Math 12, Fall 2007

Lecture 10

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Outline

1. Review and overview
   - Last class
2. Today's material
   - The gradient
3. Group Work
4. Next class

Outline

Differentiation
The chain rule

- $f(x, y), x = x(s), y = y(s)$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

- $f(x, y), x = x(s, t, r), y = y(s, t, r)$

$$f_s = f_x x_s + f_y y_s$$

$$f_t = f_x x_t + f_t y_t$$

Outline

The gradient vector field

- We have already seen the importance of the partial derivatives, f_x, f_y
- If $f : \mathbb{R}^2 \to \mathbb{R}$, then $\nabla f = f_x\vec{i} + f_y\vec{j}$ gives a vector field on $\mathbb{R}^2$ called the gradient vector field.
- If $f : \mathbb{R}^3 \to \mathbb{R}$, then $\nabla f = f_x\vec{i} + f_y\vec{j} + f_z\vec{k}$

∇f and directional derivatives

Theorem: Let (x_0, y_0) be a point in the plane and $\vec{u} =< a, b >$ a vector. Then, if $f : \mathbb{R}^2 \to \mathbb{R}$,

$$D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$$

Proof of Theorem

Let $g(t) = f(x_0 + ta, y_0 + tb)$ and compute the derivative of g at zero:

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}$$
$$= \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} = D_{\vec{u}} f(x_0, y_0)$$

But, we may also view $g(t)$ as $f(x, y)$ with $x = x_0 + ta, y = y_0 + tb$ and compute $g'(0)$ using the chain rule:

$$\frac{dg}{dt} = f_x x_t + f_y y_t$$

Since $x_t = a, y_t = b$, at $t = 0$ we have

$$g'(0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b = \nabla f(x_0, y_0) \cdot < a, b >$$

Q.E.D.

Direction of maximal ascent

The gradient point in the direction of maximal ascent of the function. In other words, out of all the unit vectors, $D_{\vec{u}}f$ is largest when $\vec{u} = \frac{\nabla f}{|\nabla f|}$.

Why is this true?

What is the direction of maximal descent? What happens at a minimum or maximum?

Cor: $D_{\vec{u}}f = 0$ for all $\vec{u}$ if and only if $\nabla f = \vec{0}$

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Rates of change using the gradient

Let $f(x,y) = xy - y^2 + x^3$

- What is the direction of maximal ascent at $(x,y) = (1,0)$?
- If you are walking along the surface alogn the curve $< x(t), y(t), f(x(t), y(t)) >$ and at $t = 1$ you are at the point $(1,1,1)$ traveling in the direction $x'(1) = -1, y'(1) = 0$, what is your instantaneous rate of change in the z direction at that time?

Tangent lines and planes

- Think of $z = f(x, y)$ as a level set $F(x, y, z) = f(x, y) - z = 0$. What is ∇F? Can we easily write the tangent plane to the surface using ∇F?
- Show that ∇f is orthogonal to the curve $f(x, y) = 0$ (or, in three variables, the surfaces $f(x, y, z) = 0$.

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Work for next class

- Reading: 15.7
- f07hw11