Math 73/103 Assignment Three
Due Friday, November 4th

CLARIFICATION: Since at least one person found some legitimate ambiguities in their notes, let me be clear about our terminology. Lebesgue measure, $(\mathbf{R}, \mathfrak{M}, m)$, is the complete measure coming from the explicit outer measure m^* we defined in lecture. In particular, $\mathfrak{M}$ is the σ-algebra of all m^*-measurable sets. A Lebesgue measurable function $f : \mathbf{R} \to \mathbf{C}$ is just a function such that $f^{-1}(V) \in \mathfrak{M}$ for any open set $V \subset \mathbf{C}$. We say f is Borel if $f^{-1}(V)$ is a Borel set in $\mathbf{R}$ for every open set V. We say $f \in \mathcal{L}^1(\mathbf{R}, \mathfrak{M}, m)$, or the f is Lebesgue integrable, if f is measurable and $\int_{\mathbf{R}} |f| \, dm < \infty$. We have also used the notation $L^+(\mathbf{R}, \mathfrak{M}, m)$ for the collection Lebesgue measurable functions f such that $f \geq 0$ everywhere.

1. Suppose that $f \in \mathcal{L}^1(X, \mathfrak{M}, m)$ is a Lebesgue integrable function on the real line. Let $\epsilon > 0$. Show that there is a continuous function g that vanishes outside a bounded interval such that $\|f - g\|_1 < \epsilon$.

ANS: On a previous assignment, we showed that given $\epsilon > 0$, there is a step function $s = \sum_{j=1}^n a_j \mathbb{I}_{I_j}$ such that $\|f - s\|_1 < \epsilon/2$ with each I_j a bounded open interval. Since s vanishes off $\bigcup I_j$, which is certainly contained in a bound interval, it is clear that it suffices to show that given $\delta > 0$ we can find a continuous function g such that $\|g - \mathbb{I}_{(a,b)}\|_1 < \delta$ and such that $g(x) = 0$ if $x \notin (a, b)$. But that is easy: just draw a picture.[1]

2. Prove Lusin's Theorem: Suppose that f is a Lebesgue measurable function on $[a, b] \subset \mathbf{R}$. Given $\epsilon > 0$, show that there is a closed subset $K \subset [a, b]$ such that $m([a, b] \setminus K) < \epsilon$ and that $f|_K$ is continuous. (And unlike the version stated in lecture, we are not assuming f is integrable.)

ANS: Let $\epsilon > 0$. Let $A_n = \{ x \in [a, b] : |f(x)| \leq n \}$. Since $A_n \subset A_{n+1}$ and $\bigcup A_n = [a, b]$, there is a n such that $m([a, b] \setminus A_n) < \epsilon/3$. Let

$$h(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq n \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Since h is bounded, $h \in \mathcal{L}^1([a, b])$. In view of the previous problem, there is a sequence $\{g_k\}$ of continuous functions such that $g_k \to h$ in L^1. But then there is a subsequence $\{g_{k_j}\}$ that converges to h almost everywhere. By Egoroff's Theorem, there is a set E such that $m([a, b] \setminus E) < \epsilon/3$ such that $\{g_{k_j}\}$ converges uniformly to h on E. Since the g_k are continuous, that means $h|_E$ is continuous. But $m([a, b] \setminus E \cap A_n) < 2\epsilon/3$. But we can find a closed set $K \subset E \cap A_n$ such that $m(E \cap A_n \setminus K) < \epsilon/3$. Then $m([a, b] \setminus K) < \epsilon$ and $f|_K$ is continuous.

[1] Let $\epsilon = \min\{(b - a)/2, \delta/2\}$. Let g be 0 off (a, b), linear from $(a, 0)$ to $(a + \epsilon, 1)$, constantly equal to 1 from $(a + \epsilon, 1)$ to $(b - \epsilon, 1)$ and linear from $(b - \epsilon, 1)$ to $(b, 0$.

3. Suppose that ρ is a premeasure on an algebra $\mathcal{A}$ of sets in X. Let ρ^* be the associated outer measure.

(a) Show that $\rho^*(E) = \rho(E)$ for all $E \in \mathcal{A}$.

(b) If $\mathfrak{M}^*$ is the σ-algebra of ρ^*-measurable sets, show that $\mathcal{A} \subset \mathfrak{M}^*$.

ANS: (a) Since we can cover $E \in \mathcal{A}$ by $\{A_n\}$ with $A_1 = E$ and $A_n = \emptyset$ if $n \geq 2$, we certainly have $\rho^*(E) \leq \rho(E)$. On the other hand, if $E \subset \bigcup A_n$ with $A_n \in \mathcal{A}$, we can let $B_n = E \cap A_n \setminus \bigcup_{i=1}^{n-1} A_i$ so that the $B_n \subset A_n$ are pairwise disjoint elements of A with union E. Then

$$\sum_n \rho(A_n) \geq \sum_n (B_n) = \rho(E).$$

This shows $\rho^*(E) \geq \rho(E)$.

(b) Suppose $A \in \mathcal{A}$. Let $F \subset X$ and $\epsilon > 0$. Let $\{A_n\} \subset \mathcal{A}$ be such that $F \subset \bigcup A_n$ and $\rho^*(F) + \epsilon \geq \sum \rho(A_n)$. But then

$$\rho^*(F) + \epsilon \geq \sum_n \rho(A_n) = \sum_n \rho(A_n \cap A) + \rho(A_n \cap A^c) \geq \rho^*(F \cap A) + \rho^*(F \cap A^c).$$

Since $\epsilon > 0$ is arbitrary, we conclude that $A \in \mathfrak{M}^*$.

4. Suppose that $f_n \to f$ in measure and that there is a $g \in \mathcal{L}^1(X, \mathfrak{M}, \mu)$ is such that $|f_n(x)| \leq g(x)$ for all $x \in X$. Show that $f_n \to f$ in $L^1(X, \mathfrak{M}, \mu)$.

ANS: Suppose that $f_n \not\to f$ in L^1. Then there is a subsequence $\{f_{n_k}\}$ and a $\epsilon > 0$ such that $\|f_{n_k} - f\|_1 \geq \epsilon$ for all k. But it is easy to check that $\{f_{n_k}\}$ still converges to f in measure. Hence there is a subsequence $\{f_{n_{k_j}}\}$ that converges to f almost everywhere. However, the LDCT then implies $\{f_{n_{k_j}}\}$ must converge to f in L^1. This gives us a contradiction.

5. Let m be Lebesgue measure on $[0, 1]$ and let μ be counting measure. Clearly, $m \ll \mu$. Show that there is no function f satisfying the conclusion of the Radon-Nikodym Theorem. Why is this not a counter-example to the Radon-Nikodym Theorem.

ANS: Suppose there were a function $h : [0, 1] \to [0, \infty)$ such that

$$m(E) = \int_E h(x)\, d\mu(x) = \sum_{x \in E} h(x)$$

for all Borel sets $E \subset [0, 1]$. Letting $E = \{x\}$, we see that $h(x) = 0$ for all $x \in [0, 1]$, which is silly.

Of course, counting measure is not σ-finite on $[0, 1]$: we can't write $[0, 1]$ as a countable union of finite sets.

6. Prove the version of Fubini and Tonelli for complete measures stated in lecture: Let $(X, \mathfrak{M}, \mu)$ and $(Y, \mathfrak{N}, \nu)$ be *complete* σ-finite measure spaces. Let $(X \times Y, \mathfrak{L}, \lambda)$ be the completion of $(X \times Y, \mathfrak{M} \otimes \mathfrak{N}, \mu \times \nu)$. Suppose that f is $\mathfrak{L}$-measurable and that either (a) $f \geq 0$ or (b) $f \in \mathcal{L}^1(\lambda)$. Show that f_x and f^y are measurable almost everywhere and in case (b), then they are integrable almost everywhere. And, with suitable modifications on null sets, $x \mapsto \int_Y f_x \, d\nu$ and $y \mapsto \int_X f^y \, d\mu$ are measurable and even integrable in case (b). Then show that the iterated integrals both agree with the double integral.

(Here is what I suggest, let g be a $\mathfrak{M} \otimes \mathfrak{N}$-measurable function that equals f almost everywhere. Then prove the following lemmas:

(a) If $E \in \mathfrak{M} \otimes \mathfrak{N}$, and $\mu \times \nu(E) = 0$, then $\nu(E_x) = 0 = \mu(E^y)$ for almost all x and y.

(b) If f is $\mathfrak{L}$-measurable and $f = 0$ λ-almost everywhere, then f_x and f^y are integrable almost everywhere and $\int_X f^y \, d\mu = 0 = \int_Y f_x \, d\nu$.)

ANS: Since $E \in \mathfrak{M} \otimes \mathfrak{N}$ implies

$$\mu \times \nu(E) = \int_X \nu(E_x) \, d\mu(x) = \int_Y \mu(E^y) \, d\nu(y),$$

Lemma (a) is immediate.

For Lemma (b), assume $f = 0$ λ-almost everywhere. Then there is a λ-null set F such that $f(x, y) = 0$ if $(x, y) \notin F$. But there is a $\mathfrak{M} \otimes \mathfrak{N}$-null set E such that $F \subset E$ (since λ is the completion of $\mu \times \nu$), and $f(x, y) = 0$ if $(x, y) \notin E$. Using Lemma (a), let $M \subset X$ and $N \subset Y$ be null sets such that $\mu(E^y) = 0$ if $y \notin N$ and $\nu(E_x) = 0$ if $x \notin M$.

Suppose that $x \notin M$. Then $f_x(y) = f(x, y)$ is zero if $y \notin E_x$. Thus $x \notin M$ implies $f_x = 0$ ν-almost everywhere. Since ν is complete, that means f_x is measurable. Of course, if f_x is measurable and equal to zero almost everywhere, then f_x is integrable with $\int_Y f_x(y) \, d\nu(y) = 0$. Similarly, if $y \notin N$, f^y is measurable with zero integral. This proves Lemma (b).

Now suppose that f is as in the statement of the theorem. We know that there is a $\mathfrak{M} \otimes \mathfrak{N}$-measurable function g such that $g = f$ λ-almost everywhere. Then $f = g + (f - g)$ and $(f - g) = 0$ λ-almost everywhere and Lemma (b) applies to $f - g$. Then, since $f_x = g_x + (f - g)_x$ and since g_x is always measurable, f_x is measurable almost everywhere. In case (b), g_x is integrable almost everywhere as is $(f - g)_x$. Hence so is f_x. But $x \mapsto \int_Y f_x \, d\nu$ is equal almost everywhere to $x \mapsto \int_Y g_x \, d\nu$, so the former is measurable (since μ is complete). In case (b), the later function is integrable, so so is the former. Of course the situation is symmetric with f^y. The rest is easy, since the f-integrals are all equal to their g-counterparts.

7. Let ν be a complex measure on $(X, \mathfrak{M})$.

(a) Show that there is a measure μ and a measurable function $\varphi : X \to \mathbf{C}$ so that $|\varphi| = 1$, and such that for all $E \in \mathfrak{M}$,

$$\nu(E) = \int_E \varphi \, d\mu. \tag{†}$$

(Hint: write $\nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$ for measures ν_i. Put $\mu_0 = \nu_1 + \nu_2 + \nu_3 + \nu_4$. Then μ_0 will satisfy (†) provided we don't require $|\varphi| = 1$. You can then use without proof the fact that any complex-valued measurable function h can be written as $h = \varphi \cdot |h|$ with φ unimodular and measurable.)

(b) [Optional: Do not turn in] Show that the measure μ above is unique, and that φ is determined almost everywhere $[\mu]$. (Hint: if μ' and φ' also satisfy (†), then show that $\mu' \ll \mu$, and that $\frac{d\mu'}{d\mu} = 1$ a.e. Also note that if φ' is unimodular and $E \in \mathfrak{M}$, then $E = \bigcup_{i=1}^4 E_i$ where $E_1 = \{\, x \in E : \operatorname{Re} \varphi' > 0 \,\}$, $E_2 = \{\, x \in E : \operatorname{Re} \varphi' < 0 \,\}$, $E_3 = \{\, x \in E : \operatorname{Im} \varphi' > 0 \,\}$, and $E_4 = \{\, x \in E : \operatorname{Im} \varphi' < 0 \,\}$.)

Comment: the measure μ in question 7 is called the *total variation* of ν, and the usual notation is $|\nu|$. It is defined by different methods in your text: see chapter 6. One can prove facts like $|\nu|(E) \geq |\nu(E)|$, although one doesn't always have $|\nu|(E) = |\nu(E)|$; this also proves that even classical notation can be unfortunate.

ANS: Let $\nu = \nu_1 - \nu_2 + i(\nu_3 - \nu_4)$ be a decomposition of ν into finite (positive) measures as in class. Define $\mu_0 = \nu_1 + \nu_2 + \nu_3 + \nu_4$. Clearly $\nu_i \ll \mu_0$ so there are non-negative functions h_i so that $\nu_i(E) = \int_E h_i \, d\mu_0$. Since each ν_i is finite, we can assume that each $h_i \in L^1(\mu_0)$ and hence that each h_i is finite-valued. Therefore we can define $h = h_1 - h_2 + i(h_3 - h_4)$ and observe that

$$\nu(E) = \int_E h \, d\mu_0. \tag{1}$$

By assumption, we can write $h = \varphi|h|$ with φ unimodular and everything in sight measurable. Then we can define μ by $\mu(E) = \int_E |h| \, d\mu_0$ and then $\nu(E) = \int_E \varphi \, d\mu$ as desired. This proves part (a).

To establish part (b), suppose that $\nu(F) = \int_F \varphi' \, d\mu'$ for all $F \in \mathfrak{M}$ for another measure μ' and unimodular function φ'. Suppose that $\mu(E) = 0$. We first need to show that $\mu'(E) = 0$. Let $E_1 = \{\, x \in E : \operatorname{Re} \varphi' > 0 \,\}$, $E_2 = \{\, x \in E : \operatorname{Re} \varphi' < 0 \,\}$, $E_3 = \{\, x \in E : \operatorname{Im} \varphi' > 0 \,\}$, and $E_4 = \{\, x \in E : \operatorname{Im} \varphi' < 0 \,\}$. Since $1 = |\varphi'|^2 = (\operatorname{Re} \varphi')^2 + (\operatorname{Im} \varphi')^2$, we must have $E = \bigcup_{i=1}^4 E_i$! On the other hand, $E_i \subseteq E$ implies that $\mu(E_i) = 0$ for all i. But then $\nu(E_i) = 0$, and

$$\int_{E_i} \varphi' d\mu' = 0 \tag{2}$$

for all i. But then $0 = \operatorname{Re} \left(\int_{E_i} \varphi' \, d\mu' \right) = \int_{E_i} \operatorname{Re} \varphi' \, d\mu'$, and we must have $\mu'(E_i) = 0$ for $i = 1, 2$.[2] Similarly, $\int_{E_i} \operatorname{Im} \varphi' \, d\mu' = 0$ and $\mu'(E_i) = 0$ for $i = 3, 4$. Therefore $\mu'(E) = 0$ and $\mu' \ll \mu$. But in that case for all $E \in \mathfrak{M}$, $\nu(E) = \int_E \varphi' \, d\mu' = \int_E \varphi' \frac{d\mu'}{d\mu} \, d\mu$. Since this holds for all E, we must have $\varphi = \varphi' \frac{d\mu'}{d\mu}$ (a.e.). Since φ and φ' are unimodular and $\frac{d\mu'}{d\mu}$ is nonnegative, we must have $\frac{d\mu'}{d\mu} = 1$ (a.e.). Thus, $\varphi = \varphi'$ (a.e.) and $\mu = \mu'$.

[2] Note that if $f(x) > 0$ for all $x \in A$ and $\int_A f \, d\mu = 0$, then $\mu(A) = 0$. To see this note that $A = \bigcup \{\, x \in A : f(x) > \frac{1}{n} \,\}$.

8. [Optional: Do NOT turn in] Suppose that $f : [a, b] \to \mathbf{R}$ is a bounded function. We want to show that f is Riemann integrable if and only if $m\big(\{ x \in [a, b] : f \text{ is not continuous at } x \}\big) = 0$. In [1, Theorem 2.28], Folland suggests the following strategy. Let

$$H(x) = \lim_{\delta \to 0}\big(\sup\{ f(y) : |y - x| \le \delta \}\big) \quad \text{and} \quad h(x) = \lim_{\delta \to 0} \inf\{ f(y) : |y - x| \le \delta \}.$$

(a) Show that f is continuous at x if and only if $H(x) = h(x)$.

(b) In the notation of our proof in lecture that Riemann integral functions are Lebesgue integrable, show that $h = \ell$ almost everywhere and $H = u$ almost everywhere.

(c) Conclude that $\int_a^b h\,dm = \mathcal{R}\underline{\int}_a^b f$ and $\int_a^b H\,dm = \mathcal{R}\overline{\int}_a^b f$.

ANS: I printed this one up separately.

References

[1] Gerald B. Folland, *Real analysis*, Second, John Wiley & Sons Inc., New York, 1999. Modern techniques and their applications, A Wiley-Interscience Publication. MR2000c:00001