# Math 63 Winter 2009

# Real Analysis

# Midterm Exam

Friday, January 30

Your name (please print): ______________________________________________

**Instructions**: This is an open book, open notes take home exam. **Use of calculators is not permitted**. **You must justify all of your answers to receive credit,** unless instructed otherwise in a given problem.

The exam is to be submitted on Wednesday February 4 during the regular Math 63 class time. The exam consists of 10 problems. Your total exam score is the sum of your scores for the 10 problems. Please do all your work in this exam booklet.

**The Honor Principle requires that you neither give nor receive any aid on this exam.**

Grader's use only

1. ________ /10
2. ________ /10
3. ________ /10
4. ________ /10
5. ________ /10
6. ________ /10
7. ________ /10
8. ________ /10
9. ________ /10
10. ________ /10

**Total:** ________ /100

(1) Prove that $\sqrt{5} - \sqrt{3}$ is irrational.

(2) Let $x_1, x_2, \cdots, x_n$ be nonzero real numbers. Prove the inequality

$$(\sum_{k=1}^{n} x_k^4)(\sum_{k=1}^{n} \frac{1}{x_k^4}) - n^2 \geq 0.$$

(3) Put $P_{2,\mathbb{Q}}(x,y)$ to be the set of polynomials in variables $x, y$ with nonnegative rational coefficients of degree at most 2. For example $\frac{1}{3}xy + 3x^2 + \frac{5}{3}y^2 + 0x + 1y + 3$ is such a polynomial. Is $P_{2,\mathbb{Q}}(x,y)$ countable? Prove your answer.

(4) Let $S$ be a nonempty bounded set of real numbers. Put $-5S$ to be the set defined by $-5S = \{-5x : x \in S\}$. Prove that $\sup(-5S) = -5 \inf S$.

(5) Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences of complex numbers, and let $x, y \in \mathbb{C}$ be such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. From the definition of the limit prove that $\lim_{n\to\infty}(x_n - y_n) = x - y$. You are not allowed to use Theorem 3.2, Theorem 3.3 or similar statements.

(6) Let $d$ be a discrete metric on $\mathbb{C}$, i.e. $d(x, y) = 1$ for all $x \neq y$ and $d(x, x) = 0$ for all $x$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences of complex numbers such that $\lim_{n\to\infty} x_n = i$ and $\lim_{n\to\infty} y_n = 3$. These limits are computed in $(\mathbb{C}, d)$. Prove that there exists $N$ such that for all $n \geq N$ we have $|x_n + y_n| = \sqrt{10}$, where $|\cdot|$ is the usual absolute value of a complex number.

(7) Let $(X, d_x)$ and $(Y, d_y)$ be metric spaces and let $f : X \to Y$ be a surjective map such that for every open $U \subset Y$ its preimage $f^{-1}(U)$ is an open subset of $X$. Prove that if $X$ is compact then $Y$ is also compact.

(8) Let $E$ be a subset of a metric space $(X, d)$ and let $E'$ be the set of all limit points of $E$. Prove that each limit point of $E'$ is a limit point of $E$ or give an example where a limit point of $E'$ is not a limit point of $E$.

(9) Consider the sequence of real numbers $1, -1, 1, -1, 1, -1, \ldots$ Is it possible to find a metric $d$ on $\mathbb{R}$ such that this sequence converges with respect to this metric? Prove your answer.

(10) Let $(X, d_x)$ and $(Y, d_y)$ be metric spaces and let $f : X \to Y$ be a surjective map such that for every open $U \subset Y$ its preimage $f^{-1}(U)$ is an open subset of $X$. Show that if $F \subset Y$ is closed then $f^{-1}(F)$ is closed in $X$. Use this to prove that if $X$ is connected, then $Y$ is also connected.