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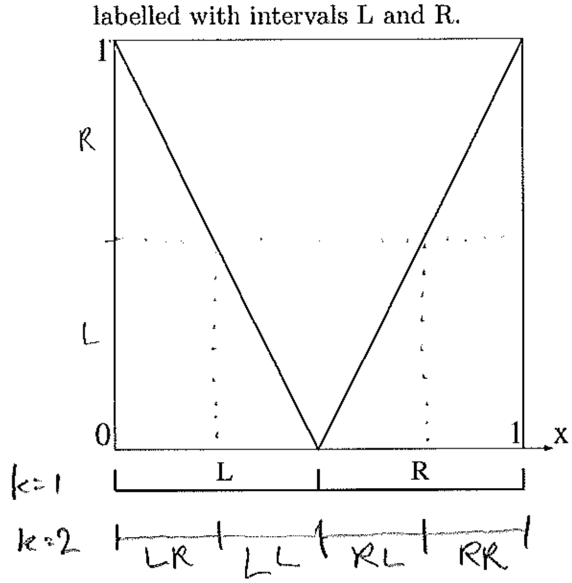
~ SOLUTIONS ~

Math 53: Chaos! 2011: Midterm 1

2 hours, 56 points total, 5 questions worth various points ($\propto$ blank space). Good luck!

1. [16 points] Consider the 1D map given by $f(x) = |2x - 1|$, as shown here, on $[0, 1]$, which has been labelled with intervals L and R.



level $k=2$ can use cobweb, or level 1 written in reverse & forward order as below.

[3] (a) Write here the subintervals down to level 3 (that is, the correct ordering of all 3-symbol itinerary subintervals on $[0, 1]$):

level-2 written in reverse order (f decreasing in L)

LRR LRL LLL LLR RLR RLL RRL RRR

level-2 written in forward order (f increasing in R)

[3] (b) Compute the Lyapunov exponent of almost all orbits of this map.

almost all orbits never hit $x = 1/2$ so have $|f'(x)| = 2$ defined.

$$h := \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \ln |f'(x_i)| \quad = \text{mean value of sequence all entries of which are } \ln 2$$

$$= \ln 2 \; > 0$$

BONUS: Describe the binary representation of all initial points in $[0,1]$ which do *not* have this exponent

Any $x_0 \in (0,1)$ with terminating binary representations will be on some subinterval boundary, so eventually mapped to $x_k = 1/2$, where f' undefined, so $h(x_0)$ undefined.

[3] (c) Imagine that a computer running a standard numerical environment (such as MATLAB) is used to iterate f starting at x_0 the only fixed point in $(0,1)$. Describe what will happen, including an estimate of how long it will take for errors to become of size $O(1)$. [Hint: numbers are represented with relative error of order 10^{-16}.]

since the fixed pt x_0 ($= 1/3$ in fact) is only represented to within error $\varepsilon \approx 10^{-16}$, MATLAB (etc) will iterate $x_0 + \varepsilon$, rather than exactly x_0. Error after k iterations will be roughly $\varepsilon e^{hk} = \varepsilon 2^k$, so $O(1)$ errors when $1 = \varepsilon 2^k$

Solve so $k = \log_2 \frac{1}{\varepsilon} \approx 53$. ← (# binary digits in mantissa!)

So numerical iterate will deviate from x_0 w/ exponentially growing error, becoming garbage w/ $O(1)$ error after ≈ 50 iters.

[3] (d) It is not hard to see that f^k has 2^k fixed points for each natural number k. Compute the number of *periodic orbits* of period 4.

[Periodic Table!] f^4 has $2^4 = 16$ fixed pts... but need to subtract off those accounted for by lower periods:

f has 2 fixed pts $\Rightarrow$ 2 p-1 orbits.

f^2 has 4 " $\Rightarrow$ 1 p-2 orbit (other 2 come from p-1's)

3 doesn't divide 4.

$\Rightarrow$ # fp's accounted by p-4 $= 16 - 2(1) - 1(2) = 12 \Rightarrow \frac{12}{4} =$ 3 period-4's

[4] (e) Sketch a proof that each point x_0 in $[0,1]$ has sensitive dependence on initial conditions.

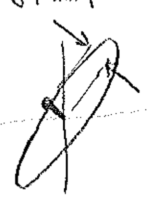
let $\varepsilon > 0$. choose $J = S_1 \cdots S_k$ the first k of itinerary of x_0, where $k > \log_2 \frac{1}{\varepsilon}$ Then since length$(J) = 2^{-k}$ we have that all pts in J are in $N_\varepsilon(x_0)$.

Read off next 2 letters of x_0. If
$$\begin{cases} LR \\ LL \end{cases} \xrightarrow{\text{choose}} y_0 \in JRR \qquad \begin{cases} RL \\ RR \end{cases} \longrightarrow y_0 \in JLR$$

Then after k iters, $|y_k - x_k| \geq 1/4$ because J has been 'eaten up' (left-shifted) by f^k.

Holds for any $\varepsilon > 0$ & $x_0 \in [0,1]$ $\Rightarrow$ all pts have sens. dep. $\square$.

$\sqrt{\lambda_{min}(AA^T)}$ 2. [11 points] Consider the two-dimensional map $f(\mathbf{x}) = A\mathbf{x}$. ie minor semiaxis of ellipse. has eigenvalues $\lambda/4$ by scalar mult.



[3] (a) Consider the case $A = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix}$. What is the closest distance to the origin that a point $\mathbf{x}$ lying on the unit circle $\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| = 1\}$ can get mapped to ? (after 1 iter!) [Hint: easier if bring out the common factor in the matrix]

$\Rightarrow$ gets mapped to ellipse. Semiaxes: $AA^T = 2^2 \begin{bmatrix} 1 & \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ & 1 \end{bmatrix} = 4\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

Find μ, eigvals of $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$: $(1-\mu)(5-\mu) - 2^2 = 0 \Rightarrow \mu^2 - 6\mu + 1 = 0$

$\Rightarrow \mu = \frac{1}{2}(6 \pm \sqrt{36-4}) = 3 \pm \sqrt{8}$ smaller one $= 3 - \sqrt{8}$

$\Rightarrow \lambda = 4\mu = 4(3-\sqrt{8})$, $\underbrace{\sqrt{\lambda} = \sqrt{4(3-2.83)}}_{\text{semiminor axis.}} \approx 0.83\ldots < 1$.

[2] (b) Your answer should be consistent with some points on the unit circle moving closer to the origin. indeed, but don't let this distract you! For this A, find all possible fate(s) of orbits starting on the unit circle (ie, to where they may tend upon repeated iteration).

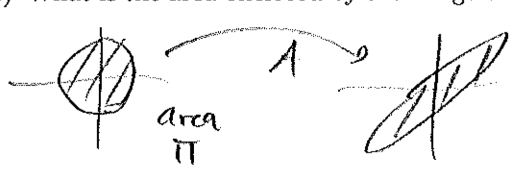
eigvals of A: A is lower-triangular so eigvals are diagonal entries

$\Rightarrow \lambda(A) = 2$, twice > 1

$\Rightarrow \vec{0}$ is a source $\Rightarrow$ all points (other than $\vec{0}$) have ∞ as limit.

All of circle gets "blown away" to ∞ (initial moving closer was red herring!)

[1] (c) What is the area enclosed by the image of the unit circle under one application of the above map?



Area growth factor $= |\det A|$

$= |2^2 - 0(4)| = 4$

Final ellipse area $= 4\pi$.

[2] (d) Now let $A = \begin{bmatrix} 1/2 & 1 \\ 1/2 & 3/2 \end{bmatrix}$. Is the origin now a *hyperbolic* fixed point?

eigvals λ: $(\frac{1}{2} - \lambda)(\frac{3}{2} - \lambda) - \frac{1}{2} = 0 \Rightarrow \lambda^2 - 2\lambda + \frac{1}{4} = 0$

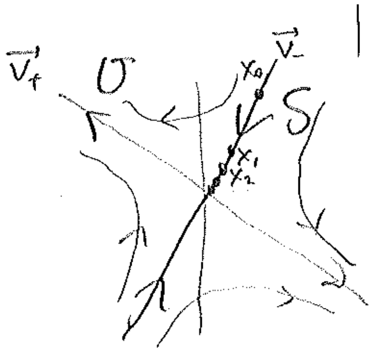
$\lambda = 1 \pm \sqrt{1 - \frac{1}{4}} = 1 \pm \frac{\sqrt{3}}{2}$ all λ's have $|\lambda| \neq 1$

$\Rightarrow$ hyperbolic.

[3] (e) Keeping A as in (d), either find the *complete* set of points which tend to the origin upon repeated iteration, or explain why this set is empty.

Notice $|\lambda_+| = \left|1 + \frac{\sqrt{3}}{2}\right| > 1$ its eigenvector would give unstable direction. $\boxed{\text{It's a Saddle}}$

$|\lambda_-| = \left|1 - \frac{\sqrt{3}}{2}\right| < 1$, so find its eigenvector for stable direction



$$A - \lambda_- I = \begin{bmatrix} 1/2 - (1 - \frac{\sqrt{3}}{2}) & 1 \\ \cdot & \cdot \end{bmatrix} = \begin{bmatrix} \frac{-1+\sqrt{3}}{2} & 1 \\ \cdot & \cdot \end{bmatrix}$$

irrelevant

$\vec{V}_- = \begin{bmatrix} a \\ b \end{bmatrix}$ is in $\mathrm{Null}(A - \lambda_- I)$ so $\frac{-1+\sqrt{3}}{2} a + 1 \cdot b = 0$

$b = \frac{1-\sqrt{3}}{2} a$ so $\vec{V}_- = \begin{bmatrix} 1 \\ \frac{1-\sqrt{3}}{2} \end{bmatrix} a = \mathrm{Span}\left\{\begin{bmatrix} 2 \\ 1-\sqrt{3} \end{bmatrix}\right\}$

Stable manifold $= \left\{\alpha \in \mathbb{R} : \vec{x} = \alpha \begin{bmatrix} 2 \\ 1-\sqrt{3} \end{bmatrix}\right\}$ is all pts x with $\lim_{n\to\infty} f^n(x) = \vec{0}$.

3. [9 points]

Consider the Henón map on $\mathbb{R}^2$ with $b = 1/2$ and general a, that is, $f(x, y) = (a - x^2 + y/2, x)$.

[3] (a) Find the fixed point(s) in terms of a. For what a does at least one fixed point exist?

Fixed pts : $f(\vec{x}) = \vec{x}$ ie $\left.\begin{aligned} a - x^2 + y/2 &= x \\ x &= y \end{aligned}\right\}$ solve simul.

sub. $y = x$

$\Rightarrow x^2 + x/2 - a = 0$

$\Rightarrow x = \frac{1}{2}\left(-1/2 \pm \sqrt{1/4 + 4a}\right) = -\frac{1}{4} \pm \sqrt{\frac{1}{16} + a}$

When $a \geq -\frac{1}{16}$ have existence of (real-valued!) fixed points.

[2] (b) Now specialize to $a = 7/4$. What is the period of the orbit from point $(3/2, -1)$?

$$\begin{pmatrix} 3/2 \\ -1 \end{pmatrix} \xrightarrow[\text{Hénon}]{a = 7/4} \begin{pmatrix} 7/4 - 3^2/2^2 + \frac{-1}{2} \\ 3/2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3/2 \end{pmatrix} \xrightarrow{\text{Hénon}} \begin{pmatrix} 7/4 - (-1)^2 + 3/4 \\ -1 \end{pmatrix}$$

$= \begin{pmatrix} 3/2 \\ -1 \end{pmatrix}$ = the original point, so period $= 2$.

[4] (c) Categorize the stability of the orbit from (b). Is it a (possibly periodic) sink, saddle, source, or none of these?

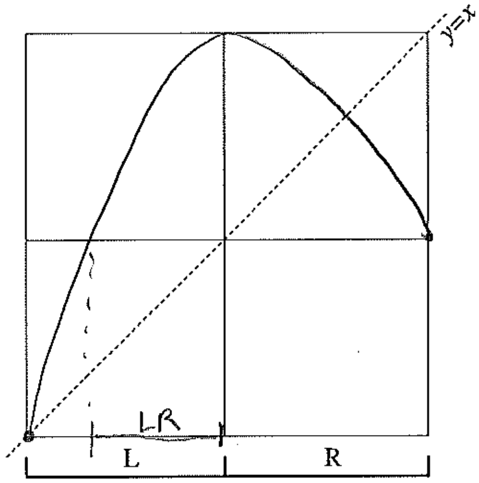
Jacobean $\vec{D}\vec{f} = \begin{bmatrix} -2x & 1/2 \\ 1 & 0 \end{bmatrix}$ ← see book, in Hénon section.

Stability matrix $A = \vec{D}\vec{f}(\vec{p_2})\vec{D}\vec{f}(\vec{p_1}) = \begin{bmatrix} -3 & 1/2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1/2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -11/2 & -3/2 \\ 2 & 1/2 \end{bmatrix}$

λ, eigenvalues(A): $\lambda^2 + 5\lambda \underbrace{- 1/4 + 3}_{1/4} = 0$

so $\lambda = \frac{1}{2}\left(-5 \pm \sqrt{25-1}\right)$; since $4 < \sqrt{24} < 5$, we have

$|\lambda_+| < 1$ but $|\lambda_-| > 1$ ⇒ a saddle.



← others are possible, eg

since we're not told f is monotonic or anything.

4. [8 points]

[2] (a) Using the axes above, with the partition L and R as shown, draw a possible graph of a smooth function f mapping $L \cup R$ into $L \cup R$ with transition graph as shown to the right.

[1] (b) Onto what interval does the subinterval LR get mapped by f?

$f(LR) = R$ by left symbol shifting rule.

[3] (c) Give a list of *all* the types of itineraries that *must* occur given the transition graph:

$\overline{L}$

$L^n \overline{R}$ $n = 0, 1, \ldots$

ie not your special choice of f graph.

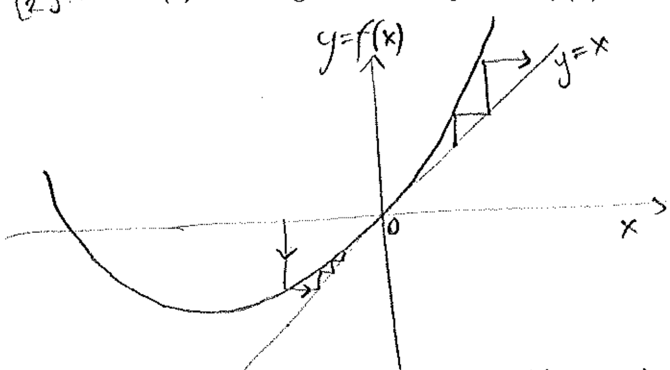
[2] (d) Given only the above information about f, is it possible that there exists a period-2 orbit? Explain.

hard!

A period-2 with $p_1 \in L$ & $p_2 \in R$ would give $\overline{LR}$ itin, not poss.
But it could be that f restricted to $R \to R$ has a period-2,
eg  ... So, yes, it's possible! (tricky! Note since subintervals don't get smaller, R doesn't define a unique x_0, unlike in $G(x)$, $2x \pmod 1$, etc)

5. [12 points] Random short questions. Please explain each briefly.

[2] (a) The origin is a fixed point of $f(x) = x + x^2$. Categorize it as a source, sink, or neither.



let $\varepsilon > 0$: $-\varepsilon < x < 0$ get attracted to 0

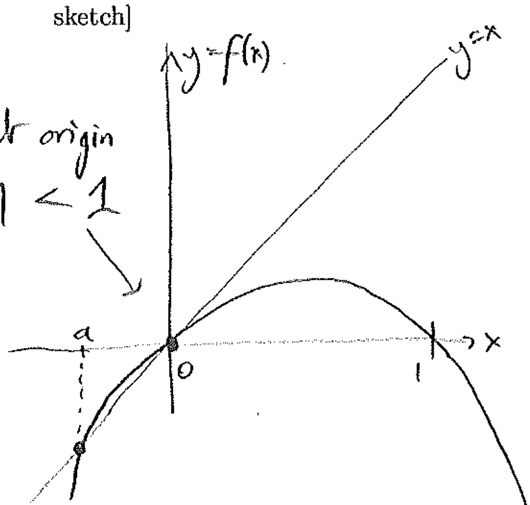
$0 < x < \varepsilon$ get repelled from 0

$\Rightarrow$ neither definition of source or sink holds.

* Many of you seem to think the stability theorem $|f'(0)| = 1$ tells you 0 is neither: this is false! (viz $f(x) = \tan x$ is source)

[2] (b) What is the Lyapunov exponent of almost all bounded orbits of $f(x) = 0.9x(1-x)$ on $\mathbb{R}$? [Hint: sketch]

slope at origin $= 0.9 < 1$



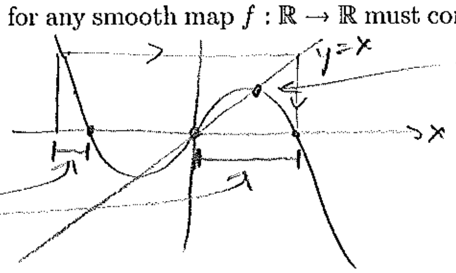
0 is a sink, with basin at least $(a, 1-a)$, $a < 0$

For all these orbits, asymptotically periodic to $p = 0$, so Lyap. exp $h = h(p)$
$= \ln|f'(0)| = \ln 9/10 < 0$.

$\{a, 1-a\}$ is p-2 source, $x < a$ or $x > 1-a$ unbounded.

[1] (c) True/False: the basin of a sink for any smooth map $f : \mathbb{R} \to \mathbb{R}$ must consist of an interval (possibly unbounded)?

No, viz:



sink fixed pt p

both these are in the basin of p, but the gap between them is not.

[3] (d) Let p be a fixed point of a map on $\mathbb{R}^m$. Give the mathematical definition of p being a source. say, f

$\vec{p}$ source if : $\exists\ \varepsilon > 0$ such that for all

$$\vec{x} \in N_\varepsilon(\vec{p}) \setminus \{\vec{p}\},$$

$$\vec{f}^{\,k}(\vec{x}) \text{ not in } N_\varepsilon(\vec{p}) \text{ for some } k \geq 1.$$

note, k must be allowed to depend on choice of $\vec{x}$! (as $\vec{x} \to \vec{p}$, k can grow without limit).

Here $N_\varepsilon(\vec{p}) := \{ \vec{x} \in \mathbb{R}^m : |\vec{x} - \vec{p}| < \varepsilon \}$ neighborhood. (a ball).

[2] (e) Compute (by hand!) the binary representation of the fraction 7/9

Use $2x \pmod 1$ map : Now repeats!

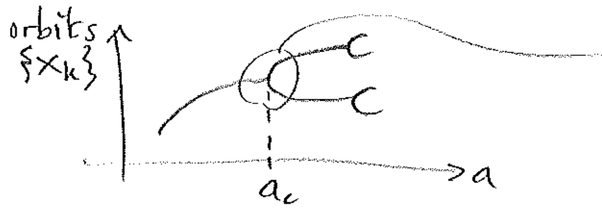
$$7/9 \to \frac{14}{9} = \frac{5}{9} \pmod 1 \to \frac{10}{9} = \frac{1}{9} \pmod 1 \to 2/9 \to 4/9 \to 8/9 \to \frac{16}{9} = 7/9 \bmod 1$$

"1" "1" "0" "0" "0" "1"

$$\frac{7}{9} = 0.\overline{110001}$$

period-6.

[2] (f) Consider a 1-dimensional map undergoing bifurcation with respect to some parameter. What is the Lyapunov *number* of the orbit at its bifurcation point?



bifurcation point : here a p-1 orbit loses stability & a p-2 is born.

As $a \to a_c$ from below, stability approaches <u>neutral</u>, ie Lyapunov number $L = 1$ (exponent is $h = 0$).