Math 3: Fall 2007
Exam 2 solutions

1. For the function $f(x) = x^2(x-2)^2$, which item correctly lists the intervals where the function is increasing?

(a) $(0, 2)$

(b) $(0, 1)$, $(2, \infty)$

(c) $(1, \infty)$

(d) $(-\infty, \sqrt{2})$, $(4, \infty)$

(e) none of the above

Answer: (b)

2. For the function $f(x) = \dfrac{1}{x} - \dfrac{1}{x^2}$, pick out the one statement that is correct.

(a) This function has a local minimum at $x = 0$.

(b) This function has an absolute minimum at $x = 2$.

(c) This function has an absolute maximum at $x = 2$.

(d) This function has a local minimum at $x = -1$.

(e) This function has a local maximum at $x = -1$.

Answer: (c)

3. Suppose $f(x)$ is continuous on the interval $[0, 5]$ and differentiable on the interval $(0, 5)$, with $f(0) = -1$ and $f(5) = 9$. Which one of the following statements holds?

(a) There must be a number c with $0 < c < 5$ with $f(c) = 10$.

(b) There must be a number c with $0 < c < 5$ with $f'(c) = 2$.

(c) There must be a number c with $0 < c < 5$ with $f(c) = -10$.

(d) There must be a number c with $0 < c < 5$ with $f'(c) = 1/2$.

(e) There is not enough information given to force any of the above to hold.

Answer: (b)

4. The equation of the tangent line to the curve $xy + x^2y^2 = 2$ at $(1, 1)$ is

(a) $y = 3x - 2$

(b) $y = x$

(c) $y = 2 - x$

(d) $x = 1$

(e) none of the above

Answer: (c)

5. Find $\dfrac{dy}{dx}$ when $y = x^{\sin(x)}$. It is

(a) $x^{\sin(x)-1}\cos(x)$

(b) $x^{\sin(x)}\cos(x)$

(c) $x^{\sin(x)}\ln(x)$

(d) $x^{\sin(x)}\cos(x)\ln(x) + x^{\sin(x)}\dfrac{\sin(x)}{x}$

(e) none of the above

Answer: (d)

6. For this problem, the following information may be useful: $\sqrt{2} \approx 1.41$, $\sqrt{3} \approx 1.73$, $e \approx 2.72$, $\pi \approx 3.14$. Suppose you use Newton's method to find a value of x where $\cos(x) = x$ starting with an initial guess of $x_0 = \pi/4$. The value of the next iteration x_1

(a) is bigger than $\pi/4$

(b) is smaller than $\pi/4$

(c) is equal to $\pi/4$

(d) impossible to compute from the given information

(e) none of the above

Answer: (b)

7. Compute $\int 2x^2 - 3x^3 - \frac{1}{x^2}\, dx$. It is

(a) $4x - 9x^2 + \frac{2}{x^3}$

(b) $4x - 9x^2 + \frac{2}{x^3} + C$

(c) $x^3 - x^4 + \frac{1}{x} + C$

(d) $\frac{2}{3}x^3 - \frac{3}{4}x^4 + \frac{1}{x} + C$

(e) none of the above

Answer: (d)

8. Find a solution $y = y(x)$ to the equation $\frac{dy}{dx} = y + 1$ and $y(0) = 1$. It is

(a) 1

(b) $\frac{1}{2}y^2 + 1$

(c) $2e^x - 1$

(d) $\frac{1}{(x+1)^2}$

(e) none of the above

Answer: (c)

9. A radioactive substance has a half life of 10 years. What fraction of this substance is still present after 15 years?

(a) $1/3$

(b) $3/8$

(c) $\sqrt{2}/4$

(d) $1/e$

(e) none of the above

Answer: (c)

10. Suppose a particle is travelling along the x-axis so that its position at time t is given by $x(t) = t^3 - 3t^2 - 24t + 5$. Which one of the following statements is true?

(a) The particle changes direction at time $t = 1$.

(b) The particle is speeding up on the intervals $(-2, 1)$ and $(4, \infty)$.

(c) The particle is moving to the right on the interval $(-2, 4)$.

(d) The particle is moving to the left on the interval $(-\infty, 1)$.

(e) There is not enough information to determine any of the above.

Answer: (b)

NON-MULTIPLE CHOICE. PLEASE SHOW ALL YOUR WORK.

11. Suppose you use the linearization method to approximate $\sqrt{10}$.

(a) What is the function $f(x)$ that you will approximate using the linearization method?

$$f(x) = \sqrt{x}$$

(b) What is the linearization $L(x)$ of your function at a convenient point for the problem?

Since 9 is near 10 and we know that $\sqrt{9} = 3$, we will find the linearization of $f(x)$ at $x = 9$. First, $f'(x) = \dfrac{1}{2\sqrt{x}}$. Then

$$\begin{aligned} L(x) &= f(9) + f'(9)(x - 9) \\ &= 3 + \frac{1}{6}(x - 9). \end{aligned}$$

(c) What is your numerical approximation of $\sqrt{10}$ by this method?

Using the linearization $L(x)$, we have

$$\sqrt{10} \approx L(10) = 3 + \frac{1}{6}(10 - 9) = 3 + \frac{1}{6} = \frac{19}{6}.$$

12. Consider the differential equation $\dfrac{dy}{dx} = y^2 + 2x$.

(a) Use Euler's method with initial point $(-1, 0)$ and step size 1 to estimate $y(1)$.

Let $F(x, y) = y^2 + 2x$. At $(-1, 0)$, the slope is $F(-1, 0) = -2$, so after one step, the new point is $(0, 0 + 1(-2)) = (0, -2)$. The slope at this point is $F(0, -2) = 4$, so the next point is $(1, -2 + 4(1)) = (1, 2)$. Therefore $y(1) \approx 2$.

(b) Now use Euler's method with initial point $(-1, 0)$ and step size 1/2 to estimate $y(1)$.

Again let $F(x, y) = y^2 + 2x$. We have the following sequence of points:

$$
\begin{aligned}
(x_0, y_0) &= (-1, 0) \\
(x_1, y_1) &= \left(-\frac{1}{2}, 0 + \frac{1}{2}F(-1, 0)\right) = \left(-\frac{1}{2}, \frac{1}{2}(-2)\right) = \left(-\frac{1}{2}, -1\right) \\
(x_2, y_2) &= \left(0, -1 + \frac{1}{2}F\left(-\frac{1}{2}, -1\right)\right) = \left(0, -1 + \frac{1}{2}(0)\right) = (0, -1) \\
(x_3, y_3) &= \left(\frac{1}{2}, -1 + \frac{1}{2}F(0, -1)\right) = \left(\frac{1}{2}, -1 + \frac{1}{2}(1)\right) = \left(\frac{1}{2}, -\frac{1}{2}\right) \\
(x_4, y_4) &= \left(1, -\frac{1}{2} + \frac{1}{2}F\left(\frac{1}{2}, -\frac{1}{2}\right)\right) = \left(1, -\frac{1}{2} + \frac{1}{2}\left(\frac{5}{4}\right)\right) = \left(1, \frac{1}{8}\right).
\end{aligned}
$$

Therefore $y(2) \approx \dfrac{1}{8}$.

13. Consider the differential equation $\dfrac{dy}{dx} = -x^2y^2$.

(a) Find the general form of the solution y as a function of x.

We use the method of separation of variables. Dividing both sides by y^2, we have

$$y^{-2}\frac{dy}{dx} = -x^2.$$

Taking the antiderivative of each side, we have

$$-y^{-1} = -\frac{1}{3}x^3 + C.$$

Then solving for y,

$$y^{-1} = \frac{1}{3}x^3 - C,$$

and

$$y = \frac{1}{\frac{1}{3}x^3 - C}.$$

This is a valid general solution, or we can multiply by $\dfrac{3}{3}$, and write $C_1 = -3C$ to get

$$y = \frac{3}{x^3 + C_1}.$$

(b) Find a particular solution to the equation using the initial value $y(1) = \frac{1}{2}$.

Using the initial point $\left(1, \frac{1}{2}\right)$, we have

$$\frac{1}{2} = \frac{3}{1^3 + C_1},$$

so $1 + C_1 = 6$, and so $C_1 = 5$. Then the particular solution is

$$y = \frac{3}{x^3 + 5}.$$