Math 12, Fall 2007

Lecture 9

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10/15/07

Outline

1. Review and overview
   - Last classes
2. Today's material
   - Matrices as linear maps
   - The chain rule
3. Group Work
4. Next class

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Differentiation

Reducing to the one variable case

- Derivatives of space curves, $\vec{r}(t) =< x(t), y(t), z(t) >$,

$$\vec{r}'(t) =< x'(t), y'(t), z'(t) >$$

- Derivatives of $f(x, y)$ in specific directions
  1. Directional derivatives, $D_{\vec{v}}f$
  2. Partial derivatives, f_x, f_y
  3. Higher order partials

Differentiation

- Tangent plane: local approximation of a function if the function is *differentiable*
- Differentiability $\iff$ tangent planes vary continuously $\iff$ f_x, f_y exist and are continuous

Outline

Matrices

- An $m \times n$ matrix, A, is a map from $\mathbb{R}^n$ to $\mathbb{R}^m$
- $A\vec{v}$ is defined via the dot product
- A is a linear map, i.e.

$$A(a\vec{v} + b\vec{w}) = aA\vec{v} + bA\vec{w}$$

The matrix definition of the derivative

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function, then the $m \times n$ matrix Df is the derivative of f at $\vec{x}_0$ if

$$\lim_{\vec{h} \to \vec{0}} \frac{f(\vec{x}_0 + \vec{h}) - f(\vec{x}_0) - Df\vec{h}}{|\vec{h}|} = 0$$

Examples:

- $n = m = 1$
- $n = 2, m = 1$

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The Chain Rule

If $f : A \subset \mathbb{R}^m \to \mathbb{R}^n$, $g : B \subset \mathbb{R}^n \to \mathbb{R}^p$ are functions differentiable at $\vec{x}_0, \vec{y}_0$ respectively with $f(A) \subset B$ and $f(\vec{x}_0) = \vec{y}_0$ then the composition function $g \circ f$ is differentiable at $\vec{x}_0$ and

$$D(g \circ f)(\vec{x}_0) = Dg(\vec{y}_0) \cdot Df(\vec{x}_0)$$

The Chain Rule

- On Variable Chain rule:

$$\frac{d}{dx}f(g(x)) = \frac{df}{dx}(g(x))\frac{dg}{dx}$$

- In more than one variable, the chain rule is more complicated: $f(x, y)$ where $x = g(s), y = h(s)$. What is $\frac{\partial f}{\partial s}$?
- Idea: changes in s produce changes in both x and y so,

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

- We've really seen this already: restrict $f(x, y)$ to a curve to produce directional derivatives.

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Chain Rule

The general chain rule: Let f be a function of variables $x_1, \ldots, x_n$ and each x_i is a function of variables $s_1, \ldots, s_m$. To find $\frac{\partial f}{\partial s_i}$:

- Differentiate f with respect to each x_j
- Differenitate each x_j with respect to s_i
- Put everything together
- Helpful to draw "tree diagram"

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Implicit Differentiation

If we consider the curve $x^2 + y^2 = 1$, what is $\frac{dy}{dx}$?

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- Also works with a surface $F(x, y, z) = 0$

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Examples

1. $f(x, y) = ln(u^2 + v^2 + w^2), u = x + 2y, v = 2x - y, w = 2xy$
2. $z = \sin(a)\tan(b), a = 3s + t, b = s - t$
3. Consider the sphere $x^2 + y^2 + z^2 - r^2 = 0$ as a function of four variables which implicitly defines z as a function of x, y, r. What is $\frac{\partial z}{\partial r}$?
4. $z = f(x, y), x = r\cos(\theta), y = r\sin(\theta)$. Find $z_r, z_\theta, z_{r\theta}$

Work for next class

- Reading: 15.6
- 15.6 # 7,8,12,13,21
- f07hw10