MATH 46 HW4 ~ SOLUTIONS ~ 4/23/07 Barnett ①

p. 100-104 #10

$$A = \begin{pmatrix} \varepsilon & 1 \\ 1 & 101 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 1/10 \\ 11 \end{pmatrix} \qquad \text{solve } A\vec{x} = \vec{b}, \quad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

For any ε an exact soln exists:
$$\varepsilon x + y = \tfrac{1}{10}$$
$$x + 101 y = 11$$

eliminate y to get $x(1-101\varepsilon) = 11 - \frac{101}{10}$ ie $x = \dfrac{0.9}{1-101\varepsilon}$

" x to get $y(1-101\varepsilon) = \frac{1}{10} - 11\varepsilon$ ie $y = \dfrac{\frac{1}{10}-11\varepsilon}{1-101\varepsilon}$ } exact for all ε

Taking $\varepsilon = 0.01$ we get $\begin{cases} x = -90 \\ y = 1 \end{cases}$ but with $\varepsilon = 0$ get $\begin{cases} x_0 = 0.9 \\ y_0 = 0.1 \end{cases}$

so the zeroth order approx. is terrible (useless)!

What went wrong? i) Well, the ε at which 'all hell breaks loose' is $\varepsilon = \frac{1}{101} = 0.0099\ldots$
($x, y \to \infty$ as $\varepsilon \to \frac{1}{101}$ using exact soln)
Our choice $\varepsilon = 0.01$ is very close to this, hence the v. large $x = 90$.

ii) 2-term expansion comes by turning the exact soln (x,y) into a power series in ε
(you may also get it by substituting $x = x_0 + \varepsilon x_1 \ldots$, $y = y_0 + \varepsilon y_1 \ldots$ in eqn).

$$x = 0.9(1-101\varepsilon)^{-1} = 0.9\left(1 + 101\varepsilon + O(\varepsilon^2)\right) = 0.9 + \underbrace{90.9}_{x_1 = \text{amplification factor}}\varepsilon + \cdots$$

$$y = \left(\tfrac{1}{10} - 11\varepsilon\right)(1-101\varepsilon)^{-1} = \tfrac{1}{10} - 11\varepsilon + 10.1\varepsilon + O(\varepsilon^2) = 0.1 \underbrace{-0.9}_{y_1 \text{ ampl. factor}}\varepsilon + \cdots$$

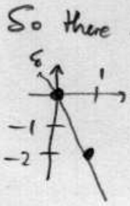
... so roughly small changes of ε about 0 amplified by 10^2.

p. 121-123 #10

$$\varepsilon u'' - a(x) u = f(x) \qquad \text{with } u(0) = 0, \quad u(1) = -\frac{f(1)}{a(1)}$$

Outer (use $\varepsilon = 0$): $u_0(x) = -\dfrac{f(x)}{a(x)}$ already matches BC at $x=1$, so no bdry layer there.

So there can only be BL @ $x=0$: set $\xi = \frac{x}{\delta}$, inner $\frac{\varepsilon}{\delta^2} U'' - a(\delta\xi) U = f(\delta\xi)$



dom. balancing:
$\delta^{-2}\varepsilon = 1$ so $\delta = \sqrt{\varepsilon}$.

so $\underbrace{U_i'' - a(0) U_i = f(0)}_{\text{leading order}}$ (set $\varepsilon = 0$).

Inner: particular solution is $U_i(\xi) = -\frac{f(0)}{a(0)}$ const.

gen. solution is $U_i(\xi) = c_1 e^{-\sqrt{a(0)}\,\xi} + \underbrace{c_2 e^{+\sqrt{a(0)}\,\xi}}_{\text{blows up as } \xi\to\infty \text{ so not valid BL function.} \Rightarrow c_2=0.} - \frac{f(0)}{a(0)}$

match BC $u(0)=0$ so $c_1 = +\frac{f(0)}{a(0)}$ ie $U_i(\xi) = \frac{f(0)}{a(0)}\left(e^{-\sqrt{a(0)}\,\xi} - 1\right)$

Note matching of asymptotic expansions is already done: $\lim_{\xi\to\infty} U_i(\xi) = \lim_{x\to 0} u_0(x) = \underbrace{-\frac{f(0)}{a(0)}}_{c_m}$

so uniform $u_u(x) = -\frac{f(x)}{a(x)} + \frac{f(0)}{a(0)} e^{-\sqrt{a(0)}\frac{x}{\sqrt{\varepsilon}}}$

p.133-135 #1 $\quad \varepsilon y' + y = e^{-t} \qquad y(0) = 2 \qquad t \geq 0$

outer $y_0(t) = e^{-t}$

inner $\tau = \frac{t}{\delta}$ so $\underbrace{\frac{\varepsilon}{\delta}Y' + Y}_{\text{balance using } \delta=\varepsilon.} = e^{-\delta\tau}$

$\Rightarrow Y' + Y = e^{-\varepsilon\tau} \rightsquigarrow e^{-0} = 1$ for leading order. $\qquad Y_i(\tau) = 1 + e^{-\tau}$, $c_m = 1$.

$y_u(t) = y_0 + y_i - c_m = e^{-t} + e^{-t/\varepsilon}$

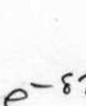
Residual $r(y_u;\varepsilon) := \varepsilon y_u' + y_u - e^{-t}$

$= -\varepsilon e^{-t} + \cancel{\varepsilon\frac{1}{\varepsilon}e^{-t/\varepsilon}} + \cancel{e^{-t}} \cancel{-e^{-t/\varepsilon}} \cancel{-e^{-t}} = -\varepsilon e^{-t} = O(\varepsilon)$

since $|e^{-t}| < 1$, uniformly on $[0,\infty)$.

#3 $\quad \varepsilon y'' + (t+1)^2 y' = 1 \qquad y(0) = 1, \quad \varepsilon y'(0) = 1 \qquad$ outer $y_0' = \frac{1}{(1+t)^2}$

Inner: $(\tau = \frac{t}{\delta})$: $\frac{\varepsilon}{\delta^2}Y'' + (\delta\tau+1)^2\frac{1}{\delta}Y' = 1$



so $\delta = \varepsilon$. for dom. balancing

mult. by ε $\to$ $Y_i'' + Y_i' + \cancel{O(\varepsilon)} = \cancel{\varepsilon}$ ignore so $Y_i'' = Ae^{-\tau} + \overbrace{B}^{c_m}$

match ICs: $y(0) = 1$ gives $A + B = 1$

$\varepsilon y'(0) = 1 \xrightarrow{\text{rescale}} \frac{\varepsilon}{\delta}Y'(0) = 1$ ie $Y'(0) = 1$ gives $-A = 1$ $\Big\}$ $A = -1$, $B = 2$

Only now can we get correct const in y_0: $y_0 = \int \frac{1}{(1+t)^2}dt + c = \frac{-1}{1+t} + c$

but $\lim_{t\to 0} y_0(t) = c_m = 2$ so $c = 3$

Combine: $y_u(t) = y_0 + y_i - c_m = 3 - \frac{1}{1+t} - e^{-t/\varepsilon}$

p. 141-142 (#1) $\varepsilon^2 y'' - \overbrace{(1+x^2)^2}^{k^2(x)} y = 0$ $\quad y(0)=0 \quad y'(0)=1$

with $\varepsilon = \lambda^{-1/2}$

gen soln. $y = c_1 \frac{1}{\sqrt{1+x^2}} e^{\frac{1}{\varepsilon}\int(1+x^2)dx} + c_2 \frac{1}{\sqrt{1+x^2}} e^{-\frac{1}{\varepsilon}\int(1+x^2)dx}$

IC $y(0)=0$ means $c_1 = -c_2$

Now $y'(x) = c_1\left[-\frac{1}{2}\cdot 2x \frac{1}{(1+x^2)^{3/2}} e^{m} + \frac{1}{(1+x^2)^{1/2}}\cdot \frac{1}{\varepsilon}(1+x^2) e^{m}\right] + c_2\left[-\text{same} - \text{same}\right]$ but with e^{-m}

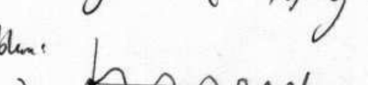
use $c_2 = -c_1$ and sub $x=0$:

$$y'(0) = 2c_1 \cdot \frac{1}{(1+0^2)^{1/2}}\cdot\frac{1}{\varepsilon}(1+0^2)e^0 = c_1\frac{2}{\varepsilon} \quad \text{so} \quad c_1 = \frac{\varepsilon}{2}$$

Solution is $y(x) = \frac{\varepsilon}{2}\frac{1}{\sqrt{1+x^2}}\underbrace{\left[e^{\frac{1}{\varepsilon}(x+\frac{x^3}{3})} - e^{-\frac{1}{\varepsilon}(x+\frac{x^3}{3})}\right]}_{\text{also known as } \sinh\left(\frac{1}{\varepsilon}[x+\frac{x^3}{3}]\right)}$

(#2) $\varepsilon^2 y'' + (\pi+x)^4 y = 0$ with $\varepsilon = \lambda^{-1/2}$ and $k(x) = (\pi+x)^2$

Eigenproblem: (high freq.)



using worksheet results WKB approx $\rightarrow$ $\varepsilon_n \approx \frac{\int_0^\pi k(x)dx}{\pi n} = \frac{[x^3/3]_\pi^{2\pi}}{\pi n}$

$= \frac{7\pi^2}{3n}$

so $\lambda_n = \frac{1}{\varepsilon_n^2} = \left(\frac{3n}{7\pi^2}\right)^2$ as given, eigenfuncs. $y_n(x) = \frac{1}{\pi+x}\sin\left(\frac{\int_0^x k(z)dz}{\varepsilon_n}\right)$

$= (\pi+x^{-1})\sin\left(\frac{3n}{7\pi^2}\left[\frac{x^3}{3}+\pi x^2+\pi^2 x\right]\right)$

(#3) $\varepsilon^2 y'' + xy = 0$ with $\lambda = \frac{1}{\varepsilon^2}$, $1<x<4$, $y(1)=y(4)=0$

$k(x)=\sqrt{x}$ $\quad \lambda_n = \left(\frac{\pi n}{\int_1^4 k(x)dx}\right)^2 = \frac{\pi^2 n^2}{\left(\underbrace{[\frac{2}{3}x^{3/2}]_1^4}_{\frac{2}{3}(8-1)}\right)^2} = \left(\frac{3}{14}\pi n\right)^2$

Eig. funcs. $y_n(x) = \frac{1}{x^{1/4}}\sin\left(\frac{1}{\varepsilon_n}\int_1^x s^{1/2}ds\right) = \frac{1}{x^{1/4}}\sin\left(\frac{3\pi n}{14}\cdot\frac{2}{3}[x^{3/2}-1]\right) = \frac{1}{x^{1/4}}\sin\left(\frac{\pi n}{7}[x^{3/2}-1]\right)$

use $\varepsilon_n = \frac{3}{14\pi n}$

note since we choose lower limit of $x=1$ (left edge) for action integral, can get correct BC by using sin (as on p. 140).

a) $my'' + ay' + kye^{-rt} = 0$ w/ ICs $y(0) = y_0$, $y'(0) = 0$

b)
$$\begin{array}{c|ccccc} & m & a & k & r & y_0 \\ \hline M & 1 & 1 & 1 & & \\ L & & & & & 1 \\ T & & -1 & -2 & -1 & \end{array}$$
5 params.

c) $k \ll 1$ so choose $t_c = \frac{m}{a}$ (it's fine to choose $t_c = r$ instead; everything is similar except β's come in different places).

we're told $y_c = y_0$

ND: $m \frac{y_c}{t_c^2} \bar{y}'' + a \frac{y_c}{t_c} \bar{y}' + k y_c \bar{y} e^{-rt_c \bar{t}} = 0$ ICs: $y_c \bar{y}(0) = y_c$, $\bar{y}'(0) = 0$

sub in t_c, y_c: $\bar{y}'' + \bar{y}' + \varepsilon \bar{y} e^{-\beta \bar{t}} = 0$ where $\varepsilon = \frac{mk}{y_0 a^2} \ll 1$, $\beta = \frac{mr}{a}$

now there's only 2 (ND) params!

d) Unperturbed (zeroth order) $y_0'' + y_0' = 0$ (dropping the 'bars') with IC $y_0(0) = 1$, $y_0'(0) = 0$

solution is $y_0(t) = 1 \quad \forall t > 0$.

1st order:

$y_0'' + \varepsilon y_1'' + \cdots + y_0' + \varepsilon y_1' + \cdots + \varepsilon y_0 e^{-\beta t} + \varepsilon^2 y_1 e^{-\beta t} + \cdots = 0$

ε^1: $\underbrace{y_1'' + y_1'}_{\text{homog. soln } c_1 + c_2 e^{-t}} = -y_0(t) e^{-\beta t} = \underbrace{-e^{-\beta t}}_{\text{driving}}$ with homog. ICs. $y_1(0) = y_1'(0) = 0$

Two cases:

$\boxed{\beta \neq 1}$: (since then driving is not a homog. soln): M.U.C. $y = Ae^{-\beta t}$

so $\beta^2 A - \beta A = -1$, ie $A = \frac{1}{\beta(1-\beta)}$

so $y_1(t) = -\frac{e^{-\beta t}}{\beta(1-\beta)} + c_1 + c_2 e^{-t}$ ICs: so $\begin{cases} y_1'(0) = \frac{-1}{1-\beta} - c_2 \text{ so } c_2 = \frac{-1}{1-\beta} \\ y(0) = \frac{1}{\beta(1-\beta)} + c_1 - \frac{1}{1-\beta} = 0 \end{cases}$

$\Rightarrow y_1(t) = \frac{e^{-\beta t}}{\beta(1-\beta)} - \frac{1}{\beta} - \frac{e^{-t}}{1-\beta}$

so $c_1 = -\frac{1}{\beta}$

$\boxed{\beta = 1}$: (driving is homog soln): $y = Ate^{-t}$ M.U.C.

$$\left.\begin{aligned} y' &= -Ate^{-t} + Ae^{-t} \\ y'' &= Ate^{-t} - 2Ae^{-t} \end{aligned}\right\} \text{ so } A = 1 \text{ matches } y'' + y' = -e^{-t}$$

$y_1(t) = te^{-t} + c_1 + c_2 e^{-t}$ ICs give $c_2 = 1$, $c_1 = -1$

$\Rightarrow y_1(t) = te^{-t} - 1 + e^{-t}$ can check this is $\lim_{\beta \to 1}$ of the $\beta \neq 1$ case!

Soln: $y(t) = 1 + \varepsilon y_1(t) + O(\varepsilon^2)$ with the 2 choices for $y_1(t)$ above.

by parts. so $v = -\cos x$

$$Ci(\lambda) = \int_\lambda^\infty \overbrace{x^{-1}}^{u} \overbrace{\cos x}^{v'}\, dx = \left[x^{-1}\sin x\right]_\lambda^\infty - \int_\lambda^\infty \overbrace{(-x^{-2})}^{u} \overbrace{\sin x}^{v'}\, dx$$

$$= 0 - \frac{\sin\lambda}{\lambda} + \left[-x^{-2}\cos x\right]_\lambda^\infty - \int_\lambda^\infty (-2x^{-3})(-\cos x)\, dx$$

$$= -\frac{\sin\lambda}{\lambda} + \frac{\cos\lambda}{\lambda^2} + O(\lambda^{-3}) \ldots$$

#13

$$I(\lambda) = \int_0^\infty \overbrace{(t+\lambda)^{-2}}^{u} \overbrace{e^{-t}}^{v'}\, dt$$

$$= \left[(t+\lambda)^{-2}(-e^{-t})\right]_0^\infty - \int_0^\infty \overbrace{(-2(t+\lambda)^{-3})}^{u} \overbrace{(-e^{-t})}^{v'}\, dt$$

$$= \frac{e^{-0}}{(0+\lambda)^2} - \left[2(t+\lambda)^{-3}(-e^{-t})\right]_0^\infty + \underbrace{\int_0^\infty \overbrace{-2\cdot 3(t+\lambda)^{-4}}^{u} \overbrace{(-e^{-t})}^{v'}\, dt}_{\left[2\cdot 3(t+\lambda)^{-4}(-e^{-t})\right]_0^\infty - \int_0^\infty 2\cdot 3\cdot 4(t+\lambda)^{-5}(-e^{-t})\, dt}$$

$$= \frac{1}{\lambda^2} - \frac{2}{\lambda^3} + O(\lambda^{-4}).$$

can see a power of λ^{-1} each time, factor of n, and a minus sign.

$$= \frac{1}{\lambda^2} - \frac{2}{\lambda^3} + \frac{2\cdot 3}{\lambda^4} - \frac{2\cdot 3\cdot 4}{\lambda^5} + \cdots + (-1)^n\frac{(n-1)!}{\lambda^n} + \cdots$$