MATH 63, CH III, PROBLEM 5

Any bounded open subset of $\mathbb{R}$ is the union of disjoint open intervals.

Proof. Let $S \subset \mathbb{R}$ be a bounded open subset. The fact that S is bounded means that there are two numbers $a, b \in \mathbb{R}$ such that $S \subset (a, b)$.

The fact that S is open means, by definition, that for every $x \in S$ there exists an interval $(x - r, x + r)$ (an "open ball" in E^1) with $r > 0$ and $(x - r, x + r) \subset S$. The key idea of the proof is that for every $x \in S$ there actually exists a *largest* open interval $(a_x, b_x) \subset S$ with $x \in (a_x, b_x)$. This is the content of the following claim.

Claim: For every element $x \in S$ there exist an open interval $(a_x, b_x) \subset S$ with $x \in (a_x, b_x)$ while $a_x \notin S, b_x \notin S$.

Proof of claim. Fix a point $x \in S$. Let A_x be the set of real numbers that are smaller than x and *not* in S,

$$A_x = \{y \in \mathbb{R} \mid y \leq x \text{ and } y \notin S\}$$

and let a_x be the least upper bound of this set,

$$a_x = \text{l.u.b.} A_x$$

Since $S \subset (a, b)$ and $x \in S$ we have $a < x$, while $a \notin S$ (because $a \notin (a, b)$). Therefore $a \in A_x$, and we see that A_x is not empty. Also, x is an upper bound for A_x, and so A_x has a least upper bound.

Similarly, let

$$B_x = \{y \in \mathbb{R} \mid y \geq x \text{ and } y \notin S\}$$

$$b_x = \text{g.l.b.} B_x$$

To check that $(a_x, b_x) \subset S$ we let $y \in (a_x, b_x)$ and prove that $y \in S$. From $a_x < y$ we see that $y \notin A_x$ (a_x is an upper bound for A_x), which implies that either $y > x$ or $y \in S$ (or both). Similarly, $y < b_x$ implies $y \notin B_x$ and so either $y < x$ or $y \in S$. Since you can't have both $y > x$ and $y < x$ it follows that $y \in S$.

We prove that $a_x \notin S$ by contradiction: assume that $a_x \in S$. Since S is open there is an open ball $B_r(a_x) = (a_x - r, a_x + r)$ with $r > 0$ that is a subset of S: $(a_x - r, a_x + r) \subset S$. Since a_x is the *least* upper bound of A_x there exists some $z \in \mathbb{R}$ with $a_x - r < z < a_x$ and $z \in A_x$ (see p.24, first paragraph). But if $z \in A_x$ then $z \notin S$, which contradicts $z \in (a_x - r, a_x + r) \subset S$.

The proof that $b_x \notin S$ is similar. This completes the proof of the claim.

We now have a collection of open intervals $I_x = (a_x, b_x)$ indexed by $x \in S$.

On the one hand, since $I_x \subset S$ for every $x \in S$, it follows that $\bigcup_{x\in S} I_x \subset S$. On the other hand, every point $x \in S$ is contained in at least one interval $x \in I_x$, and so $S \subset \bigcup_{x\in S} I_x$. This proves

$$S = \bigcup_{x\in S} I_x$$

i.e., S is the union of the open intervals I_x. If we can show that the intervals I_x are *disjoint*, then we are done. This means that we must prove that for two indices $x, y \in S$ we have either $I_x = I_y$ or $I_x \cap I_y = \emptyset$.

So assume that $I_x \cap I_y$ is not empty, i.e., there exists a number $p \in I_x \cap I_y$. We will show that $I_x = I_y$. Since $a_x < p$ (because $p \in I_x$) and $p < b_y$ (because $p \in I_y$) we always have $a_x < b_y$. Likewise $a_y < b_x$. Now if $a_y < a_x$ then $a_y < a_x < b_y$ implies that $a_x \in I_y$, which contradicts $a_x \notin S$. Therefore $a_y < a_x$ is not possible. Likewise $a_y > a_x$ gives $a_x < a_y < b_x$, or $a_y \in I_x \subset S$, contradicting $a_y \notin S$. It follows that $a_x = a_y$. The same argument proves $b_x = b_y$ and therefore $I_x = I_y$.

$\square$