# Tangent Planes and Linear Approximations

November 10, 2006

# Tangent Planes

- Let $S$ be a surface with equation $z = f(x, y)$.
- Let $P(x_0, y_0, z_0)$ be a point on $S$.
- Let $C_1$ and $C_2$ be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface $S$.
- Let $T_1$ and $T_2$ be the tangent lines to the curves $C_1$ and $C_2$.
- The **tangent plane** to the surface $S$ at the point $P$ is defined to be the plane that contains both tangent lines $T_1$ and $T_2$.

# Equations of the tangent plane

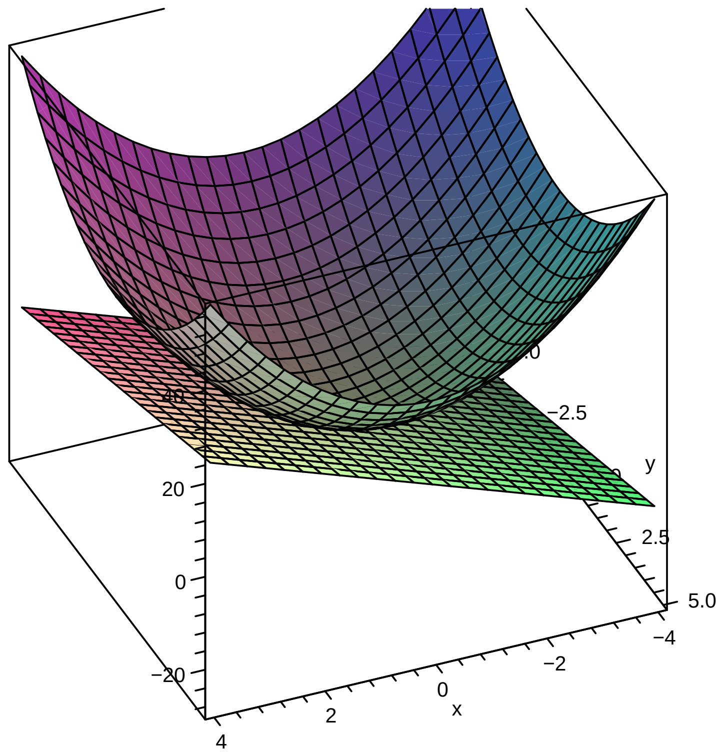
- Suppose $f$ has a continuous partial derivatives.

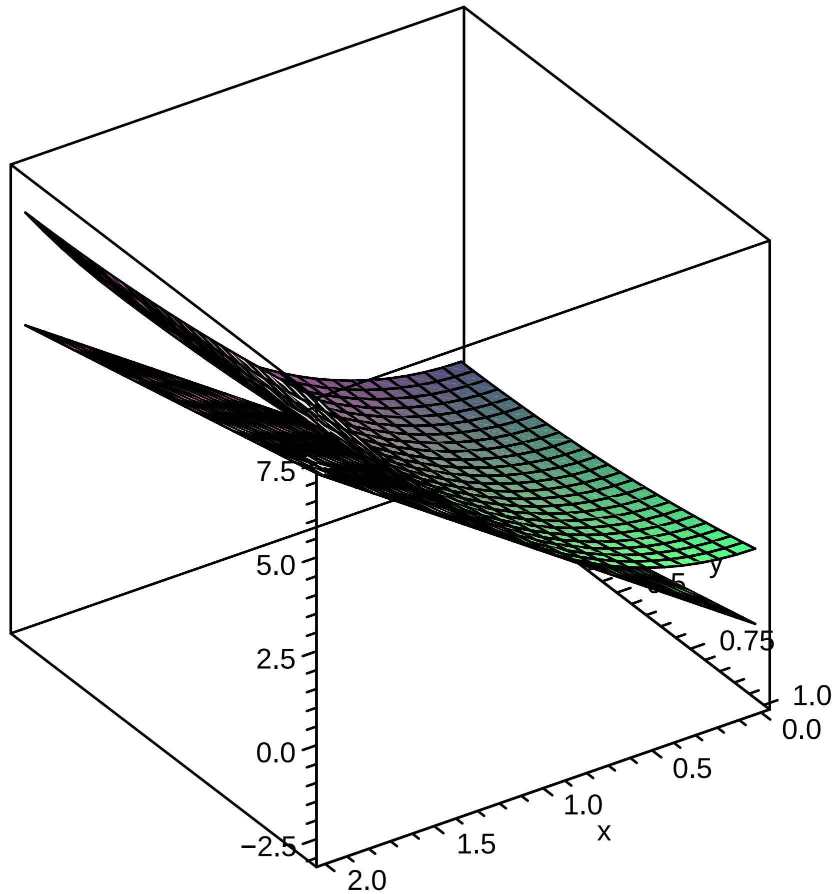
- An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

# Examples

- Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.




7.5
5.0
2.5
0.0
−2.5
2.0
1.5
1.0
x
0.5
0.0
1.0
0.75
y

# Linear Approximations

- The linear function whose graph is this tangent plane

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

is called the **linearization** of $f$ at $(a,b)$ and the approximation

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

is called the **linear approximation** or the **tangent plane approximation** of $f$ at $(a,b)$.

# Examples

- Find the linearization of the function $f(x, y) = \sqrt{xy}$ at the point $(4, 16)$.

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- Find the linearization of the function $f(x, y) = 1 + y + x\cos y$ at $P_0(0, 0)$.

# The increment of $z$

- Recall that for a function of one variable, $y = f(x)$, if $x$ changes from $a$ to $a + \Delta x$, we defined the increment of $y$ as

$$\Delta y = f(a + \Delta x) - f(a).$$

- If $f$ is differentiable at $a$, then

$$\Delta y = f^{'}(a)\Delta x + \epsilon\Delta x,$$

where $\epsilon \to 0$ as $\Delta x \to 0$.

- If $z = f(x, y)$ and $x$ changes from $(a, b)$ to $(a + \Delta x, b + \Delta y)$, then the **increment** of $z$ is

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

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- If $z = f(x, y)$, then $f$ is **differentiable** at $(a, b)$ if $\Delta z$ can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where $\epsilon_1$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

- **FACT:** If the partial derivatives $f_x$ and $f_y$ exist near $(a, b)$ and are continuous at $(a, b)$, then $f$ is differentiable at $(a, b)$.

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- Example: Show t hat $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there.

# Differentials

- For a differentiable function $z = f(x, y)$ we define the **differential** d$z$, also called the **total differential**, is defined by
$$\mathrm{d}z = f_x(x, y)\mathrm{d}x + f_y(x, y)\mathrm{d}y = \frac{\partial z}{\partial x}\mathrm{d}z + \frac{\partial z}{\partial y}\mathrm{d}y,$$
where the **differentials** d$x$ and d$y$ are independent variables.

- If d$x = \Delta x = x - a$ and d$y = \Delta y = y - b$ the the differential of $z$ is
$$\mathrm{d}z = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

# Examples

- If $f(x, y) = x^2 + 3xy - y^2$, find the differential d$z$.

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- If $f(x, y) = x^2 + 3xy - y^2$, find the differential d$z$.

- If $x$ changes from $2$ to $2.05$ and $y$ changes $3$ to $2.96$, compare the values of $\Delta z$ and d$z$.