Homework #8

1) Page 346 #6

$y u_{xx} - 2u_{xy} + x u_{yy} = 0.$

$a = y \quad b = -2 \quad c = x$

discriminant is $b^2 - 4ac = 4 - 4xy = 4(1-xy)$

The eqn is parabolic when $1 - xy = 0.$

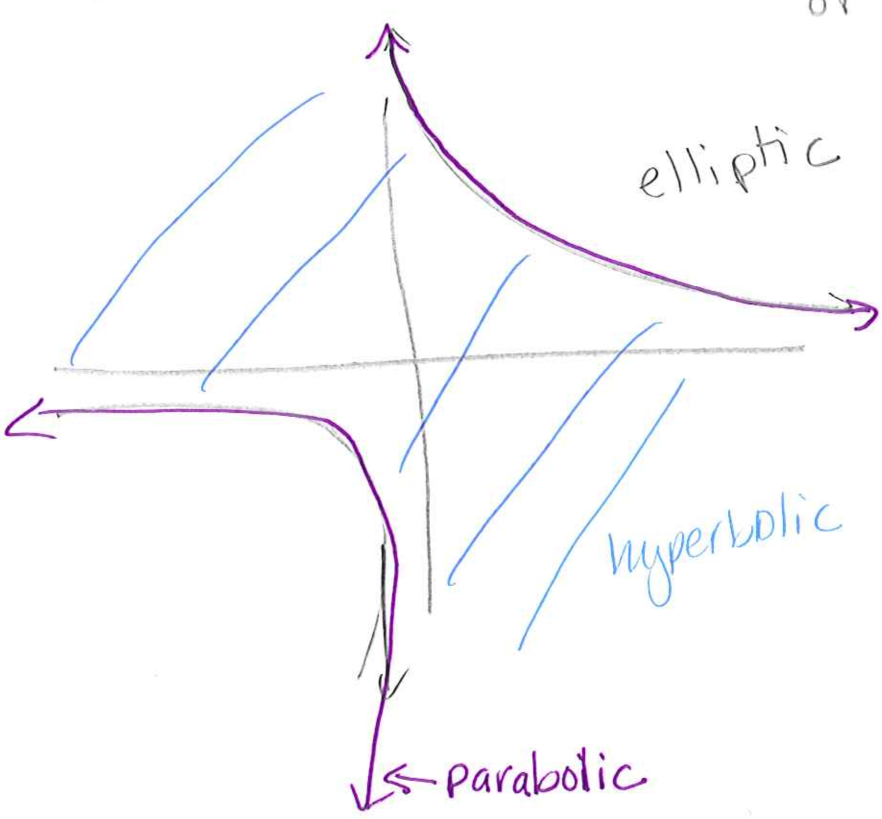
$\rightarrow y = 1/x$

The eqn is elliptic when $1 - xy < 0$

or $xy > 1$

The eqn is hyperbolic when $1 - xy > 0$

or $xy < 1.$



2) Page 345 #2 c.

(a) $u_{xx} + u = 6y$

$\underbrace{6y}_{\text{constant wrt } x.}$

1st find homogeneous solution.

$u_{xx} + u = 0$.

$\Rightarrow$ $r^2 + 1 = 0 \Rightarrow r = \pm i$

$u_h(x,y) = C_1(y)\cos(x) + C_2(y)\sin(x)$

Particular solution

$u_p(x,y) = 6y$.

$\Rightarrow$ Solution is $u = u_h + u_p = C_1(y)\cos(x) + C_2(y)\sin x + 6y$.

(d) $u_{tx} + u_x = 1$

- let $v = u_x$

$v_t + v = 1 \Rightarrow v = C(x)e^{-t} + 1$

$\Rightarrow u_x = C(x)e^{-t} + 1$

$\Rightarrow u(x,t) = x + A(x)e^{-t} + B(t)$

(e) $u u_t = x - t = \frac{1}{2}\frac{d}{dt}(u^2) = x-t$

$\Rightarrow \frac{d}{dt}(u^2) = 2(x-t)$

$\Rightarrow u^2(x,t) = 2xt - t^2 + C(x)$

$\Rightarrow u(x,t) = \pm\sqrt{2xt - t^2 + C(x)}$

3) Page 345 #3.

$(*)\ u_{xt} = f(x,t)$. $\quad x, t > 0$.

$u(x,0) = g(x) \quad x > 0$.

$u(0,t) = h(t)$

$g(0) = h(0) \quad g'(0) = h'(0)$.

1st integrate $(*)$ wrt t.

$$u_x'(x,t) = \int_0^t f(x,s)\,ds + a(x).$$

now wrt x.

$$u(x,t) = \int_0^x \int_0^t f(v,s)\,ds\,dv + A(x) + B(t).$$

IC. $u(x,0) = A(x) + B(0) = g(x)$.

$A(x) = g(x) - B(0)$.

BC: $u(0,t) = A(0) + B(t) = h(t) \rightarrow B(t) = h(t) - A(0)$.

at $t=0$ $u(0,0) = A(0) + B(0) = h(0) = g(0)$

$A(0) = h(0) - B(0)$

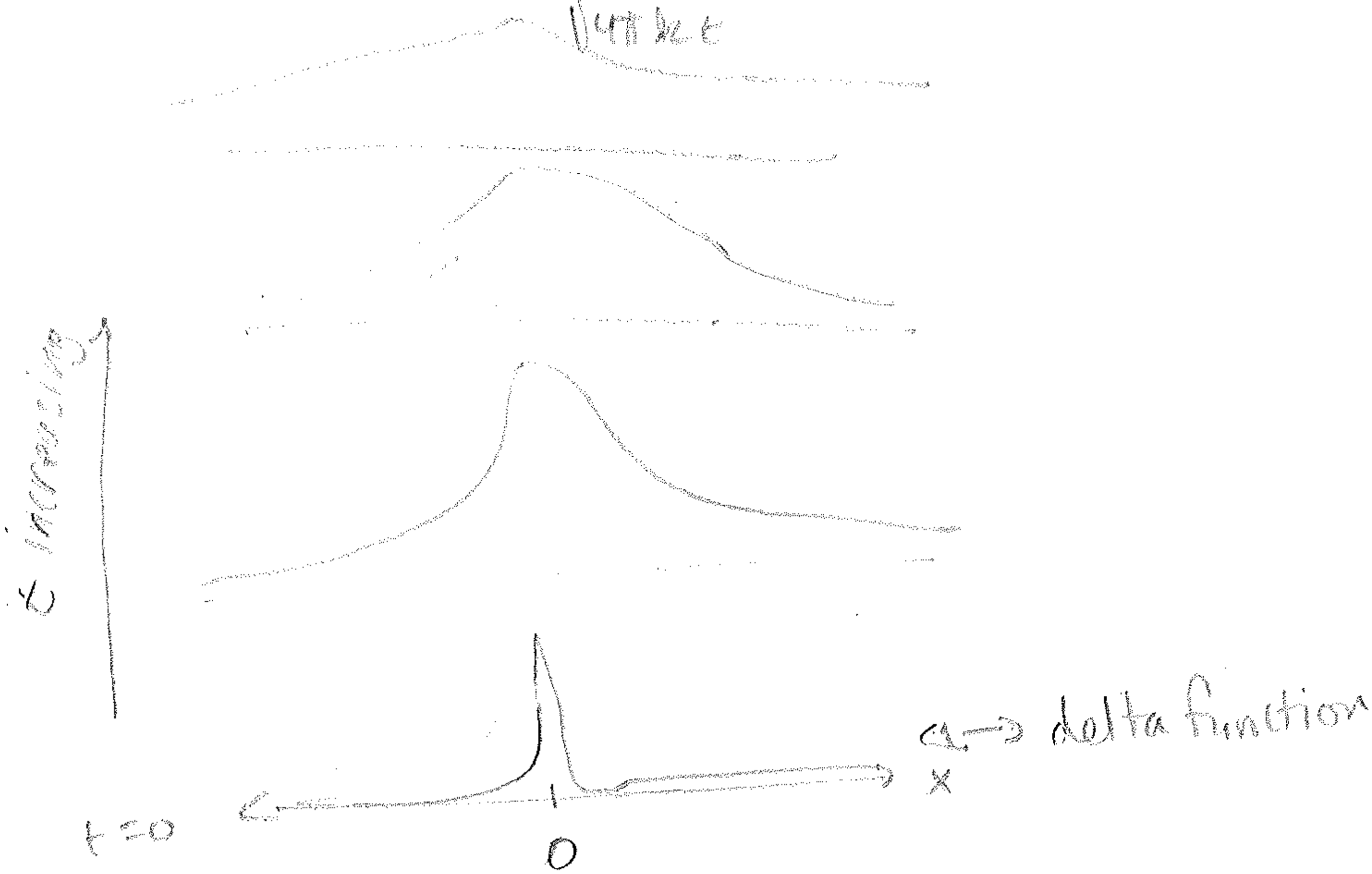
So $B(t) = h(t) - h(0) + B(0)$.

$$\text{So } u(x,t) = \int_0^x \int_0^t f(v,s)\,ds\,dv + g(x) - B(0) + h(t) - h(0) + B(0)$$

$$= \int_0^x \int_0^t f(v,s)\,ds\,dv + g(x) + h(t) - h(0)$$

4) Page 345 #1

$$u(x,t;0) = \frac{e^{-x^2/4\pi k t}}{\sqrt{4\pi k t}}$$



as $t \to 0^+$ $u(x,t;0) \to \delta(x)$.

Changing k changes the width of the gaussian.

3) Page 365 #3

$$\begin{cases} u_t = \Delta u & x \in \Omega \quad t>0 \\ u(x,0) = f(x) & x \in \Omega \\ u(x,t) = g(x) & x \in \partial\Omega \quad t>0 \end{cases}$$

Suppose $\exists$ 2 solutions u_1, u_2.
let $W = u_1 - u_2$. The W satisfies the PDE

$$\begin{cases} W_t = \Delta W & x \in \Omega \\ W(x,0) = 0 & x \in \Omega \\ W(x,t) = 0 & x \in \partial\Omega \quad t>0 \end{cases}$$

let $E(t) = \int_\Omega W^2(x,t)\,dx$ Goal: show $W=0$.

① We know $E(0) = \int_\Omega (W(x,0))^2\,dx = 0$.
would like to show $E=0$.
② We also know $E(t) \geq 0 \quad \forall t$

$$E'(t) = 2\int_\Omega W W_t\,dx$$
$$= 2\int_\Omega W \Delta W\,dx = 2\left[-\int_\Omega \nabla W \nabla W\,dx + \int_{\partial\Omega} \frac{\partial W}{\partial n} W\,ds\right]$$

$$= 2 - \int_\Omega (\nabla W)^2\,dx + 0 \text{ By BC.}$$

$\Rightarrow E(t)$ is decreasing. adding this to ①&②
mean $E(t) = 0$.
$\Rightarrow W(x,t) = 0$
$\therefore$ Solution is unique.

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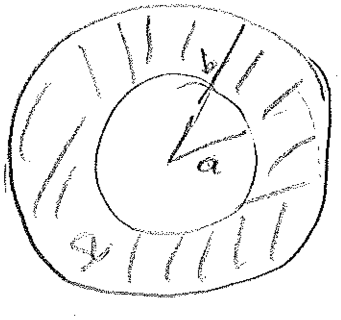
⑥ Page 365 #5

let $u(r,\theta,t)$ = density



total mass of u in Ω at time t

$$= \int_0^{2\pi}\int_a^b u(r,\theta,t)\, r\, dr\, d\theta$$

we can ignore this since rotationally symmetric

$$= 2\pi \int_a^b u(r,t)\, r\, dr$$

The net influx. Change $= \int_{\partial\Omega} \bar{J}\cdot\bar{n}\, ds = 2\pi a J(a,t) - 2\pi b J(b,t)$

Conservation law says the change in density = Change in Flux.

$$\Rightarrow \frac{d}{dt} 2\pi \int_a^b u(r,t)\, r\, dr = 2\pi\left[a J(a,t) - b J(b,t)\right]$$

$$\Rightarrow \int_a^b u_t(r,t)\, r\, dr = \int_a^b \frac{d}{dr}(r J(r,t))\, dr$$

Since this is true $\forall t$,

$$u_t(r,t)\, r = -\frac{d}{dr}(r J(r,t))$$

$$\Rightarrow u_t(r,t) = -\frac{1}{r}\frac{d}{dr}(r J(r,t)) = +\frac{D}{r}\frac{d}{dr}(r u_r)$$

7) Page 366 #11

$$u_t - k u_{xx} = 0 \qquad x > 0 \quad t > 0.$$

$$u(0,t) = 1 \qquad u(\infty, t) = 0 \qquad t > 0.$$

$$u(x,0) = 0 \qquad x > 0.$$

let $U(z) = u(x,t)$ where $z = \frac{x}{\sqrt{kt}}$

$$u_t = U'(z)\frac{dz}{dt} = U'(z)\left(\frac{x}{\sqrt{k}} \frac{-1}{2} t^{-3/2}\right)$$

$$U_x = U'(z)\frac{dz}{dx} = \frac{U'(z)}{\sqrt{kt}}$$

$$U_{xx} = \frac{d}{dx}(U_x) = \frac{U''(z)}{\sqrt{kt}}\frac{dz}{dx} = \frac{U''}{kt}$$

$$\text{so } U_t - k U_{xx} = \frac{-xU'(z)}{2\sqrt{k}\, t^{3/2}} - \frac{k}{kt}U'' = 0$$

$$\Rightarrow \quad \frac{-z}{2}U' - U'' = 0.$$

let $v = U'$

$$\frac{v'}{v} = \frac{-z}{2} \Rightarrow \ln v = -\frac{z^2}{4} + C$$

$$v = Ce^{-z^2/4}$$

$$\Rightarrow \quad U(z) = c\int_0^z e^{-s^2/4}\, ds + d$$

We need

① $\lim_{z\to\infty} U(z) = 0 = c \lim_{z\to\infty} \int_0^z e^{-s^2/4}\, ds + d.$

② & $U(0) = d = 1$

now $\int_0^\infty e^{-s^2/4}\, ds = \frac{\sqrt{\pi}}{2}$.

① + ② $\Rightarrow c = -\frac{2}{\sqrt{\pi}}$

So $U(z) = -\frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2/4}\, ds + 1$

$\Rightarrow u(x,t) = 1 - \mathrm{erf}(x/\sqrt{4kt})$

8) Page 367 #13

$$\begin{cases} u_t = u_{xx} - u^3 & 0<x<l \quad t>0 \\ u(0,t) = 0 \quad u(l,t) = 0. & t>0 \\ u(x,0) = 0 & 0<x<l. \end{cases}$$

Use an energy method to show the only solution is $u=0$.

let $E(t) = \int_0^l (u(x,t))^2 \, dx$

So $E(t) \geq 0$. $\quad E(0) = \int_0^l (u(x,0))^2 \, dx = 0.$

$$E'(t) = \int_0^l 2uu_t \, dx = 2\int_0^l u(u_{xx} - u^3) \, dx.$$

$$= 2\left[\int_0^l \underbrace{u}_{u} \underbrace{u_{xx} \, dx}_{dV} - \int_0^l u^4 \, dx\right]$$

$$= 2\left[\underbrace{uu_x\Big|_0^l}_{=0 \text{ by BC.}} - \int_0^l (u_x)^2 dx - \int_0^l u^4 dx\right]$$

$$= -2\left[\underbrace{\int_0^l [(u_x)^2 + u^4] dx}_{\geq 0}\right] \leq 0$$

So $E(t)$ is decreasing, positive starting at 0.

$\Rightarrow E(t) = 0 \Rightarrow u(x,t) = 0$.

There is no other option.

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