Killing form

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra with basis $\{b_1, b_2, \ldots b_m\}$. Then $\mathfrak{g}$ is a $\mathfrak{g}$-module under the adjoint action: x acts on $\mathfrak{g}$ by

$$\begin{array}{rcl} \mathrm{ad}_x : \mathfrak{g} & \to & \mathfrak{g} \\ y & \mapsto & [x,y]. \end{array}$$

The *Killing form* is the associative bilinear form $K(,) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defined by

$$K(x,y) = \mathrm{Tr}(\mathrm{ad}_x \mathrm{ad}_y).$$

It is associative in the sense that $K([x,y],z) = K(x,[y,z])$. This is equivalent to the property ad-invarience, that $K([x,y],z) = -K(y,[x,z])$. The Killing form is also nondegenerate, i.e.

$$S := \{x \in \mathfrak{g} \mid K(x,y) \text{ for all } y \in \mathfrak{g}\} = 0,$$

precisely when $\mathfrak{g}$ is semisimple (note: this depends on the fact that char $\mathbb{C} = 0$. When char $F = p$, we have nondegenerate $\Rightarrow$ semisimple, but semisimple $\not\Rightarrow$ nondegenerate). To show this, it is useful to note that the associative property of K implies that S is an *ideal* of $\mathfrak{g}$.

1 Humphrey's treatment of trace forms and existence of K

Recall a Lie algebra $\mathfrak{g}$ is *solvable* if $\mathfrak{g}^{(l)} = 0$ for some k, where $\mathfrak{g}^{0)} = \mathfrak{g}$ and $\mathfrak{g}^{(i)} := [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}]$.

Theorem 1.1 (Cartan's Critereon)**.** *Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{gl}(V)$, $\dim V = n$. Suppose that $\mathrm{Tr}(xy) = 0$ for all $x \in [\mathfrak{g},\mathfrak{g}]$, $y \in \mathfrak{g}$. Then $\mathfrak{g}$ is solvable.*

We can use this to show that every semisimple $\mathfrak{g}$ has a nondegenerate (non-trivial) Killing form. More generally, if $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a faithful (injective) representation of $\mathfrak{g}$, then we can define a similar form $\beta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ by

$$\beta(x,y) = \mathrm{Tr}(\varphi(x)\varphi(y)).$$

Then β is all the beautiful things we wish of it: it's symmetric (clearly), ad-invarient (associative), and nondegenerate ($\varphi(S) \cong S$ is a solvable ideal, so is in $\mathrm{Rad}\mathfrak{g} = 0$). In fact, the Killing form is just β in the special case that $\varphi = \mathrm{ad}$!

Let $\mathfrak{g}$ be one of the classical Lie algebras (type A, B ,C or D), and let x_V be the image of $x \in \mathfrak{g}$ under the defining representation. Then our favorite form on $\mathfrak{g}$ is he form $\langle , \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ by

$$\langle x,y \rangle = \mathrm{Tr}(x_V y_V),$$

It can be shown that

$$K(x,y) = \begin{cases} 2(r+1)\langle x,y\rangle & \text{in type } \mathfrak{sl}_{r+1},\ \mathfrak{sp}_{2r} \\ (2r-1)\langle x,y\rangle & \text{in type } \mathfrak{so}_{2r+1}, \\ 2(r-1)\langle x,y\rangle & \text{in type } \mathfrak{so}_{2r}. \end{cases}$$

References

[Hum] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, 1997.

[Ser] J.P. Serre, *Complex Semisimple Lie Algebras*, Springer, New York 1987.