Here are some sample solutions; no guarantees ...

1. (Optimization).

   (a) Find and classify all local extreme points of $f(x, y) = x^2 + x + 2y^2$ on the domain $x^2 + y^2 < 1$.

   Solution: The only critical point is $(-1/2, 0)$ where the value $f(-1/2, 0) = -1/4$ is a local minimum by the second derivative test.

   (b) Determine the absolute maximum and minimum of $f(x, y) = x^2 + x + 2y^2$ on the domain $x^2 + y^2 \leq 1$. Be sure to indicate both the maximum and minimum values as well as the coordinates of all points at which they occur.

   Solution: To determine the absolute extrema, we use the critical point from the first part together with points of interest on the boundary: $x^2 + y^2 = 1$. We can proceed directly or via Lagrange multipliers.

   Directly, we solve the constraint for $y^2 = 1 - x^2$ and substitute into $f(x, y)$ to obtain a function $g(x) = x^2 + x + 2(1 - x^2) = -x^2 + x + 2$ on the interval $[-1, 1]$. g has a critical point at $x = 1/2$ so we compute the values $g(-1) = 0$, $g(1/2) = 2.25$ and $g(2) = 2$. It is clear the absolute minimum value is $-1/4$ occurring at the interior critical point $(-1/2, 0)$ while the absolute maximum is 2.25 occurring when $x = 1/2$, so at the two points $(1/2, \pm\sqrt{3}/2)$.

   Alternatively, using Lagrange multipliers, we set $\nabla f = \langle 2x + 1, 4y\rangle = \lambda\langle 2x, 2y\rangle$, and obtain $2x+1 = \lambda 2x$, $4y = \lambda 2y$ and $x^2+y^2 = 1$. The second equation says that $2y(\lambda - 2) = 0$, so either $y = 0$ or $\lambda = 2$. If $y = 0$, we consider the points $(\pm 1, 0)$ while if $\lambda = 2$, the first equations gives $x = 1/2$ and we consider $(1/2, \pm\sqrt{3}/2)$. The results are of course the same as above.

2. Suppose that $z = f(x, y)$, $x = uv$ and $y = u + 3v$. Assume that when $u = 2$ and $v = 1$, $\dfrac{\partial z}{\partial u} = -2$ and $\dfrac{\partial z}{\partial v} = -1$. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

   Solution: We write down the appropriate derivatives using chain rule:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

We also compute $\dfrac{\partial x}{\partial u} = v$, $\dfrac{\partial x}{\partial v} = u$, $\dfrac{\partial y}{\partial u} = 1$, and $\dfrac{\partial y}{\partial v} = 3$. Using $u = 2$ and $v = 1$, we substitute into the chain rule formulas to get:

$$-2 = \frac{\partial z}{\partial x}(1) + \frac{\partial z}{\partial y}(1)$$
$$-1 = \frac{\partial z}{\partial y}(2) + \frac{\partial z}{\partial y}(3)$$

Solving simultaneously, yields $\dfrac{\partial z}{\partial y} = 3$ and $\dfrac{\partial z}{\partial x} = -5$.

3. Find an equation of the plane which is perpendicular to the line $x = 2 - t$, $y = 2t$, $z = 3 + t/2$, and which contains the line $x = 4 + 2s$, $y = -1 + 3s$, $z = 2 - 8s$.

   Solution: The plane has normal vector parallel to the line: $\mathbf{n} = \langle -1, 2, 1/2 \rangle$ and contains the point (on the other line) $(4, -1, 2)$, so has the form
   $(-1)(x - 4) + 2(y + 1) + (1/2)(z - 2) = 0$.

4. Consider the surface $x^2 + y^2 + z^2 = 9$. Find the point of intersection of the tangent plane to the surface at the point $(1, 2, 2)$ and the x-axis.

   Solution: To find the equation of the tangent plane to the level surface $F(x, y, z) = x^2 + y^2 + z^2 = 9$, we compute the gradient $\nabla F = \langle 2x, 2y, 2z \rangle$. At the point $(1, 2, 2)$ the gradient is $\mathbf{n} = \nabla F(1, 2, 2) = \langle 2, 4, 4 \rangle$. The tangent plane at that point is given by $2(x - 1) + 4(y - 2) + 4(z - 2) = 0$. The plane intersects the x-axis when both y and z equal zero, so substituting into the equation of the plane yields $x = 9$, so the point of intersection is $(9, 0, 0)$.

5. Find the maxima and minima of $f(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = x^2 + 2y^2 + 3z^2 = 6$.

   Solution: Setting $\nabla f = \lambda \nabla g$ yields

$$yz = \lambda 2x$$
$$xz = \lambda 4y$$
$$xy = \lambda 6z$$

We note that if any of the variables are zero, precisely 2 are zero which leads to 6 points, but they are all uninteresting since the value of f on them is zero, and zero is clearly not the max nor min. So we assume all of x, y, z are nonzero. Solving the equations for λ and equating yields $x^2 = 2y^2 = 3z^2$, so that $6 = x^2 + 2y^2 + 3z^2 = 3x^2$. This implies $x = \pm\sqrt{2}$, $y = \pm 1$, and $z = \pm\sqrt{2/3}$ which produces 8 points. Four of these points (with 0 or 2 negative coordinates) yield the absolute maximum of $2/\sqrt{3}$. The other four yield the absolute minimum value of $-2/\sqrt{3}$.

6. Write an equation for the tangent plane to the level surface $f(x, y, z) = ze^{xy} + xe^{yz} = 2$ at the point $(1, 0, 1)$.

   Solution: We compute the gradient to the level surface: $\nabla f(1, 0, 1)$.
   $\nabla f = \langle yze^{xy} + e^{yz}, xze^{xy} + xze^{yz}, e^{xy} + xye^{yz}\rangle$, so $\nabla f(1, 0, 1) = \langle 1, 2, 1\rangle$. Thus the tangent plane has equation $(x - 1) + 2y + (z - 1) = 0$.

7. What is the arclength of the curve $y = \ln(\cos(x))$ for x from 0 to $\pi/4$.

   Solution: The curve can clearly be parametrized as $\mathbf{r}(t) = \langle t, \ln(\cos(t))\rangle$. The velocity is $\mathbf{r}'(t) = \langle 1, -\tan t\rangle$ so the speed is $|\mathbf{r}'(t)| = \sqrt{1 + \tan^2 t} = |\sec(t)| = \sec(t)$. So the arclength is $\displaystyle\int_0^{\pi/4} \sec(t)\, dt = \ln(\sec t + \tan t)\Big|_0^{\pi/4} = \ln(\sqrt{2} + 1)$.

8. Find the absolute extrema of $f(x, y) = e^{xy} + e^x$ in the first quadrant of the xy-plane.

   Solution: Note that while this is a closed region, it is not a closed, bounded region, so there is no reason necessarily to expect global extrema. Indeed as x goes to infinity with $y = 0$, the function f becomes arbitrarily large, so there is no global maximum.

   We consider critical points: $f_x = ye^{xy} + e^x = 0$ and $f_y = xe^{xy} = 0$. Since the exponential never vanishes, $f_y = 0$ means $x = 0$, and this together with $f_x = y + 1 = 0$ implies $(0, -1)$ is the only critical point in the plane, but it does not lie in the first quadrant. What's a body to do? Well, there is some boundary, namely the axes.

   On the x-axis, $y = 0$ and the function $f(x, 0) = 1 + e^x$ which attains a minimum value of 2 at the origin, and increases to infinity as x increases. On the y-axis, $x = 0$, and the function $f(0, y) = 2$ for all values of y. Thus the absolute minimum value is 2 which occurs for all points $(0, y)$.

9. Express the antiderivative $\displaystyle\int \frac{\sin(t^2) - t^2}{t^6}\, dt$ as an infinite series.

   Solution: The Maclaurin series for $\sin x$ is $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, which has an infinite radius of convergence.

Thus $\sin(t^2) = t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \cdots$, so

$$\frac{\sin(t^2) - t^2}{t^6} = \frac{-1}{3!} + \frac{t^4}{5!} - \frac{t^8}{7!} + \frac{t^{12}}{9!} - \cdots = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{t^{4n}}{(3+2n)!}.$$

Thus the integral $\displaystyle\int \frac{\sin(t^2) - t^2}{t^6}\, dt = C + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{t^{4n+1}}{(4n+1)(3+2n)!}$

10. (Multiple choice — No partial credit) **Circle the correct answer.**

(a) Find the tangent plane to the surface $z = x^2y^3$ at the point $(1, 1, 1)$.

Solution: $2x + 3y - z = 4$

(b) Consider the level curve of $f(x, y) = x^2 - 3y^2$ which passes through the point $(3, 1)$. Along what vector should one go to remain on the same level curve?

Solution: $\langle -6, 6 \rangle$

(c) What is the arclength of the piece of the parabola $y = x^2$ from $(0, 0)$ to $(2, 4)$?

Solution: $\displaystyle\int_0^2 \sqrt{1 + 4t^2}\, dt$

(d) If $f(x, y) = \int_y^x \cos(t^3)\, dt$, then $\dfrac{\partial f}{\partial y} =$

Solution: None of the above, since -cos(y^3) is the correct answer.

(e) Suppose that you are given a function $f(x, y)$ and vectors $\mathbf{u} = \langle \frac{1}{2}, \frac{-\sqrt{3}}{2} \rangle$ and $\mathbf{v} = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$. If $(D_{\mathbf{u}}f)(x_0, y_0) = 2$ and $(D_{\mathbf{v}}f)(x_0, y_0) = -1$, then $\dfrac{\partial f}{\partial x}(x_0, y_0) =$

Solution: 1

(f) Suppose that the graph of $z = f(x, y)$ represents the surface of a mountain, and you are standing at a point (x_0, y_0, z_0) on the surface. You are told that the gradient of f at (x_0, y_0) is $\nabla f(x_0, y_0) = \langle 1, 3 \rangle$. If you move in the direction of the gradient, what is your initial angle of elevation?

Solution: $\tan^{-1}\sqrt{10}$