Math 25 Second Exam - Part b

November 10, 2009

Instructions: Show all work. You may feel free to consult your textbook or your notes, **but you may not consult any other source, animate or inanimate**.

You must show general procedures at all stages of a solution. Solving an equation or congruence by inspection or exhaustive search will get you no credit.

Your solutions are due at the beginning of class on Wednesday, November 11.

1. (20) The goal of this problem is to determine whether $1729 = 7 \cdot 13 \cdot 19$ is a strong pseudoprime to the base 2.

   (a) (5) As a useful lemma, show that $2^{36} \equiv 1 \pmod{1729}$.

   (b) (15) Determine whether or not 1729 is a strong pseudoprime to the base 2, giving all details. Note: you can use part (a) to make your argument a bit shorter.

2. (20) Define the arithmetic function F by $F(n) = \sum_{d|n} \phi(d)\tau(n/d)$ where ϕ is the Euler phi-function, and $\tau(n) = \sum_{d|n} 1$ is the number of divisors of n. **Assume without proof** that F is a multiplicative function.

   (a) (15) Show by induction on k that for any prime p, $F(p^k) = \sigma(p^k)$, where $\sigma(n) = \sum_{d|n} d$ is the sum of divisors function. Hint: Play with some base cases to see what is going on, and consider $F(p^{k+1}) - F(p^k)$.

   (b) (5) Prove or disprove $F(n) = \sigma(n)$ for all $n \geq 1$.