Homework for Math 103
Assignment One – Due September 30

1. Show that the *countable* union of sets of measure zero in $\mathbf{R}$ has measure zero.

ANS: Suppose that E_n has measure zero for $n = 1, 2, \dots,$ and let $E = \bigcup E_n$. Let $\epsilon > 0$. By assumption, there are intervals $I_{n,m}$ such that $E_n \subset \bigcup_{m=1}^{\infty} I_{n,m}$ and $\sum_{m=1}^{\infty} \ell(I_{n,m}) < \frac{\epsilon}{2^{-n}}$. Then $E \subset \bigcup_{n,m=1}^{\infty} I_{n,m}$ and $\sum_{n,m=1}^{\infty} \ell(I_{n.m}) < \sum_{n=1}^{\infty} \frac{\epsilon}{2^{-n}} = \epsilon$. This suffices.

2. Suppose $f : [a, b] \to \mathbf{R}$ is bounded, and let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of $[a, b]$. Prove that $L(f, \mathcal{P}) \le U(f, \mathcal{Q})$, where $L(f, \mathcal{P})$ and $U(f, \mathcal{Q})$ are the lower and upper Riemann sums, respectively, for f on $[a, b]$. (Hint: the result is trivial if $\mathcal{P} = \mathcal{Q}$; now let $\mathcal{R} = \mathcal{P} \cup \mathcal{Q}$.)

ANS: The following are relatively easy to prove for any partition $\mathcal{P}$ and any prtition $\mathcal{R}$ such that $\mathcal{P} \subset \mathcal{R}$: $L(f, \mathcal{P}) \le U(f, \mathcal{P})$, $L(f, \mathcal{P}) \le L(f, \mathcal{R})$, and $U(f, \mathcal{R}) \le U(f, \mathcal{P})$. Thus if $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{R}$ are as in the problem, then

$$\begin{aligned} L(f, \mathcal{P}) &\le L(f, \mathcal{R}) \\ &\le U(f, \mathcal{R}) \\ &\le U(f, \mathcal{Q}). \end{aligned}$$

However, the challenge is the write up your solutions elegantly. Here are two suggestions for proving that $L(f, \mathcal{R}) \ge L(f, \mathcal{P})$.

Method I—Brute Force: Let $\mathcal{P} = \{ a = t_0 < \cdots < t_n = b \}$ and $\mathcal{R} = \{ a = s_0 < \cdots < s_m \}$. Since $\mathcal{P} \subset \mathcal{R}$, for any k, there is a unique i such that $[s_{k-1}, s_k] \subset [t_{i-1}, t_i]$. Furthermore,

$$t_i - t_{i-1} = \sum_{[s_{k-1}, s_k] \subset [t_{i-1}, t_i]} s_k - s_{k-1}. \tag{†}$$

Thus if

$$m_i := \inf_{t \in [t_{i-1}, t_i]} f(t) \quad \text{and} \quad n_k = \inf_{t \in [s_{k-1}, s_k]} f(t),$$

then we have $m_i \le n_k$ whenever $[s_{k-1}, s_k] \subset [t_{i-1}, t_i]$. Thus

$$\begin{aligned} L(f, \mathcal{R}) &= \sum_{k=1}^{m} n_k (s_k - s_{k-1}) \\ &= \sum_{i=1}^{n} \Big(\sum_{[s_{k-1}, s_k] \subset [t_{i-1}, t_i]} n_k (s_k - s_{k-1}) \Big) \\ &\ge \sum_{i=1}^{n} m_i \Big(\sum_{[s_{k-1}, s_k] \subset [t_{i-1}, t_i]} s_k - s_{k-1} \Big) \end{aligned}$$

which, by (†), is

$$\begin{aligned} &= \sum_{i=1}^{n} m_i (t_i - t_{i-1}) \\ &= L(f, \mathcal{P}). \end{aligned}$$

Method II—Simple: Suppose that $\mathcal{R}$ refines $\mathcal{P} = \{ a = t_0 < \cdots < t_n = b \}$ by adding a single point s where $t_{j-1} < s < t_j$. Define m_i as in "Method I" and let

$$n_{j1} = \inf_{t \in [t_{j-1}, s]} f(t) \quad \text{and} \quad n_{j2} = \inf_{t \in [s, t_j]} f(t),$$

and note that $m_j \le n_{1j} + n_{2j}$. Then

$$\begin{aligned} L(f,\mathcal{R}) &= \sum_{i=1}^{j-1} m_i(t_i - t_{i-1}) + n_{j1}(s - t_{i-1}) + n_{2j}(t_i - s) + \sum_{i=j+1}^{n} m_i(t_i - t_{i-1}) \\ &\le \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \\ &= L(f,\mathcal{P}). \end{aligned}$$

Now the general result follows from a simple induction.

3. Prove that a bounded function $f : [a,b] \to \mathbf{R}$ is Riemann integrable on $[a,b]$ if and only if for all $\epsilon > 0$ there is a partition $\mathcal{P}$ of $[a,b]$ such that

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon.$$

ANS: Note that for any partition $\mathcal{P}$, $L(f,\mathcal{P}) \le \mathcal{R}\underline{\int}_a^b f$ and $U(f,\mathcal{P}) \ge \mathcal{R}\overline{\int}_a^b$ Suppose that f is Riemann integrable. Then given $\epsilon > 0$ there are *partitions* $\mathcal{P}$ and $\mathcal{Q}$ such that

$$\begin{aligned} \mathcal{R}\underline{\int}_a^b f - L(f,\mathcal{P}) &< \frac{\epsilon}{2}, \quad \text{and} \\ U(f,\mathcal{Q}) - \mathcal{R}\overline{\int}_a^b f &< \frac{\epsilon}{2}. \end{aligned}$$

Now let $\mathcal{R} = \mathcal{P} \cup \mathcal{Q}$. Then using the previous problem, we see that the two inequalities above hold with $\mathcal{P}$ and $\mathcal{Q}$ replaced by $\mathcal{R}$. In particular, since f is integrable, $\mathcal{R}\underline{\int}_a^b f = \mathcal{R}\overline{\int}_a^b f = \mathcal{R}\int_a^b f$ and

$$U(f,\mathcal{R}) - L(f,\mathcal{R}) < \frac{\epsilon}{2} + \mathcal{R}\int_a^b f - \mathcal{R}\int_a^b f + \frac{\epsilon}{2} = \epsilon.$$

Now assume that for all $\epsilon > 0$ a partition $\mathcal{P}$ exists as stated in the problem. The previous problem implies that

$$\mathcal{R}\underline{\int}_a^b f \le \mathcal{R}\overline{\int}_a^b f.$$

Let $\epsilon > 0$ be given, and choose $\mathcal{P}$ such that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon$. Thus

$$\mathcal{R}\underline{\int}_a^b f \ge L(f,\mathcal{P}) > U(f,\mathcal{P}) - \epsilon \ge \mathcal{R}\overline{\int}_a^b f - \epsilon.$$

Thus,

$$0 \le \mathcal{R}\overline{\int}_a^b f - \mathcal{R}\underline{\int}_a^b f < \epsilon.$$

Since ϵ is arbitrary, it follows that $\mathcal{R}\overline{\int}_a^b f = \mathcal{R}\underline{\int}_a^b f$, and f is Riemann integrable on $[a,b]$ as required.

4. (*Rudin*: page 31 #1) Suppose that $(X,\mathcal{M})$ is a measurable space. Show that if $\mathcal{M}$ is countable, then $\mathcal{M}$ is finite. (Hint: since $\mathcal{M}$ is countable, you can show that $\omega_x = \bigcap\{\, E : E \in \mathcal{M} \text{ and } x \in E \,\}$ belongs to $\mathcal{M}$. The sets $\{\, \omega_x \,\}_{x \in X}$ partition X.)

ANS: Notice that if $E \in \mathcal{M}$ and if $x \in E$, then $\omega_x \subset E$. On the other hand, if $F \in \mathcal{M}$ and if $x \notin F$, then $x \in \omega_s \setminus F$, and $\omega_x \subset \omega_x \setminus F$ so $\omega_x \cap F = \emptyset$. Thus if $\omega_x \cap \omega_y \ne \emptyset$, then $x \in \omega_y$ and $\omega_x \subset \omega_y$. By symmetry, $\omega_y \subset \omega_x$ and $\omega_x = \omega_y$. This shows that $\{\, \omega_x \,\}_{x \in X}$ partitions X. If $x \in F \in \mathcal{M}$, then $\omega_x \subset F$ and $F = \bigcup_{x \in F} \omega_x$. Thus the elements of $\mathcal{M}$ are in one-to-one correspondence with the (distinct) subsets of $\{\, \omega_x \,\}_{x \in X}$. If this set is finite, then so is $\mathcal{M}$. If it is infinite, then it has at least as many subsets as does $\mathbf{Z}$ — and there are uncountably many of these.

5. Let X be an uncountable set and let $\mathcal{M}$ be the collection of subsets E of X such that either E or E^c is countable. Prove that $\mathcal{M}$ is a σ-algebra.

ANS: Since $\mathcal{M}$ certainly contains X and is closed under taking complements, the only issue is to show that $\mathcal{M}$ is closed under countable unions. Suppose that $\{ E_n \}_{n=1}^{\infty} \subset \mathcal{M}$. If all the E_n are countable, then the countable union $E = \bigcup_n E_n$ of countable sets is countable and $E \in \mathcal{M}$. If E_k^c is countable, then note that $E^c \subset E_k^c$ must also be countable. Thus in all cases, $E \in \mathcal{M}$ and $\mathcal{M}$ is a σ-algebra.

6. Recall from calculus that if $\{ a_n \}$ is a sequence of nonnegative real numbers, then $\sum_{n=1}^{\infty} a_n = \sup_n s_n$, where $s_n = a_1 + \cdots + a_n$. (Note the value ∞ is allowed.)

(a) Show that $\sum_{n=1}^{\infty} a_n = \sup\{ \sum_{k\in F} a_k : F \text{ is a finite subset of } \mathbf{Z}^+ = 1, 2, 3, \dots \}$. (The point of this problem is that if I is a (not necessarily countable) set, and if $a_i \geq 0$ for all $i \in I$, then we can define $\sum_{i\in I} a_i = \sup\{ \sum_{k\in F} a_k : F \text{ is a finite subset of } I \}$, and our new definition coincides with the usual one when both make sense.)

ANS: Let $I = \sup\{ \sum_{k\in F} a_k : F \text{ is a finite subset of } \mathbf{Z}^+ = 1, 2, 3, \dots \}$. Since $F = \{ 1, \dots, n \}$ is finite,

$$\begin{aligned} I &:= \sup\{ \sum_{k\in F} a_k : F = \{ 1, \dots, n \} \} \\ &\geq \sup s_n = \sum_{n=1}^{\infty} a_n. \end{aligned}$$

Let $\epsilon > 0$. Choose a finite set $F \subset \mathbf{Z}^+$ such that $\sum_{k\in F} a_k > I - \epsilon$. Let $m = \max_{k\in F} k$. Then $s_m \geq \sum_{k\in F} a_k > I - \epsilon$. Thus

$$\sum_{n=1}^{\infty} a_n = \sup s_n \geq s_m > I - \epsilon.$$

Since ϵ was arbitary, $\sum_{n=1}^{\infty} a_n \geq I$. Thus $\sum_{n=1}^{\infty} a_n = I$ as claimed.

(b) Now let X be a set and $f : X \to [0, \infty)$ a function. For each $E \subset X$, define

$$\nu(E) := \sum_{x\in E} f(x).$$

Show that ν is a measure on $\bigl(X, \mathcal{P}(X)\bigr)$. (In lecture, we considered the special cases of *counting measure*, where $f(x) = 1$ for all $x \in X$, and the *delta measure at* x_0, where $f(x_0) = 1$ for some $x_0 \in X$ and $f(x) = 0$ otherwise. Another important example is the case where $\sum_{x\in X} f(x) = 1$. Then f is a (discrete) probability distribution on X and $\nu(E)$ is the probability of the event E for this distribution.)

ANS: Let $\{ E_n \}$ be disjoint sets and $E = \bigcup_n E_n$. Suppose that $\nu(E) = \infty$. Fix $M > 0$. Choose a finite set $F \subset E$ such that $M < \sum_{x\in E} f(x)$. Let $F_n = \{ x \in F : x \in E_n \}$. Since all but finitely may F_n are empty,

$$M < \sum_{x\in E} f(x) = \sum_n \sum_{x\in F_n} f(x) \leq \sum_{n=1}^{\infty} \nu(E_n).$$

Since M is arbitray, $\sum_{n=1}^{\infty} \nu(E_n) = \infty = \nu(E)$. So we can assume from here on that $\nu(E) < \infty$. Let $\epsilon > 0$. Choose a finite set $F \subset E$ such that $\nu(E) - \epsilon < \sum_{x\in E} f(x)$. Let $F_n = \{ x \in F : x \in E_n \}$. Since all but finitely may F_n are empty,

$$\nu(E) - \epsilon < \sum_{x\in E} f(x) = \sum_n \sum_{x\in F_n} f(x) \leq \sum_{n=1}^{\infty} \nu(E_n).$$

Since ϵ is arbitrary, $\nu(E) \leq \sum_{n=1}^{\infty} \nu(E_n)$.

Notice that if $\nu(E) < \infty$, the $\nu(E_k) < \infty$ for all k. Let $\epsilon > 0$. Since $\sum_n \nu(E_n) = \sup_n \sum_{k=1}^n \nu(E_k)$ it will suffice to show that for any n

$$\nu(E) + \epsilon > \sum_{k=1}^n \nu(E_k).$$

Choose finite sets $F_k \subset E_k$ such that

$$\nu(E_k) - \frac{\epsilon}{n} < \sum_{x \in F_k} f(x).$$

Put $F = \bigcup_{k=1}^n F_k$. Note that F is finite, and since the F_k are disjoint,

$$\nu(E) \geq \sum_{x \in F} f(x) = \sum_{k=1}^n \sum_{x \in F_k} f(x) > -\epsilon + \sum_{k=1}^n \nu(E_k).$$

That's it.

(c) Let X, f, and ν be as in part (b). Show that if $\nu(E) < \infty$, then $\{ x \in E : f(x) > 0 \}$ is countable. Hint: if $\{ x \in E : f(x) > 0 \}$ is uncountable, then for some $m \in \mathbf{Z}^+$, the set $\{ x \in E : f(x) > \frac{1}{m} \}$ is infinite. (Note that this last result says that discrete probability distributions "live on" countable sample spaces.)

7. (*Rudin*: page 31 #3) Prove that if f is a real-valued function on a measurable space $(X, \mathcal{M})$ such that $\{ x : f(x) \geq r \}$ is measurable for all rational r, then f is measurable.

8. (*Rudin*: page 31 #5) Suppose that $f, g : (X, \mathcal{M}) \to [-\infty, \infty]$ are measurable functions. Prove that the sets

$$\{ x : f(x) < g(x) \} \quad \text{and} \quad \{ x : f(x) = g(x) \}$$

are measurable. (Remark: if $h = f - g$ were defined, then this problem would be much easier (why?). The problem is that $\infty - \infty$ and $-\infty + \infty$ make no sense, so h may not be everywhere defined.)

ANS: Since $\{ x : f(x) = g(x) \}$ is the complement of $\{ x : f(x) < g(x) \} \cup \{ x : g(x) < f(x) \}$ it suffice to see that $\{ x : f(x) < g(x) \}$ is measurable. But

$$\{ x : f(x) < g(x) \} = \bigcup_{r \in \mathbf{Q}} \{ x : f(x) < r < g(x) \},$$

and each $\{ x : f(x) < r < g(x) \} = g^{-1}\big((r, \infty]\big) \cap f^{-1}\big([-\infty, r)\big)$ is measurable.