Workshop 4

Matrices and Linear Transformations

Instructions:

Get into groups and work on the following exercises. Each group is expected to turn in *one neatly written copy* of their solutions at the end of the class period.

Exercise 1. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ be vectors in $\mathbb{R}^n$. Show that if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are linearly dependent then $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_p)$ are linearly dependent.

Exercise 2. Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ span $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Suppose that $T(\mathbf{v}_i) = \mathbf{0}$ for $i = 1, 2, \ldots, p$. Show that $T(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

Exercise 3. Let A, B and C be matrices. Assuming that the sizes of the matrices are "compatible" in each case, use the definition of matrix multiplication to prove the following:

a. $A(B + C) = AB + AC$

b. $(A + B)C = AC + BC$

c. $r(AB) = (rA)B = A(rB)$ for any scalar r.

Exercise 4.*

a. Let A be a 2×2 matrix. Suppose that A has the following property: $AB = BA$ for *any* 2×2 matrix B. What can you say about A? *Hint:* Try taking B to be any one of the matrices
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

b. What can you say if A is a 3×3 matrix with the property that $AB = BA$ for all 3×3 matrices B?

c. Can you guess what happens in general? That is, what do you think is true of A if it is an $n \times n$ matrix with the property that $AB = BA$ for all $n \times n$ matrices B?