The Chain Rule

November 13, 2006

Lecture 24

The Chain Rule (case 1)

- Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}x}{\mathrm{d}t}.$$

Examples

- If $z = x^2y + xy^3$, where $x = \cos t$, $y = \sin t$, find dz/dx when $t = \pi/2$.

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- The pressue P (in kilopascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is $300K$ and increasing at a rate of $0.1K/s$ and the volume is $100\ L$ and increasing at a rate of $0.2\ L/s$.

The Chain Rule (Case 2)

- Suppose t hat $z = f(x, y)$ is a differentiable function of x and y, where $x = g(x, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$
\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}
\end{aligned}
$$

Examples

- Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ for the following examples:
- $z = e^{xy} \sin x$, where $x = 2s + 4t$, $y = \frac{2s}{3t}$.

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- $z = e^{xy} \sin x$, where $x = 2s + 4t$, $y = \frac{2s}{3t}$.
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- $w = xy + xz + yz$, where $x = st$, $y = e^{st}$, $z = x + t$.

Implicit differentiation (revisited)

- Suppose that
$$F(x, y) = 0$$
defines y implicitly as a differentiable function of x.

- If F is differentiable, using the Case 1 of Chain Rule
$$\frac{\partial F}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}x} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

- If $F_y \neq 0$,
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}.$$

Example

- Find $y^{'}$ if $x^3 + y^3 = 6xy$.

Implicit differentiation (Case 2)

- If z is given implicitly by an equation of the form

$$F(x, y, z) = 0,$$

  and F is differentiable, Case 2 of the Chain Rule tells us

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0.$$

- If $F_z \neq 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_x}.$$

Example

- Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.