Math 118. **Combinatorics.** Spring 2013

Problem Set 3. Due on Friday, 5/10/2013.

1. The inversion number and the major index of a permutation $\pi \in \mathcal{S}_n$ are defined as

$$\mathrm{inv}(\pi) = |\{(i,j) \ : \ 1 \leq i < j \leq n \text{ and } \pi_i > \pi_j\}|, \qquad \mathrm{maj}(\pi) = \sum_{i \in D(\pi)} i,$$

where $D(\pi) = \{i : 1 \leq i \leq n-1 \text{ and } \pi_i > \pi_{i+1}\}$. Prove that

$$\sum_{\pi \in \mathcal{S}_n} t^{\mathrm{inv}(\pi)} = \sum_{\pi \in \mathcal{S}_n} t^{\mathrm{maj}(\pi)}$$

2. Prove that

$$\sum_{\pi \in \mathcal{S}_n} \mathrm{maj}(\pi) = n! \frac{n(n-1)}{4}.$$

3. Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$ with $a_i \leq b_i$ for all i. Let $\sigma, \tau \in \mathcal{S}_n$ be such that $a_{\sigma(1)} \leq a_{\sigma(2)} \leq \cdots a_{\sigma(n)}$ and $b_{\tau(1)} \leq b_{\tau(2)} \leq \cdots b_{\tau(n)}$. Prove that $a_{\sigma(i)} \leq b_{\tau(i)}$ for all i.

4. Let $P_{n,k}$ be the 'slice' of the n-dimensional cube $0 \leq x_i \leq 1$, $i = 1, \ldots, n$, between the hyperplanes $\sum_{i=1}^n x_i = k$ and $\sum_{i=1}^n x_i = k+1$. Prove that

$$n! \, \mathrm{Vol}(P_{n,k}) = \#\{\pi \in S_n : \pi \text{ has } k \text{ descents}\}.$$

5. A proper coloring of $\mathbb{R}^2$ with k colors is a map $f : \mathbb{R}^2 \longrightarrow \{1, 2, \ldots, k\}$ such if $x, y \in \mathbb{R}^2$ are at (Euclidean) distance 1, then $f(x) \neq f(y)$. Let c be the minimum number of colors needed in a proper coloring of $\mathbb{R}^2$. Prove that $4 \leq c \leq 7$.