Math 118. Combinatorics.

Problem Set 4. Due on Friday, 3/4/11.

1. Fix $t \geq 0$. Show that $p_{n-t}(n)$ becomes eventually constant as $n \to \infty$. What is this constant c_t? What is the least value of n for which $p_{n-t}(n) = c_t$?

2. Show that for any partition λ,
$$\sum_i (i-1)\lambda_i = \sum_i \binom{\lambda_i'}{2},$$
where the λ_i' denote the parts of the conjugate partition λ'.

3. Prove that
$$\prod_{i\geq 0}(1+q^{2i+1}) = \sum_{k\geq 0} \frac{q^{k^2}}{(1-q^2)\cdots(1-q^{2k})},$$

4. How many SYT of shape (n^n) have main diagonal $(1, 4, 9, 16, \ldots, n^2)$?

5. Let $f^{\lambda/2}$ denote the number of SYT of shape λ having the entry 2 in the first row. Evaluate the sums
$$\sum_{\lambda\vdash n} f^{\lambda/2} f^{\lambda} \qquad \text{and} \qquad \sum_{\lambda\vdash n}\left(f^{\lambda/2}\right)^2.$$

6. Let M be a random $n \times n$ matrix with entries in the finite field $\mathbb{F}_q$, where each entry is chosen uniformly and independently at random. Show that with probability at least $1/4$, M is non-singular (i.e., it has nonzero determinant).
Hint: The generating function for pentagonal numbers may be surprisingly useful here.

7. Show that the (ordinary) generating function for Dyck paths D whose peak heights strictly increase from left to right is
$$\sum_D q^{|D|} = \sum_{k\geq 0} q^k [k]_q!,$$
where $[k]_q! = (1+q)(1+q+q^2)\ldots(1+q+\cdots+q^{k-1})$, and $|D|$ is half of the number of steps of D.