# Part A: Categories and Functors

CATEGORIES, TOPOI, AND LOGIC

WINTER 2013

## Introduction

In this first part of the course, we will get acquainted with categories, their properties and their structure. Three important themes will follow us as key concepts throughout the course:

- Limits, colimits, duality, and related concepts.
- Functors, natural transformations, and related concepts.
- Adjunctions, monads, and related concepts.

These three themes are multi faceted and closely intertwined. There is no hope to fully grasp these concepts all at once in a particular order: mastering these concepts will require continuous gradual work, exploring new aspects as information arrives. To start this process, it is useful to build a sufficiently large collection of examples of categories to see how these various concepts materialize in each one.

Our two primary sources, Mac Lane and Goldblatt, introduce these concepts in very different ways.

- Mac Lane introduces functors and natural transformation in Chapter I, limits and colimits in Chapter III, and adjunctions in Chapter IV; Chapter II contains some important examples and building blocks for categories.
- Goldblatt introduces limits and colimits in Chapter 3, functors and natural transformations are delayed until Chapter 9, and adjunctions are only adressed at the very end of the book in Chapter 15.

I think this general asynchronicity is actually an advantage since a lot of these concepts require revisiting when new information arrives, which will be automatic as you read through both primary sources.

For this first part, we will go through the first two themes and we will cover adjunctions later. Adjunctions are very important but I agree with Goldblatt that they can be delayed for a while. So in the next part, we will mostly follow Goldblatt and foray into topoi before we come back to look at adjoints.

# Reading

I recommend the following path:

(1) Both Goldblatt and Mac Lane start with background and other preliminaries. Read Goldblatt's prospectus and Chapter 1 together with Mac Lane's introduction.

(2) Read Goldblatt's Chapter 2, which introduces categories. He gives important basic examples in §2.5, which you should use to start your own collection of examples. Some of these basic constructs are subtle; focus on basic mechanical understanding for now.

(3) Read Mac Lane's Chapter 1. There is some overlap with Goldblatt's Chapter 2, but functors, natural transformations, and a lot of other stuff is brand new. Mac Lane's style is much dryer than Godblatt's. You can think of this as a preview of things to come. Again, focus on basic mechanical understanding for now and let these new concepts mature as you go along.

(4) Skim Mac Lane's Chapter 2. This chapter is an overview of basic tools to construct new categories from old. Some of these tools are more relevant than others. For the time being, at least record where things are in this chapter so you can go back when you need to. Whenever possible, use these tools to add one or more examples in your bag of test categories.

(5) Read Goldblatt's Chapter 3. This chapter contains the basic mechanics of categories. Every tool is important; take your time reading through this and internalize each part as much as possible.

(6) Read Mac Lane's Chapter III. This will take you through limits and colimits again but from a different perspective where functors take a more predominant role. Universal objects and the Yoneda Lemma are fundamentally important though highly abstract. These will be difficult to understand on your first reading, don't worry we will get back to them. The important parts of this chapter are §3–5, as you read these sections, keep in mind that there isn't much that is new here — limits and colimits were covered in Goldblatt's Chapter 3. What is important here is the new perspective that Mac Lane emphasizes which is your first real encounter with the Tao of Category Theory.

As you read through Mac Lane, keep in mind that his goal is much more general than Goldblatt's and this course. Categories are useful in just about every aspect of mathematics and thus Mac Lane offers a very broad spectrum of examples, some of which require very substantial background. It is perfectly safe to ignore examples for which you don't have the necessary background. There may be some places where you have to skip so much that you end up missing out on some important ideas; let me know when that happens and we will talk through these ideas in person.

# Exercises

All exercises in Goldblatt and Mac Lane are worth trying as your reading progresses. Omit exercises involving specialized categories where you may not have sufficient background; when necessary, we will talk about these in person.

Here is a guide to the most important exercises.

**Goldblatt —**

- Ch. 1 — None.
- Ch. 2 — There are no formal exercises in this section but the examples in §2.5 are very important. Write your own summary for each one. Work through concrete cases for examples 10, 11, 12.
- Ch. 3 — All: they are standard drill exercises for limits and colimits.

**Mac Lane —**

- Ch. I — §3: 2; §4: 1, 4, 5; §5: 1, 2, 3, 6, 8, 9.
- Ch. II — §3: 2; §4: 2, 4, 7, 8; §5: 1, 2, 3; §6: 2, 4; §7: 1.
- Ch. III — §3: 2, 3, 5, 6; §4: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10; §5: 1, 3, 4, 5.

# Problems

Here are the assigned problems for this part of the course. Once you have succesfully completed all of them, we will move on to the next part. The problems are of variable difficulty. They are not intended to be attacked in a linear fashion but a few later problems depend on earlier ones. Although it is acceptable to go through these without assistance, it is expected that you will ask for guidance as you work through them. You may turn in solutions to each of the five problems separately.

PROBLEM A.1. This problem is about the category $\mathbf{Rel}$ of binary relations. The objects of this category are sets. The arrows $A \to B$ are relations $R \subseteq A \times B$; it is often convenient to write $x \; R \; y$ and $x \not R \; y$ instead of $(x, y) \in R$ and $(x, y) \notin R$. The composition of $R \subseteq A \times B$ and $S \subseteq B \times C$ is

$$S \circ R = \{(x, z) \in A \times C : (\exists y \in B)(x \; R \; y \wedge y \; S \; z)\} .$$

The identity arrow on $A$ is the diagonal relation $\Delta_A = \{(x, y) \in A \times A : x = y\}$ .

(a) Verify that $\mathbf{Rel}$ is indeed a category.

(b) Show that there is a functor $\Gamma : \mathbf{Set} \to \mathbf{Rel}$ which sends every function $f : A \times B$ to its graph $\Gamma(f) = \{(x, y) \in A \times B : y = f(x)\}$ .

(c) Show that there is a functor $\Theta : \mathbf{Rel} \to \mathbf{Set}$ that sends every relation $R \subseteq A \times B$ to the function $\Theta(R)$ that maps each $X \in \mathcal{P}(A)$ to $\{y \in B : (\exists x \in X)(x \; R \; y)\} \in \mathcal{P}(B)$.

(d) Are the functors $\Gamma$ and $\Theta$ faithful? Are they full?

PROBLEM A.2. This problem is about comma categories (Example 12 in Goldblatt §2.5 and §II.6 in Mac Lane).

(a) Let $\mathbb{R}$ be poset category of real numbers. Describe $0 \downarrow \mathbb{R}$ and $\mathbb{R} \downarrow 0$.

(b) Let $2 = \{0, 1\}$ be the standard two-element set. Describe $2 \downarrow \mathbf{Set}$ and $\mathbf{Set} \downarrow 2$.

(c) Let $\mathbf{2}$ denote the category with two objects $0, 1$ with only one non-identity arrow $0 \to 1$. Decribe $\mathbf{2} \downarrow \mathbf{Cat}$ and $\mathbf{Cat} \downarrow \mathbf{2}$.

(d) Let $\mathbf{1}+\mathbf{1}$ denote the category with two objects and no arrows other than the identities. Decribe $\mathbf{1}+\mathbf{1} \downarrow \mathbf{Cat}$ and $\mathbf{Cat} \downarrow \mathbf{1}+\mathbf{1}$.

PROBLEM A.3. Consider the natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$, the rationals $\mathbb{Q}$, and the reals $\mathbb{R}$, and the real unit interval $[0, 1]$ as poset categories (with the usual ordering).

(a) When does the product/coproduct of a general indexed family $\langle x_i \rangle_{i \in I}$ exist in each of these categories? Describe what the product/coproduct is when it does exist.

(b) Which of these poset categories are finitely complete/cocomplete? Complete/cocomplete?

(c) Describe products and coproducts in the product category $\mathbb{N} \times [0, 1]$. Is this category finitely complete/cocomplete? Complete/cocomplete?

(d) Describe products and coproducts in the functor category $[0, 1]^{\mathbb{N}}$. Is this category finitely complete/cocomplete? Complete/cocomplete?

PROBLEM A.4. The goal of this problem is to compute concrete limits and colimits in $\mathbf{Set}$. As usual, let $\omega = \{0, 1, 2, \ldots\}$ denote the set of natural numbers. For each positive integer $n$, let $t_n, u_n, v_n : \omega \to \omega$ be defined by $t_n(x) = n + x$, $u_n(x) = nx$, $v_n(x) = \lfloor x/n \rfloor$.

(a) Compute the equalizers of $t_1$ and $t_2$; $u_2$ and $u_3$; $v_2$ and $v_3$.

(b) Compute the pullbacks of $t_1$ and $t_2$; $u_2$ and $u_3$; $v_2$ and $v_3$.

(c) Compute the coequalizers of $t_0$ and $t_1$; $u_2$ and $t_1 \circ u_2$; $v_1$ and $v_2$.

(d) Compute the pushout of $t_0$ and $t_1$; $s_2$ and $t_1 \circ s_2$; $v_1$ and $v_2$.

PROBLEM A.5. A **concrete category** is a category $\mathbf{C}$ together with a faithful functor $U : \mathbf{C} \to \mathbf{Set}$, called the **forgetful functor**.

Usually, the objects of a concrete category $\mathbf{C}$ are sets with some additional structure and its morphisms consist of some functions between these sets (usually those that preserve the additional structure). Then $U$ is the functor that simply "forgets" the additional structure from the set in question. For example, a poset is a set equipped with a partial ordering and the morphisms in the category $\mathbf{Pos}$ of posets are order-preserving functions.

Let $\mathbf{C}$ be a concrete category with forgetful functor $U : \mathbf{C} \to \mathbf{Set}$. Given a set $B$, the comma category $I \downarrow_U \mathbf{C}$ has for objects are pairs $(a, e)$ where $e : I \to U(a)$ and the arrows $(a, e) \to (a', e')$ are the $\mathbf{C}$-arrows $f : a \to a'$ such that the triangle

$$\begin{tikzcd} & U(a) \arrow[d, bend left, "U(f)"] \\ I \arrow[ur, bend left, "e"] \arrow[r, bend right, "e'"'] & U(a') \end{tikzcd}$$

commutes. If $I \downarrow_U \mathbf{C}$ has an initial object $(a, e)$, it is called the **free object** in $\mathbf{C}$ generated by $I$.

(a) Let $\mathbf{C}$ be a small category. Show that we can always define a faithful functor $T : \mathbf{C} \to \mathbf{Set}$ by sending every object $a$ to the set $T(a)$ of all $\mathbf{C}$-arrows with codomain $a$, and sending every arrow $f : a \to b$ to the function $T(f) : T(a) \to T(b)$ defined by $g : \bullet \to a \mapsto f \circ g : \bullet \to b$. Therefore, every small category is concrete, in some way or another.

(b) Consider the category $\mathbf{Rel}$ from Problem A.1 with the functor $\Theta : \mathbf{Rel} \to \mathbf{Set}$, viewed as a forgetful functor. Show that the free object generated by the set $I$ is $(I, e)$ where $e : I \to \mathcal{P}(I)$ is defined by $e(i) = \{i\}$ for every $i \in I$.

(c) The contravariant powerset functor $\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ sends each function $f : A \to B$ to the preimage function $f^{-1} : \mathcal{P}(A) \to \mathcal{P}(B)$ defined by $f^{-1}(Y) = \{x \in A : f(x) \in Y\}$ for every $Y \subseteq B$. Viewing this as a forgetful functor, show that the free object for the generating set $I$ is $(\mathcal{P}(I), e)$, where $e : I \to \mathcal{P}(\mathcal{P}(I))$ is defined by $e(i) = \{J \in \mathcal{P}(I) : i \in J\}$ for every $i \in I$.

(d) The covariant powerset functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ sends each function $f : A \to B$ to the image function $f``: \mathcal{P}(A) \to \mathcal{P}(B)$ defined by $f``(X) = \{f(x) : x \in X\}$ for every $X \subseteq A$. Viewing this as a forgetful functor, argue that there are no free objects for any nonempty generating set.