1. (12) Evaluate the indefinite integral $\int \sin^6 x \cos^3 x \, dx$.

$$\int \sin^6 x \cos^3 x \, dx = \int \sin^6 x \cos^2 x \cos x \, dx$$

$$= \int \sin^6 x \, (1 - \sin^2 x) \cos x \, dx$$

Substitute $u = \sin x$

$du = \cos x \, dx$

$$\int \sin^6 x \cos^3 x \, dx = \int u^6 (1 - u^2) \, du$$

$$= \int (u^6 - u^8) \, du$$

$$= \frac{u^7}{7} - \frac{u^9}{9} + C$$

$$= \boxed{\frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C}$$

2. (14) Determine whether the following integral is convergent or divergent. Evaluate if it is convergent.

$$\int_0^\infty \frac{x^2}{9+x^6}\,dx.$$

$$\int_0^\infty \frac{x^2}{9+x^6}\,dx = \lim_{t\to\infty} \int_0^{t} \frac{x^2}{9+x^6}\,dx \quad \text{—} (*)$$

Consider $\displaystyle\int_0^t \frac{x^2}{9+x^6}\,dx = \int_0^t \frac{x^2}{9+(x^3)^2}\,dx$

Substitute $u = x^3$

$du = 3x^2\,dx$

$$= \frac{1}{3}\int_0^{t^3} \frac{du}{9+u^2}$$

$$= \frac{1}{3}\left(\frac{1}{3}\arctan\left(\frac{u}{3}\right)\right]_0^{t^3}$$

$$= \frac{1}{9}\left[\arctan\frac{t^3}{3} - \arctan 0\right]$$

$$= \frac{1}{9}\arctan\left(\frac{t^3}{3}\right)$$

From $(*)$ $\displaystyle\int_0^\infty \frac{x^2}{9+x^6}\,dx = \lim_{t\to\infty} \frac{1}{9}\arctan\left(\frac{t^3}{3}\right) = \frac{1}{9}\left(\frac{\pi}{2}\right)$

Hence $\displaystyle\int^\infty \frac{x^2}{9+x^6}\,dx$ is convergent $\int = \frac{\pi}{18}$

3. (14) Evaluate $\int \frac{1}{x^2\sqrt{x^2-1}}\,dx$.

put $x = \sec\theta$.

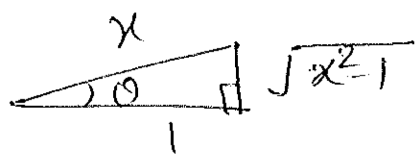
$dx = \sec\theta\tan\theta\,d\theta$

$$\int \frac{1}{x^2\sqrt{x^2-1}}\,dx = \int \frac{\sec\theta\tan\theta\,d\theta}{\sec^2\theta\sqrt{\sec^2\theta-1}}$$

$$= \int \frac{\tan\theta}{\sec\theta\tan\theta}\,d\theta.$$

$$= \int \cos\theta\,d\theta = \sin\theta + C.$$

Now $x = \sec\theta \Rightarrow \cos\theta = \frac{1}{x}$



$$\sin\theta = \frac{\sqrt{x^2-1}}{x}$$

Hence $\int \frac{1}{x^2\sqrt{x^2-1}}\,dx = \frac{\sqrt{x^2-1}}{x} + C.$

4. (14) Determine and justify whether the series $\sum_{n=1}^{\infty} \frac{\pi^n}{(n+1)\,2^{2n+1}}$ converges or diverges. Mention any test(s) that you might use and verify that it is applicable.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\pi^{n+1}}{(n+2)\,2^{2n+3}} \cdot \frac{(n+1)\,2^{2n+1}}{\pi^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\pi(n+1)}{(n+2)\,4} \right|$$

$$= \lim_{n\to\infty} \frac{\pi}{4} \frac{1+1/n}{1+2/n}$$

$$= \frac{\pi}{4} < 1$$

Hence by ratio test the given series is abs. cgt & hence cgt.

5. (14) Determine and justify whether the series $\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}} + \cos^2 n}{n^3}$ converges or diverges. Mention any test(s) that you might use and verify that it is applicable.

Consider $\sum_{n=1}^{\infty} e^{1/n}/n^3$

$0 < e^{\frac{1}{n}} \leq e$ for all n

hence $\frac{e^{1/n}}{n^3} \leq \frac{e}{n^3}$ for all n. — (1)

We know that $\sum_{n=1}^{\infty} \frac{e}{n^3} = e \sum_{n=1}^{\infty} \frac{1}{n^3}$

is cgt because $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is cgt

(p-series test, $p = 3 > 1$)

By (1) & comparison test, $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^3}$ is cgt.

Now consider $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^3}$

$0 \leq \frac{\cos^2 n}{n^3} \leq \frac{1}{n^3}$ for all n

Once again by comparison test $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^3}$ is cgt

as $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is cgt.

Sum of two cgt series is cgt & hence the given series is cgt.

6. (12) Determine and justify whether the series $\sum_{n=1}^{\infty} \frac{n-1}{3n+1}$ converges or diverges. Mention any test(s) that you might use and verify that it is applicable.

$$\lim_{n\to\infty} \frac{n-1}{3n+1} = \lim_{n\to\infty} \frac{1-1/n}{3+1/n}$$

$$= \frac{1}{3} \neq 0$$

By Test for divergence $\sum_{n=1}^{\infty} \frac{n-1}{3n+1}$ is dgt.

7. (20) For each of the following statements, fill in the blank with the letters **T** or **F** depending on whether the statement is true or false. You do not need to show your work and no partial credit will be given on this problem.

(a) Let $\{a_n\}$ be a sequence such that $\lim_{n\to\infty} |a_n| = 2$. Then $\lim_{n\to\infty} a_n = 2$ or $\lim_{n\to\infty} a_n = -2$.

take $a_n = (-2)^n$

ANS: F

(b) The series $\sum_{n=1}^{\infty} \frac{(-1)^n \ln(n)}{n}$ is conditionally convergent.

ANS: T

(c) The series $\sum_{n=1}^{\infty} (\arctan(n+1) - \arctan(n))$ is convergent.

ANS: T

(d) The series $.9 + .99 + .999 + .9999 + \cdots$ converges to 1.

ANS: F

(e) The series $\sum_{n=1}^{\infty} \frac{n^3}{n^4 - 1}$ is convergent.

ANS: F.