MATH 25, FALL 2010 MIDTERM #1 SOLUTIONS

(1) Use induction to show that

$$1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$$

for all $n \geq 1$.

Solution. The base case of $n = 1$ is trivial to check, since $1^3 = 1^2 2^2/4 = 1$.

Suppose the above formula is true for n. We wish to prove it true for $n+1$. The inductive hypothesis is the equation

$$1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}.$$

Add $(n+1)^3$ to both sides. The LHS becomes the sum of cubes from 1 to $n+1$, which is the LHS of the formula we want to prove. The RHS becomes

$$\frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4} = \frac{(n+1)^2(n^2+4n+4)}{4} = \frac{(n+1)^2(n+2)^2}{4}.$$

This is the RHS of the formula we want to prove, so we are done. □

(2) Describe all pairs of integers x, y which solve the equation $333x + 144y = 18$.

Solution. This problem is an application of the Euclidean algorithm. Applying the Euclidean algorithm to the pair of integers $333, 144$ yields

$$\begin{aligned} 333 &= 144 \cdot 2 + 45, \\ 144 &= 45 \cdot 3 + 9, \\ 45 &= 9 \cdot 5 + 0. \end{aligned}$$

Therefore, $\gcd(333, 144) = 9$, and since $9|18$, the original equation has solutions. We now reverse substitute in the steps of the Euclidean algorithm to find a solution to $333x + 144y = 18$:

$$\begin{aligned} 9 &= 144 - 45 \cdot 3, \\ 9 &= 144 - (333 - 144 \cdot 2) \cdot 3 \\ &= 333(-3) + 144(7). \end{aligned}$$

To get solutions to the original equation $333x + 144y = 18$, we multiply this equation by 2 to get a pair of solutions $x = -6, y = 14$. The general formula for solutions is given by

$$x = -6 + \frac{144}{9}n = -6 + 16n, y = 14 - \frac{333}{9}n = 14 - 37n,$$

where n is any integer. □

(3) Let $m, n > 1$ be positive integers. Recall that $\log_m n$ is the real number which satisfies the equation $m^{\log_m n} = n$.
  (a) Show that $\log_2 6$ is irrational.
  (b) More generally, show that if there is a prime p which divides m but not n, or which divides n but not m, then $\log_m n$ is irrational.
  (c) Suppose m, n have the same prime factors. Is it always true that $\log_m n$ is rational?

Solution.
  (a) We prove this (as the other two parts) by contradiction. Suppose $\log_2 6$ were rational; say $\log_2 6 = a/b$, where a, b are integers. Since $\log_2 6 > 0$, we can choose both a, b to be non-negative; as a matter of fact, both positive, since $b \neq 0$ is clear, and $a \neq 0$ since $2^0 \neq 6$. With these facts about a, b in mind, notice that we have

$$2^{a/b} = 6, \text{or } 2^b = 6^a = 2^a 3^a.$$

However, this is impossible, because 3 divides the right hand side (we are using the fact that $a, b > 0$ in an essential way here, to ensure that we are dealing with integers), but $3 \nmid 2^b$. This is a contradiction so $\log_2 6$ must be irrational.

  (b) The idea in this problem is similar to the previous. If $\log_m n$ is rational, say equal to a/b, then again we can choose $a, b > 0$, since $\log_m n > 0$ (because $n > 1$: notice that if $n = 1$ the statement we are trying to prove is false.)
We then have an equation $m^{a/b} = n$, or $m^a = n^b$. If $p|m$ but $p \nmid n$, say, then we get a contradiction, since p divides the LHS but not the RHS. Similarly if $p|n$ but $p \nmid m$. (Again, we are using the fact that $a, b > 0$ in an essential way.)

  (c) $\log_m n$ might be rational, but it also might not be rational. For example, choose $n = 6, m = 12$. If $\log_6 12 = a/b$ (again, one can choose $a, b > 0$ for the same reasons as previously), then we get an equation $6^{a/b} = 12$, or $6^a = 12^b$. The prime factorizations of the respective sides are $2^a 3^a, 2^{2b} 3^b$. Since these must be equal, we get $a = 2b, a = b$, which imply that $a = b = 0$, a contradiction.

Another nice way to see that $\log_6 12$ is irrational is to notice that $\log_6 12 = \log_6 6 + \log_6 2$. Since $\log_6 6 = 1$ is rational, $\log_6 12$ is irrational if and only if $\log_6 2$ is, but this is irrational since $\log_6 2 = 1/\log_2 6$ and we already showed $\log_2 6$ is irrational in the first part. □

(4) Recall that the Fibonacci sequence is the sequence of numbers f_n defined by $f_1 = f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for $n \geq 1$. Suppose we perform the Euclidean algorithm on the pair of integers f_{n+1} and f_n. How many Euclidean divisions have been performed when the algorithm ends?

Solution. We claim the answer is $n - 1$, at least if $n \geq 2$. (When $n = 1$, the algorithm ends in one step, not none, but this is just a minor issue.)

The first claim is that a Euclidean division of f_{k+1} by f_k gives the equation $f_{k+1} = f_k \cdot 1 + f_{k-1}$, when $k \geq 3$. This is true, because $0 \leq f_{k-1} < f_k$, and the recursive equation $f_{k+1} = f_k + f_{k-1}$ shows that f_{k-1} differs from f_{k+1} by a multiple of f_k.

So each time we apply a Euclidean division to f_{k+1} and f_k, we end up with f_k, f_{k-1}. This continues until we reach f_3, f_2, since $f_3 = 2, f_2 = 1$, and the algorithm terminates with the last equation $2 = 1 \cdot 2 + 0$. If we started with f_{n+1}, f_n, it takes $n + 1 - 3 = n - 2$ steps to reach the final Euclidean division, and the final division adds one more, for a total of $n - 1$. □

(5) The following questions are true/false. For each statement, first state whether it is true or false, and then prove that your answer is correct. (If the statement is false, you only need to find one counterexample.)
  (a) Suppose a, b, c, d are integers, and $a|b, c|d$. Then $(a + c) \mid (b + d)$.
  (b) Let p be a prime. If $\gcd(p, a) = p$ and $\gcd(p^2, b) = p^2$, then the exponent of p appearing in the prime factorization of ab is 3.
  (c) If a and b are both positive integers, and $\gcd(a, b) = 1$, then the equation $ax - by = 1$ has a pair of integer solutions x, y where x, y are both positive.
  (d) Every divisor of a number of the form $6k + 1$ must also be of the form $6k + 1$.

Solution.
  (a) This is false; there are many counterexamples, but a simple one is $a = b = c = 1, d = 2$; it is clear that $1|1, 1|2$, but $2 \nmid 3$.

  (b) This is false; for example, take $a = p^2, b = p^2$. Then $ab = p^4$ has exponent 4, not 3.

  (c) This is true. Since $\gcd(a, b) = 1$, and $1|1$, we know this equation has some integer solutions, say x_0, y_0. The general form of integer solutions is given by $x = x_0 + bn, y = y_0 + an$. Since $a, b > 0$, we can choose n to be very large and positive (say, more than twice as large as $|x_0|$ and $|y_0|$) to ensure that $x, y > 0$.

  (d) This is false. For example, $25 = 6 \cdot 4 + 1$, but $5|25$, and 5 clearly does not have form $6k + 1$. □