Homework #6

1) Page 225 #7

$-(x^2y')' = \lambda y \quad 1 < x < e$

$y(1) = y(e) = 0.$

Show eigenvalues you must be nonnegative

Find eigenvalues & eigenfunctions

multiply by y & integrate

$-\int_1^e (x^2y')'y\,dx = \lambda\int_1^e y^2\,dx$

dv u Integrate by parts.

$-\left[x^2y'y\Big|_1^e - \int_1^e x^2(y')^2\,dx\right] = \lambda\int_1^e y^2\,dx$

$\overset{\shortparallel}{0}$ by Boundary conditions

$\Rightarrow \lambda\int_1^e y^2\,dx = \int_1^e x^2(y')^2\,dx \geq 0$

Now find eigenfunction & eigenvalues

$-x^2y'' - 2xy' - \lambda y = 0.$

let $y = x^m$ plug into equation.

$-m^2 - 2m - \lambda = 0 \Rightarrow m^2 + 2m - \lambda = 0$

$m = \frac{-2 \pm \sqrt{4-4\lambda}}{2} = -1 \pm \sqrt{1-\lambda}$

We have periodic BC $\Rightarrow$ ~~const~~ oscillatory

$\Rightarrow \lambda > 1$

$y(x) = Ax^{-1}\sin(\sqrt{\lambda-1}\ln x) + Bx^{-1}\cos(\sqrt{\lambda-1}\ln x)$

$y(1) = 0 + B\cos(0) = 0 \quad \Rightarrow B = 0$

$y(e) = Ae^{-1}\sin(\sqrt{\lambda-1}) = 0$

$\Rightarrow \sqrt{\lambda-1} = n\pi \quad \Rightarrow \lambda = n^2\pi^2 + 1$

eigenfunctions are $y_n(x) = x^{-1}\sin(n\pi \ln x)$

2) Page 225 #8

$-y'' - 2by' - \lambda y = 0 \qquad 0 < x < 1 \quad b > 0.$

$y(0) = y(1) = 0$

$y = e^{rt} \Rightarrow -r^2 - 2br - \lambda = 0 \Rightarrow r^2 + 2br + \lambda = 0$

$r = \dfrac{-2b \pm \sqrt{4b^2 - 4\lambda}}{2} = -b \pm \sqrt{b^2 - \lambda}$

Case 1: $\lambda < b^2$

$y = Ae^{r_1 t} + Be^{r_2 t} \rightarrow A = B = 0$ only trivial solution

Case 2: $\lambda = b^2$

$y = Ae^{-bt} + Bte^{-bt} \qquad y(1) = Be^{-b} = 0$

$y(0) = A = 0 \qquad \Rightarrow B = 0$

Again only trivial

Case 3: $\lambda > b^2$

$y(t) = Ae^{-bt}\cos(\sqrt{\lambda - b^2}\, t) + Be^{-bt}\sin(\sqrt{\lambda - b^2}\, t)$

$y(0) = A = 0$

$y(1) = Be^{-b}\sin(\sqrt{\lambda - b^2}\, 1) = 0$ eigenvalues $\lambda_n = n^2\pi^2 + b^2$

$\Rightarrow (\lambda - b^2)^{1/2} = n\pi \rightarrow \lambda_n = n^2\pi^2 + b^2$

eigenfunctions $y_n(t) = e^{-bt}\sin(n\pi t)$

3) Page 245 #9

$$u(t) = e^t \int_0^t e^{-s} u(s)\,ds$$

$$\rightarrow u(t)\, e^{-t} = \int_0^t e^{-s} u(s)\,ds$$

Differentiate to get an IVP.

$$-u e^{-t} + u' e^{-t} = e^{-t} u(t)$$

$$\Rightarrow u' e^{-t} = 2 e^{-t} u \qquad IC: u(0) = 0.$$

$$\Rightarrow \frac{u'}{u} = 2$$

$$\rightarrow \ln u = 2t + C \Rightarrow u = Ce^{2t}$$

$$\Rightarrow u = 0.$$

For a more interesting solution

$$u(t) = e^t \int_0^t e^{-s} u(s)\,ds + 2$$

4) Page 244 #6.

$$u(t) = 1 + \int_0^t s \ln\left(\frac{s}{t}\right) u(s)\,ds$$

Derivative wrt t

$$u'(t) = t \ln\left(\frac{t}{t}\right) u(t) + \int_0^t s \left(\frac{-1}{t}\right) u(s)\,ds.$$

$$\ln\left(\tfrac{t}{t}\right) = 0$$

$$= -\frac{1}{t} \int_0^t s\, u(s)\,ds$$

And again.

$$u'' = -\frac{1}{t} t u(t) + \underbrace{+\frac{1}{t^2} \int_0^t s\, u(s)\,ds}_{= -\frac{1}{t} u'}$$

$$u'' + \frac{1}{t} u' + u(t) = 0$$

$$u(0) = = 1$$

$$u'(0) = \lim_{t \to 0^+} \frac{-1}{t} \int_0^t s\, u(s)\,ds \overset{LH}{=} \lim_{t \to 0^+} \frac{t\, u(t)}{1}$$

$$= 0.$$

5) Page 246 #24

$$u'' + u = t^2 \qquad u(0)=1 \qquad u'(0)=0$$

Integrate

$$\int_0^t u''(s)ds + \int_0^t u(s)ds = \int_0^t s^2 ds = \frac{s^3}{3}\Big|_0^t = \frac{t^3}{3}$$

$$u'(t) - \underbrace{u'(0)}_{=0} + \int_0^t u(s)ds = \frac{t^3}{3}$$

Integrate again

$$\int_0^t u'(s)ds + \int_0^t\left(\int_0^y u(s)ds\right)dy = \int_0^t \frac{s^3}{3}ds$$

$$u(t) - \underbrace{u(0)}_{=1} + \int_0^t u(y)(t-y)dy = \frac{t^4}{12}$$

$$u(t) + \int_0^t u(y)(t-y)\,dy = \underbrace{\frac{t^4}{12} - 1}_{f(t)}$$

$$(Ku)(t) = \int_0^t u(y)(t-y)dy$$

6) Page 244 #8

$$u(t) = t + \mu \int_0^t (t-s)\,u(s)\,ds$$

$$f(t) = t \qquad \lambda = \mu \qquad (K u)(t) = \int_0^t (t-s)\,u(s)\,ds.$$

$$u_0 = 0.$$

$$u_1 = t + \mu \int_0^t (t-s)\overset{u_0}{0}\,ds = t$$

$$u_2 = t + \mu \int_0^t (t-s)\,u_1\,ds = t + \mu \int_0^t (t-s)\,s\,ds$$

$$= t + \mu \left(\frac{ts^2}{2} - \frac{s^3}{3} \Big|_0^t \right) = t + \mu \left(\frac{t^3}{2} - \frac{t^3}{3} \right)$$

$$= t + \mu t^3 \left(1/6 \right)$$

$$u_3 = t + \mu \int_0^t (t-s) \left(s + \frac{\mu s^3}{6} \right) ds$$

$$= t + \mu \int_0^t \left[ts + \frac{\mu t s^3}{6} - s^2 - \frac{\mu s^4}{6} \right] ds$$

$$= t + \mu \left(\frac{ts^2}{2} + \frac{\mu t s^4}{24} - \frac{s^3}{3} - \frac{\mu s^5}{30} \Big|_0^t \right.$$

$$= t + \mu \left(\frac{t^3}{2} + \frac{\mu t^5}{24} - \frac{t^3}{3} - \frac{\mu t^5}{30} \right)$$

7) Page 245 #13

a) $u(x) = f(x) + \lambda \int_0^{1/2} u(y)dy \quad \alpha_1(x) = 1 \quad \beta_1 = 1$

multiply By β_1 & integrate over $[0, 1/2]$

$$\underbrace{\int_0^{1/2} u(y)dy}_{c} = \int_0^{1/2} f(x)dx + \lambda \int_0^{1/2} dx \underbrace{\int_0^{1/2} u(y)dy}_{c}$$

$$c = \int_0^{1/2} f(x)dx \mp \tfrac{1}{2}\lambda c$$

$$(1 - \tfrac{\lambda}{2})c = \int_0^{1/2} f(x)dx \rightarrow c = \frac{1}{(1-\lambda/2)} \int_0^{1/2} f(x)dx$$

Note $u(x) = f(x) + \lambda c = f(x) + \dfrac{\lambda}{1-\lambda/2} \displaystyle\int_0^{1/2} f(x)dx$

So solution exist is $\lambda \neq 2$.

c) $u(x) = f(x) + \int_0^1 xy\, u(y)\, dy \quad \alpha_1(x) = x \quad \beta_1(x) = x$

multiply by $\beta_1(x)$ & integrate. over $[0,1]$

$$\underbrace{\int_0^1 x u(x) dx}_{C} = \int_0^1 x f(x) dx + \int_0^1 x^2 dx \underbrace{\int_0^1 y u(y) dy}_{c}$$

$$c = \int_0^1 x f(x) dx + \frac{x^3}{3}\Big|_0^1 c$$

$$\rightarrow c = \frac{1}{1 - 1/3} \int_0^1 x f(x) dx = \frac{3}{2} \int_0^1 x f(x) dx$$

$$u(x) = f(x) + x \int_0^1 y u(y) dy = f(x) + xC = f(x) + \tfrac{3}{2}x \int_0^1 y f(y) dy$$

solution exist always.

8) Page 244 #3

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$$

$(A - 4I)u = b.$

1st Check if 4 is an eigenvalue of A.

$$|A - \lambda I| = (2-\lambda)(3-\lambda) - 2 = 6 - 5\lambda + \lambda^2 - 2$$
$$= 4 - 5\lambda + \lambda^2 = (\lambda - 4)(\lambda - 1)$$

$\lambda = 4, 1$ are eigenvalues.

$\Rightarrow$ $(A - 4I)$ has a zero eigenvalue.

$\Rightarrow$ the only way there can be a solution is if $\bar{b}$ is orthogonal to $\bar{v}$ the eigenvector associated w/ $\lambda = 4$ eigenvalue.

a) Goal Find spectrum of

$$(Ku)(x) = \int_0^1 (1 - 5x^2y^2)u(y)\,dy.$$

This is a degenerate kernel. where

$\alpha_1(x) = 1 \qquad \alpha_2(x) = -5x^2$

$\beta_1(x) = 1 \qquad \beta_2(x) = x^2$

Thus we need to look at the eigenvalues & eigenvectors of

$$A = \begin{bmatrix} \langle \beta_1, \alpha_1 \rangle & \langle \beta_1, \alpha_2 \rangle \\ \langle \beta_2, \alpha_1 \rangle & \langle \beta_2, \alpha_2 \rangle \end{bmatrix} = \begin{bmatrix} 1 & -5/3 \\ 1/3 & -1-\lambda \end{bmatrix}$$

eigenvalues are $\lambda = \pm 2/3$.

eigenvectors: for $\lambda_1 = 2/3$ $\begin{bmatrix} 1/3 & -5/3 \\ 1/3 & -5/3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\frac{c_1}{3} = \frac{5}{3}c_2 \to c_1 = 5c_2$$

$$\bar{V}_1 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

for $\lambda_2 = -2/3$ $\begin{bmatrix} 5/3 & -5/3 \\ 1/3 & 1/3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to c_1 = c_2$

$$\bar{V}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\to$ eigenfunctions are

$u_1(x) = 5 + -5x^2 = 5(1-x^2)$

$u_2(x) = 1 + x^2$

1 is not in the eigenspace of (Ku) so $Ku - u = f$ has a unique solution even if f is not in the span of α_j's $= \mathrm{Span}\{1, x^2\}$.

$f_1 = \langle x, 1 \rangle = \int_0^1 x\,dx = \frac{1}{2}$

$f_2 = \langle x, x^2 \rangle = \int_0^1 x^3 dx = \frac{1}{4}$

We need to find $\bar{c}$, the solution of

$$(A - I)\bar{c} = \bar{f}$$

$$\begin{bmatrix} 0 & -5/3 \\ 1/3 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/4 \end{bmatrix} \rightarrow -\frac{5}{3} c_2 = 1/2 \rightarrow c_2 = -3/10$$

$$\frac{1}{3} c_1 = \frac{1}{4} + 2c_2$$

$$c_1 = \frac{3}{4} + \cancel{6}^{3}\left(-3/\cancel{10}_{5}\right) = \frac{3}{4} - \frac{9}{5}$$

$$= \frac{15 - 36}{20} = \frac{-21}{20}$$

$$\Rightarrow u(x) = \frac{-1}{\lambda}\left(f(x) - \sum_{j=1}^{n} \alpha_j(x) c_j\right)$$

$$= -x + \left(\frac{-21}{20} - \frac{15}{10} x^2\right)$$

16) Page 244 #4c

Goal: Find eigenvalues & eigenfunctions ie find u & λ st

$$\circledast \quad Ku = \lambda u$$

Rewrite:

$$\lambda u = \int_0^x y(\pi - x)\, u(y)\, dy + \int_x^\pi x(\pi - y) u(y) dy.$$

take derivative of both sides.

$$\lambda u' = \int_0^x -y\, u(y)\, dy + x(\pi - x)u(x) - \underbrace{0(\pi - x)u(0) \cdot 0}_{=0}$$

$$+ \int_x^\pi (\pi - y) u(y) + \underbrace{\cancel{x}(\pi - \pi) u(\pi)}_{=0} - x(\pi - \cancel{x})u(x)$$

$$= \int_0^x -y u(y)\, dy - \int_{-\pi}^x (\pi - y) u(y) dy + 0.$$

Take derivative again.

$$\lambda u'' = -x\, u(x) - (\pi - x)\, u(x) = \pi\, u(x).$$

rewrite $u'' - \pi/\lambda\, u = 0.$ BC. $u(0) = 0$ by $\circledast$, $u(\pi) = 0$ by $\circledast$.

In order to satisfy the BC. we must have oscillatory eigenfunctions $\Rightarrow \frac{\pi}{\lambda} < 0$. let $k^2 = \frac{-\pi}{\lambda}$

$$u'' + k^2 u = 0 \Rightarrow u = c_1 \cos(kx) + c_2 \sin(kx)$$

$u(0) = c_1 = 0$ $\quad u(\pi) = \sin(k\pi)$

u $\quad \Rightarrow k\pi = n\pi \Rightarrow k = n.$

$\Rightarrow n^2 = \frac{-\pi}{\lambda}$ therefore $\lambda_n = \frac{-\pi}{n^2}$

eigenfunctions are $u_n = \sin(nx).$