Lecture Notes

Math 3 / Fall 2012

Week 6

Differential equations

A **first-order differential equation** is an equation involving a variable x, a function y and its derivative $y' = \frac{dy}{dx}$.
A **solution** to such an equation is a function $y = f(x)$ which, together with its derivative, $y' = f'(x)$ fit the equation.

Every antiderivative problem $y' = g(x)$ is a first-order differential equation whose solutions are the antiderivatives of $g(x)$

Example

The solutions to $y' = cos(x)$ are all of the form $y = \sin(x) + C$

The differential equation $y' = ay$

Example

The equation $y' = ay$ has the solution $y = e^{ax}$ and also $y = 3e^{ax}$ and $y = e^{ax+b}$.
More generally, the solutions are $y = Ke^{ax}$ for some constant K.

Because: we can rewrite the equation as $\dfrac{y'}{y} = a$

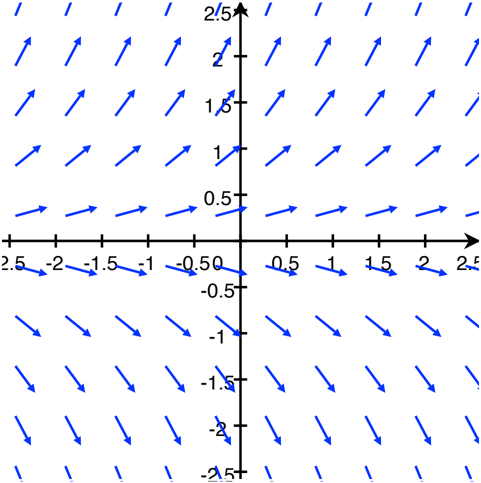
The left-hand side is the derivative of $\ln y$ (assuming $y > 0$)
The right-hand side is the derivative of $ax + b$

Therefore $\ln y = ax + b$ or $y = e^{ax+b} = Ke^{ax}$, where $K = e^b$

A similar anaysis works when $y < 0\ldots$

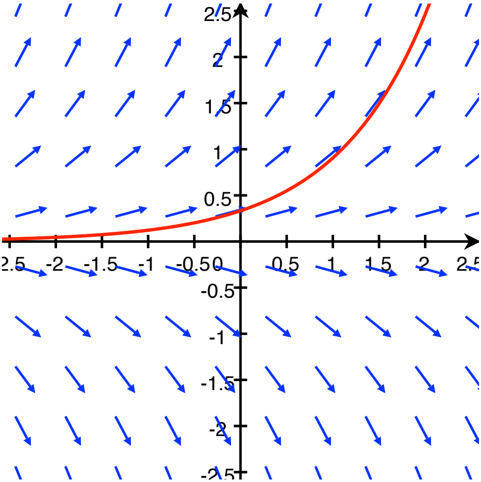
Slope fields

Solving a differential equation for y' tells you what the slope of a solution going through (x, y) ought to be...



Slope fields

To get a graphical solution, just go with the flow!



Separation of variables

Sometimes, we can find analytic solutions...

A **separable differential equation** is one that can be put in the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \qquad \text{or} \quad g(y)\,dy = f(x)\,dx$$

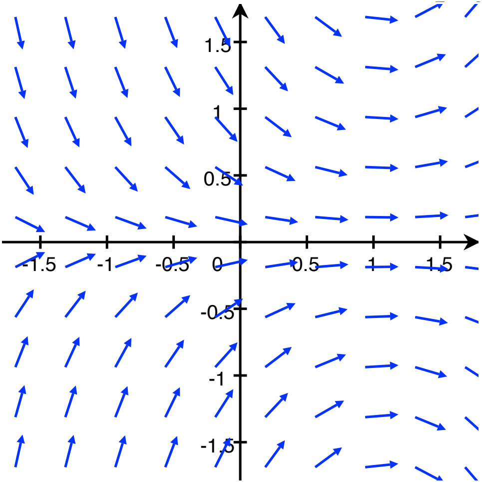
Then, we have

$$\int g(y)\,dy = \int f(x)\,dx$$

and we can find a solution by integrating both sides and solving for y...

Separation of variables: $y' = (x-1)y$

If $\dfrac{dy}{dx} = (x-1)y$ then $\dfrac{dy}{y} = (x-1)\,dx$



Separation of variables: $y' = (x-1)y$

Integrating both sides of $\dfrac{dy}{y} = (x-1)\,dx$, we get

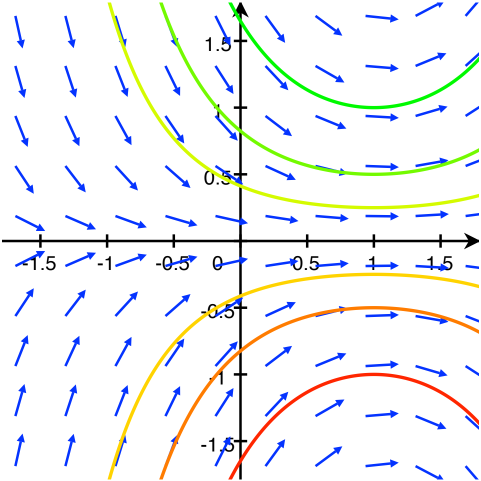
$$\ln y = \frac{x^2}{2} - x + C \qquad (y > 0)$$

Or:

$$y = e^{x^2/2 - x + C} = Ke^{(x-1)^2/2}$$

These are all the solutions to $y' = (x-1)y$

Separation of variables: $y' = (x-1)y$



Existence and uniqueness

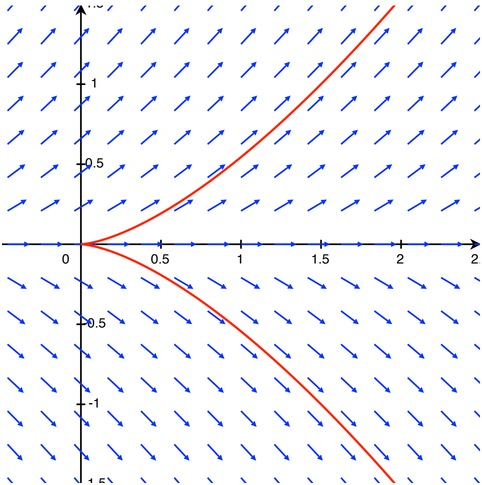
Although there is no simple recipe to find solutions of differential equations, we sometimes know that solutions do exist...

Theorem (Peano Existence/Picard Uniqueness)

Suppose $G(x, y)$ *is a continuous function of two variables. Given any point* (x_0, y_0), *there is a function* f *such that* $f(x_0) = y_0$ *and* $f'(x) = G(x, f(x))$ *when* $|x - x_0| < \delta$. *If, moreover,* $\partial G(x, y)/\partial y$ *is also continuous, then there is exactly one such function* f.

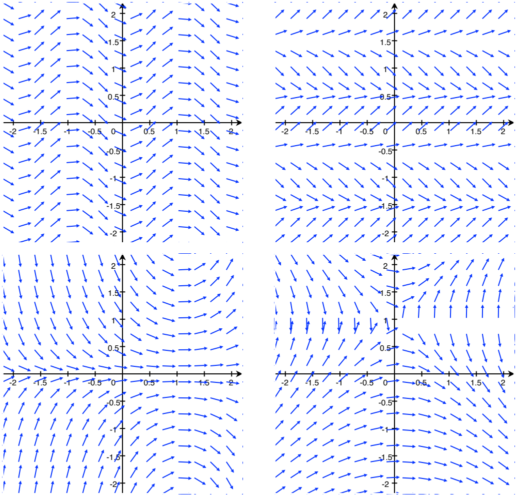
Existence and uniqueness

The equation $y' = y^{1/3}$ has several solutions with $y(0) = 0$



More slope fields...

$$y' = \sin \pi x \quad y' = \cos \pi y \quad y' = (x-1)y \quad y' = \frac{x}{y-1}$$



Separation of variables: explanation

We're looking at a separable equation

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}.$$

This is often written in the form

$$h(y)\,dy = g(x)\,dx$$

which suggests integrating both sides. Sounds reasonable but ...

$$\boxed{\text{Why does this work?}}$$

Separation of variables: explanation

Let $H(y)$ be a particular antiderivative of $h(y)$. In other words:

$$H'(y) = h(y).$$

Then, by the chain rule,

$$\frac{d}{dx}\left[H(y)\right] = h(y)\frac{dy}{dx}.$$

Our original equation is

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}.$$

Separation of variables: explanation

Combining the two, we have

$$\frac{d}{dx}\left[H(y)\right] = g(x).$$

In other words, $H(y)$ is an antiderivative of $g(x)$:

$$\frac{d}{dx}\left[H(y)\right] = g(x) \quad \text{or} \quad H(y) = \int g(x)\,dx.$$

Separation of variables: explanation

Since $H(y) = \int g(x)\,dx$, if $G(x)$ is a particular antiderivative of $g(x)$ then

$$H(y) = G(x) + C$$

for some constant C.

Therefore, every solution y is implicitly defined by

$$H(y) = G(x) + C$$

for some choice of constant C.

More separable equations...

(a) Separate the x's and the y's

(b) Integrate both sides

(c) Solve for y

1. $y' = -\pi$
2. $y' = y^2 x$
3. $y' = yx^2$
4. $y' = \sin(x)\tan(y)$
5. $y' = e^x \cos^2(y)$
6. $y' = x(x-2)/y$
7. $y' = x^2 e^{-y} - 2xe^{-y}$
8. $y'\sqrt{y - x^2 y} = -xy$
9. $x^2 + 2yy' = 3x$
10. $x^2 y' + 2xy = 3x$

Newton's Law of Cooling

$$\frac{dT}{dt} = -k(T - T_{\mathrm{ambient}})$$

Separation of variables leads to

$$\frac{dT}{(T - T_{\mathrm{ambient}})} = -k\,dt.$$

Integrating both sides:

$$\ln|T - T_{\mathrm{ambient}}| = -kt + C$$

Or

$$T = Ae^{-kt} + T_{\mathrm{ambient}}$$

for some constant A.

Exponential growth & decay

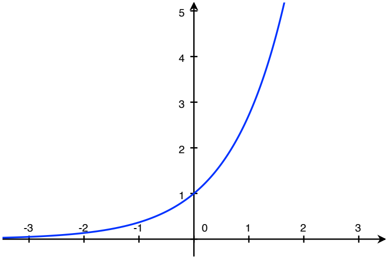
$$y' = Ay$$

$$\boxed{\text{The rate of change of } y \text{ is proportional to } y}$$

The general solution is $y = Ke^{Ax}$ for some constant K

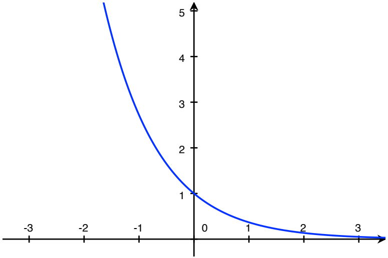
Exponential growth & decay

$A > 0$



Growth

$A < 0$



Decay

Bacterial growth

Suppose a bacteria culture grows at a rate proportional to the number of cells present. If the culture contains 700 cells initially and 900 after 12 hours, how many will be present after 24 hours?

- Model: $N(t) = N_0 e^{kt}$
- $700 = N(0) = N_0 e^{k0}$ so $N_0 = 700$
- $900 = N(12) = N_0 e^{k12}$ so $k = \frac{1}{12} \ln(900/700) = 0.021$
- Therefore: $N(24) = N_0 e^{k24} \approx 1159$

Half life & doubling time

If $f(t) = Ke^{At}$ then the offset t such that

$$B = \frac{f(t_0 + t)}{f(t_0)} = \frac{Ke^{A(t_0+t)}}{Ke^{At_0}} = e^{At}$$

depends only on A and not on t_0. Namely, $t = \dfrac{\ln B}{A}$.

- In an exponential growth scenario ($A > 0$)

  the **doubling time** is $\dfrac{\ln 2}{A}$
- In an exponential decay scenario ($A < 0$)

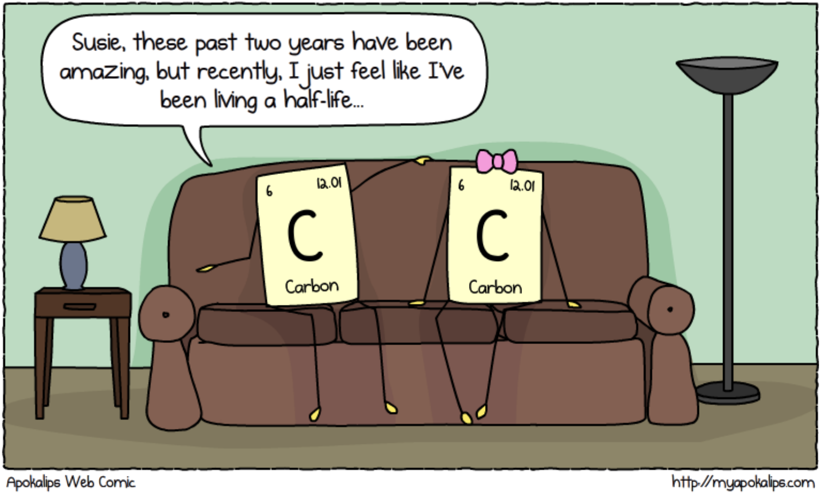
  the **half life** is $\dfrac{\ln 1/2}{A} = -\dfrac{\ln 2}{A}$

Bacterial growth revisited

Suppose a bacteria culture grows at a rate proportional to the number of cells present. If the culture contains 700 cells initially and 900 after 12 hours, how many will be present after 24 hours?

- Model: $N(t) = N_0 e^{kt}$
- $N(12)/N(0) = 900/700$
- $N(24)/N(12) = 9/7$ because $24 - 12 = 12 - 0$
- Therefore: $N(24) = \frac{9}{7} \cdot 900 \approx 1159$

Carbon dating



Carbon dating

The half life of the isotope Carbon-14 is about 5 730 years. If there is only 20% left of the Carbon-14 originally present in an object, how old is the object?

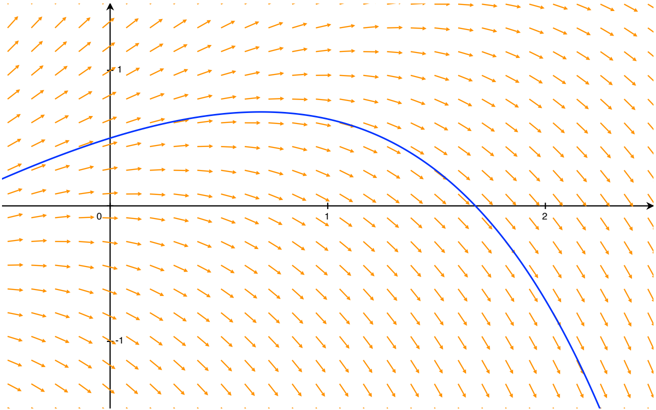
Suppose the amount of Carbon-14 at time t is Ke^{At}

First, $5\,730 = -\dfrac{\ln 2}{A}$ so $A = -1.21 \times 10^{-4}$.

Now, the age of the object is $\dfrac{\ln 0.2}{A}$, about 13 300 years.

Euler's method: graphically

To graphically solve $y' = G(x, y)$, follow the flow on the slope field.



Euler's method: numerically

To numerically solve $y' = G(x, y)$ with $y(x_0) = y_0$, successively compute

$$x_{n+1} = x_n + h, \qquad y_{n+1} = y_n + hG(x_n, y_n)$$

n	x_n	y_n
0	0.0	0.500
1	0.1	0.550
2	0.2	0.595
3	0.3	0.634
4	0.4	0.668
5	0.5	0.695
$\vdots$	$\vdots$	$\vdots$

$G(x, y) = y - x$

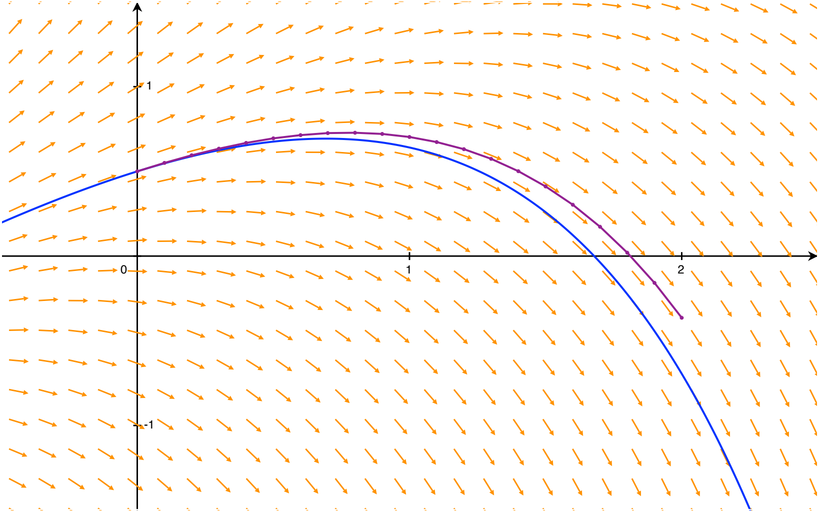
with $h = 0.1$

and $(x_0, y_0) = (0, 1/2)$

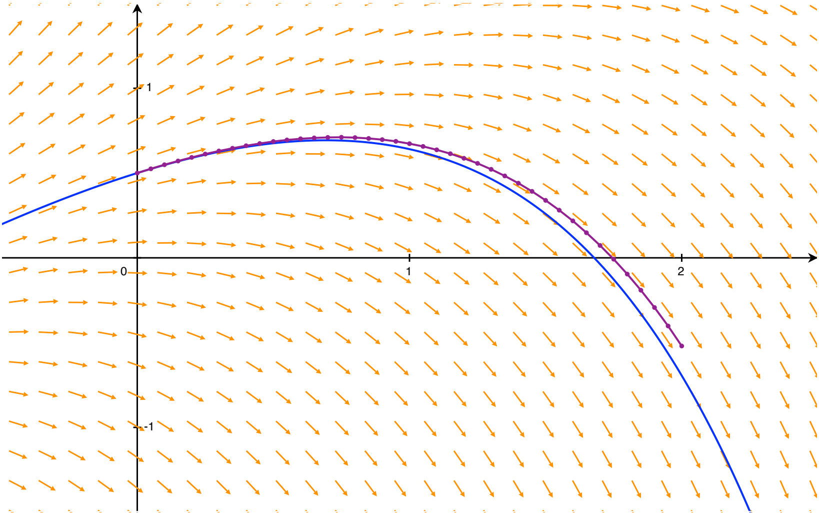
$x_3 = 0.2 + 0.1 = 0.3$

$y_3 = 0.595 + 0.1G(0.2, 0.595) = 0.634$

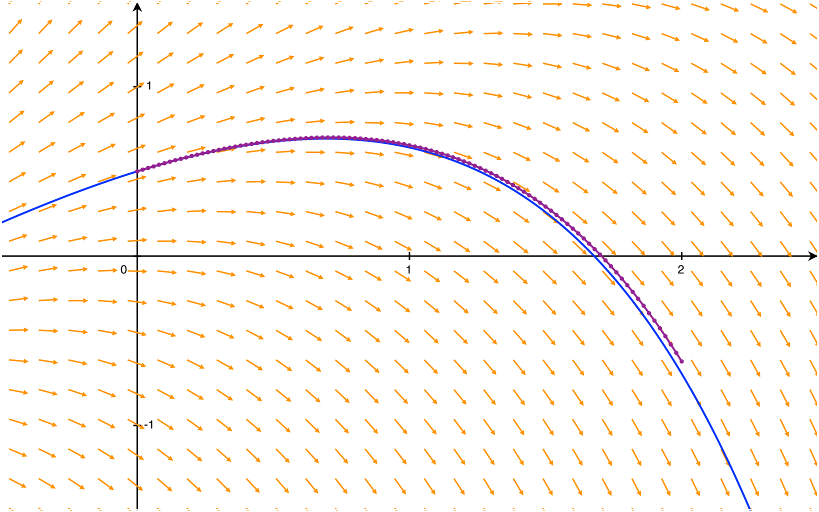
Euler's method: $y' = y - x$ with $h = 0.1$



Euler's method: $y' = y - x$ with $h = 0.05$



Euler's method: $y' = y - x$ with $h = 0.02$



Runge–Kutta method (RK2)

To numerically solve $y' = G(x, y)$ with $y(x_0) = y_0$, successively compute

$$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + hG(x_n + h/2, y_n + k/2)$$

where $k = hG(x_n, y_n)$

n	x_n	y_n
0	0.0	0.500
1	0.1	0.548
2	0.2	0.589
3	0.3	0.625
4	0.4	0.655
5	0.5	0.676
6	0.6	0.690
$\vdots$	$\vdots$	$\vdots$

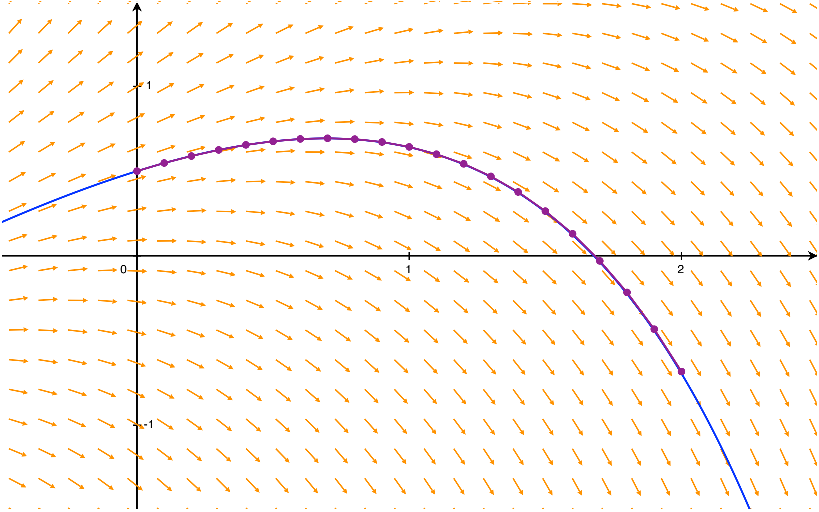
$G(x, y) = y - x$

$h = 0.1 \qquad (x_0, y_0) = (0, 1/2)$

$k = 0.1G(0.2, 0.589) = 0.039$

$$y_3 = 0.589 + 0.1G(0.2 + 0.05, 0.589 + 0.0195)$$
$$= 0.589 + 0.1 \times 0.358 = 0.625$$

RK2 method: $G(x, y) = y - x$ with $h = 0.1$



Runge–Kutta method (RK4)

To numerically solve $y' = G(x, y)$ with $y(x_0) = y_0$, successively compute

$$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6}.$$

Where:

$$\begin{aligned}
k_1 &= hG(x_n, y_n) \\
k_2 &= hG(x_n + h/2, y_n + k_1/2) \\
k_3 &= hG(x_n + h/2, y_n + k_2/2) \\
k_3 &= hG(x_n + h, y_n + k_3)
\end{aligned}$$