Algebra Final Exam

Due Friday, March 8

Instructions You may use only your class notes, texts and homework on this exam. Don't discuss the exam with anyone but me. If you find an error on the exam, let me know as soon as possible. Exams are due in my office between 1 and 2 on Friday, March 8.

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1 Let $F \subseteq E$ be a Galois field extension. Assume that $G = \mathrm{Gal}(E/F)$ is abelian of order n, and that F contains a primitive n^{th} root of unity. Show that there are elements $a_1, a_2 \ldots, a_m \in F$ and positive integers $e_1, e_2, \ldots, e_m$ such that E is a splitting field over F for the polynomial

$$f = (X^{e_1} - a_1)(X^{e_2} - a_2) \cdots (X^{e_m} - a_m).$$

(**Hint:** The fundamental theorem of abelian groups says that G is product of cyclic groups C_i. Find Galois extensions K_i of F with $\mathrm{Gal}(K_i/F) \cong C_i$. Once you have found f, show E is a splitting field for f by comparing degrees.)

2 Let $E = F[\alpha]$ where $\alpha^m \in F$. Assume that the characteristic of F does not divide m. If m is relatively prime to $|E : F|$, show that E is Galois over F. (**Hint:** Adjoin m^{th} roots of unity.)

3

(a) Let $F \subseteq E_i \subseteq L$ be a repeated radical extensions of F for $i = 1, \ldots, k$. Show that the compositum $\langle E_1, \ldots, E_n \rangle$ is also a repeated radical extension of F.

(b) Suppose that $F \subseteq K$ is a repeated radical extension. Show that there is a field $L \supseteq K$ such that L is a finite degree normal extension of F and L is a repeated radical extension of F. (**Hint:** Find a splitting field L over F for some $f \in F[X]$ such that $K \subseteq L$ (alternatively, what is the smallest possible normal extension $L \supseteq F$ which can contain K?). Show that each root of f lies in a repeated radical extension of F.)

(c) Let $f \in F[X]$ be an irreducible polynomial. Show that if one root of f is expressible in terms of radicals, then all the roots of f are expressible in terms of radicals.