Homework 5 Solutions, M31F11

(9.9) Let $H = \langle (1,1) \rangle \leq \mathbb{Z}_2 \times \mathbb{Z}_4$. Then

$$H = \{(0,0), (1,1), (0,2), (1,3)\}$$

The element $(1,0) \notin H$, so consider the coset

$$H + (1,0) = \{(1,0), (0,1), (1,2), (0,3)\}.$$

Since $[H + (1,0)] \cap H = \phi$, and $|H| = |H + (0,1)| = 4$, and $|\mathbb{Z}_2 \times \mathbb{Z}_4| = 2 \cdot 4 = 8$, it follows that

$$H \cup [H + (1,0)] = \mathbb{Z}_2 \times \mathbb{Z}_4$$

because $|H \cup [H + (1,0)]| = |H| + |H + (1,0)| = 4 + 4 = 8$ since the two sets are disjoint.

Thus, H & $H + (1,0)$ are the only cosets of H in $\mathbb{Z}_2 \times \mathbb{Z}_4$.

(Alternatively, you could observe that each element of $\mathbb{Z}_2 \times \mathbb{Z}_4$ is either in H or in $H + (1,0)$, so these are the only two cosets of H.)

(9.12) Let G be a group. Define aRb iff there exists $x \in G$ such that $a = xbx^{-1}$. Then R is an equivalence relation on G.

Proof We must show that R is reflexive, transitive, and symmetric:

Reflexive Since $e = e^{-1}$, we have aRa:

$$a = eae = eae^{-1}$$

Transitive If aRb and bRc, then there exist $x, y \in G$ such that

$$a = xbx^{-1} \qquad b = ycy^{-1}$$

Therefore, $a = xycy^{-1}x^{-1} = (xy)c(xy)^{-1}$.

Since G is a group, $xy \in G$ when $x, y \in G$, and hence aRc. Thus R is transitive.

Symmetric If aRb, then $a = xbx^{-1}$. This is equivalent to the statement $x^{-1}ax = b$. Since $x = (x^{-1})^{-1}$, it follows that bRa, so R is symmetric as claimed. $\boxtimes$

HW 5 Solutions M31 F11

Please find the left and right cosets of $\langle (12) \rangle \leq S_3$.

The subgroup $\langle (12) \rangle = \{(12), id\}$

Right cosets $\langle (12) \rangle (13) = \langle (12) \rangle (132)$

$= \{(132), (13)\}$

because $(12)(13) = (132)$.

$\langle (12) \rangle (23) = \langle (12) \rangle (123)$

$= \{(123), (23)\}$

because $(12)(23) = (123)$

Left cosets $(13)\langle (12) \rangle = \{(13), (123)\}$

because $(13)(12) = (123)$

$= (123)\langle (12) \rangle$ because $(123)(12) = (13)$

$(23)\langle (12) \rangle = \{(23), (132)\}$

$= (132)\langle (12) \rangle$

Observe that although $(13)\langle (12) \rangle = (123)\langle (12) \rangle$, we do <u>not</u> have $\langle (12) \rangle (13) = \langle (12) \rangle (123)$!

The only coset that is both a right and left coset is $\langle (12) \rangle id = \langle (12) \rangle$.

HW 5 Solutions M31 F11

(10.5) Let $X = \{1,2,3,4,5\}$ and let $Y = \{1,2,3\}$.
Observe that $(P(Y), \Delta) \leq (P(X), \Delta)$.

Claim $[P(X) : P(Y)] = 4$

Proof First, observe that $|P(Z)| = 2^{|Z|}$ for any finite set Z: If $z \in Z$, then to create a subset W of Z we either say $z \in W$ or $z \notin W$. Since we repeat this choice for all elements of Z, and since this choice is independent for each element of Z, it follows that there are $2^{|Z|}$ ways to choose a subset W of Z. In other words, $|P(Z)| = 2^{|Z|}$ as claimed.

Therefore, $|P(X)| = 2^5 = 32$ and $|P(Y)| = 2^3 = 8$.
By Lagrange's Theorem, then, $[P(X) : P(Y)] = \dfrac{|P(X)|}{|P(Y)|}$

$$= 32/8$$
$$= 4$$

as claimed. ⊠

(10.5) If $|G|=8$ and G is not cyclic, then $a^4=e$ for all $a \in G$.

Proof We know that $o(a) \mid |G|$ by Theorem 10.4.

If $o(a) \mid 8$, then $o(a) \mid 4$ also unless $o(a)=8$, since the divisors of 8 are 1, 2, 4, & 8.
So, to prove $o(a) \mid 4$ (and hence $a^4=e$ by Theorem 4.4(ii)) for all $a \in G$, we only need to show that G has no elements of order 8. We proceed by contradiction.

Suppose $o(g)=8$ for some $g \in G$. Then $\langle g \rangle = \{e, g, g^2, \ldots, g^7\} \leq G$. But since $|\langle g \rangle| = |G| = 8$ it follows that $\langle g \rangle = G$, which contradicts the hypothesis that G is not cyclic.
Thus G has no elements of order 8, and it follows that $o(a) \mid 4$ for all $a \in G$. Appealing to Theorem 4.4 (ii) completes the proof. $\boxtimes$