Math 25, Homework 8, November 21, 2008

1. For each arithmetic function determine if it is multiplicative.
   (a) $f(n) = \varphi(n)^2$,
   (b) $f(n) = \varphi(n^2)$,
   (c) $f(n) = \sigma(\varphi(n))$,
   (d) $f(n)$ is the sum of the proper divisors of n (that is, the sum of those $d \mid n$ with $1 \le d < n$).
   (e) Let p be an odd prime and let $f(n) = (n/p)$.
   (f) Let p be a prime with $p \equiv 1 \pmod 4$ and let g be a primitive root for p. If $n \equiv g^k \pmod p$ for some integer k, let $f(n) = i^k$, and if $n \equiv 0 \pmod p$, let $f(n) = 0$. (Here "i" is a complex number with $i^2 = -1$.)
   (g) $f(n) = \lambda(n)$,
   (h) $F(x)$ is a polynomial with integer coefficients and $f(n)$ is the number of solutions to the congruence $F(x) \equiv 0 \pmod n$.

2. If $S \subseteq \mathbb{N}$, let f_S be the characteristic function of S; that is, $f_S(n) = 1$ if $n \in S$ and $f_S(n) = 0$ if $n \notin S$. Under what condition on S is f_S multiplicative? In particular, determine if f_S is multiplicative if
   (a) m is a fixed integer and S is the set of natural numbers coprime to m,
   (b) S is the set of squares,
   (c) S is the set of squarefree numbers,
   (d) S is the set of primes,
   (e) S is the set of numbers n with $\omega(n)$ even (here, $\omega(n)$ is the number of primes that are divisors of n),
   (f) S is the set of numbers n with $\omega(n)$ odd,
   (g) S is the set of squarefull numbers.

3. Describe all positive integers n with $\varphi(n) + \sigma(n) = 2n$. (Hint: Use $\sigma = N * u$, $\varphi = N * \mu$, where $N(n) = n$ for all n and $u(n) = 1$ for all n.)