1 2.3.18

No, if $C\mathbf{x} = \mathbf{v}$ is consistent for all $\mathbf{v} \in \mathbb{R}^6$ then C is an 6×6 matrix which is onto, hence by theorem 8 $(i) \implies (f)$ we have that the transformation $\mathbf{x} \mapsto C\mathbf{x}$ is one-to-one.

2 4.1.2

a) if $\mathbf{u} \in W$ then $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ with $xy \geq 0$

Now $c\mathbf{u} = c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$

and $cxcy = c^2xy \geq 0$ since $c^2 \geq 0$ and $xy \geq 0$ hence $c\mathbf{u} \in W$

b)let $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $\mathbf{u}, \mathbf{v} \in W$ since $0 \times -1 = 1 \times 0 = 0 \geq 0$ but $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $1 \times -1 = -1 < 0$ hence $\mathbf{u} + \mathbf{v} \notin W$

3 4.1.8

Let W be the subset of polynomials in $\mathbb{P}^n$ such that $\mathbf{p}(0) = 0$ is a subspace of $\mathbb{P}^n$ since if $\mathbf{p}(0) = 0$ and $\mathbf{p}'(0) = 0$ then
a)$0(0) = 0$ hence $0 \in W$
b)$(\mathbf{p} + \mathbf{p}')(0) = \mathbf{p}(0) + \mathbf{p}'(0) = 0 + 0 = 0$ hence $\mathbf{p} + \mathbf{p}' \in W$ and
c)if $c \in \mathbb{R}$ then $(c\mathbf{p})(0) = c(\mathbf{p}(0)) = c \times 0 = 0$ hence $c\mathbf{p} \in W$ thus W is a subspace of $\mathbb{P}^n$

Note: b and c of the definition of subspace together imply condition a since $0\mathbf{u} = (0 + 0)\mathbf{u} = 0\mathbf{u} + 0\mathbf{u}$ which implies $\mathbf{0} = 0\mathbf{u}$

4 4.1.12

Note if$\mathbf{u} \in W$ then $\mathbf{u} = \begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix}$ for some s and t.

Hence $\mathbf{u} \in span \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} , \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix} \right\}$ Hence

$$W \subseteq span \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} , \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix} \right\}$$

Similarly, since W is the set of all vectors of the form $\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix}$ we have

$$span \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} , \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix} \right\} \subseteq W$$

hence

$$W = span \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} , \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix} \right\}$$

and by theorem 1 we have $span \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} , \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix} \right\}$ is a subspace thus W is as well.

5 4.1.22

If $A, B \in H$ and $c \in \mathbb{R}$ then
a) the zero matrix, $\mathbf{0}$ is in H since $F\mathbf{0} = 0$

b) $A+B \in H$ since $F(A+B) = FA + FB = 0 + 0 = 0$
c) $cA \in H$ since $F(cA) = cFA = c \times 0 = 0$

Hence H is a subspace of$M_{2\times 4}$

6 4.2.4

If $\mathbf{u} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is in the null space of A then

$$0 = A\mathbf{u} = \begin{bmatrix} x_1 - 6x_2 + 4x_3 \\ 2x_3 \end{bmatrix}$$

Hence we must have $x_3 = 0$ $x_1 = 6x_2$ and x_2, x_4 are free, or

$$\mathbf{u} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6x_2 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus as in example 3 we have $span \left\{ \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = NulA$

7 4.2.10

Let

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{array}{c} a + 3b = c \\ b + c + a = d \end{array} \right\}$$

Notice since $c = a + 3b$ we have $b + c + a = d \Rightarrow b + 3b + a + a = d \Rightarrow 4b + 2a = d$. Hence if $\mathbf{u} \in W$ if and only if

$$\mathbf{u} = \begin{bmatrix} a \\ b \\ a + 3b \\ 4b + 2a \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix}$$

which implies $W = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 4 \end{bmatrix} \right\}$

which implies by theorem 1 that W is a vector space.

8 4.2.28

Let $A = \begin{bmatrix} 5 & 1 & -3 \\ -9 & 2 & 5 \\ 4 & 1 & -6 \end{bmatrix}$ then the first system of equations implies that $\begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix}$ is in the column space of A. Since the column space of A is a vector space we have that for any $c \in \mathbb{R}$ that $c \begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix}$ is in the column space of A which implies that $5 \begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 45 \end{bmatrix}$ is in the column space of A. Hence $A\mathbf{x} = \begin{bmatrix} 0 \\ 5 \\ 45 \end{bmatrix}$ has a solution.