A Quick Introduction to Row Reduction

1 Gaussian Elimination

Suppose we are asked to solve the system of equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 4x_1 + 5x_2 + 6x_3 &= 7 \\ 6x_1 + 7x_2 + 8x_3 &= 9. \end{aligned}$$

That is, we want to find all values of x_1, x_2 and x_3 so that these equations are simultaneously satisfied. How do we go about this? One procedure for doing so is called Gaussian elimination. By adding multiples of the equations to each other we hope to simplify the system enough that we can just read off the solutions. For example, if we replace the second equation by itself plus -4 times the first equation we get the system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ -3x_2 - 6x_3 &= -9 \\ 6x_1 + 7x_2 + 8x_3 &= 9. \end{aligned}$$

This system is simpler than the first in that its second equation no longer involves x_1. If we now multiply the first equation by -6 and add it to the third equation we get

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ -3x_2 - 6x_3 &= -9 \\ -5x_2 - 10x_3 &= -15. \end{aligned}$$

Now we have eliminated x_1 from both the first and second equations. Now let's see if we can't get x_2 out of some of the others. We can multiply the second equation by $-1/3$ to get

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ x_2 + 2x_3 &= 3 \\ -5x_2 - 10x_3 &= -15. \end{aligned}$$

Now if we multiply the second equation by 5 and add it to the third equation the system becomes

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ x_2 + 2x_3 &= 3 \\ 0 &= 0. \end{aligned}$$

The third equation is clearly redundant, but we leave it anyway. We can eliminate x_2 from the first equation by adding -2 times the second equation:

$$\begin{aligned} x_1 - x_3 &= -2 \\ x_2 + 2x_3 &= 3 \\ 0 &= 0. \end{aligned}$$

At this point it should be clear that any choice we make for x_3 results in a solution to the system by setting $x_1 = -2 + x_3$ and $x_2 = 3 - 2x_3$. Another way to put it is:

$$\begin{aligned} x_1 &= -2 + x_3 \\ x_2 &= 3 - 2x_3 \\ x_3 &\quad \text{is free.} \end{aligned}$$

The procedure we just went through is justified because at each step we produced a system with the same solutions as the original system. This is true in general. If we are trying to solve a system of equations, the following operations on the system do not alter the solutions:

(1) Adding a multiple of one equation to another.

(2) Multiplying an equation by a *nonzero* constant.

(3) Changing the order of the equations.

The fundamental idea behind Gaussian elimination is to use these operations to successively eliminate variables from the equations until it is clear what the solutions are. In particular we try to proceed as follows: use the first equation to eliminate x_1 from all the other equations; then use the second equation to eliminate x_2 from all the other equations; then use the third equation to eliminate x_3 from the other equations; etc.

Let's look at another example. Consider the system

$$\begin{aligned} -2x_1 + 4x_2 + 5x_3 - 5x_4 &= 3 \\ x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\ 3x_1 - 6x_2 - 6x_3 + 8x_4 &= -3. \end{aligned}$$

Since it will be easier to eliminate x_1 using the second equation, we move it to the top:

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 &= 3 \\ 3x_1 - 6x_2 - 6x_3 + 8x_4 &= -3. \end{aligned}$$

We now add 2 times the first equation to the second to get

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\ 3x_3 + x_4 &= 3 \\ 3x_1 - 6x_2 - 6x_3 + 8x_4 &= -3 \end{aligned}$$

and now add -3 times the first equation to the third:

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\ 3x_3 + x_4 &= 3 \\ -3x_3 + -x_4 &= -3. \end{aligned}$$

Since x_2 is missing from the second and third equations we move on to x_3. That is, we use the second equation to get rid of x_3 in the first and third. Adding the second equation to the third gives

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\ 3x_3 + x_4 &= 3 \\ 0 &= 0. \end{aligned}$$

We now multiply the second equation by 1/3:

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\ x_3 + (1/3)x_4 &= 1 \\ 0 &= 0 \end{aligned}$$

and then add the second equation to the first:

$$\begin{aligned} x_1 - 2x_2 + (10/3)x_4 &= 1 \\ x_3 + (1/3)x_4 &= 1 \\ 0 &= 0 \end{aligned}$$

At this point we see that for any choice of x_2 and x_4 we have a solution. Therefore the solution is

$$\begin{aligned} x_1 &= 1 + 2x_2 - (10/3)x_4 \\ x_2 & \quad \text{is free} \\ x_3 &= 1 - (1/3)x_4 \\ x_4 & \quad \text{is free.} \end{aligned}$$

2 Row Reduction

We can streamline the Gaussian elimination process by collecting all of the numbers appearing in our system in a matrix and dealing exclusively with it.

The first system above

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 4x_1 + 5x_2 + 6x_3 &= 7 \\ 6x_1 + 7x_2 + 8x_3 &= 9. \end{aligned}$$

gives rise to the *augmented matrix*

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{pmatrix}$$

The first three columns correspond to the coefficients in front of the variables whereas the last column represents the numbers to the right of the equals signs. We now carry out Gaussian elimination on this matrix. The steps are the same as before, but instead of equations we talk about rows. We add -4 times the first row to the second row to get

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 6 & 7 & 8 & 9 \end{pmatrix}$$

and then add -6 times the first row to the third:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{pmatrix}.$$

Now we multiply the second row by $-1/3$, yielding

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -5 & -10 & -15 \end{pmatrix}.$$

We then add 5 times the second row to the third:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and add -2 times the second row to the first:

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

At this point we can turn this back into a system, remembering that the numbers in the first three columns are the coefficients of the variables and the last column is what goes to the right of the equals signs:

$$\begin{aligned} x_1 - x_3 &= -2 \\ x_2 + 2x_3 &= 3 \\ 0 &= 0. \end{aligned}$$

This is exactly what we ended up with before.

The process of Gaussian elimination when applied to a matrix is typically called *row reduction*. We can now rephrase the allowable steps in solving a system of equations in terms of rows of a matrix:

(1) Adding a multiple of one row to another.

(2) Multiplying a row by a *nonzero* constant.

(3) Changing the order of the rows.

These operations are called *elementary row operations*. Since they correspond to the operations that we are allowed to perform on a system of equations we have the following fact: *If a matrix A represents a system of linear equations, then any matrix gotten from A by elementary row operations represents the same system.*

3 The Matrix Equation $A\mathbf{x} = \mathbf{b}$.

Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{pmatrix}, \ , \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Recall the definition of matrix-vector multiplication:

$$A\mathbf{x} = x_1 \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 + x_3 \\ -3x_1 - x_2 + 2x_3 \\ 5x_2 + 3x_3 \end{pmatrix}.$$

We therefore see that the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

is *exactly the same* as the system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ -3x_1 - x_2 + 2x_3 &= 1 \\ 5x_2 + 3x_3 &= -1 \end{aligned}$$

Therefore, solving $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$ is exactly the same as solving this system of equations, and this we know how to do via row reduction. We form the augmented matrix and row reduce. The augmented matrix is

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ -3 & -1 & 2 & 1 \\ 0 & 5 & 3 & 0 \end{pmatrix}.$$

Add 3 times row 1 to row 2:

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 5 & 3 & -1 \end{pmatrix}.$$

Add -1 times row 2 to row 3:

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 0 & -2 & -2 \end{pmatrix}.$$

Multiply row 3 by -1/2:

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

Add -5 times row 3 to row 2:

$$\begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Add -1 times row 3 to row 1:

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 5 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Multiply row 2 by 1/5:

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -4/5 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Finally, add -2 times row 2 to row 1:

$$\begin{pmatrix} 1 & 0 & 0 & 3/5 \\ 0 & 1 & 0 & -4/5 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

This final matrix is equivalent to the system

$$\begin{aligned} x_1 &= 3/5 \\ x_2 &= -4/5 \\ x_3 &= 1. \end{aligned}$$

In other words, the solution to $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = \begin{pmatrix} 3/5 \\ -4/5 \\ 1 \end{pmatrix}.$$

This example demonstrates a slight variant of the Gaussian elimination method from the first section. Namely (thinking in terms of equations again) we try to use the first equation to eliminate x_1 from all equations below. Then we try to use the second equation to eliminate x_2 from all the equations *below*. Then we try to use the third equation to eliminate x_3 from all the equations *below*, etc.. Once we've done this, we work from the bottom row up, eliminating variables as we go.

A final note: when we try to solve a system of the form $A\mathbf{x} = \mathbf{0}$, we often just row reduce A without building an augmented matrix. This is because row operations on the augmented matrix $\begin{pmatrix} A & \mathbf{0} \end{pmatrix}$ wouldn't change the last column: it will always be a column of zeros. Therefore, we leave off the last column and just "imagine" that it's there. For example, if we wanted to solve

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{x} = \mathbf{0}$$

we would just row reduce

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

Add row 1 to row 2:

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Now we remember that we're supposed to imagine an extra column of zeros. Therefore, this is equivalent to the system

$$\begin{aligned} x_1 + x_2 &= 0 \\ 0 &= 0. \end{aligned}$$

It is clear that x_2 can be anything and $x_1 = -x_2$. In other words, the solutions to $\mathbf{Ax} = \mathbf{0}$ are

$$\mathbf{x} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

where x_2 is free to take on any value.