Section B: Answer ALL questions.

Problem B1: [15 pts]

For which values of the constant λ in the range $[0, \pi]$ will the various solutions of the ODE

$$y' + \cos\left(\lambda + \frac{1}{t}\right) y = 1$$

spread apart as $t \to \infty$.

Hint: do not try to solve the equation, instead look carefully at the integrating factor.

Solution:

This is a first order linear equation in standard form with $p(t) = \cos(\lambda + \frac{1}{t})$. The long term behaviour of the integrating factor $R(t) = e^{\int p(t)dt}$ determines whether solutions diverge ($R \to 0$) or converge ($R \to \infty$). Since we cannot compute R explicitly we must deduce the behaviour of the integral by studying the integrand $p(t)$.

When $\frac{\pi}{2} < \lambda \leq \pi$ we have $p(t) \to \cos\lambda < 0$. Therefore for large t, $p(t) < \frac{1}{2}\cos\lambda < 0$ which implies $\int p(t)dt \to -\infty$ and hence $R \to 0$. Thus solutions will diverge in this range.

When $0 \leq \lambda < \frac{\pi}{2}$ a similar argument shows $p(t) > \frac{1}{2}\cos\lambda > 0$ and so $\int p(t)dt \to \infty$. Thus $R \to \infty$ and the solutions converge.

When $\lambda = \frac{\pi}{2}$ the above arguments breakdown as we can only deduce that $p(t) < 0$. On its own this is only sufficient to deduce that $\int p(t)dt < M$ for some constant M. Thus we can see that while solutions cannot converge, they may remain with a fixed distance.

It is possible to go further an deduce that they actually spread apart when $\lambda = \frac{\pi}{2}$. Since $\cos(\pi + x) = -\sin x$ we can apply a Taylor series argument to see that $p(t) \approx -\frac{1}{t}$ for large t with error bounded by a multiple of $\frac{1}{t^2}$. Thus $\int p(t)dt \to -\infty$. The answer is thus $\lambda \in [\frac{\pi}{2}, \pi]$.

Important Note: arguments based upon the idea that $\int p(t)dt \to \int \cos\lambda dt$ don't really make sense without introducing of what it means for a function to converge to another function. When $\lambda = \frac{\pi}{2}$ this argument completely collapses as $|p(t) - \cos\lambda|$ is much larger than $|\cos\lambda| = 0$, i.e. the error from your approximation dominates.

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Problem B2: [10 pts] When everything is working properly, a particular pendulum oscillates according to the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{1}{10}\frac{d\theta}{dt} + \sin(\theta) = 0$$

where θ is the angle counter-clockwise from the stable equilibrium position. (see section 9.2)

The pivot point has unfortunately rusted and now gives extra resistance against counter-clockwise movement. The new ODE is

$$\frac{d^2\theta}{dt^2} + f\left(\frac{d\theta}{dt}\right) + \sin(\theta) = 0$$

where f is the function defined by

$$f(y) = \begin{cases} \frac{y}{10}(1+y^2), & y \geq 0 \\ \frac{y}{10}, & y < 0. \end{cases}$$

Supposing that the pendulum starts at rest, hanging vertically down. **Estimate** the minimum instantaneous angular velocity that would be needed to make the pendulum rotate once completely counter-clockwise before coming to rest again.

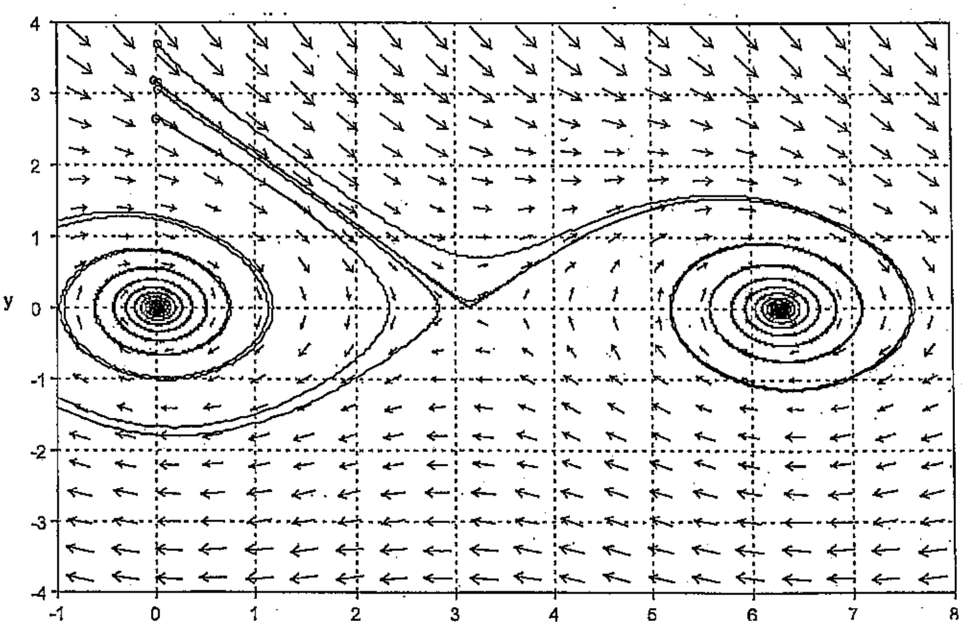
Hint: in the plotting software on the website $f(y)$ would be written $0.1y(1+(y>0)y^2)$.

Solution:

Transform the second order equation into a system of first order equations using the new variables $x = \theta$, $y = \theta'$.

$$x' = y$$
$$x' = -\sin(x) - f(y).$$

The system has critical points at $(0, 2n\pi)$ for integer n. Since positive y values correspond to counter-clockwise rotation, physically the parameter n indicates the number of counter-clockwise rotations by the pendulum before it comes to a rest. We are thus looking for the smallest v so that the trajectory starting at $(0, v)$ spirals into the critical point at $(0, 2\pi)$. Examining the phase-plane diagram produced using the software on the website, we see this occurs at approximately $v = 3.16$.



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Problem B3: [10 pts]

The 2×2 matrices A, B, C and D have real, constant components and the following eigenvalues

$$\begin{array}{ll} A & r = -1+2i, -1-2i \\ B & r = 1+2i, 1-2i \\ C & r = -1, -3 \\ D & r = 3i, -3i. \end{array}$$

The second order ODE $mx'' + \gamma x' + kx = 0$ describing an unforced spring-mass system can always be turned into a system of linear equations by defining a new variable $y = x'$. The resulting system is of the form

$$x' = Mx$$

for some 2×2 matrix M. Of the 4 matrices A, B, C and D above, one describes an undamped spring-mass, one a damped spring-mass, one an overdamped spring-mass and one doesn't describe a spring-mass at all. Which is which?

Solution:

The easiest way to do this is by critical point analysis. For each system of equations $(A..D)$ the only critical point is $(0,0)$ corresponding to the potential spring-mass being at rest at the equilibrium position.

For system A $(0,0)$ is a spiral sink critical point. Physically this would be small oscillations dying out: a damped spring-mass.

For system B $(0,0)$ is unstable: not a spring-mass.

For system C $(0,0)$ is a nodal sink. No oscillation, but the spring-mass settles quickly to equilibrium: over-damped spring-mass.

For system D $(0,0)$ is a stable center. The spring-mass would be locked into a repeating pattern: undamped spring-mass.

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Problem B4: [15 pts]

You have just inherited an old orchard, but unfortunately it is infested with a nasty species of pest. This pest is known to obey the "logistic growth" (see section 2.5) model of population growth with constants $r = 0.5$, $K = 1000$. You decide to reduce the number of pests by introducing a toxin to the orchard. Suppose that the death rate due to the toxin is proportional to the population of pest with proportionality constant α. How large must α be for the pest population to eventually drop below an acceptable level of 100?

Solution:

The logistic equation is

$$y' = ry(1 - \frac{y}{K}.$$

Adding in the death-rate due to toxin we get instead

$$y' = ry(1 - \frac{y}{K}) - \alpha y = y(r - \alpha - \frac{r}{K}y).$$

The critical points of this new system are $y_1 = 0$ and $y_2 = \frac{K(r-\alpha)}{r}$.

If $\alpha \geq r$ then the only critical point with physical meaning is y_1 and this is stable as $y' < 0$ for $y > 0$ (the population can't dip below $y = 0$). Thus all these values ensure eventually extinction.

If $\alpha < r$ then y_1 is unstable and y_2 is stable as $y' > 0$ for $0 < y < y_2$ and $y' < 0$ for $y > y_2$. Thus the population will eventually approach y_2. Thus we need to find α such that $y_2 < 100$. Now if $y_2 = \frac{K(r-\alpha)}{r} < 100$ then $r - \alpha < \frac{100r}{K}$ and $\alpha > r(1 - \frac{100}{K}) = 0.45$.