Math 12, Fall 2007

Lecture 15

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Outline

1. Review and overview
   - Last class
2. Today's material
   - The change of variables formula
   - Integration in two variables in polar coordinates
3. Group Work
4. Next class

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Integration of a function of two variables

General domains

- Iterated integrals and Fubini's theorem

$$\int\int_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dx\,dy$$

- Non-rectangular domains: parameterize boundary and introduce variables into the bounds of integration

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Transformations

- Dealing with non-rectangular regions: parameterizations
- New idea: create a map which transforms the region into a rectangle
- Example: $T(u, v) = (u, uv)$ on $[0, 1] \times [0, 1]$

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Jacobians

Transformations deform rectangles into new shapes. How can we measure this distortion?

- If $T(u, v) = (x(u, v), y(u, v))$, then DT is a 2×2 matrix

$$DT = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

- We can use DT to tell us how the map is stretching the rectangle: $T_u = (x_u, y_u)$ and $T_v = (x_v, y_v)$ are tangent vectors to the image of T. We can approximate the area of the image by the area of the parallelogram generated by these two vectors.
- $|T_u \times T_v| = det(DT)$
- This is called the Jacobian of the transformation T

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COV

Suppose T is a smooth transformation whose Jacobian is nonzero that maps a region S in the uv-plane to a region R in the xy-plane. Suppose tht f is continuous on R and that T is one-to-one on the interior of S. Then

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v))J\,du\,dv$$

where $T(u,v) = ((x(u,v), y(u,v))$ and J is the Jacobian.

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Polar coordinates

- $(x, y) \to (r, \theta)$
- $T(r, \theta) = (r\cos(\theta), r\sin(\theta))$
- $detDT = r$
- $r^2 = x^2 + y^2, \theta = \tan^{-1}(y/x)$

Change of variables

If f is a continuous function defined on a polar rectangle $R = [a, b] \times [\alpha, \beta] = \{(r, \theta) | a \leq r \leq b, \alpha \leq \theta \leq \beta\}$ where $0 \leq \beta - \alpha \leq 2\pi$ then

$$\int\int_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta$$

General change of variables

If we are trying to integrate f over a region D, we may always change to polar coordinates:

$$\int \int_D f \, dA = \int \int_{D^*} f \, r dr d\theta$$

where D^* is the same region of D, described with respect to the polar variables.

Examples

1. $$\int\int_D (1 - x^2 - y^2)\, dA$$
   where D is the circle of radius 2 centered at the origin.
2. Let $R = \{(x, y) | 1 \leq x^2 + y^2 \leq 4, 0 \leq y \leq x\}$ and find
   $$\int\int_R \tan^{-1}(y/x)\, dA$$
3. Find the volume of a sphere is radius a centered at the origin.
4. Find
   $$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\, dy\, dx$$

Examples

A cylindrical drill with radius r_1 is used to bore a hole through the center of a solid ball of radius r_2. Find the volume of the ring-shpaed solid that remains. Express the volume in terms of the height, h, of the ring.

Work for next class

- Reading: 16.6
- f07hw16