Math 11

Section 3

Monday, September 29, 2008

Example Find the distance between the point $P = (4,\ -3,\ 5)$ and the plane containing points $Q = (3,\ 0,\ 0)$, $R = (0,\ -3,\ 0)$, and $S = (0,\ 0,\ 3)$ using two different methods:

A. Let T be the point on the plane that is closest to point P; that is, the vector $\vec{PT}$ is perpendicular to the plane. Draw a picture showing the plane, the points P and T, and another point on the plane, say Q. Notice that the vector $\vec{PT}$ is the projection of the vector PQ onto a vector perpendicular to the plane; use this fact to find the length of the vector $\vec{PT}$. That length is the distance between P and the plane.

Solution: Two vectors parallel to the plane are $\vec{QR} = \langle -3,\ -3,\ 0\rangle$ and $\vec{QS} = \langle -3,\ 0,\ 3\rangle$. A vector perpendicular to the plane is their cross product

$$\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & -3 & 0 \\ -3 & 0 & 3 \end{vmatrix} = \langle -9, 9, -9\rangle .$$

The projection of $\vec{PQ}$ on $\vec{v}$ is either $\vec{PT}$ or $-\vec{PT}$, depending on whether $\vec{v}$ points from the plane toward or away from P. In either case, the length of $\vec{PT}$ is the length of the projection of $\vec{PQ}$ on $\vec{v}$, which is the absolute value of the component of $\vec{PQ}$ in the direction of $\vec{v}$. This is

$$\left| \frac{\vec{PQ} \cdot \vec{v}}{|\vec{v}|} \right| = \left| \frac{\langle -1,\ 3,\ -5\rangle \cdot \langle -9,\ 9,\ -9\rangle}{|\langle -9,\ 9,\ -9\rangle|} \right| = \frac{81}{9\sqrt{3}} = 3\sqrt{3}.$$

B. Again, let T be the point on the plane that is closest to point P; that is, the vector $\vec{PT}$ is perpendicular to the plane. Find a (scalar) equation for the plane and a (vector parametric) equation for the line through P perpendicular to the plane. Notice that T is the point of intersection of this line with the plane. Use this fact to find T and the distance between P and T. That distance is the distance between P and the plane.

Solution: In part (A) we found a vector perpendicular to the plane, $\langle -9,\ 9,\ -9\rangle$. Another vector perpendicular to the plane (and one that is easier to deal with) is $\langle a,\ b,\ c\rangle = \langle -1,\ 1,\ -1\rangle$.

We know an equation for the plane has the form $ax + by + cz = d$, where a, b and c are the components of a vector perpendicular to the plane. Also, any point on the plane, say $Q = (3,\ 0,\ 0,\ )$ must satisfy any equation of the plane, so an equation of the plane is

$$ax + by + cz = a(3) + b(0) + c(0),$$

$$-x + y - z = -3.$$

An equation for the line through $P = (4,\ -3,\ 5)$ perpendicular to the plane, that is, parallel to a vector perpendicular to the plane, is

$$\langle x,\ y,\ z\rangle = \langle 4,\ -3,\ 5\rangle + t\,\langle -1,\ 1,\ -1\rangle\,,$$

$$\langle x,\ y,\ z\rangle = \langle 4 - t,\ -3 + t,\ 5 - t\rangle\,.$$

To find T we find the point on the line that also satisfies the equation of the plane:

$$-x + y - z = -3$$

$$-(4 - t) + (-3 + t) - (5 - t) = 3$$

$$-12 + 3t - 3$$

$$t = 3$$

$$T = (x, y, z) = (4 - 3,\ -3 + 3,\ 5 - 3) = (1,\ 0,\ 2).$$

Now the distance between P and T is

$$\sqrt{(4-1)^2 + (-3-0)^2 + (5-2)^2} = 3\sqrt{3}.$$