Math 31 Lesson Plan

Day 12: Logic and Proofwriting

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Supplies needed:

- Colored chalk
- Quizzes!

Goals for students: Students will:

- Get more practice with rephrasing mathematical statements (including statements that require two things)
- Improve their understanding of what is (and isn't) the negation of a statement
- Develop their proofwriting skills.

[**Lecture Notes: Write everything in blue, and every equation, on the board. [Square brackets] indicate anticipated student responses. *Italics* are instructions to myself.**]

Return quizzes. Explain why I always want somebody new to participate in class.

Today I want to talk about three things:

- Strategies for starting a proof
- Negating mathematical statements
- Statements whose proof requires two pieces

These, and other hints for learning how to write good proofs, are also discussed on http://www.d.umn.edu/ jgallian/Proofs.html, which is a website written by the author of the supplementary textbook. I'd really recommend that you check it out. I meant to assign this as reading last night and I forgot – oops! So, please read it, maybe before the next homework is due.

Discussion When you're presented with a statement to prove, how do you start? What do you do? Start in small gps? 1:10

The things I think are most important: *write their ideas on board too*

- Rephrase what you have to prove (conclusions) as many ways as possible
- Rewrite hypotheses as many ways as possible
- Apply theorems!

Often, a helpful way to rewrite the hypotheses and conclusions is to use their negative, their opposite statement. We've seen this when writing proofs by contradiction: you assume the desired conclusion is false and see where you can go from there. So let's talk about negating mathematical statements.

First, though, what do I mean by a mathematical statement? A mathematical statement is a sentence, that can be either true or false. So, is $n^3 - n$ a statement? [no] Is "3 divides $n^3 - n$" a statement? [yes]

Here are several statements to negate. Grab a partner – someone you haven't worked with before – and try to write down the opposite or falsification of these statements. Try to rephrase them as many ways as possible.

- A group G has an element of order $|G|$.
- The order $o(g)$ of an element of a group G satisfies $o(g) \leq |G|$.
- Either a group G is infinite or G has an element of finite order.
- H is a subgroup of G and $|H| < |G|$.
- For every element g of a group G, $o(g) \mid |G|$.
- The integer k is the smallest integer such that $x^k \in H$.
- A group G has a subgroup of order g if $g \mid |G|$.

After groups have worked for a while, have the groups pair up and compare their negation statements.

As a class, discuss negations of any contentious statements. 1:30

- In math, "or" means "Either one or the other, or possibly both."
- The negative of "or" is "Neither."
- The negative of "and" is "Not both" – ie, "At least one of them is false."
- The negative of "there exists" is "for all," and vice versa.

We've seen examples of where negations are useful when doing proofs by contradiction. Does anyone know of any other proof techniques where being able to negate a statement is useful? [Proof by contrapositive.]

To prove the statement $A \Rightarrow B$ by contrapositive, we show that "Not B implies Not A." Can someone explain to me why this proves the original statement? *Think-pair-share if necessary* [If A does not imply B, then we would have to have some situation when A is true but B is false. However, if we have proved that "Not B implies Not A," this can never happen, since B false implies A false.]

So let's do an example. This is part of Theorem 5.4.

Claim: If H, K are subgroups of G and $H \cup K \leq G$, then either $H \subseteq K$ or $K \subseteq H$.

If we want to prove this by contrapositive, how would we rephrase this statement to do that? [We will use proof by contrapositive. In other words, we will assume that $H \not\subseteq K$ and $K \not\subseteq H$, and try to show that $H \cup K \not\leq G$.]

Please try to finish this proof in your groups.

Are there any questions about proof by contrapositive?

I want to talk about one more type of proof before class is over.

Sometimes, when trying to prove a statement, it looks like there's only one thing to prove, but really when you rewrite the question it turns out that there are two. An example would be "Show that $S = T$." What are the two things that we have to show here?

Can you think of any more sorts of statements where you have to prove two things? *Think-pair-share*

Statements whose proof requires proving two things:

- The order of $x \in G$ is n (must show $x^n = e$ and that if $x^k = e$ then $k \geq n$)
- For two statements A and B, $A \Leftrightarrow B$ (must show $A \Rightarrow B$ and $B \Rightarrow A$)
- g is the unique element such that $P(g)$ is true for some statement P (must show $P(g)$ is true, and that if $P(h)$ is also true then $h = g$.)

Here's an example for you to prove: This is problem 5.6(a) in the textbook.

Let G be a cyclic group of size n. Prove that if $m \in \mathbb{Z}^+$, then G has an element of order m iff $m|n$.