Math 118. Combinatorics.
Problem Set 1. Due on Wednesday, 1/19/11.

1. Prove that the number of compositions of n with an even number of parts is 2^{n-2}.

2. Give a bijection between the set of rooted binary trees with n internal vertices and the set of rooted plane trees on $n+1$ vertices.

3. Show that the generating function for the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ is given by the following continued fraction:

$$\sum_{n=0}^{\infty} C_n x^n = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \dots}}}}.$$

4. (a) Find the (ordinary) generating function for the number of paths from $(0,0)$ to $(n,0)$ using steps $U = (1,1)$, $D = (1,-1)$, and $H = (1,0)$.

   (b) The same problem, but now with the restriction that the paths cannot go below the x-axis.

   Hint: Try doing (b) before (a).

5. Prove that

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2(n-k)}{n-k} = 4^n.$$

6. (a) In how many ways can we choose k points, no two consecutive, from a collection of n points arranged in a line? (b) What if the n points are arranged in a circle?

7. Let h_n be the number of ways to choose a permutation π of $[n]$ and a subset S of $[n]$ such that if $i \in S$, then $\pi(i) \notin S$. Find an expression for the exponential generating function $\sum_{n \geq 0} h_n \frac{z^n}{n!}$.