HOMEWORK 3

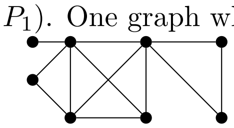
SOLUTIONS

1.3.8 Which of the following are graphic sequences? Provide a construction of a proof of impossibility for each.

(a) $(\mathbf{5}, \mathbf{5}, \mathbf{4}, \mathbf{3}, \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1})$: Using the iteration from Theorem 1.3.31, you get

$$\begin{aligned}(5,5,4,3,2,2,2,1) &\to (4,3,2,1,1,2,1) = (4,3,2,2,1,1,1)\\ &\to (2,1,1,0,1,1) = (2,1,1,1,1,0)\end{aligned}$$

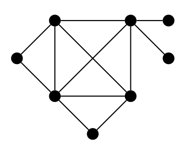
the last of which is graphic ($P_3 + P_2 + P_1$). One graph which has this sequence is



(b) $(\mathbf{5}, \mathbf{5}, \mathbf{4}, \mathbf{4}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1})$: Using the iteration from Theorem 1.3.31, you get

$$\begin{aligned}(5,5,4,4,2,2,1,1) &\to (4,3,3,1,1,1,1)\\ &\to (2,2,0,0,1,1) = (2,2,1,1,0,0)\end{aligned}$$

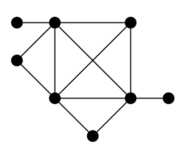
the last of which is graphic ($P_4 + P_1 + P_1$). One graph which has this sequence is



(c) $(\mathbf{5}, \mathbf{5}, \mathbf{5}, \mathbf{3}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1})$: Using the iteration from Theorem 1.3.31, you get

$$\begin{aligned}(5,5,5,3,2,2,1,1) &\to (4,4,2,1,1,1,1) \to (3,1,0,0,1,1) = (3,1,1,1,0,0)\\ &\to (0,0,0,0,0)\end{aligned}$$

the last of which is graphic (5 vertices w. no edges). One graph which has this sequence is



(d) $(\mathbf{5}, \mathbf{5}, \mathbf{5}, \mathbf{4}, \mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1})$: Using the iteration from Theorem 1.3.31, you get

$$\begin{aligned}(5,5,5,4,2,1,1,1) &\to (4,4,3,1,0,1,1) = (4,4,3,1,1,1,0)\\ &\to (3,2,0,0,1,0) = (3,2,1,0,0,0)\end{aligned}$$

which is not graphic since there are not three vertices of non-zero degree to connect the degree vertex to. So (5,5,5,4,2,1,1,1) is not graphic.

1.3.18 For $k \geq 2$, prove that a k-regular bipartite graph has no cut-edge.

Proof. Notice that any component of G is also k-regular and bipartite, so we'll assume, without loss of generality, that G is connected. Suppose there is a cut edge e of G, and consider a H component of $G - e$. Then H is also bipartite, and has $n(H) - 1$ vertices of degree k and one vertex v of degree $k - 1$. But if H has partites X and Y (say $v \in Y$), this means that H simultaneously has $|X| * k$ and $|Y| * k - 1$ edges. So

$$k(|Y| - |X|) = 1$$

which is a contradiction since $k \geq 2$. □

1.3.25 Prove that every cycle of length $2r$ in a hypercube is contained in a subcube of dimension at most r. Can a cycle of length $2r$ be contained in a subcube of dimension less than r?

Proof. Consider the set of bits which are not shared amongst all of the vertices of the cycle C. The cycle sits in the subcube generated by varying only those bits which vary in C. There are at most r of them since the farthest a vertex can be from any other vertex in C half the length of the cycle. So C sits inside some copy of Q_r.

On the other hand,
(0,0,0) — (1,0,0) — (1,1,0) — (1,1,1) — (1,0,1) — (0,0,1) — (0,1,1) — (0,1,0) — (0,0,0)
is a cycle of length $8 = 2 * 4$ in a hypercube of dimension 3. (So yes, but not as an induced subgraph. Also, this is the smallest example since $Q_2 \cong C_4$.)

□

1.3.32 Prove that the number of simple even graphs with vertex set $[n]$ is $2^{\binom{n-1}{2}}$.
(Hint: establish a bijection with the set of all simple graphs on $V(G) = [n-1]$.)

Proof. There is a bijection between simple graphs on $[n-1]$ and even simple graphs on $[n]$ given by

$$\{\text{ simple graphs on } [n-1]\} \to \{\text{ simple even graphs on } [n]\}$$
$$G \mapsto G + v_n + \{v_n v_i \mid d(v_i) \text{ is odd}\}$$

(connect the new vertex to every odd vertex in G; since there were an even number of these, not only is the new degree on each odd v_i, but the degree of v_n is also even). The inverse of this map is

$$\{\text{ simple even graphs on } [n]\} \to \{\text{ simple graphs on } [n-1]\}$$
$$G \mapsto G - v_n.$$

(Notice that we're interested in *graphs, not isomorphism classes of graphs,* so both of these maps are one-to-one). Since there are $2^{\binom{n-1}{2}}$ simple graphs on $[n-1]$, there are also $2^{\binom{n-1}{2}}$ simple even graphs on $[n]$

□

1.3.44 Let G be a loopless graph with average vertex degree $a = 2e(G)/n(G)$.

(a) **Prove that $G - x$ has average degree at least a if and only if $d(x) \leq a/2$.**

Proof. Let $n = n(G)$, $k = e(G)$, and $d = d_G(x)$. Since $n(G - x) = n - 1$ and $e(G - x) = k - d$, the average degree in $G - e$ is $2(k - d)/(n - 1)$. This is greater than or equal to a exactly when

$$n(k - d) \geq (n - 1)k \quad \Leftrightarrow \quad nd \leq k \quad \Leftrightarrow \quad d \leq k/n = a/2.$$

□

(b) **Use part (a) to give an algorithmic proof that if $a > 0$, then G has a subgraph with minimum degree greater than $a/2$.**

If G is a regular graph, then we're done. If not, then there is some vertex v of degree less than $a/2$. By (a), $G - v$ has average degree at least $a/2$. Continue deleting vertices of degree less than $a/2$; each time the average degree of the remaining graph G' will go up since we are only deleting vertices of degree $d < \frac{1}{2}a \leq \frac{1}{2}\left(2e(G')/n(G')\right)$. Iterate until there are no more vertices of degree less than $a/2$; we will not run out of vertices to remove since, at every step,

$$n(G') > \Delta(G') \geq 2e(G')/n(G') \geq a/2.$$

(c) **Show that there is no constant c greater than $1/2$ such that G must have a subgraph with minimum degree greater than ca; this proves that the bound in part (b) is best possible.** (Hint: use $K_{1,n-1}$.)

Proof. Consider the star on n vertices. This has average degree $2(n - 1)/n < 2$, and so it's not possible to remove a vertex and raise the average degree: by removing vertices, it is only possible to drop the average degree. Moreover, every subgraph of G has $\delta(G) \leq 1$. Since

$$\lim_{n\to\infty} 2\frac{n-1}{n} = 2, \qquad \text{we have } \lim_{n\to\infty} ca = 2c,$$

and so for any $c \geq \frac{1}{2}$, there is some n for which $ca > 1$. □

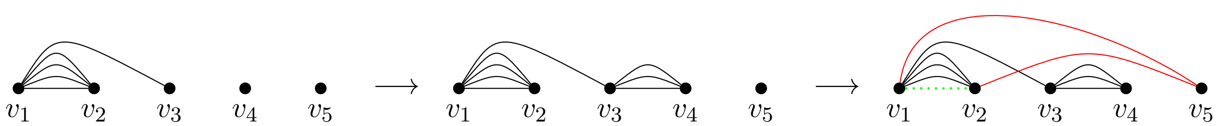
1.3.63 Let $d_1, \ldots, d_n$ be integers such that $d_1 \geq \cdots \geq d_n \geq 0$. Prove that there is a loopless graph (multiple edges allowed) with degree sequence $d_1, \ldots, d_n$ if and only if $\sum_i d_i$ is even and $d_1 \leq d_2 + \cdots + d_n$. (Hakimi [1962])

Proof. If $\sum_i d_i$ is odd, then $d_1, \ldots, d_n$ cannot be a degree sequence. Since there are no loops, there can be at most $D = d_2 + \cdots + d_n$ edges in a graph with degree sequence $d_1, \ldots, d_n$, so $d_1 \leq D$.

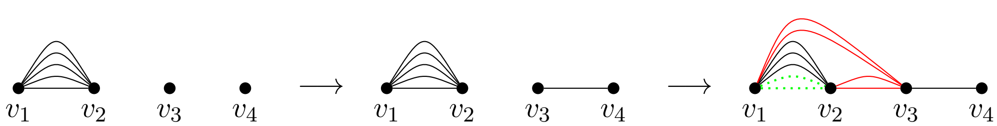
Now assume that $\sum_i d_i$ is even and $d_1 \leq d_2 + \cdots + d_n$. Start with vertices $v_1, \ldots, v_n$. Add d_2 edges incident to v_1 and v_2, then up to d_3 edges to v_3, and so on, until we've added d_1 edges in total. This is possible by our hypothesis.

Now take the last vertex v_j which is not "full" and add edges from v_j to v_{j+1}, and so on until v_j is "full". Iterate this process until there is at most one vertex v_ℓ which is not full (maybe $\ell = j$, but $\ell \neq 2$ since $d_2 \leq d_1$). The number of edges N remaining to be added to v_ℓ is even (since $\sum_i d_i$ is even, and N is $\sum_i d_i$ minus the sum of the degrees in the graph so far). There are at least N edges joining v_1 to v_2 since $N \leq d_\ell \leq d_2$; to avoid loops, remove N edges connecting v_1 and v_2 and add $N/2$ edges connecting v_ℓ to each of v_1 and v_2.

For example, consider the sequence $(5, 4, 4, 3, 2)$, the first few iterations look like



Alternatively, consider $(5, 5, 5, 1)$ (to see why ℓ isn't always n):

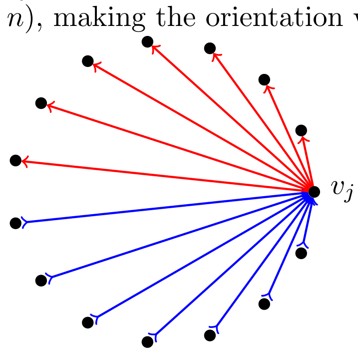


$\square$

1.4.8 Prove that there is an n-vertex tournament with in-degree equal to out-degree at every vertex if and only if n is odd.

Proof. If the in-degree is equal to the out-degree, then, in particular, the underlying graph of the tournament is even. This only happens when n is odd. (The underlying graph is a complete graph, and is regular with degree $n - 1$.

Now assume n is odd. One tournament with the same in and out degrees at each vertex is as follows: Label the vertices $v_1, \ldots, v_n$. For each v_i, direct the edges in common with $v_{i+1 \pmod n}, v_{i+2 \pmod n}, \ldots, v_{i+(n-1)/2 \pmod n}$ away (giving it an out-degree of half the available edges). So any vertex v_j points to its $(n-1)/2$ successors and is pointed to by its $(n-1)/2$ predecessors (mod n), making the orientation well-defined.



□

1.4.10 Prove that a digraph is strongly connected if and only if for each partition of the vertex set into non-empty sets S and T, there is an edge from S to T.

Proof. If a digraph is strongly connected, then every pair of vertices u and v has a u, v-path. Now take any two partitions of $V(G)$ into sets S and T, and select vertices $s \in S$ and $t \in T$. Then let P be a s, t-path, and consider the last vertex in P which is not in T. That vertex and its successor in P are joined by an edge from S to T (it has a successor since P's last vertex is in T).

Now assume that for each partition of the vertex set into non-empty sets S and T, there is an edge from S to T. Consider any two vertices u and v; we can construct a u, v-path as follows. Partition $V(G)$ into $S = \{u\}$ and $T = V(G) - u$. Then there is an edge out of u by our assumption. If v is at the tip of one such edge, then we are done. At each step, add vertices which are successors of S, removing them from T (there is at least one, since there is at least one by our hypothesis). As soon as an edge connects to v, we have recovered a u, v-path. □

1.4.14 Let G be an n-vertex digraph with no cycles. Prove that the vertices of G can be ordered as $v_1, \ldots, v_n$ so that if $v_iv_j \in E(G)$, then $i < j$.

Proof. What we are concerned with is the transitivity of order, i.e. a sequence of inequalities produces a new inequality. A sting of inequalities produced by our criteria is equivalent to a walk in G; since G is acyclic, every walk is a u, v-path. So our criteria for the ordering is equivalent to requiring that whenever v_i appears before v_j in any path, i must be less than j. If there is no such order, then there must be some pair of vertices u and v for which there is both a u, v-path and a v, u-path. However, this would produce a cycle (follow the u, v path until it intersects with the v, u-path and then follow the v, u-path back). So there must be some ordering which agrees with all paths. □

2.1.2 Let G be a graph.

(a) **Prove that G is a tree if and only if G is connected and every edge is a cut edge.**

Proof. Recall that an edge is a cut edge if an only if it is part of a cycle. Then every edge is a cut edge if and only if G is acyclic. Since G is a tree if and only if G is connected and acyclic, G is a tree if and only if it is connected and every edge is a cut edge. □

(b) **Prove that G is a tree if and only if G is loopless and adding any edge with endpoints in $V(G)$ creates exactly one cycle.**

Proof. By cor ???, adding any edge to a tree produces a cycle. Now suppose adding any edge e produces a cycle. Then G must be connected (since e is not a cut edge of $G + e$, and an edge with one endpoint in one component and the other in a separate component would be a cut edge). Now suppose G already has a cycle C. Then adding an edge e with both endpoints adds at least two cycles: the edge separates in C into two paths; each of the new cycles is one of those two paths, closed off by e.

□

2.1.18 Prove that every tree with maximum degree $\Delta > 1$ has at least Δ leaves. Show that this is the best possible by constructing an n-vertex tree with exactly Δ leaves foe each choice of n, Δ with $n > \Delta \geq 2$.

Proof. If $\Delta = 0$ or 1, then $G = P_1$ or P_2. Otherwise, pick a vertex v of degree Δ. Then $G - v$ contains all of the leaves of G. It also has (at least) Δ components since there is at most one path connecting any two of the vertices adjacent to v, which must be the path going through v. Each of those components has at least one leaf (or one isolated point) and is itself a tree. Since v removed (at most) one edge incident to each resulting component, each component has at least one leaf (or isolated vertex) which was also a leaf of G. So G has at least Δ leaves.

□