Combinations

6/28/2006

Binomial Coefficients

Definition. *The number of distinct subsets with* j *elements that can be chosen from a set with* n *elements is denoted by* $\binom{n}{j}$. *The number* $\binom{n}{j}$ *is called a binomial* coefficient.

Recurrence Relation

Theorem. *For integers n and j, with $0 < j < n$, the binomial coefficients satisfy:*

$$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}.$$

Pascal's triangle

	j = 0	1	2	3	4	5	6	7	8	9	10
n = 0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

Theorem. *The binomial coefficients are given by the formula*

$$\binom{n}{j} = \frac{(n)_j}{j!} = \frac{n!}{j!(n-j)!}.$$

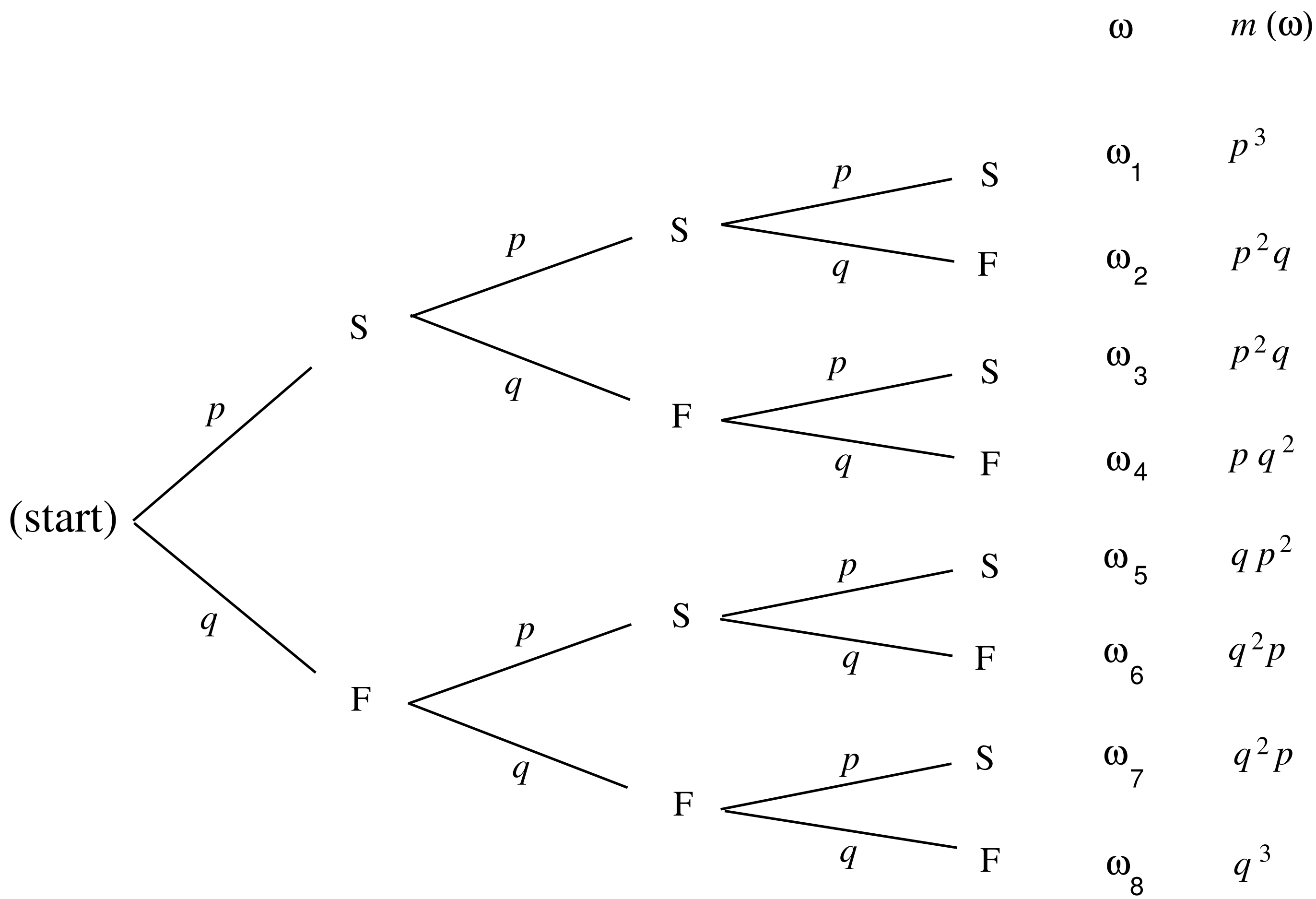
Bernoulli Trials

Definition. *A* Bernoulli trials process *is a sequence of n chance experiments such that*

1. Each experiment has two possible outcomes, which we may call *success* and *failure*.

2. The probability p of success on each experiment is the same for each experiment, and this probability is not affected by any knowledge of previous outcomes. The probability q of failure is given by $q = 1 - p$.

Tree diagram



Binomial Probabilities

We denote by $b(n, p, j)$ the probability that in n Bernoulli trials there are exactly j successes.

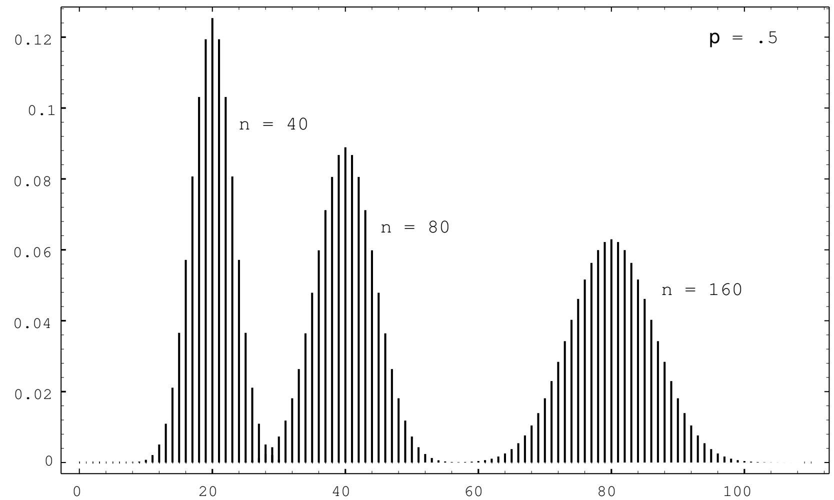
Theorem. *Given n Bernoulli trials with probability p of success on each experiment, the probability of exactly j successes is*

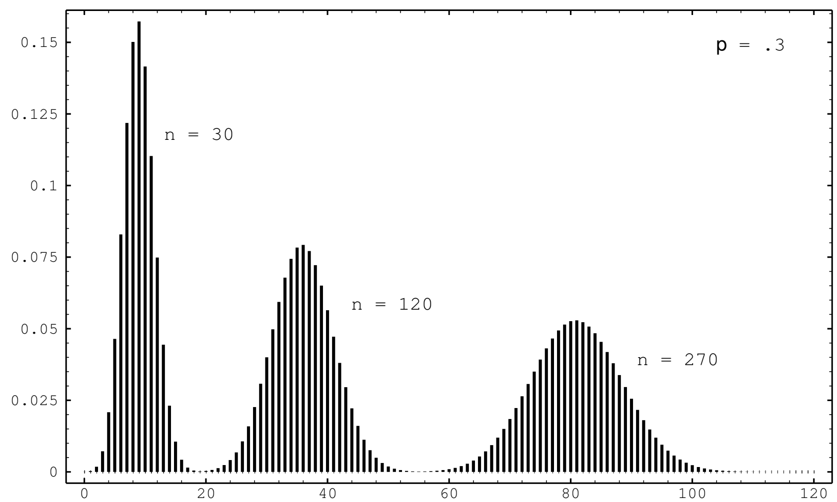
$$b(n, p, j) = \binom{n}{j} p^j q^{n-j}$$

where $q = 1 - p$.

Binomial Distributions

Definition. *Let n be a positive integer, and let p be a real number between 0 and 1. Let B be the random variable which counts the number of successes in a Bernoulli trials process with parameters n and p. Then the distribution $b(n, p, k)$ of B is called the* binomial distribution*.*


p = .5
n = 40
n = 80
n = 160
0.12
0.1
0.08
0.06
0.04
0.02
0
0
20
40
60
80
100


p = .3
n = 30
n = 120
n = 270
0.15
0.125
0.1
0.075
0.05
0.025
0
0
20
40
60
80
100
120

Binomial Expansion

Theorem. *The quantity* $(a+b)^n$ *can be expressed in the form*

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j} \ .$$

Corollary. *The sum of the elements in the* n*th row of Pascal's triangle is* 2^n*. If the elements in the* n*th row of Pascal's triangle are added with alternating signs, the sum is 0.*

Inclusion-Exclusion Principle

Theorem. *Let P be a probability distribution on a sample space Ω, and let $\{A_1,\ A_2,\ \ldots,\ A_n\}$ be a finite set of events. Then*

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i) \ - \sum_{1 \le i < j \le n} P(A_i \cap A_j)$$

$$+ \sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k) - \cdots \ .$$

That is, to find the probability that at least one of n events A_i occurs, first add the probability of each event, then subtract the probabilities of all possible two-way intersections, add the probability of all three-way intersections, and so forth.

Hat Check Problem (revisited)

Find the probability that a random permutation contains at least one fixed point.

- If A_i is the event that the ith element a_i remains fixed under this map, then
$$P(A_i) = \frac{1}{n}.$$
- If we fix a particular pair (a_i, a_j), then
$$P(A_i \bigcap A_j) = \frac{1}{n(n-1)}.$$
- The number of terms of the form $P(A_i \bigcap A_j)$ is $\left(\begin{array}{c} n \\ 2 \end{array} \right)$.

- For any three events A_1, A_2, A_3

$$P(A_i \cap A_j \cap A_k) = \frac{(n-3)!}{n!} = \frac{1}{n(n-1)(n-2)} \ ,$$

and the number of such terms is

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{3!} \ .$$

- Hence

$$P(\text{at least one fixed point}) = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots (-1)^{n-1}\frac{1}{n!}$$

and

$$P(\text{no fixed point}) = \frac{1}{2!} - \frac{1}{3!} + \cdots (-1)^{n}\frac{1}{n!} \ .$$

n	Probability that no one gets his own hat back
3	.333333
4	.375
5	.366667
6	.368056
7	.367857
8	.367882
9	.367879
10	.367879

Problem

Show that the number of ways that one can put n different objects into three boxes with a in the first, b in the second, and c in the third is $n!/(a!\,b!\,c!)$.

Problem

Prove that the probability of exactly n heads in $2n$ tosses of a fair coin is given by the product of the odd numbers up to $2n - 1$ divided by the product of the even numbers up to $2n$.