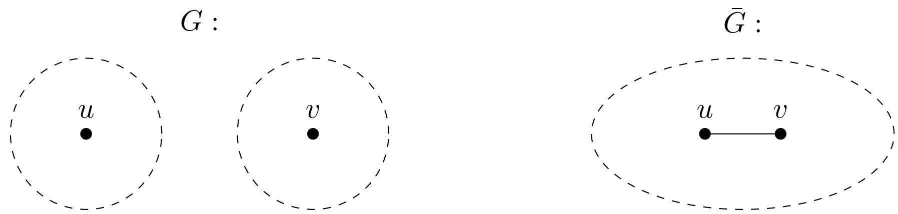
# Homework 1 solutions

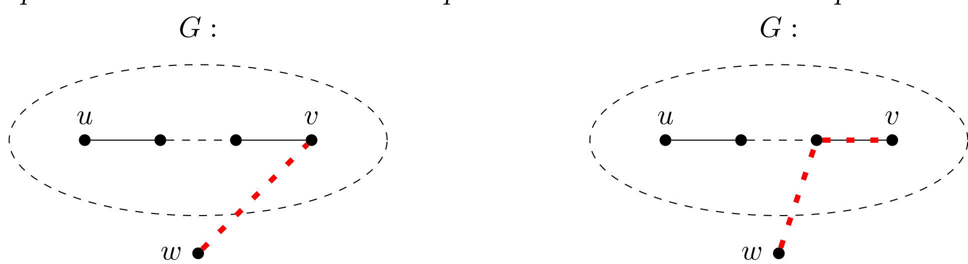
1.1.10 **If $G$ is simple and disconnected, then $\bar{G}$ is connected.**

*Proof.* Consider any two vertices $u, v \in V(\bar{G})$. We'll use the fact that $G$ is disconnected to show that $u$ and $v$ lie on a path in $\bar{G}$. Since this will be true for *any* two vertices in $\bar{G}$, this will imply that $\bar{G}$ is connected.

<u>Case 1: $u$ and $v$ are not connected in $G$.</u> If two vertices are not connected, then they're certainly not adjacent. So $uv \notin E(G)$, and therefore $uv \in E(\bar{G})$, and so $u$ and $v$ are connected in $\bar{G}$.



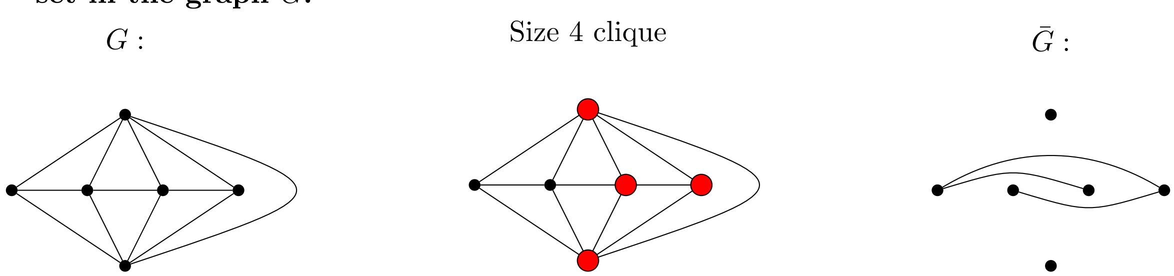

<u>Case 2: $u$ and $v$ are connected in $G$.</u> Since $G$ is not connected, there must be some third vertex $w$ which is not connected to either $u$ or $v$: *If $w$ is connected to $v$ by a path, then the subgraph which is the union of the two paths $w$ to $v$ and $u$ to $v$ has a path from $w$ to $u$.*




Then $w$ is not adjacent to $u$ or $v$ in $G$, so $uw$ and $uv$ are both in $E(\bar{G})$. So $u, uw, w, wv, v$ is a path connecting $u$ and $v$ in $\bar{G}$.

□

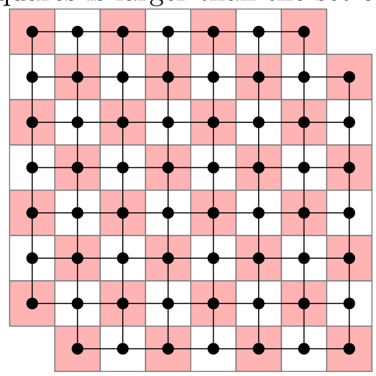
1.1.11 **Determine the maximum size of a clique and the maximum size of an independent set in the graph $G$:**




*Proof.* In a clique of size $n$, every vertex has degree $n-1$. In this graph, the maximal degree is 5. However, there are only 2 vertices with degree 5, so there is no clique of size 5. Similarly, there are only four vertices of degree 4, so there is no clique of size 5. There is, however a clique of size 4 (drawn above). An independent set is a clique in $\bar{G}$ (also drawn above). Since $\bar{G}$ is a path, any maximal independent set in $G$ is of size 2. □

1.1.14 **Prove that removing opposite corner squares from an 8-by-8 checkerboard leaves a subboard that cannot be tiled with 1-by-2 rectangles. Using the same argument, make a general statement about all bipartite graphs.**

*Proof.* Notice that the opposite corners are of the same color. So when you remove those two squares, there are more squares left of one color (say red) than of the other (say white). A tiling of the board is a perfect matching between red squares and white squares, which we can't have if the set of red squares is larger than the set of white squares.



To frame this question in terms of bipartite graphs, we can think about the checkerboard as a map that is two-colorable. Draw vertices for each square and connect two vertices if the squares share a border. Since this adjacency graph is two-colorable it is bipartite (the two partites being the set of white and red squares, respectively). A tiling is a choice of $k$ disjoint edges which, as a set, cover all vertices. In general, this cannot be done if the two partites are of different sizes.

□

1.1.15 **Consider the following four families of graphs:**

$$A = \{\textbf{paths}\}, B = \{\textbf{cycles}\}, C = \{\textbf{complete graphs}\}, D = \{\textbf{bipartite graphs}\}.$$

**For each pairs of families, determine all isomorphism classes of graphs belonging to both families.**

*Answer:* Isomorphism classes are just unlabeled graphs. For a fixed number of vertices $n$ the size of the edge sets are as follows:

$$|E(P_n)| = n - 1, \qquad |E(C_n)| = n, \qquad |E(K_n)| = \binom{n}{2} = \tfrac{1}{2}n(n-1).$$

A partite graph with partites of sizes $r \leq s$ has at most $r^2$ edges.

$\{\textbf{paths}\} \cap \{\textbf{cycles}\} = \emptyset$**:** The edge sets of $C_n$ and $P_n$ are always of different sized (there is no $C_1$ or $C_2$ if we restrict to simple graphs).

$\{\textbf{paths}\} \cap \{\textbf{complete graphs}\} = \{P_1 = K_1, P_2 = K_2\}$**:** Counting edges again, $n - 1 = \frac{1}{2}n(n-1)$ has exactly two solutions: $n = 1$ and $n = 2$, so $P_1$ and $P_2$ are the only two candidates. Checking each individually, we see that these are, in fact, both complete.

**$\{$paths$\} \cap \{$bipartite graphs$\} = \{$paths$\}$:** By alternating, all paths are two-colorable, and so are bipartite.

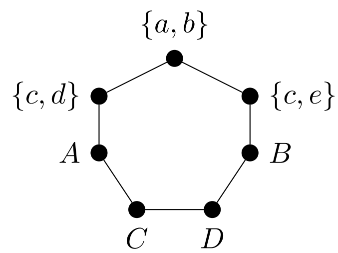
**$\{$cycles$\} \cap \{$complete graphs$\} = \{C_3 = K_3\}$:** Back to counting edges, the only solutions to $n = \frac{1}{2}n(n-1)$ are $n = 1$ and $n = 3$. However, there is not simple cycle with one vertex, so $C_3$ is the only candidate. Upon inspection, $C_3 = K_3$.

**$\{$cycles$\} \cap \{$bipartite graphs$\} = \{C_{2k}\}_{k \in \mathbb{Z}_{\geq 2}}$:** A cycle is a type walk from a vertex back to itself. In a bipartite graph, any walk must alternate between the parites, so any walk from a vertex back to itself must be of even length (have an even number of edges). So the even cycles are the only candidates. And, in fact, by alternating colors, we can always 2-color an even cycle, and so every even cycle is bipartite.

**$\{$complete graphs$\} \cap \{$bipartite graphs$\} = \{K_1, K_2\}$:** Any bipartite graph with more than two vertices must be missing at least one edge, so is not complete. However, since $K_2$ is a path, it's bipartite. The single vertex, $K_1$ is also bipartite with one empty partite.

1.1.25 **Argue that the Petersen graph $G$ has no 7-cycles.**

*Proof.* Suppose there is some 7-cycle in $G$. Pick some vertex in $G$ to start at, and for the sake of generality, call it $\{a, b\}$. Its two neighbors cannot have any elements in common, but there are only three numbers left to choose from, so those two neighbors must have an element in common; call it $c$.

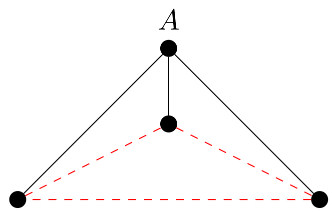



The girth of $G$ is 5. Therefore $\{a, b\}$ must not be adjacent to and of $A, B, C$, or $D$. So all of the unknown vertices must have exactly one intersection with $\{a, b\}$. Similarly, $C$ and $D$ must both have exactly one intersection with $\{c, d\}$ and with $\{c, e\}$. So $C$ and $D$ must both contain $c$, which is a contradiction. □

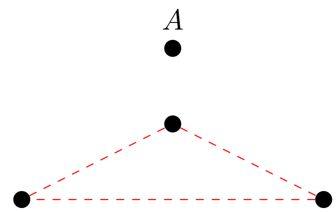
1.1.29 **Prove that every set of six people contains at at least three mutual acquaintances of three mutual strangers.**

*Proof.* Pick a person $A$. From the remaining 5 people, $A$ either has at least three people who she knows or three people that she doesn't know. If $A$ knows three people, then either there's some pair of those people who also know each other (forming a clique of size at least three), or no two of them know each other (in which case they form an independent set of size three. If $A$ has three strangers, then either there is some pair of those strangers who also don't know each other (forming an independent set of size three with $A$), or they all know eachother (forming a clique of size three.

Case 1:




Case 2:

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