4.8: Volumes of Solids of Revolution

Mathematics 3
Lecture 25
Dartmouth College

March 03, 2010

Example 1

Before we begin, let's test our knowledge of integration. Consider the indefinite integral:

$$\int x\sqrt{x+1}\,dx$$

a.) Evaluate by using integration by parts:

$$\int u\,dv = uv - \int v, du$$

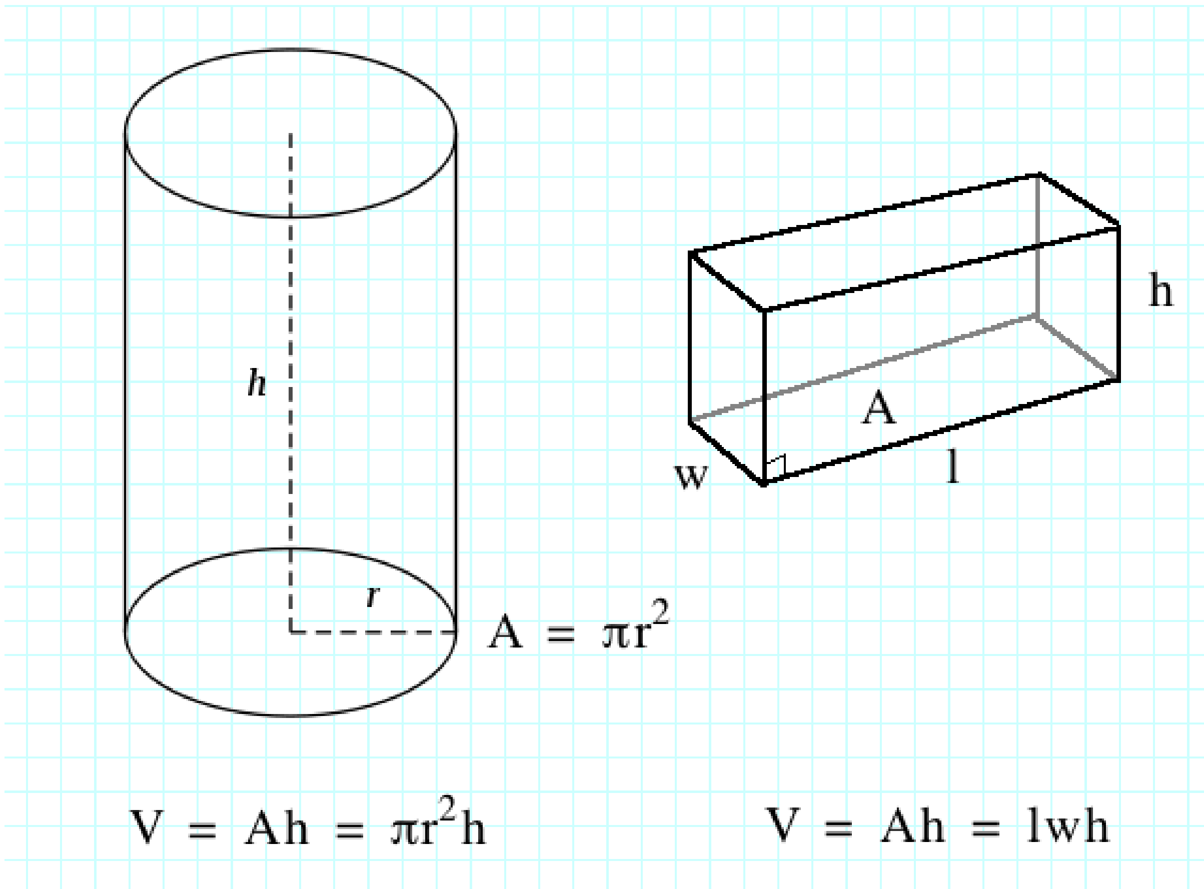
$$\begin{cases} u = f(x) & v = g(x) \\ du = f'(x)\,dx & dv = g'(x)\,dx \end{cases}$$

b.) Evaluate by using a substitution $u = g(x)$:

$$\int f'(g(x))g'(x)\,dx = \int f(u)\,du$$

Volume of a Right Prism

Right Prism with base (cross-sectional) area A and height h: $V = Ah$



Cavalieri's Principle

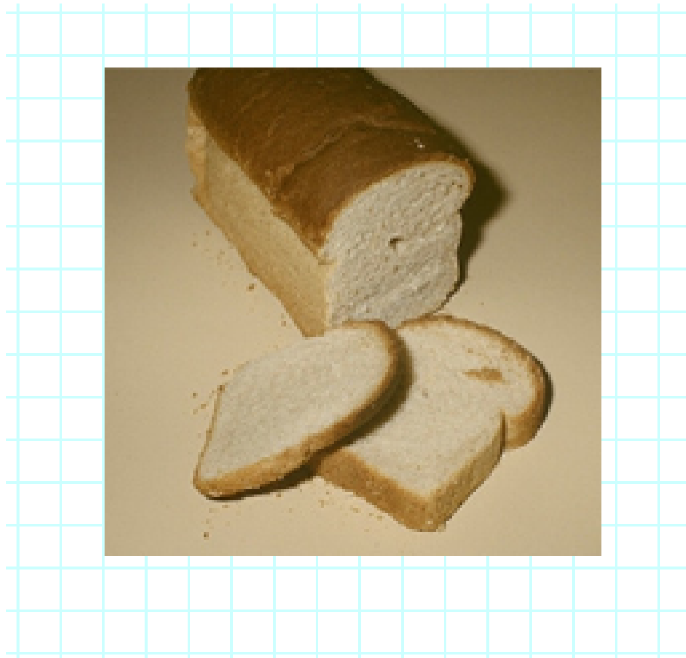
Bonaventura Cavalieri (1598 - 1647) stated a principle for volumes of solids that anticipated the integral calculus:

Suppose two solids in three dimensions are included between two parallel planes. If every plane parallel to these two planes intersects both solids in cross-sections of equal area, then the two solids have the **same volumes.**

NOTE: *Cavalieri's principle* also holds for regions in 2D which have the same cross-sectional lengths and, thus, have the **same areas**.

Volume by Slicing (Loafbread)

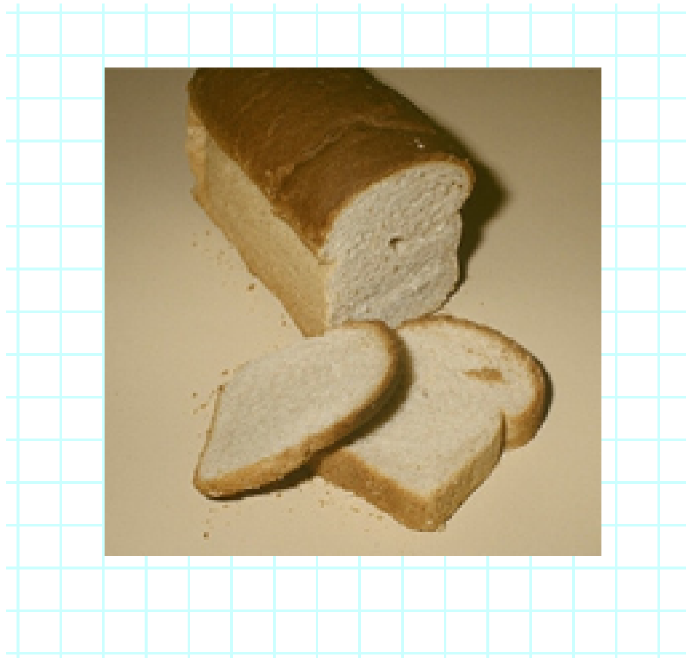
A loaf of bread is sliced into n thin slices (of equal width Δx) which we approx as prisms:



$$\text{Volume } V = \sum_{i=1}^{n} V(x_i) \approx \sum_{i=1}^{n} A(x_i)\Delta x$$

Volume by Slicing (Loafbread)

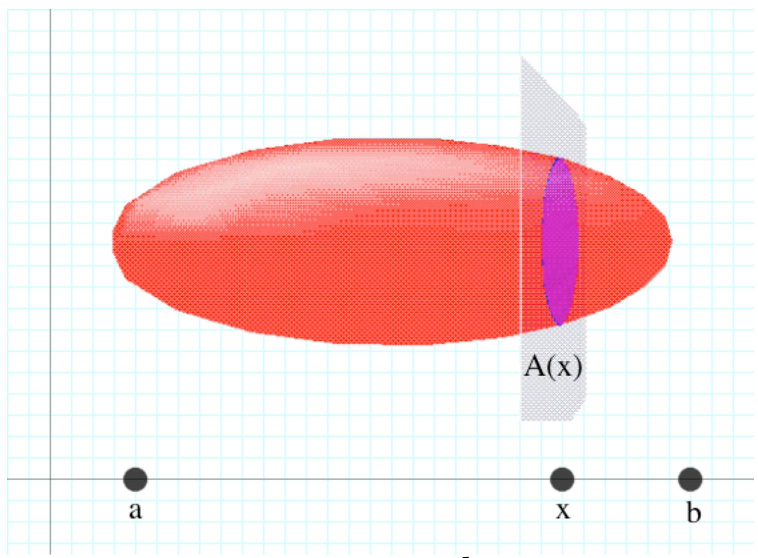
The actual volume should be the limit as $n \to \infty$ ($\Delta x \to 0$):



$$\text{Volume } V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\Delta x = \int_a^b A(x)\,dx$$

Volume by Slicing (General Solid)

Suppose that a three-dimensional solid lies along the x-axis covering the inteval $[a, b]$ and the cross-sectional area at x is a continuous function, call it it $A(x)$. How do we define/compute it's volume V?

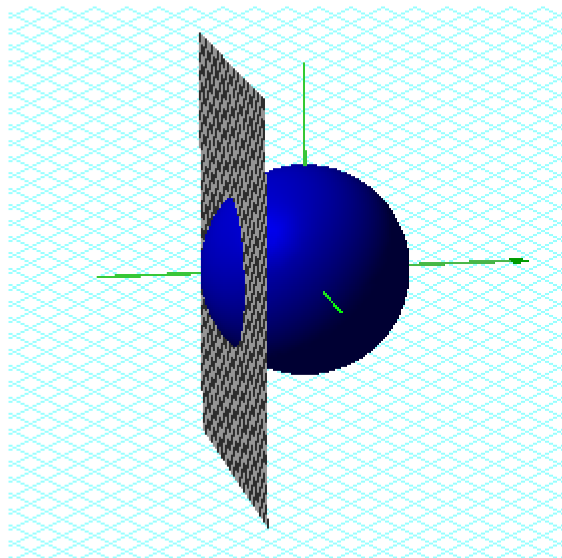


$$\text{Volume } V = \int_a^b A(x)\,dx$$

NB: This works in full generality for ANY Solid Object in 3D!

Example 2

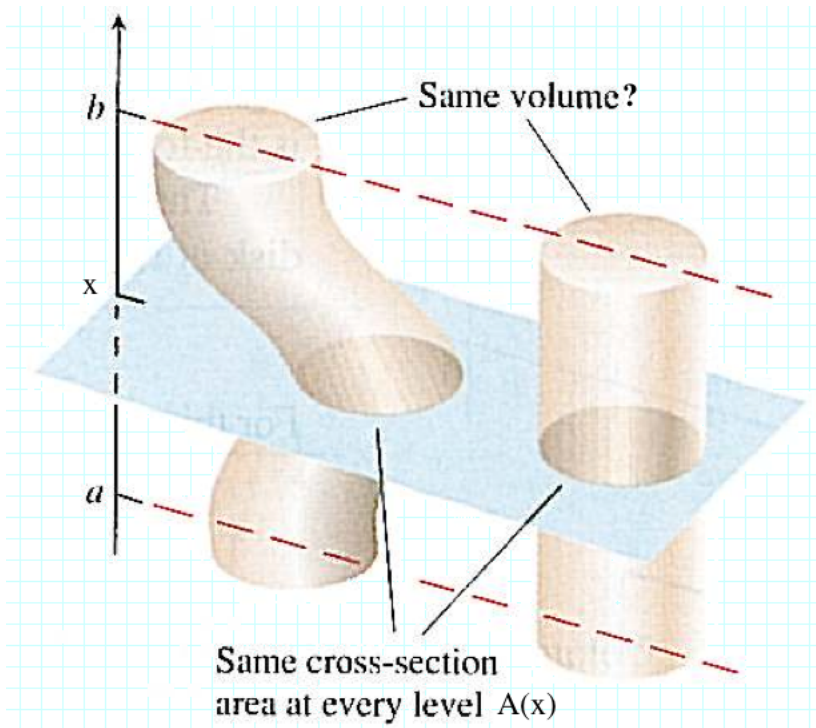
Compute the volume of a sphere of radius $r = 2$ at the origin by the Volume by Slicing method.



$$V = \int_{-2}^{2} A(x)\,dx = \int_{-2}^{2} \pi(4 - x^2) = \frac{32}{3}\pi = \frac{4}{3}\pi 2^3$$

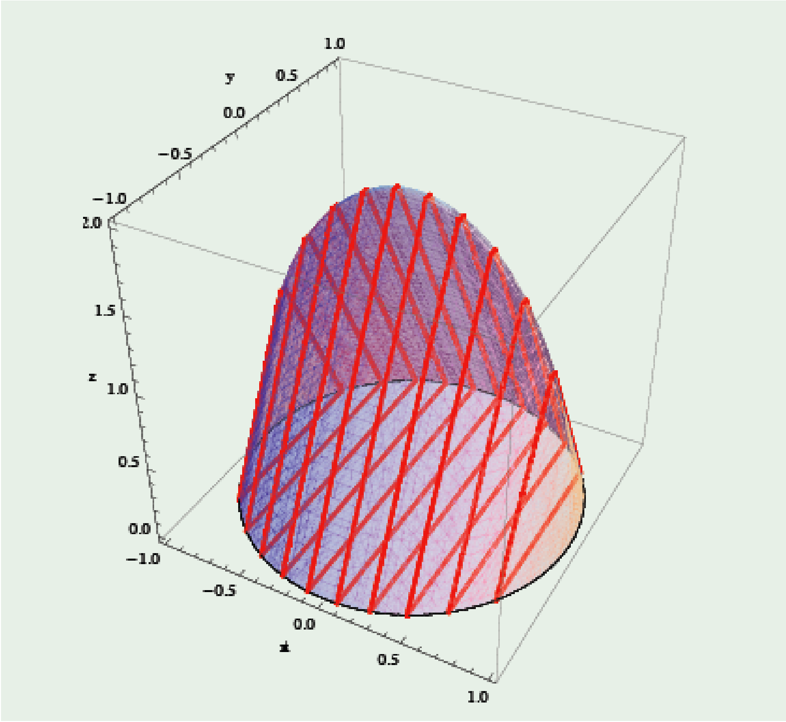
Proof of Cavalieri's Principle

Suppose two solids S_1 and S_2 have the same height and cross-sectional areas:



$$\text{Volumes by slicing} \Rightarrow V_1 = \int_a^b A(x)\,dx = V_2 \qquad \surd \smile$$

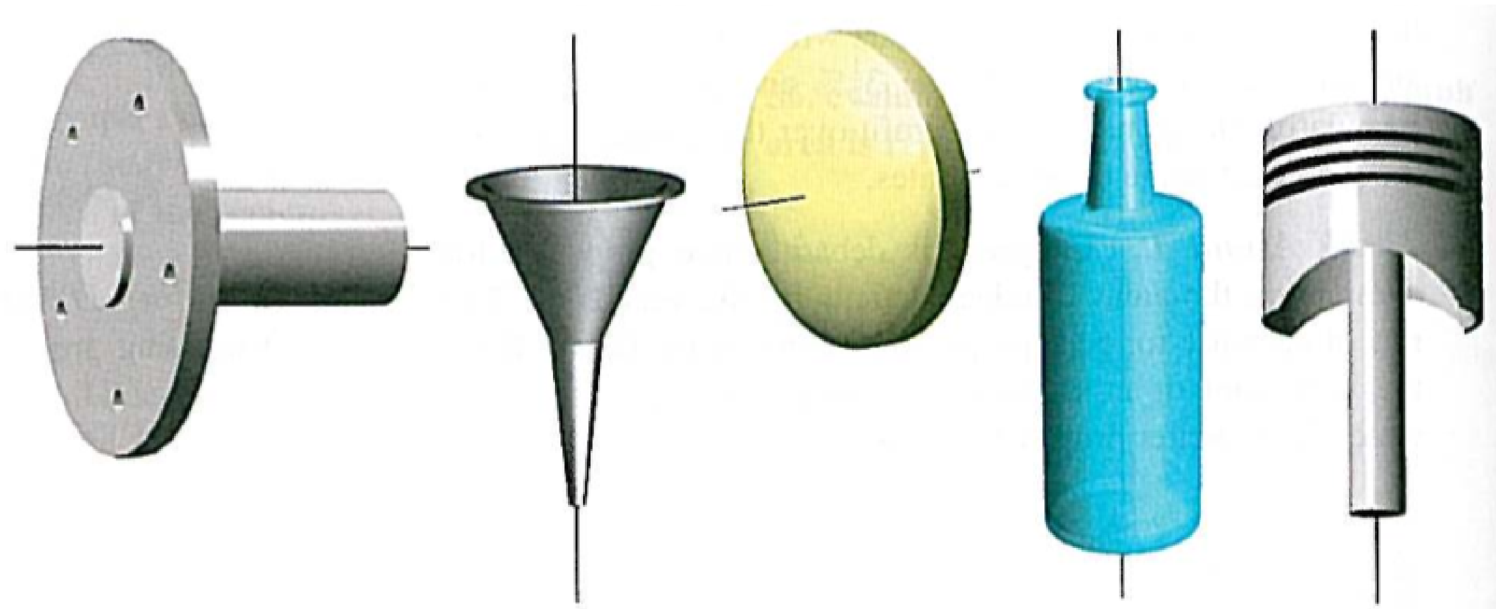
Example 3 A solid object has as its base the circular region defined by the unit circle. Every cross section of the object perpendicular to the x-axis is a triangle whose base vertices are on the circle and whose height equals the length of the base. Find the volume of this object.



$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} 2(1-x^2)\,dx = \frac{8}{3}$$

Solids of Revolution

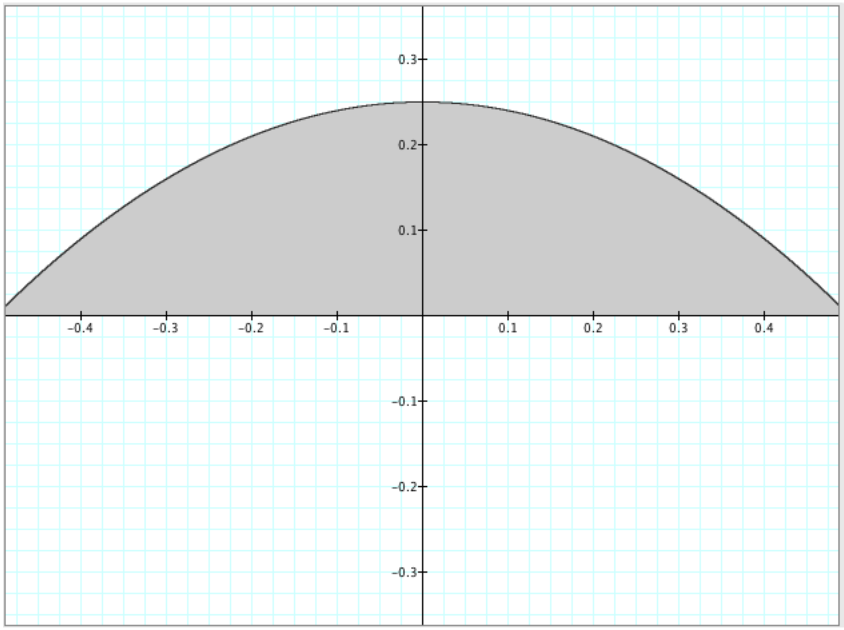
Solids of Revolution are commonly used in engineering and manufacturing, such as axles, funnels, pills, bottles, and pistons.

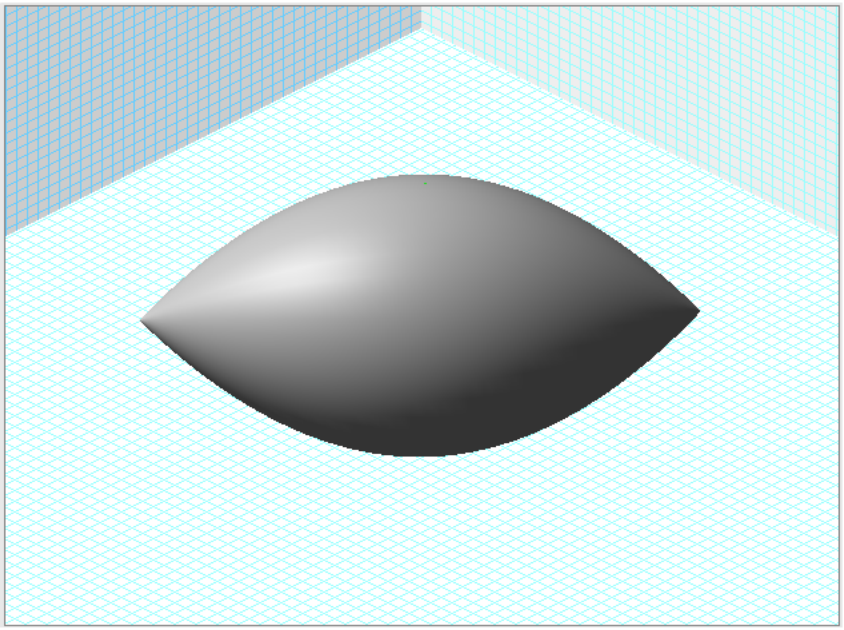


Solids of revolution

Solids of Revolution

A Solid of Revolution is generated by taking a region in the plane, say the area under the graph of a function $y = f(x)$ over $[a, b]$, and **rotating** it about an axis (e.g., x-axis or another line) in three dimensions.



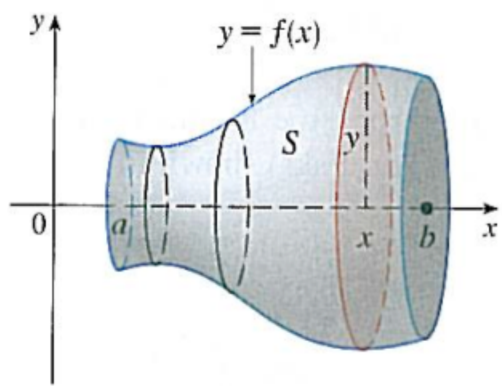


Solids of Revolution

Every perpendicular cross-section at x is a circle of radius $r = f(x)$, so the area function $A(x)$ is given by:

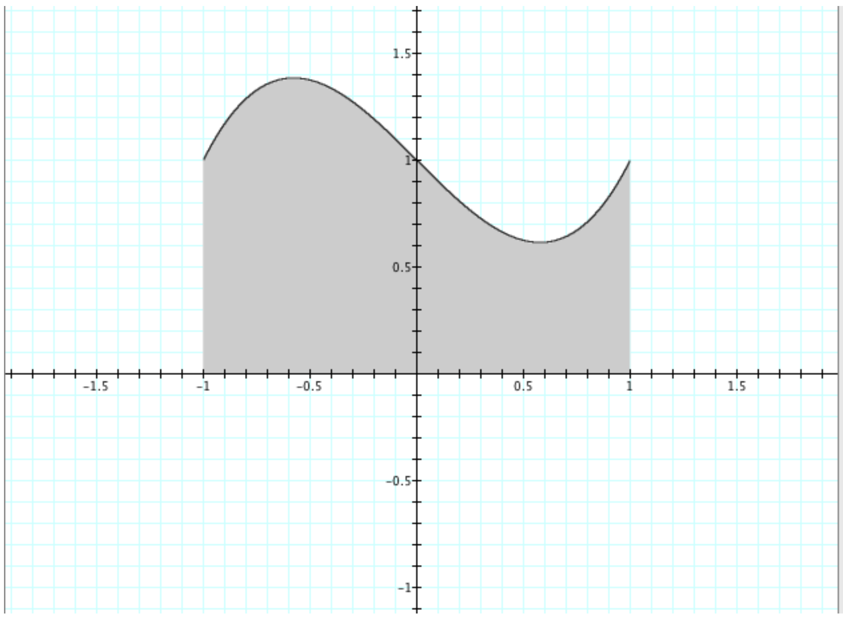
$$A(x) = \pi r^2 = \pi[f(x)]^2.$$

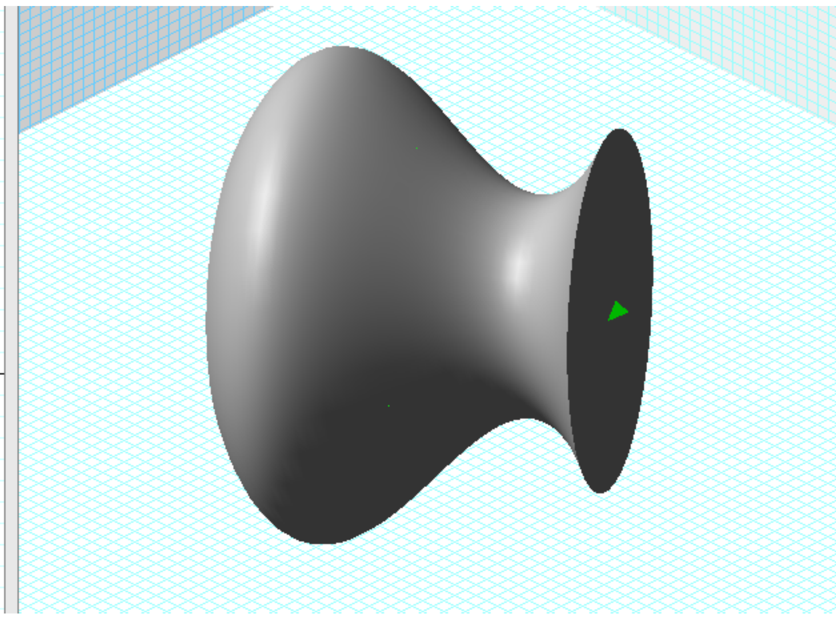
Thus, from Volumes by Slicing, the volume V is the definite integral:



$$V = \int_a^b A(x)\,dx = \int_a^b \pi[f(x)]^2\,dx$$

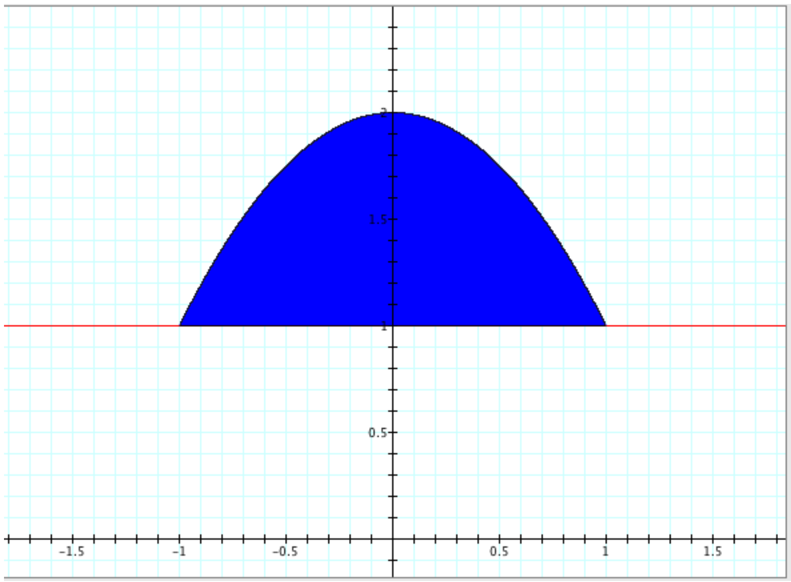
Example 4: Find the volume of the solid of revolution generated by revolving the region bounded by the x-axis, the curve $y = x^3 - x + 1$ and the vertical lines $x = -1$ and $x = 1$ around the x-axis.

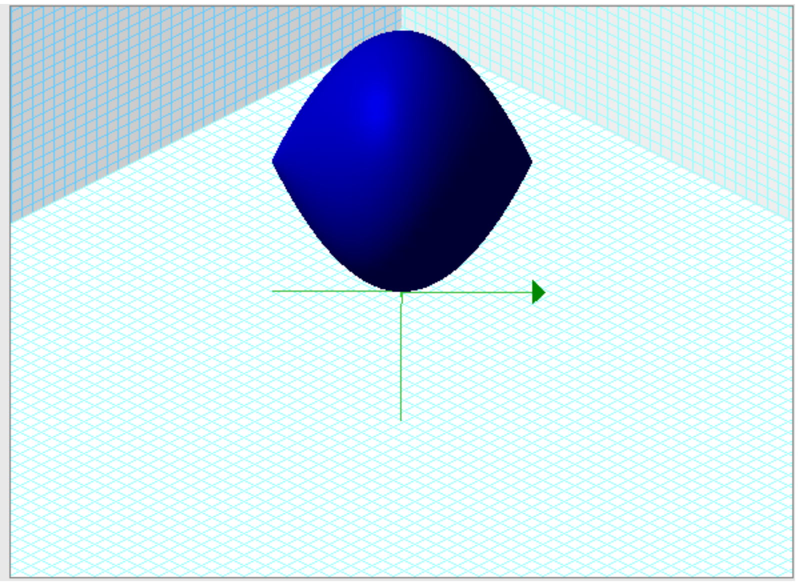




$$V = \int_{-1}^{1} \pi(x^3 - x + 1)^2 \, dx = \frac{226\pi}{105}$$

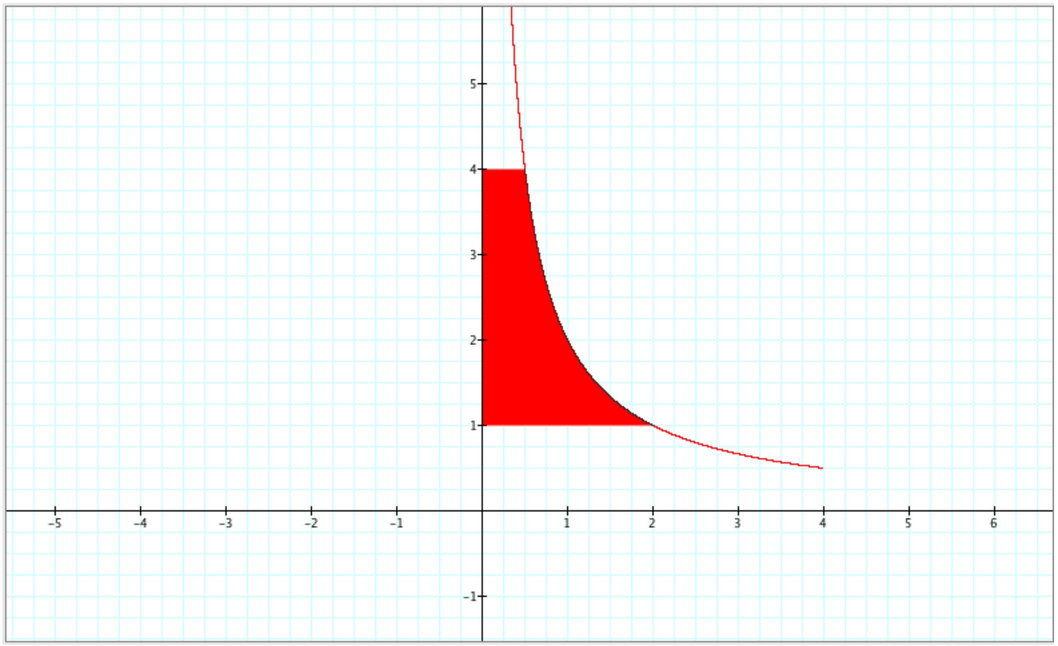
Example 5: Find the volume of the solid of revolution generated by revolving the region bounded by $f(x) = 2 - x^2$ and $g(x) = 1$ about the line $y = 1$.





$$V = \int_{-1}^{1} A(x)\, dx = \int_{-1}^{1} \pi((2-x^2)-1)^2\, dx = \int_{-1}^{1} \pi(1-x^2)^2\, dx = \frac{16\pi}{5}$$

Example 6: Find the volume of the solid of revolution generated by revolving the region between the y-axis and the curve $xy = 2, \quad 1 \leq y \leq 4$, around the y-axis.



$$V = \int_1^4 A(y)\,dy = \int_1^4 \pi(r(y))^2\,dy = \pi \int_1^4 \frac{4}{y^2}\,dy = 3\pi$$