Centralizers

Zajj Daugherty

April 8, 2008

The big idea: If A is a semisimple algebra acting on a vector space V, and B is the full centralizer of A in $\mathrm{End}(V)$, then A is also the full centralizer of B. This duality can tell us a lot about the representation theory about both algebras.

1 Duality

Let V be a finite dimensional vector space and $S \subset \mathrm{End}(V)$. Define the *centralizer* of S by

$$\mathcal{C}(S) = \{ \ x \in \mathrm{End}(V) \mid xs = sx \text{ for all } s \in S\}.$$

Note that $\mathcal{C}(S)$ is an associative algebra with unit id_V.

Now let $A \subset \mathrm{End}(V)$ be a semisimple algebra containing id_V, and set $B = \mathcal{C}(A)$. Then the vector space $A \otimes B$ is an associative algebra with multiplication

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2 \otimes b_1 b_2),$$

and $A \simeq A \otimes 1$ and $B \simeq 1 \otimes B$.

By Wedderburn's theorem, there exist finite dimensional vector spaces V^λ, with λ running over some finite index set, such that

$$A \xrightarrow{\sim} \bigoplus_\lambda \mathrm{End}(V^\lambda).$$

Let π^λ be the projection of A onto $\mathrm{End}(V^\lambda)$. Then π^λ is an irrep of A, is distinct from each other projection $\pi^{\lambda'}$, and every irrep of A is equivalent to some π^λ [GW, Prop 3.3.1].

Lemma 1. *[GW, Prop 3.3.2] Suppose (ρ, W) is a finite-dimensional representation of A. Set $U^\lambda = \mathrm{Hom}_A(V^\lambda, W)$ and define a linear map*

$$\varphi : \bigoplus_\lambda U^\lambda \otimes V^\lambda \to W, \qquad u_\lambda \otimes v_\lambda \mapsto u_\lambda(v_\lambda).$$

Then φ is an A-module homomorphism and $\varphi^{-1}\rho(x)\varphi = \bigoplus_\lambda \mathrm{id}_{U^\lambda} \otimes \pi^\lambda(x)$.

In other words, there is an A-module isomorphism

$$V \simeq \bigoplus_{i=1}^{r} V_i \otimes U_i,$$

where V_i is an irreducible A-module, $V_i \not\simeq V_j$ for $i \neq j$, and $U_i = \mathrm{Hom}_A(V_i, V)$. Under this isomorphism,

$$A \simeq \bigoplus_{i=1}^{r} \mathrm{End}(V_i) \otimes \mathrm{id}_{U_i}.$$

Theorem 2 (The Big Centralizer Theorem)**.** *[GW, Thm 3.3.7] Let V be a finite-dimensional vector space and $A \subset \mathrm{End}(V)$ a semisimple algebra. Then the algebra $B = \mathcal{C}(A)$ is semisimple, one has $\mathcal{C}(B) = A$, and*

$$B \simeq \bigoplus_{i=1}^{r} \mathrm{id}_{V_i} \otimes \mathrm{End}(U_i).$$

Hence the subspace $V_i \otimes U_i$ are irreducible and distinct representations of $A \otimes B$.

This theorem does two things. First, this simultaneously decomposes V as an A module and as a B module. As an A module, V_i occurs with multiplicity $\dim(U)i$, and vice versa. This duality also a correspondence between irreducible representations of A and irreducible representation of B.

1.1 Example: matrices meet permutations

Let ρ be the defining representation of $G = GL(V)$, and define $\rho_k = \rho^{\otimes k}$ on $V^{\otimes k}$:

$$g \cdot (v_1 \otimes \cdots \otimes v_k) = \rho_k(g)(v_1 \otimes \cdots \otimes v_k) = gv_1 \otimes \cdots \otimes gv_k, \quad g \in G.$$

Notice that we can permute the positions of the vectors in the tensor product without changing the action of G. So the action of the symmetric group by place permutations, given by

$$s \cdot (v_1 \otimes \cdots \otimes v_k) = \sigma_k(s)(v_1 \otimes \cdots \otimes v_k) = v_{s^{-1}(1)} \otimes \cdots \otimes v_{s^{-1}(k)}, \quad s \in S_n$$

(s moves the i^{th} vector to the $s(i)^{\text{th}}$ position), commutes with the action of G. And indeed, this is the full centralizer of the G-action:

Theorem 3. *Set $A = \rho_k(\mathbb{C}GL(V))$ and $B = \sigma_k(\mathbb{C}S_k)$. Then $A = \mathcal{C}(B)$ and $B = \mathcal{C}(A)$.*

2 Induction and Restriction

Let $B \subset A$ be algebras, and M an A-module. Let $\mathcal{A}$ and $\mathcal{B}$ be the full centralizers of A and B (respectively) in $\mathrm{End}(M)$. So $\mathcal{B} \supset \mathcal{A}$, M decomposes into irreducible $A \otimes \mathcal{A}$ modules as

$$M = \bigoplus_{\lambda} A^{\lambda} \otimes \mathcal{A}^{\lambda}$$

(each with multaplicity 1) and into irreducible $B \otimes \mathcal{B}$ modules as

$$M = \bigoplus_{\mu} B^{\mu} \otimes \mathcal{B}^{\mu}.$$

Futhermore

$$\mathrm{Res}^A_B(A^{\lambda}) = \bigoplus_{\mu} c^{\mu}_{\lambda} B^{\mu} \quad \text{and} \quad \mathrm{Res}^{\mathcal{B}}_{\mathcal{A}}(\mathcal{B}^{\mu}) = \bigoplus_{\mu} c^{\lambda}_{\mu} \mathcal{A}^{\lambda} \quad \text{imply} \quad c^{\mu}_{\lambda} = c^{\lambda}_{\mu}.$$

Example. Let

$$U = U\mathfrak{sl}_n, \quad M = L(2\varepsilon_1), \quad N = L(\varepsilon_1+\varepsilon_2), \quad \text{and} \quad V = L(\varepsilon_1).$$

Let $\mathcal{B}$ be the centralizer of U in $\mathrm{End}(M\otimes N\otimes V)$ and $\mathcal{A}$ be the centralizer of U in $\mathrm{End}(N\otimes V)$. Since M is an irreducible U module, the centralizer of $U\otimes U$ in $\mathrm{End}(M\otimes(N\otimes V))$ is then $\mathbb{C}\otimes\mathcal{A}$, whose irreducible representations take the form $\mathbb{1}\otimes\mathcal{A}^\mu\cong\mathcal{A}^\mu$. Note $U\overset{\Delta}{\hookrightarrow}U\otimes U$. So

$$\mathrm{Res}_U^{U\otimes U}(M\otimes L(\mu)) = \bigoplus_\lambda c_{M,\mu}^\lambda L(\lambda)$$

$$\mathrm{Res}_{\mathbb{C}\otimes\mathcal{A}}^{\mathcal{B}}(\mathcal{B}^\lambda) = \bigoplus_\mu c_{M,\mu}^\lambda \mathcal{A}^\mu.$$

So the coefficient on $\mathcal{A}^\mu$ in $\mathrm{Res}_{\mathbb{C}\otimes\mathcal{A}}^{\mathcal{B}}(\mathcal{B}^\lambda) = \bigoplus_\mu c_{M,\mu}^\lambda\mathcal{A}^\mu$ is the multiplicity of λ in the tensor of $L(\varepsilon_1+\varepsilon_2)$

References

[GW] R. Goodman, N. R. Wallach, *Representations and invariants of the classical groups*, Cambridge University Press, 1998.