Quantum ergodicity and billiards

Courant Instructor Day—September 10, 2004

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Research areas

I. Eigenfunctions of the Laplacian ('drum problem')

- *waves:* elliptic PDE, eigenproblem
- properties of eigenfunctions $\longrightarrow$ 'quantum chaos'
- fast numerical methods to handle short-wavelength limit

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- *diffusion*: parabolic PDE, time-dependent
- fast forward models for messy 3D geometry
- ill-posed inverse problem $\longrightarrow$ Bayesian statistics

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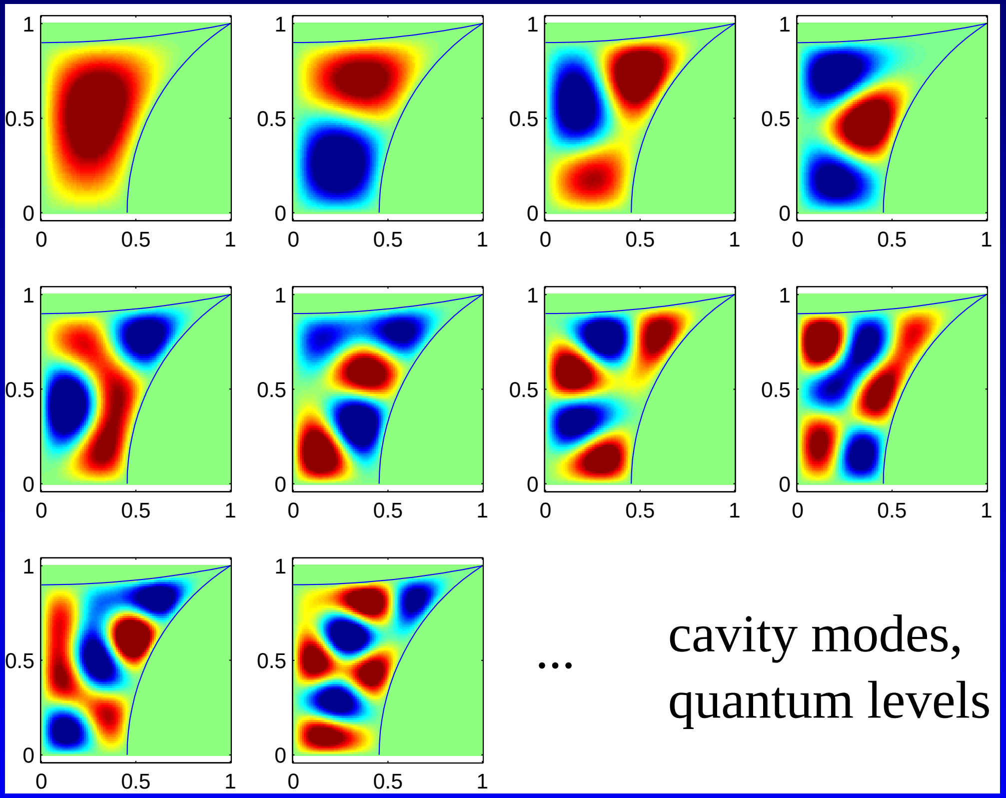
Simple linear 2^{nd}-order PDEs

…novel: behaviour, numerical methods, applications

Dirichlet eigenproblem

normal modes of elastic membrane (drum):
eigenfunctions ϕ_j of Laplacian Δ in domain $\Omega \subset \mathbb{R}^2$

$$\begin{aligned} -\Delta\phi_j &= E_j\phi_j \qquad \text{inside } \Omega \\ \phi_j &= 0 \qquad\quad \text{on } \partial\Omega \end{aligned}$$



'energy' eigenvalue E_j

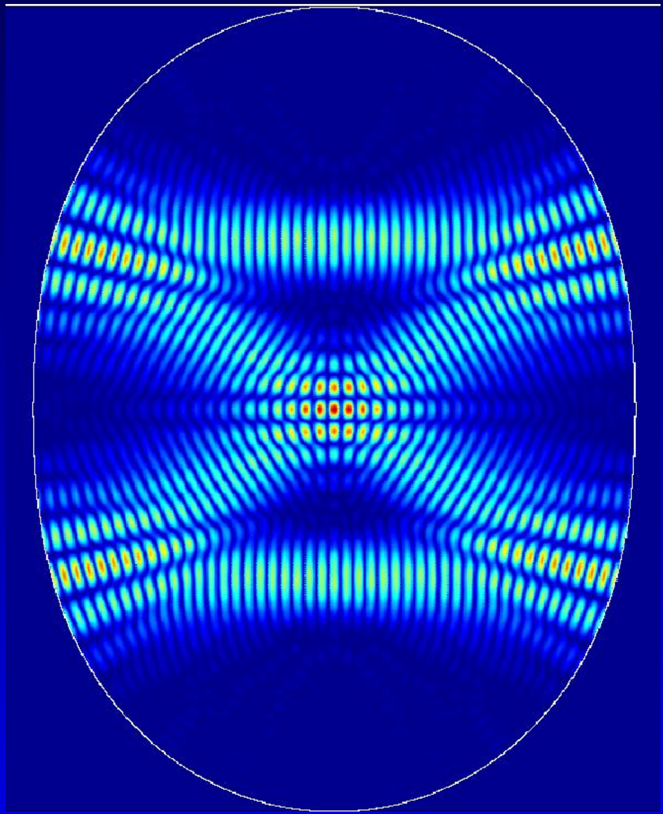
'level' $j = 1 \cdots \infty$

$\langle \phi_i, \phi_j \rangle_\Omega = \delta_{ij}$

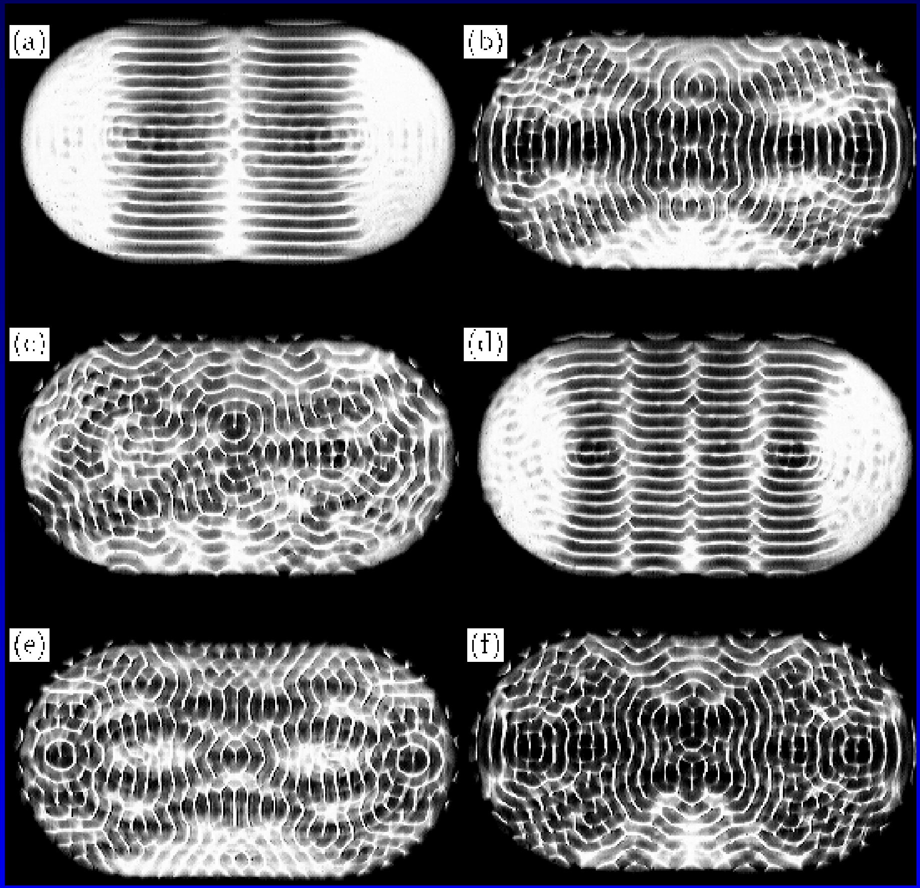
MOVIE $j \approx 3000$

Physical applications

- electromagnetic modes in waveguide ($\partial\Omega$ = metallic wall)
- acoustic resonances (Neumann BCs)
- quantum particles, *e.g.* cold electrons in ‘quantum dots’



dielectric laser resonators Tureci



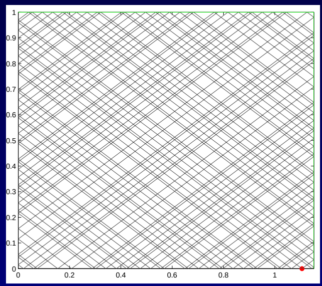
liquid surfaces Kudrolli

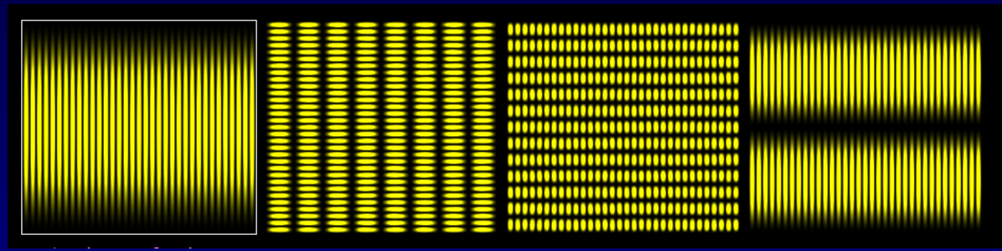
Connection to dynamics

classical ‘billiard’ system: point particle bouncing inside Ω

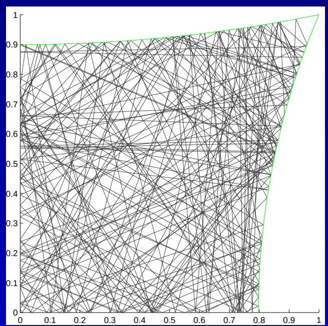
classical | quantum (PDE)

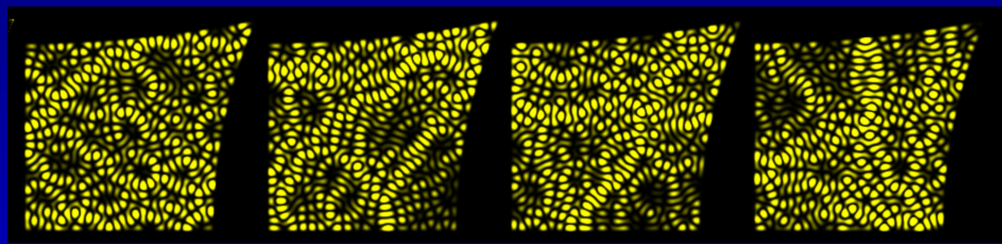
Integrable:





Ergodic:





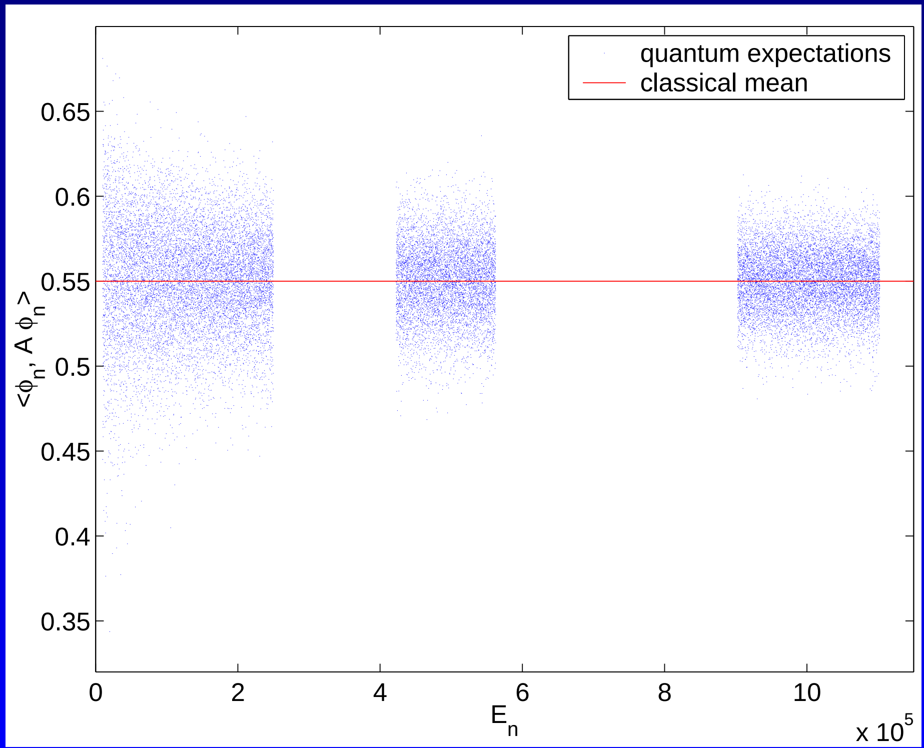
Quantum chaos: classical effects on ϕ_j, semiclassical trace formula (periodic orbits), random matrix theory…

- What happens in semiclassical ($E_j \to \infty$) limit? (Einstein 1917!)

Quantum ergodicity

- Quantum Ergodicity Theorem (80's): proven *almost all* ϕ_j become uniformly distributed as $E \to \infty$
- Quantum Unique Ergodicity (Sarnak '94): conjectured true for *every single* level j (in 'arithmetic manifolds'; proven Lindenstrauss '03)

Numerical test via matrix elements $A_{ij} := \langle \phi_i, \hat{A}\phi_j \rangle_\Omega$
Strong evidence for QUE in ergodic billiard at unprecedented E:



(B '04)

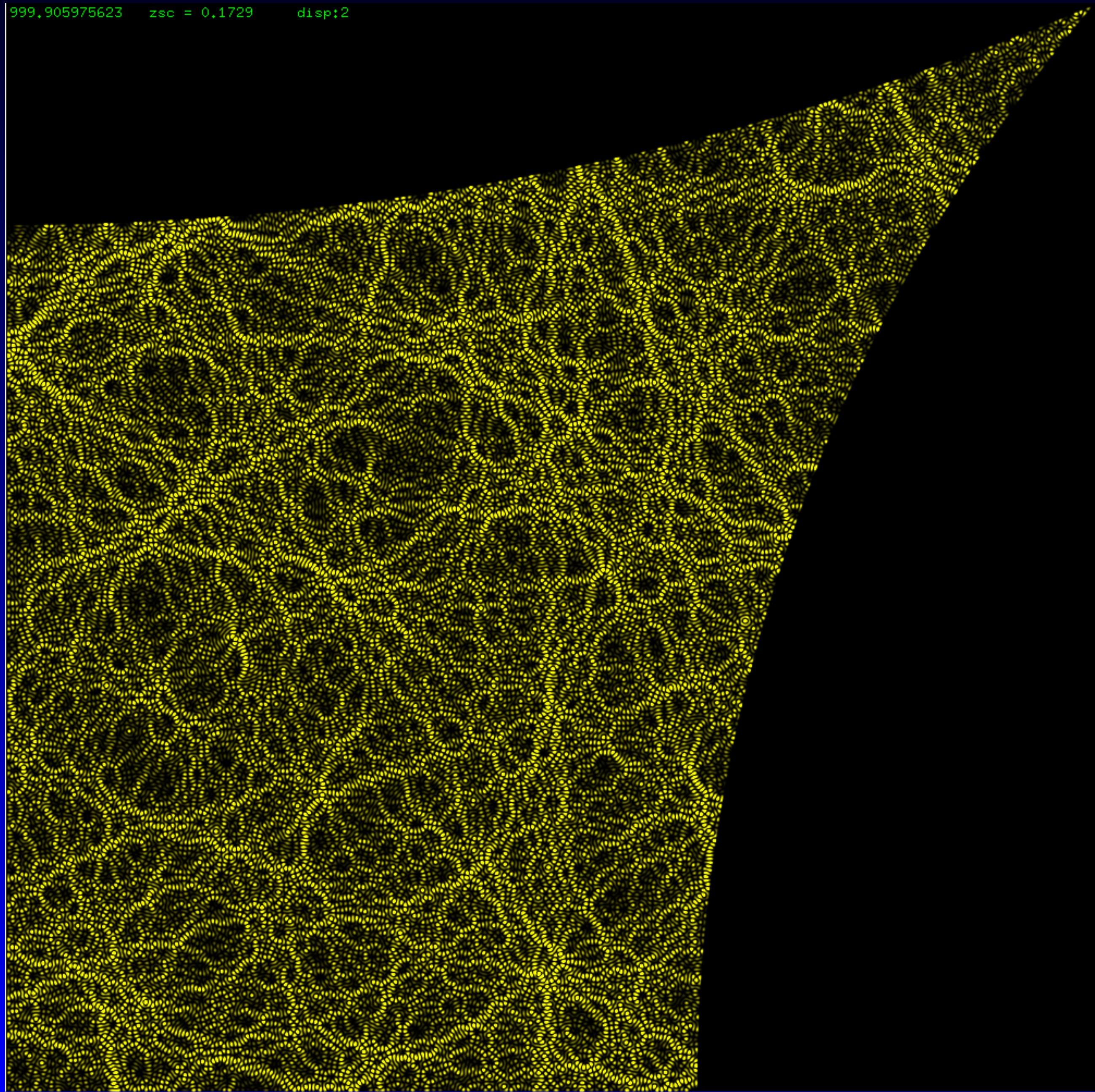
Can predict slow $E^{-1/2}$ decay of variance semiclassically!

High-energy eigenfunction ϕ_j

$E \approx 10^6$
$j \approx 5 \times 10^4$

collect 3×10^4 of these...

$\longrightarrow$ statistical properties



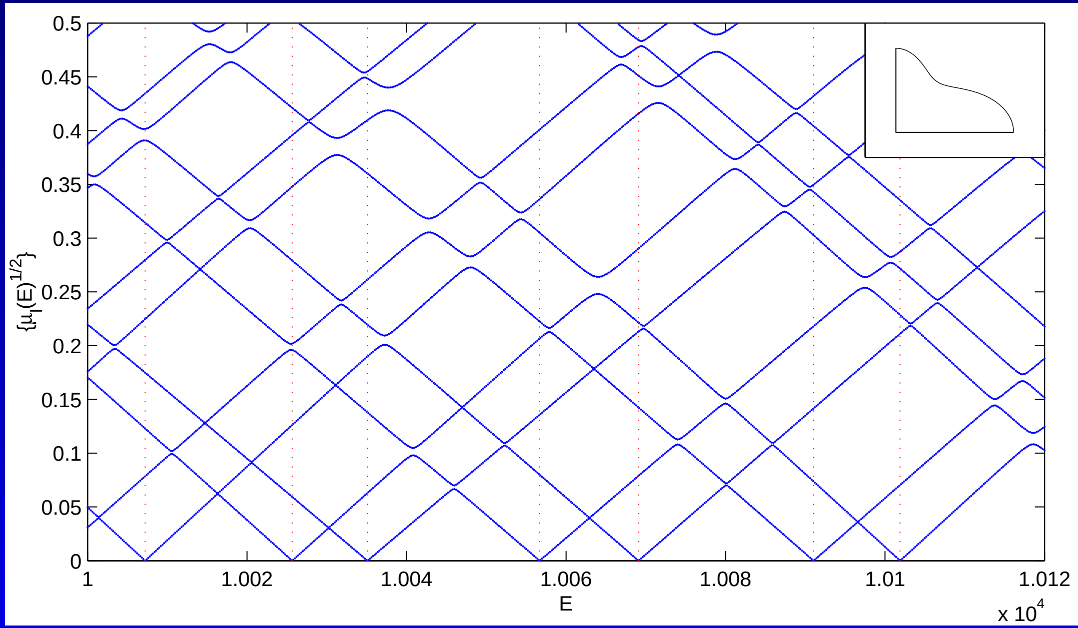
Numerical methods for eigenproblem

Finite Elements, Finite Differences will not help you—need boundary method!

'Standard' boundary method: (similar to integral equation)

- linear space $\mathcal{H}_\Omega(E) = \{u : -\Delta u = Eu \text{ in } \Omega\}$ obeys PDE not BCs
- Guess an E, compute $\lambda_1(E) = \min_{u \in \mathcal{H}_\Omega(E), u \neq 0} ||u||^2_{\partial\Omega} / ||u||^2_\Omega$

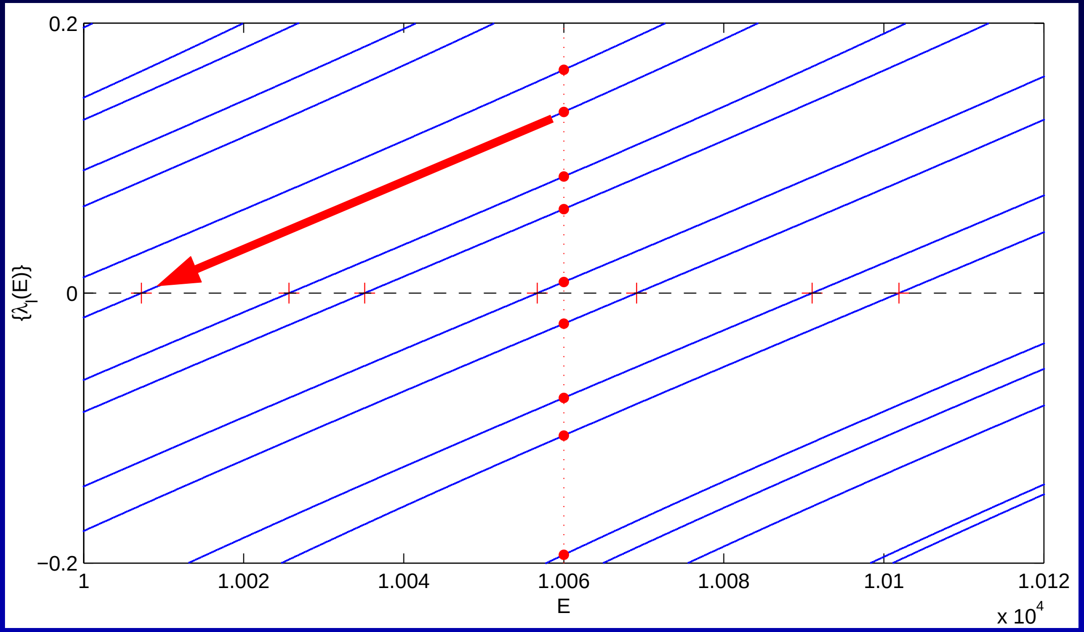
using generalized eigenproblem in numerical basis for $\mathcal{H}_\Omega(E)$



- Must search E-axis: where $\lambda_1(E) \to 0$ gives levels $E = E_j$

Acceleration: Vergini's scaling method

Special boundary operator $K_E : \partial\Omega \to \partial\Omega$, its spectral problem at *single* E approximates all nearby levels E_j and ϕ_j (no search!)



- relies on quasi-orthogonality on $\partial\Omega$ of normal derivatives $\partial_n \phi_j$
- factor $O(E^{1/2}) \sim 10^3$ faster than any other known method
- questions: error analysis, basis sets, Ω shapes, corners…

Preprints/software: `http://www.cims.nyu.edu/~barnett`

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