HW 4

12.5

#10 $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{1+2\sqrt{n}}$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{n}}{1+2\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\frac{1}{\sqrt{n}}+2} = \frac{1}{2}$$

Hence $\displaystyle\lim_{n\to\infty} (-1)^n \frac{\sqrt{n}}{1+2\sqrt{n}} \neq 0$ (it does not exist.)

By Test for divergence $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{1+2\sqrt{n}}$ is dgt//.

#12 $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^{1/n}}{n}$

$$b_n = \frac{e^{1/n}}{n}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{e^{1/n}}{n}$$

Now $e^{1/n} \leq e$ for all n

Hence we get $0 \leq \frac{e^{1/n}}{n} \leq \frac{e}{n}$ for all n

& by Squeeze th^m $\displaystyle\lim_{n\to\infty} \frac{e^{1/n}}{n} = 0$ (PTO)

[Alternatively: $\lim_{n\to\infty} e^{1/n} = 1$

& hence $\lim_{n\to\infty} \frac{e^{1/n}}{n} = 0$]

$\left\{\frac{e^{1/n}}{n}\right\}$ is a decreasing seq^n because

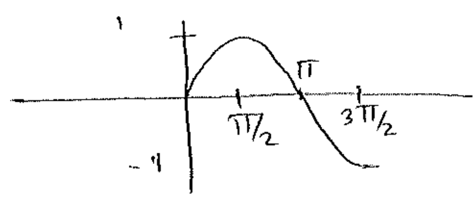
when you go from n to $n+1$, the numerator decreases & the denominator increases.

[Alternatively: show that $\left(\frac{e^{1/x}}{x}\right)' < 0$ for $x > 0$]

Hence by Alternating series test, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^{1/n}}{n}$

is cgt.

#16 $\quad \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n!}$



$\sin\left(\frac{n\pi}{2}\right) = 0$ if n is even

$= \pm 1$ if n is odd.

$\sin\left(\frac{\pi}{2}\right) = 1$, $\sin\left(\frac{3\pi}{2}\right) = -1$, $\sin\left(\frac{5\pi}{2}\right) = 1$, ...

& hence $\sin\left(\frac{(2k+1)\pi}{2}\right) = (-1)^k$ for $k = 0, 1, \ldots$

Hence $\sum_{n=1}^{\infty} \frac{\sin(\frac{n\pi}{2})}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$

$b_n = \frac{1}{(2n+1)!}$

$\{b_n\}$ is a decreasing seq^n & $\lim_{n\to\infty} b_n = 0$

Hence $\sum_{n=1}^{\infty} \frac{\sin(\frac{n\pi}{2})}{n!}$ converges by Alternating series test

32 For what values of p is ~~each~~ the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ cgt?

If $\underline{p>0}$ $\{b_n = \frac{1}{n^p}\}$ is a decreasing seq^n

& $\lim_{n\to\infty} \frac{1}{n^p} = 0$ & hence $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ converges by Alternating series test.

If $p \leq 0$ $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$ ~~Pdoe~~

$\lim_{n\to\infty} \frac{(-1)^n}{n^p}$ does not exit if $p \leq 0$ ($\& \neq 0$) & hence by Test for divergence $\sum \frac{(-1)^{n-1}}{n^p}$ is dgt. Serie converges if and only if $p>0$

12.6

#6 $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$

Consider $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^4} \right| = \sum_{n=1}^{\infty} \frac{1}{n^4}$ is convergent by p-series test $(p = 4 > 1)$

Hence $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ is abs. cgt.

#12 $\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$

Consider $\sum_{n=1}^{\infty} \left| \frac{\sin 4n}{4^n} \right|$

$$\left| \frac{\sin 4n}{4^n} \right| \leq \frac{1}{4^n} \quad \text{for all } n$$

The series $\sum_{n=1}^{\infty} \frac{1}{4^n}$ is cgt because it is a geometric series with common ratio $r = \frac{1}{4} < 1$

$\Rightarrow \sum_{n=1}^{\infty} \left| \frac{\sin 4n}{4^n} \right|$ is cgt by comparison test.

Hence the series $\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$ is abs. cgt.

#14 $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2 2^n}{n!}$

Ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left(\frac{(n+1)^2 \, 2^{n+1}}{(n+1)!} \cdot \frac{n!}{(n)^2 \, 2^n} \right)$$

$$= \lim_{n\to\infty} \frac{(n+1)^2 \, 2}{(n+1) \, n^2}$$

$$= \lim_{n\to\infty} \frac{2(n+1)}{n^2}$$

$$= \lim_{n\to\infty} \frac{\frac{2}{n} + 2/n^2}{1}$$

$$= 0 < 1$$

So by Ratio test the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2 2^n}{n!}$

is cgt.

12.8

#4 $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+2)} \frac{(n+1)}{x^n}\right|$$

$$= \lim_{n\to\infty} \left| \frac{x\,(n+1)}{n+2} \right|$$

$$= \lim_{n\to\infty} |x| \left(\frac{1+1/n}{1+2/n}\right)$$

$$= |x|$$

By Ratio test if $|x|<1$ the series converges

& if $|x|>1$ the series diverges

Hence the radius of convergence $\underline{\underline{R=1}}$.

To check the endpts:

If $x=1$ the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

& this series converges by Alternating series test. $\left[\left\{\frac{1}{n+1}\right\} \text{ is decreasing with limit} = 0 \text{ as } n\to\infty. \right]$

If $x = -1$ the series becomes $\sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n}$

& is dgt (Harmonic series or p-series with $p=1$)

Hence the interval of convergence $= [-1, 1]$

#14 $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{x^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^{2n+2}\,(2n)!}{(2n+2)!\; x^{2n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(2n+1)} \right|$$

$$= \lim_{n\to\infty} \frac{|x|^2}{(2n+2)(2n+1)} = 0 < 1 \quad \text{for all } x.$$

By ratio test the series cges for all x.

Hence radius $R = \infty$

& interval of convergence $= (-\infty, \infty)$

\# 18 $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4^n}(x+1)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x+1)^{n+1}}{4^{n+1}}\ \frac{4^n}{n(x+1)^n}\right|$$

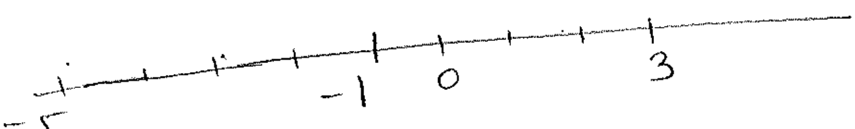
$$= \lim_{n\to\infty}\left|\frac{(n+1)(x+1)}{4\ n}\right|$$

$$= \frac{|x+1|}{4}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$

$$= \frac{|x+1|}{4}$$

If $|x+1| < 4$, the series is cgt by Ratio test.
If $|x+1| > 4$, the series diverges by Ratio test.

~~ie.~~ Hence the radius of convergence $\underline{\underline{R=4}}$ [Note: the series is centered at -1]



If $-5 < x < 3$ the series converges.

End pts: If $x=-5$, $\displaystyle\sum_{n=1}^{\infty}\frac{n}{4^n}(-4)^n$

$$= \sum_{n=1}^{\infty}(-1)^n n$$

$\lim_{n\to\infty} (-1)^n n$ does not exist.

Hence by Test for divergence, $\sum_{n=1}^{\infty} (-1)^n n$ diverges.

If $x = 3$, $\sum_{n=1}^{\infty} \frac{n}{4^n} 4^n = \sum_{n=1}^{\infty} n$ is divergent by Test for divergence.

Hence interval of convergence $= \underline{(-5, 3)}$.

12.9

\# 6. $f(x) = \frac{1}{x+10} = \frac{1}{10+x} = \frac{1}{10\left(1+\frac{x}{10}\right)}$

$$= \frac{1}{10\left(1-\left(-\frac{x}{10}\right)\right)}$$

$$\Rightarrow f(x) = \frac{1}{10}\left[\frac{1}{1-\left(-\frac{x}{10}\right)}\right]$$

$$= \frac{1}{10}\sum_{n=0}^{\infty}\left(\frac{-x}{10}\right)^n \quad \text{for } \left|\frac{x}{10}\right| < 1$$

i.e. if $|x| < 10$

(Optional: $= \frac{1}{10}\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{10^n} = \cancel{\frac{1}{10}\sum}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{10^{n+1}} \quad)$$

Series converges if $|x| < 10$ & diverges if $|x| \geq 10$ [geometric series with common ratio $r = \frac{-x}{10}$]

($R = 10$ ← optional)

Interval of convergence $= (-10, 10)$

#16. $f(x) = \dfrac{x^2}{(1-2x)^2}$

Consider $\dfrac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n$ if $|2x| < 1$
i.e. if $|x| < 1/2$
with $R = \frac{1}{2}$

differentiating $\frac{1}{1-2x}$ & its power series term by term

$$\frac{2}{(1-2x)^2} = \sum_{n=1}^{\infty} 2^n n x^{n-1} \quad \text{with } R = \tfrac{1}{2}$$

Hence $\dfrac{1}{(1-2x)^2} = \sum_{n=1}^{\infty} 2^{n-1} n x^{n-1}$

So $f(x) = \dfrac{x^2}{(1-2x)^2} = \sum_{n=0}^{\infty} 2^{n-1} \cancel{n} x^{n+1}$

& Radius of convergence $= \frac{1}{2}$.

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First consider $\ln(1-t)$ & find out a power series exp^n of $\ln(1-t)$

$$\ln(1-t) = \int \frac{-1}{1-t}\,dt = -\int \sum_{n=0}^{\infty} t^n \,dt \quad \text{if } |t|<1 \text{ with } R=1$$

term by term integration

$$= C - \sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1} \quad \text{with } R=1.$$

If $t=0$, $\ln 1 = C$

$\Rightarrow C=0$

Hence $$\ln(1-t) = -\sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1} \quad \text{with } R=1.$$

Now $$\frac{\ln(1-t)}{t} = -\sum_{n=0}^{\infty} \frac{t^n}{n+1} \quad \text{with } R=1.$$

Integrating

$$\int \frac{\ln(1-t)}{t}\,dt = -\sum_{n=0}^{\infty} \frac{t^{n+1}}{(n+1)^2} \quad \underline{\underline{\text{with } R=1}}$$

[Alternatively: Use Example 6 directly (p 767) for $\ln(1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n}$ $R=1$]