Math 31 Lesson Plan

Day 18: Section 12, Take 2

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Supplies needed:

- Colored chalk
- Homework

Goals for students: Students will:

- Develop their proof-writing skills, both by seeing good proofs modeled and by working through tricky proofs (why are two groups not isomorphic?)

[**Lecture Notes: Write everything in blue, and every equation, on the board. [Square brackets] indicate anticipated student responses. *Italics* are instructions to myself.**]

return homework

Today I'm going to prove a few of the theorems from Section 12 that were confusing. Then I'd like you to work in groups to check whether certain groups are isomorphic or not. If two groups are isomorphic, that means they have the same number of elements and they have exactly the same group-theoretic properties – order of elements, being abelian or not, having the same subgroup lattice. In other words, they represent the same information. They might look different but for all intents and purposes, they're the same. This is why isomorphism is a useful concept!

At the end, we'll prove Cayley's Theorem, and also connect homomorphisms with that set $N(H)$ that you had to prove was a subgroup for last week's starred homework.

Speaking of starred problems, I made a mistake on this week's starred problem: Correction to starred problem: Prove that any group G with $|G| > 2$ has a nontrivial automorphism.

A quick Reminder about the midterm: Friday, October 28. I'm not going to give you a practice exam, because if I were to write a practice exam, it would consist of me choosing problems from the textbook that I think are about the difficulty level of the homework problems, and that reflect the important points of the material we've covered. Thinking it over, it seems to me that this is something that you can do, and that in fact doing it would help you study for the midterm – it requires going back over your homeworks, and also picking out the most important aspects of things we've been learning. So I've created a forum on Blackboard: Create Your Own Practice Midterm. You can post problems there that you think would be good practice midterm problems, so that you and your classmates can work on them; if I see some posted that I don't think are appropriate, I'll say so.

Why do we care about transpositions/A_n? Sometimes it's nice (and helpful) to break things down into small pieces, and study them one by one. A_n is important for Galois Theory (Math 81) – A_n is one of the links used in that class to connect fields, polynomials, and groups.

A_n is also what's called a simple group, which is a definition we'll come back to later. 12:40

THEOREM 12.4 (I) Let $\phi : G \to H$ be a homomorphism. Then $\phi(e_G) = e_H$.

Proof: Write $x = \phi(e_G) \in H$. Because $e_G e_G = e_G$, the fact that ϕ is a homomorphism tells us that

$$\phi(e_G) = \phi(e_G e_G) = \phi(e_G)\phi(e_G).$$

In other words, $x = x^2$. But then, multiplying on the left by x^{-1} we see that $e_H = x = \phi(e_G)$ as claimed. □

THEOREM 12.5 (IV) Let $\phi : G \to H$ be a monomorphism. Then $o(x) = o(\phi(x))$ for any $x \in G$.

Proof: Suppose $o(x) = n < \infty$. Then $x^n = e_G$, so $\phi(x^n) = (\phi(x))^n = e_H$. Therefore, by Theorem 4.4(ii), what can we conclude about $o(\phi(x))$? [$o(\phi(x))$ must divide n.] However, if ϕ is a monomorphism, we will use proof by contradiction to prove that $o(\phi(x)) = n$. So what do we do? [Suppose $o(\phi(x)) = k < n$. Then,

$$\phi(e_G) = e_H = (\phi(x))^k = \phi(x^k).$$

Since ϕ is 1-1, this implies that $e_G = x^k$ for some $k < n$, which contradicts the fact that $o(x) = n$. Are we done? [no] What else do we have to prove? [We have to check the case $o(x) = \infty$.]

Suppose $o(x) = \infty$. Again, we use proof by contradiction to show that $o(\phi(x)) = \infty$ also. So, suppose $o(\phi(x)) = n < \infty$. Then

$$\phi(e_G) = e_H = \phi(x)^n = \phi(x^n).$$

Since ϕ is 1-1 by hypothesis, this tells us that $x^n = e_G$, which contradicts the fact that $o(x) = \infty$.

Therefore $o(\phi(x)) = o(x)$ for any x if ϕ is a monomorphism. □

THEOREM 12.6 (I) Let $\phi : G \to K$ be a homomorphism and let $H \leq G$. Then $\phi(H) \leq K$.

What do we have to show? [We must show that $\phi(H)$ is closed under multiplication and under inverses.] In other words, we must show that if $k_1, k_2 \in \phi(H)$, then $k_1 k_2 \in \phi(H)$; and if $k \in \phi(H)$ then $k^{-1} \in \phi(H)$. So let's do this.

Proof: Suppose $k_1, k_2 \in \phi(H)$. This means what? [that there exist $h_1, h_2 \in H$ such that $\phi(h_1) = k_1, \phi(h_2) = k_2$.] Therefore,

$$k_1 k_2 = \phi(h_1)\phi(h_2) = \phi(h_1 h_2) \in \phi(H)$$

because why? [ϕ is a homomorphism and H is a subgroup of G, hence it's closed under multiplication.] Similarly, if $k \in \phi(H)$, we can write $k = \phi(h)$ for some $h \in H$. Then,

$$k^{-1} = \phi(h)^{-1} = \phi(h^{-1}) \in \phi(H).$$

Thus $\phi(H)$ is a subgroup as claimed. □

1:00

discuss in groups Are D_4 and S_4 isomorphic? What about D_3 and S_3? What about D_4 and $\mathbb{Z}_8$? What about $D_3 \times \mathbb{Z}_4$ and $D_4 \times \mathbb{Z}_3$?

discuss as a class: Ways to show that two groups are not isomorphic:

- Show the groups have different numbers of elements (size, cardinality)
- Show that they have different numbers of elements of a certain order
- Show that one is abelian and one isn't

Cardinality just means size. Cardinality is the word from logic, where they focus on the different sizes of infinity, not just different finite sizes and the difference between infinite and finite sets. If you think this is cool, take Math 89 in the winter!

1:20

I want to talk about Cayley's Theorem now. Since Cayley's Theorem relies on the symmetric group, how do we define S_X for a set X? [S_X is the set of bijections from X to itself.]

$$S_X = \{f : X \to X \text{ bijections}\}$$

CAYLEY'S THEOREM: *Every group $(G, *)$ is isomorphic to a subgroup of S_G, the symmetric group on the set G.*

Proof: Let $x \in G$. Define $f_x : G \to G$ by $f_x(y) = x * y$ for any $y \in G$. I claim that f_x is a bijection. *Have class check in pairs* If $f_x(y) = f_x(z)$, then $x * y = x * z$, and therefore $y = z$. So f_x is 1-1. To see that f_x is onto, let $z \in G$. Observe that $x^{-1} * z \in G$ as well, and $f_x(x^{-1}z) = x * x^{-1} * z = z$. Therefore f_x is onto.

I claim that the map $\phi : G \to S_G$ given by $\phi(x) = f_x$ is a monomorphism. *check in pairs* First, we check that ϕ is a homomorphism; that is, $\phi(x * y) = f_x \circ f_y$. But, for any $z \in G$,

$$\phi(x * y)(z) = f_{x*y}(z) = x * y * z = x * (y * z) = f_x(y * z) = f_x(f_y(z)).$$

Thus ϕ is a homomorphism. Moreover, ϕ is 1-1: If $\phi(x) = \phi(y)$, then that means that $f_x(z) = f_y(z)$ for every $z \in G$. In particular, this holds for $e \in G$; so

$$f_x(e) = f_y(e) = x * e = y * e$$

and hence $y = x$.

Since $\phi : G \to S_G$ is a 1-1 homomorphism, $\phi : G \to \phi(G)$ is an isomorphism onto its range. Since $G \leq G$, Theorem 12.6 tells us that $\phi(G) \leq S_G$, and hence G is isomorphic to a subgroup of S_G as claimed. □

If only 5 mins V_4 via $(P(\{1, 2\}), \Delta)$ and Cayley table.

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if at least 10 mins DEFINITION: The *kernel* of a homomorphism $\phi : G \to H$ is

$$\ker \phi = \{g \in G : \phi(g) = e_H\}.$$

CLAIM: The kernel of a homomorphism is a subgroup of G. *Prove in groups*

In fact, PROPOSITION: The normalizer $N(\ker \phi) = G$ for any homomorphism ϕ.

Who can remind me of the definition of the normalizer?

$$N(\ker \phi) = \{a \in G : aga^{-1} \in \ker \phi \ \forall \ g \in \ker \phi\}$$

So what do I have to show? **Proof:** If $a \in G$ is arbitrary, I want to show that $\phi(aga^{-1}) = e_H$ for any $g \in \ker \phi$. But,

$$\phi(aga^{-1}) = \phi(a)\phi(g)\phi(a)^{-1} = \phi(a)e_H\phi(a)^{-1} = e_H.$$

Therefore, $a \in N(\ker \phi)$, and since $a \in G$ was arbitrary, we have that $N(\ker \phi) = G$ as claimed. □