Homework #3

1) Page 101 #7

Goal: find a 3 term approximation of the solution to $(x+1)^3 = \varepsilon x$

let $x = x_0 + \varepsilon^{1/3} x_1 + \varepsilon^{2/3} x_2 + \cdots$

Plug into polynomial.

$$(x_0 + \varepsilon^{1/3} x_1 + \varepsilon^{2/3} x_2 + \cdots + 1)^3 = \varepsilon(x_0 + \varepsilon^{1/3} x_1 + \cdots)$$

Collect terms

ε^0: $(x_0+1)^3 = 0 \Rightarrow x_0 = -1$

ε^1: $x_1^3 = x_0 \Rightarrow x_1^3 = -1 \Rightarrow x_1 = -1$ $\left(-e^{\pm 2\pi i/3}\right.$ Not needed only wanted real root$\left.\right)$

(Next term on RHS)

$\varepsilon^{4/3}$: $3x_2 x_1^2 = x_1 \Rightarrow x_2 = 1/(3x_1) = -1/3$

$\Rightarrow x \approx -1 - \varepsilon^{1/3} - \frac{1}{3}\varepsilon^{2/3}$

2) Page 103 #14

Need leading order approximation & IVP's for $O(\varepsilon)$ & $O(\varepsilon^2)$ terms.

We have $\frac{dy}{dt} = e^{-\varepsilon/y} \overset{\text{maclaurin series}}{=} 1 - \frac{\varepsilon}{y} + \frac{1}{2!}\frac{\varepsilon^2}{y^2} + \cdots$

now let $y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$
and plug in

$$y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots = 1 - \frac{\varepsilon}{y_0 + \varepsilon y_1 + \varepsilon^2 y_2} + \frac{\varepsilon^2}{2!}\left(\frac{1}{y_0 + \varepsilon y_1 + \varepsilon^2 y_2}\right)^2$$

$$= 1 - \frac{\varepsilon}{y_0}\left(1 + \frac{\varepsilon y_1}{y_0} + \cdots\right)^{-1} + \frac{1}{2}\left(\frac{\varepsilon}{y_0}\right)^2\left(1 + \frac{\varepsilon y_1}{y_0} + \cdots\right)^{-2}$$

$" 1 - \frac{\varepsilon y_1}{y_0} +$ via binomial series

Initial condition
$y(0) = y_0(0) + \varepsilon y_1(0) + \varepsilon^2 y_2(0) + \cdots = 1$
Collect terms
ε^0: $y_0' = 1 \Rightarrow y_0(t) = t + C$
IC $y_0(0) = 1 \Rightarrow C = 1$ so $y_0(t) = 1 + t$

ε^1: $y_1' = \frac{-1}{1+t}$ $\quad y_1(0) = 0.$

ε^2: $y_2' = \frac{1}{2}\cdot\left(\frac{1}{1+t}\right)^2 - \frac{y_1}{(1+t)^2}$ $\quad y_2(0) = 0.$

3) Page 103 # 16

$$y'' \pm \varepsilon t y \qquad 0<\varepsilon \ll 1$$
$$y(0)=0$$
$$y'(0)=1$$

let $y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$
Plug into DE

$$\begin{cases} y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \cdots = \varepsilon t (y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots) \\ y_0(0) + \varepsilon y_1(0) + \varepsilon^2 y_2(0) + \cdots = 0 \\ y_0'(0) + \varepsilon y_1'(0) + \varepsilon^2 y_2'(0) + \cdots = 1 \end{cases}$$

Collect equations:

ε^0: $y_0'' = 0 \rightarrow y_0(t) = At + B$

$y_0(0) = 0 \Rightarrow B = 0$

$y_0'(0) = 1 \Rightarrow A = 1$

$y_0(t) = t$

ε^1: $y_1'' = t^2 \Rightarrow y_1' = \frac{t^3}{3} + B$

$\rightarrow y_1 = \frac{t^4}{12} + Bt + C$

$y_1(0) = C = 0$

$y_1'(0) = B = 0 \rightarrow y_1(t) = t^4/12$

ε^2: $y_2'' = t y_1 = t^5/12 \rightarrow y_2' = t^6/6 \cdot 12 + B$

$y_2 = t^7/(6 \cdot 12 \cdot 7) + Bt + C$

$y_2(0) = C = 0$

$y_2'(0) = B = 0$

$\Rightarrow y_2(t) = \frac{t^7}{6 \cdot 12 \cdot 7}$

$y_a = t + \varepsilon\, t^4/12 + \varepsilon^2 t^7/(12\cdot 6\cdot 7) \cdots$

the residual is

$$y_a'' - t\varepsilon y_a = \varepsilon \frac{4\cdot 3}{12} t^2 + \varepsilon^2 t^5/12 - \left(t\varepsilon\left(t + \frac{\varepsilon t^4}{12} + \frac{\varepsilon^2 t^7}{12\cdot 6\cdot 7} + \cdots\right)\right)$$

$$= -\frac{\varepsilon^3 t^8}{12\cdot 7\cdot 6}$$

This does not converge uniformly to 0 for $t \geq 0$, as $\varepsilon \to 0^+$.

4) Page 111 # 1 b,c

b) $\varepsilon x^3 + x - 2 = 0$

let $w = x\delta(\varepsilon)$ Plug into equation

$\frac{\varepsilon}{\delta^3} w^3 + \frac{w}{\delta} - 2 = 0$

let $\frac{\varepsilon}{\delta^3} \sim \frac{1}{\delta} \Rightarrow \delta = O(\sqrt{\varepsilon})$

new equation

$\frac{\varepsilon}{\varepsilon^{3/2}} w^3 + \frac{w}{\sqrt{\varepsilon}} - 2 = 0$

$w^3 + w - \sqrt{\varepsilon}(2) = 0$

leading order equation is $w^3 + w = 0$

roots are $w = 0, \pm i$

$\Rightarrow$ leading order behavior of roots of original equation are $x = 0, \pm i/\sqrt{\varepsilon}$

c) $\varepsilon^2 x^6 - \varepsilon x^4 - x^3 + 8 = 0$

let $w = x/\delta$ Plug into equation

$\frac{\varepsilon^2}{\delta^6} w^6 - \frac{\varepsilon}{\delta^4} w^4 - \frac{w^3}{\delta^3} + 8 = 0$

Try scaling $\frac{\varepsilon^2}{\delta^6} \sim \frac{\varepsilon}{\delta^4} \rightarrow \delta^2 = O(\varepsilon) \Rightarrow \delta = O(\sqrt{\varepsilon})$

but $1/\delta^3$ will be large $\Rightarrow$ BAD!

Try Scaling $\frac{\varepsilon^2}{\delta^6} \sim \frac{1}{\delta^3} \rightarrow \delta = O(\varepsilon^{2/3})$

then $\varepsilon/\delta^4 = \varepsilon/\varepsilon^{8/3} = 1/\varepsilon^{5/3}$

$$\frac{\varepsilon^2}{(\varepsilon^{-2/3})^6} W^6 - \frac{\varepsilon}{(\varepsilon^{2/3})^4} W^4 - \frac{W^3}{(\varepsilon^{2/3})^3} + 8 = 0$$

$$\frac{\varepsilon^2}{\varepsilon^4} W^6 - \frac{\varepsilon}{\varepsilon^{8/3}} W^4 - \frac{W^3}{\varepsilon^2} + 8 = 0$$

multiply by ε^2

$$W^6 - \frac{\varepsilon^2}{\varepsilon^{5/3}} W^4 - W^3 + 8\varepsilon^2 = 0$$

$$W^6 - \varepsilon^{1/3} W^4 - W^3 + 8\varepsilon^2 = 0$$

leading order equation

$$W^6 - W^3 = 0$$

$$W^3(W^3 - 1) = 0 \quad \Rightarrow \quad W^3 = 1$$

The roots are $W = 0, 1, e^{\pm 2\pi i/3}$

$\Rightarrow$ leading order roots of original equation are
$X = 0, \ \varepsilon^{-2/3}, \ \varepsilon^{-2/3} e^{\pm 2\pi i/3}$

5) Page 112 # 2

Find the correction term for $\varepsilon^2 x^3 + x - 2 = 0$

The scaled equation was $w^3 + w - 2\sqrt{\varepsilon} = 0$

Use regular perturbation

let $w = w_0 + \varepsilon^{1/2} w_1 + \varepsilon w_2 + \cdots$

Plug into equation

$$(w_0 + \varepsilon^{1/2} w_1 + \varepsilon w_2 + \cdots)^3 + w_0 + \varepsilon w_1 + \cdots - 2\sqrt{\varepsilon} = 0.$$

ε^0: $w_0^3 + w_0 = 0$. $\rightarrow$ From previous problem.

Next term involves $\sqrt{\varepsilon}$.

$$3w_0^2 w_1 + w_1 - 2 = 0.$$

$$\rightarrow w_1 = \frac{2}{1+3w_0^2}$$

$\Rightarrow$ The correction term is $\boxed{\frac{2}{1+3w_0^2}}$

6) Page 121 #1 $\qquad 0<\varepsilon \ll 1 \quad 0<x<1$

a) $\varepsilon y'' + 2y' + y = 0 \quad y(0) = 0,\ y(1) = 1$

$P(x) = 2 > 0.$

$\rightarrow$ layer is near 0.

outer layer solution

$2y' + y = 0 \qquad y(1) = 1$

$\frac{y'}{y} = -\frac{1}{2} \Rightarrow y = Ce^{-1/2 t}$

$y(1) = Ce^{-1/2} = 1 \rightarrow C = e^{1/2}$

$\Rightarrow y_0(t) = e^{1/2(1-t)}$

Inner layer solution.

let $\quad W \equiv \frac{t}{\delta(\varepsilon)} \quad Y = y(W\delta)$

Plug into Differential equation

$\frac{\varepsilon}{\delta^2} Y'' + \frac{2Y'}{\delta} + Y = 0 \qquad Y(0) = 0 \quad Y(1) =$

let $\varepsilon/\delta^2 \sim 1/\delta \Rightarrow \delta = \varepsilon$

$\frac{\varepsilon}{\varepsilon^2} Y'' + \frac{2Y'}{\varepsilon} + Y = 0$

$\rightarrow Y'' + 2Y' + \varepsilon Y = 0$

leading order equation

$Y'' + 2Y' = 0$

$\frac{Y''}{Y'} = -2 \Rightarrow Y' = Ce^{-2W}$

$\Rightarrow Y = Ce^{-2W} + B$

$Y(0) = C + B = 0 \Rightarrow C = -B$

$\Rightarrow Y(W) = B(1-e^{-2W})$

We choose the constant so that

$$\lim_{t\to 0} y_0(t) = \lim_{W\to\infty} Y(W)$$

$$e^{1/2} = B$$

$$\Rightarrow y_i(t) = e^{1/2}(1-e^{-2(t/\varepsilon)})$$

$$\Rightarrow y(t) = y_0(t) + y_i(t) - B$$

$$= e^{1/2(1-t)} + e^{1/2}(1-e^{-2t/\varepsilon}) - e^{1/2}$$

h) $\varepsilon y'' - (2-x^2)y = -1 \quad y'(0)=0 \quad y(1)=1$

outer layer $-(2-x^2)y = -1$

$$\Rightarrow y = \frac{1}{2-x^2}$$

This satisfies both conditions $\Rightarrow$ no boundary layer.

9) $\varepsilon y'' - b(x)y' = 0 \quad y(0)=\alpha \quad y(1)=\beta \quad \alpha\neq\beta \quad b(x)>0$

$-b(x) < 0 \Rightarrow$ boundary layer is near 1.

outer layer solution

$$-b(x)y'(x) = 0 \qquad y(0)=\alpha$$

$$\Rightarrow y(x) = C \Rightarrow y(x) = \alpha.$$

Inner layer solution

let $W = (1-x)/\delta \Rightarrow (1-x) = \delta W \Rightarrow x = 1 - \delta W$

$Y(W) = y(1 - W\delta)$

Plug into ODE.

BC. $y(1) = Y(0) = \beta$

$$\frac{\varepsilon}{\delta^2} Y'' - \frac{b(1-\delta W)}{\delta} Y' = 0$$

since δ small $b(1-\delta W) \sim b(1)$

$\Rightarrow$ balance $\frac{\varepsilon}{\delta^2} \sim \frac{1}{\delta} \Rightarrow \delta = \varepsilon$

ODE Becomes

$$Y'' - b(1-\varepsilon W) Y' = 0$$

let $V = Y'$

$$V' - b(1-\varepsilon W) V = 0$$

Since we are doing a leading order approximation. We look at

$$V' - b(1) V = 0$$

$$\Rightarrow V = Ce^{-b(1)W}$$

$$\Rightarrow Y_i(W) = Ce^{-b(1)W} + B$$

BC $Y_i(0) = C + B = \beta$

Find BC by matching

$$\lim_{x \to 1} y_0(x) = \lim_{W \to \infty} Y_i(W)$$

$$\alpha = B \Rightarrow C = \beta - \alpha$$

$$\Rightarrow y(x) = y_0(x) + y_i(x) - B$$

$$= \alpha + (\beta - \alpha) e^{-b(1)(1-x)/\varepsilon}$$

7) Page 121 #2

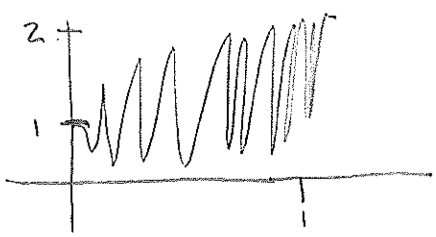
$$\varepsilon u'' + u = 0 \qquad 0 < x < 1$$
$$u(0) = 1 \quad u(1) = 2$$

The exact solution is

$$u = C_1 \cos(\varepsilon^{-1/2} x) + C_2 \sin(\varepsilon^{-1/2} x)$$

$$u(0) = C_1 = 1$$
$$u(1) = 1\cos(\varepsilon^{-1/2}) + C_2 \sin(\varepsilon^{-1/2}) = 2$$
$$\Rightarrow \quad C_2 = \frac{2 - \cos(\varepsilon^{-1/2})}{\sin(\varepsilon^{-1/2})}$$

Approximate plot of the solution.



Singular perturbation will not work because there is no separation of scales. Also "matching" would be impossible with the highly oscillatory nature of the solution.

8) Page 122 # 3.

$$\varepsilon y'' + y' + ae^{y} = 0 \qquad y(0)=0 \qquad y(1)=0.$$

Outer layer solution

$$y' = -ae^{y} \Rightarrow -e^{-y} = ax + c$$
$$\Rightarrow y_0 = -\ln(ax+c)$$

if Boundary layer is at 0.

$$y_0(1) = -\ln(a+c) = 0 \Rightarrow a+c=1$$
$$\Rightarrow c = 1-a$$
$$y_0(x) = -\ln(ax + (1-a))$$

If Boundary layer is at 1

$$y_0(0) = -\ln c = 0 \Rightarrow c=1$$
$$y_0(x) = -\ln(ax+1)$$

Innerlayer Solution: $W = x/\delta \quad Y(w) = y(w\delta)$

$$\frac{\varepsilon}{\delta^2} Y'' + \frac{Y'}{\delta} + ae^{Y} = 0.$$

let $\varepsilon/\delta^2 \sim 1/\delta \Rightarrow \delta = \varepsilon.$

$$Y'' + Y' + \varepsilon a e^{Y} = 0.$$

unperturbed problem $\quad \varepsilon = 0$

$$Y'' + Y' = 0 \quad \Rightarrow Y_0 = Ae^{-w} + B$$

This is well behaved independent of a.

look at outer solutions. we need $y_0(x)$ to be bounded in $[0,1)$.

for boundarly layer at $x=1$

$\Rightarrow$ need $ax+1 > 0 \Rightarrow a > -1/x \quad \forall x \in [0,1)$

$\Rightarrow a>0$ is sufficant.

For boundary layer at $x=0$.

We need $ax + (1-a) > 0 \Rightarrow a(x-1)+1 > 0$

$\Rightarrow a \leq 0$ is sufficent.

So, one gets a boundary layer as long as $a \neq 0$.

9) Page 122 4

$$\varepsilon u'' - (2x+1)u' + 2u = 0 \qquad 0<x<1$$

$$u(0)=1 \quad u(1)=0$$

$p(x) = -(2x+1) \leq 0 \rightarrow$ boundary layer near $x=1$.

Outer layer solution (near $x=0$)

$$-(2x+1)u' + 2u = 0$$

$$\frac{u'}{u} = \frac{+2}{2x+1} = \frac{1}{x+1/2}$$

$$\ln u = 1\ln(x+1/2)+C \Rightarrow u(x) = C(x+1/2)^1$$

$$u(0) = 1/2\, C = 1 \rightarrow c=2.$$

$$u_0(x) = 2x+1$$

Inner layer $\rightarrow x = 1 - w\delta$

let $w = \frac{1-x}{\delta}$ $\quad Y(w) = u_i(1-w\delta)$

Plugin

$$\frac{\varepsilon}{\delta^2} Y'' + \frac{(2(1-w\delta)+1)}{\delta} Y' + 2Y = 0$$

Coefficients $\quad \varepsilon/\delta^2 \quad 2 \quad 1/\delta$

if $\varepsilon/\delta^2 \sim 1/\delta \rightarrow \delta \sim \varepsilon$. everything scales well.

$\Rightarrow$ ODE Becomes

$$Y'' + (3-2\varepsilon w)Y' + 2\varepsilon Y = 0$$

leading order equation is

$$Y'' + 3Y' = 0$$

$$\Rightarrow Y(w) = Ae^{-3w} + B$$

$$Y(0) = A+B = 0. \quad \rightarrow A = -B$$

$$Y_i(w) = B(1-e^{-3w})$$

Now do matching

$$\lim_{w\to\infty} y_i(w) = \lim_{x\to 1} y_0(x)$$

$$B = 2(1)+1=3$$

$\Rightarrow y_i(w) = 3(1-e^{-3w})$

$\Rightarrow y_i(x) = 3(1-e^{-3(1-x)/\varepsilon})$

$\Rightarrow$ The uniform approximation is

$$u_u(x) = u_0(x) + u_i(x) - B$$
$$= 2x+1 + 3(1-e^{-3(1-x)/\varepsilon}) - 3.$$