# Math 68. Algebraic Combinatorics.

**Problem Set 3.** Due on Thursday, 11/8/2007.

1. Prove that
$$\frac{1}{1-z} = \prod_{j\geq 0}(1+z^{2^j}).$$

2. A $(0,1)$-necklace of length $n$ and weight $i$ is a circular arrangement of $i$ 1's and $n-i$ 0's. For instance, the $(0,1)$-necklaces of length 6 and weight 3 are (writing a circular arrangement linearly) 000111, 001011, 010011, and 010101. (Cyclic shifts of a linear word represent the same necklace, e.g., 000111 is the same as 110001.) Let $N_n$ denote the set of all $(0,1)$-necklaces of length $n$. Define a partial order on $N_n$ by letting $u \leq v$ if we can obtain $v$ from $u$ by changing some 0's to 1's. It's easy to see (you may assume it) that $N_n$ is graded of rank $n$, with the rank of a necklace being its weight. Show that $N_n$ is rank-symmetric, rank-unimodal, and Sperner.

3. Let $q$ be a prime power, and let $V$ be an $n$-dimensional vector space over $\mathbb{F}_q$. Let $B_n(q)$ denote the poset of all subspaces of $V$, ordered by inclusion. It's easy to see that $B_n(q)$ is graded of rank $n$, the rank of a subspace of $V$ being its dimension.

   (a) Show that the number of elements of $B_n(q)$ of rank $k$ is given by the $q$-binomial coefficient
   $$\left[\begin{array}{c} n \\ k \end{array}\right] = \frac{(q^n-1)(q^{n-1}-1)\dots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\dots(q-1)}.$$
   (One way to do this is to count in two ways the number of $k$-tuples $(v_1,\dots,v_k)$ of linearly independent elements from $\mathbb{F}_q^n$: (1) first choose $v_1$, then $v_2$, etc., and (2) first choose the subspace $W$ spanned by $v_1,\dots,v_k$, and then choose $v_1$, $v_2$, etc.)

   (b) Show that $B_n(q)$ is rank-symmetric. (You can use (a) if you want.)

   (c) Show that every element $x \in B_n(q)_k$ covers $[k] = 1+q+\cdots+q^{k-1}$ elements and is covered by $[n-k] = 1+q+\cdots+q^{n-k-1}$ elements.

   (d) Define operators $U_i : \mathbb{R}B_n(q)_i \to \mathbb{R}B_n(q)_{i+1}$ and $D_i : \mathbb{R}B_n(q)_i \to \mathbb{R}B_n(q)_{i-1}$ by
   $$U_i(x) = \sum_{\substack{y\in B_n(q)_{i+1}\\ y>x}} y, \qquad D_i(x) = \sum_{\substack{z\in B_n(q)_{i-1}\\ z<x}} z.$$
   Show that $D_{i+1}U_i - U_{i-1}D_i = ([n-i]-[i])I_i$.

   (e) Deduce that $B_n(q)$ is rank-unimodal and Sperner.

   (f) ** Let $0 \leq i < n/2$. Find an explicit order-matching $\mu : B_n(q)_i \to B_n(q)_{i+1}$.

4. Let $M(n)$ be the set of all subsets of $[n]$, with the ordering $A \leq B$ if the elements of $A$ are $a_1 > a_2 > \cdots > a_j$ and the elements of $B$ are $b_1 > b_2 > \cdots > b_k$, where $j \leq k$ and $a_i \leq b_i$ for $1 \leq i \leq j$. (The empty set $\emptyset$ is the bottom element of $M(n)$.)

(a) Draw the Hasse diagrams (with vertices labeled by the subsets they represent) of $M(1)$, $M(2)$, $M(3)$, and $M(4)$.

(b) Show that $M(n)$ is graded of rank $\binom{n+1}{2}$. What is $\mathrm{rank}(\{a_1, \ldots, a_k\})$?

(c) Define the *rank-generating function* of a graded poset $P$ to be

$$F(P, q) := \sum_{x \in P} q^{\mathrm{rank}(x)}.$$

Show that the rank-generating function of $M(n)$ is given by

$$F(M(n), q) = (1+q)(1+q^2)\cdots(1+q^n).$$

(d) * Show that $M(n)$ is rank-symmetric, rank-unimodal, and Sperner.

5. (a) Find an explicit injection $\mu : L(m,n)_i \to L(m,n)_{i+1}$ for $0 \le i < \frac{1}{2}mn$.

(b) ** Find $\mu$ as in (a) such that $\mu$ is also an order-matching.

(c) ** We say that a graded rank-symmetric poset $P$ of rank $n$ has a *symmetric chain decomposition* if we can write $P$ as a disjoint union of saturated chains $C$, such that each $C$ starts at some $P_i$ and ends at $P_{n-i}$. Show that $L(m,n)$ has a symmetric chain decomposition.