

Math 11
Section 3
Friday, October 10, 2008

Example: Approximate the distance between the origin and the point $(3.02, 3.99, 12.02)$.

Solution: Consider the function $f(x, y, z)$ that gives the distance between the origin and the point (x, y, z) :

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}.$$

We want to approximate the value $f(3.02, 3.99, 12.02)$.

We recognize f as a differentiable function (except possibly at the origin). Its derivative is

$$f'(x, y, z) = \left(\frac{\partial f}{\partial x}(x, y, z), \frac{\partial f}{\partial y}(x, y, z), \frac{\partial f}{\partial z}(x, y, z) \right) = \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right).$$

(We know f is differentiable because its partial derivatives are continuous.)

Because f is differentiable, we can approximate it using a tangent approximation or using differentials.

Method 1: Use the tangent approximation to f near $(3, 4, 12)$:

$$\begin{aligned} f(x, y, z) &\approx f'(3, 4, 12) \cdot ((x, y, z) - (3, 4, 12)) + f(3, 4, 12) = \\ &\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13} \right) \cdot (x - 3, y - 4, z - 12) + 13 = \frac{3x + 4y + 12z - 169}{13} + 13 = \\ &\frac{3x + 4y + 12z}{13}. \\ f(3.02, 3.99, 12.02) &\approx \frac{9.06 + 15.96 + 144.24}{13} = \frac{169.26}{13} = 13.02. \end{aligned}$$

Method 2: Use differentials:

Setting $w = f(x, y, z)$, we have:

$$dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz =$$
$$\frac{x}{\sqrt{x^2 + y^2 + z^2}} dx + \frac{y}{\sqrt{x^2 + y^2 + z^2}} dy + \frac{z}{\sqrt{x^2 + y^2 + z^2}} dz.$$

Near the point $(3, 4, 12)$, we have

$$dw = \frac{3 dx}{13} + \frac{4 dy}{13} + \frac{5 dz}{13};$$
$$\Delta w \approx \frac{3 \Delta x}{13} + \frac{4 \Delta y}{13} + \frac{5 \Delta z}{13}.$$

Using this approximation at $(x, y, z) = (3.02, 3.99, 12.02)$, we have

$$\Delta x = 3.02 - 3 = .02 \quad \Delta y = 3.99 - 4 = -.01 \quad \Delta z = 12.02 - 12 = .02,$$

and so

$$\Delta w \approx \frac{.06}{13} + \frac{-.04}{13} + \frac{.4}{13} = .02,$$

and

$$f(3.02, 3.99, 12.02) = w = 13 + \Delta w \approx 13 + .02 = 13.02.$$