Classical Lie Algebras

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Each finite dimensional complex simple Lie algebras belongs to one of four families, $\{A_r \mid r \geq 1\}$, $\{B_r \mid r \geq 3\}$, $\{C_r \mid r \geq 3\}$, or $\{D_r \mid r \geq 4\}$, where r is the *rank* of each algebra, or is one of five exceptional Lie algebras, G_2, F_4, E_6, E_7, or E_8. The *classical* Lie algebras are those belonging to the four families.

Each Lie algebra will be represented faithfully as subsets of

$$\mathfrak{gl}_n = \{n \times n \text{ matrices with complex entries }\}$$

with multiplication given by the commutator product, or *Lie bracket*:

$$[x, y] = xy - yx.$$

Let V be a $\mathfrak{g}$-module of dimension n with orthonormal basis $\{v_1, \ldots, v_n\}$. Many of the following calculations rest on the fact that the Lie algebras $\mathfrak{sl}_n$, $\mathfrak{so}_n$, and $\mathfrak{sp}_n$ are isomorphic to subalgebras of $\mathrm{End}(V)$, and V is a faithful V module. Let $E_{ij} \in \mathrm{End}(V)$ be defined by

$$E_{ij} v_\ell = \delta_{j\ell} v_i.$$

Type A A_r can be realized as $\mathfrak{sl}_{r+1}$, the set of all traceless matrices in $\mathfrak{gl}_{r+1}$. Thus it admits a faithful representation on the space $V = \mathbb{C}^{r+1}$. In this representation, $\mathfrak{sl}(V) = \{x \in \mathrm{End}(V) \mid \mathrm{tr}(x) = 0\}$ has basis

$$\{b_{ij}, h_k \mid 1 \leq i \neq j \leq n, 1 \leq k < n\},$$

where $b_{ij} = E_{ij}$ and $h_k = E_k - E_{k+1.k+1}$. Thus the dual basis with respect to $\langle , \rangle$ is given by $b^*_{ij} = b_{ji}$ and $h^*_k = \frac{1}{2} h_k$. The triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

is given by $\mathfrak{h} = \langle h_i \mid 1 \leq i \leq n-1\rangle$, $\mathfrak{n}^+ = \langle b_{ij} \mid 1 \leq i < j \leq n\rangle$, and $\mathfrak{n}^- = \langle b^*_{ij} \mid b_{ij} \in \mathfrak{n}^+\rangle$.

For type B, C, and D, define a bilinear form $\langle , \rangle_J$ on V by specifying $n \times n$ matrix J, where

$$n = \begin{cases} r+1 & \text{for type A,} \\ 2r+1 & \text{for type B,} \\ 2r & \text{for type C and D,} \end{cases}$$

and setting

$$\langle v, w\rangle_J = v^t J w.$$

Type B B_r can be realized as $\mathfrak{so}_{2r+1}$, the set of all matrices in $\mathfrak{sl}_{2r+1}$ such that

$$\langle xu, v\rangle_J + \langle u, xv\rangle_J = 0 \text{ for all } u, v \in V,$$

where $V = \mathbb{C}^{2r+1}$ and

$$J = \begin{pmatrix} 0 & \mathbb{1} & {\scriptstyle 0} \\ \mathbb{1} & 0 & \vdots \\ & & {\scriptstyle 0} \\ {\scriptstyle 0\cdots 0} & & {\scriptstyle 1} \end{pmatrix}$$

Thus B_r has basis

$$\{b_{ij} \mid 1 \le i < j \le n\},$$

where

$$b_{ij} = E_{ij} - E_{ji},$$

Define b_{ij} for $1 \le i, j \le n$ by

$$b_{ij} = \begin{cases} b_{ij} & \text{if } i < j, \\ -b_{ji} & \text{if } i > j, \\ 0 & \text{if } i = j. \end{cases}$$

So $\mathfrak{so}(V)$ has bracket

$$[b_{i_1,j_1}, b_{i_2,j_2}] = \begin{cases} b_{i_1,j_2}, & j_1 = i_2, \\ b_{j_1,i_2}, & i_1 = j_2, \\ -b_{i_1,i_2}, & j_1 = j_2, \\ -b_{j_1,j_2}, & i_1 = i_2, \\ 0, & \text{otherwise.} \end{cases},$$

and dual basis (with respect to $\langle , \rangle$) defined by

$$b_{ij}^* = b_{ij}.$$

Let $\mathfrak{h}$ be generated by $\{b_{2i-1,2i} \mid 1 \le i \le \frac{n}{2}\}$.

Type C C_r can be realized as $\mathfrak{sp}_{2r}$, the set of all matrices in $\mathfrak{sl}_{2r+1}$ such that

$$\langle xu, v\rangle_J + \langle u, xv\rangle_J = 0 \text{ for all } u, v \in V,$$

where $V = \mathbb{C}^{2r+1}$ and

$$J = \left(\begin{array}{c|c} 0 & -\mathbb{1} \\ \hline \mathbb{1} & 0 \end{array}\right)$$

So $\mathfrak{sp}(V)$ has basis (in $\mathrm{End}(V)$)

$$\left\{ \begin{array}{rcll} E_{ii} - E_{r+i,r+i} & = & A_i, & 1 \le i \le r, \\ E_{ij} - E_{r+j,r+i} & = & B_{i,j}, & 1 \le i \ne j \le r, \\ E_{i,r+i} & = & C_i, & 1 \le i \le r, \\ E_{i,r+j} + E_{j,r+i} & = & D_{ij}, & 1 \le i < j \le r, \\ E_{r+i,i} & = & E_i, & 1 \le i \le r, \\ E_{r+i,j} + E_{r+j,i} & = & F_{ij}, & 1 \le i < j \le r \end{array} \right\}.$$

With respect to $\langle , \rangle$, $\mathfrak{sp}(V)$ has dual basis

$$A_i^* = A_i, \quad B_{i,j}^* = B_{j,i},$$

$$C_i^* = 2E_i, \quad D_{ij}^* = F_{ij},$$
$$E_i^* = 2C_i, \quad \text{and} \quad F_{ij}^* = D_{ij}.$$

The triangular decomposition
$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$
is given by $\mathfrak{h} = \langle A_i \rangle$, $\mathfrak{n}^+ = \langle C_i, D_{ij}, B_{i<j} \rangle$, and $\mathfrak{n}^- = \langle b_i^* \mid b_i \in \mathfrak{n}^+ \rangle$. The skew symmetric form evaluated on the basis of V is

$$\langle v_i, v_j \rangle_J = \begin{cases} 1, & \text{if } j = i + r, \\ -1, & \text{if } j = i - r, \\ 0, & \text{otherwise.} \end{cases}$$

Type D D_r can be realized as $\mathfrak{sl}_{2r}$, , the set of all matrices in $\mathfrak{sl}_{2r}$ such that

$$\langle xu, v \rangle_J + \langle u, xv \rangle_J = 0 \text{ for all } u, v \in V,$$

where $V = \mathbb{C}^{2r+1}$ and
$$J = \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline \mathbb{1} & 0 \end{array} \right).$$

All basis and bracket calculations follow just as in type B.