Math 22 Workshop I
8 July 2010

1. Let A be a $m \times n$ matrix, let $\mathbf{b}$ and $\mathbf{b}'$ be vectors in $\mathbf{R}^m$ and let c be a scalar. Prove the following statements.

(a) If $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{b}'$ are both consistent, then $A\mathbf{x} = \mathbf{b} + \mathbf{b}'$ is consistent.

ANS: By assumption there are vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{R}^n$ such that $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_2 = \mathbf{b}'$. If $\mathbf{x}_3 := \mathbf{x}_1 + \mathbf{x}_2$, then we have $A\mathbf{x}_3 = A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{b} + \mathbf{b}'$. Therefore $A\mathbf{x} = \mathbf{b} + \mathbf{b}'$ is consistent as claimed.

(b) If $A\mathbf{x} = \mathbf{b}$ is consistent, then so is $A\mathbf{x} = c\mathbf{b}$.

ANS: By assumption there is an $\mathbf{x}_1 \in \mathbf{R}^n$ such that $A\mathbf{x}_1 = \mathbf{b}$. If $\mathbf{x}_2 := c\mathbf{x}_1$, then we have $A\mathbf{x}_2 = A(c\mathbf{x}_1) = cA\mathbf{x}_1 = c\mathbf{b}$. Therefore $A\mathbf{x} = c\mathbf{b}$ is consistent.

2. Let A be a $m \times n$ matrix, let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbf{R}^n$ and let c be a scalar.

(a) If $\mathbf{u}$ and $\mathbf{v}$ are solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$, then so is $\mathbf{u} + \mathbf{v}$.

ANS: By assumption, $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$. Therefore, $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$. Therefore $\mathbf{u} + \mathbf{v}$ is a solution to $A\mathbf{x} = \mathbf{0}$ as required.

(b) If $\mathbf{u}$ is a solution to $A\mathbf{x} = \mathbf{0}$, then $c\mathbf{u}$ is too.

3. A variation on problem 2 (with the same hypotheses).

(a) Is it true that $\mathbf{u}$ and $\mathbf{v}$ are solutions to $A\mathbf{x} = \mathbf{0}$ *if and only if* $\mathbf{u} + \mathbf{v}$ is?

ANS: This is false. Although we proved the "$\Longrightarrow$" direction in problem 2 (also called the "only if" direction), the converse is false. ***For example***, suppose that $\mathbf{u}$ is a solution to the non-homogeneous system $A\mathbf{x} = \mathbf{b}$ (so that $\mathbf{b} \neq \mathbf{0}$). Let $\mathbf{v} := -\mathbf{u}$. Then $\mathbf{u} + \mathbf{v} = \mathbf{0}$ is a solution to $A\mathbf{x} = \mathbf{0}$, but neither $\mathbf{u}$ nor $\mathbf{v}$ is. For a more specific example, let $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\mathbf{u} = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$.

(b) Suppose that $c \neq 0$. Then is it true that $\mathbf{u}$ is a solution to $A\mathbf{x} = \mathbf{0}$ *if and only if* $c\mathbf{u}$ is?

ANS: Suppose that $A(c\mathbf{v}) = \mathbf{0}$. Then $cA\mathbf{v} = \mathbf{0}$. Since $c \neq 0$, we can divide both sides by c to obtain $A\mathbf{v} = \mathbf{0}$. We proved the other direction above. (COMMENT: your solution must mention where you use $c \neq 0$. The statement is false otherwise, so it is important to emphasize where the hypothesis is used.)

4. Let A be a $m \times n$ matrix. Show that if $\mathbf{u}_1, \dots, \mathbf{u}_p$ are all solutions to $A\mathbf{x} = \mathbf{0}$ and if $\mathbf{v} \in \operatorname{Span}(\{\mathbf{u}_1, \dots, \mathbf{u}_p\})$, then $\mathbf{v}$ is a solution to $A\mathbf{x} = \mathbf{0}$.

ANS: If $v \in \operatorname{Span}(\{\mathbf{u}_1, \dots, \mathbf{u}_p\})$, then there are scalars $c_1, \dots, c_p$ such that $v = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p$. But then

$$\begin{aligned} A\mathbf{v} &= A(c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p) \\ &= A(c_1\mathbf{u}_1) + \cdots + A(c_p\mathbf{u}_p) \\ &= c_1 A\mathbf{u}_1 + \cdots + c_p A\mathbf{u}_p \\ &= c_1\mathbf{0} + \cdots + c_p\mathbf{0} \\ &= \mathbf{0}. \end{aligned}$$

Therefore $\mathbf{v}$ is a solution to $A\mathbf{x} = \mathbf{0}$ as claimed.

5. Prove or disprove the following statements.

(a) If the vectors $\mathbf{u}$ and $\mathbf{v}$ are solutions to $A\mathbf{x} = \mathbf{b}$, then so is $\mathbf{u} + \mathbf{v}$.

ANS: This is false. Let $\mathbf{u} = \mathbf{v} = \mathbf{b} = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and let $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $A\mathbf{u} = A\mathbf{v} = \mathbf{b}$, but $A(\mathbf{u} + \mathbf{v}) = 2\mathbf{b} \neq \mathbf{b}$. In other words, $\mathbf{u} + \mathbf{v}$ is not a solution to $A\mathbf{x} = \mathbf{b}$.

(b) If A and B are 2×2 matrices and if $\mathbf{u} \in \mathbf{R}^2$, then $A(B\mathbf{u}) = B(A\mathbf{u})$.

ANS: This is false. Let $A = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Then if $u = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, we have $A\mathbf{u} = \mathbf{0}$ so that $B(A\mathbf{u}) = \mathbf{0}$. On the other hand $B\mathbf{u} = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, so that $A(B\mathbf{u}) = A\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Thus $A(B\mathbf{u}) \neq B(A\mathbf{u})$ in this case.