Lecture Notes

Math 3 / Fall 2012

Week 1

Polynomials

- A **linear function** is of the form $ax + b$
- A **quadratic function** is of the form $ax^2 + bx + c$
- A **cubic function** is of the form $ax^3 + bx^2 + cx + d$
- A ***n*th degree polynomial** is of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are constants called the **coefficients**

Polynomial Interpolation Theorem

Theorem

Given $n+1$ *data points*

$$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$$

with different x-coordinates, there is exactly one polynomial of degree at most n that passes through all of them.

- 2 points determine a unique linear function
- 3 points determine a unique quadratic function

Polynomial Interpolation Theorem

Example

Find the quadratic function $q(x) = ax^2 + bx + c$ that interpolates the three data points $(0, 1)$, $(-1, 2)$, and $(3, 1)$.

- We first set up the three equations:

$$\begin{aligned} 1 = \quad q(0) \quad &= \quad a \cdot (0)^2 \quad + \quad b \cdot (0) \quad + c = c, \\ 2 = q(-1) &= a \cdot (-1)^2 + b \cdot (-1) + c = a - b + c, \\ 1 = \quad q(3) \quad &= \quad a \cdot (3)^2 \quad + \quad b \cdot (3) \quad + c = 9a + 3b + c. \end{aligned}$$

- Since $c = 1$, this simplifies to two equations:

$$1 = a - b, \qquad 9a + 3b = 0.$$

- We then find that $a = \frac{1}{4}$, $b = -\frac{3}{4}$, and $c = 1$.

Least Squares Approximation Theorem

Theorem

Given any number of data points

$$(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$$

with different x-coordinates, there is exactly one polynomial $p(x)$ of degree at most n that minimizes the **sum of squared errors***:*

$$SSE = (p(x_1) - y_1)^2 + (p(x_2) - y_2)^2 + \cdots + (p(x_m) - y_m)^2.$$

- The case $n = 1$ is called **linear regression**
- Use applet to find the least squares best fit polynomial...

Slope of a line

- Two points (x_0, y_0) and (x_1, y_1) determine a unique line
- The **slope** of that line is

$$m = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\text{rise}}{\text{run}}$$

  - If $m > 0$ then the line is increasing
  - If $m = 0$ then the line is constant
  - If $m < 0$ then the line is decreasing
- Vertical lines do not have a well defined slope
- Parallel lines have the same slope
  Perpendicular lines have slopes that multiply to -1

Equations for lines

- **General Form**

$$Ax + By = C$$

- **Slope–Intercept Form**

$$y = mx + b$$

- **Point–Slope Form**

$$y = m(x - x_0) + y_0 \quad \text{or} \quad \frac{y - y_0}{x - x_0} = m$$

Functions

A **function** f is a rule that takes an input x and returns a unique output $f(x)$

- Algebraic:

$$f(x) = \pi x^2$$

- Prosaic:

$f(x)$ is the area of a circle with radius x

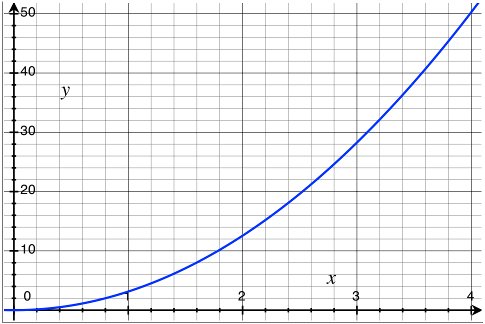
- Tabular:

x	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$	0.8	3.1	7.1	12.6	19.6	28.3	38.5	50.3

Functions

A **function** f is a rule that takes an input x and returns a unique output $f(x)$

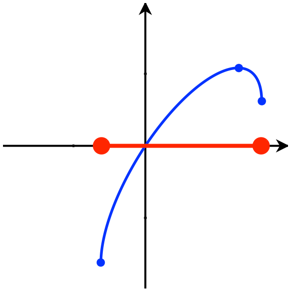
- Graphical:



Domain of a function

The **domain** of a function is the set of all sensible inputs x for the function

- The domain is the "shadow" of the graph on the x-axis:

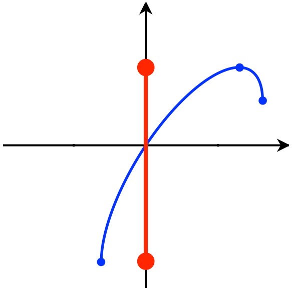


- The number a is in the domain of f when the vertical line $x = a$ meets the graph

Range of a function

The **range** of a function is the set of all possible outputs $y = f(x)$ for the function

- The range is the "shadow" of the graph on the y-axis:



- The number b is in the range of f when the horizontal line $y = b$ meets the graph

Interval Notation

Domains and ranges are often described using intervals:

- $[a, b]$ = all numbers x such that $a \leq x \leq b$
- $(a, b]$ = all numbers x such that $a < x \leq b$
- $[a, b)$ = all numbers x such that $a \leq x < b$
- (a, b) = all numbers x such that $a < x < b$

Finding the domain

Example

Find the domain of $f(x) = \dfrac{\sqrt{25 - x^2}}{(x-1)(x+2)}$

- We need $x - 1 \neq 0$ so $x \neq 1$
- We need $x + 2 \neq 0$ so $x \neq -2$
- We need $25 - x^2 \geq 0$ so $-5 \leq x \leq 5$
- The domain is $[-5, -2) \cup (-2, 1) \cup (1, 5]$

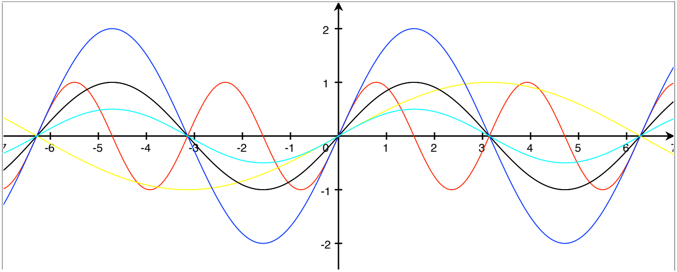
Scaling a graph

Given a function f and a positive constant a:

- The graph of $g(x) = af(x)$ is that of f except that it is **scaled vertically**
  - dilated when $a > 1$
  - compressed when $a < 1$
- The graph of $h(x) = f(ax)$ is that of f except that it is **scaled horizontally**
  - compressed when $a > 1$
  - dilated when $a < 1$

Scaling a graph

Horizontal and vertical scalings of $f(x) = \sin(x)$ (black curve) with $a = 2$ and $a = \frac{1}{2}$



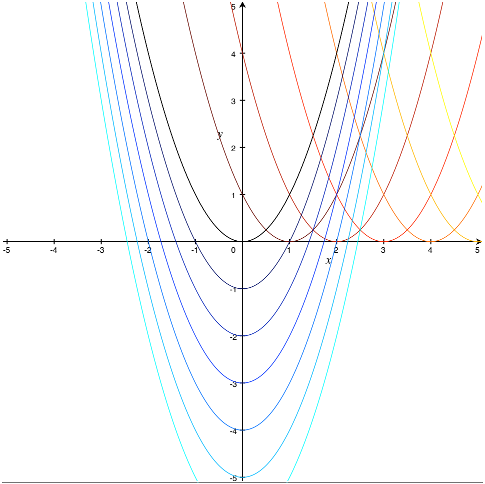
Shifting a graph

Given a function f and an arbitrary constant b:

- The graph of $g(x) = f(x) + b$ is that of f except that it is **translated vertically**
  - up when $b > 0$
  - down when $b < 0$
- The graph of $h(x) = f(x + b)$ is that of f except that it is **translated horizontally**
  - left when $b > 0$
  - right when $b < 0$

Shifting a graph

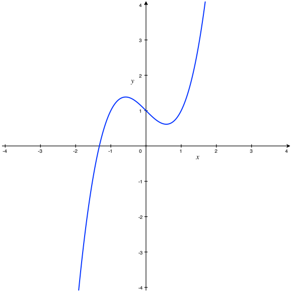
Horizontal and vertical shifts of $f(x) = x^2$ (black curve) by $b = 1, 2, 3, \ldots$

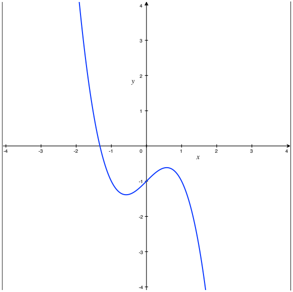


Reflecting a graph

- The graph of $g(x) = -f(x)$ is that of $f(x)$ except that it is **reflected across the x-axis**

$f(x) = x^3 - x + 1$ $\qquad$ $-f(x) = -x^3 + x - 1$

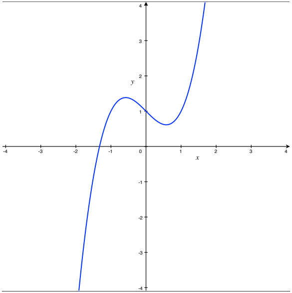




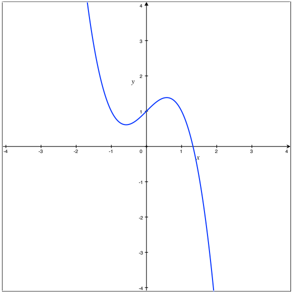
Reflecting a graph

- The graph of $g(x) = f(-x)$ is that of $f(x)$ except that it is **reflected across the** y**-axis**

$f(x) = x^3 - x + 1$



$f(-x) = -x^3 + x + 1$

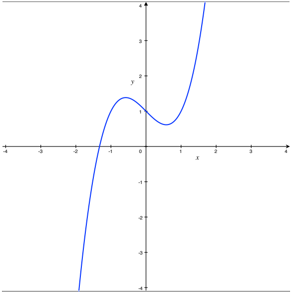


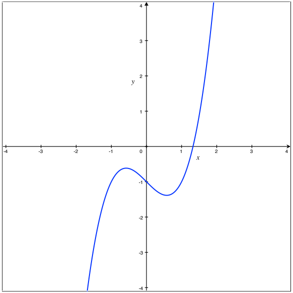
- A function that remains the same when reflected across the y-axis is called **even**

Reflecting a graph

- The graph of $g(x) = -f(-x)$ is that of $f(x)$ except that it is **reflected across the origin**

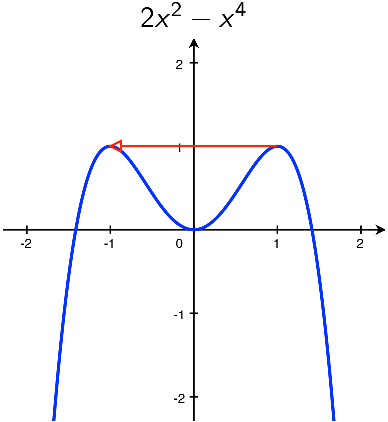
$f(x) = x^3 - x + 1$ $\qquad$ $-f(-x) = x^3 - x - 1$





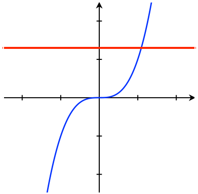
- A function that remains the same when reflected across the origin is called **odd**

Even and Odd



Inverse function

A function f is **one-to-one** if the equation $f(x) = b$ never has more than one solution.



The **inverse** of f is the function f^{-1} such that

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x$$

for every number x

Finding the inverse

Example

Find the inverse of $f(x) = \dfrac{2x}{x-1}$

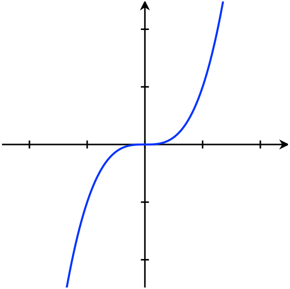
Set $f(y) = x$ and solve for $y \ldots$

- $x = \dfrac{2y}{y-1}$
- $x(y-1) = 2y$
- $xy - x = 2y$
- $xy - 2y = x$
- $(x-2)y = x$
- $y = \dfrac{x}{x-2}$

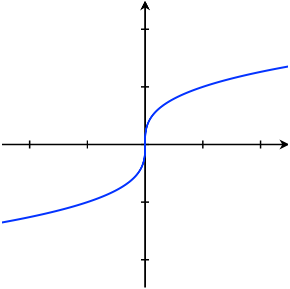
Graph of the inverse

The graph of the inverse f^{-1} is that of f but reflected across the diagonal $x = y$

$f(x) = x^3$



$f^{-1}(x) = \sqrt[3]{x}$



Function composition

The **composition** $f \circ g$ of two functions f and g is obtained by feeding the output of g as the input of f:

$$(f \circ g)(x) = f(g(x))$$

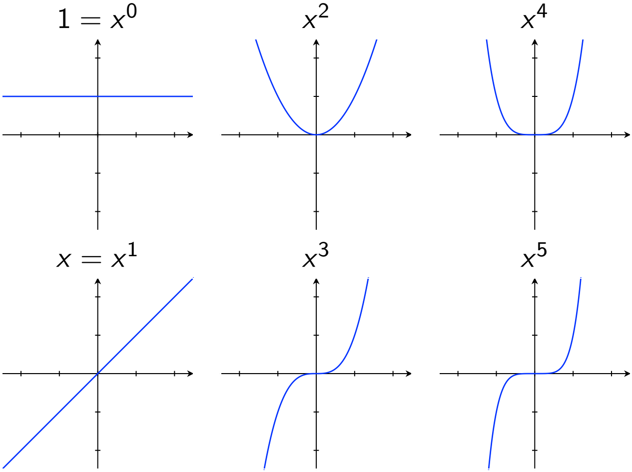
For example, if $f(x) = \dfrac{x-2}{x+2}$ and $g(x) = x^3$ then

$$(f \circ g)(x) = f(g(x)) = \frac{x^3 - 2}{x^3 + 2}$$

Note that this is different from

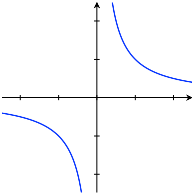
$$(g \circ f)(x) = g(f(x)) = \left(\frac{x-2}{x+2}\right)^3$$

Power functions

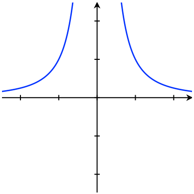


Power functions

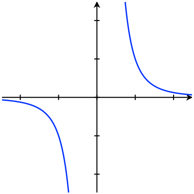
$$x^{-1} = \frac{1}{x}$$



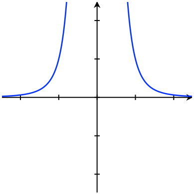
$$x^{-2} = \frac{1}{x^2}$$



$$x^{-3} = \frac{1}{x^3}$$

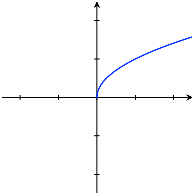


$$x^{-4} = \frac{1}{x^4}$$

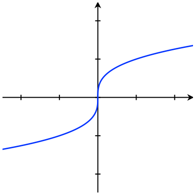


Power functions

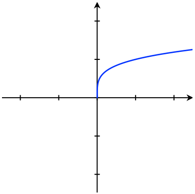
$x^{1/2} = \sqrt{x}$



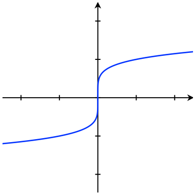
$x^{1/3} = \sqrt[3]{x}$



$x^{1/4} = \sqrt[4]{x}$



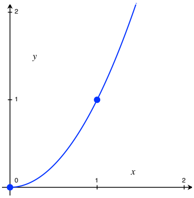
$x^{1/5} = \sqrt[5]{x}$



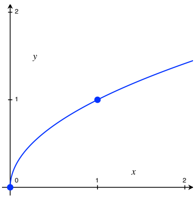
Power functions

For general exponents b, we can only make sense of x^b when $x > 0$

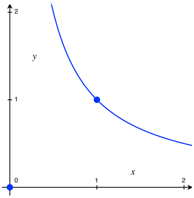
$b > 1$



$0 < b < 1$



$b < 0$



Quickly sketching complex functions

Example

Sketch the function $f_0(x) = -\dfrac{1}{x-1} + \dfrac{1}{3}$

Decompose the function...

- $f_0(x)$ is $f_1(x) = -\dfrac{1}{x-1}$ shifted up by $\dfrac{1}{3}$
- $f_1(x)$ is $f_2(x) = \dfrac{1}{x-1}$ reflected across the x-axis
- $f_2(x)$ is $f_3(x) = \dfrac{1}{x}$ shifted right by 1

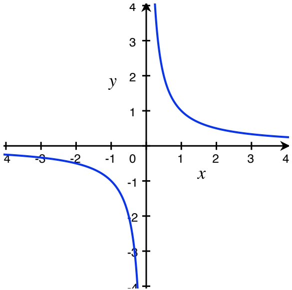
Then recompose the graph...

Quickly sketching complex functions

Example

Sketch the function $f_0(x) = -\frac{1}{x-1} + \frac{1}{3}$

$$f_3(x) = \frac{1}{x}$$

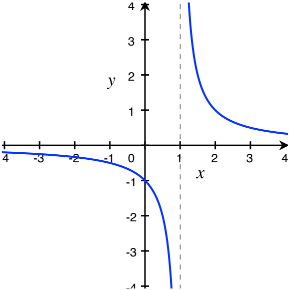


Quickly sketching complex functions

Example

Sketch the function $f_0(x) = -\frac{1}{x-1} + \frac{1}{3}$

$$f_2(x) = \frac{1}{x-1}$$

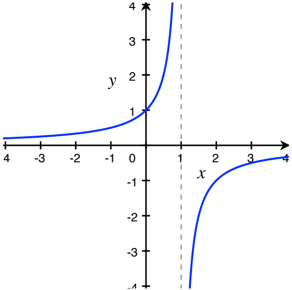


Quickly sketching complex functions

Example

Sketch the function $f_0(x) = -\dfrac{1}{x-1} + \dfrac{1}{3}$

$$f_1(x) = -\frac{1}{x-1}$$



Quickly sketching complex functions

Example

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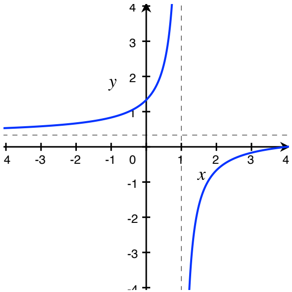
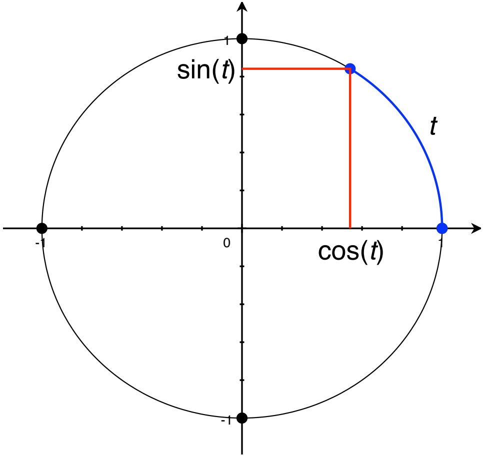
$$f_0(x) = -\frac{1}{x-1} + \frac{1}{3}$$



Radians

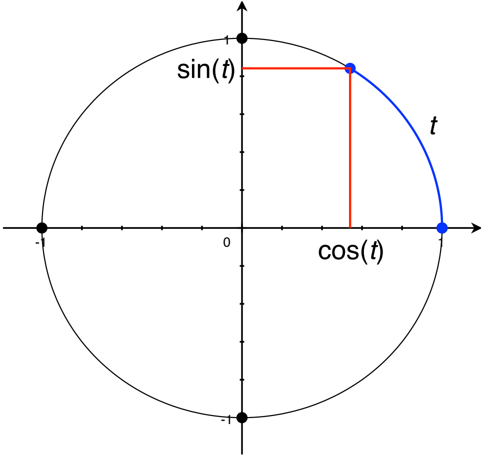
The angle with radian measure t is obtained by walking distance t along the unit circle starting at $(1, 0)$, counterclockwise if $t > 0$ and clockwise if $t < 0$



Radians

The coordinates of the end point of the arc are:

$$x = \cos(t) \quad \text{and} \quad y = \sin(t)$$

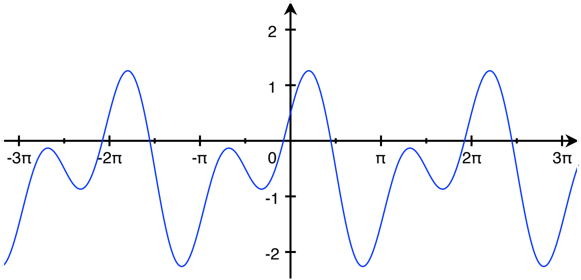


Periodic functions

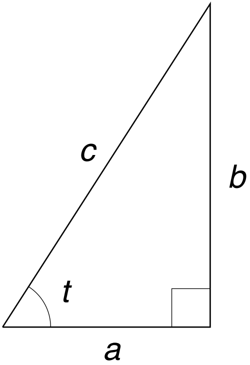
A function f has **period** p if it is unchanged when translated horizontally by p:

$$f(x) = f(x \pm p) = f(x \pm 2p) = f(x \pm 3p) = \cdots$$

Both sin and cos have period 2π since that is the total circumference of the unit circle

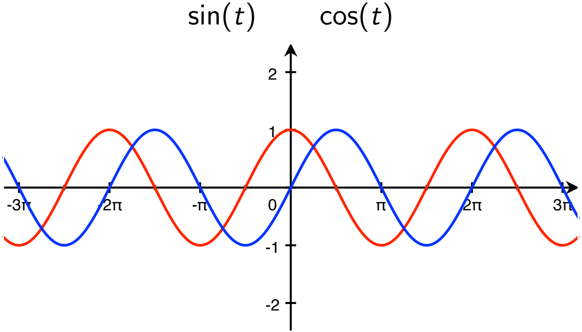


Trigonometric functions



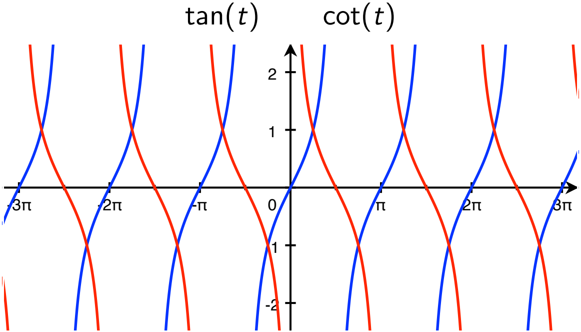
$$a^2 + b^2 = c^2$$
$$\cos(t) = \frac{a}{c}$$
$$\sin(t) = \frac{b}{c}$$
$$\tan(t) = \frac{b}{a} = \frac{\sin(t)}{\cos(t)}$$
$$\cot(t) = \frac{a}{b} = \frac{\cos(t)}{\sin(t)}$$
$$\sec(t) = \frac{c}{a} = \frac{1}{\cos(t)}$$
$$\csc(t) = \frac{c}{b} = \frac{1}{\sin(t)}$$

Trigonometric functions



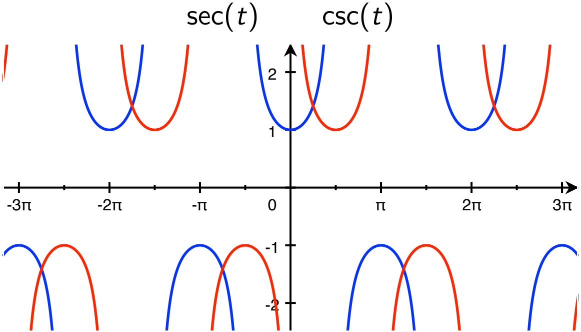
$$\cos(t) = \sin(t + \frac{\pi}{2})$$

Trigonometric functions



$$\cot(t) = -\tan(t + \tfrac{\pi}{2})$$

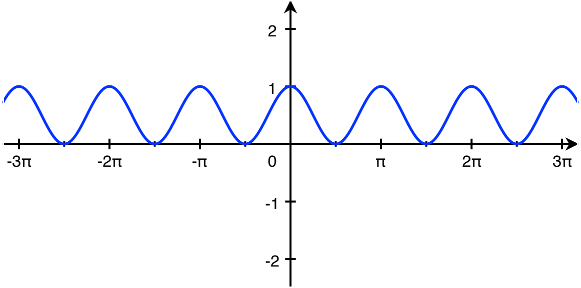
Trigonometric functions



$$\csc(t) = \sec(t + \tfrac{\pi}{2})$$

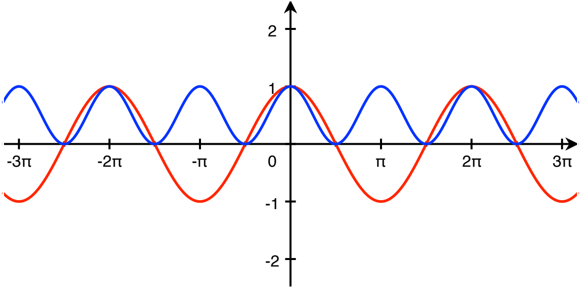
Trigonometric identities

$\cos^2(x)$



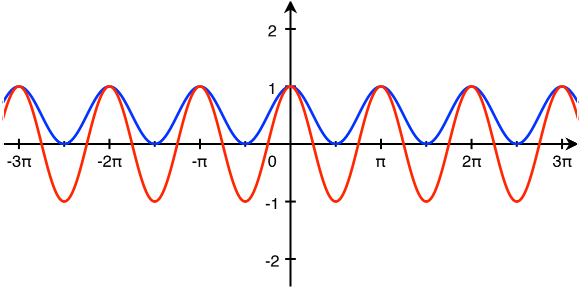
Trigonometric identities

$\cos^2(x) \quad \text{vs} \quad \cos(x)$



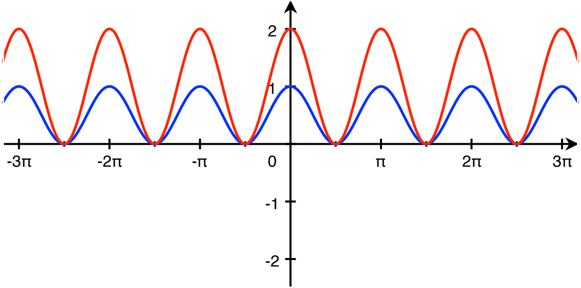
Trigonometric identities

$\cos^2(x) \quad$ vs $\quad \cos(2x)$



Trigonometric identities

$\cos^2(x) \quad \text{vs} \quad \cos(2x)+1$



Trigonometric identities

$$\cos^2(x) = \frac{\cos(2x) + 1}{2}$$