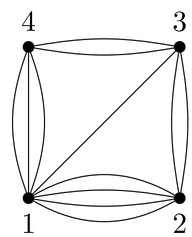
HOMEWORK 5

SOLUTIONS

2.2.3 Let G be the graph below. Use the Matrix Tree Theorem to find a matrix whose determinant is $\tau(G)$.



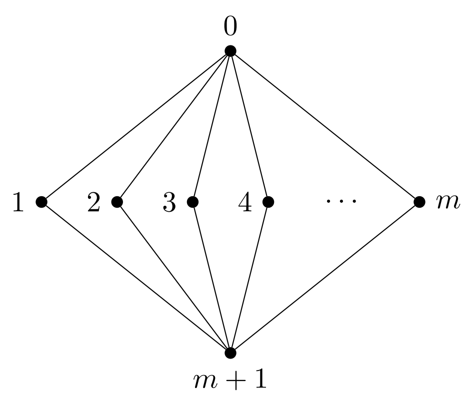
Answer:

$$D = \begin{pmatrix} 8 & & & \\ & 6 & & \\ & & 5 & \\ & & & 5 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 4 & 1 & 3 \\ 4 & 0 & 2 & 0 \\ 1 & 2 & 0 & 2 \\ 3 & 0 & 2 & 0 \end{pmatrix} \quad \text{so} \quad Q = \begin{pmatrix} 8 & -4 & -1 & -3 \\ -4 & 6 & -2 & 0 \\ -1 & -2 & 5 & -2 \\ -3 & 0 & -2 & 5 \end{pmatrix}$$

So, for example

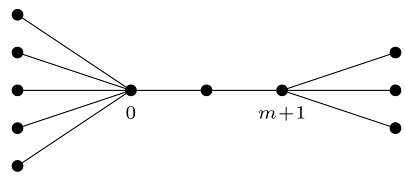
$$\tau(G) = \det \begin{pmatrix} 6 & -2 & 0 \\ -2 & 5 & -2 \\ 0 & -2 & 5 \end{pmatrix} = \det \begin{pmatrix} 8 & -4 & -1 \\ -1 & -2 & 5 \\ -3 & 0 & -2 \end{pmatrix} = -\det \begin{pmatrix} -4 & -2 & 0 \\ -1 & 5 & -2 \\ -3 & -2 & 5 \end{pmatrix} = 106$$

2.2.10 Compute $\tau(K_{2,m})$. Also, compute the number of isomorphism classes of spanning trees of $K_{2,m}$.



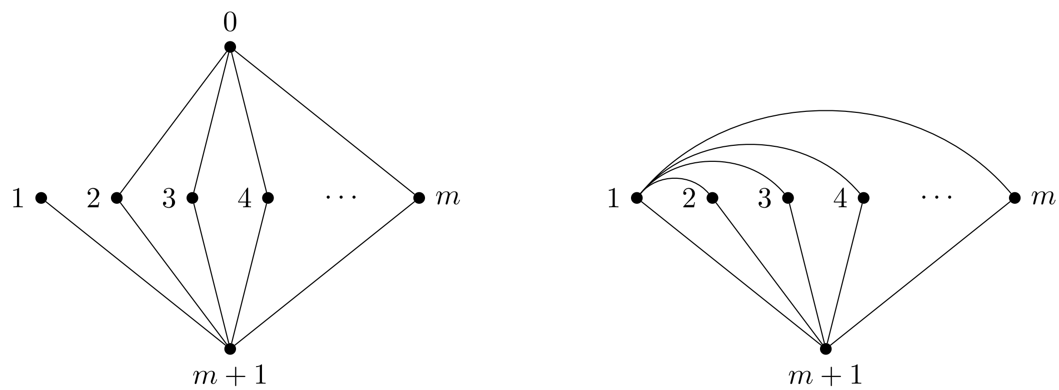
Answer. **Method 1:** Since any spanning tree T must be connected, there has to be some minimal path from 0 to $m+1$. Since T is acyclic, there can only be one such path. Pick a vertex amongst $\{1, \ldots, m\}$ for the unique path to go through. For every other vertex, pick one edge (connecting to 0 or $m+1$). This gives $m * 2^{m-1}$ spanning trees.

Each one of the resulting trees looks like

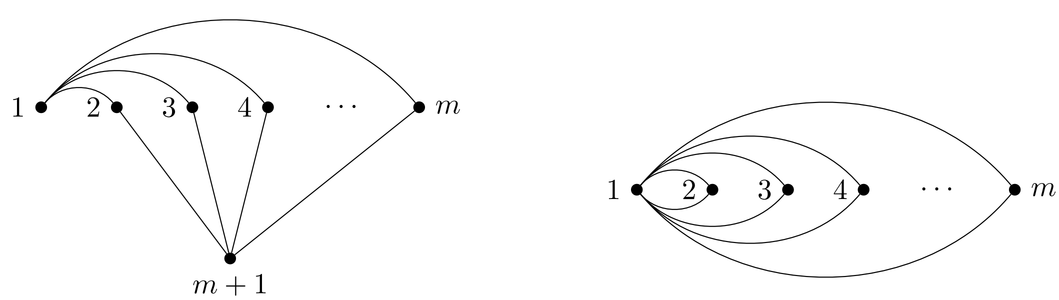


There are $\lceil \frac{m-1}{2} \rceil$ isomorphism classes of these ([1 and $m-2$ leaves]+[2 and $m-3$ leaves]+$\cdots$).

Method 2: Using the formula $\tau(G) = \tau(G-e) + \tau(G \cdot e)$, the number of spanning trees of $K_{2,m}$ is the same as the sum of the number of spanning trees for the two graphs



The one on the left has the same number of spanning trees at $K_{2,m-1}$. The one of the right simplifies again to the sum of the number of spanning trees for the two graphs



Again, the graph on the left is $K_{2,m-1}$. The graph on the right is a an m-start with doubled edges. So

$$\tau(K_{2,m}) = 2\tau(K_{2,m-1}) + 2^{m-1}.$$

Using induction, you can see that $n2^{n-1}$ satisfies this recursion relation.

Fun fact: if you have a recursion relation that you can't seem to work out on your own, you can work out the first few cases on your own and plug those first few values into the Online

Encyclopedia of Integer Sequences (http://oeis.org/). For example, when $m = 1$, $K_{2,1} = P_2$. So

$$\tau(K_{2,1}) = 1, \quad \tau(K_{2,2}) = 2 * 1 + 2^1 = 4, \quad \tau(K_{2,3}) = 2 * 4 + 2^2 = 12,$$
$$\tau(K_{2,4}) = 2 * 12 + 2^3 = 32, \quad \tau(K_{2,5}) = 2 * 32 + 2^4 = 80, \ldots$$

Try plugging in 1,4,12,32,80 into the OEIS.

To count isomorphism classes, notice that in the recursion, we're implying that at some step I'm taking a tree from a doubled star (where some pair of edges incident from the middle vertices are chosen for the tree – those were the contracted edges which caused the doubled star), and all other middle vertices were adjacent to exactly one of the vertices from the partite of size 2. See above.

2.2.23 Prove that if the Graceful Tree Conjecture is true and T is a tree with m edges, then K_{2m} decomposes into $2m-1$ copies of T. (Hint: Apply the cyclically invariant decomposition of K_{2m-1} for trees with $m-1$ edges from the proof of Theorem 2.2.16.)

Proof. Let $T' = T - v$ where v is some leaf of v. Decompose the complete graph G on $[2m-1]$ into $2m-1$ copies of T'. Each vertex in G is represented by the neighbor of v in T exactly once in this decomposition, so the complete graph on $[2m]$ can be built by adding one edge from that neighbor to vertex $2m$ for each copy of T'. This presents a decomposition of K_{2m} into $2m-1$ copies of T. □

2.2.31 An *up-down labeling* is a graceful labeling for which there exists a critical value α such that every edge joins vertices with labels above and below α. Prove that ecery caterpillar has an up-down labeling. Prove that the 7-vertex tree that is not a caterpillar has no up-down labeling.

Proof.

(1) Caterpillars have up-down labelings. Order the vertices of the spine (*vertebrae*) from left to right, noting that the endpoints are leaves. We'll show that there's an up-down labeling so that the right-most vertebrae is labeled with a 0, by induction on the number k of vertebrae.

Base case: If $k = 2$, the caterpillar is P_2, which trivially has up-down labeling with $\alpha = \frac{1}{2}$. Put the 0 on the right.

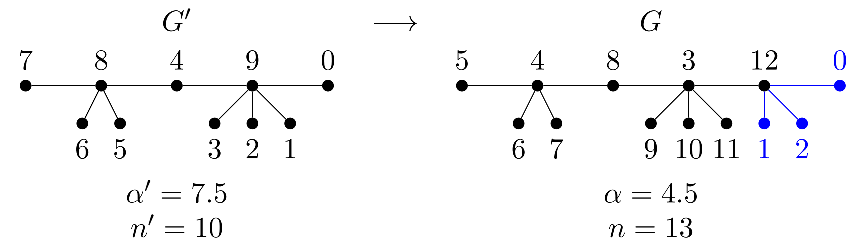
Induction hypothesis: Any caterpillar with spine of length k has an up-down labeling with the right-most vertex labeled with 0.

Induction step: Let G be a caterpillar with spine of length $k+1$. Consider the second to last vertebrae on the right v, and let G' be the subgraph induced by deleting all of the leaves adjacent to v (except possibly one if G is a star). G' is a caterpillar with spine of length k, so by the induction hypothesis has an up-down labeling with a 0 on v; let α' be the critical value of that labeling. Now label G as follows:

(a) label the deleted leaves with $0, 1, \ldots, d(v) - 2$ (there are $d(v) - 1$ leaves), and

(b) for every vertex of G also in G', if a' is the label in G', replace with the label $a = n(G) - 1 - a'$.

For example,



Claim: This labeling of G is an up-down labeling with $\alpha = n(G) - 1 - \alpha'$.
First, this labeling is graceful since the edge differences in G' are fixed by their relabelings ($|(n-1-a')-(n-1-b')| = |a'-b'|$), and so lend differences $0, 1, \ldots, n'-1 = n-(d(v)-2)-1 = n+d(v)+1$. The differences contributed by the new leaves are $n-1, n-2, \ldots, n-(d(v)-2) = n - d(v) + 2$. So the new labeling is graceful.

Take any two vertices in G'. They are labeled in G' with labels a', b' so that

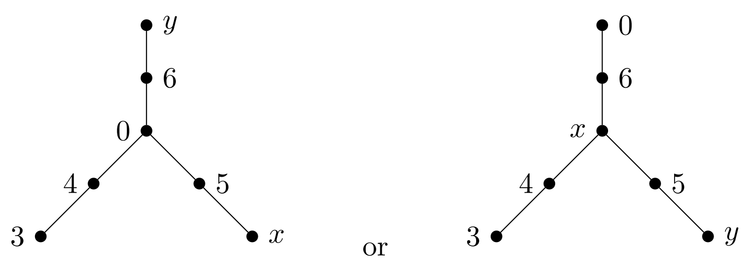
$$a' < \alpha' < b', \qquad \text{so} \qquad a = n-1-a' > n-1-\alpha' > n-1-b' = b.$$

Therefore every edge joining two vertices in $V(G')$ have the up-down property (that one label in G is above $\alpha = n - 1 - \alpha'$ and one is below). Furthermore, the largest label applied to the new vertices is $d(v) - 2$, so since $n = n' + d(v) - 1$ and $0 < \alpha < n' - 1$, we have

$$d(v) - 2 < n - 1 - \alpha' < n - 1$$

so all of the new leaves in G also have the up-down property.

(2) Y does not have and up-down labeling. Let's show by contradiction and suppose Y does have an up-down labeling with critical value α. Without loss of generality, suppose the center vertex has label less than α. Then the three adjacent to the center vertex are greater than α and the three leaves are less. So $\{0, 1, 2, 3\}$ cover the middle and the leaves and $3 < \alpha < 4$. In order for some edge difference to be 1, this means that $3 \leftrightarrow 4$. Also, since there has to be an edge difference of 6, $6 \leftrightarrow 0$. So we either have



In either case, there is no way to assign 1 and 2 to x and y so that the labeling is graceful. $\square$

2.2.33 Let T be an orientation of a tree such that the heads of the edges are all distinct; the one vertex that is not a head is the *root*. Prove that T is a union of paths from the root. Prove that for each vertex of T, exactly one path reaches it from

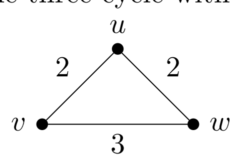
the root.

Proof. Let r be the root in T and let T' be the tree without the orientation. For any vertex u in T, there is a unique r, u path P in T'. The last vertex reachable along P from r in T is the first vertex from which there is no arrow out of v after v in P; but the edge preceding v in P pointed in to v, so $v = u$. Furthermore, P is the only such path because an r, u-path in T is an r, u-path in T'.

Since every vertex is reachable from a r, u path, and T' is acyclic, T is the union of all of these paths. □

2.3.2 Prove or disprove: if T is a minimum-weight spanning tree of a weighted graph G, then a u, v-path in T is a minimum-weight u, v-path in G.

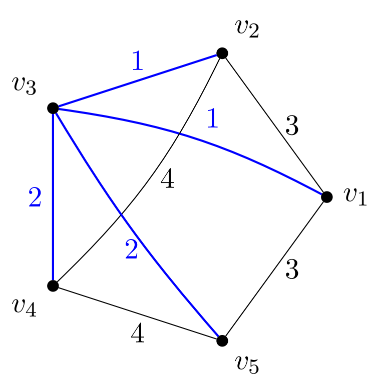
Answer. This is false. Consider the three cycle with weights 2, 2, and 3.



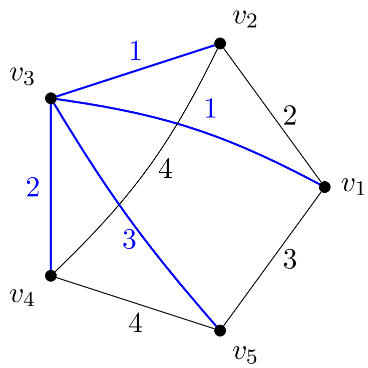
The minimal spanning tree is the set of edges uv and uw, but the minimal v, w path is along the edge vw.

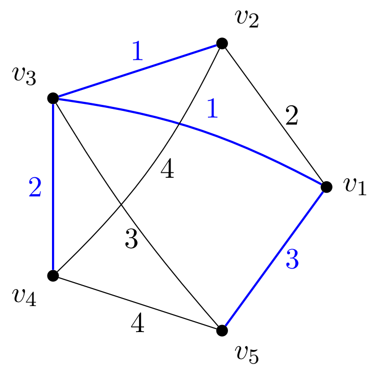
2.3.4 In the graph obtained from K_5 by deleting two non-incident edges, assign weights $(1, 1, 2, 2, 3, 3, 4, 4)$ to the edges in two ways: one way so that the minimum-weight spanning tree is unique, and another way so a minimum-weight spanning tree is not unique.

Unique:



Non-unique:





2.3.10 Prim's Algorithm grows a spanning tree from a given vertex of a connected weighted graph G, iteratively adding the cheapest edge from a vertex already reached to a vertex not yet reached, finishing when all the vertices of $V(G)$ have been reached. (Ties are broken arbitrarily.) Prove that Prim's Algorithm produces a minimum-weight spanning tree of G.

Proof. Since G is connected, the algorithm terminates when every vertex is covered, and each step covers a new vertex with one new edge. So the result is a connected graph with n vertices and n-1 edges, and is therefore a tree.

Now suppose T is the result and T^* is a minimum spanning tree. Let e be the first edge added to T which is not in T^*, and let S be the set of vertices added to T up until this point. Say e has endpoints u in S and v not in S. Then not only does $T^* + e$ have a cycle, but that cycle is created by the two paths that now exist from u to v (there was a unique u, v path in $T*$, and now there's the one along e). So there is some edge e^* in that cycle which is incident to exactly one vertex S; since e is amongst the cheapest edges incident to exactly one vertex in S, $w(e) \leq w(e^*)$, and so $T^* - e + e^*$ is a spanning tree with weight less than or equal to that of $T*$ (and therefore equal to) with one more edge in common with T. By iteration, we find that $\text{wt}(T) = \text{wt}(T^*)$ and is therefore a minimum.

$\square$