Second Homework Assignment
Math 73/103
Due Wednesday, October 19$^{\text{th}}$

1. Page 32 of the text, problem #6. (Note that we have already shown that $\mathfrak{M}$ is a σ-algebra so there is no need to show it again.)

ANS: We already know $\mathfrak{M}$ is a σ-algebra. Let $\{E_i\}$ be a countable pairwise disjoint family of measurable subsets with $E := \bigcup_i E_i$. If all the E_i are countable then so is E. Thus we clearly have

$$\mu(E) = \sum_i \mu(E_i).$$

On the other hand, if one the E_i — say E_k is uncountable, then E_k^c is countable and contains all the other E_i with $i \neq k$. Thus E is uncountable, $\mu(E) = 1$ and

$$\sum_i \mu(E_i) = \mu(E_k) = 1.$$

Thus μ is a measure.

The key to the rest of the problem is to realize that $f : (X, \mathfrak{M}) \to \mathbf{C}$ is measurable if and only if f is constant μ-almost everywhere; that is, f is measurable if and only if there is a $c \in \mathbf{C}$ such that $f^{-1}(X \setminus \{c\})$ is countable. Of course, if this assertion is correct, then f is equal to the constant function $g(x) = c$ almost everywhere and

$$\int_X f(x)\, d\mu(x) = \int_X c\, d\mu(x) = c\mu(X) = c.$$

It is fairly clear that if f is constant almost everywhere, then f is measurable. So, assume that f is measurable. Then for any open set V, either $f^{-1}(V)$ is uncountable or $f^{-1}(V)^c = f^{-1}(V^c)$ is uncountable. Let $\{V_n\}$ be a *countable* basis for the topology of $\mathbf{C}$. In view of the above, let

$$B_n := \begin{cases} V_n & \text{if } f^{-1}(V_n) \text{ is uncountable, and} \\ V_n^c & \text{if } f^{-1}(V_n^c) \text{ is uncountable.} \end{cases}$$

Let $A = \bigcup B_n$. I claim that A can consist of at most one point. If $x \neq y$, then there is a n such that $x \in V_n$ and $y \in V_n^c$. Thus at most one of x and y belong to B_n. Thus at most one of x and y can belong to A. Now it will suffice to see that $f^{-1}(A)$ is uncountable. (This implies its complement is countable.) For this, it suffices to see that $\mu(f^{-1}(A)) = 1$.

But since $C \cup D$ is the disjoint union of $C \setminus D$, $C \cap D$ and $D \setminus C$, it follows that if both $f^{-1}(C)$ and $f^{-1}(D)$ are uncountable, then so is $f^{-1}(C \cap D)$. But

$$A = \bigcap_n B_n = \bigcap_n F_n \quad \text{where, } F_n = B_1 \cap \cdots \cap B_n.$$

Then $f^{-1}(F_n)$ is uncountable and

$$\mu(f^{-1}(A)) = \lim_n \mu(f^{-1}(F_n)) = 1.$$

This completes the proof.

2. Page 32 of the text, problem #7.

ANS: Note that $f_n(x) = |f_n(x)| \le f_1(x)$ for all $x \in X$, and $f_1 \in \mathcal{L}^1(\mu)$. Therefore, the conclusion follows from the LDCT. For a counterexample, take $f_k := \mathbb{I}_{[k,\infty)}$. Then $f_k \searrow 0$, but all the f_k have infinite integrals.

3. Page 32 of the text, problem #10.

ANS: Since constant functions are summable if $\mu(X) < \infty$, use the LDCT on the sequence $g_n = |f_n - f|$ together with the observation that f_n and $f_n - f$ integrable implies f is too.[1]

For a counterexample when $\mu(X) = \infty$, consider Lebesgue measure on $\mathbf{R}$ and set $f_n = \frac{1}{n}\mathbb{I}_{[-n,n]}$. Then $f_n \to 0$ uniformly, while $\int_{\mathbf{R}} f_n \, du = 2$ for all n.

4. Page 32 of the text, problem #12. (This is easy if f is bounded.)

ANS: First notice that the conclusion is obvious if f is bounded[2]. In general, let $f_n = \max\{|f|, n\}$. Since $f_n \nearrow |f|$, the MCT implies that $\int_X f_n \, d\mu \nearrow \int_X |f| \, d\mu$. In particular, we can choose N such that

$$\left| \int_X f_N \, d\mu - \int_X |f| \, d\mu \right| < \frac{\epsilon}{2}.$$

Now since f_N is bounded, choose $\delta > 0$ so that $\mu(E) < \delta$ implies that $\int_E f_N \, d\mu < \epsilon/2$. The point being that

$$\int_E |f| \, d\mu \le \left| \int_E f_N \, d\mu \right| + \left| \int_X (|f| - f_N) \, d\mu \right| < \epsilon.$$

(We've used $|f| \ge f_N$ for the second to last inequality.)

[1] Alternatively, you can show that the $\{ f_n \}$ are *uniformly bounded*; that is, there exists M such that $\|f_n\|_\infty \le M$ for all n. However, you must prove this. By assumption, we only know that for each n, $M_n := \|f_n\|_\infty < \infty$. But by assumption, there is a N such that $n \ge N$ implies $\|f_n - f\|_\infty < 1$. It follows that $\|f\|_\infty \le \|f_N\|_\infty + 1$ and *for all* n

$$\|f_n\|_\infty \le M := \max\{ \|f\|_1, \dots, \|f_N\|_\infty, \|f_N\|_\infty + 2 \}.$$

Now we can apply the LDCT with $g \equiv M$.

[2] This technique is used quite often—reduce the problem to a simplier situation (e.g., a characteristic function, simple function, or, as here, a bounded function.

5. Suppose that Y is a topological space and that $\mathfrak{M}$ is a σ-algebra in Y containing all the Borel sets. Suppose in addition, μ is a measure on $(Y, \mathfrak{M})$ such that for all $E \in \mathfrak{M}$ we have

$$\mu(E) = \inf\{\, \mu(V) : V \text{ is open and } E \subset V \,\}. \tag{1}$$

Suppose also that

$$Y = \bigcup_{n=1}^{\infty} Y_n \quad \text{with } \mu(Y_n) < \infty \text{ for all } n \geq 1. \tag{2}$$

In \$25 words, μ is a *σ-finite outer regular measure* on $(Y, \mathfrak{M})$.

(a) Show that Lebesgue measure m is a σ-finite outer regular measure on $(\mathbf{R}, \mathfrak{M})$.

ANS: Since (1) is obviously satisfied if $m(E) = \infty$, we can assume that $m(E) < \infty$. If $\epsilon > 0$, then by definition of m (as the restriction of m^*), there are open intervals $\{I_n\})$ such that

$$E \subset \bigcup_n I_n \quad \text{and} \quad m(E) + \epsilon > \sum_n \ell(I_n).$$

But $V := \bigcup_n I_n$ is an open set containing E and

$$m(E) + \epsilon > \sum_n \ell(I_n) = \sum_n m(I_n) \leq m(V).$$

This implies (1). Since $\mathbf{R} = \bigcup_n [-n, n]$, Lebesgue measure is also σ-finite. This proves (a).

(b) Suppose E is a μ-measurable subset of Y.

(i) Given $\epsilon > 0$, show that there is an open set $V \subset Y$ and a closed set $F \subset Y$ such that $F \subset E \subset V$ and $\mu(V \setminus F) < \epsilon$.

ANS: Suppose $\mu(E) < \infty$. Then in view of (1), there is an open set $V \supset E$ such that $\mu(V) - \mu(E) < \epsilon/2$. Since $\mu(E) < \infty$, $\mu(V \setminus E) < \epsilon/2$. Now in general, $X = \bigcup_n X_n$ with $\mu(X_n) < \infty$ for each n. Let $E_n = E \cap X_n$. Then there are open sets $V_n \subset E_n$ such that $\mu(V_n \setminus E_n) < \frac{\epsilon}{2^{n=1}}$. Let $V = \bigcup V_n$. Then V is open and contains E. Furthermore,

$$\mu(V \setminus E) = \mu\Big(\bigcup V_n \setminus \bigcup E_n\Big) \leq \mu\Big(\bigcup_n (V_n \setminus E_n)\Big) \leq \epsilon/2.$$

But the above reasoning shows that there is an open set W containing E^c such that $\mu(W \setminus E^c) < \epsilon/2$. Then $F = W^c$ is a closed subset of E, and $\mu(E \setminus F) < \epsilon/2$. Then, since $V \setminus F = V \setminus E \cup E \setminus F$, we have $\mu(V \setminus F) < \epsilon$ as required.

(ii) Show that there is a G_δ-set $G \subset Y$ and a F_σ-set $A \subset Y$ such that $A \subset E \subset G$ and $\mu(G \setminus A) = 0$.

ANS: Let V_n and F_n be as in part (i) such that $\mu(V_n \setminus F_n) < \frac{1}{n}$. Let $G = \bigcap V_n$ and $A = \bigcup F_n$. But for any n, $\mu(G \setminus A) \leq \mu(V_n \setminus F_n) < \frac{1}{n}$. Hence $\mu(G \setminus A) = 0$.

(c) Argue that $(\mathbf{R}, \mathfrak{M}, m)$ is the completion of the restriction of Lebesgue measure to the Borel sets in $\mathbf{R}$.

ANS: Let $\mathfrak{B} = \mathfrak{B}(\mathbf{R})$ be the Borel sets in $\mathbf{R}$. Let $(\mathbf{R}, \mathfrak{B}_0, m_0)$ be the completion. Since G_δ sets and F_σ sets are Borel, part (b)(ii) shows that $\mathfrak{M} \subset \mathfrak{B}_0$. But if $E \in \mathfrak{B}_0$, then $E = B \cup N$ where B is Borel and N is a subset of a Borel m-null set. Since Lebesgue measure is complete, $N \in \mathfrak{M}$, and hence, $E \in \mathfrak{M}$. Thus $\mathfrak{M} = \mathfrak{B}_0$ and it is clear that $m = m_0$.

6. Let m be Lebesgue measure on $\mathbf{R}$ and suppose that E is a set of finite measure. Given $\epsilon > 0$, show that there is a finite *disjoint* union F of open intervals such that $m(E \triangle F) < \epsilon$ where $E \triangle F := (E \setminus F) \cup (F \setminus E)$ is the symmetric difference. (This illustrates the first of Littlewood's three principles: "Every Lebesgue measurable set is nearly a disjoint union of open intervals".)

ANS: Since $m(E) < \infty$, we can find, in view of problem 5a, and open set $V \subset \mathbf{R}$ containing E such that $m(V \setminus E) < \epsilon/2$. But V is a countable *disjoint union* of intervals: $V = \bigcup_n I_n$. Since

$$\infty > m(V) = \sum_{n=1}^{\infty} m(I_n),$$

there is a N such that $\sum_{n>N} m(I_n) < \epsilon/2$. Let $F = \bigcup_{n=1}^{N} I_n$. (Then F is a disjoint union of intervals.) Also

$$m(E \Delta F) = m(E \setminus F) + m(F \setminus E) \leq m(V \setminus F) + m(V \setminus E) < \epsilon.$$

That's it folks.

7. Let $(X, \mathfrak{M}, \mu)$ be a measure space, and let $(X, \mathfrak{M}_0, \mu_0)$ be its completion.

(a) If $f : X \to \mathbf{C}$ is μ_0-measurable, show that there is a μ-measurable function $g : X \to \mathbf{C}$ such that $f = g$ a.e. $[\mu_0]$.

ANS: See the answer to part (b).

(b) In part (a), is there necessarily a μ-null set N such that $f(x) = g(x)$ for all $x \notin N$?

ANS: I asked these questions in a poor order. If g is μ-measurable, then $\{\, x \in X : f(x) \neq g(x) \,\}$ is only guaranteed to belong to $\mathfrak{M}_0$. But if it is a μ_0-null set then it is contained in a μ-null set $N \in \mathfrak{M}$. This proves the statement in part (b).

Now let's look at part (a). Clearly, it suffices to consider only functions $f : X \to [0, \infty)$. I claim it is enough to prove that the result is true for simple functions. In view of the remark above, this means that given μ_0-measurable simple function s, there is a μ-measurable simple function s' which agrees with s off a μ-null set in $\mathfrak{B}$. Recall that there are nonnegative μ_0-measurable simple functions $s_n \nearrow f$. If there nonnegative μ-measurable simple functions s'_n and a null set $N_n \in \mathfrak{B}$ so that $s'_n = s_n$ off N_n, then $s'_n \nearrow f$ except possibly on the null set $N = \bigcup N_n \in \mathfrak{B}$. Replacing the s'_n by $s''_n = \mathbb{I}_{X \setminus N} \cdot s'_n$, then the sequence $\{ s''_n \}$ converges everywhere to a function g which is necessarily μ-measurable. Of course, $g = f$ off N. This proves the claim.

However, to prove the result for a simple function, it surely suffices to prove it only for a characteristic function of a measurable set $D \in \mathfrak{M}_0$. By definition, D is μ_0-measurable if and only if there are sets $A, B \in \mathfrak{B}$ so that $A \subseteq D \subseteq B$ with $\mu(B \setminus A) = 0$. In particular, $\mathbb{I}_A = \mathbb{I}_D$ off of $N = B \setminus A$; this completes the proof.

(c) What does this result say about Lebesgue measurable functions and Borel functions on $\mathbf{R}$? (Compare with problem #14 on page 59 of the text.)

ANS: Since the Lebesgue measurable sets (with Lebesgue measure) result from the completion of Lebesgue measure on the Borel sets, we obtain, as a special case, the fact that a Lebesgue measurable function is equal to a Borel function almost everywhere.

8. Suppose that $(X, \mathfrak{M}, \mu)$ is a measure space. Recall that $E \in \mathfrak{M}$ is called σ-finite if E is the countable union of sets of finite measure. Let $f \in \mathcal{L}^1(\mu)$.

(a) Show that $\{ x \in X : f(x) \neq 0 \}$ is σ-finite.

ANS: Let $A\{ x \in X : f(x) \neq 0 \}$ and $A_n = \{ x \in X : |f(x)| \geq \frac{1}{n} \}$. Since

$$\infty > \|f\|_1 \geq \int_{A_n} |f(x)| \, d\mu(x) \geq \mu(A_n)/n,$$

we must have $\mu(A_n) < \infty$. But $A = \bigcup_n A_n$.

(b) Suppose that $f \geq 0$. Show that there are (measurable) simple functions φ_n such that $\varphi_n \nearrow f$ everywhere and there is a single σ-finite set outside of which the φ_n vanish.

ANS: Sigh. This was silly (and the result of my being sloppy in borrowing old homework assignments). There are simple functions $\varphi_n \nearrow f$ by general nonsense. Of course, the have to vanish where f does. Now apply part (a).

(c) Given $\epsilon > 0$ show that there is simple function such that

$$\int_X |f - \varphi| \, d\mu < \epsilon.$$

ANS: If $f \geq 0$ and $\{\varphi_n\}$ is the sequence from part (b), then $\|f - \varphi_n\|_1 \to 0$ by the LDCT. We get the general result by applying this to $\mathrm{Re}(f)^{\pm}$ and $\mathrm{Im}(f)^{\pm}$.

(d) If $(X, \mathfrak{M}, \mu) = (\mathbf{R}, \mathfrak{M}, m)$ is Lebesgue measure, show that we can take the simple function φ in part (c) to be a step function — that is, a finite linear combination of characteristic functions of *intervals*.

ANS: In view of part (c), we just have to show that we can approximate a simple function by a step function the L^1-norm. For this, it suffices to consider a single characteristic function $\mathbb{I}_E$ with $\mathbb{I}_E \in \mathcal{L}^1$.[3] But then $m(E) < \infty$ and given $\epsilon > 0$ there is a finite disjoint union $F = \bigcup_n I_n$ of intervals such that $m(E \Delta F) < \epsilon$. But $\mathbb{I}_F$ *is* a step function and

$$\|\mathbb{I}_E - \mathbb{I}_F\|_1 = m(E \Delta F) < \epsilon.$$

[3]If a simple function is in $\mathcal{L}^1$, then we can write it in canonical form $\sum_i \alpha_i \mathbb{I}_{E_i}$ with each $\alpha_i \neq 0$. Then we must have $m(E_i) < \infty$.