Math 11
Section 3
Monday, October 13, 2008

Example: The temperature at a point (x, y, z) in space is given by the function $f(x, y, z) = x^2 + xy - z^3$. A fly is buzzing around, with its position at time t given by the function $\vec{r}(t) = (\cos t,\, \sin t,\, t)$. How fast is the fly's (external) temperature changing when $t = \frac{\pi}{2}$?

For concreteness, the units in this problem are seconds, meters and degrees Celsius.

Let us set $w = f(x, y, z) = f(\vec{r}(t)) = (f \circ \vec{r})(t)$. We want the rate of change of w with respect to t when $t = \frac{\pi}{2}$. Or, in other words, we want $(f \circ \vec{r})' \left(\frac{\pi}{2}\right)$.

Using the Chain Rule and the fact that

$$f'(x, y, z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2x + y,\, x,\, -3z^2)$$

we get that

$$(f \circ \vec{r})'(t) = f'(\vec{r}(t)) \cdot \vec{r}\,'(t) = (2\cos t + \sin t,\, \cos t,\, -3t^2) \cdot (-\sin t, \cos t, 1)$$

$$(f \circ \vec{r})' \left(\frac{\pi}{2}\right) = \left(1, 0, \frac{-3\pi^2}{4}\right) \cdot (-1, 0, 1) = -1 - \frac{3\pi^2}{4}.$$

We can also use the Chain Rule in its alternative form,

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t} = (2x + y)(-\sin t) + (x)(\cos t) + (-3z^2)(1).$$

Substituting values $t = \frac{\pi}{2}$, $x = \cos t = 0$, $y = \sin t = 1$, $z = t = \frac{\pi}{2}$, we get

$$\frac{\partial w}{\partial t} = (1)(-1) + (0)(0) + \left(-\frac{3\pi^2}{4}\right)(1) = -1 - \frac{3\pi^2}{4}.$$

Example: The quantity w is a function of position on the plane, $w = f(x, y)$, and $f'(1, 1) = (2, 4)$. We can express x and y in terms of polar coordinates r and θ. Find the partial derivatives of w with respect to r and to θ at the point $(x, y) = (1, 1)$.

We can express x and y as functions of r and θ,

$$x = r\cos\theta \qquad y = r\sin\theta.$$

We can compute

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta.$$

Now we can use the chain rule to write

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial w}{\partial x}\cos\theta + \frac{\partial w}{\partial y}\sin\theta$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial w}{\partial x}r\sin\theta + \frac{\partial w}{\partial y}r\cos\theta.$$

Because the components of the total derivative of f are its partial derivatives, and $f'(1, 1) = (2, 4)$, we know that at the point $(1, 1)$ we have

$$\frac{\partial w}{\partial x} = 2 \qquad \frac{\partial w}{\partial y} = 4.$$

Also, at the point $(x, y) = (1, 1)$, the polar coordinates are $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$, and so $\cos\theta = \sin\theta = \frac{\sqrt{2}}{2}$. Substituting these values into the above equations, we get

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cos\theta + \frac{\partial w}{\partial y}\sin\theta = 2\frac{\sqrt{2}}{2} + 4\frac{\sqrt{2}}{2} = 3\sqrt{2}$$

$$\frac{\partial w}{\partial \theta} = -\frac{\partial w}{\partial x}r\sin\theta + \frac{\partial w}{\partial y}r\cos\theta = -2\sqrt{2}\frac{\sqrt{2}}{2} + 4\sqrt{2}\frac{\sqrt{2}}{2} = 2.$$

Example: Suppose that the position of a buzzing fly is given by a differentiable function of time, and that the temperature at a point in space is given by a differentiable function of position. Show that the rate of change of the fly's (external) temperature with respect to time is the product of the fly's speed with the rate of change of temperature in the direction of the fly's motion.

Note that this makes sense. If, in the direction of motion, temperature is increasing at .1 degrees per meter, and the fly is moving at 3 meters per second, then the fly's temperature ought to be increasing at .3 degrees per second.

Hint: Use the chain rule to compute the rate of change of the fly's temperature with respect to time. Note that the rate of change of temperature in the direction of the fly's motion is a directional derivative.

Suppose the temperature function is $f(x, y, z)$ and the fly's position function is $\vec{r}(t)$. By the Chain Rule, the rate of change of the fly's temperature with respect to time is

$$f'(\vec{r}(t)) \cdot \vec{r}'(t).$$

We can write the fly's velocity $\vec{r}'(t)$ as the product of the fly's speed $\frac{ds}{dt} = |\vec{r}'(t)|$ with the unit vector giving the direction of motion $\vec{u} = \frac{1}{|\vec{r}'(t)|}\vec{r}'(t)$, and so we see the rate of change of the fly's temperature with respect to time is

$$f'(\vec{r}(t)) \cdot \vec{r}'(t) = f'(\vec{r}(t)) \cdot \left(\frac{ds}{dt}\right) \vec{u} = \left(\frac{ds}{dt}\right) (f'(\vec{r}(t)) \cdot \vec{u}).$$

Finally, we see that the factor $\frac{ds}{dt}$ is the fly's speed, and the factor $f'(\vec{r}(t)) \cdot \vec{u}$ is the directional derivative of f at the point $\vec{r}(t)$ (the fly's position) in the direction $\vec{u}$ (the direction of the fly's motion).

Putting all this together: The rate of change of the fly's temperature with respect to time is the product of the fly's speed with the rate of change of the temperature (with respect to distance) in the direction of the fly's motion.