Math 31 Winter 2004

Topics in Algebra

Final Exam

Friday March 12, 2004

**The exam is due at 11:50 PM on Tuesday March 16
in the Instructor's office 414 Bradley Hall.**

Your name (please print): _____

Instructions: This is an open book open notes exam. You can consult any printed matter you like, but **you can not consult other humans. Use of calculators is not permitted**. **You must justify all of your answers to receive credit,** unless instructed otherwise. If I am not in my office when you submit your exam, then please write the time you finished working on it and slide it under my office door.

The exam total score is the sum of the 10 best (out of 11) problem scores. Please do all your work on the paper provided.

The Honor Principle requires that you neither give nor receive any aid on this exam.
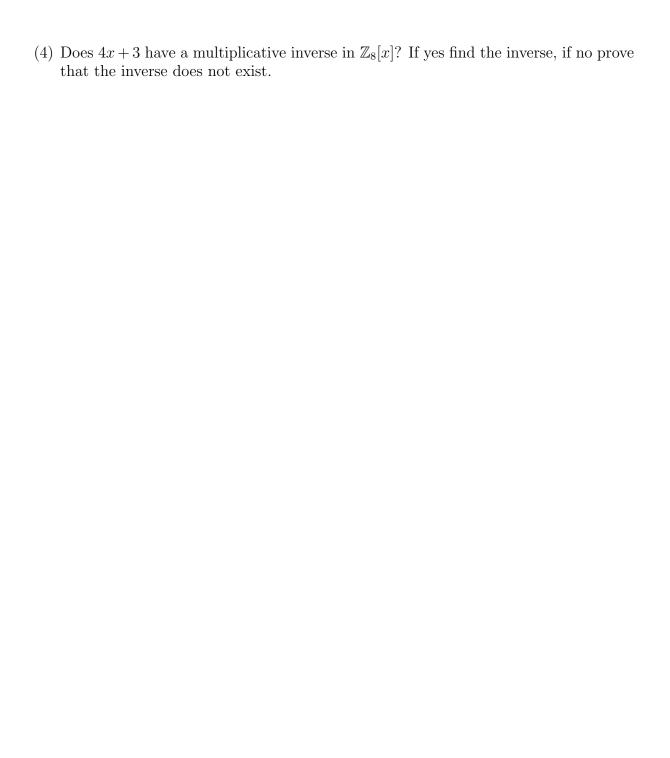
1. _____ /12

2. _____ /12

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Total: _____ /120

(1) Let G be a group with 77 elements. Prove that G is isomorphic to $\mathbb{Z}_{77}$.

(2) Let $\text{Gl}(3, \mathbb{Q})$ be the group of 3×3 matrices with rational entries and nonzero determinant with the group operation being multiplication of matrices, and let $\mathbb{Q}^+$ be the group of positive rational numbers with the group operation being multiplication of rational numbers. Construct the surjective homomorphism from the group $\text{Gl}(3, \mathbb{Q})$ onto $\mathbb{Q}^+$.

(3) Let F be a field with $3^4 = 81$ elements. Show that the mapping $\phi : F \to F$ that maps $x \in F$ to x^3 is a ring homomorphism.

(4) Does $4x + 3$ have a multiplicative inverse in $\mathbb{Z}_8[x]$? If yes find the inverse, if no prove that the inverse does not exist.

(5) Find an ideal $I \subset \mathbb{Q}[x]$ such that factor ring $F = \mathbb{Q}[x]/I$ is a field and such that the polynomial $f(x) = 4x^4 - 4x^2 + 1$ has a zero in F.

(6) Prove that every group of order 54 has an element of order three. (Hint: you can try to show that a group of order 54 with no elments of order three has to be Abelian, and then get a contradiction with one of the theorems from the course.)

(7) Prove that $\mathbb{Q}[x]/<2x^3+x+1>$ is a field. Find the dimension of this field considered as a vector space over $\mathbb{Q}$.

(8) Let G be a (not necessarily Abelian) group of order 96 and let $\phi : G \rightarrow \mathbb{Z}_{12}$ be a surjective homomorphism. Prove that G has a normal subgroup of **order** 48.

(9) Find all the non-isomorphic Abelian groups of order 36. Find the number of elements of order 9 in each one of those groups.

(10) Are $2\mathbb{Z}/10\mathbb{Z}$ and $\mathbb{Z}_5$ isomorphic as groups? Are they isomorphic as rings?

(11) Let $\beta = (1,2,3,4)(6,7,8,9,10,11)(12,13,14) \in S_{23}$. Find the disjoint cycle presentation for β^{2004}.