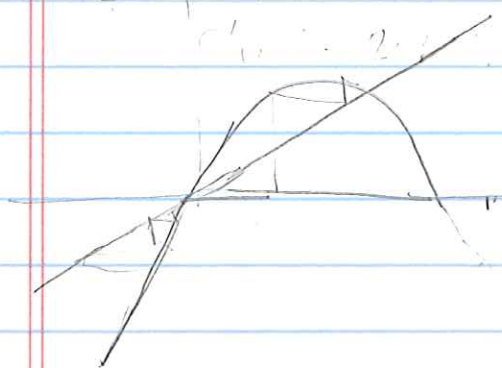
HW #4

T3.2 $g(x) = 2.5x(1-x)$

fixedpts of $g(x)$:

$2.5x - 2.5x^2 = x \rightarrow \frac{5}{2}x^2 - \frac{3}{2}x = 0$

$x = 0, \frac{3}{5}$



$g'(x) = \frac{5}{2} - 5x$

$|g'(3/5)| = |\frac{5}{2} - 5(\frac{3}{5})| = |2.5 - 3| = \frac{1}{2} < 1$

$\rightarrow$ sink. basin of attraction is $(0,1)$.

Thus all bounded orbits converge to the sinks.

The Lyaponov exponent is

$h(x) = \ln|g'(3/5)| = \ln|1/2|$.

T3.3 $f(x) = (x+q)(\text{mod } 1)$ q is irrational

- verify f has no periodic orbits.

Suppose $\exists$ a k-periodic orbit $\{x_1, \ldots x_k\}$
$f^k(x_1) = x_1$

$\Rightarrow x_1 = x_1 + kq \ (\text{mod } 1)$
$\Rightarrow 0 = kq \ (\text{mod } 1)$ Thi [illegible]
This cannot be true since q irrational.
$\therefore$ Thus there are no periodic orbits.

- Next we need to verify that the Lyapunov exponent is 0.
$f'(x) = 1 (\text{mod } 1) \quad \forall x.$
So $h(x) = \ln(1) = 0.$ ✓

3.1 $f(x) = a - x^2$ $\quad$ a is a constant.

(a) $a - x^2 = x \Rightarrow x^2 + x - a = 0$

$$x = \frac{-1 \pm \sqrt{1+4a}}{2}$$

only one fixedpt if $1+4a = 0 \Rightarrow a = -1/4$.

(b) if $a < -1/4$, there are no real fixed pts.
all iterations will go to $-\infty$.

c) $f'(x) = -2x$ $\quad$ $a > -1/4$
need to find a st one of the fixed pts is stable.
the upper fixed pt will be the stable one.
So we need $-1 + \sqrt{1+4a} \leq 1$
$\Rightarrow \sqrt{1+4a} < 2$
$\Rightarrow 1+4a < 4 \Rightarrow 4a < 3 \Rightarrow a < 3/4$

d) we want a double periodic orbit.
$\Rightarrow$ need to find a st $f^2(x) = x$ & $|f^{2\prime}(x)| < 1$.

$$f^2(x) = a - (a-x^2)^2 = \underbrace{a - (a^2 - 2ax^2 + x^4)}_{a - a^2 + 2ax^2 - x^4} = x$$

$\Rightarrow x^4 - 2ax^2 + x + a^2 - a = 0$.
We know fixed pts are roots of this polynomial.
$\Rightarrow$ We most do long division.

$$
\begin{array}{r}
x^2 - x \quad + (a-1) \\
x^2 + x - a \; \overline{\big) \, x^4 + 0x^3 - 2ax^2 + x + a^2 - a} \\
-(x^4 + x^3 - ax^2) \\
\hline
-x^3 + ax^2 + x \\
-(x^3 - x^2 + ax) \\
\hline
(a+1)x^2 + (a-1)x + a^2 - a \\
-[(a-1)x^2 + (a-1)x + a^2 + a) \\
\hline
0 \qquad 0 \qquad 0
\end{array}
$$

So we need to find the roots of $x^2 - x + (a-1)$

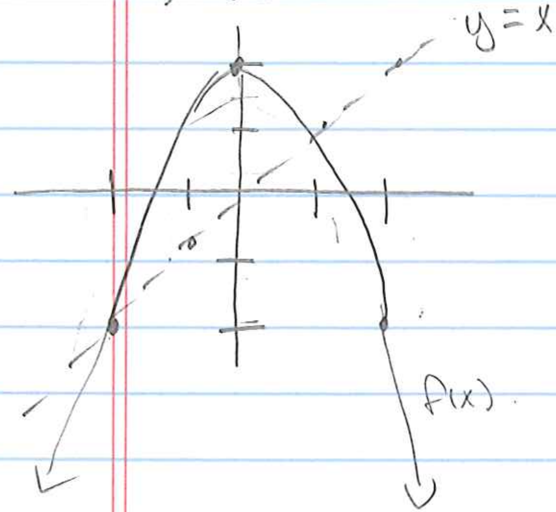
$$\rightarrow \quad x = \frac{1 \pm \sqrt{1-4(a-1)}}{2} = p_1, p_2 \quad = 1/2 \text{ when } a = 5/4$$

$$
\begin{aligned}
(f^{2\,\prime})(p_1) &= f'(p_1) f'(p_2) \\
&= -2\left(\frac{1+\sqrt{1-4(a-1)}}{2}\right)\left[-2\left(\frac{1-\sqrt{1-4(a-1)}}{2}\right)\right] \\
&= \left(1+\sqrt{1-4(a-1)}\right)\left(1-\sqrt{1-4(a-1)}\right) \\
&= 1 - (1-4(a-1)) \\
&= 1 - 1 + 4(a-1) = 4(a-1)
\end{aligned}
$$

we want this <1. $\Rightarrow 4(a-1) < 1 \Rightarrow a - 1 < 1/4$

$$\Rightarrow a < 1 + 1/4 = 5/4.$$

e) When $a=2$



Notice that $f:[-2,2]\to[-2,2]$
infact this is simply a shifted
& rescaled logistic function
with $a=4$.
$\Rightarrow$ it has all periodic orbits.

T3.8

Thm 3.11 Thm 3.11

f, g are conjugate maps $g C(x) = C(f(x))\ \forall x$.

If x_1 is a period-k pt for f, then (i) $C(x_1)$ is a period-k pt for g. (ii) $(g^k)'(C(x_1)) = (f^k)'(x_1)$

let $\{x_1, x_2, \cdots, x_k\}$ be the k-periodic orbit of f.

$\Rightarrow (f^k)'(x_1) = f'(x_1) f'(x_2) \cdots f'(x_k)$

(ii) $\frac{d}{dx}\left(g^k(C(x))\right) = \frac{d}{dx}\left(C(f^k(x))\right)$

$g^{k\,\prime}(C(x))\, C'(x) = C'(\overbrace{f^k(x)}^{x})\,(f^k)'(x)$

$\Rightarrow g^{k\,\prime}(C(x)) = (f^k)'(x).$

(i) we know $f^k(x_1) = x_1$

$\Rightarrow C(f^k(x_1)) = C(x_1)$

We also know $C(f^k(x_1)) = g^k(C(x_1))$ by conjugate map

$\Rightarrow g^k(C(x_1)) = C(x_1)$

$\Rightarrow C(x_1)$ is a k-periodic pt of g.

T3.10

(a) $L\underbrace{RR\cdots R}_{k-1}$ is not periodic for period less than k.

(b) The interval $L\underbrace{R\cdots RL}_{k-1}$ must contain a ~~fixedidic~~ fixed pt in L by cor. 3.18.
by part (a) this pt must be k-periodic

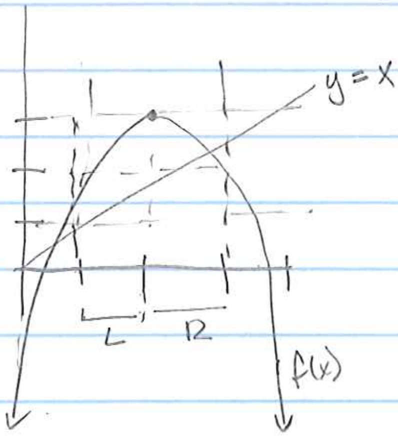
3.4

In T3.2 we showed all bounded orbits have the same Lyapunov exponent.
since there was a sink all bounded orbit are asymptotically periodic to the sink.
the Lyapunov exponent is $\ln|1/2| < 1$.
$\Rightarrow$ There are no Chaotic orbits.

3.5

a) find roots $x = \frac{-8 \pm \sqrt{64-4(5)(-2)}}{2(-2)} = 2 \pm \frac{1}{4}\sqrt{24}$

$= 2 \pm \frac{1}{2}\sqrt{7} \sim 0.6771,\ 3.322.$



$f'(x) = -4x + 8 \rightarrow \begin{cases} x = 2 \text{ is the vertex} \\ y = 3 \end{cases}$

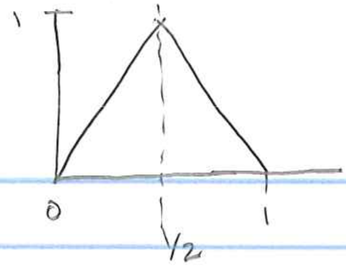
b) The interval [1,3] maps to itself.
we will partition it into [1,2] ∪ [2,3]

c)

$$\begin{tikzcd} L \arrow[loop left] \arrow[r, bend right] & R \arrow[l, bend right] \arrow[loop right] \end{tikzcd}$$

This is a fully connected graph so all periods are possible.

3.7



Goal: show periodic pts are dense in $I = [0,1]$.

This means we need to show that for any x_0 not periodic in I $\exists$ a periodic pt within ε from it.

let x_0 have the itinerary LRRRRLRLLR...
Then $\overline{LRRRRLR}$ will have a periodic point in it. and the distance between the periodic pt & x_0 is at most $(1/2)^7 = \varepsilon$.

If we wanted to find a periodic pt closer we could truncate the interval & force periodicity later.