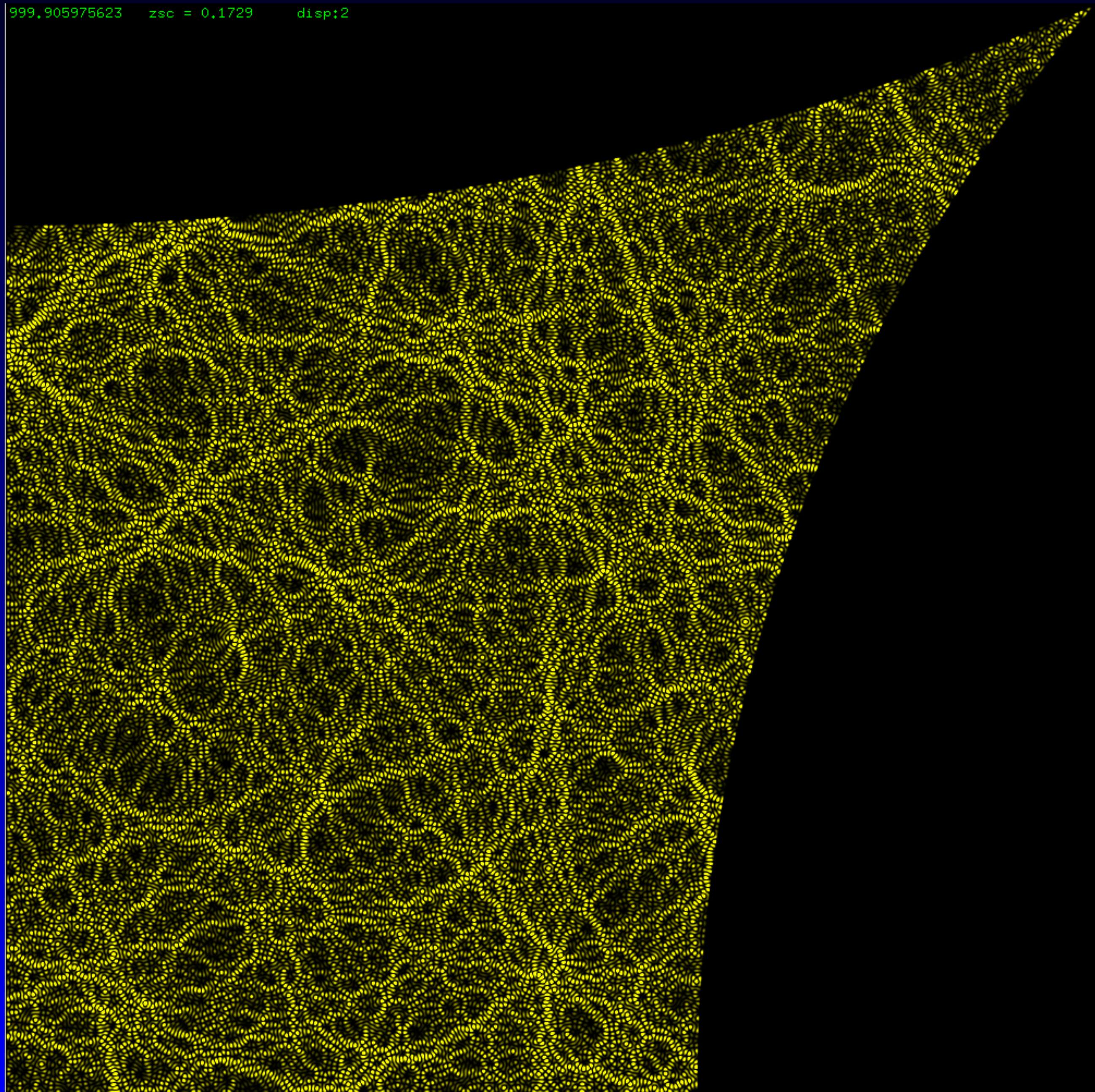
Wavevectors $\mathbf{k}_j$, spaced uniformly in direction, $|\mathbf{k}_j| = k$.

- Ray analogue of classical ergodicity.
- Predicts equidistribution as $E \to \infty$:

  deviations die like $\left| \langle \phi_n, \hat{A}\phi_n \rangle - \bar{A} \right| \sim E^{-1/2}$

High-energy eigenfunction ϕ_n

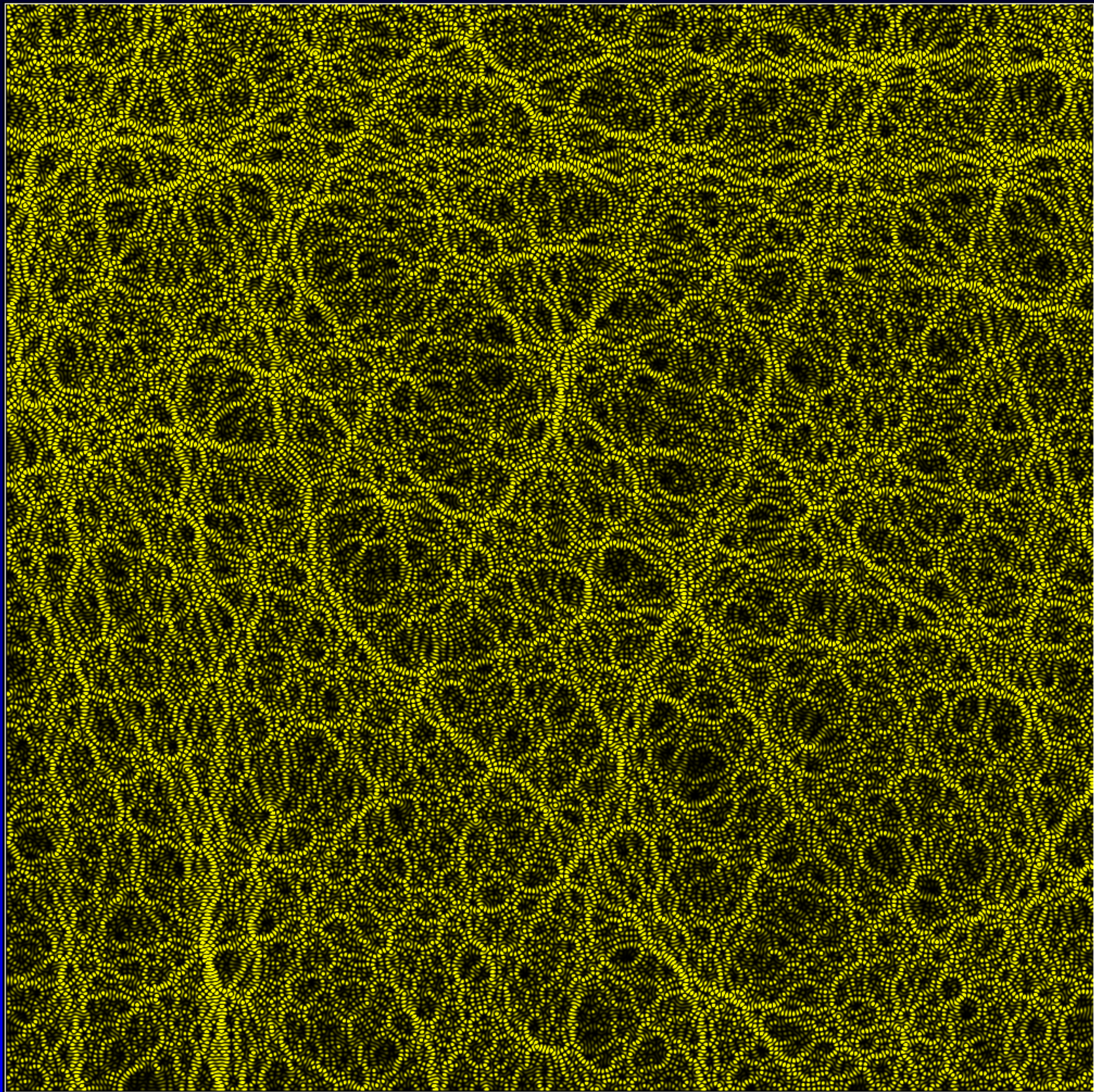
$k \approx 10^3$
$E \approx 10^6$
$n \approx 5 \times 10^4$



Random plane waves

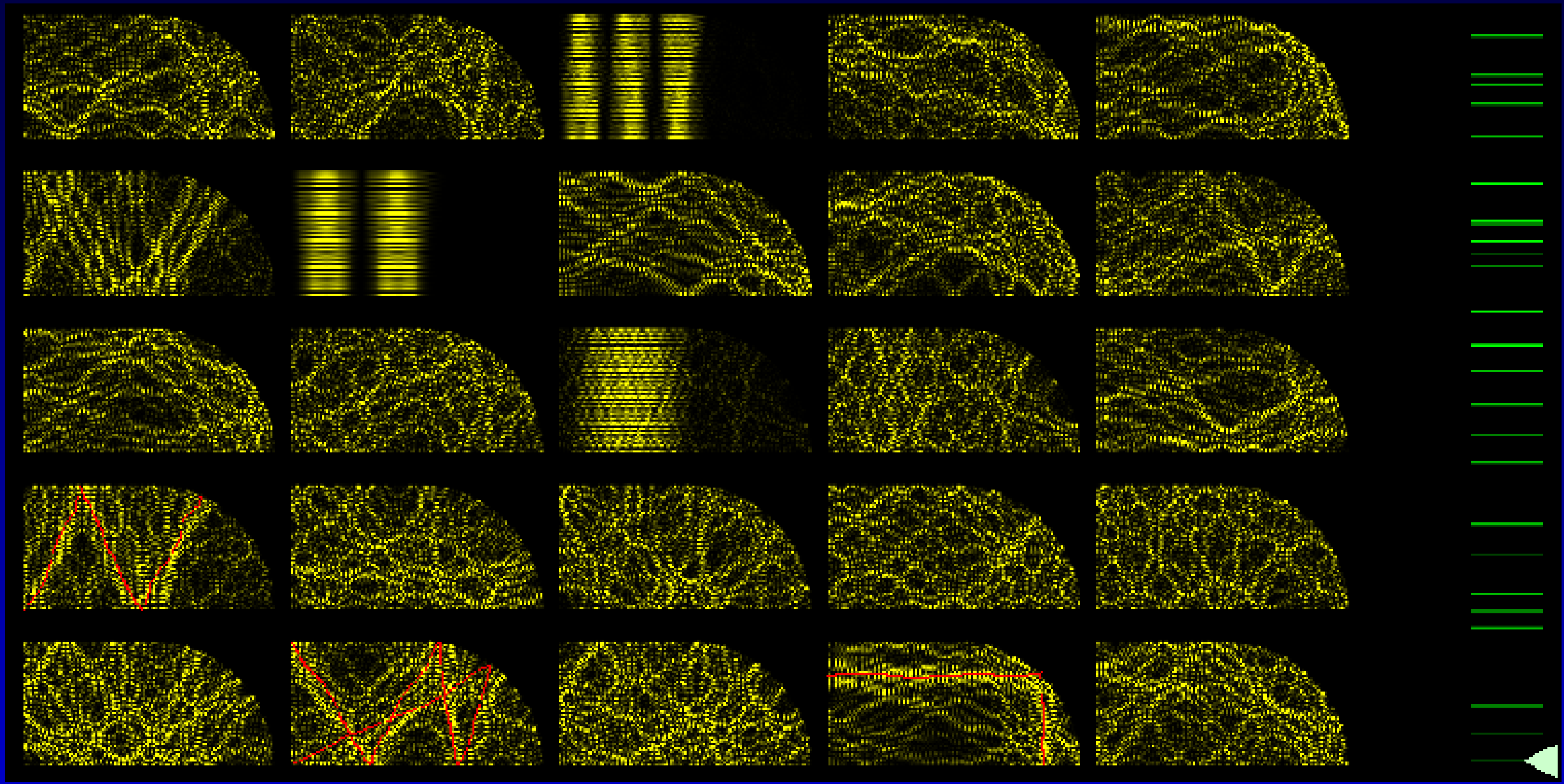
stringy structures appear due to $|\mathbf{k}| =$ const.

Interesting …to the eye only?



Motivation: 'Scars'

Heller '84 observed: often mass concentrates (*localizes*) on short classical unstable periodic orbits (UPOs)...



Theory (Heller, Kaplan): on UPO higher classical return prob.

- Strong scars were thought to persist as $E \to \infty$. (No longer!)
- For certain A, our $\langle \phi_n, \hat{A}\phi_n \rangle$ is a measure of scarring

Quantum Ergodicity Theorem

QET (Schnirelman '74, Colin de Verdière '85, Zelditch '87…):

For ergodic systems and well-behaved A,

$$\lim_{E_n \to \infty} \langle \phi_n, \hat{A}\phi_n \rangle - \bar{A} \;=\; 0$$

is true for almost all ϕ_n.

- Could exist an exceptional set (*scars?*) which are not ergodic
- This set has to be a vanishing fraction of the total number

QET makes physicists happy: 'Correspondence Principle'
all quantum & classical answers agree as $\lambda \to 0$

Quantum Unique Ergodicity

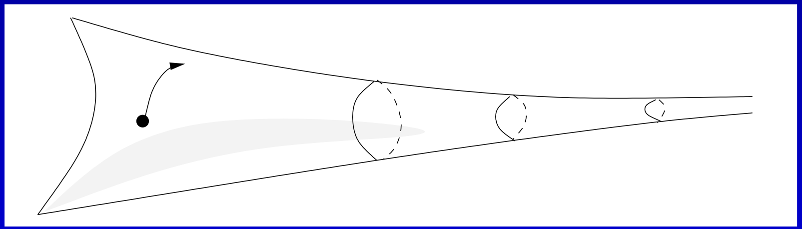
QUE conjecture (Rudnick & Sarnak '94)
For every single eigenfunction,

$$\lim_{E_n \to \infty} \langle \phi_n, \hat{A}\phi_n \rangle - \bar{A} \;=\; 0$$

All converge to unique measure: $d\mu_\phi =$ uniform. (no scars)

- Contrast classical flow has many invariant measures: each UPO

QUE was in context of hyperbolic manifolds…



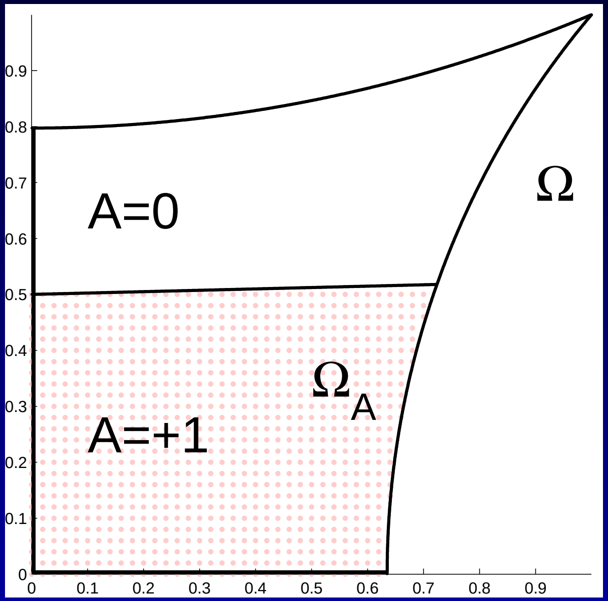
negative curvature
causes chaos

Constant-curvature arithmetic case: recent analytic progress…

- Lindenstrauss '03: measure can't collapse on to UPO
- Luo & Sarnak '03: bounds on sums $\Rightarrow$ deviations $\sim E^{-1/4}$

Numerical tests

Analytics only for special systems (symmetries, all $\Lambda = 1$)



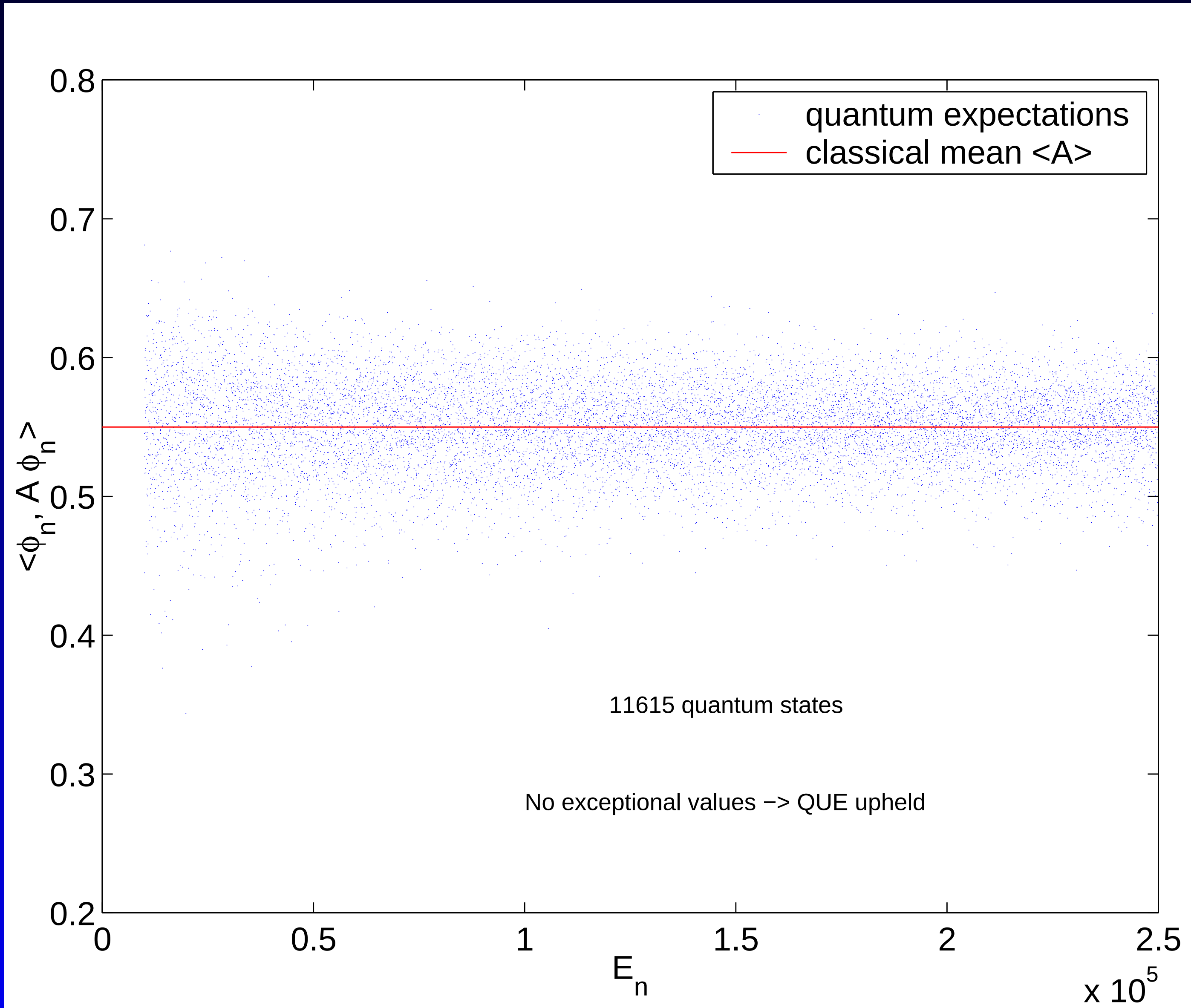
Test generic chaotic system (Λ's differ)

e.g. Sinai-type billiard: concave walls $\Rightarrow$ Anosov

$A(\mathbf{r}) =$ piecewise const: fast quantum calc using boundary

- classical $\bar{A} = \dfrac{\mathrm{vol}(\Omega_A)}{\mathrm{vol}(\Omega)}$
- quantum $\langle \phi_n, \hat{A}\phi_n \rangle =$ 'probability mass' inside Ω_A

Results: Expectation values

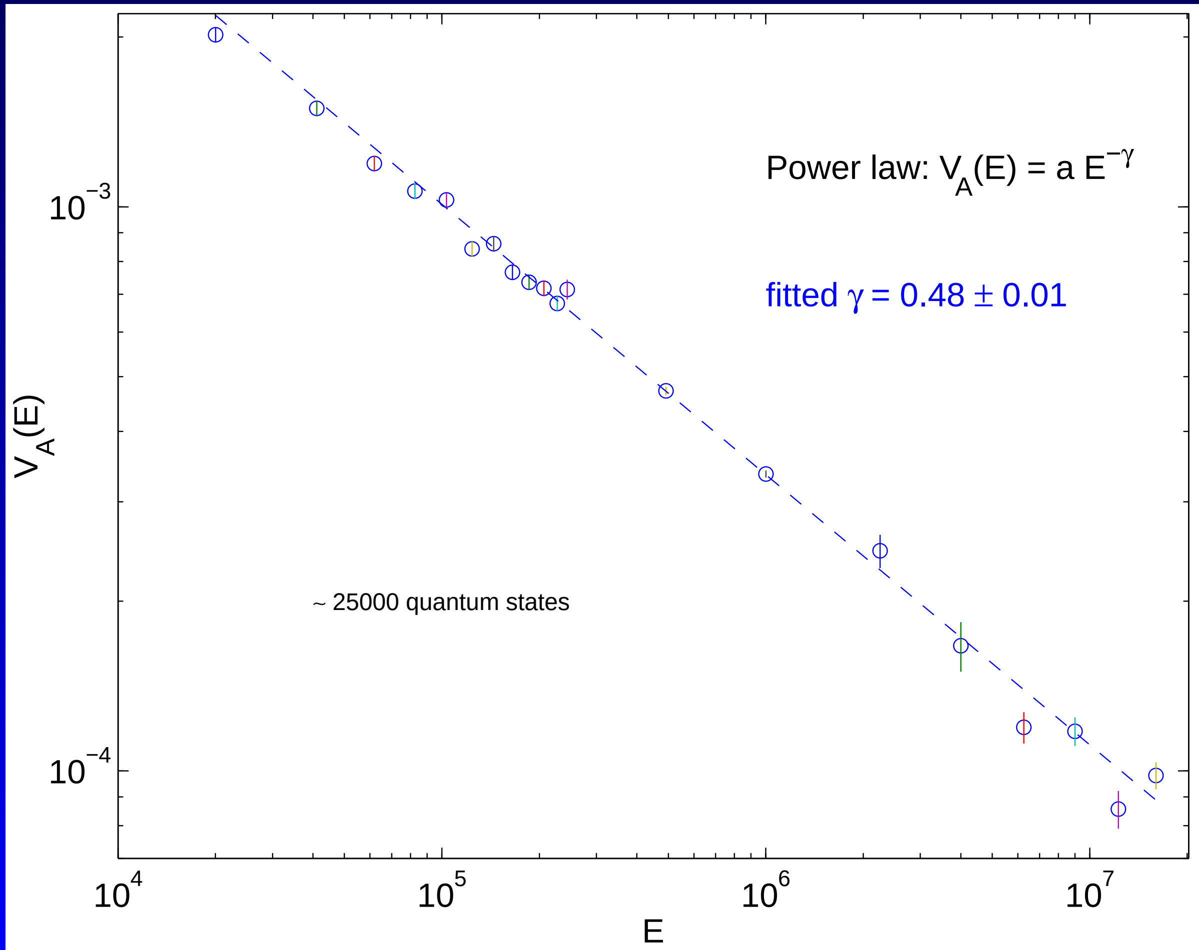


mean $\langle \phi_n, \hat{A}\phi_n \rangle \to \bar{A}$.   Variance slowly decreasing, but how?

Results: Equidistribution rate

Quantum variance: $$V_A(E) \equiv \frac{1}{m} \sum_{E_n \approx E}^{N \le n < N+m} \left| \langle \phi_n, \hat{A}\phi_n \rangle - \bar{A} \right|^2$$

- Hard to measure: *e.g.* 1% needs $m \sim 2 \times 10^4$ indep samples!



Results: Power law

Variance $V_A(E) = aE^{-\gamma}$, found $\gamma = 0.48 \pm 0.01$

Consistent with conjecture that deviations $\sim E^{-1/4}$ (*i.e.* $\gamma = 1/2$)

Previous experiments also used piecewise-constant $A(\mathbf{r})$:

- Aurich & Taglieber '98: negatively-curved surfaces, lowest $n < 6000$ only
- Bäcker '98: billiards, $n < 6000$, but many choices of A

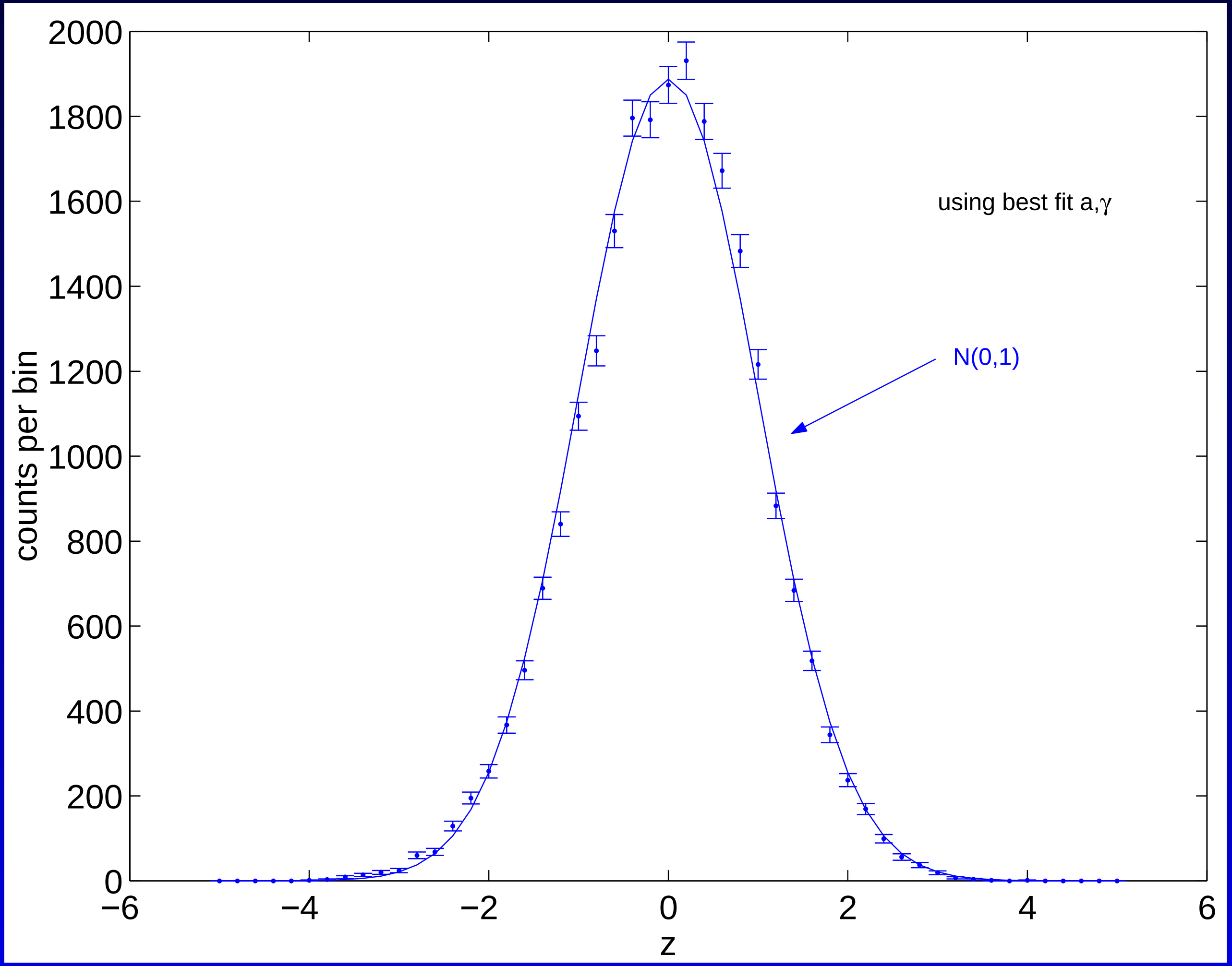
Can see power-law not asymptotic until $n \gtrsim 10^4$

$\ldots$we go 100 times higher!

up to level $n \approx 8 \times 10^5$, $E \approx 1.6 \times 10^7$

Results: Distribution of deviations

Plot deviations scaled by $\sqrt{V_A(E)}$:



- Consistent with Gaussian (*i.e.* random wave model), convincing

Theory: Semiclassical variance estimate I

Feingold & Peres '86

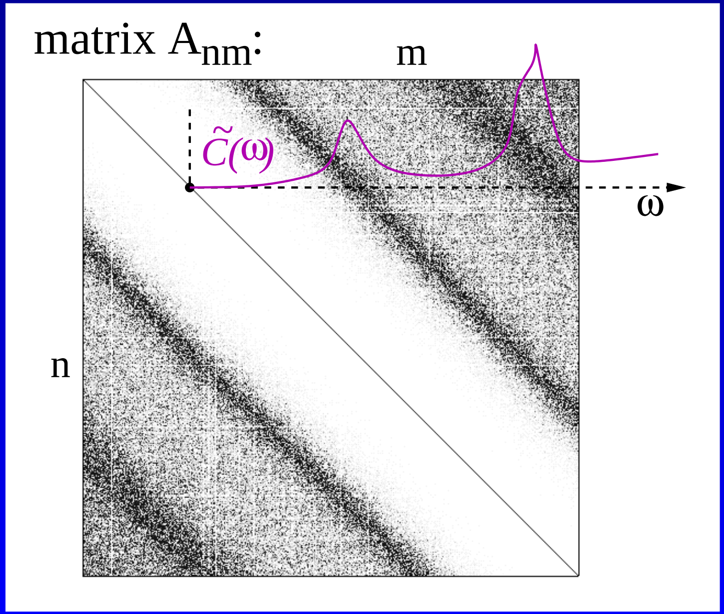
Signal $A(t)$ = follow A along particle trajectory $\mathbf{r}(t)$
Consider autocorrelation of this signal:

$$\langle A(t)A(t+\tau)\rangle_t \overset{\text{ergod}}{=} \overline{A(0)A(\tau)} \overset{\text{QET}}{\approx} \langle \phi_n, \hat{A}(0)\hat{A}(\tau)\phi_n\rangle$$

$\downarrow$ fourier transform $\downarrow$

noise power spectrum $\tilde{C}_A(\omega)$

band profile of matrix $A_{nm} \equiv \langle \phi_n, \hat{A}\phi_m\rangle$



$\omega \equiv$ distance from diagonal

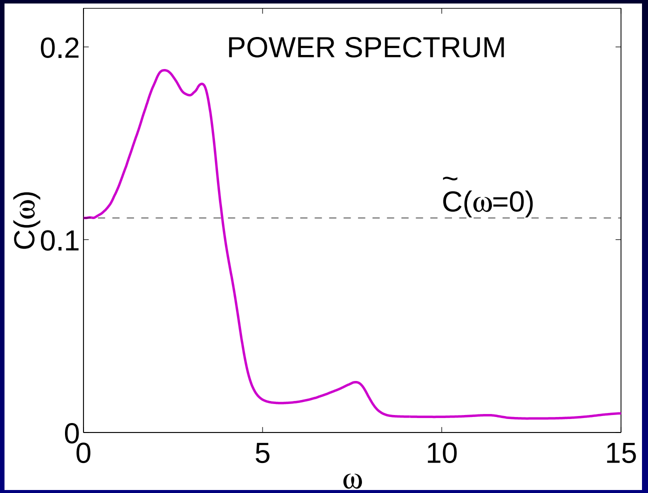
Barnett *et al.* '00: verified in stadium billiard

- Note: the diagonal is our quantum expectation!

Theory: Semiclassical variance estimate II

Estimate $\tilde{C}_A(\omega)$ numerically:

Measure power spectrum of $A(t)$ along long trajectories $\boxed{\text{LISTEN to } A(t)}$



Physics: $\tilde{C}_A(\omega)$ is heating (dissipation) rate under external driving by field A.

(Cohen '99: fluctuation-dissipation)

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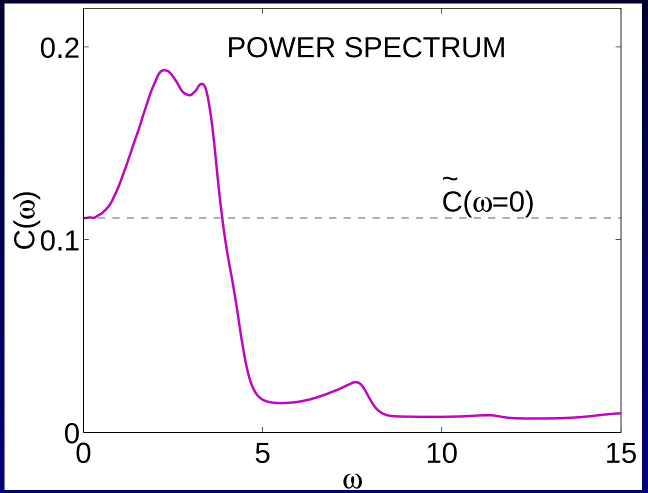

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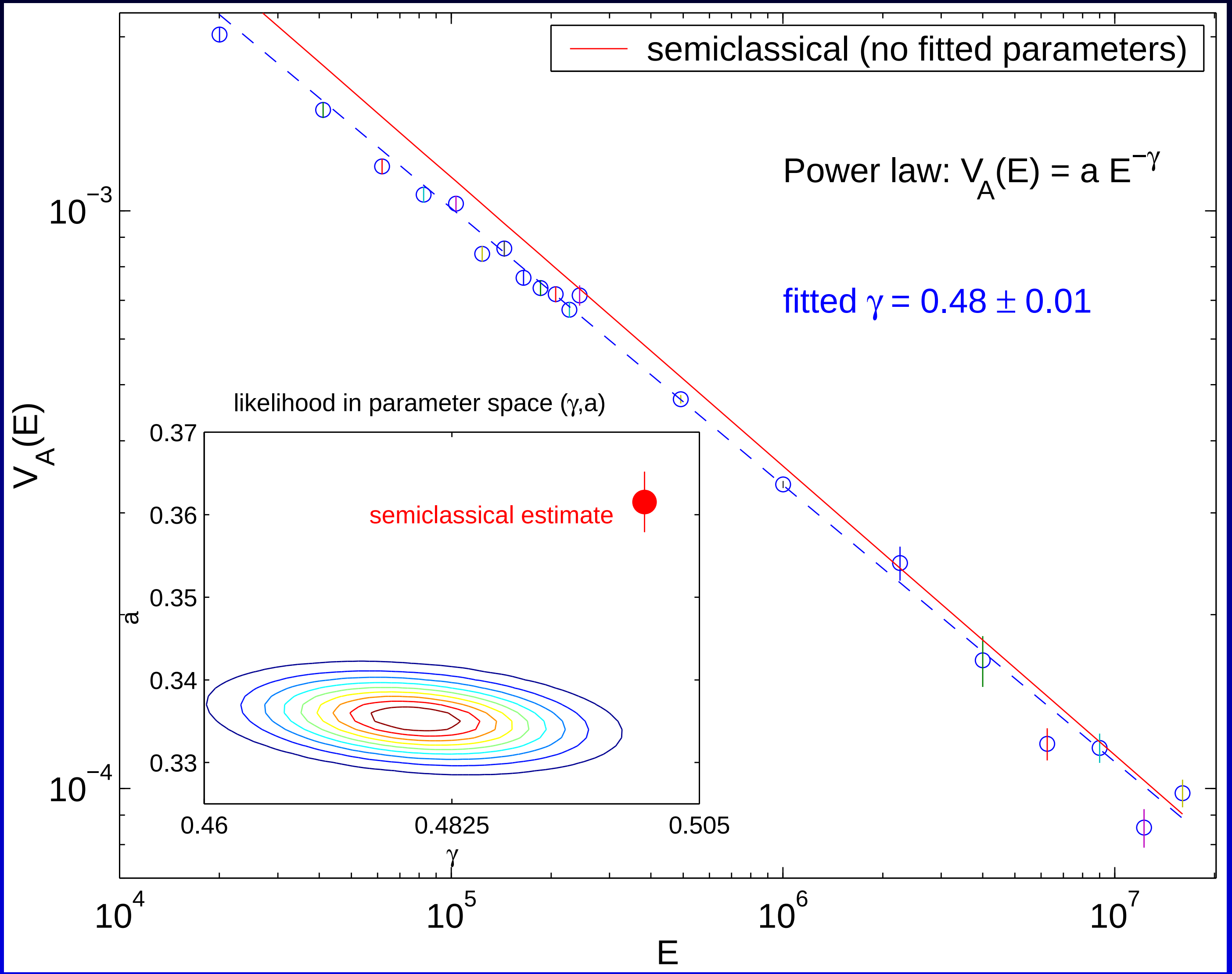
DC limit $\omega \to 0$ gives diagonal variance:

$$V_A(E) \equiv \text{var}(A_{nn}) \to \underbrace{\frac{2}{\text{vol}(\Omega)} \tilde{C}_A(\omega = 0)}_{\text{prefactor } a} E^{-1/2}, \qquad \leftarrow \gamma = 1/2$$

Time-reversal symmetry: for diagonal, extra factor 2

Results: semiclassical estimate



Good agreement, no fitted params. (Estimate a is 8% too big)

- Compare arithmetic surfaces: $a_{\text{classical}} \neq a_{\text{quantum}}$ provably

Numerical methods sketch

$\boxed{1}$ Compute eigenfunctions ϕ_n via scaling method :

(Vergini & Saraceno ’94; correct explanation (QET) Barnett, Cohen & Heller ’00)

If: find A s.t. $\bar{A} \neq 0$ but dynamics gives $\lim_{\omega \to 0} \tilde{C}_A(\omega) = 0$

Then: matrix $A_{nm} \approx$ diagonal: Eigenvectors of $A \approx \{\phi_n\}$.

Put into a basis, size $N \sim 1/\lambda \sim \sqrt{E}$. (*e.g.* $N \sim 4000$)

- One dense matrix diagonalization returns $O(N)$ cluster of ϕ_n’s
- $O(N) \sim 10^3$ faster than boundary integral equation methods!

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$\boxed{3}$ Norm formula for Helmholtz solutions (little known?):

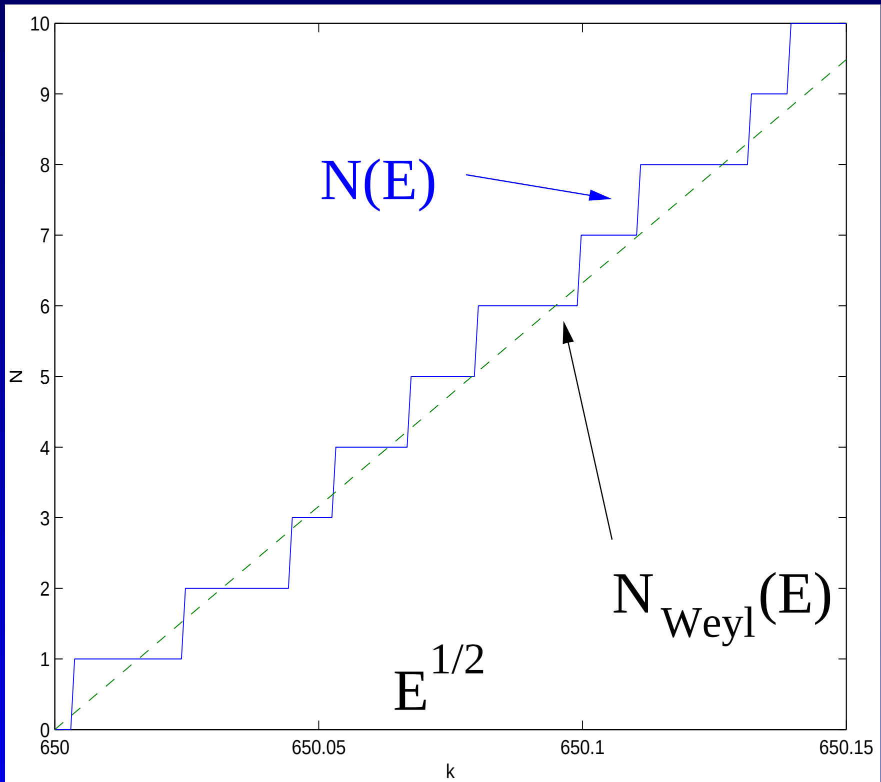
$$\langle \phi, \phi \rangle_{\Omega_A} = \frac{1}{2k^2} \oint_{\partial\Omega_A} (\mathbf{n} \cdot \nabla\phi)(\mathbf{r} \cdot \nabla\phi) - \phi\, \mathbf{n} \cdot \nabla(\mathbf{r} \cdot \nabla\phi)\, ds$$

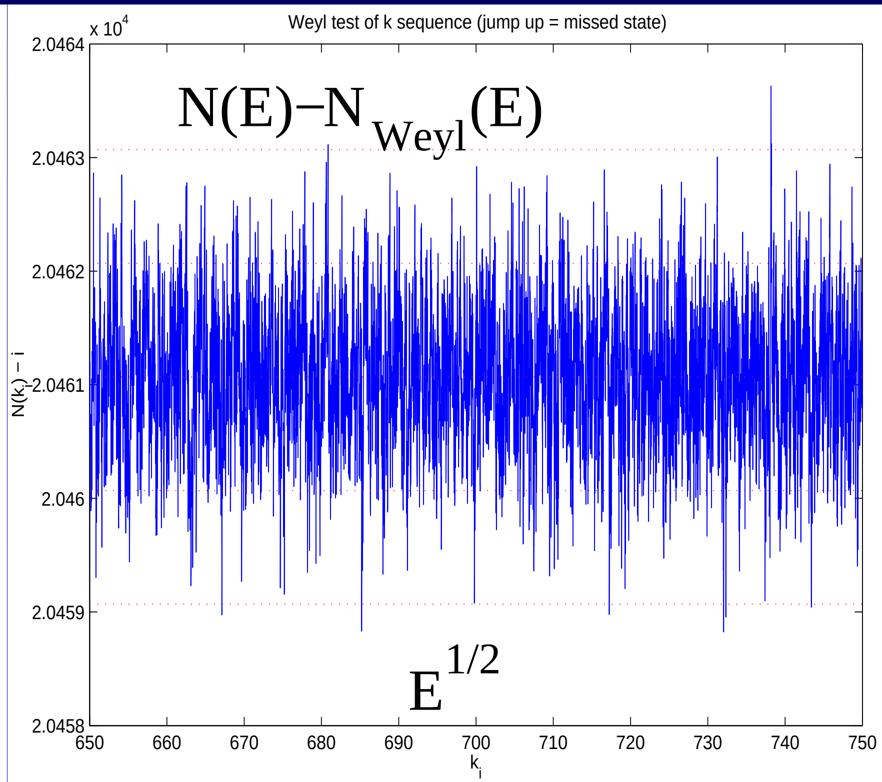
- Overall effort scales $O(N^2)$ per state (few CPU-days total)

Missing levels?

Weyl's estimate for $N(E)$, the # eigenvalues $E_n < E$:

$$N_{\mathrm{Weyl}}(E) \;=\; \frac{\mathrm{vol}(\Omega)}{4\pi} E - \frac{L}{4\pi}\sqrt{E} + O(1)\cdots$$





- not 1 state missing in sequence of 6812 states

Conclusion

- Are quantum (laplacian) eigenfunctions spatially uniform in chaotic systems as $E \to \infty$?
  - Measured rate of equidistribution in generic billiard
  - Unprecendented range in E & sample size
  - Strong support for QUE conjecture (no scars)
  - Power law consistent with conjectured $\gamma = 1/2$
  - Semiclassical estimate good, not perfect
- Directions
  - Study prefactor a for other choices of A (does it vary?)
  - Variant of QUE: off-diagonal matrix elements?
  - Scaling method: basis sets, rigor, fast Bessels...