Newton's Law of Gravitation

Newton's Law of Gravitation states that if we a *point* mass m_1 located at $\mathbf{x} = \langle x, y, z \rangle$ and a another point mass m_2 located at $\mathbf{p} = \langle p_1, p_2, p_3 \rangle$, then the force acting on m_2 is given by

$$\mathbf{F} = -G \frac{m_1 m_2}{|\mathbf{x} - \mathbf{p}|^3} \mathbf{p},$$

where G is Newton's gravitational constant.

Newton developed his law of gravitation to understand the motion of the planets. Treating planets as point masses is not controversial when they are billions of miles apart. Unfortunately, up close, planets are not exactly point masses! For example, most satellites orbit at between 100-400 miles from the surface of Earth. But it is reasonable to assume that our planets are large spheres B_R of radius R which are spherically homogeneous in that their density is constant on spherical shells centered at the center of B_R.

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The Mass of a Planet

Example

If B_R is a ball of radius R centered at the origin with density $\mu(|\mathbf{x}|)$ at the point $\mathbf{x} = \langle x, y, z \rangle$. Then the mass of B_R is, using spherical coordinates,

$$\begin{aligned}
\iiint_{B_R} \mu(|\mathbf{x}|)\, dV &= \int_0^{\pi} \int_0^{2\pi} \int_0^{R} \mu(\rho)\rho^2 \sin(\varphi)\, d\rho\, d\theta\, d\varphi \\
&= 2\pi \int_0^{\pi} \left(\sin(\varphi)\, d\varphi\right) \int_0^{R} \rho^2 \mu(\rho)\, d\rho \\
&= 4\pi \int_0^{R} \rho^2 \mu(\rho)\, d\rho.
\end{aligned}$$

The Gravitational Force of a Planet on a Point Mass

Now if we think of our planetary ball B_R as a bunch of little bits and use a Riemann sum, the force of B_R pulling on our point mass m_2 is given by the "vector integral"

$$\iiint_{B_R} \mu(|\mathbf{x}|) \frac{\mathbf{x}-\mathbf{p}}{|\mathbf{x}-\mathbf{p}|^3}\, dV \cdot m_2 = m_2 \mathbf{F}$$

where $\mathbf{F}$ is just a vector $\langle a, b, c \rangle$ with, for example,

$$a = \iiint_{B_R} \frac{(x-p_1)\mu(\sqrt{x^2+y^2+z^2})}{\left(\sqrt{(x-p_1)^2+(y-p_2)^2+(z-p_3)^2}\right)^3}\, dV.$$

Simplifications

Now, for convenience, we assume that our planet is centered at the origin and that $\mathbf{p} = \langle 0, 0, D \rangle$.

Then, for example,

$$a = \iiint_{B_R} \mu(|\mathbf{x}|) \frac{x}{|\mathbf{x} - \mathbf{p}|^3} \, dV = 0,$$

since the contribution on each side of the yz-plane cancel each other.

Similarly, $b = 0$.

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Newton's Formula

Let $M(D)$ be the mass of only the portion of B_R which lies within D units of the center of B_R. Therefore,

$$M(D) = \begin{cases} 4\pi \displaystyle\int_0^R \rho^2 \mu(\rho)\, d\rho & \text{if } D \geq R, \text{ and} \\ 4\pi \displaystyle\int_0^D \rho^2 \mu(\rho)\, d\rho & \text{if } D \leq R. \end{cases}$$

Newton's Formula asserts that the attraction of our spherical planet on m_2 was exactly the same as if B_R was a point mass *with mass* $M(D)$. Proving this amounts to showing that our constant c is given by

$$c = -\frac{M(D)}{D^3} D = -\frac{1}{D^2} M(D).$$

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Computing c

But

$$c = \iiint_{B_R} \mu(|\rho|) \frac{z - D}{|\mathbf{x} - \mathbf{p}|^3} \, dV.$$

But in spherical coordinates,

$$\begin{aligned} |\mathbf{x} - \mathbf{p}| &= \sqrt{x^2 + y^2 + (z - D)^2} = \sqrt{x^2 + y^2 + x^2 - 2zD + D^2} \\ &= \sqrt{\rho^2 - 2D\rho\cos(\varphi) + D^2}. \end{aligned}$$

Thus, in spherical coordinates,

$$\begin{aligned} c &= \int_0^{\pi} \int_0^{2\pi} \int_0^{R} \mu(\rho) \frac{\rho\cos(\varphi) - D}{(\rho^2 - 2D\rho\cos(\varphi) + D^2)^{3/2}} \rho^2 \sin(\varphi) \, d\rho \, d\theta \, d\varphi \\ &= 2\pi \int_0^{R} \rho^2 \mu(\rho) \underbrace{\int_0^{\pi} \frac{\rho\cos(\varphi) - D}{(\rho^2 - 2D\rho\cos(\varphi) + D^2)^{3/2}} \sin(\varphi) \, d\varphi}_{A} \, d\rho. \end{aligned}$$

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A Nasty Integral

To evaluate the interior integral "A", we first make the substitution $u = \cos(\varphi)$, and then integrate by parts:

$$
\begin{aligned}
A &= \int_{-1}^{1} (\rho u - D)(\rho^2 - 2D\rho u + D^2)^{-3/2}\, du \\
&= \frac{\rho - D}{D^2\sqrt{\rho^2 - 2D\rho u + D^2}}\,\Big|_{-1}^{1} \\
&= \frac{1}{D^2}\Big(\frac{\rho - D}{\sqrt{(\rho - D)^2}} - \frac{\rho + D}{\sqrt{(r + D)^2}}\Big) = \begin{cases} 0 & \text{if } \rho > D \text{, and} \\ -\dfrac{2}{D^2} & \text{if } \rho < D. \end{cases}
\end{aligned}
$$

Plugging, into the nasty integral above:

$$
c = \begin{cases} \frac{-4\pi}{D^2} \int_0^R \rho^2 \mu(\rho) & \text{if } D > R \text{, and} \\ \frac{-4\pi}{D^2} \int_0^D \rho^2 \mu(\rho)\, d\rho & \text{if } D \leq R. \end{cases}
$$

This proves Newton's Formula!