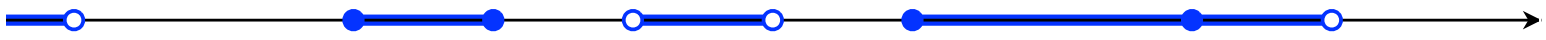
Lecture Notes

Math 3 / Fall 2012

Week 3

Interior points, isolated points, and endpoints

Suppose D is a set of real numbers and a is a point of D



- a is an **interior point** of D when there is an $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon)$ entirely contained in D
- a is an **isolated point** of D when there is an $\varepsilon > 0$ such that a is the only point of $(a - \varepsilon, a + \varepsilon)$ contained in D
- a is a **left endpoint** of D if there is an $\varepsilon > 0$ such that $[a, a + \varepsilon)$ is contained in D but no point of $(a - \varepsilon, a)$ is in D
- a is a **right endpoint** of D if there is an $\varepsilon > 0$ such that $(a - \varepsilon, a]$ is contained in D but no point of $(a, a + \varepsilon)$ is in D

Continuity: interior points

If a is an **interior point** of the domain of f then we say that f is **continuous at** a if $\lim_{x \to a} f(x) = f(a)$

Checklist:

1. $f(a)$ exists
2. a is an interior point of the domain of f
3. $\lim_{x \to a} f(x)$ exists
4. $\lim_{x \to a} f(x) = f(a)$

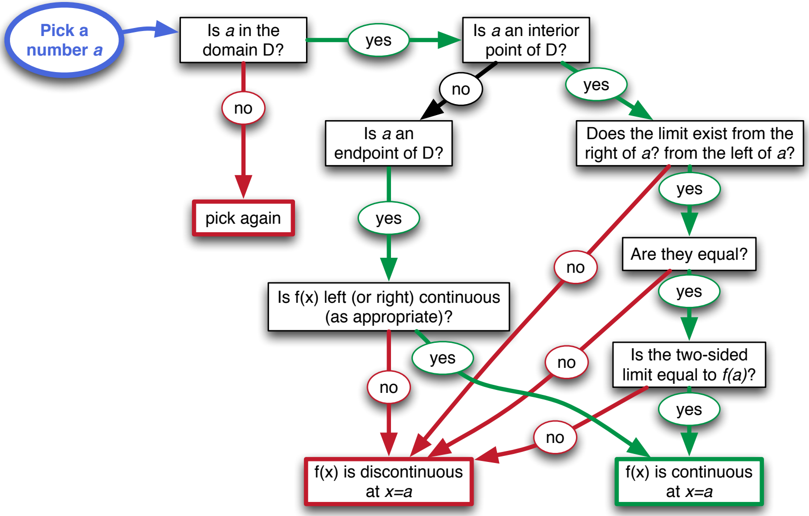
Continuity: endpoints

If a is a $\begin{bmatrix} \textbf{left} \\ \textbf{right} \end{bmatrix}$ **endpoint** of the domain of f then we say that f is **continuous at** a if $\lim\limits_{x \to a^{[\pm]}} f(x) = f(a)$

Checklist:

1. $f(a)$ exists
2. a is a $\begin{bmatrix} \text{left} \\ \text{right} \end{bmatrix}$ endpoint of the domain of f
3. $\lim\limits_{x \to a^{[\pm]}} f(x)$ exists
4. $\lim\limits_{x \to a^{[\pm]}} f(x) = f(a)$

Continuity: flowchart



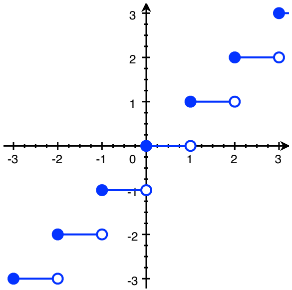
Continuity: common functions

All functions built only using power functions, exponential functions and trigonometric functions are continuous at every point of their domain.

The **floor function**

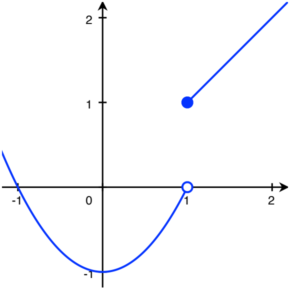
$$\lfloor x \rfloor = \text{the largest integer} \leq x$$

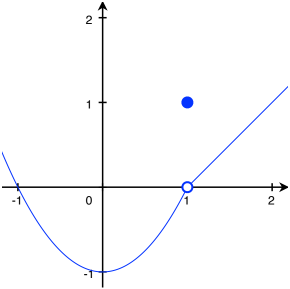
is a common function with many discontinuities



Types of discontinuities

- A **jump discontinuity** occurs when $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ both exist but they are not equal
- A **removable discontinuity** occurs when $\lim_{x \to a} f(x)$ exists but does not equal $f(a)$ (or $f(a)$ is undefined)





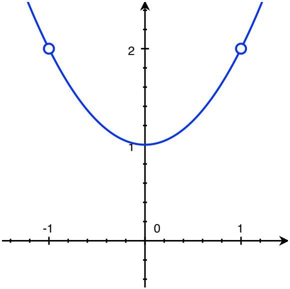
Fixing functions

If f has a removable discontinuity at a then

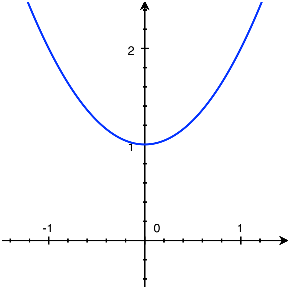
$$\bar{f}(x) = \begin{cases} f(x) & \text{if } x \neq a \\ \lim\limits_{x \to a} f(x) & \text{if } x = a \end{cases}$$

is continuous at a

$$\frac{x^4 - 1}{x^2 - 1}$$



$$x^2 + 1$$



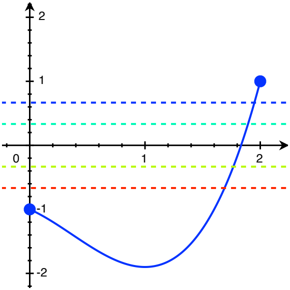
Intermediate value theorem

Theorem

Suppose f is continuous on the closed interval $[a, b]$. If

$$f(a) \leq y \leq f(b) \quad \text{or} \quad f(a) \geq y \geq f(b)$$

then there is at least one x in $[a, b]$ such that $f(x) = y$.



Finding roots

Theorem

Every polynomial of odd degree has at least one real root.

Because:

$$\begin{aligned} p(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\ &= x^n(a_n + a_{n-1}/x + \cdots + a_1/x^{n-1} + a_0/x^n) \end{aligned}$$

Therefore $\lim_{x\to\pm\infty} p(x) = \lim_{x\to\pm\infty} a_n x^n$

If n is odd and $a_n > 0$ then $\lim_{x\to\pm\infty} p(x) = \pm\infty$

If n is odd and $a_n < 0$ then $\lim_{x\to\pm\infty} p(x) = \mp\infty$

Finding roots

$$f(x) = x^3 - 3x + 1 \qquad root = 0.347296355\ldots$$

x	$f(x)$
0.0	$+1.000$
0.1	$+0.701$
0.2	$+0.408$
0.3	$+0.127$
0.4	-0.136
0.5	-0.375
0.6	-0.584
0.7	-0.757
0.8	-0.888
0.9	-0.971

x	$f(x)$
0.30	$+0.127$
0.31	$+0.100$
0.32	$+0.073$
0.33	$+0.046$
0.34	$+0.019$
0.35	-0.007
0.36	-0.033
0.37	-0.059
0.38	-0.085
0.39	-0.111

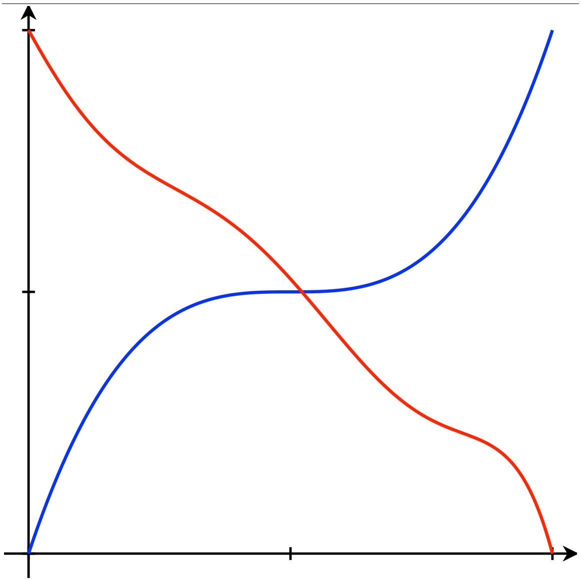
x	$f(x)$
0.340	$+0.019$
0.341	$+0.017$
0.342	$+0.014$
0.343	$+0.011$
0.344	$+0.009$
0.345	$+0.006$
0.346	$+0.003$
0.347	$+0.001$
0.348	-0.002
0.349	-0.004

Buddhist monk problem

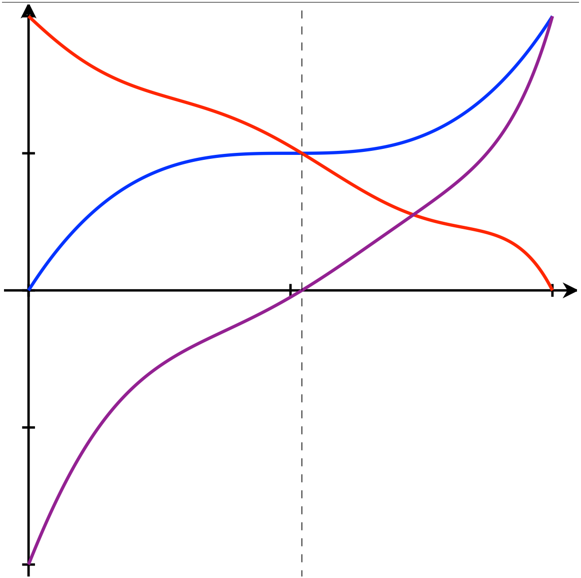
Exactly at sunrise one morning, a buddhist monk set out to climb a tall mountain along a very narrow path leading to a beautiful temple at the peak. The monk climbed the path at varying speed, stopping once in a while to rest and enjoy the beautiful scenery. The monk reached the temple just before sunset. After meditating for a few days, the monk began his journey back along the same path, walking at varying speed from sunrise to sunset.

Argue that there must be a spot along the path that the monk will pass on both trips at exactly the same time of day.

Buddhist monk problem



Buddhist monk problem



Equation of the tangent line

The **slope** of the tangent line to f at $(a, f(a))$ is the **derivative**

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

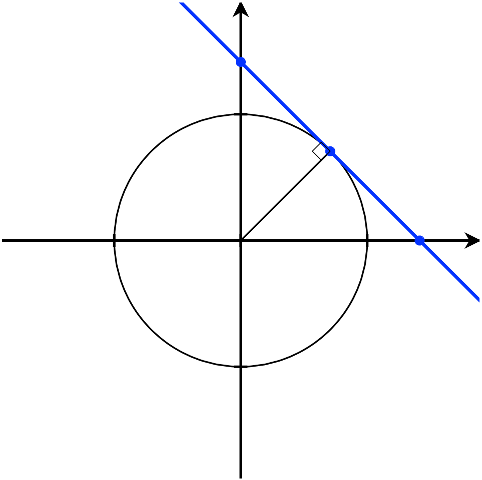
. . . when this limit exists!

The **point-slope equation** of the tangent line to f at $(a, f(a))$ is then

$$y = f'(a)(x - a) + f(a)$$

Tangent to a circle

$$f(x) = \sqrt{1-x^2} \quad \text{at} \quad \left(\sqrt{\tfrac{1}{2}}, \sqrt{\tfrac{1}{2}}\right)$$



Tangent to a circle

$$f(x) = \sqrt{1 - x^2} \quad \text{at} \quad \left(a, \sqrt{1 - a^2}\right)$$

- The slope of the line from $(0, 0)$ to $(a, \sqrt{1 - a^2})$ is $\dfrac{\sqrt{1 - a^2}}{a}$
- The perpendicular line has slope $-\dfrac{a}{\sqrt{1 - a^2}}$
- Therefore $f'(a) = -\dfrac{a}{\sqrt{1 - a^2}}$

$$\boxed{\text{The derivative of } \sqrt{1 - x^2} \text{ is } -\dfrac{x}{\sqrt{1 - x^2}}}$$

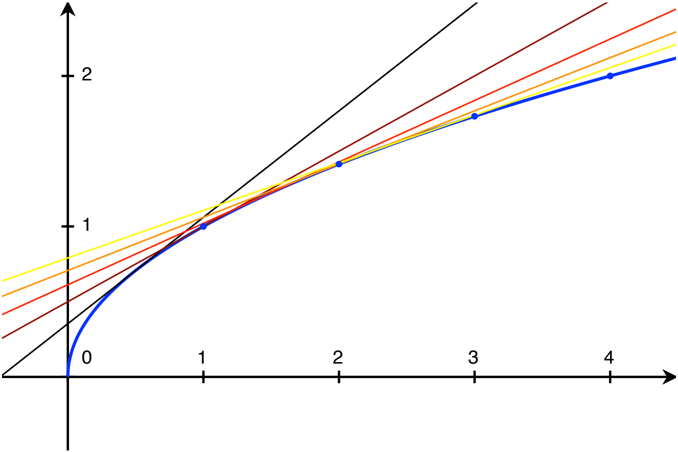
Tangent to $\sqrt{x}$ at $(a, \sqrt{a})$

$$\begin{aligned}
\text{slope} &= \lim_{h\to 0} \frac{\sqrt{a+h}-\sqrt{a}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{a+h}-\sqrt{a}}{h} \cdot \frac{\sqrt{a+h}+\sqrt{a}}{\sqrt{a+h}+\sqrt{a}} \\
&= \lim_{h\to 0} \frac{(a+h)-a}{h} \cdot \frac{1}{\sqrt{a+h}+\sqrt{a}} \\
&= 1 \cdot \frac{1}{\sqrt{a}+\sqrt{a}} = \frac{1}{2\sqrt{a}}
\end{aligned}$$

$$A^2 - B^2 = (A-B)(A+B)$$

$$\text{Tangent line:} \quad y = \frac{1}{2\sqrt{a}}(x-a) + \sqrt{a}$$

Tangent to $\sqrt{x}$ at $(a, \sqrt{a})$

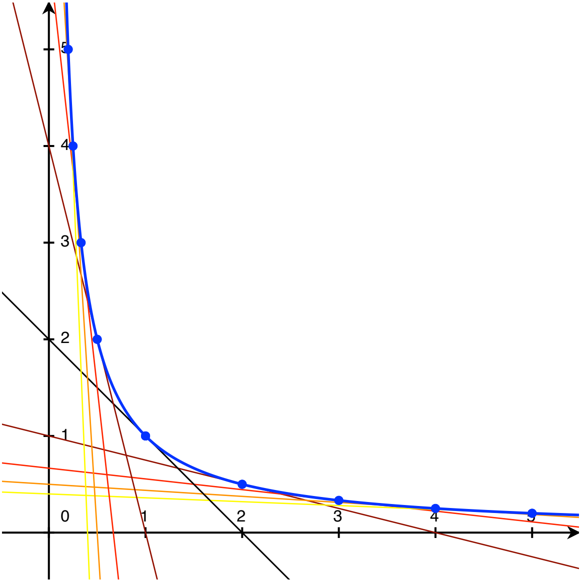


Tangent to $1/x$ at $(a, 1/a)$

$$\begin{aligned} \text{slope} &= \lim_{h\to 0} \frac{\dfrac{1}{a+h} - \dfrac{1}{a}}{h} \\ &= \lim_{h\to 0} \frac{\left(\dfrac{a-(a+h)}{a(a+h)}\right)}{h} \\ &= \lim_{h\to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2} \end{aligned}$$

$$\text{Tangent line:} \quad y = -\frac{1}{a^2}(x-a) + \frac{1}{a}$$

Tangent to $1/x$ at $(a, 1/a)$

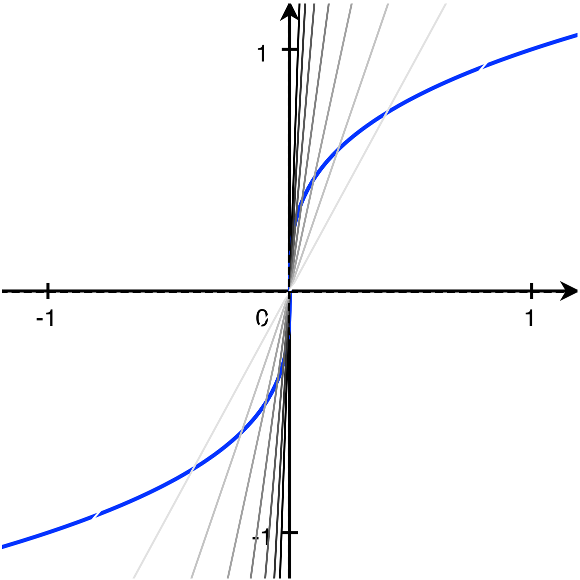


Tangent to $\sqrt[3]{x}$ at $(0,0)$

$$\begin{aligned}\text{slope} &= \lim_{h\to 0}\frac{\sqrt[3]{0+h}-\sqrt[3]{0}}{h} = \lim_{h\to 0}\frac{\sqrt[3]{h}}{h}\\ &= \lim_{h\to 0}\frac{h^{1/3}}{h^{1}} = \lim_{h\to 0} h^{-2/3} \quad \text{DNE!}\end{aligned}$$

$$\boxed{\text{The tangent line to } \sqrt[3]{x} \text{ at } (0,0) \text{ is the vertical line } x=0}$$

Tangent to $\sqrt[3]{x}$ at $(0, 0)$



Some derivatives...

- The derivative of a linear function is its slope
- The derivative of e^x is e^x
- The derivative of x^2 is $2x$
- The derivative of $\sqrt{x}$ is $1/2\sqrt{x}$
- The derivative of $1/x$ is $-1/x^2$

Translation, stretching, reflection

If we translate, stretch or reflect the graph of f together with its tangent line at a point, the transformed line remains tangent to the transformed graph at the transformed point

- **Vertical transformations:**
  - The derivative of $f(x) + b$ is $f'(x)$
  - The derivative of $af(x)$ is $af'(x)$
- **Horizontal transformations:**
  - The derivative of $f(x + b)$ is $f'(x + b)$
  - The derivative of $f(ax)$ is $af'(ax)$

Tangent line to b^x

Recall the change of base formula $b^x = e^{\ln(b)x}$

Since the derivative of $f(x) = e^x$ is $f(x) = e^x$ ...
... the derivative of $b^x = f(\ln(b)x)$ is

$$\ln(b)f(\ln(b)x) = \ln(b)b^x$$

Example

The tangent line to 10^x at $(0, 1)$ is

$$y = \ln(10)x + 1 \quad (\ln(10) = 2.302585092994\ldots)$$

Tangent line to $\ln x$

Recall that the graph of $\ln x$ is that of e^x reflected across the diagonal $x = y$

Suppose $b = e^a$ or equivalently $a = \ln b$
The tangent line to e^x at (a, b) has slope $e^a = b \ldots$
$\ldots$ and its reflection across $x = y$ has slope $1/b$

$$\boxed{\text{The derivative of } \ln x \text{ is } \frac{1}{x} \text{ (when } x > 0\text{)}}$$

What is the derivative of $\log_b x$?

(Answer: $1/\ln(b)x$)

Derivatives

The **derivative** of f is the function f' defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Other notations...

- **Leibniz style:**

$$\frac{dy}{dx} \quad \frac{df}{dx} \quad \frac{d}{dx}[x^2] \quad \cdots$$

- **Modern style:**

$$D_x y \quad D_x f \quad D_x[x^2] \quad \cdots$$

- **Postmodern style...**

Power rule

We saw that:

- The derivative of $x = x^1$ is $1 = 1 \cdot x^0$
- The derivative of x^2 is $2x = 2 \cdot x^1$
- The derivative of $\sqrt{x} = x^{1/2}$ is $1/2\sqrt{x} = \frac{1}{2} \cdot x^{-1/2}$
- The derivative of $1/x = x^{-1}$ is $-1/x^2 = (-1) \cdot x^{-2}$

Theorem

For every exponent p, the derivative of x^p is px^{p-1}.

Derivative of $f(x) = x^4$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h} \\
&= \lim_{h\to 0} \frac{(x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4) - x^4}{h} \\
&= \lim_{h\to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} \\
&= \lim_{h\to 0} 4x^3 + 6x^2h + 4xh^2 + h^3 = 4x^3
\end{aligned}
$$

Derivative of $f(x) = \sqrt[3]{x}$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt[3]{x+h} - \sqrt[3]{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt[3]{x+h} - \sqrt[3]{x}}{h} \cdot \frac{(\sqrt[3]{x+h})^2 + \sqrt[3]{x+h}\sqrt[3]{x} + (\sqrt[3]{x})^2}{(\sqrt[3]{x+h})^2 + \sqrt[3]{x+h}\sqrt[3]{x} + (\sqrt[3]{x})^2} \\
&= \lim_{h\to 0} \frac{(x+h) - x}{h} \cdot \frac{1}{(\sqrt[3]{x+h})^2 + \sqrt[3]{x+h}\sqrt[3]{x} + (\sqrt[3]{x})^2} \\
&= \lim_{h\to 0} \frac{1}{(\sqrt[3]{x+h})^2 + \sqrt[3]{x+h}\sqrt[3]{x} + (\sqrt[3]{x})^2} \\
&= \frac{1}{3(\sqrt[3]{x})^2} = \frac{1}{3(x^{1/3})^2} = \frac{1}{3x^{2/3}} = \frac{1}{3} \cdot x^{-2/3}
\end{aligned}
$$

$$A^3 - B^3 = (A-B)(A^2 + AB + B^2)$$

Differentiability and continuity

Theorem

If f is differentiable at x then f is continuous at x.

Because:

$$\lim_{h\to 0} f(x+h) - f(x) = \lim_{h\to 0} h \cdot \frac{f(x+h) - f(x)}{h} = 0 \cdot f'(x) = 0$$

Therefore $\lim_{h\to 0} f(x+h) = f(x)$

Sum rule

Theorem

If f and g are both differentiable at x then so is the sum $f+g$ and

$$(f+g)'(x) = f'(x) + g'(x).$$

Because:

$$\begin{aligned}
(f+g)'(x) &= \lim_{h\to 0} \frac{(f(x+h)+g(x+h)) - (f(x)+g(x))}{h} \\
&= \lim_{h\to 0} \left(\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h} \right) \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= f'(x) + g'(x)
\end{aligned}$$

Product rule

Theorem

If f and g are both differentiable at x then so is the product $f \cdot g$ and

$$(f \cdot g)'(x) = f(x)g'(x) + f'(x)g(x).$$

Example

If $f(x) = 17$ and $g(x) = 1/x$ then

$$(f \cdot g)'(x) = f(x)g'(x) + f'(x)g(x) = 17 \cdot \frac{-1}{x^2} + 0 \cdot \frac{1}{x} = -\frac{17}{x^2}$$

Example

If $f(x) = x$ and $g(x) = x^2$ then

$$(f \cdot g)'(x) = f(x)g'(x) + f'(x)g(x) = x \cdot 2x + 1 \cdot x^2 = 3x^2$$

Because:

$$
\begin{aligned}
(f \cdot g)'(x) &= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \frac{\begin{array}{r} f(x+h)g(x+h) - f(x+h)g(x) \\ + f(x+h)g(x) - f(x)g(x) \end{array}}{h} \\
&= \begin{array}{c} \left(\lim_{h \to 0} f(x+h)\right) \left(\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}\right) \\ + \left(\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\right) \left(\lim_{h \to 0} g(x)\right) \end{array} \\
&= f(x)g'(x) + f'(x)g(x)
\end{aligned}
$$

Chain rule

Theorem

If g is differentiable at x and f is differentiable at $g(x)$ then the composite $f \circ g$ is differentiable at x and

$$(f \circ g)'(x) = g'(x)f'(g(x)).$$

We've already seen examples of this:

$$\frac{d}{dx}[f(ax)] = af'(ax) \quad \text{and} \quad \frac{d}{dx}[f(x+b)] = f'(x+b)$$

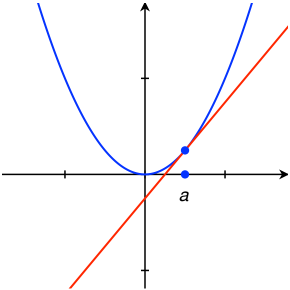
Combining these into one:

$$\frac{d}{dx}[f(ax+b)] = af'(ax+b)$$

Chain rule: an explanation

Suppose g is differentiable at a and f is differentiable at $g(a)$.
The tangent line to g at $(a, g(a))$ is

$$\ell(x) = g'(a)(x - a) + g(a).$$



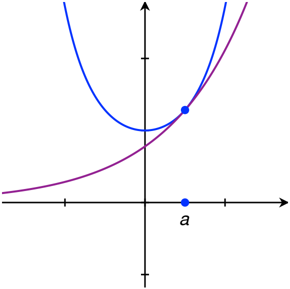
Note how $g(x) \approx \ell(x)$ near a.

Chain rule: an explanation

Because $g(x) \approx \ell(x)$ near a, the compositions

$$f(g(x)) \qquad \text{and} \qquad f(\ell(x))$$

are tangent at a.



The tangent line to $f(\ell(x)) = f(g'(a)(x-a) + g(a))$ is has slope

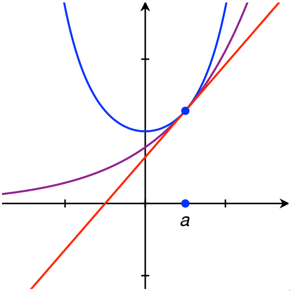
$$g'(a)f'(g'(a)(a-a) + g(a)) = g'(a)f'(g(a)).$$

Chain rule: an explanation

Since $f(g(x))$ is tangent to $f(\ell(x))$ at a and $f(\ell(x))$ is tangent to the line

$$k(x) = g'(a)f'(g(a))(x-a) + f(g(a))$$

at a, this line $k(x)$ is tangent to $f(g(x))$ at a.



Therefore the derivative of $f(g(x))$ at a is $g'(a)f'(g(a))$.