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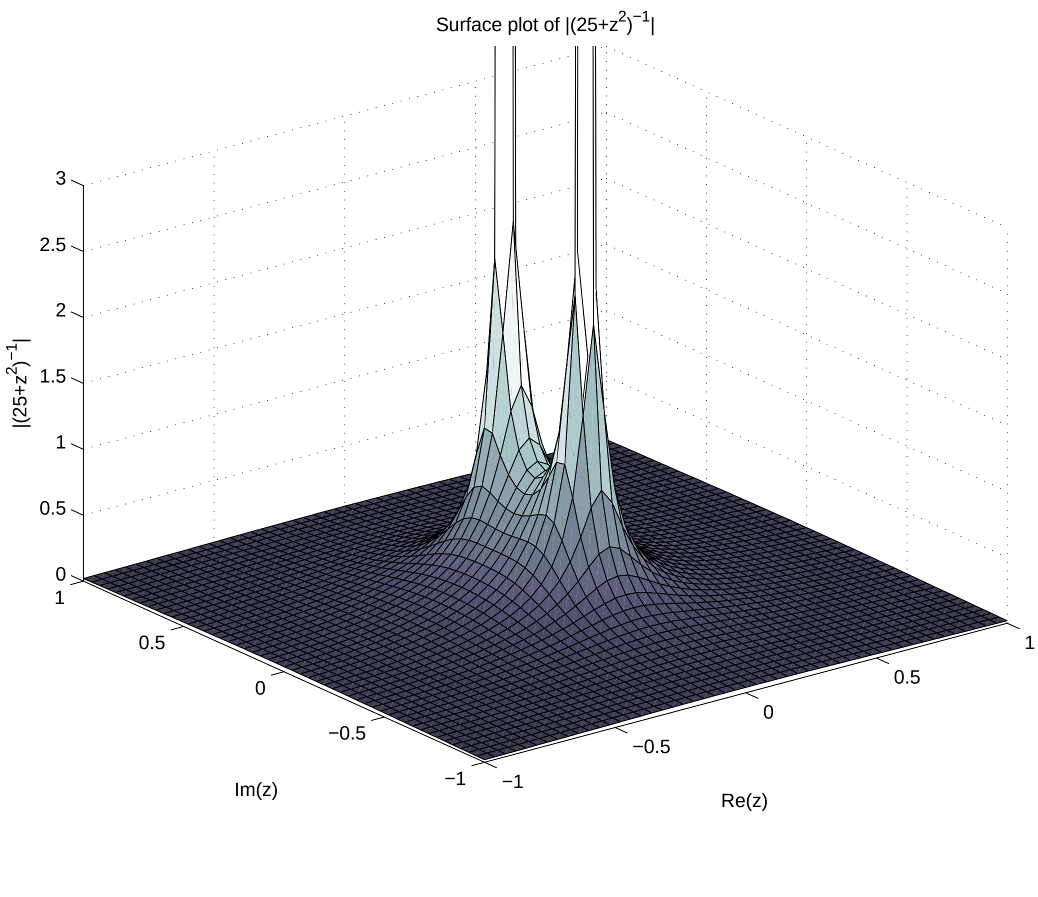
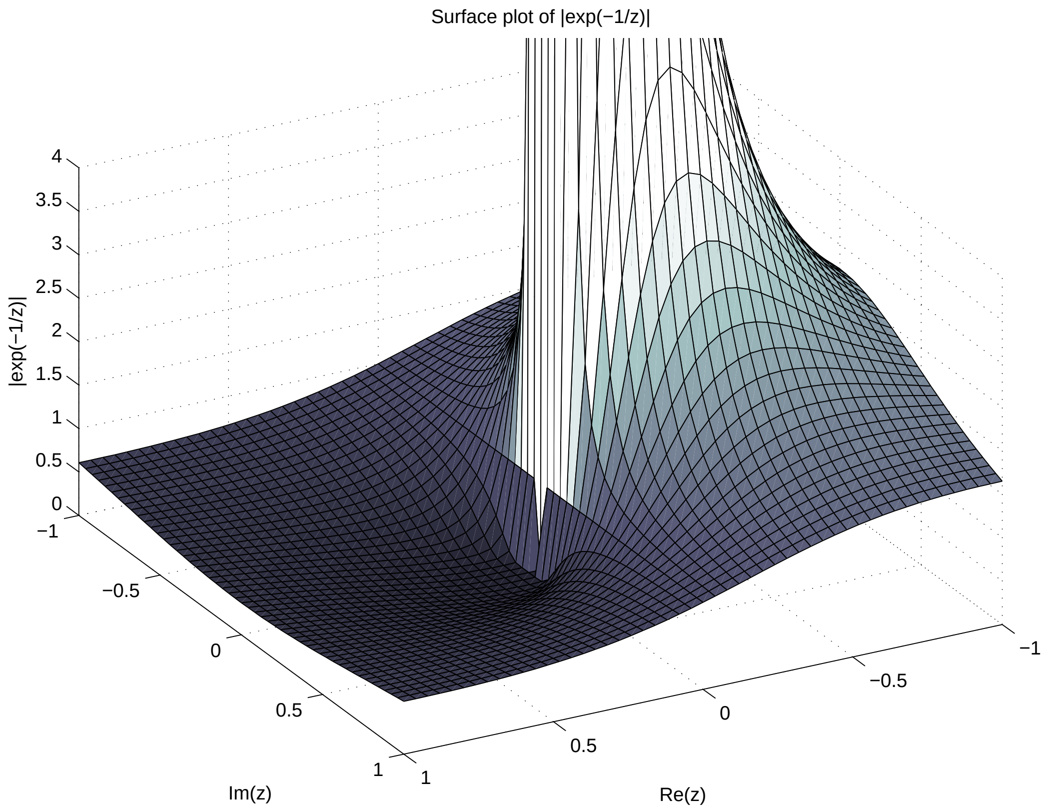
MATH 126 - Dr. Barnett
Homework 3
due January 30, 2011

Problem 1. Given a complex-analytic function defined on some punctured domain, say $\Omega \setminus \{p\}$, we consider p to be an isolated singularity. The classification theorem of isolated singularity then says that any such singularity must be either removable, a pole, or an essential singularity. Intuitively speaking, we say that a removable singularity is one such that we can "patch up" the function so that it will be analytic on all of Ω. We say that a pole is one such that its modulus spikes up to infinity as we approach it, in more mathematical language, p is a pole if $|f(z)| \to \infty$ as $z \to p$. Lastly, an essential singularity is one such it is neither removable, nor a pole; these singularities have the amazing property that the image of any punctured neighborhood around them is dense in $\mathbb{C}$.

For part (a), we are given the function $f(z) = 1/(25 + z^2)$, to which we can easily show that there are isolated singularities at $\pm i/5$. As the figure below illustrates, these singularities are poles by the observation that $|f(\pm i/5)| \to \infty$ as $z \to \pm i/5$.



Similarly, given the function $f(z) = \exp(-1/z)$, we have a singularity at $z = 0$; however, the figure below demonstrates that this is neither a removable singularity, nor does it have the nice property of a pole. This means that the singularity at $z = 0$ must be an essential singularity.



Problem 2. Observe that

$$\int_a^b x^m \, dx = (b^{m+1} - a^{m+1})/(m+1).$$

Supposing there were weights such that Newton-Cotes integrated x^m exactly over $[a,b]$, we would have

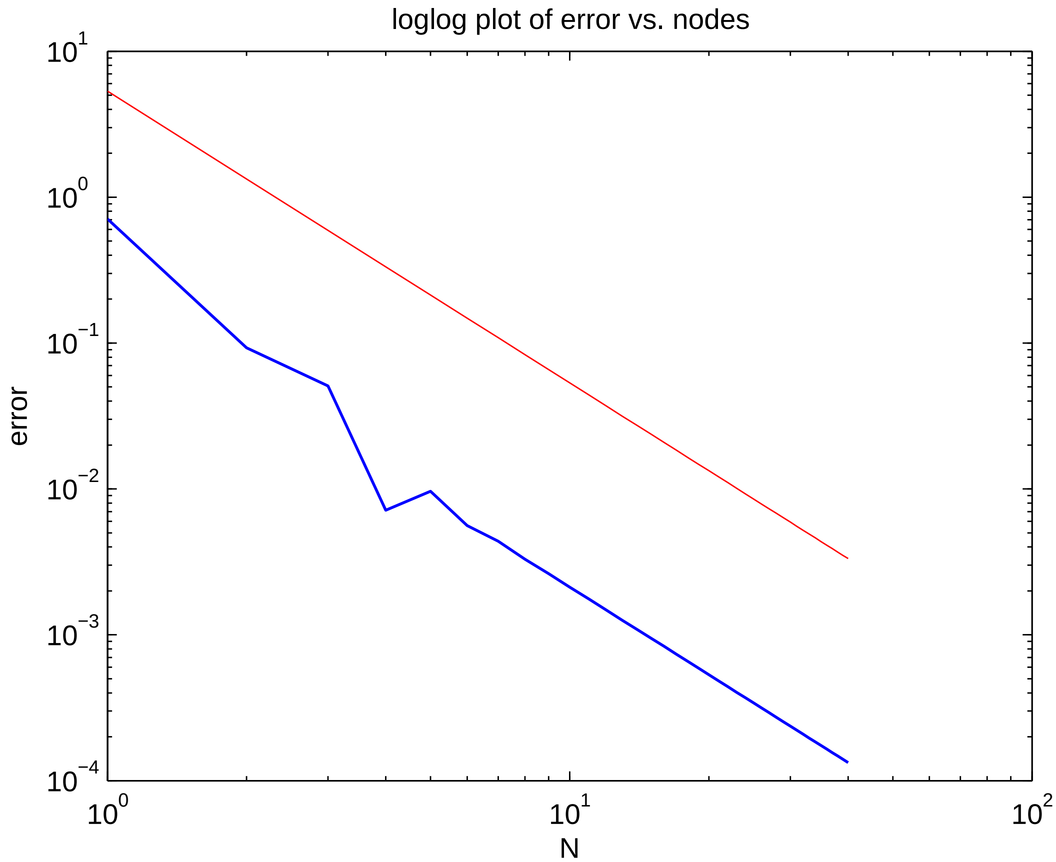
$$\sum_{j=0}^{m} w_j x_j^m = (b^{m+1} - a^{m+1})/(m+1).$$

Given distinct points $\{x_j\}_{j=0}^n \subset [a,b]$, we have a system of equations $Aw = c$ for $m = 0, 1, \ldots, n$ which translates to

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_0 & x_1 & x_2 & \cdots & x_n \\ x_0^2 & x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_0^n & x_1^n & x_2^n & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} b - a \\ (b^2 - a^2)/2 \\ (b^3 - a^3)/3 \\ \vdots \\ (b^{n+1} - a^{n+1})/(n+1) \end{bmatrix}$$

where the matrix is nonsingular since it is Vandermonde. The solution to this system exists and is *unique*, which says precisely that there are unique weights which integrate any polynomial up to degree n over the interval $[a,b]$.

Problem 3. For part (a), we calculated the integral of $f(x) = (1 + 4x^2)^{-1}$ with the trapezoid rule for $N + 1$ equally-spaced nodes.



We showed in class that

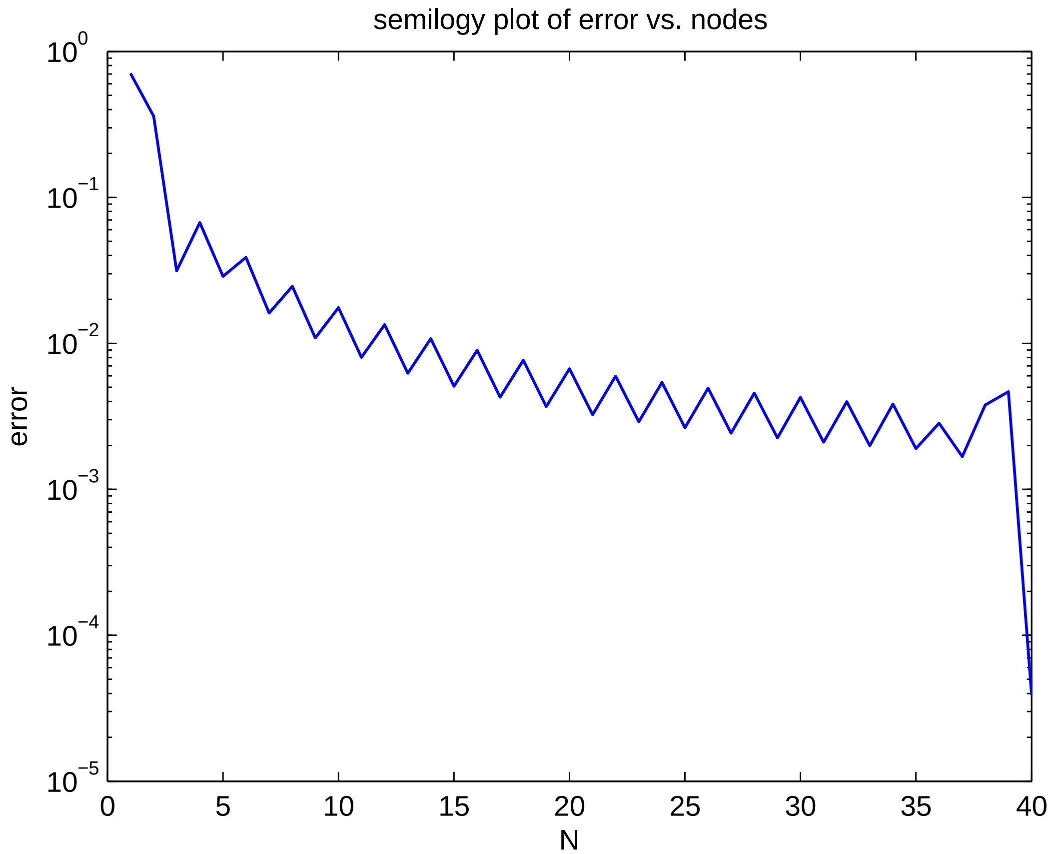
$$\text{Error} \leq \frac{(b-a)^3}{12N^2}||f''||_\infty.$$

In this instance, it is a trivial, yet tedious, exercise to show that $||f''||_\infty = 8$, and so

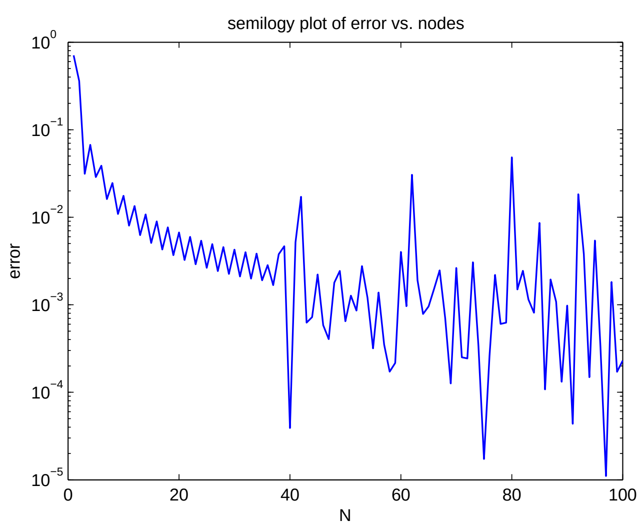
$$\text{Error} \leq \frac{16}{3N^2}$$

on the interval $[-1, 1]$. This error bound is illustrated by the red line in the above figure.

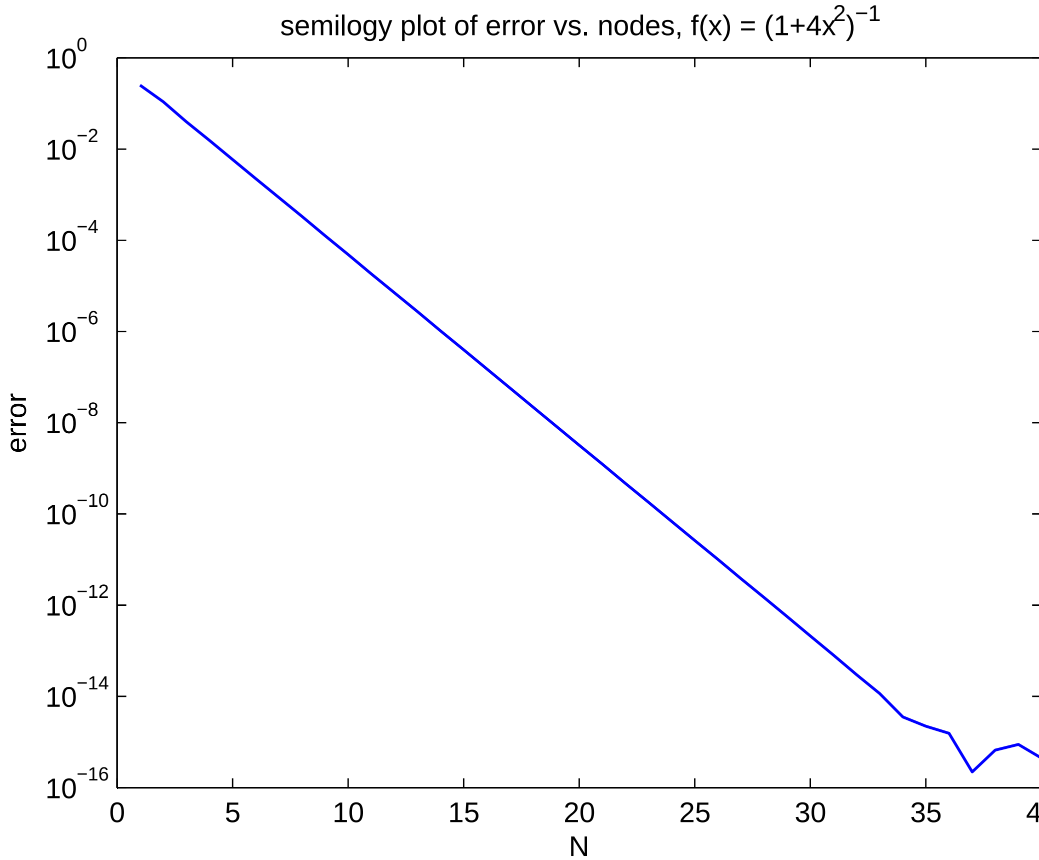
For part (b), we calculated the same integral using Newton-Cotes by solving the system of equations presented in Problem 2.



This approach posed some problems, namely that around $N = 33$, the Vandermonde matrix became singular. At this point, we can no longer trust the weights found by solving the system of equations, and so we expect no better than approximately 10^{-3} error. In fact, this caused our error to become erratic for $N > 40$, as the figure below demonstrates.



Problem 4. For part (a), we have $f(x) = (1 + 4x^2)^{-1}$, whose error converges exponentially using Gaussian quadrature. Using $(10, 4.855 \times 10^{-5})$ and $(25, 2.624 \times 10^{-11})$ as sample points, it follows that the error fits approximately $E(N) = 0.7317e^{-0.9621N}$. This says that this quadrature method converges exponentially to the analytic answer at a rate of $\alpha \approx -0.9621$.

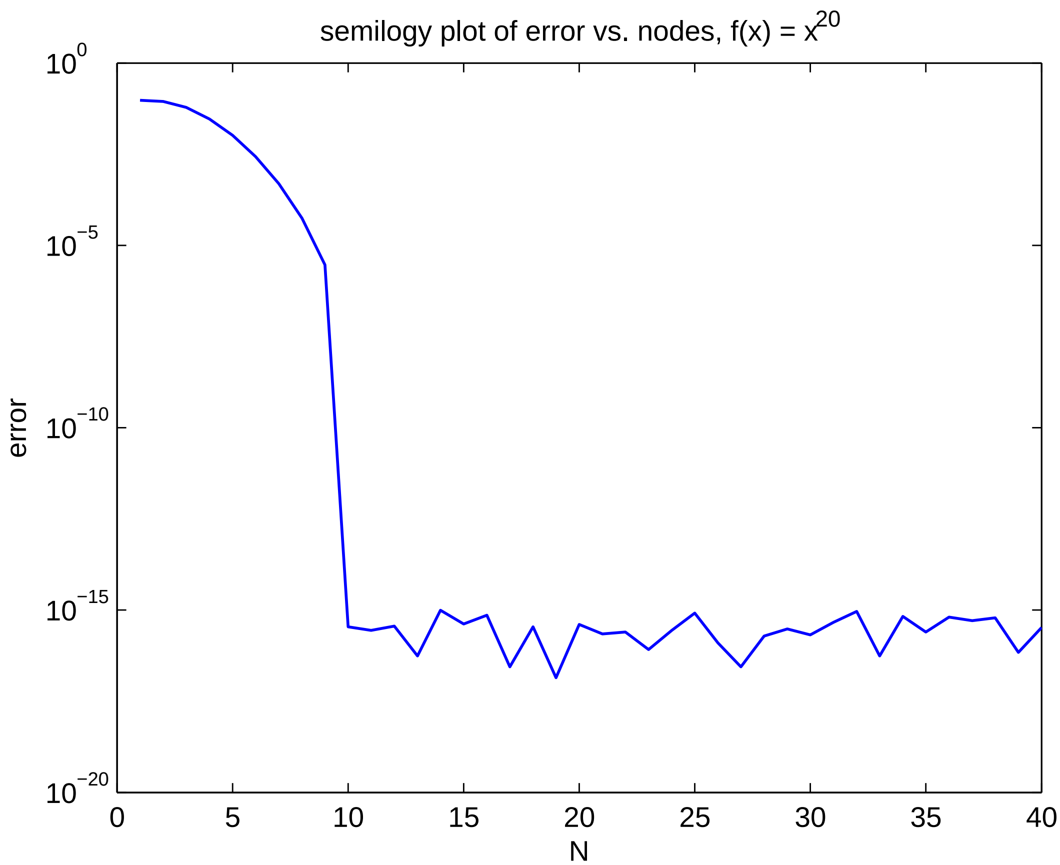


It is plainly seen by comparing this figure to those in Problem 3 that this method has vastly superior convergence. For $N = 33$ or so, we basically obtain error that is near machine precision, which is about 10-12 orders of magnitude better than for either method used in Problem 3.

For part (b), we have $f(x) = x^{20}$. For Gaussian quadrature, we know that

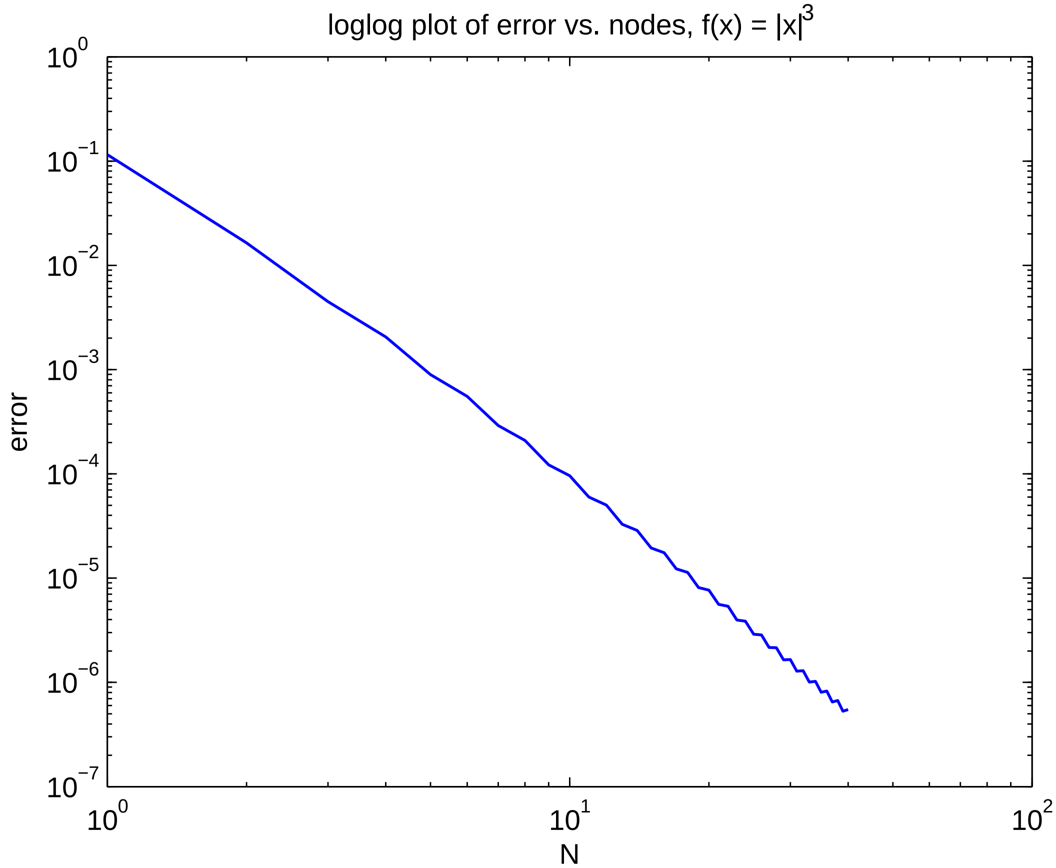
$$\text{Error} \leq \frac{||f^{(2n+2)}||_\infty}{(2n+2)!} \int_{-1}^{1} q_{n+1}^2(x)\,dx \qquad \text{where} \qquad q_{n+1}(x) = \prod_{j=0}^{n}(x - x_j).$$

This means that when $N = 10$, we need to take the 22nd derivative of x^{20}, which is zero, and so the convergence is neither exponential nor algebraic, but rather *exact*.



Even though this scheme predicts that we should obtain the exact value at $N = 10$, we actually observe that this does not happen. This occurs due to the floating point error inherent in our calculations. Despite not having exact convergence, this demonstrates the fact that we can calculate the exact integral of a monomial very easily with Gaussian quadrature.

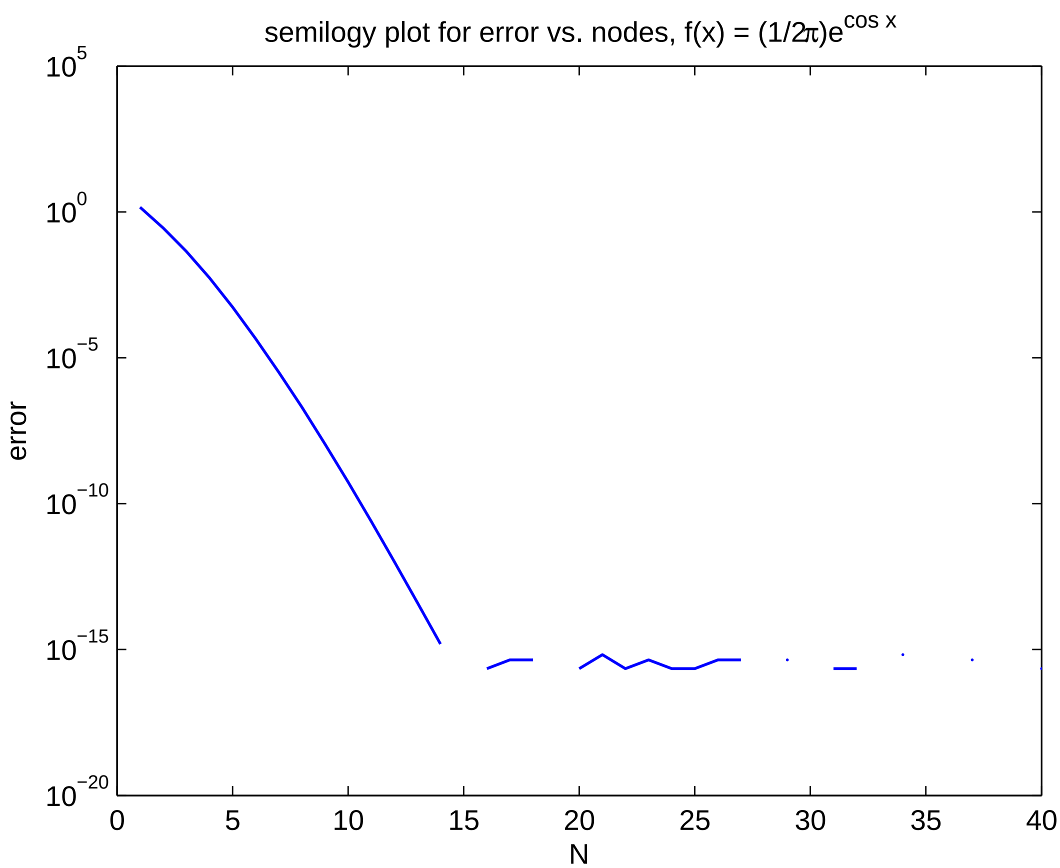
For part (c), we considered $f(x) = |x|^3$. As the figure below suggests, we obtain algebraic convergence since the error is linear in a loglog plot.



Using $(7, 2.909 \times 10^{-4})$ and $(38, 6.687 \times 10^{-7})$ as sample points, we can approximate the error in this figure by $E(N) = 0.3153N^{-3.591}$. This says that the algebraic order is ≈ -3.591.

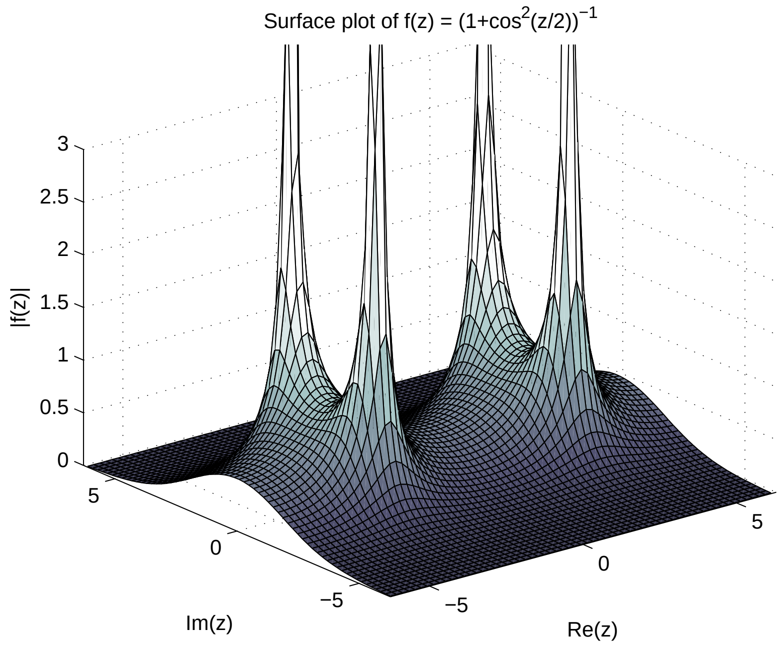
Problem 5. I seriously tried to do this problem, but none of the hints, nor any of the email hints helped me solve this problem. I'm sure I'm missing some obvious trick or some intuitive way of simplifying matters... I just don't see it.

Problem 6. For part (a), since exp and cos are both entire functions, then $f(z)$ is likewise entire in $\mathbb{C}$, and so there are no singularities. In this case, $f(z)$ has no singularities and we expect very fast exponential convergence.

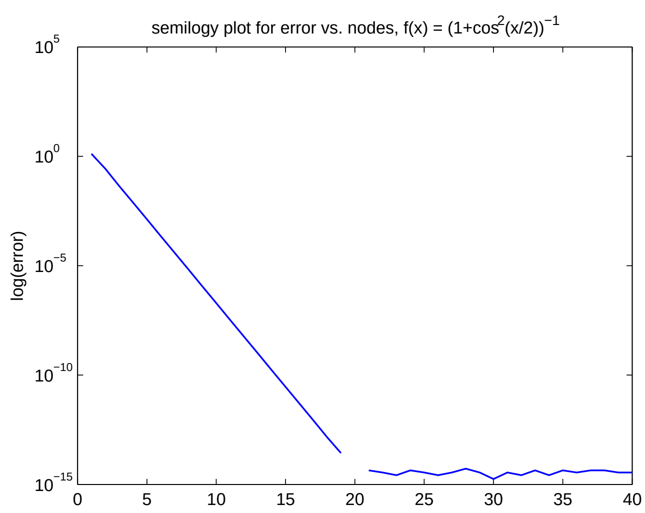


For $N < 15$, the error is nearly linear in the logplot, hence we have exponential convergence as expected. While the plot may look bizarre, the gaps in the figure above, in fact, are the points where there is no error! Again, for all $N > 15$, we either have no error or error on the order of machine precision.

For part (b), we consider $f(x) = (1+\cos^2(x/2))^{-1}$. As the following figure illustrates, we expect there to be poles at approximately $\pm\pi \pm 1.76i$.

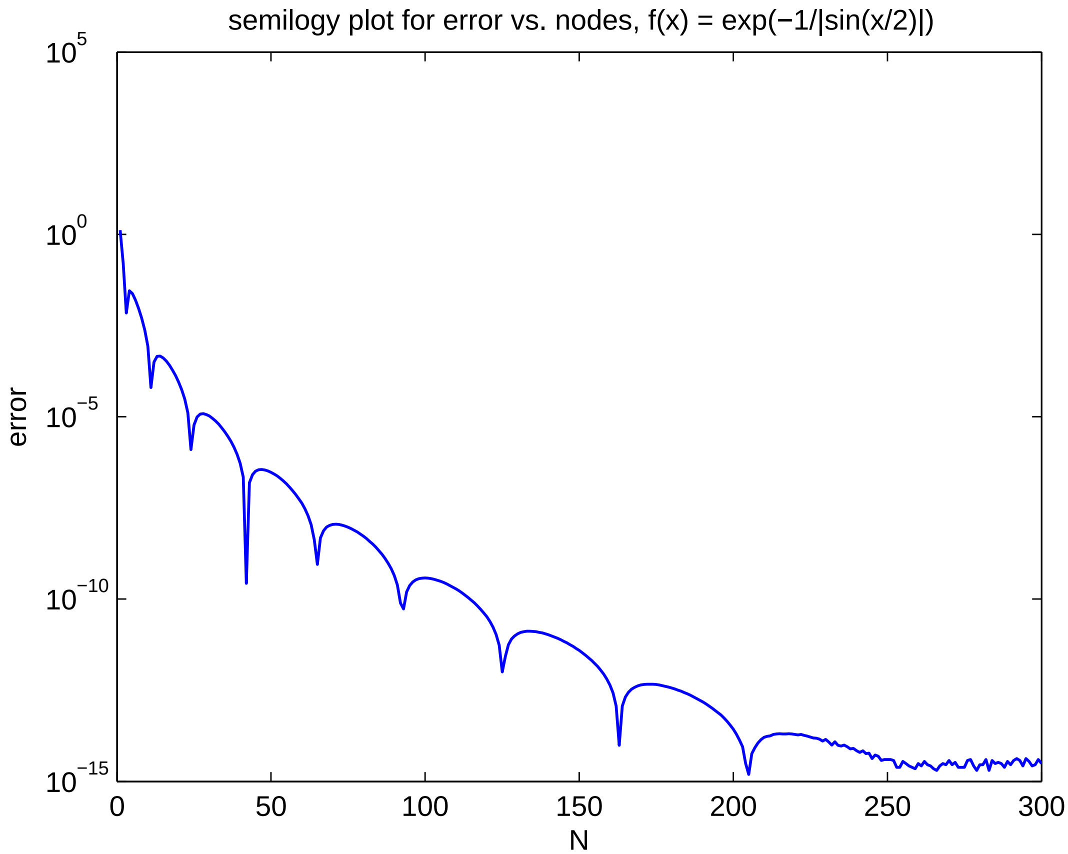


Since we are integrating over the real line, this tells us that our rate of exponential convergence should be approximately -1.76. As we see in the next figure, we do in fact have exponential convergence.

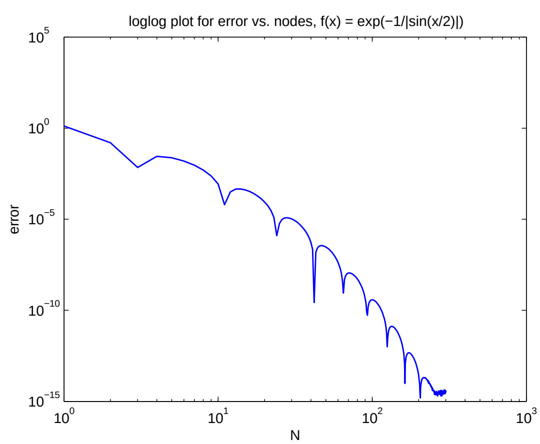


Using $(5, 1.321 \times 10^{-3})$ and $(18, 1.439 \times 10^{-15})$ as sample points, the error can be given by $E(N) \approx 8.969e^{-1.765N}$. This says that our rate of convergence is ≈ -1.765, which is precisely what we predicted by identifying the poles of $f(z)$!

Lastly, for part (c), we have $f(x) = \exp(-1/|\sin(z/2)|)$, which has a removable singularities at 0. As such, this function is not real analytic, yet it is C^∞, and so we expect there to be algebraic convergence.



Since this is not linear in the semilogy plot, we certainly don't have exponential convergence; however, the loglog plot does appear to converge algebraically for $N > 50$.



Using $(53, 2.043 \times 10^{-7})$ and $(185, 2.531 \times 10^{-13})$ as sample points, I found the algebraic rate of convergence to be approximately -10.88, i.e. $E(N) = \mathcal{O}(N^{-10.88})$.