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MATH 126 - Dr. Barnett
Homework 4
due February 6, 2012

Problem 1.

Proof of (a). Given a sequence of orthogonal polynomials $\{q_n\}_{n=1}^{\infty}$ over the interval $[a, b]$ such that $\deg(q_n) = n$, we know that each q_n has all n roots in $[a, b]$. Suppose q_n has multiple roots and let $x_1, \ldots, x_k$ be the distinct roots at which q_n changes sign; in particular, this means that $k < n$. Define a polynomial

$$g(x) = \prod_{i=1}^{k}(x - x_i)$$

and note that g alternates signs at each x_i, and so it alternates in tandem with q_n as well. This means that $q_n(x)g(x)$ is a continuous nonzero function which is either nonnegative or nonpositive, and so

$$\int_a^b q_n(x)g(x)\,dx \neq 0.$$

However, since $\deg(g) = k < n$, then $g \perp q_n$, hence

$$\int_a^b q_n(x)g(x)\,dx = 0,$$

a contradiction; therefore q_n has n simple roots. □

Proof of (b). The Cauchy-Schwarz inequality states that

$$|\langle f, h\rangle| \leq ||f||_2 \cdot ||h||_2 \qquad \text{where} \qquad \langle f, h\rangle = \int_a^b f(x)h(x)\,dx$$

Now, let K be an integral operator and let $k(t, s)$ be its kernel function on the square $[a, b]^2$, the 2-norm of K is then defined by

$$||K||_2 = \sup_{||u||_2=1} ||Ku||_2.$$

Fix $t_0 \in [a, b]$ and consider the following:

$$\begin{aligned} |(Ku)(t_0)| &= \left|\int_a^b k(t_0, s)u(s)\,ds\right| \\ &= |\langle k(t_0), u\rangle| \\ &\leq ||k(t_0)||_2 \cdot ||u||_2 \qquad \text{(Cauchy-Schwarz Inequality).} \end{aligned}$$

Supposing $||u||_2 = 1$, this says that $||Ku||_2 \leq ||k||_2$ since we chose $t_0 \in [a, b]$ arbitrarily; therefore $||K||_2 \leq ||k||_2$. □

Problem 2. Taking the derivative with respect to t of the integral equation tells us that $u'(t)$ is constant, hence $u(t) = at + b$ for some $a, b \in \mathbb{R}$. In fact, this constant tells us the value of the leading coefficient:

$$a = -\int_0^1 s^3(as + b)\, ds.$$

Plugging these values into our integral equation, we come up with the following dependence relation

$$\left(a + \int_0^1 s^3(as + b)\, dx\right) t + (b - 1) = 0,$$

which further says that $b = 1$. By linearity of the integral, we then know that

$$a\left(1 + \int_0^1 s^4\, ds\right) = -\int_0^1 s^3\, ds,$$

to which it is easily computed that $a = -\frac{5}{24}$. We then conclude that

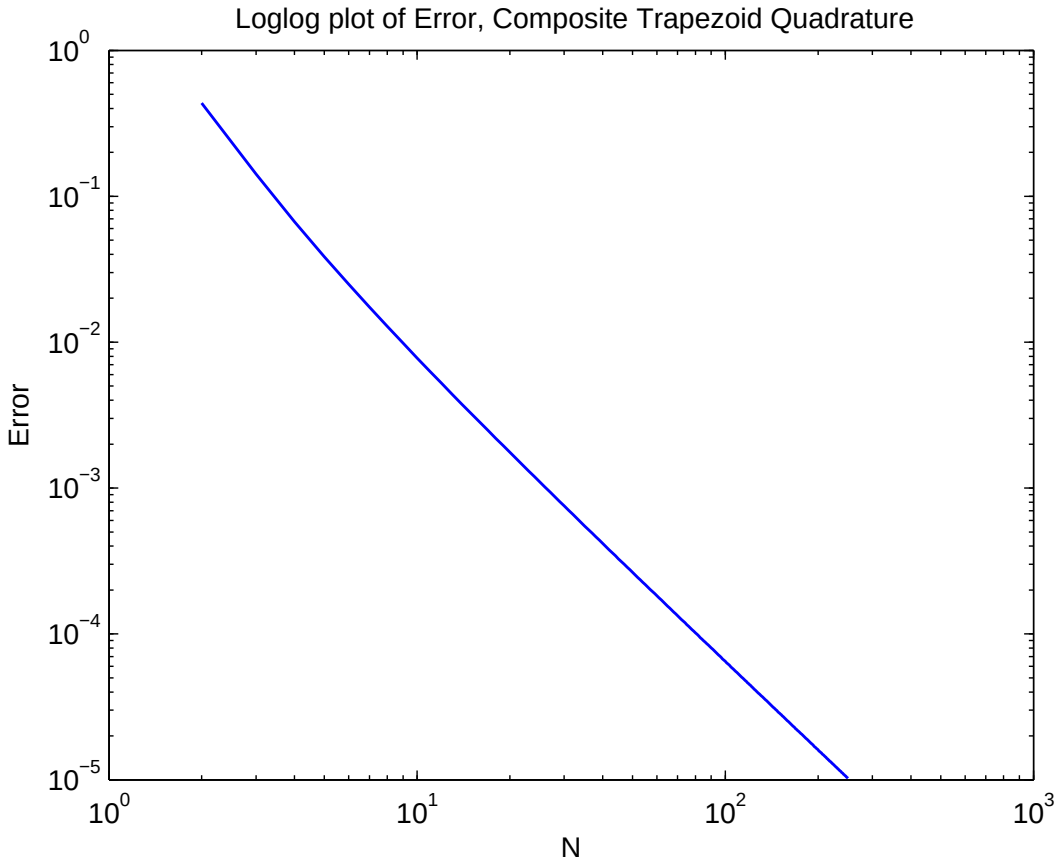
$$u(t) = -(5/24)t + 1$$

is the analytic solution to the integral equation, a fact which is easily verified by plugging $u(t)$ back into the given equation.

Problem 3. In part(a) of this problem, we compare sup norm error at nodes using Gaussian Quadrature and Composite Trapezoid Quadrature in computing the solution to the integral equation

$$u(t) + \int_0^1 e^{ts} u(s)\, ds = e^t + \frac{1}{t+1}(e^{t+1} - 1) \qquad \text{for all } t \in [0, 1],$$

via the Nyström method, which we know has analytic solution $u(t) = e^t$. The first method we used is the Composite Trapezoid Quadrature, which we already know to yield algebraic convergence of the maximum error.



As the figure above demonstrates, this property carries over via the Nyström method on the nodes; note that this method requires approximately 250 nodes to yield error that is approximately 10^{-5}.

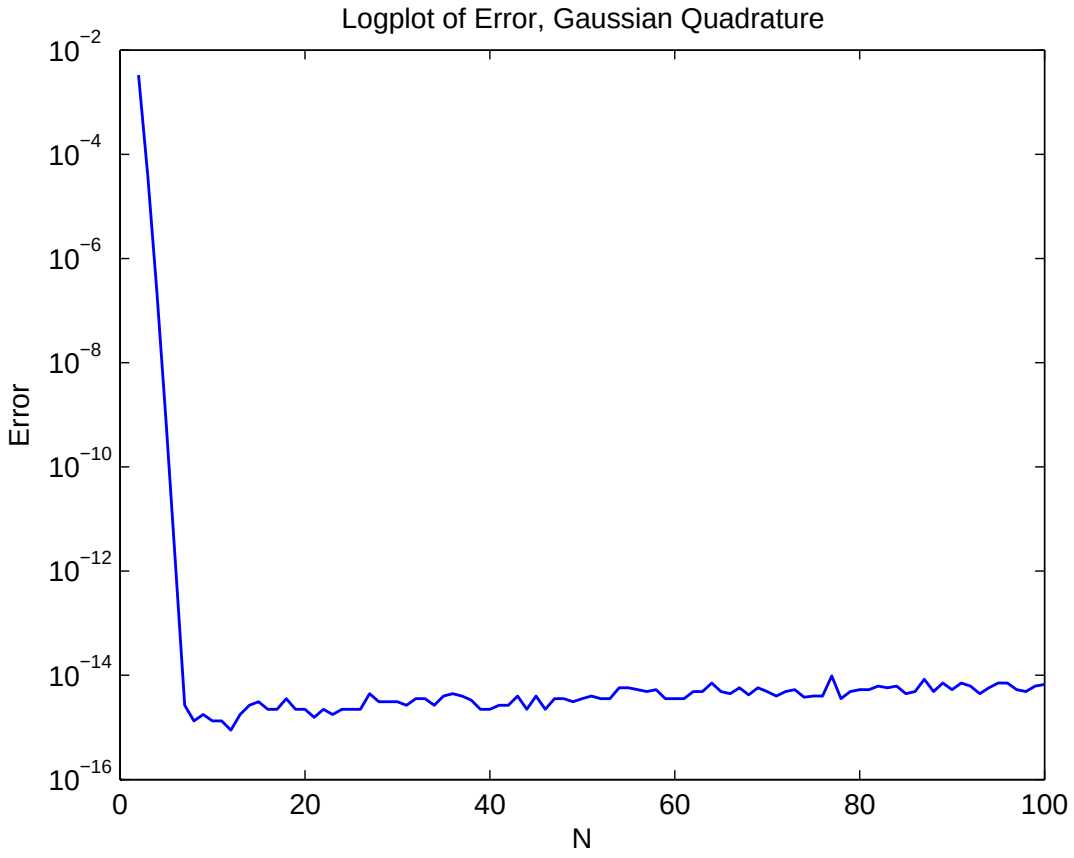
Next, we consider Gaussian Quadrature. In order to obtain the weights and nodes via "gauss.m" (provided in the previous homework) which only calculates them over the interval $[-1, 1]$, we need to perform a transformation on them so as to be valid in the interval $[0, 1]$. Given weights w_i and nodes x_i from the interval $[-1, 1]$, we can transform these values into weights and nodes over an arbitrary interval $[a, b]$ as follows:

$$\tilde{w}_i = \frac{b-a}{2} w_i \qquad \text{and} \qquad \tilde{x}_i = \frac{b-a}{2} x_i + \frac{a+b}{2}.$$

In our case, we are working over the interval $[0, 1]$, and so our adjusted weights and nodes were

$$\tilde{w}_i = \frac{w_i}{2} \qquad \text{and} \qquad \tilde{x}_i = \frac{x_i + 1}{2}.$$

After applying the Gaussian Quadrature scheme, we observed the following convergence:

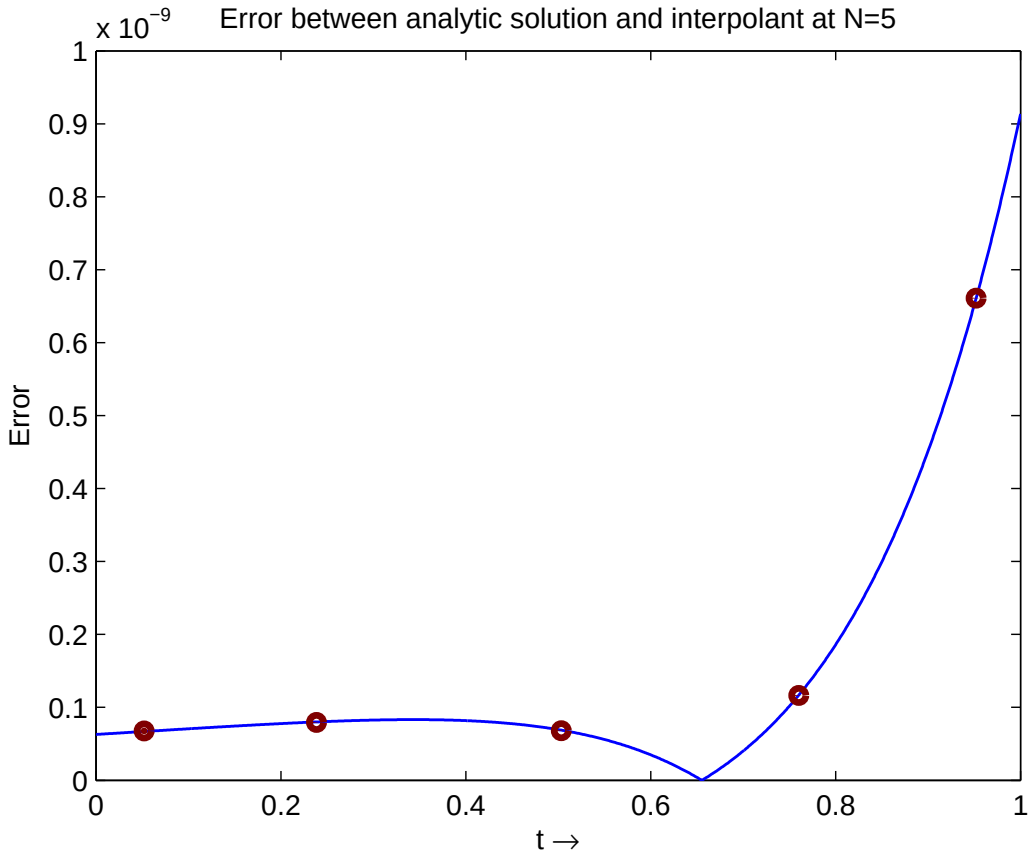


As expected, this scheme converged exponentially fast on the nodes, yielding machine precision with as few as seven nodes! In fact, we could obtain error smaller than 10^{-5} with as few as four nodes.

In applying the Nyström method, we are required to invert an $N \times N$ matrix to solve the linear system. One important question to ask ourselves is whether this matrix is well-conditioned or ill-conditioned. In this case, we find that the condition number stabilizes at around 2.356, which says that the matrix is well-conditioned. This tells us that we have a matrix which is full rank, unlike when we are solving large linear systems in MATLAB involving Vandermonde matrices.

Having only looked at the error on the nodes, we now turn our attention to the error incurred on the entire interpolated function obtained from the kernel. The following plot demonstrates this error for $N = 5$, with the positions of the nodes highlighted for

emphasis.



As is clearly visible in the above figure, the true error is not dictated by the error merely at the nodes. This tells us that it is not enough to look only at the error at the nodes of our quadrature scheme; we must also look at the error of the interpolant we extract from the kernel.

Problem 4. For part (a), we need to solve the equation $Ke^{imt} = \lambda_m e^{imt}$ for λ_m. To do so, we consider the following:

$$\begin{aligned} Ke^{imt} &= \int_0^{2\pi} \frac{1}{2\pi}\tilde{k}(s-t)e^{ims}\,ds \\ &= \frac{1}{2\pi}\int_{-t}^{2\pi-t} \tilde{k}(u)e^{im(u+t)}\,du \qquad \text{(substitution: } u = s-t) \\ &= \frac{1}{2\pi}e^{imt}\int_{-t}^{2\pi-t} \tilde{k}(u)e^{imu}\,du \\ &= \left(\frac{1}{2\pi}\int_0^{2\pi} \tilde{k}(u)e^{imu}\,du\right)e^{imt}. \end{aligned}$$

We were able to shift the bounds of integration forward by t since $\tilde{k}(s-t)$ is assumed to be 2π-periodic and it is easily seen that e^{ims} is 2π-periodic as well. What this says is that e^{imt} is an eigenfunction of K with eigenvalue

$$\lambda_m = \frac{1}{2\pi}\int_0^{2\pi} \tilde{k}(u)e^{imu}\,du.$$

For part (b), let

$$\tilde{k}(t) = \sum_{n\in\mathbb{Z}} \tilde{k}_n e^{-int}, \quad f(t) = \sum_{m\in\mathbb{Z}} f_m e^{-imt}, \quad \text{and} \quad u(t) = \sum_{p\in\mathbb{Z}} u_p e^{-ipt}$$

be Fourier series over $[0, 2\pi)$. Our goal will be to relate the coefficients which solve the integral equation

$$\int_0^{2\pi} \frac{1}{2\pi}\tilde{k}(s-t)u(s)\,ds = f(t).$$

First off, we will apply part (a) by observing that

$$\begin{aligned} f(t) &= \int_0^{2\pi} \frac{1}{2\pi}\tilde{k}(s-t)\left(\sum_{p\in\mathbb{Z}} u_p e^{-ips}\right)ds \\ &= \sum_{p\in\mathbb{Z}} u_p \int_0^{2\pi} \frac{1}{2\pi}\tilde{k}(s-t)e^{-ips}\,ds \\ &= \sum_{p\in\mathbb{Z}} u_p \lambda_{-p} e^{-ipt} \qquad \text{(by part (a))} \\ &= \sum_{p\in\mathbb{Z}} u_p \left(\frac{1}{2\pi}\int_0^{2\pi} \tilde{k}(s)e^{-ips}\,ds\right)e^{-ipt}. \end{aligned}$$

Substituting our Fourier series expansion of $\tilde{k}$ into this form and simplifying, we have

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\sum_{p\in\mathbb{Z}}\left(u_p\int_0^{2\pi}\sum_{n\in\mathbb{Z}}\tilde{k}_n e^{-ins}e^{-ips}\,ds\right)e^{-ipt} \\ &= \frac{1}{2\pi}\sum_{p\in\mathbb{Z}}\left(\sum_{n\in\mathbb{Z}}u_p\tilde{k}_n\int_0^{2\pi}e^{-i(n+p)s}\,ds\right)e^{-ipt} \\ &= \sum_{p\in\mathbb{Z}}u_p\tilde{k}_{-p}e^{-ipt}, \end{aligned}$$

where the last line follows by orthogonality of $\{e^{imt}\}$ over the interval $[0, 2\pi)$, and the fact that when $n = -p$ the integral evaluates to 2π. Multiplying both sides by e^{iqt} for some $q \in \mathbb{Z}$, writing f in its Fourier series expansion, and integrating over $[0, 2\pi)$, we have

$$\sum_{m\in\mathbb{Z}}f_m\int_0^{2\pi}e^{i(q-m)t}\,dt = \sum_{p\in\mathbb{Z}}u_p\tilde{k}_{-p}\int_0^{2\pi}e^{i(q-p)t}\,dt,$$

which implies that

$$f_q = u_q\tilde{k}_{-q},$$

an amazingly simple relationship!

To compute the 2-norm of K, we need to interpret exactly what this relationship is actually saying. If we view K as an infinite-dimensional representation matrix over the Fourier basis, then this relationship can be captured by the following:

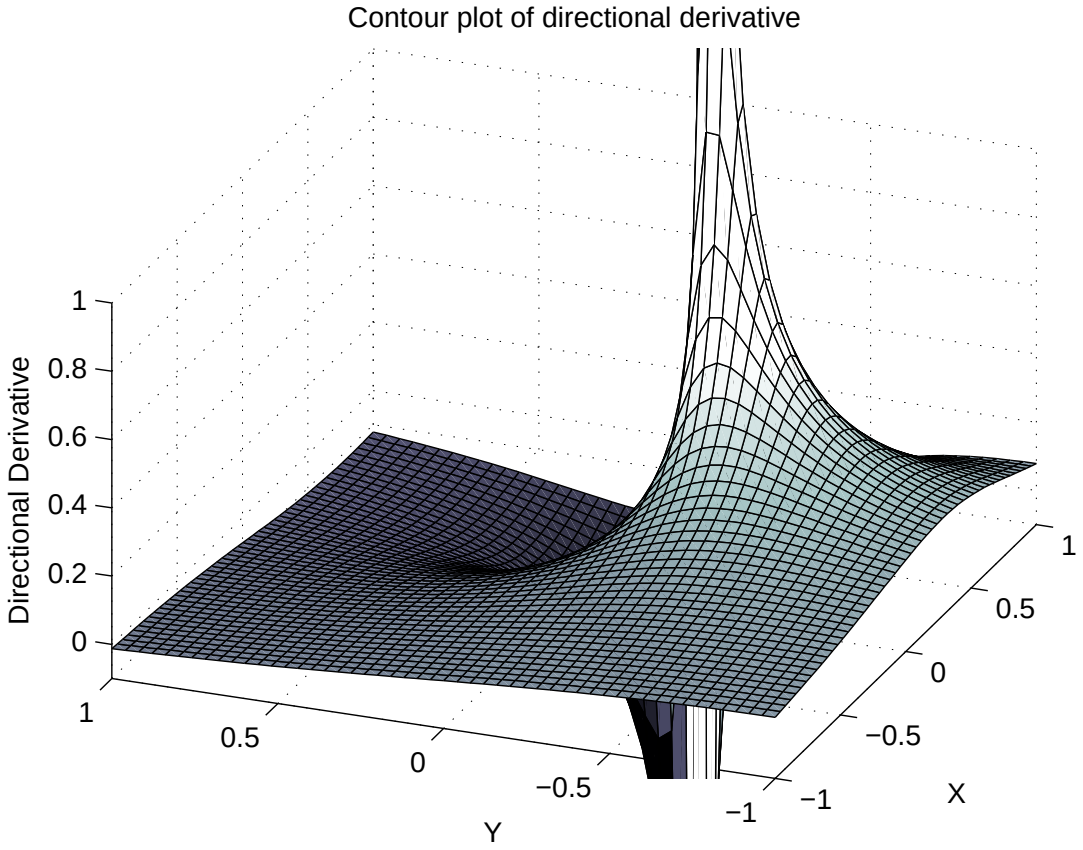
$$\begin{bmatrix} \ddots & & & & 0 \\ & \tilde{k}_1 & & & \\ & & \tilde{k}_0 & & \\ & & & \tilde{k}_{-1} & \\ 0 & & & & \ddots \end{bmatrix}\begin{bmatrix} \vdots \\ u_{-1} \\ u_0 \\ u_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ f_{-1} \\ f_0 \\ f_1 \\ \vdots \end{bmatrix}$$

This is highly convenient, since the singular values of a diagonal matrix are precisely the moduli of its entries. Furthermore, thinking of K as a matrix allows us to conclude that its 2-norm is the largest of these singular values. In effect, this says that $||K||_2$ is equal to the dominant coefficient (in modulus) of its kernel representation in the Fourier basis.

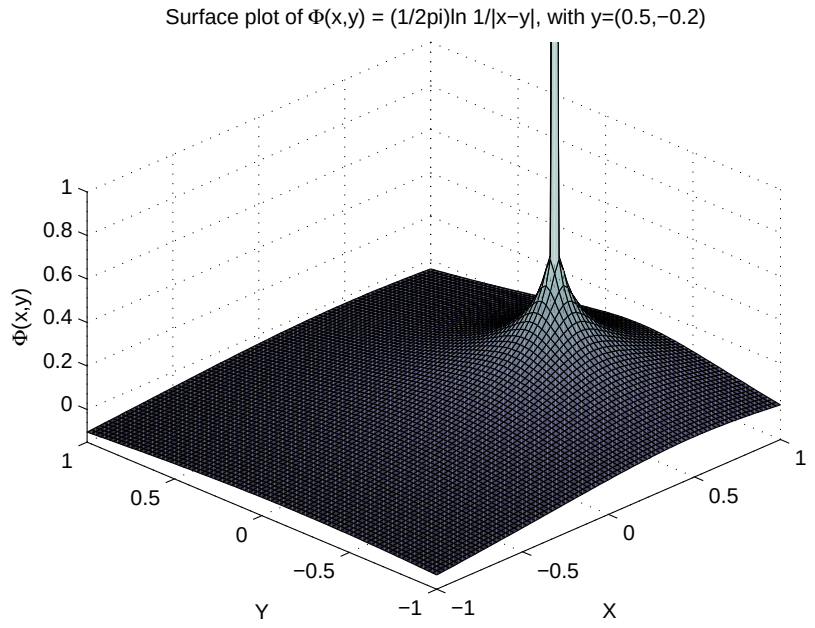
Lastly, since Parseval's equality says that the Fourier coefficients tend to zero as $|m| \to \infty$, then the smallest singular value tends to zero as well. This tells us that $||K^{-1}||_2$, which is defined as the inverse of the smallest singular value, must be infinite. Consequently, the condition number of the 1st-kind Fredholm problem is likewise infinite.

As for the 2nd-kind Fredholm problem, the condition number is infinite if and only if there is a Fourier coefficient which is exactly one. To see this, observe that the equation $u - Ku = f$ can be written as $(I - K)u = f$; this means that the singular values of $I - K$ have the form $|1 - \tilde{k}_m|$. If there is some n such that $\tilde{k}_n = 1$, then the smallest singular value will be zero, hence $||K^{-1}||_2 = \infty$. Likewise, if no Fourier coefficient is exactly one, then the reciprocal singular values are bounded below by some nonzero value, hence $||K^{-1}||$ is finite, since $\tilde{k}_m \to 0$ as $|m| \to \infty$ implies that the singular values tend to one.

Problem 5. See "dirdiv.m" and "hw4_5a.m" for the scripts which generate the following figure:



This graph makes sense in light of its actual surface plot:



If we fix a direction in which to compute the directional derivative, and then travel in a circle centered at the dipole, the calculated derivative will oscillate between its maximum value and its minimum value (the negative of the maximum). The closer we get to the dipole, the larger the oscillations; the further away we get, the milder the oscillations. Note that the plot of the actual function $\Phi(x, y) = (1/2\pi) \ln 1/|x - y|$ looks much like the plot of a complex function with a pole at y. Similarly, the directional derivative plot kinda looks like the surface plot of a complex function around an essential singularity.