Part D: Adjoint Functors and Quantifiers

CATEGORIES, TOPOI, AND LOGIC

WINTER 2013

Introduction

In the last part, we saw how to interpret propositional logic in a topos. We will now see how to interpret predicate logic in a topos. When we interpreted propositional logic we used truth arrows $1 \to \Omega$ to serve as truth values for propositions. The truth value of a predicate $p(x_1, \ldots, x_n)$ depends on the values of the variables $x_1, \ldots, x_n$. This observation leads us to interpreting predicates as characteristic arrows $p : X_1 \times \cdots \times X_n \to \Omega$ where X_i is the object over which the variable x_i ranges.[1] Given two predicates $p, q : X_1 \times \cdots \times X_n \to \Omega$, we can compose with the arrows $\wedge, \vee, \supset : \Omega \times \Omega \to \Omega$ and $\neg : \Omega \to \Omega$ to form the compound predicates $p \wedge q, p \vee q, p \supset q$, and $\neg p$. So the only remaining step to is the interpretation of the quantifiers $\forall$ and $\exists$.

The main difference between quantifiers and logical connectives is that quantifiers change the dependent variables of a predicate. Given a predicate $p(x, y)$, the variable y becomes bound in $\forall y\, p(x, y)$ and $\exists y\, p(x, y)$.[2] So if $p : X \times Y \to \Omega$ is the interpretation of $p(x, y)$ then the interpretations of $\forall y\, p(x, y)$ and $\exists y\, p(x, y)$ are arrows $X \to \Omega$ since they don't depend on y. The meaning of these quantifiers tells us that for any possible value a for the variable y we should have $\forall y\, p(x, y) \leq p(x, a)$ and $p(x, a) \leq \exists y\, p(x, y)$. Moreover, $\forall y\, p(x, y)$ and $\exists y\, p(x, y)$ should be optimal with these two properties in the sense that if $q(x) \leq p(x, a)$ for all possible values a for y then $q(x) \leq \forall y\, p(x, y)$ and similarly if $p(x, a) \leq r(x)$ for all possible values a for y then $\exists y\, p(x, y) \leq r(x)$. This makes perfect sense in $\mathbf{Set}$ where we have legitimate possible values a for y. To make sense of this in topoi, the trick is to realize that we don't want to make individual comparisons $q(x) \leq p(x, a)$ and $p(x, a) \leq r(x)$ but we want to make all of them. We can do this by composing the projection $\pi_X : X \times Y \to X$ with q and r to introduce an artificial dependency on y. Then the comparisons $q \circ \pi_X \leq p$ and $p \leq r \circ \pi_X$ make perfect sense since these are all subobjects of $X \times Y$.

We have now found the required properties of $\forall$ and $\exists$, but we haven't shown that there are objects meeting those requirements. To address this it helps to generalize the above situation. Given an arrow $f : U \to V$, we obtain a functor $f^{-1} : \mathbf{Sub}(V) \to \mathbf{Sub}(U)$ where the subobject

[1] In classical predicate logic, one often makes the assumption that all variables range over the same domain of discourse. The interpretation of predicate logic in topoi actually works better without this assumption. (See §11.8 of Goldblatt for some consequences of this choice.)

[2] There is no loss of generality in considering a predicate of only two variables since any tuple of variables $x_1, \ldots, x_n$ ranging over $X_1, \ldots, X_n$ can be thought as a single variable $\bar{x}$ ranging over the product $X_1 \times \cdots \times X_n$. This even works for the empty tuple since the empty product is the terminal 1.

$p : V \to \Omega$ gets sent to the subobject $p \circ f : U \to \Omega$.[3] We want to find functors $\forall_f, \exists_f : \mathbf{Sub}(U) \to \mathbf{Sub}(V)$ with the properties

$$p \leq \forall_f(q) \iff p \circ f \leq q \quad \text{for all } q : U \to \Omega \text{ and } p : V \to \Omega,$$

and

$$\exists_f(q) \leq r \iff q \leq r \circ f \quad \text{for all } q : U \to \Omega \text{ and } r : V \to \Omega.$$

Such functors $\forall_f, \exists_f : \mathbf{Sub}(U) \to \mathbf{Sub}(V)$ with the above properties are respectively called the left and right adjoints of the functor $f^{-1} : \mathbf{Sub}(V) \to \mathbf{Sub}(U)$. The $\forall y\, p(x, y)$ and $\exists y\, p(x, y)$ from the above discussion correspond to the special case where $U = X \times Y$, $V = X$, and $f = \pi_X$.

In a still more general setting, two functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ form an **adjoint pair** if for all objects $c \in \mathbf{C}$ and $d \in \mathbf{D}$ there is a bijection

$$\theta_{c,d} : \hom_{\mathbf{C}}(c, G(d)) \cong \hom_{\mathbf{D}}(F(c), d)$$

and these bijections are natural in the sense that if $\gamma : c' \to c$ in $\mathbf{C}$ and $\delta : d \to d'$ in $\mathbf{D}$, we have

$$\theta_{c',d'}(\delta \circ \xi \circ F(\gamma)) = G(\delta) \circ \theta_{c,d}(\xi) \circ \gamma$$

for all $\xi : c \to F(d)$. We then say that F is **left adjoint** to G or that G is **right adjoint** to F. (The case of $\forall_f$ and $\exists_f$ are somewhat simpler than this because $\mathbf{Sub}(U)$ and $\mathbf{Sub}(V)$ are poset categories.)

The left adjoint $\exists_f : \mathbf{Sub}(U) \to \mathbf{Sub}(V)$ of $f^{-1} : \mathbf{Sub}(V) \to \mathbf{Sub}(U)$ is the **image functor** which sends each subobject $s : \bullet \rightarrowtail U$ to the image part of the mono-epi factorization of $f \circ s$.[4] Thus the mono-epi factorization takes care of the existence of $\exists_f$. The existence proof for the right adjoint $\forall_f : \mathbf{Sub}(U) \to \mathbf{Sub}(V)$ uses a different trick that we will discuss after Problem D.2.

Reading

Adjoints and quantifiers are covered by Goldblatt in Chapter 15, which is sufficently independent from other chapters to read right away. Mac Lane covers adjoints in Chapter IV and it is interesting to revisit limits in Chapter V immediately after. In Chapter 11, Goldblatt uses a slightly different approach to interpret predicate logic in topoi; §11.8 on existence and free logic is important to read to understand some of the more important differences between the classical and the categorial approaches to predicate logic.

Problems

Here are the assigned problems for this part of the course. Once you have succesfully completed all these problems, we will finish our discussion of quantifiers as adjoints.

[3] In $\mathbf{Set}$, this is the contravariant powerset functor which sends a $A \subseteq V$ to its preimage $f^{-1}(A) \subseteq U$.

[4] In $\mathbf{Set}$, this is the covariant powerset functor which sends each $A \subseteq U$ to its image $f``(A) = \mathbf{Sub}(V)$.

PROBLEM D.1. Recall that a functor between poset categories is simply an order preserving function. In this context, $F : \mathbf{P} \to \mathbf{Q}$ and $G : \mathbf{Q} \to \mathbf{P}$ form an adjoint pair exactly when the equivalence

$$p \leq G(q) \quad \iff \quad F(p) \leq q$$

holds for all $p \in \mathbf{P}$ and $q \in \mathbf{Q}$. Indeed, since hom-sets have at most one element, this is enough to ensure the existence of appropriate bijections $\theta_{p,q} : \hom_{\mathbf{P}}(p, G(q)) \cong \hom_{\mathbf{Q}}(F(p), q)$ and the naturality condition is automatically satisfied.

(a) Show that if $p = \sup_{i \in I} p_i$ in $\mathbf{P}$ then $\sup_{i \in I} F(p_i) = F(p)$ in $\mathbf{Q}$.

(b) Show that if $q = \inf_{i \in I} q_i$ in $\mathbf{Q}$ then $\inf_{i \in I} G(q_i) = G(q)$ in $\mathbf{P}$.

(c) Show that the left adjoint of the diagonal functor $\Delta : \mathbf{P} \to \mathbf{P}^I$, when there is one, must be the least upper bound functor $(p_i)_{i \in I} \mapsto \sup_{i \in I} p_i$.

(d) Show that the right adjoint of the diagonal functor $\Delta : \mathbf{Q} \to \mathbf{Q}^I$, when there is one, must be the greatest lower bound functor $(p_i)_{i \in I} \mapsto \inf_{i \in I} p_i$.

PROBLEM D.2. Let $\mathbf{E}$ be a topos. Given an object I in $\mathbf{E}$ the comma category $\mathbf{E} \downarrow I$ is the category whose objects are pairs (A, p) where A is an object of $\mathbf{E}$ and $p : A \to I$ is an arrow of $\mathbf{E}$, and whose arrows $(A, p) \to (B, q)$ are arrows $f : A \to B$ in $\mathbf{E}$ such that $q \circ f = p$. In this problem, we will find the left and right adjoints of the functor $I^* : \mathbf{E} \to \mathbf{E} \downarrow I$ which sends every object A of $\mathbf{E}$ to the object $I^*(A) = (A \times I, \mathrm{pr}_I)$ of $\mathbf{E} \downarrow I$ and every arrow $f : A \to B$ in $\mathbf{E}$ to the arrow $f \times 1_I : (A \times I, \mathrm{pr}_I) \to (B \times I, \mathrm{pr}_I)$ in $\mathbf{E} \downarrow I$.

(a) Show that the left adjoint to $I^* : \mathbf{E} \to \mathbf{E} \downarrow I$ is the **dependent sum functor** $\Sigma_I : \mathbf{E} \downarrow I \to \mathbf{E}$ which sends every object (A, p) in $\mathbf{E} \downarrow I$ to the underlying object A in $\mathbf{E}$ and every arrow $f : (A, p) \to (B, q)$ in $\mathbf{E} \downarrow I$ to the underlying arrow $f : A \to B$ in $\mathbf{E}$.

(b) The functor $I^* : \mathbf{E} \to \mathbf{E} \downarrow I$ also has a right adjoint $\Pi_I : \mathbf{E} \downarrow I \to \mathbf{E}$, which called the **dependent product functor**. The object $\Pi_I(A, p)$ in $\mathbf{E}$ is the equalizer of the composite

$$I \times A^I \xrightarrow{\mathrm{ev}} A \xrightarrow{p} I$$

and the projection $\mathrm{pr}_I : I \times A^I \to I$. Given an arrow $f : (A, p) \to (B, q)$ in $\mathbf{E} \downarrow I$ the arrow $\Pi_I(f) : \Pi_I(A) \to \Pi_I(B)$ is the unique arrow such that

$$\begin{tikzcd}
\Pi_I(A) \arrow[r, tail] \arrow[d, "\Pi_I(f)"'] & I \times A^I \arrow[r, "\mathrm{ev}"] & A \arrow[d, "f"] \\
\Pi_I(B) \arrow[r, tail] & I \times B^I \arrow[r, "\mathrm{ev}"] & B
\end{tikzcd}$$

commutes.[5]

[5]In $\mathbf{Set}$, $\Pi_I(A, p)$ is the set of all functions $s : I \to A$ such that $p \circ s = 1_I$ (i.e., the sections of $p : A \to I$) and $\Pi_I(f)$ composes each such s with f to obtain a section $f \circ s : I \to B$ for $q : B \to I$.