The graded Birman-Murakami-Wenzl algebra

February 7, 2008

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1 Introduction

The Brauer algebras were introduced by Richard Brauer in 1937 in his study of the symplectic and orthogonal groups [Br]. His work was spurred by that of Frobenius and Schur, who provided the link between the representation theory of the symmetric group S_n and the general linear group via their commuting actions on the tensor space $V^{\otimes k}$. More generally, a *tensor power centralizer algebra* is an algebra of operators which preserves symmetries in a tensor space.

The graded Birman-Murakami-Wenzl algebra was defined by Nazarov [Naz] to capture the action of Jucys-Murphy operators on the irreducible representations of Brauer algebras. This algebra is the analog to the graded Hecke algebra of type A. Ariki, Mathas, and Rui [AMR] study cyclotomic quotients of $\mathcal{W}_k$.

The paper of Orellana and Ram [OR] provided a unified approach to studying tensor power centralizer algebras, including the affine and cyclotomic Birman-Murakami-Wenzl algebras. Here, we will define both the graded braid group and the graded BMW algebra, and explore representations of each.

Fix $k \in \mathbb{Z}_{\geq 0}$. The *group algebra of the symetric group* $\mathbb{C}S_k$ is the algebra over $\mathbb{C}$ given by generators $t_1, \dots, t_{k-1}$ and relations

$$t_i^2 = 1, \qquad \text{for } 1 \leq i \leq k-1, \tag{1.1}$$

$$\begin{aligned} t_i t_{i+1} t_i &= t_{i+1} t_i t_{i+1}, && \text{for } 1 \leq i \leq k-2, && (1.2)\\ t_i t_j &= t_j t_i, && \text{if } j \neq i \pm 1. && (1.3) \end{aligned}$$

The *graded braid group* $\mathbb{B}_k$ is the algebra generated by $\mathbb{C}S_k$ and $\mathbb{C}[y_1, \dots y_k]$ with additional relations

$$y_i y_j = y_j y_i, \qquad \text{for } 1 \le i, j \le k, \tag{1.4}$$
$$t_i y_j = y_j t_i \qquad \text{if } j \ne i,\ i+1, \tag{1.5}$$

$$t_i(y_i + y_{i+1}) = (y_i + y_{i+1})t_i \qquad \text{for } 1 \le i \le k-1. \tag{1.6}$$

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra with Casimir element

$$\kappa = \sum bb^*,$$

where b ranges over a basis of $\mathfrak{g}$, and $\{b^*\}$ is a dual basis to $\{b\}$. Let M and V be $\mathfrak{g}$-modules. In section 3, we will explore the action of the braid group on the tensor space $M \otimes V^{\otimes k}$ which commutes with the action of $\mathfrak{g}$. For this action, S_k acts by permutation of the factors and the y_i acts by the Casimir on the first i factors in the tensor space.

Fx a set of constants $\{ z^{(i)} \in \mathbb{C} \mid i \in \mathbb{Z}_{\ge 0} \}$ The *graded Birman-Murakami-Wenzl algebra* $\mathcal{W}_k$ is the quotient of the algebra $\mathbb{B}_k$ by the relations

$$t_i e_j = e_j t_i, \quad e_i e_j = e_j e_i, \quad \text{if } j \ne i,\ i \pm 1, \tag{1.7}$$
$$e_i y_j = y_j e_i, \quad \text{if } j \ne i,\ i+1, \tag{1.8}$$

$$e_i t_i = e_i = t_i e_i, \tag{1.9}$$
$$t_i e_{i+1} e_i = t_{i+1} e_i, \tag{1.10}$$
$$e_{i+1} e_i t_{i+1} = e_{i+1} t_i, \tag{1.11}$$

$$e_i e_{i+1} e_i = e_i \tag{1.12}$$
$$e_{i+1} e_i e_{i+1} = e_{i+1} \tag{1.13}$$

$$e_i(y_i + y_{i+1}) = 0 = (y_i + y_{i+1})e_i, \tag{1.14}$$

$$e_1 y_1^\ell e_1 = z^{(\ell)} e_1, \qquad \text{for } \ell \in \mathbb{Z}_{\ge 0}, \tag{1.15}$$

where e_i is defined by the relation

$$e_i - 1 = t_i y_i - y_{i+1} t_i. \tag{1.16}$$

Due to the relation (1.6), we also have $e_i - 1 = y_i t_i - t_i y_{i+1}$ The *Brauer algebra* $\mathbb{C}B_k(z^{(0)})$ is the subalgebra of $\mathcal{W}_k$ generated by the elements $t_1, \dots, t_{k-1}, e_1, \dots, e_{k-1}$.

In section 4, we describe an action of $\mathcal{W}_k$ on $M \otimes V^{\otimes k}$ for $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{g} = \mathfrak{so}_{2n}$.

Finally, in section **??**, we will discuss how this theory may be extended to explore the so-called two-boundary case.

2 Lie theoretic background

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra with basis $\{b_1, b_2, \dots b_m\}$. Then $\mathfrak{g}$ is a $\mathfrak{g}$-module under the adjoint action: x acts on $\mathfrak{g}$ by

$$\begin{array}{rcl} \mathrm{ad}_x : \mathfrak{g} & \to & \mathfrak{g} \\ y & \mapsto & [x, y]. \end{array}$$

The *Killing form* is the non-degenerate form $\langle , \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defined by

$$\langle x, y \rangle = \mathrm{Tr}(\mathrm{ad}_x \mathrm{ad}_y).$$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\mathfrak{h}^* = \{\mu : \mathfrak{h} \to \mathbb{C}\}$ be its dual. If M is a $\mathfrak{g}$-module and $\mu \in \mathfrak{h}^*$ is a weight, then the μ *weight space* of M is

$$M_\mu = \{m \in M \mid hm = \mu(h)m, \text{ for all } h \in \mathfrak{h}\}.)$$

Since $\mathfrak{g}$ is semisimple, the adjoint action induces a decomposition of $\mathfrak{g}$ into weight spaces

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right),$$

where R is the set of *roots*, the weights in $\mathfrak{h}^*$ which have nontrivial weight spaces in $\mathfrak{g}$. Let R^+ be the set of positive roots, so $R^- = \{-\alpha \mid \alpha \in R^+\}$ and $R = R^+ \cup R^-$. Then we can fix the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \qquad \text{where } \mathfrak{n}^\pm = \bigoplus_{\alpha \in R^\pm} \mathfrak{g}_\alpha.$$

Since $\langle , \rangle$ is ad-invariant, if $x \in \mathfrak{g}_\alpha$ for $y \in \mathfrak{g}_\beta$, where $\alpha, \beta \in \mathfrak{h}^*$, then

$$\alpha(h)\langle x, y \rangle = \langle [h, x], y \rangle = -\langle x, [h, y] \rangle = -\beta(h)\langle x, y \rangle.$$

So

$$\text{if } \alpha \neq -\beta, \text{ then } \langle x, y \rangle = 0, \tag{2.1}$$

i.e. the subspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal precisely when $\alpha + \beta \neq 0$.

For each $\alpha \in R^+$, there is a unique element $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ for which $\alpha(h_\alpha) = 2$. If $y_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \neq 0$, there is a unique $y_\alpha \in \mathfrak{g}_{-\alpha}$ for which

$$[y_\alpha, y_\alpha] = h_\alpha.$$

Moreover,

$$[h_\alpha, y_\alpha] = 2y_\alpha, \quad \text{ and } [h_\alpha, y_\alpha] = -2y_\alpha,$$

and so $\mathrm{span}\{y_\alpha, y_\alpha, h_\alpha\}$ is isomorphic to $\mathfrak{sl}_2$ (see [Ser, Ch.6, §1]).

The form $\langle , \rangle$ is non-degenerate, so the map

$$\begin{array}{c} \mathfrak{h} \to \mathfrak{h}^* \\ h \mapsto \langle h, \cdot \rangle \end{array} \qquad \text{is an isomorphism.}$$

By definition, $\langle h_\alpha, h \rangle = \alpha(h)$ for $h \in \mathfrak{h}$, so this isomorphism induces the form $\langle , \rangle$ on $\mathfrak{h}^*$ defined by

$$\langle \lambda, \mu \rangle = \langle h_\lambda, h_\mu \rangle, \qquad \text{where } \langle h_\lambda, h \rangle = \lambda(h).$$

Let $\{\alpha_i\}$ be a base of simple roots for R (see [Ser, Ch.5, §8]). The *fundamental weights* $\omega_i \in \mathfrak{h}^*$ are those weights for which

$$\langle \omega_i, \alpha_j \rangle = \delta_{i,j}, \qquad \text{i.e.} \quad \omega_i(h_{\alpha_j}) = \delta_{i.j}.$$

The irreducible $\mathfrak{g}$-modules are indexed by the *dominant integral weights*, $\lambda \in P^+ \subseteq \mathfrak{h}^*$, where P^+ is the $\mathbb{Z}_{\geq 0}$-module generated by the ω_i:

$$P^+ = \sum_i \mathbb{Z}_{\geq 0}\omega_i.$$

Finally, let

$$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha. \tag{2.2}$$

2.1 Specifics for classical Lie algebras $\mathfrak{sl}_n$, $\mathfrak{so}_n$, and $\mathfrak{sp}_n$

Let V be a $\mathfrak{g}$-module of dimension n with orthonormal basis $\{v_1, \ldots, v_n\}$. Many of the following calculations rest on the fact that the Lie algebras $\mathfrak{sl}_n$, $\mathfrak{so}_n$, and $\mathfrak{sp}_n$ are isomorphic to subalgebras of $\mathrm{End}(V)$, and V is a faithful V module. For the details of many explicit calculations for this section, see appendix A

2.1.1 Bases and structure

Let $E_{ij} \in \mathrm{End}(V)$ be defined by

$$E_{ij} v_\ell = \delta_{j\ell} v_i.$$

$\underline{\mathfrak{sl}(V)}$**:** The Lie algebra $\mathfrak{sl}_n \cong \mathfrak{sl}(V) = \{x \in \mathrm{End}(V) \mid \mathrm{tr}(x) = 0\}$ has basis

$$\{b_{ij}, h_k \mid 1 \leq i \neq j \leq n, 1 \leq k < n\},$$

where

$$b_{ij} = E_{ij} \text{ and } h_k = E_k - E_{k+1.k+1}.$$

Thus the dual basis with respect to $\langle, \rangle$ is given by

$$b_{ij}^* = b_{ji} \text{ and } h_k^* = \frac{1}{2} h_k.$$

The triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

is given by $\mathfrak{h} = \langle h_i \mid 1 \leq i \leq n-1 \rangle$, $\mathfrak{n}^+ = \langle b_{ij} \mid 1 \leq i < j \leq n \rangle$, and $\mathfrak{n}^- = \langle b_{ij}^* \mid b_{ij} \in \mathfrak{n}^+ \rangle$.

$\underline{\mathfrak{so}(V)}$**:** The Lie algebra $\mathfrak{so}(V) \cong \mathfrak{so}_n$ is the subalgebra of $\mathfrak{sl}(V)$ for which the inner product on V is ad-invariant :

$$\mathfrak{so}(V) = \{x \in \mathfrak{sl}(V) | \langle xu, v \rangle + \langle u, xv \rangle = 0 \text{ for all } u, v \in V\}.$$

Thus $\mathfrak{so}(V)$ has basis

$$\{b_{ij} \mid 1 \leq i < j \leq n\},$$

where

$$b_{ij} = E_{ij} - E_{ji},$$

Define b_{ij} for $1 \le i, j \le n$ by

$$b_{ij} = \begin{cases} b_{ij} & \text{if } i < j, \\ -b_{ji} & \text{if } i > j, \\ 0 & \text{if } i = j. \end{cases}$$

So $\mathfrak{so}(V)$ has bracket

$$[b_{i_1,j_1}, b_{i_2,j_2}] = \begin{cases} b_{i_1,j_2}, & j_1 = i_2, \\ b_{j_1,i_2}, & i_1 = j_2, \\ -b_{i_1,i_2}, & j_1 = j_2, \\ -b_{j_1,j_2}, & i_1 = i_2, \\ 0, & \text{otherwise.} \end{cases},$$

and dual basis (with respect to $\langle , \rangle$) defined by

$$b_{ij}^* = b_{ij}.$$

Let $\mathfrak{h}$ be generated by $\{b_{2i-1,2i} \mid 1 \le i \le \frac{n}{2}\}$.

$\underline{\mathfrak{sp}(V)}$**:** If $n = 2r$ is even, $\mathfrak{sp}(V) \cong \mathfrak{sp}_n$ is the subalgebra of $\mathfrak{sl}(V)$ for which the skew-symmetric form $\langle , \rangle_J$ on V, given by the matrix

$$J = \left(\begin{array}{c|c} 0 & -\mathbb{1} \\ \hline \mathbb{1} & 0 \end{array} \right)$$

is ad-invariant:

$$\mathfrak{sp}(V) = \{a \in \mathfrak{sl}(V) | \langle au, v \rangle_J + \langle u, av \rangle_J = 0 \text{ for all } u, v \in V\},$$

where $\langle u, v \rangle_J = u^t J v$. Then $\mathfrak{sp}(V)$ has basis (in $\mathrm{End}(V)$)

$$\left\{ \begin{array}{rcll} E_{ii} - E_{r+i,r+i} & = & A_i, & 1 \le i \le r, \\ E_{ij} - E_{r+j,r+i} & = & B_{i,j}, & 1 \le i \ne j \le r, \\ E_{i,r+i} & = & C_i, & 1 \le i \le r, \\ E_{i,r+j} + E_{j,r+i} & = & D_{ij}, & 1 \le i < j \le r, \\ E_{r+i,i} & = & E_i, & 1 \le i \le r, \\ E_{r+i,j} + E_{r+j,i} & = & F_{ij}, & 1 \le i < j \le r \end{array} \right\}.$$

With respect to $\langle , \rangle$, $\mathfrak{sp}(V)$ has dual basis

$$A_i^* = A_i, \quad B_{i,j}^* = B_{j,i},$$

$$C_i^* = 2E_i, \quad D_{ij}^* = F_{ij},$$

$$E_i^* = 2C_i, \quad \text{and} \quad F_{ij}^* = D_{ij}.$$

The triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

is given by $\mathfrak{h} = \langle A_i \rangle$, $\mathfrak{n}^+ = \langle C_i, D_{ij}, B_{i<j} \rangle$, and $\mathfrak{n}^- = \langle b_i^* \mid b_i \in \mathfrak{n}^+ \rangle$. The skew symmetric form evaluated on the basis of V is

$$\langle v_i, v_j \rangle_J = \begin{cases} 1, & \text{if } j = i + r, \\ -1, & \text{if } j = i - r, \\ 0, & \text{otherwise.} \end{cases}$$

2.1.2 Weights

Let $\varepsilon_1, \dots, \varepsilon_n$ be an orthonormal basis of the vector space $\mathbb{R}^n$. Then

$$\mathfrak{h}^* = \begin{cases} \{\ \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \mid \lambda_i \in \mathbb{R}, \lambda_1 + \cdots + \lambda_n = 0\ \}, & \mathfrak{g} = \mathfrak{sl}_n \\ \{\ \lambda_1\varepsilon_1 + \cdots + \varepsilon_{\lfloor \frac{n}{2} \rfloor} \mid \lambda_i \in \mathbb{R}\ \}, & \mathfrak{g} = \mathfrak{so}_n \text{ or } \mathfrak{sp}_n. \end{cases}$$

The fundamental weights (the generators of P^+) are given by

$$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i - \tfrac{i}{n}(\varepsilon_1 + \cdots \varepsilon_n), \quad 1 \le i \le n-1, \qquad \text{if } \mathfrak{g} = \mathfrak{sl}_n,$$

$$\begin{array}{c} \omega_i = \varepsilon_1 + \cdots + \varepsilon_i, \quad 1 \le i \le r-2, \\ \omega_{r-1} = \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} - \varepsilon_r), \\ \omega_r = \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} + \varepsilon_r), \end{array} \qquad \text{if } \mathfrak{g} = \mathfrak{so}_{2r},$$

$$\begin{array}{c} \omega_i = \varepsilon_1 + \cdots + \varepsilon_i, \quad 1 \le i \le r-1, \\ \omega_r = \tfrac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_r), \end{array} \qquad \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1},$$

$$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i, \quad 1 \le i \le r, \qquad \text{if } \mathfrak{g} = \mathfrak{sp}_{2r}.$$

Thus, the dominant integral weights (which index finite dimensional irreducible $\mathfrak{g}$-modules) are all $\lambda \in \mathfrak{h}^*$ such that

$$\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_{n-1}\varepsilon_{n-1} - \tfrac{|\lambda|}{n}(\varepsilon_1 + \cdots \varepsilon_n), \qquad \text{if } \mathfrak{g} = \mathfrak{sl}_n,$$

$$\text{where} \quad \begin{array}{c} \lambda_i \in \mathbb{Z}, \\ 0 \le \lambda_{i+i} \le \lambda_i, \\ |\lambda| = \lambda_1 + \cdots + \lambda_{n-1}, \end{array}$$

$$\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r, \qquad \text{if } \mathfrak{g} = \mathfrak{so}_{2r},$$

$$\text{where} \quad \begin{array}{c} \lambda_i \in \mathbb{Z} \ \text{ for all } i \ \text{ or } \lambda_i \in \mathbb{Z} + \tfrac{1}{2} \ \text{ for all } \ i, \\ 0 \le |\lambda_r| \le \lambda_{r-1} \le \cdots \le \lambda_1, \end{array}$$

$$\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r, \qquad \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1},$$

$$\text{where} \quad \begin{array}{c} \lambda_i \in \mathbb{Z} \ \text{ for all } i \ \text{ or } \lambda_i \in \mathbb{Z} + \tfrac{1}{2} \ \text{ for all } \ i, \\ 0 \le \lambda_{i+i} \le \lambda_i, \end{array}$$

$$\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_r\varepsilon_r, \qquad \text{if } \mathfrak{g} = \mathfrak{sp}_{2r},$$

$$\text{where} \quad \begin{array}{c} \lambda_i \in \mathbb{Z}, \\ 0 \le \lambda_{i+i} \le \lambda_i. \end{array}$$

Finally, recall

$$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha = \sum_j \omega_j \in \mathfrak{h}^*.$$

Specifically, in each of these cases, this element is given by

$$2\rho = \sum_i (y-2i)\varepsilon_i, \qquad \text{where } y = \begin{cases} n+1, & \text{if } \mathfrak{g} = \mathfrak{sl}_n, \\ n, & \text{if } \mathfrak{g} = \mathfrak{so}_n, \\ n+2, & \text{if } \mathfrak{g} = \mathfrak{sp}_n. \end{cases}$$

2.1.3 Partitions

It is convenient at times to think of these weights as partitions. Identify each dominant integral weight λ with the configuration of boxes with $|\lambda_i|$ boxes in for i. In the case of $\mathfrak{g} = \mathfrak{so}_{2r}$, if $\lambda_r < 0$, mark the boxes with a minus sign. For example, if $\lambda = 3\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3 - \varepsilon_4$, the associated partition is

$$\begin{array}{|c|c|c|} \hline & & \\ \hline & \\ \cline{1-2} & \\ \cline{1-2} - \\ \cline{1-1} \end{array}$$

If b is j^{th} box of row i of λ, the *content* of b is

$$c(b) = (j-i)\frac{\lambda_i}{|\lambda_i|}.$$

For example, if we fill in the boxes in the above λ with their respective contents, we get

$$\begin{array}{|c|c|c|} \hline 0 & 1 & 2 \\ \hline -1 & 0 \\ \cline{1-2} -2 & -1 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$

Let $L(\lambda)$ be the highest-weight $\mathfrak{g}$-module of weight λ. Since $\mathfrak{g}$ is semisimple, every finite dimensional $\mathfrak{g}$-module M decomposes as the direct sum of simple modules:

$$M = \bigoplus_{\lambda \in P^+} c_\lambda L(\lambda).$$

In particular, for all $\mu \in P^+$,

$$L(\mu) \otimes L(\omega_1) = \bigoplus_{\lambda \in \Lambda(\mu)} L(\lambda), \tag{2.3}$$

where

$$\Lambda(\mu) = \begin{cases} \mu^+ = \left\{ \begin{array}{c} \text{partitions of height} < n \\ \text{obtained by adding a box to } \mu \end{array} \right\} & \text{if } \mathfrak{g} = \mathfrak{sl}_n \\[2ex] \mu^\pm = \left\{ \begin{array}{c} \text{partitions of height} \leq r \\ \text{obtained by adding a box to } \mu \end{array} \right\} & \text{if } \begin{array}{l} \mathfrak{g} = \mathfrak{so}_{2r},\ \mathfrak{sp}_{2r}, \text{ or} \\ \mathfrak{g} = \mathfrak{so}_{2r+1} \text{ and } \lambda_r \leq 0 \end{array} \\[2ex] \mu^\pm \cup \{\mu\} & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1} \text{ and } \lambda_r > 0. \end{cases}$$

Note that if $\mathfrak{g} = \mathfrak{so}_{2r}$, then there is a possibility of adding or removing a negative box, and the removal of a box from row r when $\mu_r = \frac{1}{2}$ or 0 will result in $\lambda_r = -\frac{1}{2}$ or -1, respectively.

2.2 The Casimir element and the operator T

The *Casimir element* of $\mathfrak{g}$ is

$$\kappa = \sum_i b_i b_i^*,$$

where b_i^* is the unique element of $\mathfrak{g}$ for which $\langle b_i^*, b_j \rangle = \delta_{ij}$. Since the Casimir is central in $U\mathfrak{g}$, it acts as a constant on each irreducible component of a $\mathfrak{g}$-module M:

Theorem 2.1. *The Casimir element κ acts on a $\mathfrak{g}$-module $L(\lambda)$ of highest weight λ by the constant*

$$\langle \lambda, \lambda + 2\rho \rangle.$$

Proof. As at the beginning of this section, choose a basis of $\mathfrak{g}$ compatible with the triangular decomposition, and let $y_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$, and $h_\alpha \in \mathfrak{h}$ generate the subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$ corresponding to $\alpha \in R^+$. So

$$\langle y_\alpha, y_\alpha \rangle = \langle -\frac{1}{2}[y_\alpha, h_\alpha], y_\alpha \rangle = \frac{1}{2}\langle h_\alpha, [y_\alpha, y_\alpha] \rangle = \frac{1}{2}\alpha(h_\alpha) = \frac{1}{2}2 = 1.$$

Let $h_1^*, \ldots, h_\ell^*$ be a basis of $\mathfrak{h}$, and let $h_1^*, \ldots, h_\ell^*$ be the dual basis with respect to $\langle , \rangle$. Then

$$\{h_1, \ldots, h_\ell\} \cup \left\{ y_\alpha, y_\alpha \mid \alpha \in R^+ \right\} \text{ is a basis of } \mathfrak{g}, \text{ and}$$

$$\{h_1^*, \ldots, h_\ell^*\} \cup \left\{ y_\alpha, y_\alpha \mid \alpha \in R^+ \right\} \text{ is a dual basis of } \mathfrak{g},$$

with respect to $\langle , \rangle$.

Now we compute the constant by which κ acts on $L(\lambda)$: If $L(\lambda)$ is a $\mathfrak{g}$-module generated by highest weight vector v_λ^+ of weight λ, i.e.

$$h_\alpha v_\lambda^+ = \lambda(h_\alpha) v_\lambda^+ \quad \text{and } y_\alpha v_\lambda^+ = 0 \quad \text{for } \alpha \in R^+,$$

then

$$\begin{aligned}
\kappa v_\lambda^+ &= \left(\sum_{i=1}^{\ell} h_i h_i^* + \sum_{\alpha \in R^+} y_\alpha y_\alpha + \sum_{\alpha \in R^+} y_\alpha y_\alpha \right) v_\lambda^+ \\
&= \left(\sum_{i=1}^{\ell} h_i h_i^* + \sum_{\alpha \in R^+} ([y_\alpha, y_\alpha] + y_\alpha y_\alpha + y_\alpha y_\alpha) \right) v_\lambda^+ \\
&= \left(\sum_{i=1}^{\ell} h_i \lambda(h_i^*) + \sum_{\alpha \in R^+} (h_\alpha + 2 y_\alpha y_\alpha) \right) v_\lambda^+ \\
&= \left(\langle \lambda, \lambda \rangle + \sum_{\alpha \in R^+} (\langle \lambda, \alpha \rangle + 0) \right) v_\lambda^+ \\
&= \langle \lambda, \lambda + 2\rho \rangle
\end{aligned}$$

□

Let $T \in \mathfrak{g} \otimes \mathfrak{g}$ be the operator on $V \otimes V$ given by

$$T = \sum_\ell b_\ell \otimes b_\ell^*.$$

So if M and V are $\mathfrak{g}$-modules,

$$\kappa \cdot (M \otimes V) = ((\kappa \otimes 1) + (1 \otimes \kappa) + 2T) \cdot (M \otimes V).$$

Corollary 2.2. *The element T acts on the λ-component of $L(\mu) \otimes L(\nu)$ by the constant*

$$T_{\mu\nu}^{\lambda} = \frac{1}{2}\big(\langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \nu, \nu + 2\rho \rangle\big).$$

From equation (2.3), if $\mathfrak{g} = \mathfrak{sl}_n$, $\mathfrak{so}_n$, or $\mathfrak{sp}_n$ and $\nu = \omega_1$, we know exactly how $L(\mu) \otimes L(\omega_1)$ decomposes. In fact, we can express these constants in terms of contents of boxes added or removed in the associated partitions.

Theorem 2.3. *Let $V = L(\omega_1)$.*

1. *If $\mathfrak{g} = \mathfrak{sl}(V)$, then*
$$T_{\mu\omega_1}^{\lambda} = c(b) - \frac{|\mu|}{n},$$
where b is the box added to obtain λ from μ.

2. *If $\mathfrak{g} = \mathfrak{so}(V)$, then*
$$T_{\mu\omega_1}^{\lambda} = \begin{cases} c(b) & \text{if } b \text{ is added,} \\ -c(b) - (n-1) & \text{if } b \text{ is removed,} \end{cases}$$
where b is the box added or removed to obtain λ from μ.

3. *If $\mathfrak{g} = \mathfrak{sp}(V)$, then*
$$T_{\mu\omega_1}^{\lambda} = \begin{cases} c(b) & \text{if } b \text{ is added,} \\ -c(b) - (n+1) & \text{if } b \text{ is removed,} \end{cases}$$
where b is the box added or removed to obtain λ from μ.

Proof.

Case 1: $\mathfrak{g} = \mathfrak{sl}(V)$. Adding a box to μ in the i^{th} row is equivalent to adding $\varepsilon_i - \frac{1}{n}(\varepsilon_1 + \cdots + \varepsilon_n)$. Note that $\omega_1 = \varepsilon_1 - \frac{1}{n}(\varepsilon_1 + \cdots + \varepsilon_n)$. So

$$\lambda = \mu + (\varepsilon_i - \varepsilon_1 + \omega_1).$$

Finally, recall $2\rho = \sum_i \omega_i = \sum_j (n + 1 - 2j)\varepsilon_j$.
If $i = 1$, then $\lambda = \mu + \omega_1$, so

$$\begin{aligned}
2T_{\mu\omega_1}^{\lambda} &= \langle \lambda, \lambda + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \omega_1, \omega_1 + 2\rho \rangle \\
&= \langle \mu + \omega_1, \mu + \omega_1 + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \omega_1, \omega_1 + 2\rho \rangle \\
&= \langle \mu, \omega_1 \rangle + \langle \omega_1, \omega_1 + 2\rho \rangle - \langle \omega_1, \omega_1 + 2\rho \rangle \\
&= 2(\langle \mu, \varepsilon_1 \rangle - \langle \mu, \frac{1}{n}(\varepsilon_1 + \cdots + \varepsilon_n) \rangle) \\
&= 2\left(\mu_1 - \frac{|\mu|}{n}\right) \qquad = \qquad 2\big((\mu_1 + 1) - 1\big) - 2\frac{|\mu|}{n}
\end{aligned}$$

Othewise, $i \neq 1$, so $\lambda = \mu + (\varepsilon_i - \varepsilon_1) + \omega_1$, so

$$\begin{aligned}
2T_{\mu\omega_1}^{\lambda} &= \big(\langle \mu + \varepsilon_i - \varepsilon_1 + \omega_1, \mu + \varepsilon_i - \varepsilon_1 + \omega_1 + 2\rho \rangle - \langle \mu, \mu + 2\rho \rangle - \langle \omega_1, \omega_1 + 2\rho \rangle\big) \\
&= \Big(2\langle \mu, \omega_1 \rangle + 2\langle \varepsilon_i - \varepsilon_1, \mu \rangle + 2\langle \varepsilon_i - \varepsilon_1, \omega_1 \rangle + \langle \varepsilon_i - \varepsilon_1, \varepsilon_i - \varepsilon_1 + 2\rho \rangle\Big) \\
&= 2\left(\left(\mu_1 - \frac{|\mu|}{n}\right) + (\mu_i - \mu_1) + \left(-\frac{1}{n} - 1 + \frac{1}{n}\right) + \frac{1}{2}\Big((1 + n + 1 - 2i) - (-1 + n + 1 - 2)\Big)\right) \\
&= 2\left(\mu_i - \frac{|\mu|}{n} - 1 + 2 - i\right) \qquad = \qquad 2\big((\mu_i + 1) - i\big) - 2\frac{|\mu|}{n}
\end{aligned}$$

The content of a box added rom row i of μ is $(\mu_i + 1) - i$. Likewise, the content of a box removed from row i of μ is $(\mu_i - 1) - i$.

Case 2: $\mathfrak{g} = \mathfrak{so}(V)$. Adding or removing a box from μ in the i^{th} row is equivalent to adding or subtracting ε_i, i.e. $\lambda = \mu \pm \varepsilon_i$. Recall $\omega_1 = \varepsilon_1$, and let $y = n$, so $2\rho = \sum_i (y - 2i)\varepsilon_i$. Thus

$$\begin{aligned}
2T^{\lambda}_{\mu\nu} &= \big(\langle \lambda, \lambda + 2\rho\rangle - \langle \mu, \mu + 2\rho\rangle - \langle \omega_1, \omega_1 + 2\rho\rangle\big) \\
&= \big(\langle \mu \pm \varepsilon_i, \mu \pm \varepsilon_i + 2\rho\rangle - \langle \mu, \mu + 2\rho\rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho\rangle\big) \\
&= \big(\pm 2\langle \varepsilon_i, \mu\rangle + \langle \varepsilon_i, \varepsilon_i \pm 2\rho\rangle - \langle \varepsilon_1, \varepsilon_1 + 2\rho\rangle\big) \\
&= \pm 2\mu_i + \Big(1 \pm (y - 2i) - (1 + y - 2)\Big) \\
&= \begin{cases} 2(\mu_i + 1 - i), & \text{if a box is added,} \\ -2\left(\mu_i - i + y - 1\right), & \text{if a box is removed,} \end{cases} \\
&= \begin{cases} 2c(b), & \text{if } b \text{ is added,} \\ 2(-c(b) - (n-1)), & \text{if } b \text{ is removed,} \end{cases}
\end{aligned}$$

since a box added would be in column $\mu_i + 1$ and row i, where a box removed would be in column μ_i and row i.

Case 3: $\mathfrak{g} = \mathfrak{sp}(V)$. Just as in case 2, $\lambda = \mu \pm \varepsilon_i$ and $\omega_1 = \varepsilon_1$. However, in this case $2\rho = \sum_i (y - 2i)\varepsilon_i$ if we let $y = n + 2$. So

$$\begin{aligned}
2T^{\lambda}_{\mu\nu} &= \begin{cases} 2(\mu_i + 1 - i) & \text{if a box is added,} \\ -2\left(\mu_i - i + y + 1\right) & \text{if a box is removed,} \end{cases} \\
&= \begin{cases} 2c(b) & \text{if } b \text{ is added,} \\ 2(-c(b) - (n+1)) & \text{if } b \text{ is removed,} \end{cases}
\end{aligned}$$

$\square$

Finally, we will denote the operator T acting on the j and j' factors of $V^{\otimes k}$ by $T_{j,j'}$, i.e.

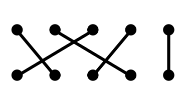
$$T_{j,j'} \cdot (m \otimes v_{i_1} \otimes \cdots v_{i_k}) = \sum_b (m \otimes v_{i_1} \otimes \cdots \otimes b v_{i_j} \otimes \cdots \otimes b^* v_{i_{j'}} \otimes \cdots \otimes v_{i_k}). \tag{2.4}$$

If $\{b\}$ is a basis of $\mathfrak{g}$, then $\{b^*\}$ presents an alternate basis for $\mathfrak{g}$, with dual basis $\{b\}$. Therefore,

$$T_{i,j} = T_{j,i}. \tag{2.5}$$

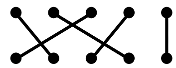
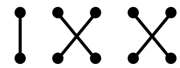
3 The graded braid group and its action on the tensor space

A *permutation* has representation as a bipartite graph with k dots in the top row, k dots in the bottom row, and k edges pairing them. For example,

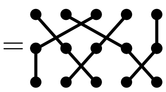
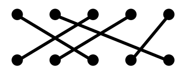


is a permutation diagram on 5 dots.

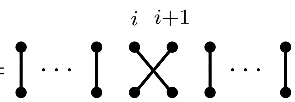
The group algebra $\mathbb{C}S_k$ is the span of the permutation diagrams with multiplication given by concatenation of diagrams. For example, if

$s_1 =$  and $s_2 =$ ,

then

$s_1 s_2 =$  $=$ 

The generators of $\mathbb{C}S_k$ are

$t_i =$ 

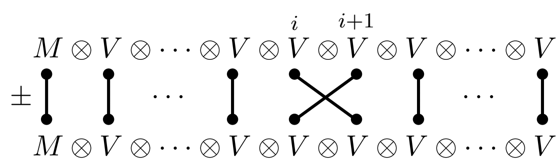
, $\qquad 1 \leq i \leq k-1.$

Define the action of $\mathbb{C}S_k$ on $M \otimes V^{\otimes k}$ via

$$t_i \cdot (m \otimes v_{j_1} \otimes \cdots \otimes v_{j_k}) = \epsilon \ (m \otimes v_{j_1} \otimes \cdots \otimes v_{j_{i-1}}) \otimes (v_{j_{i+1}} \otimes v_{j_i}) \otimes (v_{j_{i+2}} \otimes \cdots \otimes v_{j_k}),$$

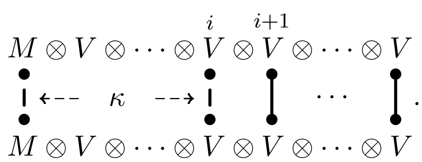
where

$$\epsilon = \begin{cases} \ \ 1 & \text{if } \mathfrak{g} = \mathfrak{so}(V), \\ -1 & \text{if } \mathfrak{g} = \mathfrak{sp}(V). \end{cases}$$



Let κ_j be the operator on $M \otimes V^{\otimes k}$ defined by

$$\kappa_i = \kappa \otimes \mathrm{id}_V^{\otimes(k-i)} \qquad \text{on} \qquad \left(M \otimes V^{\otimes i}\right) \otimes V^{\otimes(k-i)},$$



.

So $\kappa_0 \cdot \left(M \otimes V^{\otimes k}\right) = (\kappa \cdot M) \otimes V^{\otimes k}$.

Theorem 3.1. *$M \otimes V^{\otimes k}$ is a $\mathbb{B}_k$-module with action determined by*

$$t_j \cdot (m \otimes v_{i_1} \otimes \cdots \otimes v_{i_k}) = \epsilon \ (m \otimes v_{i_1} \otimes \cdots \otimes v_{i_{j-1}}) \otimes (v_{i_{j+1}} \otimes v_{i_j}) \otimes (v_{i_{j+2}} \otimes \cdots \otimes v_{i_k}) \tag{3.1}$$

$$y_j \cdot (m \otimes v_{i_1} \otimes \cdots \otimes v_{i_k}) = \epsilon \ \frac{1}{2}(\kappa_j - \kappa_{j-1}) \cdot (m \otimes v_{i_1} \otimes \cdots \otimes v_{i_k}) \tag{3.2}$$

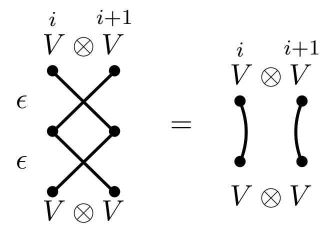
where

$$\epsilon = \begin{cases} \ \ 1 & \textit{if } \mathfrak{g} = \mathfrak{so}(V) \\ -1 & \textit{if } \mathfrak{g} = \mathfrak{sp}(V) \end{cases},$$

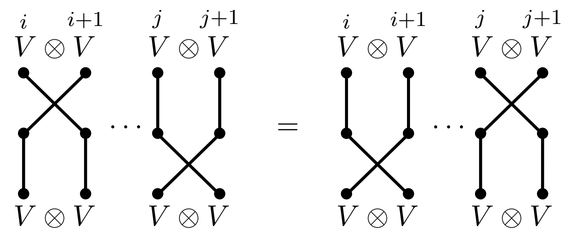
and this action commutes with the action of $\mathfrak{g}$.

Proof. The relations (1.1)-(1.6) are visible from the following diagrams:

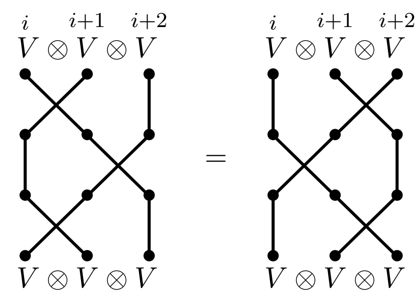
$t_i^2 = 1 \quad$ for $1 \le i \le k-1$:



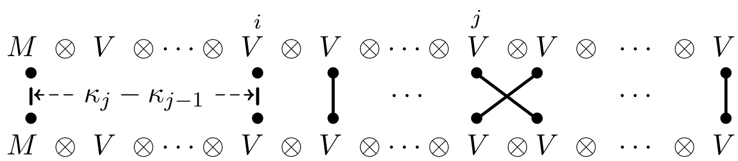
$t_i t_j = t_j t_i, \quad$ for $\begin{array}{r} 1 \le i, j \le k-1, \\ i \ne j,\ j \pm 1: \end{array}$



$t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \quad$ for $1 \le i \le i-2$:



Similarly, the relation (1.5) for $j < i$ is visible from the diagram

$y_j t_i =$  $= t_i y_j.$

The relation (1.6) follows from direct computation:

$$
\begin{aligned}
& t_i(y_i + y_{i+1}) \cdot (m \otimes v_{j_1} \otimes \cdots \otimes v_{j_{i+2}}) \\
&\qquad = \frac{\epsilon}{2}\, t_i(\kappa_{i+1} - \kappa_{i-1}) \cdot (m \otimes v_{j_1} \otimes \cdots \otimes v_{j_{i+2}}) \\
&\qquad = \frac{\epsilon}{2}\, t_i \left(2\sum_{\ell=1}^{i-1} (T_{\ell,i} + T_{\ell,i+1}) + 2T_{i,i+1} + 1^{\otimes(i-1)} \otimes (\kappa \otimes 1 + 1 \otimes \kappa) \right) \\
&\qquad\qquad\qquad \cdot (m \otimes v_{j_1} \otimes \cdots \otimes v_{j_i} \otimes v_{j_{i+1}} \otimes v_{j_{i+2}}) \\
&\qquad = \frac{1}{2} \left(2\sum_{\ell=1}^{i-1} (T_{\ell,i+1} + T_{\ell,i}) + 2T_{i+1,i} + 1^{\otimes(i-1)} \otimes (1 \otimes \kappa + \kappa \otimes 1) \right) \\
&\qquad\qquad\qquad \cdot (m \otimes v_{j_1} \otimes \cdots \otimes v_{j_{i+1}} \otimes v_{j_i} \otimes v_{j_{i+2}}) \\
&\qquad = (y_i + y_{i+1}) t_i \cdot (m \otimes v_{j_1} \otimes \cdots \otimes v_{j_{i+2}}).
\end{aligned}
$$

Finally, to verify the relation (1.5) for $i < j$, rewrite the action in (4.5) as

$$y_0 + y_1 + \cdots + y_j = \frac{\epsilon}{2}\, \kappa_j, \qquad \text{where } y_0 = \frac{\epsilon}{2}\, \kappa_0,$$

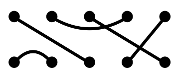
as operators on $M \otimes V^{\otimes k}$. So we can combine (1.6) and (1.5) to find the equivalent relation

$$t_i \kappa_j = \kappa_j t_i, \text{ if } i \neq j.$$

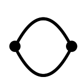
Now certainly the action of the symmetric group, i.e. the action of t_i, commutes with the the action of $\mathfrak{g}$, and thus the action of $U\mathfrak{g}$. $\square$

4 The graded BMW algebra and its action on the tensor space

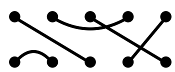
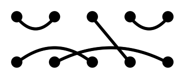
A *Brauer diagram* on k dots is a graph with k dots in the top row, k dots in the bottom row and k edges pairing the dots. For example,



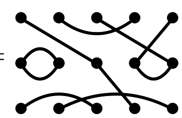
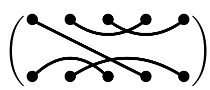
is a diagram on 7 dots. The *Brauer algebra* B_k is the span of the Brauer diagrams with k dots and multiplication given by concatenation (as for $\mathbb{C}S_k$), with the relation

 $= z^{(0)}.$

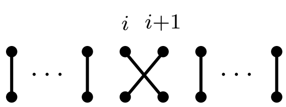
For example, if

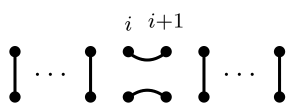
$d_1 =$  and $d_2 =$  ,

then

$d_1 d_2 =$  $= z^{(0)}$ 

A *vertical edge* is and edge pairing a top dot with a bottom dot, and a *horizontal edge* is and edge pairing a top dot with a top dot, or a bottom dot with a bottom dot. For $1 \leq i \leq k-1$, let

$t_i =$ 

, and $e_i =$ 

.

The *Brauer algebra* is the subalgebra of $\mathcal{W}_k$ generated by $t_1, \ldots, t_{k-1}$ and $e_1, \ldots, e_{k-1}$. Theorem **??** below shows that if $\mathfrak{g} = \mathfrak{so}(V)$ or $\mathfrak{g} = \mathfrak{sp}(V)$ and $V = L(\omega_1)$, then the action of $\mathbb{B}_k$ on $M \otimes V^{\otimes k}$ factors through $\mathcal{W}_k$. As in (1.16), let

$$e_i = 1 + t_i y_i - y_{i+1} t_i.$$

Recall, as in equation (2.4), $T_{j,j'}$ is the operator $t = \sum_b b \otimes b^*$ acting on the j and j' factors of $V^{\otimes k}$. From (4.5) and (4.6),

$$\begin{aligned}
t_i y_{i+1} t_i \cdot (m \otimes v_{j_1} \otimes \cdots \otimes v_{j_i} \otimes v_{j_{i+1}}) \\
&= \epsilon\, t_i y_{i+1} \cdot (m \otimes v_{j_1} \otimes \cdots \otimes v_{j_{i+1}} \otimes v_{j_i}) \\
&= \frac{1}{2}\, t_i \left((1^{\otimes i+1} \otimes \kappa) + 2 \sum_{\ell=1}^{i} T_{\ell,i+1} \right) \cdot (m \otimes v_{j_1} \otimes \cdots \otimes v_{j_{i+1}} \otimes v_{j_i}) \\
&= \frac{\epsilon}{2} \left((1^{\otimes i} \otimes \kappa \otimes 1) + 2T_{i,i+1} + 2 \sum_{\ell=1}^{i-1} T_{\ell,i} \right) \cdot (m \otimes v_{j_1} \otimes \cdots \otimes v_{j_i} \otimes v_{j_{i+1}}) \\
&= (y_i + \epsilon\, T_{i,i+1})(m \otimes v_{j_1} \otimes \cdots \otimes v_{j_i} \otimes v_{j_{i+1}})
\end{aligned}$$

Since $t_i^2 = 1$,

$$t_i y_i - y_{i+1} t_i = -\epsilon \; t_i T_{i,i+1} \tag{4.1}$$

as operators on $M \otimes V^{\otimes k}$.

From (2.3), if $\mathfrak{g} = \mathfrak{so}(V)$ or $\mathfrak{sp}(V)$, then

$$L(\omega_1) \otimes L(\omega_1) = L(0) \oplus L(2\varepsilon_1) \oplus L(\varepsilon_1 + \varepsilon_2).$$

Define p_i as the operator

$$p_i = 1^{\otimes i} \otimes \mathrm{pr}_{L(0)} \otimes 1^{\otimes (k-i-1)}.$$

Proposition 4.1. *If $V = L(\omega_1)$ and $\mathfrak{g} = \mathfrak{so}(V)$ or $\mathfrak{g} = \mathfrak{sp}(V)$, then*

$$e_i = \begin{cases} n \; p_i, & \text{if } \; \mathfrak{g} = \mathfrak{so}(V), \\ -n \; p_i, & \text{if } \; \mathfrak{g} = \mathfrak{sp}(V), \end{cases}$$

as an operator on $M \otimes V^{\otimes k}$.

Proof. As in equation (4.1),

$$e_i = 1 + t_i y_i - y_{i+1} t_i = 1 - \epsilon \; t_i T_{i,i+1}.$$

If $\mathfrak{g} = \mathfrak{so}(V)$, then the simple modules $L(0)$ and $L(2\varepsilon_1)$ correspond to the symmetric elements of $V \otimes V$, while $L(\varepsilon_1 + \varepsilon_2)$ corresponds to the anti-symmetric elements of $V \otimes V$. Otherwise, if $\mathfrak{g} = \mathfrak{sp}(V)$, then the simple modules $L(0)$ and $L(\varepsilon_1 + \varepsilon_2)$ correspond to the anti-symmetric elements of $V \otimes V$, while $L(2\varepsilon_1)$ corresponds to the symmetric elements of $V \otimes V$. Thus $\epsilon \cdot t_i$ acts on the i and $i+1$ copy of $L(\omega_1) \otimes L(\omega_1)$ by the constant

$$\begin{array}{rl} \epsilon & \text{in the 0 component,} \\ 1 & \text{in the } 2\varepsilon_1 \text{ component, and} \\ -1 & \text{in the } \varepsilon_1 + \varepsilon_2 \text{ component.} \end{array} \tag{4.2}$$

By theorem 2.3, $T_{i,i+1}$ acts on the i and $i+1$ copy of $L(\omega_1) \otimes L(\omega_1)$ by the constant

$$\begin{array}{rl} -(n-\epsilon) & \text{in the 0 component,} \\ 1 & \text{in the } 2\varepsilon_1 \text{ component, and} \\ -1 & \text{in the } \varepsilon_1 + \varepsilon_2 \text{ component.} \end{array} \tag{4.3}$$

Thus, $e_i = 1 - 2 t_i T_{i,i+1}$ acts on the i and $i+1$ copy of $L(\omega_1) \otimes L(\omega_1)$ by the constant

$$\begin{array}{rl} \epsilon \, n & \text{in the 0 component,} \\ 0 & \text{in the } 2\varepsilon_1 \text{ component, and} \\ 0 & \text{in the } \varepsilon_1 + \varepsilon_2 \text{ component.} \end{array} \tag{4.4}$$

□

Corollary 4.2.

1. *If $\mathfrak{g} = \mathfrak{so}(V)$, then e_i acts on $M \otimes V^{\otimes k}$ via*

$$e \cdot (v_a \otimes v_b) = \delta_{ab} \sum_{\ell=1}^{n} v_\ell \otimes v_\ell.$$

2. If $\mathfrak{g} = \mathfrak{sp}(V)$, $\dim(V) = 2r$, then e_i acts on $M \otimes V^{\otimes k}$ via

$$e \cdot (v_a \otimes v_b) = \langle v_b, v_a \rangle_J \sum_{\ell=1}^{r} (v_\ell \otimes v_{r+\ell} - v_{r+\ell} \otimes v_\ell)$$

Now we are ready to describe the action of $\mathcal{W}_k$ on the tensor space.
Let $\mathfrak{g} = \mathfrak{so}_n$ or let $\mathfrak{g} = \mathfrak{sp}_n$and n be even . Let $M = L(\mu)$ and $V = L(\omega_1)$.

Theorem 4.3. *$M \otimes V^{\otimes k}$ is a $\mathcal{W}_k$-module via the action*

$$t_j \cdot (m \otimes v_{i_1} \otimes \cdots \otimes v_{i_k}) = (m \otimes v_{i_1} \otimes \cdots \otimes v_{i_{j-1}}) \otimes (v_{i_{j+1}} \otimes v_{i_j}) \otimes (v_{i_{j+2}} \otimes \cdots \otimes v_{i_k}) \quad (4.5)$$

$$y_j \cdot (m \otimes v_{i_1} \otimes \cdots \otimes v_{i_k}) = \frac{\epsilon}{2}(\kappa_j - \kappa_{j-1}) \cdot (m \otimes v_{i_1} \otimes \cdots \otimes v_{i_k}) \quad (4.6)$$

and this action commutes with the action of $\mathfrak{g}$, if

$$z^{(\ell)} = \begin{cases} n\left(\langle \mu, \mu + 2\rho \rangle\right)^\ell, & \text{for } \mathfrak{g} = \mathfrak{so}(V), \\ -n\left(\langle \mu, \mu + 2\rho \rangle\right)^\ell, & \text{for } \mathfrak{g} = \mathfrak{sp}(V). \end{cases}$$

Proof. The commutation relations

$$\begin{aligned} t_i e_j &= e_j t_i, \quad \text{if } i \neq j, j \pm 1 \\ e_i e_j &= e_j e_i, \quad \text{if } i \neq j, j \pm 1 \end{aligned}$$

and the idempotent relation

$$e_i^2 = z^{(0)} e_i$$

follow immediately from Theorem 4.3. The first tangle relation (1.9)

$$e_i t_i = e_i = t_i e_i$$

follows from (4.2) and (4.4). Since κ acts on $L(\mu) \otimes L(0)$ by $\langle \mu, \mu + 2\rho \rangle$, the unwrapping relation (1.15)

$$e_1 y_1^\ell e_1 = z^{(\ell)} e_1, \quad \text{for } \ell \in \mathbb{Z}_{\geq 0}$$

holds.

Relation (1.6)

$$t_i(y_i + y_{i+1}) = (y_i + y_{i+1}) t_i,$$

together with the commutation relations implies

$$e_i(y_i + y_{i+1}) = (y_i + y_{i+1}) e_i.$$

Since

$$\begin{aligned} &(y_i + y_{i+1}) \cdot (m \otimes v_{j_1} \otimes \cdots \otimes v_{j_{i+1}}) \\ &= \frac{\epsilon}{2}(\kappa_{i+1} - \kappa_{i-1}) \cdot (m \otimes v_{j_1} \otimes \cdots \otimes v_{j_{i+1}}) \\ &= \frac{\epsilon}{2}(m \otimes v_{j_1} \otimes \cdots \otimes v_{j_{i-1}}) \otimes \left(\kappa_{i,i+1} \cdot (v_{j_i} \otimes v_{j_{i+1}})\right) \\ &\quad + \frac{\epsilon}{2} \sum_b \left((bm \otimes v_{j_i} \otimes \cdots \otimes v_{j_{i-1}}) \right. \\ &\qquad \left. + \sum_{\ell=1}^{i-1} (m \otimes \cdots \otimes bv_{j_\ell} \otimes \cdots \otimes v_{j_{i-1}}) \right) \otimes \left(b^* \cdot (v_{j_i} \otimes v_{j_{i+1}})\right). \end{aligned}$$

Now, since $e_i = c\, p_i$,

$$
\begin{aligned}
(y_i + y_{i+1})e_i \cdot (m \otimes v_{j_1} \otimes \cdots \otimes v_{j_{i+1}}) \\
&= \frac{\epsilon}{2}(\kappa_{i+1} - \kappa_{i-1}) \cdot \Big((m \otimes v_{j_1} \otimes \cdots \otimes v_{j_{i-1}}) \otimes c\, \mathrm{pr}_0(v_{j_i} \otimes v_{j_{i+1}}) \Big) \\
&= \frac{\epsilon}{2}(m \otimes v_{j_1} \otimes \cdots \otimes v_{j_{i-1}}) \otimes 0 \\
&\quad + \frac{\epsilon}{2} \sum_b \Bigg((bm \otimes v_{j_i} \otimes \cdots \otimes v_{j_{i-1}}) \\
&\qquad + \sum_{\ell=1}^{i-1} (m \otimes \cdots \otimes b v_{j_\ell} \otimes \cdots \otimes v_{j_{i-1}}) \Bigg) \otimes 0 \\
&= 0,
\end{aligned}
$$

because $x \in \mathfrak{g}$ acs on $L(0)$ by 0.

It remains to prove relations 1.10, 1.11, 1.12, and 1.13.

Consider the subalgebra generated by e_i, e_{i+1}, t_i, and t_{i+1}, and its action on $V^{\otimes 3}$ alone.

Case 1: $\mathfrak{g} = \mathfrak{so}(V)$ Recall, that $e_i \cdot (v_{j_i} \otimes v_{j_{i+1}}) = \delta_{j_i, j_{i+1}} \sum_\ell v_\ell \otimes v_\ell$. Since

$$
\begin{aligned}
t_i e_{i+1} e_i \cdot (v_{j_i} \otimes v_{j_{i+1}} \otimes v_{j_{i+2}}) &= t_i e_{i+1} \cdot \delta_{j_i, j_{i+1}} \sum_\ell (v_\ell \otimes v_\ell \otimes v_{j_{i+2}}) \\
&= t_i \cdot \delta_{j_i, j_{i+1}} \sum_\ell (v_{j_{i+2}} \otimes v_\ell \otimes v_\ell) \\
&= \delta_{j_i, j_{i+1}} \sum_\ell (v_\ell \otimes v_{j_{i+2}} \otimes v_\ell) \\
&= t_{i+1} e_i \cdot (v_{j_i} \otimes v_{j_{i+1}} \otimes v_{j_{i+2}}),
\end{aligned}
$$

and

$$
\begin{aligned}
e_{i+1} e_i t_{i+1} \cdot (v_{j_i} \otimes v_{j_{i+1}} \otimes v_{j_{i+2}}) &= e_{i+1} e_i \cdot (v_{j_i} \otimes v_{j_{i+2}} \otimes v_{j_{i+1}}) \\
&= e_{i+1} \cdot \delta_{j_i, j_{i+2}} \sum_\ell (v_\ell \otimes v_\ell \otimes v_{j_{i+1}}) \\
&= \delta_{j_i, j_{i+2}} \sum_\ell (v_{j_{i+1}} \otimes v_\ell \otimes v_\ell) \\
&= e_{i+1} t_i \cdot (v_{j_i} \otimes v_{j_{i+1}} \otimes v_{j_{i+2}}),
\end{aligned}
$$

1.10 and 1.11 hold. Similarly, since

$$
\begin{aligned}
e_i e_{i+1} e_i \cdot (v_{j_i} \otimes v_{j_{i+1}} \otimes v_{j_{i+2}}) &= e_i \cdot \delta_{j_i, j_{i+1}} \sum_\ell (v_{j_{i+2}} \otimes v_\ell \otimes v_\ell) \\
&= \delta_{j_i, j_{i+1}} \sum_\ell (v_\ell \otimes v_\ell \otimes v_{j_{i+2}}) \\
&= e_i \cdot (v_{j_i} \otimes v_{j_{i+1}} \otimes v_{j_{i+2}}),
\end{aligned}
$$

and

$$\begin{aligned} e_{i+1}e_ie_{i+1}\cdot(v_{j_i}\otimes v_{j_{i+1}}\otimes v_{j_{i+2}}) &= e_{i+1}e_i\cdot\delta_{j_{i+1},j_{i+2}}\sum_\ell(v_{j_i}\otimes v_\ell\otimes v_\ell) \\ &= e_{i+1}\cdot\delta_{j_{i+1},j_{i+2}}\sum_\ell(v_\ell\otimes v_\ell\otimes v_{j_i}) \\ &= \delta_{j_{i+1},j_{i+2}}\sum_\ell(v_{j_i}\otimes v_\ell\otimes v_\ell) \\ &= e_{i+1}\cdot(v_{j_i}\otimes v_{j_{i+1}}\otimes v_{j_{i+2}}), \end{aligned}$$

1.12, and 1.13 hold.

Case 2: $\mathfrak{g}=\mathfrak{sp}(V)$ Here, $e_i\cdot(v_{j_i}\otimes v_{j_{i+1}})=-\langle v_{j_i},v_{j_{i+1}}\rangle_J\sum_{\ell=1}^r(v_\ell\otimes v_{r+\ell}-v_{r+\ell}\otimes v_\ell)$. So since

$$\begin{aligned} t_ie_{i+1}e_i\cdot(v_{j_i}\otimes v_{j_{i+1}}\otimes v_{j_{i+2}}) &= -t_ie_{i+1}\cdot\langle v_{j_i},v_{j_{i+1}}\rangle_J\sum_{\ell=1}^r(v_\ell\otimes v_{\ell+r}\otimes v_{j_{i+2}}-v_{\ell+r}\otimes v_\ell\otimes v_{j_{i+2}}) \\ &= -t_i\langle v_{j_i},v_{j_{i+1}}\rangle_J\sum_{\ell=1}^r(v_{j_{i+2}}\otimes v_\ell\otimes v_{\ell+r}-v_{j_{i+2}}\otimes v_{\ell+r}\otimes v_\ell) \\ &= \langle v_{j_i},v_{j_{i+1}}\rangle_J\sum_{\ell=1}^r(v_\ell\otimes v_{j_{i+2}}\otimes v_{\ell+r}-v_{\ell+r}\otimes v_{j_{i+2}}\otimes v_\ell) \\ &= t_{i+1}e_i\cdot(v_{j_i}\otimes v_{j_{i+1}}\otimes v_{j_{i+2}}), \end{aligned}$$

and

$$\begin{aligned} e_{i+1}e_it_{i+1}\cdot(v_{j_i}\otimes v_{j_{i+1}}\otimes v_{j_{i+2}}) &= -e_{i+1}e_i\cdot(v_{j_i}\otimes v_{j_{i+2}}\otimes v_{j_{i+1}}) \\ &= -\langle v_{j_i},v_{j_{i+2}}\rangle_J\sum_{\ell=1}^r(v_{j_{i+1}}\otimes v_\ell\otimes v_{\ell+r}-v_{j_{i+1}}\otimes v_{\ell+r}\otimes v_\ell) \\ &= e_{i+1}\cdot(v_{j_{i+1}}\otimes v_{j_i}\otimes v_{j_{i+2}}) \\ &= e_{i+1}t_i\cdot(v_{j_i}\otimes v_{j_{i+1}}\otimes v_{j_{i+2}}), \end{aligned}$$

1.10 and 1.11 hold. Similarly, since

$$\begin{aligned} e_ie_{i+1}e_i\cdot(v_{j_i}\otimes v_{j_{i+1}}\otimes v_{j_{i+2}}) &= -e_ie_{i+1}\cdot\langle v_{j_i},v_{j_{i+1}}\rangle_J\sum_{\ell=1}^r(v_\ell\otimes v_{\ell+r}\otimes v_{j_{i+2}}-v_{\ell+r}\otimes v_\ell\otimes v_{j_{i+2}}) \\ &= -e_i\langle v_{j_i},v_{j_{i+1}}\rangle_J\sum_{\ell=1}^r(v_{j_{i+2}}\otimes v_\ell\otimes v_{\ell+r}-v_{j_{i+2}}\otimes v_{\ell+r}\otimes v_\ell) \\ &= -\langle v_{j_i},v_{j_{i+1}}\rangle_J\sum_{\ell=1}^r(v_\ell\otimes v_{\ell+r}\otimes v_{j_{i+2}}-v_{\ell+r}\otimes v_\ell\otimes v_{j_{i+2}}) \\ &= e_i\cdot(v_{j_i}\otimes v_{j_{i+1}}\otimes v_{j_{i+2}}), \end{aligned}$$

and

$$
\begin{aligned}
e_{i+1}e_ie_{i+1}\cdot(v_{j_i}\otimes v_{j_{i+1}}\otimes v_{j_{i+2}}) &= -e_{i+1}e_i\cdot\langle v_{j_{i+1}},v_{j_{i+2}}\rangle_J\sum_{\ell=1}^{r}(v_{j_i}\otimes v_\ell\otimes v_{\ell+r}-v_{j_i}\otimes v_{\ell+r}\otimes v_\ell)\\
&= -e_{i+1}\langle v_{j_{i+1}},v_{j_{i+2}}\rangle_J\sum_{\ell=1}^{r}(v_\ell\otimes v_{\ell+r}\otimes v_{j_i}-v_{\ell+r}\otimes v_\ell\otimes v_{j_i})\\
&= -\langle v_{j_{i+1}},v_{j_{i+2}}\rangle_J\sum_{\ell=1}^{r}(v_{j_i}\otimes v_\ell\otimes v_{\ell+r}-v_{j_i}\otimes v_{\ell+r}\otimes v_\ell)\\
&= e_{i+1}\cdot(v_{j_i}\otimes v_{j_{i+1}}\otimes v_{j_{i+2}}).
\end{aligned}
$$

1.12, and 1.13 hold. $\square$

A Explicit calculations

A.1 The Lie bracket for $\mathfrak{sp}(V)$

Since the representation $\mathfrak{so}_n \to \mathrm{End}(V)$ via $b_{ij} \to E_{ij} - E_{ji}$ is faithful, we can utilize known matrix multiplication to calculate Lie brackets of basis elements: for basis elements b_{i_i,j_1}, b_{i_2,j_2},

$$\begin{aligned}
[b_{i_1,j_1}, b_{i_2,j_2}] &= b_{i_1,j_1}b_{i_2,j_2} - b_{i_2,j_2}b_{i_1,j_1} \\
&= (E_{i_1,j_1} - E_{j_1,i_1})(E_{i_2,j_2} - E_{j_2,i_2}) - (E_{i_2,j_2} - E_{j_2,i_2})(E_{i_1,j_1} - E_{j_1,i_1}) \\
&= \delta_{j_1,i_2}(E_{i_1,j_2} - E_{j_2,i_1}) - \delta_{j_1,j_2}(E_{i_1,i_2} - E_{i_2,i_1}) - \delta_{i_1,i_2}(E_{j_1,j_2} - E_{j_2,j_1}) + \delta_{i_1,j_2}(E_{j_i,i_2} - E_{i_2,j_1}) \\
&= \begin{cases} b_{i_1,j_2} & j_1 = i_2, \\ b_{j_1,i_2} & i_1 = j_2, \\ -b_{i_1,i_2} & j_1 = j_2, \\ -b_{j_1,j_2} & i_1 = i_2, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

A.2 The dual basis for $\mathfrak{sp}(V)$

Recall that $\mathfrak{sp}(V)$ has basis (in $\mathrm{End}(V)$)

$$\left\{ \begin{array}{rcll} E_{ii} - E_{r+i,r+i} &=& A_i, & 1 \le i \le r, \\ E_{ij} - E_{r+j,r+i} &=& B_{i,j}, & 1 \le i \ne j \le r, \\ E_{i,r+i} &=& C_i, & 1 \le i \le r, \\ E_{i,r+j} + E_{j,r+i} &=& D_{ij}, & 1 \le i < j \le r, \\ E_{r+i,i} &=& E_i, & 1 \le i \le r, \\ E_{r+i,j} + E_{r+j,i} &=& F_{ij}, & 1 \le i < j \le r \end{array} \right\}.$$

With respect to the form on $\mathfrak{sp}(V)$ defined by $\langle x, y\rangle = \mathrm{Tr}(xy)$, we have

$$A_i^* = \frac{1}{2}A_i, \quad B_{i,j}^* = \frac{1}{2}B_{j,i},$$

$$C_i^* = E_i, \quad D_{ij}^* = \frac{1}{2}F_{ij},$$

$$E_i^* = C_i, \quad \text{and} \quad F_{ij}^* = \frac{1}{2}D_{ij}.$$

Motivated by (2.1), we will calculate the weight spaces containing each basis element of $\mathfrak{sp}(V)$: Let $1 \le i \le r$. For all $1 \le j \ne k \le r$,

$$\begin{aligned}
[A_i, B_{jk}] &= [E_{ii} - E_{r+i,r+i}, E_{jk} - E_{r+k,r+j}] \\
&= \delta_{ij}E_{ik} - \delta_{ik}E_{ji} + 0 + 0 + \delta_{ik}E_{r+i,r+j} - \delta_{ij}E_{r+k,r+i} \\
&= \begin{cases} E_{jk} - E_{r+k,r+j} & \text{if } i = j \\ -(E_{jk} - E_{r+k,r+j}) & \text{if } i = k \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} B_{jk} & \text{if } i = j \\ -B_{jk} & \text{if } i = k \\ 0 & \text{otherwise} \end{cases} \\
&= (\varepsilon_j - \varepsilon_k)\varepsilon_i^\vee B_{jk},
\end{aligned}$$

where $\varepsilon_i(\varepsilon_j^\vee) = \delta_{ij}$. So $B_{jk} \in \mathfrak{g}_{\varepsilon_j - \varepsilon_k}$, and $\varepsilon_i^\vee = A_i$

For all $1 \leq j \leq r$,

$$\begin{aligned} [A_i, C_j] &= [E_{ii} - E_{r+i,r+i}, E_{j,r+j}] \\ &= 2\delta_{ij} E_{j,r+j} \\ &= 2\varepsilon_j \varepsilon_i^\vee C_j. \end{aligned}$$

So $C_j \in \mathfrak{g}_{2\varepsilon_j}$.

For $1 \leq j < k \leq r$,

$$\begin{aligned} [A_i, D_{jk}] &= [E_{ii} - E_{r+i,r+i}, E_{j,r+k} + E_{k,r+j}] \\ &= \delta_{ij}(E_{j,r+k} + E_{k,r+j}) + \delta_{ik}(E_{j,r+k} + E_{k,r+j}) \\ &= (\varepsilon_j + \varepsilon_k)\varepsilon_i^\vee D_{jk} \end{aligned}$$

So $D_{jk} \in \mathfrak{g}_{\varepsilon_j+\varepsilon_k}$.

For $1 \leq j \leq r$,

$$\begin{aligned} [A_i, E_j] &= [E_{ii} - E_{r+i,r+i}, E_{r+j,j}] \\ &= -2\delta_{ij} E_{j,r+j} \\ &= -2\varepsilon_j \varepsilon_i^\vee E_j. \end{aligned}$$

So $E_j \in \mathfrak{g}_{-2\varepsilon_j}$.

For $1 \leq j < k \leq r$,

$$\begin{aligned} [A_i, F_{jk}] &= [E_{ii} - E_{r+i,r+i}, E_{r+j,k} + E_{r+j,k}] \\ &= -\delta_{ij}(E_{r+j,k} + E_{r+k,j}) - \delta_{ik}(E_{r+j,k} + E_{r+k,j}) \\ &= -(\varepsilon_j + \varepsilon_k)\varepsilon_i^\vee F_{jk} \end{aligned}$$

So $F_{jk} \in \mathfrak{g}_{-(\varepsilon_j+\varepsilon_k)}$.

Finally, since $A_i, A_j \in \mathfrak{h}$ for all $1 \leq i, j \leq r$, we have $[A_i, A_j] = 0$, so $A_i \in \mathfrak{g}_0$.

Now, by (2.1), we can compare weight spaces to conclude that

$$A_i^* = c_{A_i} A_i, \quad B_{ij}^* = c_{B_{ij}} B_{ji}, \quad C_i^* = c_{C_i} E_i,$$

$$D_{ij}^* = c_{D_{ij}} F_{ij}, \quad E_i^* = c_{E_i} C_i, \quad \text{and} \quad F_{ij}^* = c_{F_{ij}} D_{ij},$$

for some $c_\star \in \mathbb{C}$. It can furthermore be checked that

$$\mathrm{Tr}(A_i A_i) = 2, \quad \mathrm{Tr}(B_{ij} B_{ji}) = 2,$$

$$\mathrm{Tr}(C_i E_i) = \mathrm{Tr}(E_i C_i) = 1, \text{ and } \mathrm{Tr}(F_{ij} D_{ij}) = \mathrm{Tr}(D_{ij} F_{ij}) = 2.$$

So since $\mathrm{Tr}(cX) = c\mathrm{Tr}(X)$ for any square matrix X and scalar c, we have

$$A_i^* = \frac{1}{2} A_i, \quad B_{i,j}^* = \frac{1}{2} B_{j,i},$$

$$C_i^* = E_i, \quad D_{ij}^* = \frac{1}{2} F_{ij},$$

$$E_i^* = C_i, \quad \text{and} \quad F_{ij}^* = \frac{1}{2} D_{ij},$$

as desired.

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