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1. **Inversion polynomials for 321-avoiding permutations** (with S.-E. Cheng, A. Kasraoui and B. Sagan), *Discrete Math.* 313 (2013), 2552–2565.

   We prove a generalization of a conjecture of Dokos, Dwyer, Johnson, Sagan, and Selsor giving a recursion for the inversion polynomial of 321-avoiding permutations. We also answer a question they posed about finding a recursive formula for the major index polynomial of 321-avoiding permutations. Other properties of these polynomials are investigated as well. Our tools include Dyck and 2-Motzkin paths, polyominoes, and continued fractions.

2. **Arc permutations** (with Y. Roichman), *J. Algebraic Combin.*, to appear.

   Arc permutations and unimodal permutations were introduced in the study of triangulations and characters. In this paper we describe combinatorial properties of these permutations, including characterizations in terms of pattern avoidance, connections to Young tableaux, and an affine Weyl group action on them.

3. **Arc Permutations (extended abstract)** (with Y. Roichman), *Discrete Math. Theor. Comput. Sci. proc. AR* (2012), 259–270.

4. **Pattern avoidance in matchings and partitions** (with J. Bloom), *Electron. J. Combin.* 20 (2013), #P5.

   Extending the notion of pattern avoidance in permutations, we study matchings and set partitions whose arc diagram representation avoids a given configuration of three arcs. These configurations, which generalize 3-crossings and 3-nestings, have an interpretation, in the case of matchings, in terms of patterns in full rook placements on Ferrers boards.

   We enumerate 312-avoiding matchings and partitions, obtaining algebraic generating functions, in contrast with the known D-finite generating functions for the 321-avoiding (i.e., 3-noncrossing) case. Our approach provides a more direct proof of a formula of Bóna for the number of 1342-avoiding permutations. We also give a bijective proof of the shape-Wilf-equivalence of the patterns 321 and 213 which greatly simplifies existing proofs by Backelin–West–Xin and Jelínek, and provides an extension of work of Gouyou-Beauchamps for matchings with fixed points. Finally, we classify pairs of patterns of length 3 according to shape-Wilf-equivalence, and enumerate matchings and partitions avoiding a pair in most of the resulting equivalence classes.

5. **Patterns in matchings and rook placements** (with J. Bloom), *Discrete Math. Theor. Comput. Sci. proc. AS* (2013), 909–920.

6. **Periodic patterns of signed shifts** (with K. Archer), *Discrete Math. Theor. Comput. Sci. proc. AS* (2013), 873–884.

   The periodic (ordinal) patterns of a map are the permutations realized by the relative order of the points in its periodic orbits. We give a combinatorial characterization of the periodic patterns of an arbitrary signed shift, in terms of the structure of the descent set of a certain cyclic permutation associated to the pattern. Signed shifts are an important family of one-dimensional dynamical systems

that includes shift maps and the tent map as particular cases. Defined as a function on the set of infinite words on a finite alphabet, a signed shift deletes the first letter and, depending on its value, possibly applies the complementation operation on the remaining word. For shift maps, reverse shift maps, and the tent map, we give exact formulas for their number of periodic patterns. As a byproduct of our work, we recover results of Gessel–Reutenauer and Weiss–Rogers and obtain new enumeration formulas for pattern-avoiding cycles.

7. **The most and the least avoided consecutive patterns**, *Proc. Lond. Math. Soc.* 106 (2013), 957–979.

   We prove that the number of permutations avoiding the consecutive pattern $12\ldots m$ —that is, containing no m adjacent entries in increasing order— is asymptotically larger than the number of permutations avoiding any other consecutive pattern of length m. This settles a conjecture of the author and Noy from 2001. We also prove a recent conjecture of Nakamura stating that, at the other end of the spectrum, the number of permutations avoiding $12\ldots(m-2)m(m-1)$ is asymptotically smaller than for any other pattern. Finally, we consider non-overlapping patterns and obtain analogous results describing the most and least avoided ones.

   The techniques used include the cluster method of Goulden and Jackson, an interpretation of clusters as linear extensions of posets, and singularity analysis of generating functions.

8. **Clusters, generating functions and asymptotics for consecutive patterns in permutations** (with M. Noy), *Adv. in Appl. Math.* 49 (2012) 351–374.

   We use the cluster method to enumerate permutations avoiding consecutive patterns. We reprove and generalize in a unified way several known results and obtain new ones, including some patterns of length 4 and 5, as well as some infinite families of patterns of a given shape. By enumerating linear extensions of certain posets, we find a differential equation satisfied by the inverse of the exponential generating function counting occurrences of the pattern. We prove that for a large class of patterns, this inverse is always an entire function.

   We also complete the classification of consecutive patterns of length up to 6 into equivalence classes, proving a conjecture of Nakamura. Finally, we show that the monotone pattern asymptotically dominates (in the sense that it is easiest to avoid) all non-overlapping patterns of the same length, thus proving a conjecture of Elizalde and Noy for a positive fraction of all patterns.

9. **Consecutive patterns in permutations: clusters and generating functions** (with M. Noy), *Discrete Math. Theor. Comput. Sci. proc. AR* (2012), 247–258.

10. **Total occurrence statistics on restricted permutations** (with A. Burstein), Pure Math. Appl. (PU.M.A.), to appear.

    We study the total number of occurrences of several vincular (also called generalized) patterns and other statistics, such as the major index and the Denert statistic, on permutations avoiding a pattern of length 3, extending results of Bóna and Homberger. In particular, for $2-3-1$-avoiding permutations, we find the total number of occurrences of any vincular pattern of length 3. In some cases the answer is given by simple expressions involving binomial coefficients. The tools we use are bijections with Dyck paths, generating functions, and block decompositions of permutations.

11. **Bijections for lattice paths between two boundaries** (with M. Rubey), *Discrete Math. Theor. Comput. Sci. proc. AR* (2012), 827–838.

    We prove that on the set of lattice paths with steps $N = (0,1)$ and $E = (1,0)$ that lie between two boundaries B and T, the two statistics 'number of E steps shared with B' and 'number of E steps shared with T' have a symmetric joint distribution. We give an involution that switches these statistics,

preserves additional parameters, and generalizes to paths that contain steps $S = (0, -1)$ at prescribed x-coordinates. We also show that a similar equidistribution result for other path statistics follows from the fact that the Tutte polynomial of a matroid is independent of the order of its ground set. Finally, we extend the two theorems to k-tuples of paths between two boundaries, and we give some applications to Dyck paths, generalizing a result of Deutsch, and to pattern-avoiding permutations.

12. **Generating trees for partitions and permutations with no k-nestings** (with S. Burrill, M. Mishna, L. Yen), *Discrete Math. Theor. Comput. Sci. proc. AR* (2012), 409–420.

We describe a generating tree approach to the enumeration and exhaustive generation of k-nonnesting set partitions and permutations. Unlike previous work in the literature using the connections of these objects to Young tableaux and restricted lattice walks, our approach deals directly with partition and permutation diagrams. We provide explicit functional equations for the generating functions, with k as a parameter.

13. **Permutations and β-shifts**, *J. Combin. Theory Ser. A* 118 (2011), 2474–2497.

Given a real number $\beta > 1$, a permutation π of length n is realized by the β-shift if there is some $x \in [0, 1]$ such that the relative order of the sequence $x, f(x), \ldots, f^{n-1}(x)$, where $f(x)$ is the fractional part of βx, is the same as that of the entries of π. Widely studied from such diverse fields as number theory and automata theory, β-shifts are prototypical examples of one-dimensional chaotic dynamical systems. When β is an integer, permutations realized by shifts were studied in [*SIAM J. Discrete Math.* 23 (2009), 765–786]. In this paper we generalize some of the results to arbitrary β-shifts. We describe a method to compute, for any given permutation π, the smallest β such that π is realized by the β-shift. We also give a way to determine the length of the shortest forbidden (i.e., not realized) pattern of an arbitrary β-shift.

14. **Allowed patterns of β-shifts**, *Discrete Math. Theor. Comput. Sci. proc. AO* (2011), 293–304.

15. **Descent sets of cyclic permutations**, *Adv. in Appl. Math.* 47 (2011), 688–709.

We present a bijection between cyclic permutations of $\{1, 2, \ldots, n+1\}$ and permutations of $\{1, 2, \ldots, n\}$ that preserves the descent set of the first n entries and the set of weak excedances. This non-trivial bijection involves a Foata-like transformation on the cyclic notation of the permutation, followed by certain conjugations. We also give an alternate derivation of the consequent result about the equidistribution of descent sets using work of Gessel and Reutenauer. Finally, we prove a conjecture of the author in [*SIAM J. Discrete Math.* 23 (2009), 765–786] and a conjecture of Eriksen, Freij and Wästlund.

16. **Restricted simsun permutations** (with E. Deutsch), *Ann. Comb.* 16 (2012), 253–269.

A permutation is simsun if for all k, the subword of the one-line notation consisting of the k smallest entries does not have three consecutive decreasing elements. Simsun permutations were introduced by Simion and Sundaram, who showed that they are counted by the Euler numbers. In this paper we enumerate simsun permutations avoiding a pattern or a set of patterns of length 3. The results involve Motzkin, Fibonacci, and secondary structure numbers. The techniques in the proofs include generating functions, bijections into lattice paths and generating trees.

17. **On basic forbidden patterns of functions** (with Y. Liu), *Discrete Appl. Math.* 159 (2011), 1207–1216.

The allowed patterns of a map on a one-dimensional interval are those permutations that are realized by the relative order of the elements in its orbits. The set of allowed patterns is completely determined by the minimal patterns that are not allowed. These are called basic forbidden patterns. In this paper we study basic forbidden patterns of several functions. We show that the logistic map $L_r(x) = rx(1-x)$ and some generalizations have infinitely many of them for $1 < r \leq 4$, and we give a lower bound on the

number of basic forbidden patterns of L_4 of each length. Next, we give an upper bound on the length of the shortest forbidden pattern of a piecewise monotone map. Finally, we provide some necessary conditions for a set of permutations to be the set of basic forbidden patterns of such a map.

18. **Cycle-up-down permutations** (with E. Deutsch), *Australas. J. Combin.* 50 (2011), 187–199.

A permutation is defined to be *cycle-up-down* if it is a product of cycles that, when written starting with their smallest element, have an up-down pattern. We prove bijectively and analytically that these permutations are enumerated by the Euler numbers, and we study the distribution of some statistics on them, as well as on up-down permutations and on all permutations. The statistics include the number of cycles of even and odd length, the number of left-to-right minima, and the number of extreme elements.

19. **The X-class and almost-increasing permutations**, *Ann. Comb.* 15 (2011), 51–68.

In this paper we give a bijection between the class of permutations that can be drawn on an X-shape and a certain set of permutations that appears in [Knuth, The Art of Computer Programming, Vol. III], in connection to sorting algorithms. A natural generalization of this set leads us to the definition of *almost-increasing permutations*, which is a one-parameter family of permutations that can be characterized in terms of forbidden patterns. We find generating functions for almost-increasing permutations by using their cycle structure to map them to colored Motzkin paths. We also give refined enumerations with respect to the number of cycles, fixed points, excedances, and inversions.

20. **Improved bounds on the number of numerical semigroups of a given genus**, *J. Pure Appl. Algebra* 214 (2010), 1862–1873.

We improve the previously best known lower and upper bounds on the number n_g of numerical semi-groups of genus g. Starting from a known recursive description of the tree $\mathcal{T}$ of numerical semigroups, we analyze some of its properties and use them to construct approximations of $\mathcal{T}$ by generating trees whose nodes are labeled by certain parameters of the semigroups. We then translate the succession rules of these trees into functional equations for the generating functions that enumerate their nodes, and solve these equations to obtain the bounds. Some of our bounds involve the Fibonacci numbers, and the others are expressed as generating functions.

We also give upper bounds on the number of numerical semigroups having an infinite number of descendants in $\mathcal{T}$.

21. **The largest and the smallest fixed points of permutations** (with E. Deutsch), *European J. Combin.* 31 (2010), 1404–1409.

We give a new interpretation of the derangement numbers d_n as the sum of the values of the largest fixed points of all non-derangements of length $n - 1$. We also show that the analogous sum for the smallest fixed points equals the number of permutations of length n with at least two fixed points. We provide analytic and bijective proofs of both results, as well as a new recurrence for the derangement numbers.

22. **The number of permutations realized by a shift**, *SIAM J. Discrete Math.* 23 (2009), 765–786.

A permutation π is realized by the shift on N symbols if there is an infinite word on an N-letter alphabet whose successive left shifts by one position are lexicographically in the same relative order as π. The set of realized permutations is closed under consecutive pattern containment. Permutations that cannot be realized are called forbidden patterns. It was shown by Amigó–Elizalde–Kennel that the shortest forbidden patterns of the shift on N symbols have length $N + 2$. In this paper we give a characterization of the set of permutations that are realized by the shift on N symbols, and we enumerate them according to their length.

23. **Permutations realized by shifts**, *Discrete Math. Theor. Comput. Sci. proc. AK* (2009), 361–372.

24. **Sorting by Placement and Shift** (with P. Winkler), *Proceedings of the Twentieth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2009*, `arXiv:0809.2957v1`.

    In sorting situations where the final destination of each item is known, it is natural to repeatedly choose items and place them where they belong, allowing the intervening items to shift by one to make room. (In fact, a special case of this algorithm is commonly used to hand-sort files.) However, it is not obvious that this algorithm necessarily terminates.

    We show that in fact the algorithm terminates after at most $2^{n-1}-1$ steps in the worst case (confirming a conjecture of L. Larson), and that there are super-exponentially many permutations for which this exact bound can be achieved. The proof involves a curious symmetrical binary representation.

25. **Forbidden patterns and shift systems** (with J.M. Amigó, M.B. Kennel), *J. Combin. Theory Ser. A*, 115 (2008), 485–504.

    The scope of this paper is two-fold. First, to present an interesting implementation of permutations avoiding generalized patterns in the framework of discrete-time dynamical systems. Indeed, the orbits generated by piecewise monotone maps on one-dimensional intervals have forbidden order patterns, which do not occur in any orbit. The allowed patterns are then those patterns avoiding the so-called forbidden root patterns and their translates. The second scope is to study forbidden patterns in shift systems, which are universal models in information theory, dynamical systems and stochastic processes. Due to its simple structure, shift systems are accessible to a more detailed analysis and, at the same time, exhibit all important properties of complex dynamical systems, allowing to export the results to other dynamical systems via order-isomorphisms.

26. **Pattern avoidance in dynamical systems** (with J.M. Amigó, M.B. Kennel), *Discrete Math. Theor. Comput. Sci. proc. AJ* (2008), 71–82.

27. **Generating trees for permutations avoiding generalized patterns**, *Ann. Comb.* 11 (2007), 435–458.

    We construct generating trees with with one and two labels for some classes of permutations avoiding generalized patterns of length 3 and 4. These trees are built by adding at each level an entry to the right end of the permutation, instead of inserting always the largest entry. This allows us to incorporate the adjacency condition about some entries in an occurrence of a generalized pattern. We find functional equations for the generating functions enumerating these classes of permutations with respect to different parameters, and in a few cases we solve them using some techniques of Bousquet-Mélou, recovering known enumerative results and finding new ones.

28. **A bijection between 2-triangulations and pairs of non-crossing Dyck paths**, *J. Combin. Theory Ser. A*, 114/8 (2007), 1481–1503.

    A k-triangulation of a convex polygon is a maximal set of diagonals so that no $k+1$ of them mutually cross in their interiors. We present a bijection between 2-triangulations of a convex n-gon and pairs of non-crossing Dyck paths of length $2(n-4)$. This solves the problem of finding a bijective proof of a result of Jonsson for the case $k=2$. We obtain the bijection by constructing isomorphic generating trees for the sets of 2-triangulations and pairs of non-crossing Dyck paths.

29. **Bounds on the number of inference functions of a graphical model** (with K. Woods), *Statist. Sinica* 17 (2007), 1395–1415.

    We give an upper bound on the number of inference functions of any directed graphical model. This bound is polynomial on the size of the model, for a fixed number of parameters, thus improving the

exponential upper bound given by Pachter and Sturmfels. We also show that our bound is tight up to a constant factor, by constructing a family of hidden Markov models whose number of inference functions agrees asymptotically with the upper bound. Finally, we apply this bound to a model for sequence alignment that is used in computational biology.

30. **The probability of choosing primitive sets** (with K. Woods), *J. Number Theory* 125 (2007), 39–49.

We generalize a theorem of Nymann that the density of points in $\mathbb{Z}^d$ that are visible from the origin is $1/\zeta(d)$, where $\zeta(a)$ is the Riemann zeta function $\sum_{i=1}^{\infty} 1/i^a$. A subset $S \subset \mathbb{Z}^d$ is called primitive if it is a $\mathbb{Z}$-basis for the lattice $\mathbb{Z}^d \cap span_{\mathbb{R}}(S)$, or, equivalently, if S can be completed to a $\mathbb{Z}$-basis of $\mathbb{Z}^d$. We prove that if m points in $\mathbb{Z}^d$ are chosen uniformly and independently at random from a large box, then as the size of the box goes to infinity, the probability that the points form a primitive set approaches $1/(\zeta(d)\zeta(d-1)\cdots\zeta(d-m+1))$.

31. **Restricted Dumont permutations, Dyck paths, and noncrossing partitions** (with A. Burstein and T. Mansour), *Discrete Math.* 306 (2006), 2851-2869.

We complete the enumeration of Dumont permutations of the second kind avoiding a pattern of length 4 which is itself a Dumont permutation of the second kind. We also consider some combinatorial statistics on Dumont permutations avoiding certain patterns of length 3 and 4 and give a natural bijection between 3142-avoiding Dumont permutations of the second kind and noncrossing partitions that uses cycle decomposition, as well as bijections between 132-, 231- and 321-avoiding Dumont permutations and Dyck paths. Finally, we enumerate Dumont permutations of the first kind simultaneously avoiding certain pairs of 4-letter patterns and another pattern of arbitrary length.

32. **Asymptotic enumeration of permutations avoiding generalized patterns**, *Adv. in Appl. Math.* 36 (2006), 138–155.

Motivated by the recent proof of the Stanley-Wilf conjecture, we study the asymptotic behavior of the number of permutations avoiding a generalized pattern. Generalized patterns allow the requirement that some pairs of letters must be adjacent in an occurrence of the pattern in the permutation, and consecutive patterns are a particular case of them. We determine the asymptotic behavior of the number of permutations avoiding a consecutive pattern, showing that they are an exponentially small proportion of the total number of permutations. For some other generalized patterns we give partial results, showing that the number of permutations avoiding them grows faster than for classical patterns but more slowly than for consecutive patterns.

33. **Inference functions**, chapter of the book *Algebraic Statistics for Computational Biology*, edited by Lior Pachter and Bernd Sturmfels, Cambridge University Press, 2005.

34. **Bounds for optimal sequence alignment** (with Fumei Lam), chapter of the book *Algebraic Statistics for Computational Biology*, edited by Lior Pachter and Bernd Sturmfels, Cambridge University Press, 2005.

35. **Old and young leaves on plane trees** (with William Y.C. Chen and Emeric Deutsch), *European J. Combin.* 27 (2006), Issue 3, 414–427.

A leaf of a plane tree is called an old leaf if it is the leftmost child of its parent, and it is called a young leaf otherwise. In this paper we enumerate plane trees with a given number of old leaves and young leaves. The formula is obtained combinatorially by presenting two bijections between plane trees and 2-Motzkin paths which map young leaves to red horizontal steps, and old leaves to up steps plus one. We derive some implications to the enumeration of restricted permutations with respect to certain statistics such as pairs of consecutive deficiencies, double descents, and ascending runs. Finally, our

main bijection is applied to obtain refinements of two identities of Coker, involving refined Narayana numbers and the Catalan numbers.

36. **Multiple pattern avoidance with respect to fixed points and excedances**, *Electron. J. Combin.* 11 (2004), #R51.

    We study the distribution of the statistics 'number of fixed points' and 'number of excedances' in permutations avoiding subsets of patterns of length 3. We solve all the cases of simultaneous avoidance of more than one pattern, giving generating functions enumerating these two statistics. Some cases are generalized to patterns of arbitrary length. For avoidance of one single pattern we give partial results. We also describe the distribution of these statistics in involutions avoiding any subset of patterns of length 3.

    The main technique is to use bijections between pattern-avoiding permutations and certain kinds of Dyck paths, in such a way that the statistics in permutations that we study correspond to statistics on Dyck paths that are easy to enumerate.

37. **Restricted Motzkin permutations, Motzkin paths, continued fractions, and Chebyshev polynomials** (with Toufik Mansour), *Discrete Math.* 305 (2005), 170–189.

    We say that a permutation π is a *Motzkin permutation* if it avoids 132 and there do not exist $a < b$ such that $\pi_a < \pi_b < \pi_{b+1}$. We study the distribution of several statistics in Motzkin permutations, including the length of the longest increasing and decreasing subsequences and the number of rises and descents. We also enumerate Motzkin permutations with additional restrictions, and study the distribution of occurrences of fairly general patterns in this class of permutations.

38. **A simple and unusual bijection for Dyck paths and its consequences** (with Emeric Deutsch), *Ann. Comb.* 7 (2003), 281–297.

    In this paper we introduce a new bijection from the set of Dyck paths to itself. This bijection has the property that it maps statistics that appeared recently in the study of pattern-avoiding permutations into classical statistics on Dyck paths, whose distribution is easy to obtain.

    We also present a generalization of the bijection, as well as several applications of it to enumeration problems of statistics in restricted permutations.

39. **Bijections for refined restricted permutations** (with Igor Pak), *J. Combin. Theory Ser. A* 105 (2004), 207–219.   *(This paper made it to the top three downloaded articles from JCTA in 2004.)*

    We present a bijection between 321- and 132-avoiding permutations that preserves the number of fixed points and the number of excedances. This gives a simple combinatorial proof of recent results of Robertson, Saracino and Zeilberger [RSZ], and the first author [Eli]. We also show that our bijection preserves additional statistics, which extends the previous results.

40. **Fixed points and excedances in restricted permutations**, *Proceedings FPSAC 2003, Electron. J. Combin.* 18 (2012), #P29.

    Using an unprecedented technique involving diagonals of non-rational generating functions, we prove that among the permutations of length n with i fixed points and j excedances, the number of 321-avoiding ones equals the number of 132-avoiding ones, for any given i, j.

    This theorem generalizes a result of Robertson, Saracino and Zeilberger, for which we also give another, more direct proof.

41. **Consecutive patterns in permutations** (with Marc Noy), *Adv. in Appl. Math.* 30 (2003), 110–125.

    In this paper we study the distribution of the number of occurrences of a permutation σ as a subword among all permutations in $\mathcal{S}_n$. We solve the problem in several cases depending on the shape of

σ by obtaining the corresponding bivariate exponential generating functions as solutions of certain linear differential equations with polynomial coefficients. Our method is based on the representation of permutations as increasing binary trees and on symbolic methods.

Others

42. **Combinatòria i biologia: funcions d'inferència i alineació de seqüències**, *Butl. Soc. Catalana Mat.* 21 (2006), n. 1, 39-52 (in Catalan).

    This article in the Bulletin of the Catalan Mathematical Society gives some applications of combinatorial tools to problems that arise in computational biology, such as determining what parts of the genome are translated to proteins, or how a DNA sequence evolved into another one via a series of mutations, insertions and deletions.

43. **Statistics on pattern-avoiding permutations**, Ph.D. Thesis, MIT, 2004, available at `http://www.math.dartmouth.edu/~sergi/papers/thesis.pdf`.

44. **Games and Invariants**, chapter of the book **Training Sessions for the International Mathematical Olympiad** (in Catalan), Catalan Mathematical Society, Barcelona, 2000.

    This book, which has also a version in CD-ROM, is used every year by the students in Spain that are preparing for the International Mathematical Olympiad. Each chapter, written by a different author, covers some particular types of problems that the students are likely to encounter in the exam.