FALL 2016-2017, EEE-202
— FINAL EXAM SOLUTIONS —

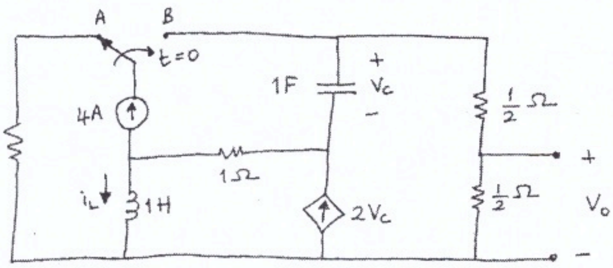
Question 1. [25 points]

Consider the following circuit. The switch is kept in position **A** for a very long time, and moved to position **B** at $t = 0$.

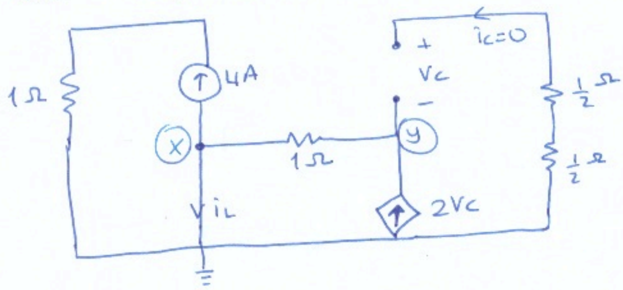
a) Find the values of $i_L(0^-)$ and $v_C(0^-)$.

b) Find a second-order differential equation for v_C for $t \geq 0$.

c) Using the result from part (b), find $v_C(t)$. Then, find $v_o(t)$.



a) at $t=0^-$, the circuit looks like the following:



$V_x = 0$

$V_y = -V_c$

KCL at x:

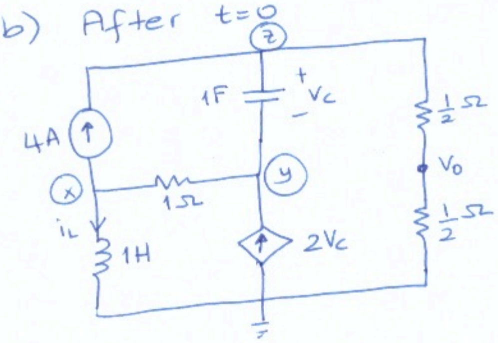
$4 + i_L - 2V_c \Rightarrow 4 + i_L + 2V_y = 0$

KCL at y:

$\frac{V_y}{1} - 2V_c = 0 \Rightarrow 3V_y = 0 \Rightarrow V_y = 0$

So, $\boxed{V_c(0^-) = 0\,V}$ $\boxed{i_L(0^-) = -4\,A}$

b) After $t=0$



KCL at x: $(V_x = L\dot{i}_L)$

$i_L + \frac{\dot{i}_L - V_y}{1} + 4 = 0$

KCL at y:

$V_y - i_L - 2V_c - \dot{V}_c = 0$

KCL at z:

$\frac{V_y + V_c}{1} + \dot{V}_c - 4 = 0$

$\boxed{i_L = \dot{V}_c + 2V_c - 4}$

$\Downarrow$

$\dot{i}_L = \ddot{V}_c + 2\dot{V}_c$

$V_c + \dot{V}_c - 4 + i_L + 2V_c + \dot{V}_c = 0$

$2\dot{V}_c + 3V_c - 4 + \dot{i}_L = 0$

$2\dot{V}_c + 3V_c - 4 + \ddot{V}_c + 2\dot{V}_c = 0$

$$\boxed{\ddot{V}_c + 4\dot{V}_c + 3V_c = 4}$$

c) characteristic polynomial:

$s^2 + 4s + 3 = 0$

$(s+1)(s+3) = 0 \Rightarrow \boxed{s_1 = -1,\ s_2 = -3}$

$V_N(t) = k_1 . e^{-t} + k_2 . e^{-3t}$ (natural response

* Forced response: $V_F = A \Rightarrow$ insert into ODE: $3A = 4 \Rightarrow \boxed{A = \frac{4}{3} = V_F}$

So, $V_c(t) = V_N(t) + V_F = k_1 . e^{-t} + k_2 e^{-3t} + \frac{4}{3}\,V$

* Now use initial conditions.

$V_c(0^-) = V_c(0) = 0 \Rightarrow \boxed{K_1 + K_2 + \frac{4}{3} = 0}$

* using $i_L = \dot{V}_c + 2V_c - 4$ and $i_L(0^-) = i_L(0) = -4A$ $\Big\}$ $i_L(0) = \dot{V}_c(0) + 2V_c(0) - 4 = -4$

$\boxed{\dot{V}_c(0) = 0}$

$\dot{V}_c(t) = -K_1 . e^{-t} - 3K_2 . e^{-3t}$

$\dot{V}_c(0) = -K_1 - 3K_2 = 0 \Rightarrow \boxed{K_1 = -3K_2}$

then, $-3K_2 + K_2 + \frac{4}{3} = 0 \Rightarrow \boxed{K_2 = \frac{2}{3}, \quad K_1 = -2}$

So,

$$\boxed{V_c(t) = -2 . e^{-t} + \frac{2}{3} e^{-3t} + \frac{4}{3} \ V}$$

(check i.c. $V_c(0) = -2 + \frac{2}{3} + \frac{4}{3} = 0$. o.k.)

+ From voltage divider, $V_o = \frac{V_z}{2}$

KCL at z:

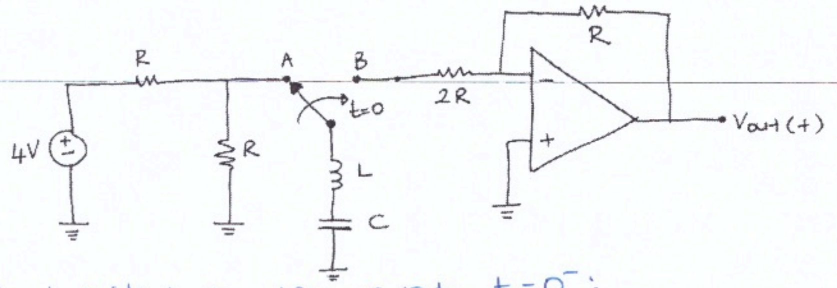
$\frac{V_z}{1} + \dot{V}_c - 4 = 0 \Rightarrow V_o(t) = \frac{4 - \dot{V}_c}{2}$

$$= \frac{4 - (2e^{-t} - 2e^{-3t})}{2}$$

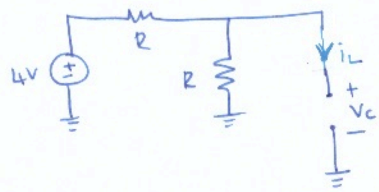
$$\boxed{V_o(t) = 2 - e^{-t} + e^{-3t} \ V}$$

Question 2. [25 points]

Consider the circuit below. The OPAMP is ideal and operating in linear mode. $R = 4\Omega$, $L = 6H$ and $C = 0.5F$. The switch is kept in position **A** for a very long time, and moved to position B at $t = 0$. Using Laplace-domain circuit analysis, find $v_o(t)$.



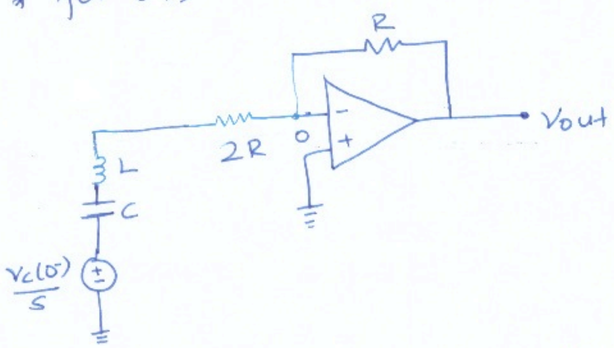
* Find initial conditions at $t = 0^-$:



$i_L(0^-) = 0$

$V_C(0^-) = 4 \cdot \frac{R}{R+R} = 2V$ (voltage divider)

* for $t \geq 0$, initial condition transformation simplifies the analysis:



$V_+ = V_- = 0$

KCL at (-) input of OPAMP

$$\frac{0 - \frac{V_c(0^-)}{s}}{2R + sL + \frac{1}{sC}} + \frac{0 - V_{out}}{R} = 0$$

$$V_{out} = R \cdot \frac{-V_c(0^-) \cdot \cancel{s}C}{\cancel{s}(s^2LC + 2sRC + 1)} = \frac{-V_c(0^-)RC}{s^2LC + 2sRC + 1}$$

$$\boxed{V_{out}(s) = \frac{-4}{3s^2 + 4s + 1}}$$

* use partial fraction expansion:

$$V_{out} = \frac{-4}{(3s+1)(s+1)} = \frac{-4}{3(s+\frac{1}{3})(s+1)} = \frac{k_1}{s+\frac{1}{3}} + \frac{k_2}{s+1}$$

$$k_1 = \left(s+\tfrac{1}{3}\right) \cdot V_{out}\Big|_{s=-\frac{1}{3}} = \frac{-4}{3(s+1)}\Big|_{s=-\frac{1}{3}} = -2$$

$$k_2 = (s+1) \cdot V_{out}\Big|_{s=-1} = \frac{-4}{3(s+\frac{1}{3})}\Big|_{s=-1} = 2$$

* So, $\boxed{v_{out}(t) = \left[-2 \cdot e^{-\frac{1}{3}t} + 2e^{-t}\right] \cdot u(t) \text{ Volts}}$

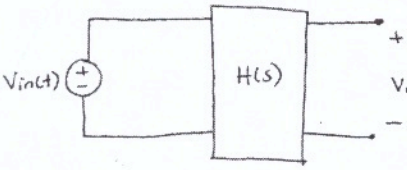
Question 3. [25 points]

Consider the following circuit. Assume zero initial conditions. Here, the block given by $H(s)$ represents a linear circuit and $H(s) = \frac{V_o(s)}{V_{in}(s)}$.

a) What is $H(s)$ if the unit step response of the circuit is $v_o(t) = [9 - 10e^{-t} + e^{-10t}]\, u(t)$?
Hint: Unit step response is the output when the input is the unit step function.

b) Find $v_o(t)$ if the input is given as $v_{in}(t) = 3e^{-t}u(t)$ V.

c) Using asymptotic lines, draw the Bode plot of $|H(jw)|$ in logarithmic scale for the transfer function found in part (a). Mark all important frequency points and denote all slopes.



a) $V_{in}(s) = \frac{1}{s}$

$$V_{out}(s) = \underset{(s+1)(s+10)}{\frac{9}{s}} - \underset{s(s+10)}{\frac{10}{s+1}} + \underset{s(s+1)}{\frac{1}{s+10}} = \frac{9(s^2+11s+10) - 10(s^2+10s) + s^2+s}{s(s+1)(s+10)} = \frac{90}{s(s+1)(s+10)}$$

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \boxed{\frac{90}{(s+1)(s+10)}}$$

b) $V_{in}(s) = \frac{3}{s+1} \Rightarrow V_{out}(s) = H(s).V_{in}(s) = \frac{270}{(s+1)^2(s+10)}$

$$= \frac{k_{11}}{(s+1)^2} + \frac{k_{12}}{s+1} + \frac{k_2}{s+10}$$

$$k_{11} = V_{out}.(s+1)^2\Big|_{s=-1} = \frac{270}{s+10}\Big|_{s=-1} = 30$$

$$k_{12} = \frac{d}{ds}\left\{V_{out}.(s+1)^2\right\}\Big|_{s=-1} = \frac{d}{ds}\left(\frac{270}{s+10}\right)\Big|_{s=-1} = -\frac{270}{(s+1)^2}\Big|_{s=-1} = -\frac{10}{3}$$

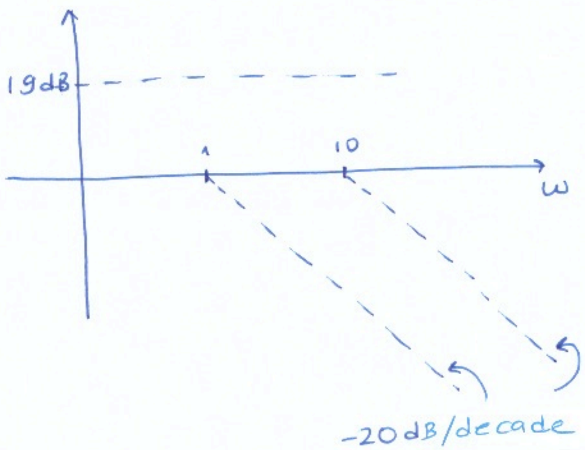
$$k_2 = V_{out}.(s+10)\Big|_{s=-10} = \frac{270}{(s+1)^2}\Big|_{s=-10} = \frac{10}{3}$$

$$\boxed{V_{out}(t) = \left[30\,t.e^{-t} - \frac{10}{3}e^{-t} + \frac{10}{3}e^{-10t}\right]u(t)}$$

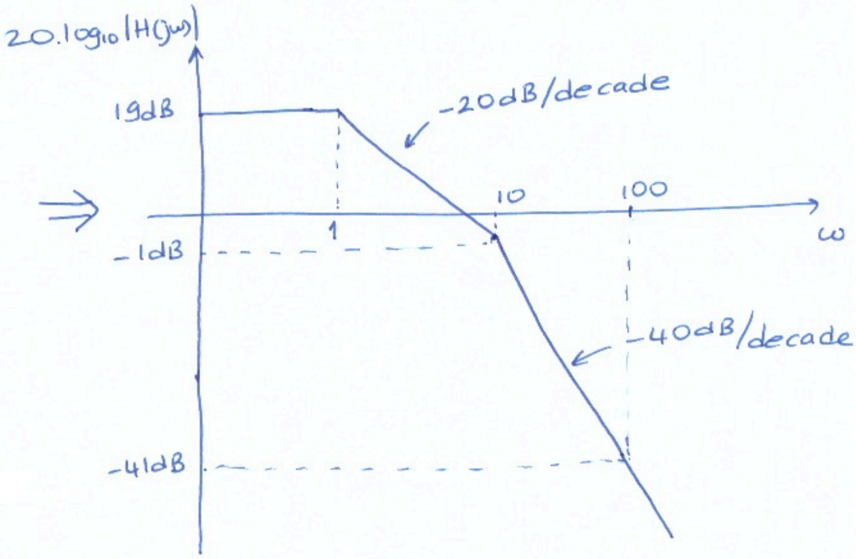
c) $H(jw) = \frac{90}{(jw+1)(jw+10)} = \frac{9}{(1+jw)(1+j\frac{w}{10})}$

$$|H(jw)| = \frac{9}{\sqrt{1+w^2}\,\sqrt{1+\left(\frac{w}{10}\right)^2}}$$ →

$$20\log_{10}|H(j\omega)| = \underbrace{20.\log_{10} 9}_{\simeq 19\,dB} + 20\log_{10}\left(\frac{1}{\sqrt{1+\omega^2}}\right) + 20\cdot\log_{10}\left(\frac{1}{\sqrt{1+\left(\frac{\omega}{10}\right)^2}}\right)$$



$\Rightarrow$



Question 4. [25 points]

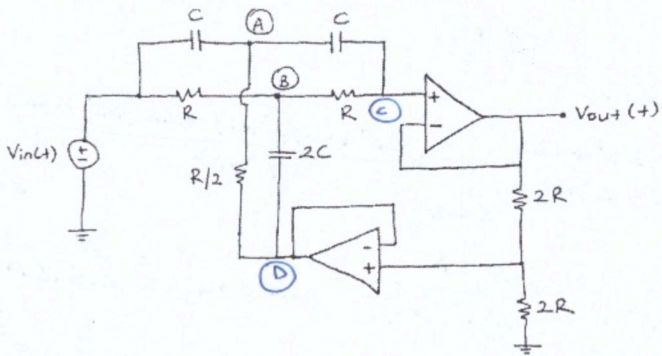
Consider the circuit below. All OPAMPs are ideal and operating in linear mode. Assume zero initial conditions for all voltages and currents. This circuit represent an implementation of a band-stop filter.

a) Find the transfer function in s-domain, $H(s) = \frac{V_{out}(s)}{V_{in}(s)}$, as a function of R and C.

b) Check that you found a band-stop filter in part (a). What is the center frequency of the stop band, in terms of R and C?

c) Assume that $R = 2\Omega$ and $C = 0.25F$. Find the output voltage $v_{out}(t)$ if $v_{in}(t) = 3\,cos(t)\,u(t)$ Volts, where $u(t)$ is the unit step function.

d) Assume that the circuit is in sinusoidal steady state. Using phasor analysis and your answer from part (a), find $v_{out}(t)$ if $v_{in}(t) = 3\,cos(t)$ Volts. Is this result consistent with what you found in part (c)?



$V_+ = V_-$ for the OPAMPS.

$V_C = V_{out}$

$V_D = \frac{V_{out} \cdot 2R}{2R + 2R} = \frac{V_{out}}{2}$

a) KCL at A: $(V_A - V_{in})sC + \left(V_A - \frac{V_{out}}{2}\right)\cdot\frac{2}{R} + (V_A - V_{out})sC = 0$

$V_A(sRC + 2 + sRC) - V_{in} \cdot sRC - V_{out}(1 + sRC) = 0$

$$\boxed{V_A = \frac{V_{in} \cdot sRC + V_{out}(1 + sRC)}{2 + 2sRC}}$$

KCL at B: $\frac{V_B - V_{in}}{R} + \frac{V_B - V_{out}}{R} + \left(V_B - \frac{V_{out}}{2}\right)\cdot 2sC = 0$

$V_B(2 + 2sRC) - V_{in} - V_{out}(1 + sRC) = 0$

$$\boxed{V_B = \frac{V_{in} + V_{out}(1 + sRC)}{2 + 2sRC}}$$

KCL at C: $\frac{V_{out} - V_B}{R} + (V_{out} - V_A)sC = 0$

$V_{out}(1 + sRC) - V_B - V_A \cdot sRC = 0$

$V_{out}(1 + sRC) - \frac{[V_{in} + V_{out}(1 + sRC)]}{2 + 2sRC} - \frac{[V_{in} \cdot sRC + V_{out}(1 + sRC)] \cdot sRC}{2 + 2sRC} = 0$

$V_{out}\left[2 + 4sRC + 2s^2R^2C^2 - 1 - sRC - sRC - s^2R^2C^2\right] = V_{in}\left[1 + s^2R^2C^2\right]$

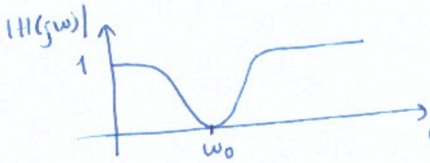
$$H(s) = \frac{V_{out}}{V_{in}} = \frac{1 + s^2R^2C^2}{1 + 2sRC + s^2R^2C^2} = \boxed{\frac{1 + s^2R^2C^2}{(1 + sRC)^2}}$$

b) $$H(j\omega) = \frac{1+(j\omega RC)^2}{(1+j\omega RC)^2} = \frac{1-\omega^2R^2C^2}{(1+j\omega RC)^2}$$

$$|H(j\omega)| = \frac{|1-\omega^2R^2C^2|}{1+\omega^2R^2C^2} \Rightarrow \text{at } \omega_0 = \frac{1}{RC},\ |H(j\omega_0)| = 0$$

at $\omega = 0$, $|H(j0)| = 1$

at $\omega \to \infty$, $|H(j\omega)| = 1$



So, this is a band-stop filter.

c) $R = 2\Omega$, $C = 0.25F$ $\Rightarrow$ $$H(s) = \frac{1+s^2/4}{(1+s/2)^2} = \frac{s^2+4}{s^2+4s+4}$$

$$V_{in} = \frac{3s}{s^2+1} \Rightarrow V_{out} = H(s).V_{in} = \frac{3s(s^2+4)}{(s^2+1)(s^2+4s+4)} = \frac{3s(s^2+4)}{(s+j)(s-j)(s+2)^2}$$

$$= \frac{k_1}{(s+2)^2} + \frac{k_2}{s+2} + \frac{k}{s-j} + \frac{k^*}{s+j}$$

$$k_1 = V_{out}.(s+2)^2\Big|_{s=-2} = \frac{3s(s^2+4)}{s^2+1}\Big|_{s=-2} = -\frac{48}{5}$$

$$k_2 = \frac{d}{ds}\left(V_{out}.(s+2)^2\right)\Big|_{s=-2} = \frac{d}{ds}\left(\frac{3s^3+12s}{s^2+1}\right)\Big|_{s=-2} = \frac{(9s^2+12)(s^2+1)-2s(3s^3+12s)}{(s^2+1)^2}\Big|_{s=-2} = \frac{48}{25}$$

$$k = V_{out}(s-j)\Big|_{s=j} = \frac{3s(s^2+4)}{(s+j)(s+2)^2}\Big|_{s=j} = \frac{3j.3}{2j.(j+2)^2} = \frac{9}{2.(3+4j)} = \frac{9}{2.5.\angle\operatorname{atan}(\frac{4}{3})}$$
$$= \frac{9}{10}.e^{-j.\tan^{-1}(\frac{4}{3})}$$

So,

$$\boxed{V_{out}(t) = \Big[\underbrace{-\frac{48}{5}.t.e^{-2t} + \frac{48}{25}.e^{-2t}}_{\text{transient response}} + \underbrace{\frac{9}{5}.\cos\big(t-\tan^{-1}(\tfrac{4}{3})\big)}_{\text{steady-state response}}\Big].u(t) \text{ Volts}}$$

$\tan^{-1}(\frac{4}{3})$: 0.93 rad, OR 53°

d) $$H(j\omega) = \frac{4-\omega^2}{-\omega^2+4j\omega+4}, \text{ at } \omega = 1 \Rightarrow H(j) = \frac{3}{3+4j}$$

$$V_{in} = 3, \quad V_{out} = H(j).V_{in} = \frac{9}{3+4j} = \frac{9}{5}e^{-j.\tan^{-1}(4/3)}$$

$$\boxed{V_{out}(t) = \frac{9}{5}.\cos(t-\tan^{-1}(4/3)) \text{ Volts}}$$

This result is consistent with part (c). Phasor analysis does not give the transient response, but only gives the steady-state response. So, as $t \to \infty$, the result in (c) converges to the result in (d), as $t.e^{-2t}$ and e^{-2t} terms converge to zero.