Math 22 Workshop III
27 April 2006

1. Suppose that V and W are vector spaces and that $T : V \to W$ is a linear transformation. If $\{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \}$ are vectors in V and if $\{ T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_p) \}$ is linearly independent, then show that $\{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \}$ is linearly independent.

2. Suppose that V and W are vector spaces and that $T : V \to W$ is a linear transformation. Suppose that $\{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \}$ is a linearly independent set of vectors in V. Must it be the case that $\{ T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_p) \}$ is linearly independent?

3. Suppose that V and W are vector spaces and that $T : V \to W$ is a *one-to-one* linear transformation. Suppose that $\{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \}$ is a linearly independent set of vectors in V. Must it be the case that $\{ T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_p) \}$ is linearly independent?

Let V and W be vector spaces. A linear transformation $T : V \to W$ which is both one-to-one and onto is called an *isomorphism of* V *onto* W. An isomorphism T is invertible, and we proved its inverse, $T^{-1} : W \to V$, is also a linear map. Note that T^{-1} is also one-to-one and onto.

4. Suppose that $T : V \to W$ is an isomorphism of V onto W.

   (a) Show that H is a subspace of V if and only if $T(H) := \{ T(\mathbf{v}) \in W : \mathbf{v} \in H \}$ is a subspace of W.

   (b) Let H be a subspace of V. Show that $\{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \}$ is a basis for H if and only if $\{ T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_p) \}$ is a basis for $T(H)$.