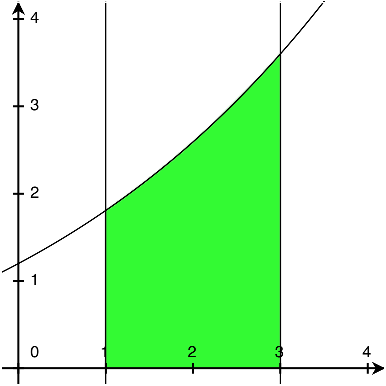
Lecture Notes

Math 3 / Fall 2012

Week 8

The area problem

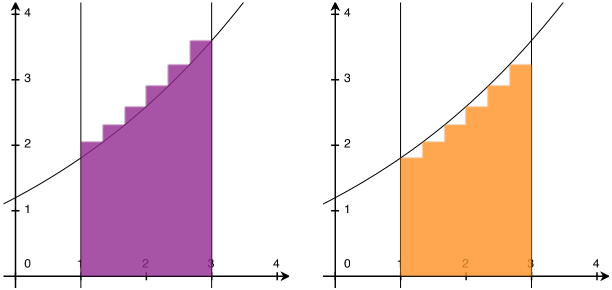
Given a function f on $[a, b]$, find the area between the graph $y = f(x)$ and the x-axis between the vertical lines $x = a$ and $x = b$.



$$\int_a^b f(x)\,dx$$

Riemann sums

To approximate the area, we **partition** the interval $[a, b]$ into smaller intervals $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.
To get an upper bound we add up areas rectangles above each interval with height equal to the maximum of f on that interval.
To get a lower bound we add up areas rectangles above each interval with height equal to the minimum of f on that interval.



Riemann sums

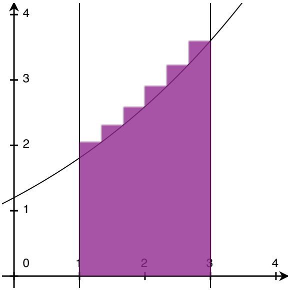
Given a partition P of $[a, b]$, these are called the **upper** and **lower Riemann sums** for f. These are denoted $U(P; f)$ and $L(P; f)$, respectively, where P is the partition of $[a, b]$.

So

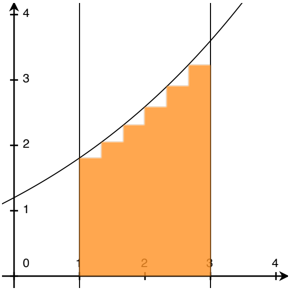
$$L(P; f) \leq \int_a^b f(x)\, dx \leq U(P; f).$$

Provided $\int_a^b f(x)\, dx$ is well defined...

Upper Sum



Lower Sum



Riemann sums

Given the partition $P : a = x_0 < x_1 < \cdots < x_n = b$, we have

$$U(P; f) = \max_1 \Delta x_1 + \max_2 \Delta x_2 + \cdots + \max_n \Delta x_n$$

$$L(P; f) = \min_1 \Delta x_1 + \min_2 \Delta x_2 + \cdots + \min_n \Delta x_n$$

Where:

- $\Delta x_i = x_i - x_{i-1}$ is the length of the ith interval $[x_{i-1}, x_i]$
- $\max_i$ is the maximum of $f(x)$ on $[x_{i-1}, x_i]$
- $\min_i$ is the minimum of $f(x)$ on $[x_{i-1}, x_i]$

In other words:

$$U(P; f) = \sum_{i=1}^{n} \max_i \Delta x_i \qquad L(P; f) = \sum_{i=1}^{n} \min_i \Delta x_i$$

Power sums

$$\sum_{i=1}^{n} 1 = 1 + 1 + 1 + \cdots + 1 = n = \tfrac{1}{1} n^1$$

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} = \tfrac{1}{2} n^2 + \tfrac{1}{2} n$$

$$\sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} = \tfrac{1}{3} n^3 + \tfrac{1}{2} n^2 + \tfrac{1}{6} n$$

$$\sum_{i=1}^{n} i^3 = 1 + 8 + 27 + \cdots + n^3 = \frac{n^2(n+1)^2}{4} = \tfrac{1}{4} n^4 + \tfrac{1}{2} n^3 + \tfrac{1}{4} n^2$$

$$\sum_{i=1}^{n} i^k = \frac{1}{k+1} n^{k+1} + (\text{lower degree terms})$$

Geometric sums

$$\sum_{i=1}^{n} b^{i-1} = b^0 + b^1 + b^2 + \cdots + b^{n-1} = \frac{b^n - 1}{b - 1} \quad (b \neq 1)$$

Because:

$$\begin{aligned}
(b-1)(b^0 &+ b^1 + \cdots + b^{n-1}) \\
&= b(b^0 + b^1 + \cdots + b^{n-1}) - (b^0 + b^1 + \cdots + b^{n-1}) \\
&= (b^1 + b^2 + \cdots + b^n) - (b^0 + b^1 + \cdots + b^{n-1}) \\
&= b^n - b^0 = b^n - 1
\end{aligned}$$

Computing sums

Example

Evaluate $\displaystyle\sum_{i=1}^{n}(i+5)^2 = 36 + 49 + 64 + \cdots + (n+5)^2$.

$$
\begin{aligned}
\sum_{i=1}^{n}(i+5)^2 &= \sum_{i=1}^{n}(i^2+10i+25) \\
&= \left(\sum_{i=1}^{n} i^2\right) + 10\left(\sum_{i=1}^{n} i\right) + 25\left(\sum_{i=1}^{n} 1\right) \\
&= \frac{n(n+1)(2n+1)}{6} + 10\frac{n(n+1)}{2} + 25n
\end{aligned}
$$

Computing sums

Example

Evaluate

$$\sum_{i=1}^{38} 3 \cdot 2^{i-5} = \frac{3}{16} + \frac{3}{8} + \frac{3}{4} + \frac{3}{2} + 3 + 6 + \cdots + 412\,316\,860\,416.$$

$$\begin{aligned}\sum_{i=1}^{38} 3 \cdot 2^{i-5} &= \sum_{i=1}^{38} 3 \cdot 2^{-4} \cdot 2^{i-1} \\ &= \frac{3}{16} \sum_{i=1}^{38} 2^{i-1} \\ &= \frac{3}{16} \frac{2^{38}-1}{2-1} = \frac{824\,633\,720\,829}{16}\end{aligned}$$

Evaluating $\int_1^2 x^2$

The partition $P_4 : 1 = 4/4, 5/4, 6/4, 7/4, 2 = 8/4$ divides $[1, 2]$ into 4 parts of length $1/4$.

$$U(P_4; x^2) = \left(\frac{5}{4}\right)^2 \frac{1}{4} + \left(\frac{6}{4}\right)^2 \frac{1}{4} + \left(\frac{7}{4}\right)^2 \frac{1}{4} + \left(\frac{8}{4}\right)^2 \frac{1}{4} = \frac{87}{32}$$

$$\begin{aligned}
L(P_4; x^2) &= \left(\frac{4}{4}\right)^2 \frac{1}{4} + \left(\frac{5}{4}\right)^2 \frac{1}{4} + \left(\frac{6}{4}\right)^2 \frac{1}{4} + \left(\frac{7}{4}\right)^2 \frac{1}{4} \\
&= \sum_{i=4}^{7} \frac{i^2}{4^2} \cdot \frac{1}{4} = \frac{1}{64} \left(\sum_{i=4}^{7} i^2\right) \\
&= \frac{1}{64} \left(\frac{7 \cdot (7+1)(2 \cdot 7 + 1)}{2} - \frac{3 \cdot (3+1)(2 \cdot 3 + 1)}{2}\right) = \frac{63}{32}
\end{aligned}$$

Evaluating $\int_1^2 x^2$

The partition $P_n : 1 = n/n, (n+1)/n, (n+2)/n, \ldots, 2 = (2n)/n$ divides $[1,2]$ into n parts of length $1/n$.

The ith part is $\left[\dfrac{n+i-1}{n}, \dfrac{n+i}{n}\right]$.

$$U(P_n; x^2) = \sum_{i=1}^{n} \left(\frac{n+i}{n}\right)^2 \frac{1}{n} = \sum_{i=1}^{n} (n^2 + 2ni + i^2)\frac{1}{n^3}$$

$$= \frac{n^2}{n^3}\left(\sum_{i=1}^{n} 1\right) + \frac{2n}{n^3}\left(\sum_{i=1}^{n} i\right) + \frac{1}{n^3}\left(\sum_{i=1}^{n} i^2\right)$$

$$= \frac{n^2}{n^3}(n) + \frac{2n}{n^3}\left(\frac{n(n+1)}{2}\right) + \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= 1 + \left(1 + \frac{1}{n}\right) + \left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) \to 1 + 1 + \frac{1}{3} = \frac{7}{3}$$

Evaluating $\int_1^2 x^2$

The partition $P_n : 1 = n/n, (n+1)/n, (n+2)/n, \ldots, 2 = (2n)/n$ divides $[1,2]$ into n parts of length $1/n$.
The ith part is $\left[\frac{n+i-1}{n}, \frac{n+i}{n}\right]$.

$$L(P_n; x^2) = \sum_{i=n}^{2n-1} \left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3}\left(\sum_{i=n}^{2n-1} i^2\right)$$

$$= \frac{1}{n^3}\left(\frac{(2n-1)((2n-1)+1)(2(2n-1)+1)}{6} - \frac{(n-1)((n-1)+1)(2(n-1)+1)}{6}\right)$$

$$= \frac{1}{n^3}\left(\left(\frac{8}{3}n^3 - 2n^2 + \frac{1}{3}n\right) - \left(\frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n\right)\right) \to \frac{7}{3}$$

Signed area

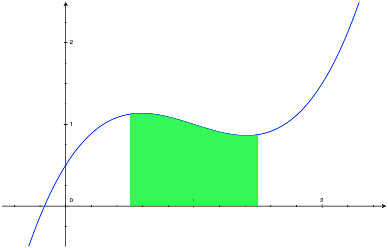
The definite integral $\int_a^b f(x)\,dx$ is the **signed area** between the graph $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$.

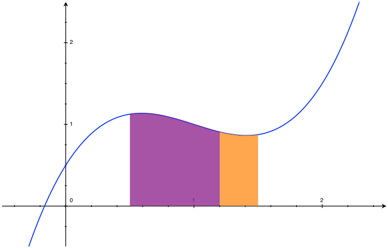
- **When** $a < b$**:** areas above the x-axis count as positive and areas below the x-axis count as negative.
- **When** $a > b$**:** areas above the x-axis count as negative and areas below the x-axis count as positive.

Properties of definite integrals

- **Reversal rule**:
  $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

- **Splitting rule**:
  $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

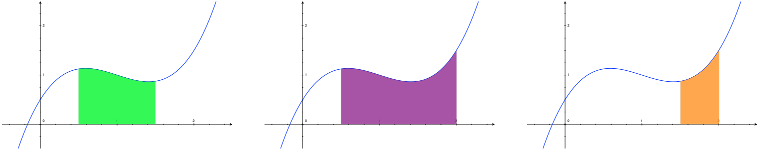




Properties of definite integrals

- **Reversal rule**:
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

- **Splitting rule**:
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



Fundamental Theorem of Calculus

Theorem (Part I)

Suppose f is continuous on an open interval I containing a and that F is defined by

$$F(x) = \int_a^x f(t)\,dt.$$

Then, F is an antiderivative of f on I.

Theorem (Part II)

Suppose that F is an antiderivative of f on the open interval I. Then,

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

for all a, b in I.

Using part I

Example

Suppose $H(x) = \int_0^{\ln(x)} e^{-t^2}\, dt$. What is $H'(x)$?

First, we know that $F(x) = \int_0^x e^{-t^2}\, dt$ is an antiderivative of e^{-x^2}. So $F'(x) = e^{-x^2}$.

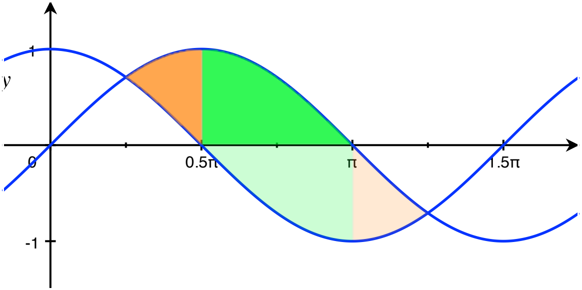
Now $H(x) = F(\ln(x))$. So, by the chain rule,

$$H'(x) = F'(\ln(x))\frac{1}{x} = e^{-(\ln(x))^2}\frac{1}{x}.$$

Using part II

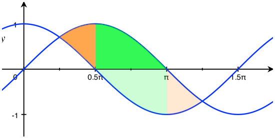
Example

Find the area contained below the graph $y = \cos x$ and above the graph $y = \sin x$ between $x = \pi/4$ and $x = 5\pi/4$.



$$\boxed{\text{Total Area}} = 2 \times \left(\boxed{\text{Orange Area}} + \boxed{\text{Green Area}} \right)$$

Using part II



$$\boxed{\text{Green Area}} = \int_{\pi/2}^{\pi} \sin(x)\,dx = -\cos(\pi) + \cos(\pi/2) = 1$$

$$\begin{aligned}
\boxed{\text{Orange Area}} &= \int_{pi/4}^{\pi/2} \sin(x)\,dx - \int_{\pi/4}^{\pi/2} \cos(x)\,dx \\
&= (-\cos(\pi/2) + \cos(\pi/4)) - (\sin(\pi/2) - \sin(\pi/4)) \\
&= (0 + \sqrt{2}/2) - (1 - \sqrt{2}/2) = \sqrt{2} - 1
\end{aligned}$$

$$\boxed{\text{Total area}} = 2(1 + \sqrt{2} - 1) = 2\sqrt{2}$$

Substitution

If $F(u) = \int f(u)\,du$ then $F(g(x)) = \int f(g(x))g'(x)\,dx$.

Because, the chain rule says that if $F'(u) = f(u)$ then

$$\frac{d}{dx}\left[F(g(x))\right] = F'(g(x))g'(x) = f(g(x))g'(x).$$

Therefore, $F(g(x)) + C = \displaystyle\int f(g(x))g'(x)\,dx.$

Substitution: method

$$\int \frac{\cos(x)}{1+\sin(x)}dx = \int \frac{1}{u}du = \ln|u| + C = \ln|1+\sin(x)| + C$$

- Find a composition with inside function $g(x)$...

  $$g(x) = 1 + \sin(x)$$

- Let $u = g(x)$ and compute the **differential** $du = g'(x)\,dx$...

  $$u = 1 + \sin(x),\ du = \cos(x)\,dx$$

- Locate du, then replace <u>all</u> remaining x's using u's...

  $$\frac{\cos(x)}{1+\sin(x)}dx \to \frac{1}{1+\sin(x)}du \to \frac{1}{u}du$$

- Integrate with respect to u, then replace <u>all</u> the u's by $g(x)$...

Substitution: examples

1. $\int e^{7x}\, dx$
2. $\int \cos(x/2)\, dx$
3. $\int e^{x^2} x\, dx$
4. $\int x\sqrt{1-x^2}\, dx$
5. $\int \frac{\ln x}{x} dx$
6. $\int \frac{x}{1+x^2}\, dx$
7. $\int \cos^3(2x)\sin(2x)\, dx$
8. $\int \frac{e^x}{1+e^x}\, dx$
9. $\int \frac{\sec(3x)\tan(3x)}{\cos(3x)} dx$

1. $u = 7x$, $du = 7\, dx$
2. $u = x/2$, $du = \frac{1}{2}\, dx$
3. $u = x^2$, $du = 2x\, dx$
4. $u = 1 - x^2$, $du = -2x\, dx$
5. $u = \ln x$, $du = \frac{1}{x} dx$
6. $u = 1 + x^2$, $du = 2x\, dx$
7. $u = \cos(2x)$,
   $du = -2\sin(2x)\, dx$
8. $u = 1 + e^x$, $du = e^x\, dx$
9. $u = \tan(3x)$,
   $du = 3\sec^2(3x)\, dx$

Trapezoid rule

The n-**point trapezoid approximation** to $\int_a^b f(x)\,dx$ is obtained by:

- Dividing $[a, b]$ into n equal parts of length $h = (b - a)/n$.
- Linking pairs of data points $(x_i, y_i) = (a + ih, f(a + ih))$ with straight line segments.
- Adding the areas of the resulting trapezoids.

Trapezoid rule: formula

$$
\begin{aligned}
T_n &= h\frac{y_0 + y_1}{2} + h\frac{y_1 + y_2}{2} + h\frac{y_2 + y_3}{2} + \cdots + h\frac{y_{n-1} + y_n}{2} \\
&= h\left(\frac{y_0}{2} + y_1 + y_2 + y_3 + \cdots + y_{n-1} + \frac{y_n}{2}\right)
\end{aligned}
$$

Example

For $\int_1^3 \cos(x^2)\,dx$, we have

x_i	1.0	1.4	1.8	2.2	2.6	3.0
y_i	0.54	-0.38	-1.00	0.13	0.99	-0.91

$$T_5 = 0.4(0.27 - 0.38 - 1.00 + 0.13 + 0.99 - 0.46) = -0.22$$

The actual value is about -0.20.

Trapezoid rule: error

Theorem

Suppose f is continuous on $[a, b]$ and twice differentiable on (a, b). If $|f''(x)| \leq M_2$ for every x in (a, b) then

$$\left| T_n - \int_a^b f(x)\,dx \right| \leq M_2 \frac{(b-a)^3}{12n^2}.$$

Example

For $\int_1^3 \cos(x^2)\,dx$ we have

$$\left| \frac{d^2}{dx^2}[\cos(x^2)] \right| = \left| 2\sin(x^2) + 4x^2\cos(x^2) \right| \leq 2 + 4x^2 \leq 38.$$

So $\displaystyle \left| T_5 - \int_1^3 \cos(x^2)\,dx \right| \leq 38 \frac{(3-1)^3}{12 \times 5^2} \leq 1.02.$

Simpson's rule

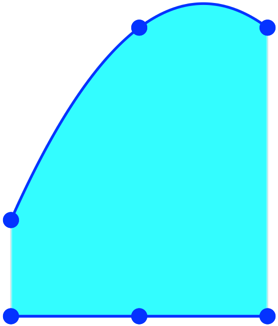
The $2n$-**point Simpson approximation** to $\displaystyle\int_a^b f(x)\,dx$ is obtained by:

- Dividing $[a, b]$ into $2n$ equal parts of length $h = (b - a)/2n$,
- Linking $\underline{\text{triplets}}$ of data points $(x_i, y_i) = (a + ih, f(a + ih))$ with parabolic segments.
- Adding the areas of the resulting pieces.

Simpson's rule: pieces

The unique parabola that goes through the three evenly spaced points $(x - h, A), (x, B), (x + h, C)$ has area

$$2h\frac{A + 4B + C}{6}.$$



Simpson's rule: formula

$$S_{2n} = 2h\frac{y_0+4y_1+y_2}{6} + 2h\frac{y_2+4y_3+y_4}{6} + \cdots + 2h\frac{y_{2n-2}+4y_{2n-1}+y_{2n}}{6}$$
$$= \frac{2}{3}h\left(\frac{y_0}{2} + 2y_1 + y_2 + 2y_3 + \cdots + y_{2n-2} + 2y_{2n-1} + \frac{y_{2n}}{2}\right)$$

Example

For $\int_1^3 \cos(x^2)\,dx$, we have

x_i	1.0	1.2	1.4	1.6	1.8	2.0
y_i	0.54	0.13	−0.38	−0.83	−1.00	−0.65

x_i	$\cdots$	2.2	2.4	2.6	2.8	3.0
y_i	$\cdots$	0.13	0.87	0.89	0.02	−0.91

$$S_{10} = \tfrac{2}{3}0.2\begin{pmatrix} 0.27 + 0.26 - 0.38 - 1.66 - 1.00 - 1.30 \\ +0.13 + 1.94 + 0.89 + 0.04 - 0.46 \end{pmatrix} = -0.20$$

Simpson's rule: error

Theorem

Suppose f is continuous on $[a,b]$ and four times differentiable on (a,b). If $|f''''(x)| \leq M_4$ for every x in (a,b) then

$$\left| S_n - \int_a^b f(x)\,dx \right| \leq M_4 \frac{(b-a)^5}{180n^4}.$$

Example

For $\int_1^3 \cos(x^2)\,dx$ we have

$$\left| \frac{d^4}{dx^4}[\cos(x^2)] \right| = \left| 48x^2 \sin(x^2) + (16x^4 - 12)\cos(x^2) \right| \leq 1000.$$

So $\left| S_{10} - \displaystyle\int_1^3 \cos(x^2)\,dx \right| \leq 1000 \dfrac{(3-1)^5}{180 \times 10^4} \leq 0.02.$