Math 22 Practice Problems

NOTE: This is not meant to represent a sample exam either in difficulty or in length. These are problems collected from old exams and/or problems left over during the preparation of the exam. I hope they will give a good indication of the general level of expectation.

1. Define what it means for a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ to be one-to-one.

2. Let
$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} -2 \\ 3 \\ \frac{1}{2} \end{pmatrix}.$$

   (a) Write $\mathbf{v}$ as a linear combination of the $\mathbf{e}_i$.

   (b) Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be a linear transformation which satisfies
$$T(\mathbf{e}_1) = \begin{pmatrix} -4 \\ 3 \end{pmatrix}, \quad T(\mathbf{e}_2) = \begin{pmatrix} -\frac{2}{3} \\ 5 \end{pmatrix} \quad \text{and} \quad T(\mathbf{v}) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$
   Use part (a) to find the standard matrix for T.

   (c) Is T one-to-one? Is T onto?

3. Let $A = \begin{pmatrix} -4 & 1 & 0 \\ -2 & -1 & -2 \\ 4 & 1 & -5 \end{pmatrix}$.

   (a) Are the columns of A linearly independent?

   (b) Do the columns of A span all of $\mathbf{R}^3$?

4. Let $A = \begin{pmatrix} 1 & 3 & 8 \\ 2 & 4 & 11 \\ 1 & 2 & 5 \end{pmatrix}$. Find A^{-1}, and use A^{-1} to solve $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

5. Fill in the blank below with a choice from the following list so that the resulting statement is always true.

(**A**) No solutions
(**B**) Exactly one solution
(**C**) At least one solution
(**D**) Infinitely many solutions
(**E**) None of the above is appropriate

(a) If $T : \mathbf{R}^n \to \mathbf{R}^m$ is not onto, then there is a $\mathbf{b} \in \mathbf{R}^m$ such that $T(\mathbf{x}) = \mathbf{b}$ has ____________.

(b) If a matrix A has a column which is not a pivot column, then $A\mathbf{x} = \mathbf{0}$ has ____________.

(c) If $\mathbf{b}$ is a linear combination of the columns of the matrix A, then $A\mathbf{x} = \mathbf{b}$ has ____________.

(d) The matrix equation $A\mathbf{x} = \mathbf{0}$ always has ____________.

(e) If the matrix equation $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, then $A\mathbf{x} = \mathbf{b}$ cannot have ____________.

(f) If the columns of A are linearly independent, then $A\mathbf{x} = \mathbf{0}$ has ____________ with $\mathbf{x} \neq \mathbf{0}$.

(g) If T is a linear transformation, then T is one-to-one if and only if $T\mathbf{x} = \mathbf{0}$ has ____________.

6. Determine the values of k and h such that the system of equations

$$\begin{aligned} x_1 + 3x_2 &= k \\ 4x_1 + hx_2 &= 8 \end{aligned}$$

has

(a) no solution,

(b) exactly one solution and

(c) infinitely many solutions.

In cases (b) and (c), write the solutions in parametric form.

7. Suppose that B is a $m \times n$ matrix and that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are vectors in $\mathbf{R}^n$ such that $\{ B\mathbf{v}_1, \dots, B\mathbf{v}_n \}$ is linearly independent. Prove that $\{ \mathbf{v}_1, \dots, \mathbf{v}_n \}$ is also linearly independent.

8. Write $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ as a product of elementary matrices.

9. Let $\{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \}$ be linearly independent in $\mathbf{R}^4$. Suppose that $\mathbf{v}_4$ is not in $\operatorname{Span}\bigl(\{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \}\bigr)$. Must $S = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \}$ be linearly independent?