Math 31 Lesson Plan

Day 14: Functions; The Symmetric Group

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Supplies needed:

- Colored chalk
- Quizzes!
- Patricia's triangle

Goals for students: Students will:

- Learn how to multiply permutations

[**Lecture Notes: Write everything in blue, and every equation, on the board. [Square brackets] indicate anticipated student responses. *Italics* are instructions to myself.**]

Collect homework. Return quizzes etc. Apologize that I don't have HW 2 back for them yet. Write Cayley table for D_3 on board.

Are there any questions about the homework? *Be prepared to explain the D_4 problem.*

With respect to the D_4 problem, some people pointed out that an ordered pair of elements is not a single element of D_4; but can you think of a way to take [for example] an element of $\langle 90 \rangle \times \langle V \rangle$ – that is, a pair (r, f) where r is a rotation and f is a flip – and combine them to get an element of D_4? *Think-pair-share*

This problem was trying to get at the idea of isomorphism, which we will be learning next week. The idea is that even though D_4 can't be written as a product of some of its subgroups (why? [because they're all abelian and D_4 isn't]), some groups might be <u>isomorphic</u> to a direct product of some of their subgroups. In the D_4 case, the isomorphism would be the map you came up with just now, that takes a pair (r, f) to a single element of D_4 by composing the operations.

For general direct products, you can't mix up the operations this way; but since all the groups in this case are subgroups of the same big group, the operations are the same, so you <u>can</u> just multiply elements from the two factor groups.

Today I want to quickly talk about Section 7: Functions, and then start on Section 8. I want to spend Monday catching up and reviewing, so your Reading Assignment for Monday is to review your homeworks, quizzes, class notes, textbook, etc and find the things that you still have questions about. Please post your lingering questions/concerns (about Sections 0-7) on Blackboard by Sunday, 10 PM. Then, I would like you to Read Section 8 by Monday, 10 PM and post on Blackboard.

I will post solutions to all three homework assignments on the course website later today, so you can look at those this weekend as you're reviewing. 12:45

Section 7: Functions

Definition: A function $f : S \to T$ is one-to-one or injective if what? [every element of S is mapped to a different element of T. Alternatively, if $f(s_1) = f(s_2)$ then $s_1 = s_2$, which is equivalent to the statement "If $s_1 \neq s_2$,, then $f(s_1) \neq f(s_2)$."]

Can someone give me an example of an injective function?

Definition: A function $f : S \to T$ is onto or surjective if what? [every element of T is the image of some element of S. In other words, for all $t \in T$, there exists $s \in S$ such that $f(s) = t$.]

Can someone give me an example of an onto function?

Note that to define the concept of injectivity and surjectivity, you need to specify the domain and range of the function (the sets S and T in the definitions). For example, the map $x \mapsto e^x$ is surjective from $\mathbb{R}$ to $\mathbb{R}^+$, but it's not surjective when considered as a map from $\mathbb{R}^+$ to $\mathbb{R}^+$.

Is there any relationship between injectivity and surjectivity in general? [no] Let's come up with examples: skip if pressed for time

- A function that is injective and surjective $f(x) = x$ *from* $\mathbb{R}$ *to* $\mathbb{R}$.
- A function that is injective but not surjective e^x *from* $\mathbb{R}$ *to* $\mathbb{R}$.
- A function that is not injective but is surjective $f(x) = x^2$ *from* $\mathbb{R}$ *to* $\mathbb{R}^+$
- A function that is not injective and not surjective $f(x) = x^2$ *from* $\mathbb{R}$ *to* $\mathbb{R}$.

Definition: A function that is both one-to-one and onto is called a bijection. What's an example of a bijection? Remember, in order to say that something is or isn't a bijection, we have to specify the domain and range.

If a function $f : S \to T$ is a bijection, then we can define the inverse function $f^{-1} : T \to S$ how? 1:00

$$f^{-1}(t) = s \Leftrightarrow f(s) = t.$$

Grab a partner; please check:

- f^{-1} is a function from T to S. What does that mean? [In other words, we need to check that f^{-1} assigns exactly one element of S to each element of T.]
- For all $s \in S$ and all $t \in T$, $f^{-1} \circ f(s) = s$; $f \circ f^{-1}(t) = t$.

DEFINITION/THEOREM: Let X be any set. The collection of all bijections $f : X \to X$ forms a group under the operation of function composition. This group is written S_X, and it's called the symmetric group or group of permutations of X.

THEOREM (CAYLEY): Any group can be written as a subgroup of S_X for some X. We won't have time to prove this today, but I'll prove next week. Today we'll look at how this works in the case of some familiar groups, though.

Let's start by looking at D_3. *Draw triangle with vertices on the board; label vertices 1, 2, 3 in pink.* We can think of a move in D_3 as a function $f : \{1, 2, 3\} \to \{1, 2, 3\}$, where the first set indicate the vertices triangle itself, and the second set indicate the corners on the board. *that is, second set is also written in pink.* So, what function is associated to each element of D_3? *Write them as a list:* $f(1) = 2, f(2) = 3, f(3) = 1$, *etc. Write the collection of lists as a column, leaving space to the right.*

For us in this class, there are two ways to write a permutation: two-line notation and cycle notation. *Write elements of* D_3 *in this way.*

By the Cayley table for D_3, we know how to multiply moves in D_3; let's see how that corresponds to multiplying permutations.

Somebody tell me two elements from the Cayley table and their product. *compute the product of the corresponding 2-line and cycle notation for the permutations.*

Do several examples (maybe even all of D_3) as a class. Write product computing instructions on the board? 1:15

Please get into groups of 3 or 4, with at least one person you haven't worked with yet.

$f = (134)(26)(587)$; $g = (1)(2457)(368)$; $h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$; $k = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 5 & 2 & 3 & 6 \end{pmatrix}$.

1. Compute $f \circ g$, $g \circ f$, $h \circ k$, $k \circ h$.
2. Rewrite the permutations in 2-line notation in cycle notation, and vice versa.

Discuss results as a class if possible.

Answers:

$$f \circ g = (132)(48)(5)(67);$$
$$g \circ f = (164)(28)(35)(7);$$
$$h \circ k = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 5 & 6 & 3 \end{pmatrix}$$
$$k \circ h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 6 & 4 & 1 & 5 \end{pmatrix}$$
$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix}$$
$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 4 & 6 & 5 & 7 & 8 & 2 & 3 \end{pmatrix}$$
$$h = (14)(25)(36)$$
$$k = (142)(35)(6)$$