# The Casimir Invariant

Zajj Daugherty

April 24, 2008

Let $\mathfrak{g}$ be an $n$-dimensional semisimple Lie algebra. Let $\{b_i\}_i$ be any basis of $\mathfrak{g}$ and let $\{b^i\}$ be the dual basis of $\mathfrak{g}$ with respect to a fixed invariant (nondegenerate symmetric associative?) bilinear form on $\mathfrak{g}$ (the Killing form is a good example). To be precise, each $b^i$ is the unique element of $\mathfrak{g}$ for which $\langle b^i, b_j \rangle = \delta_{i,j}$. So in $\mathbb{R}^n$ the standard basis is its own dual with respect to the standard inner product.

The *casimir element* is an element of the universal enveloping algebra $U\mathfrak{g}$ given by the formula

$$\kappa = \sum_i b_i b^i.$$

Although $\kappa$ seems to depend on the chosen basis of $\mathfrak{g}$, one can show that it is indeed independent of this choice. Moreover, the invarience of $\langle , \rangle$ puts $\kappa$ in the center of $U\mathfrak{g}$ as it commutes with each element of $\mathfrak{g}$.

Given a representation $\rho$ of $\mathfrak{g}$ on $V$, the corresponding casimir is $\rho(\kappa)$, the linear operator on $V$ given (appropriately enough) by

$$\rho(\kappa) = \sum_i \rho(b_i)\rho(b^i).$$

The *quantum Casimir* is

$$\kappa_q = e^{-h\rho} \sum S(b_i) a_i \in \mathcal{Z}(U_q\mathfrak{g}).$$

So $\Delta(\kappa_q) = (R_{21}R_{12})^{-1}(\kappa_q \otimes \kappa_q)$. This is one reason why we care about $R_{21}R_{12}$. [Recall $g$ is *group-like* if $\Delta(g) = g \otimes g$. ]

**Theorem 1.** *The Casimir element $\kappa$ acts on a $\mathfrak{g}$-module $L(\lambda)$ of highest weight $\lambda$ by the constant*

$$\langle \lambda, \lambda + 2\rho \rangle.$$

*Proof.* Choose a basis of $\mathfrak{g}$ compatible with the triangular decomposition, and let $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$, and $h_\alpha \in \mathfrak{h}$ generate the subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$ corresponding to $\alpha \in R^+$. So

$$\langle e_\alpha, f_\alpha \rangle = \langle -\frac{1}{2}[e_\alpha, h_\alpha], f_\alpha \rangle = \frac{1}{2}\langle h_\alpha, [e_\alpha, f_\alpha] \rangle = \frac{1}{2}\alpha(h_\alpha) = \frac{1}{2}2 = 1.$$

Let $h_1^*, \ldots, h_\ell^*$ be a basis of $\mathfrak{h}$, and let $h_1^*, \ldots, h_\ell^*$ be the dual basis with respect to $\langle , \rangle$. Then

$$\{h_1, \ldots, h_\ell\} \cup \left\{e_\alpha, f_\alpha \mid \alpha \in R^+\right\} \text{ is a basis of } \mathfrak{g}, \text{ and}$$

$$\{h_1^*, \ldots, h_\ell^*\} \cup \left\{f_\alpha, e_\alpha \mid \alpha \in R^+\right\} \text{ is a dual basis of } \mathfrak{g},$$

with respect to $\langle , \rangle$.

Now we compute the constant by which $\kappa$ acts on $L(\lambda)$: If $L(\lambda)$ is a $\mathfrak{g}$-module generated by highest weight vector $v_\lambda^+$ of weight $\lambda$, i.e.

$$h_\alpha v_\lambda^+ = \lambda(h_\alpha) v_\lambda^+ \quad \text{and } e_\alpha v_\lambda^+ = 0 \quad \text{for } \alpha \in R^+,$$

then

$$\begin{aligned}
\kappa v_\lambda^+ &= \left( \sum_{i=1}^{\ell} h_i h_i^* + \sum_{\alpha \in R^+} e_\alpha f_\alpha + \sum_{\alpha \in R^+} f_\alpha e_\alpha \right) v_\lambda^+ \\
&= \left( \sum_{i=1}^{\ell} h_i h_i^* + \sum_{\alpha \in R^+} ([e_\alpha, f_\alpha] + f_\alpha e_\alpha + f_\alpha e_\alpha) \right) v_\lambda^+ \\
&= \left( \sum_{i=1}^{\ell} h_i \lambda(h_i^*) + \sum_{\alpha \in R^+} (h_\alpha + 2 f_\alpha e_\alpha) \right) v_\lambda^+ \\
&= \left( \langle \lambda, \lambda \rangle + \sum_{\alpha \in R^+} (\langle \lambda, \alpha \rangle + 0) \right) v_\lambda^+ \\
&= \langle \lambda, \lambda + 2\rho \rangle
\end{aligned}$$

□

So if $M$ and $V$ are $\mathfrak{g}$-modules,

$$\kappa \cdot (M \otimes V) = ((\kappa \otimes 1) + (1 \otimes \kappa) + 2t) \cdot (M \otimes V), \qquad \text{where} \quad t = \sum_i b_i \otimes b_i^*. \tag{0.1}$$

Thus the element $t$ acts on the $L(\lambda)$ isotypic component of $L(\mu) \otimes L(\nu)$ by the constant

$$\tfrac{1}{2}(\langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \nu, \nu + 2\rho \rangle). \tag{0.2}$$

## Some $\kappa$ lemmas:

Let $\mathfrak{g} = \mathfrak{sl}_n$, $\mathfrak{so}_n$, or $\mathfrak{sp}_n$ and $\nu = \omega_1$.

**Lemma 2.** *Let $V = L(\omega_1)$.*

1. *If $\mathfrak{g} = \mathfrak{sl}(V)$, then*
$$t^\lambda_{\mu\omega_1} = c(b),$$
*where $b$ is the box added to obtain $\lambda$ from $\mu$.*

2. *If $\mathfrak{g} = \mathfrak{so}(V)$, then*
$$t^\lambda_{\mu\omega_1} = \begin{cases} c(b) & \text{if } b \text{ is added,} \\ -c(b) - (n-1) & \text{if } b \text{ is removed,} \end{cases}$$
*where $b$ is the box added or removed to obtain $\lambda$ from $\mu$.*

3. *If* $\mathfrak{g} = \mathfrak{sp}(V)$*, then*

$$t^{\lambda}_{\mu\omega_1} = \begin{cases} c(b) & \text{if } b \text{ is added,} \\ -c(b) - (n+1) & \text{if } b \text{ is removed,} \end{cases}$$

*where* $b$ *is the box added or removed to obtain* $\lambda$ *from* $\mu$*.*

*Proof.*

**Case 1:** $\mathfrak{g} = \mathfrak{sl}(V)$**.** Adding a box to $\mu$ in the $i^{\text{th}}$ row is equivalent to adding $\varepsilon_i - \frac{1}{n}(\varepsilon_1 + \cdots + \varepsilon_n)$. Note that $\omega_1 = \varepsilon_1 - \frac{1}{n}(\varepsilon_1 + \cdots + \varepsilon_n)$. So

$$\lambda = \mu + (\varepsilon_i - \varepsilon_1 + \omega_1).$$

Finally, recall $2\rho = \sum_i \omega_i = \sum_j (n + 1 - 2j)\varepsilon_j$.
If $i = 1$, then $\lambda = \mu + \omega_1$, so

$$\begin{aligned} 2t^{\lambda}_{\mu\omega_1} &= \langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \omega_1, \omega_1 + 2\rho \rangle \\ &= \langle \mu + \omega_1, \mu + \omega_1 + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \omega_1, \omega_1 + 2\rho \rangle \\ &= \langle \mu, \omega_1 \rangle + \langle \omega_1, \omega_1 + 2\rho \rangle - \langle \omega_1, \omega_1 + 2\rho \rangle \\ &= 2(\langle \mu, \varepsilon_1 \rangle - \langle \mu, \frac{1}{n}(\varepsilon_1 + \cdots + \varepsilon_n) \rangle) \\ &= 2\left(\mu_1 - \frac{|\mu|}{n}\right) \quad = \quad 2\big((\mu_1 + 1) - 1\big) - 2\frac{|\mu|}{n} \end{aligned}$$

Othewise, $i \neq 1$, so $\lambda = \mu + (\varepsilon_i - \varepsilon_1) + \omega_1$, so

$$\begin{aligned} 2t^{\lambda}_{\mu\omega_1} &= \big(\langle \mu + \varepsilon_i - \varepsilon_1 + \omega_1, \mu + \varepsilon_i - \varepsilon_1 + \omega_1 + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \omega_1, \omega_1 + 2\rho \rangle\big) \\ &= \Big(2\langle \mu, \omega_1 \rangle + 2\langle \varepsilon_i - \varepsilon_1, \mu \rangle + 2\langle \varepsilon_i - \varepsilon_1, \omega_1 \rangle + \langle \varepsilon_i - \varepsilon_1, \varepsilon_i - \varepsilon_1 + 2\rho \rangle\Big) \\ &= 2\left(\left(\mu_1 - \frac{|\mu|}{n}\right) + (\mu_i - \mu_1) + \left(-\frac{1}{n} - 1 + \frac{1}{n}\right) + \frac{1}{2}\Big((1 + n + 1 - 2i) - (-1 + n + 1 - 2)\Big)\right) \\ &= 2\left(\mu_i - \frac{|\mu|}{n} - 1 + 2 - i\right) \quad = \quad 2\big((\mu_i + 1) - i\big) - 2\frac{|\mu|}{n} \end{aligned}$$

The content of a box added rom row $i$ of $\mu$ is $(\mu_i + 1) - i$. Likewise, the content of a box removed from row $i$ of $\mu$ is $(\mu_i - 1) - i$.

**Case 2:** $\mathfrak{g} = \mathfrak{so}(V)$**.** Adding or removing a box from $\mu$ in the $i^{\text{th}}$ row is equivalent to adding or subtracting $\varepsilon_i$, i.e. $\lambda = \mu \pm \varepsilon_i$. Recall $\omega_1 = \varepsilon_1$, and let $y = n$, so $2\rho = \sum_i (y - 2i)\varepsilon_i$. Thus

$$\begin{aligned} 2t^{\lambda}_{\mu\nu} &= \big(\langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \omega_1, \omega_1 + 2\rho \rangle\big) \\ &= \big(\langle \mu \pm \varepsilon_i, \mu \pm \varepsilon_i + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle\big) \\ &= \big(\pm 2\langle \varepsilon_i, \mu \rangle + \langle \varepsilon_i, \varepsilon_i \pm 2\rho \rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho \rangle\big) \\ &= \pm 2\mu_i + \Big(1 \pm (y - 2i) - (1 + y - 2)\Big) \\ &= \begin{cases} 2(\mu_i + 1 - i), & \text{if a box is added,} \\ -2\left(\mu_i - i + y - 1\right), & \text{if a box is removed,} \end{cases} \\ &= \begin{cases} 2c(b), & \text{if } b \text{ is added,} \\ 2(-c(b) - (n - 1)), & \text{if } b \text{ is removed,} \end{cases} \end{aligned}$$

since a box added would be in column $\mu_i+1$ and row $i$, where a box removed would be in column $\mu_i$ and row $i$.

**Case 3:** $\mathfrak{g} = \mathfrak{sp}(V)$**.** Just as in case 2, $\lambda = \mu \pm \varepsilon_i$ and $\omega_1 = \varepsilon_1$. However, in this case $2\rho = \sum_i (y-2i)\varepsilon_i$ if we let $y = n+2$. So

$$\begin{aligned} 2t^{\lambda}_{\mu\nu} &= \begin{cases} 2(\mu_i + 1 - i) & \text{if a box is added,} \\ -2\left(\mu_i - i + y + 1\right) & \text{if a box is removed,} \end{cases} \\ &= \begin{cases} 2c(b) & \text{if } b \text{ is added,} \\ 2(-c(b) - (n+1)) & \text{if } b \text{ is removed,} \end{cases} \end{aligned}$$

□

The remaining lemmas can be shown similarly as in lemma 2.

**Lemma 3.** *If $M = L((s)^m)$, then*

$$\begin{aligned} \kappa_M &= (sm)y + sm(s-m) \\ &= \text{area}(s \times m \text{ rectangle}) \cdot (y + \text{ content of corner box }), \end{aligned}$$

*where*

$$y = \begin{cases} 2N-1 & \mathfrak{gl}_N \\ N & \mathfrak{sl}_N \\ N-1 & \mathfrak{so}_N \\ N+1 & \mathfrak{sp}_N \end{cases}$$

*So $\kappa$ "recovers" your rectangle.*

**Lemma 4.** *If $M = L(\mu) = L((s)^m)$, $N = L(\nu) = L((t)^n)$, $n \leq m$, then*

$$\kappa_{\mu+\nu} - \kappa_\mu - \kappa_\nu = 2stn.$$

**Lemma 5.** *If $\lambda = \mu - \varepsilon_i + \varepsilon_j$ is obtained from $\mu$ by moving a box from one row and into another, then*

$$\begin{aligned} \kappa_\lambda &= \kappa_\mu - 2\Big((\lambda_i - i) - (\lambda_j + 1 - j)\Big) \\ &= \kappa_\mu - 2(\text{content of old box} - \text{content of new box}) \end{aligned}$$