Homework 2 solutions

1.1.30 **Let G be a simple graph with adjacency matrix A and incidence matrix M. Prove that the degree of v_i is the ith diagonal entry of A^2 and MM^T. What do the entries in position (i,j) of A^2 and MM^T say about G?**

Proof. Since A is symmetric, the ith diagonal entry of A is the dot product square of the vector

$$\mathrm{v}^{(i)} \qquad \text{where } \mathrm{w}_j^{(i)} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

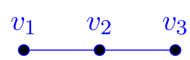
In general, the dot product square of a vector is the sum of the squares of the entries in the vector. Since all of the entries of v are 1 or 0, $\mathrm{v} \cdot \mathrm{v}$ is just the sum of the entries of v, which is the degree of v_i. The ith diagonal entry of MM^T is the dot product square of

$$\mathrm{w}^{(i)} \qquad \text{where } \mathrm{w}_j^{(i)} = \begin{cases} 1 & \text{if } e_j \text{ is incident to } v_i \\ 0 & \text{otherwise.} \end{cases}$$

Again, this is just the sum of the entries in $\mathrm{w}^{(i)}$, which is also the degree of v_i.

In A^2, the (i,j) entry is the dot-product $\mathrm{v}^{(i)} \cdot \mathrm{v}^{(j)}$, which will again be the sum of 1's and 0's: a 1 occurs in the kth summand if v_i and v_j share v_k as a neighbor. So (i,j) entry is the number of neighbors v_i and v_j have in common!

(**Fun fact:** In general, the (i,j) entry of A^k will be the number of length k walks from v_i to v_j. For example, if

$G =$ 

Then

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}, \quad A^4 = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}, \cdots)$$

Similarly, the (i,j) entry of MM^T is the dot product $\mathrm{w}^{(i)} \cdot \mathrm{w}^{(j)}$, which will again be the sum of 1's and 0's: a 1 occurs in the kth summand if v_i and v_j share e_k as an edge. So (i,j) entry is the number of edges joining v_i to v_j. □

1.2.1 **Determine whether the statements below are true or false.**

(a) **Every disconnected graph has an isolated vertex.**

False. For example, is disconnected and has no isolated vertex.

(b) **A graph is connected if and only if some vertex is connected to all other vertices.**

True. IF the graph is connected, then by definition, any vertex is connected to every other vertex in G. On the other hand, if there is a vertex that is connected to every other vertex in G, then by transitivity (of the connection relation), every other pair of vertices is also connected (it P is a u,v-path and P' is a v,w-path, then P followed by P' is a u,w-walk, which contains a u,w-path).

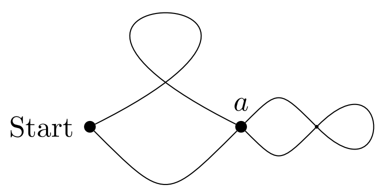
(c) **The edge set of every closed trail can be partitioned into edge sets of cycles.**

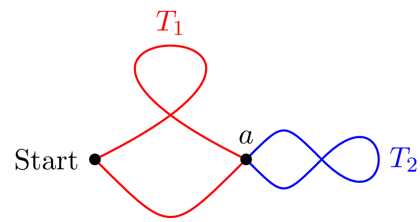
True. This is certainly true for a closed trail which is a single vertex or any trail which is itself a cycle. If a closed trail T is not a cycle, then either (1) the first/last vertex is repeated more that just the twice, or (2) there is some other vertex which is repeated; call this special vertex a. Decompose the trail T into

T_1, the trail which walks to the first occurrence of a and skips everything between then and the second occurence of a and then completes the rest of T, and

T_2, which starts at the first occurrence of a and walks to the second occurrence of a.

Both T_1 and T_2 are closed trails of shorter length than T. Strongly induct on the length of T.





(d) **If a maximal trail in a graph is not closed, then its endpoints have odd degree.**

True. If a trail is not closed, then for each endpoint, the number of edges in the trail which are incident to that endpoint must be odd. If a trail is maximal, then there are no edges incident to the endpoint(s) which are not already in the trail; so the number of edges in the trail which are incident to each endpoint is equal to the degree of that vertex.

1.2.4 **Let G be a loopless graph. For $v \in V(G)$ and $e \in E(G)$, describe the adjacency and incidence matrics of $G - v$ and $G - e$ in terms of the corresponding matrices for G.**

$\underline{G - e}$: Suppose the endpoints of e are u and v. The adjacency matrix is the same as that of G, except in the (u, v) and the (v, u) entries, which are each reduced by exactly 1. The incidence matrix has one fewer columns than $I(G)$ (the one corresponding to e), and is otherwise unchanged.

$\underline{G - v}$: The adjacency matrix is one dimension smaller, and is achieved by deleting the row and column of $A(G)$ corresponding to v. The incidence matrix has one fewer row and degree(v) fewer columns and is achieved by deleting the row corresponding to v and the columns corresponding to all incident edges to v.

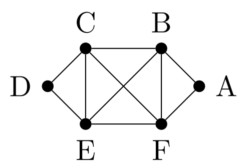
1.2.12 **Convert the proof given in item 1.2.32 to a procedure for finding an Eulerian circuit in a connected even graph.**

Start by trying to draw an Eulerian trail (recording the order of the edges and vertices). If you get stuck in one direction, try to extend the the other end (adding edges and vertices in revers order to the beginning). When you run out of room, you must have a closed trail by Lemma 1.2.31. Since it's closed, think of it as a circuit C, which you can start at any of its vertices.

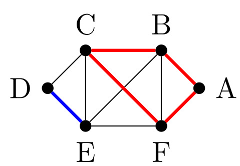
If you missed an edge e, find a path from e to your circuit (which doesn't intersect unnecessarily with C). Then start a new trail which starts at a far endpoint of e, follows the path to C, and then continues around C. If this new trail is not closed, you can close it again by Lemma 1.2.31. Iterate until there are no edges left.

For example:

An even connected graph

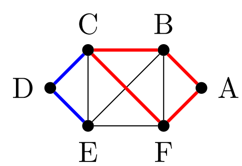


Try 1:



Trail: A–B–C–F–A
Missing edge: DE

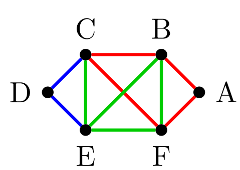
Try 2:



Original Trail: A–B–C–F–A
Path with edge: E–D–C

New Trail: E–D–$\underbrace{\text{C–F–A–B–C}}_{\text{same order as before, but new start point}}$

Closing off try 2:



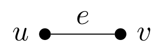
Last Trail: E–D–C–F–A–B–C
Closing it off (in reverse): E–B–F–E–C

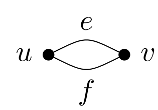
New Trail: C–E–F–B–E–D–C–F–A–B–C

1.2.15 **Let W be a closed walk of length at least 1 that does not contain a cycle. Prove that some edge of W repeats immediately (once in each direction).**

Proof. (By strong induction)
Let $\ell(W)$ denote the length of the walk.
Suppose $\ell(W) = 1$. If W is closed, then it must be a loop (a cycle of length 1), which contradicts the hypothesis. If $\ell(W) = 2$ and W is closed, then W is one of the following:

$$W = u, e, v, e, u \qquad\qquad W = u, e, v, f, u$$





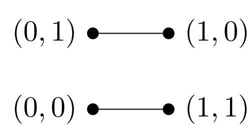
pro The latter has a cycle, so it can only be the first, in which the edge e immediately repeats.

Now assume that any closed walk of length $k < n$ which does not contain a cycle has an edge which immediately repeats. Any closed walk with only one repeated vertex (the first) is itself a cycle, so if $W = v_0, e_1, v_1, \ldots, e_\ell v_\ell$ does not contain a cycle, then it must have another vertex w which is repeated. Now consider the subwalk U of W which starts at the first occurrence of w, continues on W until the second occurrence of w (like T_2 in 1.2.1 (c)). This is a closed walk of length less than $\ell(W)$, and since W does not have a cycle, neither does this subwalk. So, by the induction hypothesis, this subwalk has an edge which immediately repeats. Since consecutive edges of this subwalk also occur consecutively in W, that means that there is an edge of W immediately repeats. □

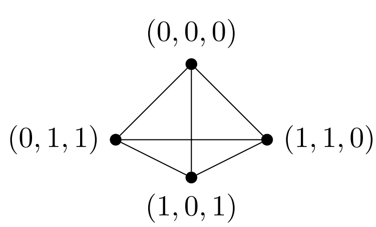
1.2.18 **Let G be the graph whose vertex set is the set of k-tuples with elements in $\{0, 1\}$. with x adjacent to y if x and y differ in exactly two positions. Determine the number of components of G.**
Answer:

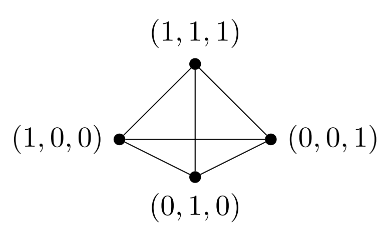
First, some examples:

$k = 2$:



$k = 3$:





In general, there will be **two connected components**:

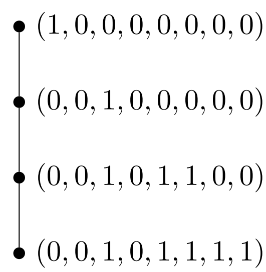
(a) one containing all points with an **odd** number of 1's, and

(b) one containing all points with an **even** number of 1's.

We prove this in two steps:

A. If two points do not share the same parity of 1's, then they're not in the same component. This can be seen by observing that an edge will only connect two vertices if those vertices have the same parity of 1's. Therefore, there is no walk (and therefore no path) in G which contains vertices of both parity's.

B. Any two points with the same parity of 1's are in the same component. We'll apply transitivity and show that every vertex is connected to one of two special vertices. **Even:** you can get to any point with an even number of 1's from the vertex with all 0's by following edges which changes the necessary 0's to 1's two at a time. **Odd:** similarly, you can get to any point with an odd number of 1's from the vertex with 1 in the first place followed by all 0's as follows: if the desired point has a 1 in the first coordinate, then change 0's to 1's as needed in pairs; if it doesn't then switch the 1 to a 0 at the same time as switching a desired 0 to a 1, and then switch 0's to 1's as needed in pairs. For example, (1,0,0,0,0,0,0,0) is connected to (0,0,1,0,1,1,1,1) by the following path:



1.2.38 **Prove that every n-vertex graph with at least n edges contains a cycle.**

Proof. If $n = 1$, and G has one vertex and at least one edge, then it has at least one loop, which is a cycle.

Now suppose $n > 1$ and G has n vertices and at least n edges. Either

(a) **every vertex in G has degree at least 2**, in which case Proposition 1.2.28 implies G contains a cycles, or

(b) G **has a vertex v of degree 1 or 0**. Then $G - v$ has $n - 1$ vertices and at least $n - 1$ edges (we deleted at most 1 edge in removing v). By strong induction, $G - v$ (and therefore G) has a cycle.

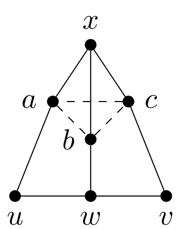
□

1.2.42 **Let G be a connected simple graph not having P_4 or C_4 as an induced graph. Prove that G has a vertex adjacent to all other vertices.**

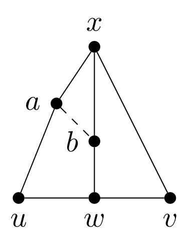
Proof. Since G is connected, any two vertices are connected by a path. Consider two vertices u and v, and pick a minimal u,v-path $P = u, e_1, v_1, \ldots, v_{\ell-1}, e_\ell, v$. Since P is minimal, no two v_i, v_j with $i \neq j \pm 1$ can be neighbors. So $G[V(P)] \cong P_{\ell+1}$ (and so G has P_k as an induced subgraph for all $k \leq \ell$ as well). Thus $\ell \leq 2$, and any pair of vertices in G must either be adjacent, or mutually adjacent to at least one vertex.

Now, either G is a complete graph (in which case every vertex is connected to every other vertex), or there are two non-adjacent vertices u and v. We'll show that any two such vertices have a common neighbor which is adjacent to all other vertices in G.

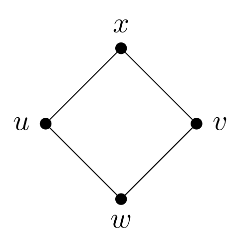
Choose a minimal u,v-path, and call the intermediate vertex w. Pick *any* other vertex x in $V(G)$. Since any minimal path in G connecting two vertices must be of length at most 2, and since G is connected, x is at most 2 away from u, v, and w. If x is not adjacent to w, then G would have one of the following as an induced subgraph (where the dashed edges are unknown):



I. x is not adjacent to u,v, or w.
Contradiction: $G[\{x, a, u, w\}] \cong P_4$



II. x is not adjacent to u or w.
(same as x not adjacent to v or w)
Contradiction: $G[\{x, a, u, w\}] \cong P_4$



III. x is adjacent to u and v, but not to w.
Contradiction: $G[\{u, v, w, x\}] \cong C_4$

Therefore, x (any other vertex in G) is adjacent to w.

□

1.3.9 **In a league with two divisions of 13 teams each, determine whether it is possible to schedule a season with each team playing 9 games against teams within its division and four games against teams in the other division.**
Answer: Formulating this as a graph theory problem, we need a 26-vertex graph which decomposes into a 4-regular bipartite graph with partites X and Y both of order 13, and two 9-regular graphs of order 13 (one on each partite of X and Y). Unfortunately, the latter isn't possible because a graph cannot have an odd number odd-degree vertices!

It is, however, possible to schedule the four games each across the divisions: Call the teams in division X $x_1, x_2, \ldots, x_{13}$, and the teams in division Y $y_1, y_2, \ldots, y_{13}$. For team x_i, schedule games with

$$y_i, \quad y_{i+1} \pmod{13}, \quad y_{i+2} \pmod{13}, \quad \text{and} \quad y_{i+3} \pmod{13}.$$

Then, conversely, y_j plays games with

$$x_j, \quad x_{i-1} \pmod{13}, \quad x_{i-2} \pmod{13}, \quad \text{and} \quad x_{i-3} \pmod{13}.$$

1.3.22 **Let G be a nonbipartite triangle-free simple graph with n vertices and minimum degree k. Let ℓ be the minimum length of an odd cycle in G.**

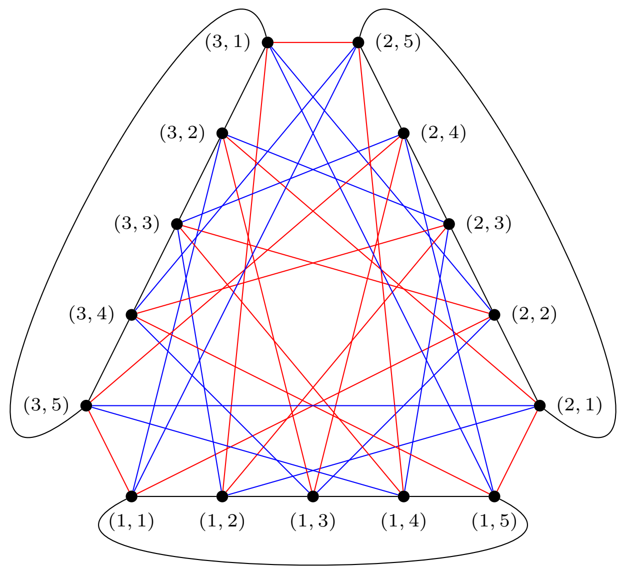
(a) **Let C be a cycle of length ℓ in G. Prove that every vertex not in $V(C)$ has at most two neighbors in $V(C)$.**
Suppose $x \in V(G) - V(C)$ has three neighbors in $V(C)$. Then those neighbors partition C into three pieces. Since ℓ is odd, at least one of those parts has odd length. And since there are no triangles in G, that part has length at most $\ell - 3$. Then by walking from x to one of the endpoints of that part, along that part, and then back to x, you've built an odd cycle of length less than ℓ, a contradiction.

(b) **By counting the edges joining $V(C)$ and $V(G) - V(C)$ in two ways, prove that $n \geq k\ell/2$ (and thus $\ell \leq 2n/k$).**
Since $\delta(G) = k$, the edges coming out of $V(C)$ is at least $k\ell - 2\ell$ (the sum of the degrees minus the edges contributing to C, which were all double-counted). On the other hand, the number of edges coming into $V(C)$ is at most $(|V(G)| - |V(C)|) * 2 = (n - \ell) * 2$. So
$$k\ell - 2\ell \leq (n-\ell) * 2 \implies k\ell \leq 2n \implies k\ell/2 \leq n.$$

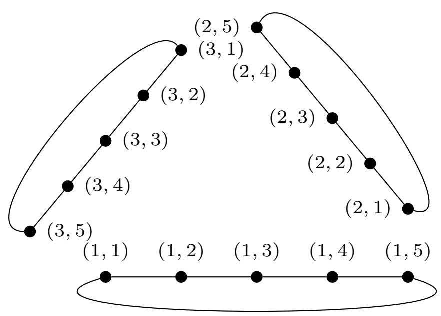
(c) **When k is even, prove that the inequality of part (b) is best possible.**
Let k be even and $\ell > 3$ odd. Then the criteria above are satisfied by the graph G with $V(G) = \{(a,b) \mid 1 \leq a \leq k/2, 1 \leq b \leq \ell\}$ and $E(G) = \{(a,b) - (c,d) \mid b = d \pm 1 \pmod{\ell}\}$ (see the example below). For this graph, $n = (k/2) * \ell$.

Proof. (1) Any cycle which has an even number of edges (a, ℓ)–$(b, 1)$ must be of even length (e.g. a cycle with no such edge will have the property that every second coordinate which appears must appear an even number of times). So the shortest odd cycle is of length ℓ.
(2) Every vertex has degree k since (a,b) will connect to $(c, b+1 \pmod{\ell})$ and $(c, b-1 \pmod{\ell})$ for $c = 1, \ldots, k$. .
(3) G is not bipartite since it has an odd cycle, and has no triangles by (1). ∎

For example, if $k = 6$, $\ell = 5$, then G is



which decomposes into the three 5-cycles



and the two 15-cycles