Math 31 Lesson Plan

Day 2: Sets; Binary Operations

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Supplies needed:

- 30 worksheets.
- Scratch paper?
- Sign in sheet

Goals for myself:

- Tell them what you're going to tell them; tell it to them; tell them what you told them.
- Watch the time! Don't worry if not all students finish the worksheet.

Goals for students: Students will:

- Feel like they understand, and can use to solve problems, the material from Sections 0 and 1 of Saracino.
- Get to know some of their classmates via collaborative activities.

[**Lecture Notes: Write everything in blue, and every equation, on the board. [Square brackets] indicate anticipated student responses. *Italics* are instructions to myself.**]

Pass around sign-in sheet

Happy Friday, everyone! Today we're going to start by discussing your questionsfrom Sections 0 & 1. Then I'd like you to work in groups on some exercises to check your understanding of the material from the text. At the end of class, if we have time, we'll start talking about symmetry groups, to get a jump start on Monday's class.

Tell students that any notation used in the book is also listed inside the front cover.

Subset can include the set itself; Proper subset means strictly smaller.

Any volunteers to explain the flaw in the proof that all horses are the same color?

- *Make sure everyone's convinced.*
- *Emphasize that if the proof falls apart for <u>any</u> m, then it doesn't hold.*
- *Why can't we skip a step in the induction hypothesis? ie, why can't we go from m to $m + 9$?*

A lot of people had questions about the second form of Mathematical Induction, so we're going to prove its validity as a class, and then we're going to talk about the proof of the Fundamental Theorem of Arithmetic. I know some of you asked for another example, but I don't think we're going to have time; so if you're still confused after we talk about the proof of the Fundamental Theorem of Arithmetic, please come see me in office hours.

Someone brought up the point on Blackboard that we proved that Induction was a valid proof method using the method of proof by contradiction. So, then, we should prove that contradiction is also a valid proof method – but then you have to prove that the tools you used in that proof are also valid. It seems like a never-ending chain, right?

Well, to be honest, it is. In order to do mathematics, you have to make some assumptions. You can't prove everything. If you're interested in what statements mathematicians usually assume to be true, and what you can or can't prove by making different types of foundational assumptions, you should take a class in logic.

The assumptions that Saracino (and I) are making are that direct proof, and proof by contradiction, are valid proof methods. If you're not convinced of that, please come see me in office hours! We feel that those are reasonable assumptions and that you should be able to convince yourself of them without too much help.

THEOREM 0.3: (Mathematical Induction, second form) *Suppose $P(n)$ is a statement about positive integers, and suppose we know two things:*

- *$P(1)$ is true;*
- *For every positive integer m, if $P(k)$ is true for <u>every</u> $k < m$, then $P(m)$ is true.*

Then it follows that $P(n)$ is true for every integer n.

Proof: The text says that this proof is similar to the proof of Theorem 0.2, the first form of Mathematical Induction. So, how do you think we should start the proof of Theorem 0.3? [proof by contradiction]

Suppose $P(n)$ is not true for some n, and let $S = \{n \in \mathbb{Z}^+ : P(n) \text{ is false}\}$. What can we say about S? [Then S is nonempty, and so S has a smallest element n_0.] What do we know about n_0? [We know $n_0 > 1$.]

By definition of n_0, we know that for all positive integers $k < n_0$, what? [$P(k)$ is true for each such k.] Therefore, by hypothesis, what? [$P(n_0)$ must also be true, which is a contradiction.] $\square$

Any questions about this proof?

When to use Form 1 vs Form 2 of induction:

- Use Form 1 when you can easily reduce $P(n)$ to a statement involving $P(n-1)$.
- Use Form 2 when you can reduce $P(n)$ to a statement involving $P(m)$ for some smaller m, but not necessarily $n-1$.

Thus, often when you have a formula that you want to prove is true for all n, the first form is easier to use. It's not hard to rewrite a formula to involve $n-1$ or $n+1$ instead of n.

However, sometimes we have statements about the positive integers that don't involve formulas. The Fundamental Theorem of Arithmetic is one of those, and since it will be important later in this course, I'm going to go through the proof together. If you still have questions about when to use the second form of induction after class, please come see me!

THEOREM 0.4 (Fundamental Theorem of Arithmetic, existence) *Let $n > 1$ be a positive integer greater than 1. Then we can write n as a product of prime numbers.*

Proof: We prove this Theorem using the second form of mathematical induction. So, what do we have to do? [We must check the base case, $n = 2$] *Discuss why the base case isn't $n = 1$. Make sure everyone's OK with that.* So what happens in the base case? [Since 2 is prime, the base case is true.] So now what? [Suppose $m > 2$ and assume that if $2 \leq k < m$, then $P(k)$ is true.] What does that mean? [In other words, we can write each such k as a product of primes.]

Now what? [Consider the integer m. If m is prime, we're done.] If m is not prime, what do we know? we can write $m = ab$, What do we know about a and b? [where $1 < a, b < m$.] What does the inductive hypothesis tell us? Then we can write both a and b as products of primes. Hence $m = ab$ is also a product of primes, and we're done. $\square$

Why can't we use the first form of induction here? [When you factor $m = ab$, you don't have any idea how big a and b are – but you know for certain that they're both less than $m - 1$!]

Someone asked on Blackboard about why, in the first form of induction, you have to move by steps of size 1.

Let $P(n)$ be a statement about the positive integers. Suppose $P(1)$ is true, and suppose that if $P(m)$ is true, then $P(m+9)$ is true. Is $P(n)$ true for all integers n? *Think-pair-share* [no, because we have no way to check what happens for $P(2), P(3), \ldots, P(9)$.]

More questions about induction, or sets?

Let's move on to binary operations, then.

In Section 1, when we're talking about binary operations, Saracino gives some examples of a binary operation, and some non-examples. Were you all convinced that the operations he listed were binary? Why?

Any time you have something that you think is an example, be sure to check the definition to make certain! If you have something that you think is <u>not</u> an example, try to show why it doesn't fit the definition.

<u>Example:</u> On $\mathbb{Z}$, subtraction is a binary operation. Why? [If a and b are both integers, so is $a - b$.] However, on $\mathbb{Z}^+$, subtraction is <u>not</u> a binary operation. It's always best to give an explicit counterexample to show why something is <u>not</u> true. So, can anyone give me a counterexample to see why subtraction isn't a binary operation on $\mathbb{Z}^+$? *Think-Pair-Share if no immediate volunteers* [Both 1 and 2 are in $\mathbb{Z}^+$, but $1 - 2 = -1 \notin \mathbb{Z}^+$.]

Also discuss division on $\mathbb{R}$?

Ask students to divide themselves into groups of 2-4, and pass out worksheet. In the last problem on the second page of the worksheet, the question asks about $\bigcap_{i=1}^{n} A_i$, where the A_i are sets. What do you think that means? *Think-pair-share if no one volunteers.* [If $A_1, A_2, \ldots, A_n$ are sets, then $\bigcap_{i=1}^{n} A_i$ is the intersection of all the sets A_i.]

The starred problem is a challenge problem, if you have extra time. 30 mins

Move around the room to answer questions and check on student progress. If some students finish more quickly, suggest that they check whether matrix addition is binary/commutative/associative. Alternatively, ask them to generalize the proof that matrix multiplication is a binary operation from 2×2 *matrices to* $n \times n$.

Any questions about the material from the worksheets? Wrap up by 1.30

The First Homework will be posted later today, on the course website. It will be due next week Friday, at the beginning of class.

The first quiz will be Monday, the first five minutes of class. Don't be late!

but, there's no reading assignment for Monday. Enjoy your weekend!