Math 13: Homework #8

(17.8.2) The boundary of S is the circle of radius 3 centered at the origin in the xy-plane. A positively oriented parameterization of the circle C is

$$r(t) = (3\cos t, 3\sin t, 0), \quad \text{for } t \in [0, 2\pi].$$

We compute $r'(t) = (-3\sin t, 3\cos t, 0)$, $F(r(t)) = (6\sin t, 0, 3\cos t\, e^{3\sin t})$

and $F(r(t)) \cdot r'(t) = -18\sin^2 t$. Then

$$\iint_S \text{curl}\, F \cdot dS \overset{\text{Stokes}}{=} \int_C F \cdot dr \overset{\text{defn}}{=} \int_0^{2\pi} F(r(t)) \cdot r'(t)\, dt = \int_0^{2\pi} -18\sin^2 t\, dt$$

$$\overset{\text{trig identity}}{=} -18 \int_0^{2\pi} \frac{1-\cos 2t}{2}\, dt = -9\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi} = \boxed{-18\pi.}$$

(17.8.4) The boundary of S is the circle C given by $x^2+z^2=9$ and $y=3$. A positively oriented parameterization of C is given by

$$r(t) = (3\sin t, 3, 3\cos t), \quad \text{for } t \in [0, 2\pi].$$

Therefore $r'(t) = (3\cos t, 0, -3\sin t)$, $F(r(t)) = (3^6 \sin^4 t \cos t, \sin(3^3 \sin t \cos t),$
$3^3 \sin t \cos t),$

and $F(r(t))\cdot r'(t) = 3^7 \sin^2 t\cos^2 t - 3^4 \sin^2 t \cos t \overset{\text{trig identity}}{=} 3^7\left(\frac{\sin 2t}{2}\right)^2 - 3^4 \sin^2 t \cos t.$

Then $\iint_S \text{curl } F \cdot dS \overset{\text{Stokes}}{=} \int_C F\cdot dr \overset{\text{defn}}{=} \int_0^{2\pi} F(r(t))\cdot r'(t)\, dt$

$$= \int_0^{2\pi} 3^7\left(\frac{\sin 2t}{2}\right)^2 - 3^4 \sin^2 t\cos t\, dt = \int_0^{2\pi} \frac{3^7}{4}(\sin 2t)^2 - 3^4 \sin^2 t \cos t\, dt$$

$$= \left[\frac{3^7}{4}\left(\frac{1}{2}t - \frac{1}{8}\sin 4t\right) - 3^4\left(\frac{1}{3}\sin^3 t\right)\right]_0^{2\pi} = \boxed{\frac{3^7}{4}\pi.}$$

(17.8.6) The boundary of S is the unit circle C in the yz-plane. We can positively orient C with the parameterization $r(t) = (0, \cos t, \sin t)$ for $t\in[0,2\pi]$. As usual, $r'(t) = (0, -\sin t, \cos t)$, $F(r(t)) = (e\cos(\sin t), 0, 0)$, and $F(r(t))\cdot r'(t) = 0$. Therefore $\iint_S \text{curl } F\cdot dS \overset{\text{Stokes}}{=} \int_C F\cdot dr = \int_0^{2\pi} 0\, dt = \boxed{0.}$

(17.8.8) Let S be the surface bounded by C. S can be described by the equation $z = 1 - x - \frac{1}{2}y$, where $x\in[0,1]$ and $y\in[0, 2-2x]$. We compute

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{-x} & e^x & e^z \end{vmatrix} = e^x \vec{k} = (0,0,e^x).$$

Therefore, $\int_C F\cdot dr \overset{\text{Stokes}}{=} \iint_S \text{curl} F\cdot dS \overset{\text{equation 17.7.10}}{=} \iint_D e^x \, dA = \int_0^1 \int_0^{2-2x} e^x \, dy\, dx = \int_0^1 (2-2x)e^x \, dx$

$= \int_0^1 2e^x dx - \int_0^1 2xe^x dx \overset{\text{integration by parts}}{=} 2e-2 - 2xe^x\Big|_0^1 + \int_0^1 2e^x dx$

$= 2e-2-2e+2e-2 = \boxed{2e-4.}$

(17.8.10) Let S be the surface enclosed by C. So S is a subset of the plane $z=5-x$, specifically every point on S satisfies $x^2+y^2 \le 9$. We compute

$$\text{Curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2z & 3y \end{vmatrix} = \vec{i} - x\vec{k} = (1, 0, -x).$$

Therefore, $\int_C F\cdot dr \overset{\text{Stokes}}{=} \iint_S \text{curl} F\cdot dS \overset{\text{equation 17.7.10}}{=} \iint_D (1-x)\, dA = \iint_{x^2+y^2\le 9} 1-x \, dA$

$\overset{\text{polar coor.}}{=} \int_0^{2\pi}\int_0^3 (1-r\cos\theta) r\, dr\, d\theta = \int_0^{2\pi}\int_0^3 r - r^2\cos\theta \, dr\, d\theta = \int_0^{2\pi} \frac{9}{2} - 9\cos\theta\, d\theta$

$= 2\pi\cdot\frac{9}{2} = \boxed{9\pi}$

17.9.2) We compute: $\text{div} F = 3x+1$. Therefore

$$\iiint_E \text{div} F \, dV = \iiint_E 3x+1 \, dV \overset{\substack{\text{cylindrical coord} \\ \downarrow}}{=} \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} (3r\cos\theta + 1) r \, dz \, dr \, d\theta$$

$$= \int_0^2 \int_0^{2\pi} r(3r\cos\theta + 1)(4-r^2) \, d\theta \, dr = \int_0^2 r(4-r^2) \left[3r\sin\theta + \theta \right]_{\theta=0}^{\theta=2\pi} dr$$

$$= 2\pi \int_0^2 r(4-r^2) \, dr = 2\pi \int_0^2 4r - r^3 \, dr = 2\pi(8-4) = \boxed{8\pi}$$

On the other hand, let S_1 be the surface given by the paraboloid $z = 4 - x^2 - y^2$ for $z \geq 0$. So $x^2+y^2 \leq 4$. We have

$$\iint_{S_1} F \cdot dS \overset{\substack{\text{equation 17.7.10} \\ \downarrow}}{=} \iint_D \left(-(x^2)(-2x) - (xy)(-2y) + (4-x^2-y^2) \right) dA$$

$$= \iint_D 2x(x^2+y^2) + 4 - (x^2+y^2) \, dA \overset{\substack{\text{polar} \\ \downarrow}}{=} \int_0^{2\pi} \int_0^2 (2r\cos\theta \, r^2 + 4 - r^2) r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{2}{5}\cos\theta \, r^5 + 2r^2 - \frac{1}{4}r^4 \right]_{r=0}^{r=2} d\theta = \int_0^{2\pi} \frac{64}{5}\cos\theta + 4 \, d\theta$$

$$= 8\pi.$$

Let S_2 be the surface given by $z=0$. The normal $\vec{n} = -\vec{k}$. Therefore

$$\iint_{S_2} F\cdot dS = \iint_{S_2} F\cdot n\, dS \overset{F=(x^2,\,xy,\,0) \text{ on } S_2}{=} \iint_{S_2} 0\, dS = 0.$$

Putting these two pieces together, $\iint_S F\cdot dS = \iint_{S_1} F\cdot dS + \iint_{S_2} F\cdot dS = 8\pi + 0 = \boxed{8\pi.}$

(17.9.4) $\operatorname{div} F = 3$, so $\iiint_E \operatorname{div} F\, dV = \iiint_E 3\, dV = 3 \operatorname{vol}(\text{unit ball}) = \boxed{4\pi.}$

On the other hand, we can parameterize the unit sphere using spherical coordinates $r(\varphi,\theta)$.

Then $r_\varphi \times r_\theta = \begin{vmatrix} i & j & k \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0 \end{vmatrix} = \sin^2\varphi\cos\theta\,\vec{i} + \sin^2\varphi\sin\theta\,\vec{j} + \sin\varphi\cos\varphi\,\vec{k}$,

and $F(r(\varphi,\theta)) = \sin\varphi\cos\theta\,\vec{i} + \sin\varphi\sin\theta\,\vec{j} + \cos\varphi\,\vec{k}$. We have

$$\iint_S F\cdot dS = \iint_D F(r(\varphi,\theta))\cdot r_\varphi\times r_\theta\, dA = \int_0^{2\pi}\int_0^{\pi} \left(\sin^3\varphi\cos^2\theta + \sin^3\varphi\sin^2\theta + \sin\varphi\cos^2\varphi\right) d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^3\varphi + \sin\varphi\cos^2\varphi\ d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi} \sin\varphi\ d\varphi\, d\theta = 2\pi\cdot 2 = \boxed{4\pi.}$$

(17.9.6) $\operatorname{div} F = 2xz^3 + 2xz^3 + 4xz^3 = 8xz^3$. Thus

$$\iint_S F\cdot dS \overset{\text{div thm}}{=} \iiint_E 8xz^3\, dV = \int_{-1}^{1}\int_{-2}^{2}\int_{-3}^{3} 8xz^3\, dz\, dy\, dx = 8\int_{-1}^{1} x\, dx \int_{-2}^{2} dy \int_{-3}^{3} z^3\, dz$$

$= \boxed{0}$, since $\int_{-1}^{1} x\,dx = 0$.

(17.9.8) $\operatorname{div} F = 3x^2y - 2x^2y - x^2y = 0$, so $\iint_S F \cdot dS \overset{\text{div thm}}{=} \iiint_E 0\,dV = \boxed{0}$.

(17.9.17) On S_1, $\vec{n} = -\vec{k}$, so $F \cdot \vec{n} = F \cdot (-\vec{k}) = -x^2z - y^2 = -y^2$ since $z = 0$ on S_1.

Therefore $\iint_{S_1} F \cdot dS = \iint_{S_1} F \cdot \vec{n}\,dS = \iint_D (-y^2)\,dA \overset{\text{polar}}{=} \int_0^{2\pi} \int_0^1 -r^2 \sin^2\theta \, r\,dr\,d\theta$

$= \int_0^{2\pi} \sin^2\theta\,d\theta \int_0^1 -r^3\,dr = -\frac{1}{4}\pi$.

Using the book's hint, S_2 is closed, so we apply the divergence theorem to compute $\iint_{S_2} F \cdot dS$.

We have $\operatorname{div} F = z^2 + y^2 + x^2$, so

$\iint_{S_2} F \cdot dS = \iiint_E \operatorname{div} F\,dV \overset{\text{spherical coord.}}{=} \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^2 \rho^2 \sin\varphi \,d\rho\,d\varphi\,d\theta = \frac{2}{5}\pi$.

Therefore $\iint_S F \cdot dS = \iint_{S_2} F \cdot dS - \iint_{S_1} F \cdot dS = \frac{2}{5}\pi + \frac{1}{4}\pi = \frac{13}{20}\pi$.

17.9.24) The unit normal to the sphere S is $\vec{n} = x\vec{i} + y\vec{j} + z\vec{k}$. If we let $F = 2\vec{i} + 2\vec{j} + z\vec{k}$, then $F \cdot \vec{n} = 2x + 2y + z^2$. Therefore

divergence thm, E = unit ball

$$\iint_S (2x+2y+z^2)\,dS = \iint_S F \cdot \vec{n}\,dS = \iint_S F \cdot dS = \iiint_E \operatorname{div} F\,dV$$

$$= \iiint_E 1\,dV = \operatorname{vol}(\text{unit ball}) = \frac{4}{3}\pi.$$