Workshop 2
"$\mathcal{A}$ implies $\mathcal{B}$" Exercises

Instructions:

Get into groups and work on the following exercises. Each group is expected to turn in *one neatly written copy* of their solutions at the end of the class period.

The exercises in this set ask you to prove or disprove various statements of the form "$\mathcal{A}$ implies $\mathcal{B}$". In each case, identify the hypothesis ($\mathcal{A}$) and the conclusion ($\mathcal{B}$) of the statement. Then prove or disprove the statement (whichever is indicated).

Exercise 1. Let A be an $m \times n$ matrix, let $\mathbf{b}, \mathbf{c} \in \mathbb{R}^m$ and let c be a scalar. Prove the following statements.

a. If $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{c}$ are both consistent then $A\mathbf{x} = \mathbf{b} + \mathbf{c}$ is consistent.

b. If $A\mathbf{x} = \mathbf{b}$ is consistent then $A\mathbf{x} = c\mathbf{b}$ is consistent.

Exercise 2. [*The warm up*] Let A be an $m \times n$ matrix, let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and let c be a scalar. Prove the following.

a. If $\mathbf{u}$ and $\mathbf{v}$ are solutions to $A\mathbf{x} = \mathbf{0}$ then so is $\mathbf{u} + \mathbf{v}$.

b. If $\mathbf{u}$ is a solution to $A\mathbf{x} = \mathbf{0}$ then so is $c\mathbf{u}$.

Exercise 3. [*The main event*] Let A be an $m \times n$ matrix. Show that if $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are solutions to $A\mathbf{x} = \mathbf{0}$ and $\mathbf{v} \in \text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ then $\mathbf{v}$ is also a solution to $A\mathbf{x} = \mathbf{0}$.

Exercise 4. Prove that the following statement is *false*: If the vectors $\mathbf{u}$ and $\mathbf{v}$ are solutions to the system $A\mathbf{x} = \mathbf{b}$ then so is $\mathbf{u} + \mathbf{v}$.

Exercise 5. Prove or disprove: If A and B are 2×2 matrices and $\mathbf{u} \in \mathbb{R}^2$ then $A(B\mathbf{u}) = B(A\mathbf{u})$.