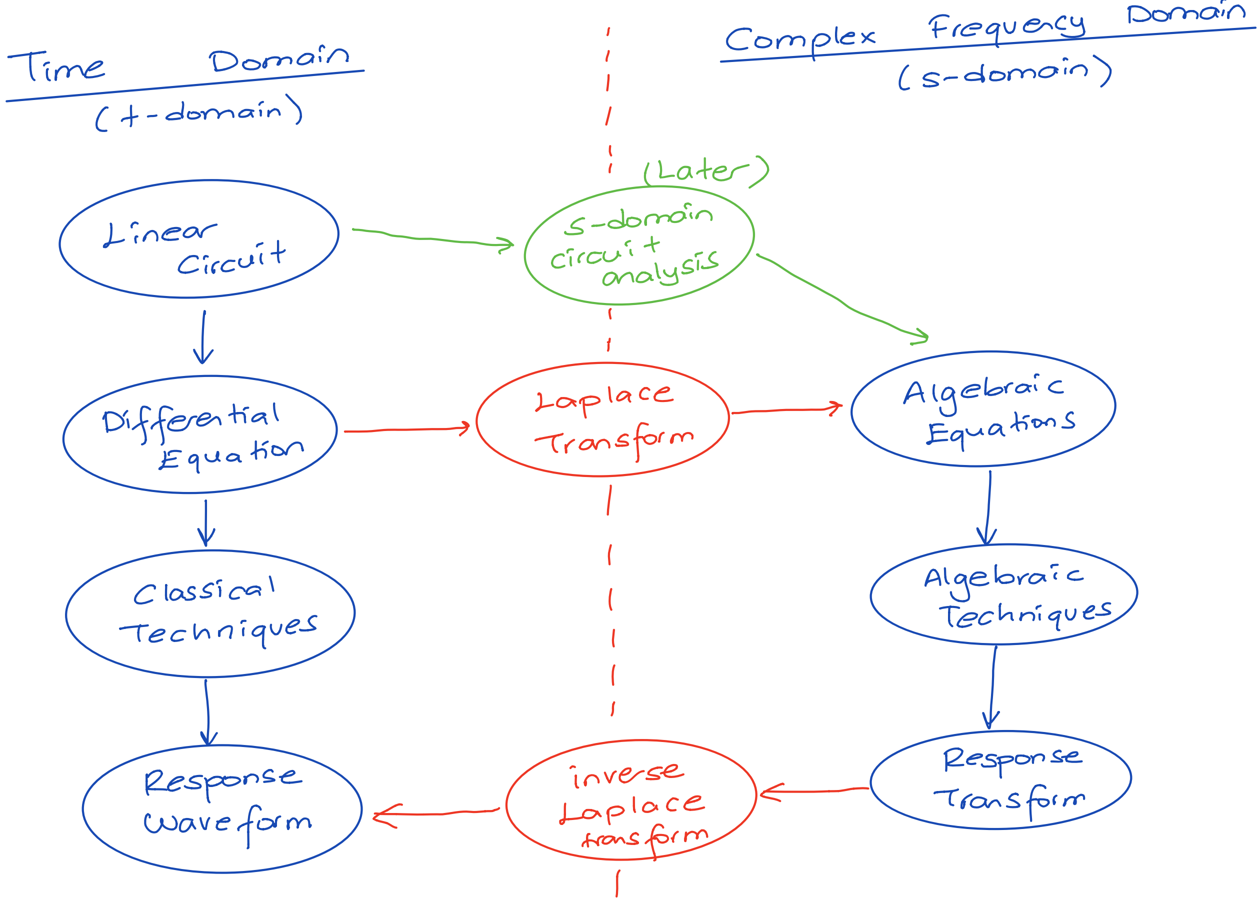
CHAPTER 9 - LAPLACE TRANSFORM

* The difficulties of finding transient response using classical differential equations are avoided when we apply Laplace transform.
* Gives us a deeper insight into the circuit behavior.



Definition:

$$\boxed{F(s) = \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt = \mathcal{L}\{f(t)\}}$$

Existence: $F(s)$ exists if $f(t)$ is piecewise continuous,

and $\underbrace{|f(t)| < K . e^{\omega t}}$ for all $t > 0$,

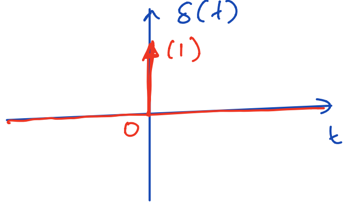
for some K, ω.

function should grow slower than an exponential

$\Rightarrow$ practical signals meet these criteria.

* Lower limit 0^- is to capture an impulse at $t = 0$.

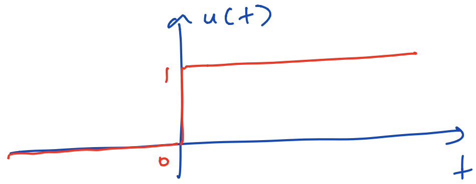
* $f(t) = \delta(t)$, $F(s) = \int_{0^-}^{\infty} \delta(t) e^{-st} dt = \int_{0^-}^{0^+} \delta(t) e^{-st} dt$



$$= \int_{0^-}^{0^+} \delta(t) \cdot e^{-s.0} \cdot dt = \int_{0^-}^{0^+} \delta(t) dt = 1$$

$$\boxed{F(s) = 1}$$

* $f(t) = u(t) = \begin{cases} 1 & , \quad t \geq 0 \\ 0 & , \quad t < 0 \end{cases}$



$$F(s) = \int_{0^-}^{\infty} u(t) e^{-st} dt = \int_{0}^{\infty} u(t) e^{-st} dt = \int_{0}^{\infty} e^{-st} dt$$

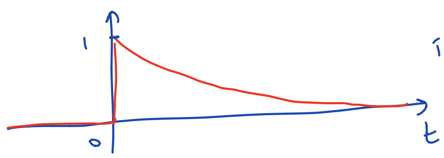
$$= - \left. \frac{e^{-st}}{s} \right|_0^{\infty} = - \left(0 - \frac{e^{-0}}{s} \right) = \frac{1}{s}$$

if $Re\{s\} > 0$

$\boxed{F(s) = \frac{1}{s}}$ for $Re\{s\} > 0$

* $f(t) = e^{-\alpha t} u(t)$, α could be real or complex

if α is real valued and $\alpha > 0$



$$F(s) = \int_{0^-}^{\infty} e^{-\alpha t} u(t) e^{-st} dt = \int_{0}^{\infty} e^{-(s+\alpha)t} dt = \left. \frac{e^{-(s+\alpha)t}}{-(s+\alpha)} \right|_0^{\infty}$$

$$= - \frac{1}{s+\alpha} \left[0 - e^{-0} \right] = \frac{1}{s+\alpha}$$

if $Re\{s+\alpha\} > 0$

$\boxed{F(s) = \frac{1}{s+\alpha}}$ for $Re\{s+\alpha\} > 0$

* ALL time functions are defined for $t \geq 0$. !

Basic Properties

1) Uniqueness : $x(t) = y(t)$ for $t \geq 0$ $\Longleftrightarrow$ $X(s) = Y(s)$

2) Linearity : $\alpha\, x(t) + \beta\, y(t)$ for $t \geq 0$ $\Longleftrightarrow$ $\displaystyle\int_{0^-}^{\infty} [\alpha x(t) + \beta y(t)]\, e^{-st}\, dt$

$$= \alpha \int_{0^-}^{\infty} x(t) e^{-st} dt + \beta \cdot \int_{0^-}^{\infty} y(t) e^{-st} dt$$

$$= \alpha X(s) + \beta Y(s)$$

α and β may be real or complex valued

Example :

$$f(t) = e^{-\alpha t} u(t) \longleftrightarrow F(s) = \frac{1}{s+\alpha}$$

$$* \quad e^{j\omega t} u(t) \longleftrightarrow \frac{1}{s - j\omega}$$

$$e^{-j\omega t} u(t) \longleftrightarrow \frac{1}{s + j\omega}$$

$$\cos(\omega t) . u(t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2} . u(t) \longleftrightarrow \frac{1}{2} \frac{1}{s - j\omega} + \frac{1}{2} \frac{1}{s + j\omega} = \frac{s}{s^2 + \omega^2}$$

$$\sin(\omega t) . u(t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} . u(t) \longleftrightarrow \frac{1}{2j} \frac{1}{s - j\omega} - \frac{1}{2j} \frac{1}{s + j\omega} = \frac{\omega}{s^2 + \omega^2}$$

3) Differentiation Rule : $f(t) \longleftrightarrow F(s)$

what is $\mathcal{L}\left\{ \dfrac{df}{dt} \right\} = ?$

$$\mathcal{L}\left\{ \frac{df(t)}{dt} \right\} = \int_{0^-}^{\infty} \frac{df(t)}{dt} \underbrace{e^{-st}}_{u} dt$$

$\Leftarrow$ use integration by parts

$$\left(\begin{array}{ll} u = e^{-st} , & dv = \dfrac{df(t)}{dt} . dt \\ du = -s e^{-st} dt , & v = f(t) \end{array} \right)$$

$$= u . v . \Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} v . du$$

$$= e^{-st} f(t) \Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} f(t) \left(-s \cdot e^{-st}\right) dt$$

$$= 0 - e^{-0} \cdot f(0^-) + s \underbrace{\int_{0^-}^{\infty} f(t) e^{-st} dt}_{F(s)}$$

$$\boxed{\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-)}$$

initial condition for $f(t)$

$$\mathcal{L}\left\{\frac{d^2 f}{dt^2}\right\} = s\,\mathcal{L}\left\{\frac{df}{dt}\right\} - f'(0^-) = s\left(sF(s) - f(0^-)\right) - f'(0^-)$$

$$= s^2 F(s) - s f(0^-) - f'(0^-)$$

$$\vdots$$

$$\mathcal{L}\left\{\frac{d^n f}{dt^n}\right\} = s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \cdots - f^{(n-1)}(0^-)$$

$(n-1)^{th}$ derivative

* This property is very useful in solving differential equations.

4) <u>Integration Rule</u>: $f(t) \longleftrightarrow F(s)$

What is $\mathcal{L}\left\{\int_{0^-}^{t} f(\tau) d\tau\right\} = ?$

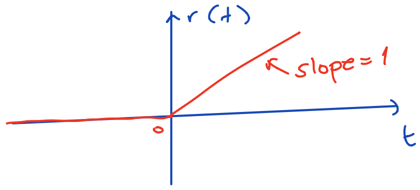
$$\int_{0^-}^{\infty} \underbrace{\left(\int_{0^-}^{t} f(\tau) d\tau\right)}_{u} \cdot \underbrace{e^{-st} dt}_{dv} = u \cdot v \Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} v \cdot du$$

$$\left(du = f(t)\,dt \, , \quad v = -\frac{e^{-st}}{s}\right) \quad = \left[-\frac{e^{-st}}{s} \cdot \int_{0^-}^{t} f(\tau) d\tau\right] \Bigg|_{0^-}^{\infty} - \int_{0^-}^{\infty} \left(-\frac{e^{-st}}{s}\right) f(t) dt$$

$$= 0 - 0 + \frac{1}{s} \underbrace{\int_{0^-}^{\infty} f(t) e^{-st} dt}_{F(s)}$$

$$\boxed{\mathcal{L}\left\{\int_{0^-}^{t} f(\tau) d\tau\right\} = \frac{F(s)}{s}}$$

$* \quad r(t) = t \cdot u(t) = \int_0^t u(\tau)\,d\tau$



$$\mathcal{L}\{r(t)\} = \frac{1}{s} \cdot \mathcal{L}\{u(t)\} = \frac{1}{s^2}$$

$* \quad f(t) = t^2 , \quad t \geq 0 \quad \Longrightarrow \quad f(t) = t^2 . u(t) = 2\int_0^t \tau\, d\tau = 2\int_0^t r(\tau)\,d\tau$

$$\mathcal{L}\{f(t)\} = \frac{2}{s}\,\mathcal{L}\{r(t)\} = \frac{2}{s^3}$$

$\vdots$

$$\boxed{\mathcal{L}\{t^n u(t)\} = \frac{n!}{s^{n+1}}}$$

5) **s-domain translation :**

$$\mathcal{L}\{f(t)\} = F(s) \iff \mathcal{L}\{\underbrace{e^{-\alpha t} f(t)}_{\text{damped version of } f(t)}\} = \int_{0^-}^{\infty} f(t)\, e^{-\alpha t} \cdot e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} f(t)\, e^{-(s+\alpha)t}\, dt = F(s+\alpha)$$

$$* \; \mathcal{L}\{t . u(t)\} = \frac{1}{s^2} \iff \mathcal{L}\{\underbrace{t . e^{-\alpha t} u(t)}_{\text{damped ramp}}\} = \frac{1}{(s+\alpha)^2}$$

$$\mathcal{L}\{\cos(\beta t) . u(t)\} = \frac{s}{s^2+\beta^2} \iff \mathcal{L}\{\underbrace{e^{-\alpha t} \cos(\beta t) u(t)}_{\text{damped cosine}}\} = \frac{s+\alpha}{(s+\alpha)^2+\beta^2}$$

$$\mathcal{L}\{\sin(\beta t)\, u(t)\} = \frac{\beta}{s^2+\beta^2} \iff \mathcal{L}\{\underbrace{e^{-\alpha t} \sin(\beta t) u(t)}_{\text{damped sine}}\} = \frac{\beta}{(s+\alpha)^2+\beta^2}$$

6) **Time-domain translation :**

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{f(t) . u(t)\} = F(s)$$

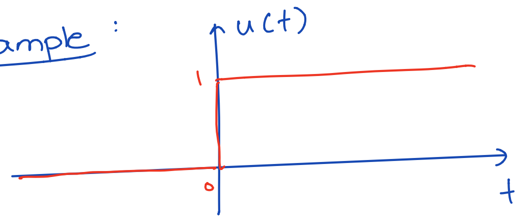
$$\mathcal{L}\{f(t-T)\, u(t-T)\} = \int_{0^-}^{\infty} f(t-T)\, u(t-T)\, e^{-st}\, dt , \qquad t' = t - T, \quad dt' = dt$$

$$= \int_{-T^-}^{\infty} f(t')\, \underbrace{u(t')}_{\text{zero for } t' < 0}\, e^{-s(t'+T)}\, dt'$$

$$= \int_{0^-}^{\infty} f(t)\, e^{-s(t+T)}\, dt = e^{-sT} \underbrace{\int_{0^-}^{\infty} f(t)\, e^{-st}\, dt}_{F(s)}$$

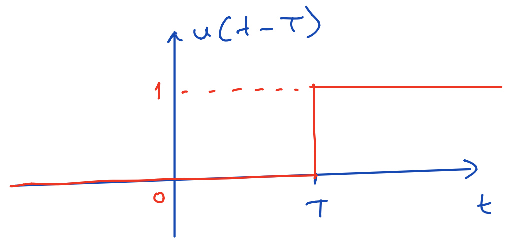
$$\boxed{\mathcal{L}\{f(t-T)\,u(t-T)\} = e^{-sT} F(s)}$$

Example :



$\Downarrow$

$F(s) = \frac{1}{s}$



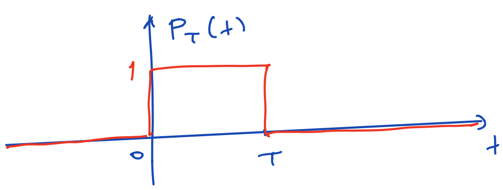
$\Downarrow$

$e^{-sT} \cdot \frac{1}{s}$

Example : pulse .

$$p_T(t) = u(t) - u(t-T)$$

$$\mathcal{L}\{p_T(t)\} = ?$$



$$F(s) = \frac{1}{s} - e^{-sT} \cdot \frac{1}{s} = \frac{1 - e^{-sT}}{s}$$

Example : $f(t) = (e^{-2t} + 4t - 1)\,u(t)$, $F(s) = ?$

$$F(s) = \frac{1}{s+2} + 4 \cdot \frac{1}{s^2} - \frac{1}{s}$$

$$(s^2) \qquad (s+2) \qquad s(s+2)$$

$$= \frac{\cancel{s^2} + 4s + 8 - \cancel{s^2} - 2s}{s^2(s+2)} = \frac{2(s+4)}{s^2(s+2)}$$

Note : For linear circuits, Laplace transforms of signal waveforms are ratios of polynomials in s.

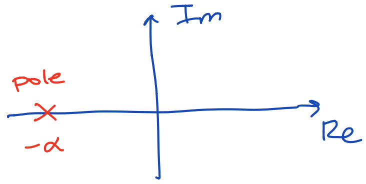
Poles - Zeroes :

$$F(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

$$= K \cdot \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \quad \begin{matrix} \leftarrow m \text{ zeroes} \\ \leftarrow n \text{ poles} \end{matrix}$$

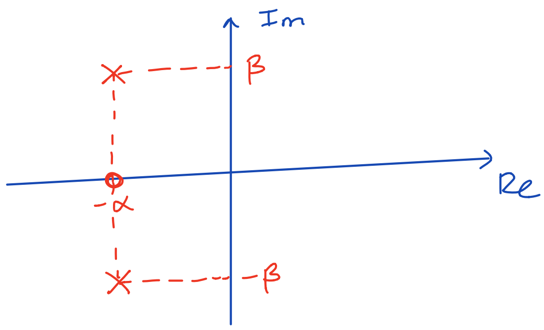
Example :

$f(t) = e^{-\alpha t} u(t) \implies F(s) = \frac{1}{s+\alpha}$ ← pole at $s = -\alpha$



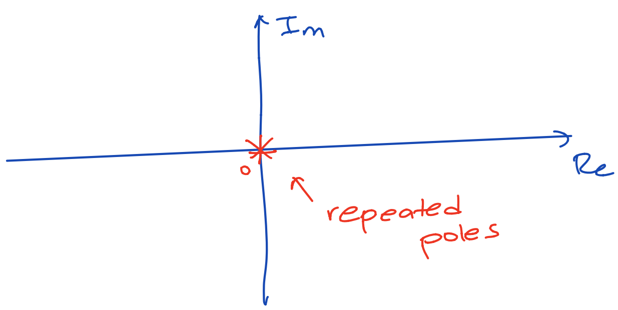
Example : $f(t) = e^{-\alpha t} \cos(\beta t)\, u(t)$

$F(s) = \frac{s+\alpha}{(s+\alpha)^2 + \beta^2}$ ← zero at $s = -\alpha$ ← poles at $s = -\alpha \pm j\beta$



Example : $f(t) = t\, u(t)$

$F(s) = \frac{1}{s^2}$ ← 2 poles at $s=0$



Inverse Transformation by Partial Fraction Expansion

$$F(s) = \frac{p(s)}{q(s)} = \frac{p(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

Assume proper rational function : $\deg(q) > \deg(p)$

i.e., $n > m$

Simple Poles : no repeated roots

$$F(s) = \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \cdots + \frac{k_n}{s-p_n}$$ ⇐ decompose into this form

$$f(t) = k_1 . e^{p_1 t} + k_2 . e^{p_2 t} + \cdots + k_n . e^{p_n t}$$

$$k_1 = (s-p_1)\, F(s)\Big|_{s=p_1} = \left[k_1 + k_2 . \frac{(s-p_1)}{s-p_2} + \cdots \; k_n . \frac{(s-p_1)}{s-p_n}\right]\Bigg|_{s=p_1} = k_1$$

So,

$$\boxed{k_i = (s-p_i)\, F(s)\Big|_{s=p_i}}$$

, where k_i is called "residue"

Example : $$F(s) = \frac{2(s+3)}{s(s+1)(s+2)} = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2}$$

$$k_1 = s . F(s)\Big|_{s=0} = \frac{2(s+3)}{(s+1)(s+2)}\Bigg|_{s=0} = \frac{2.3}{1.2} = 3$$

$$k_2 = (s+1)\, F(s)\Big|_{s=-1} = \frac{2(s+3)}{s(s+2)}\Bigg|_{s=-1} = \frac{2.2}{-1.1} = -4$$

$$k_3 = (s+2)\, F(s)\Big|_{s=-2} = \frac{2(s+3)}{s(s+1)}\Bigg|_{s=-2} = \frac{2.1}{-2.(-1)} = 1$$

$$f(t) = 3.u(t) - 4.e^{-t}\, u(t) + e^{-2t}\, u(t)$$

$$= \left[3 - 4e^{-t} + e^{-2t}\right] u(t)$$

Complex Roots : come from damped/undamped cosines and sines

$$\underbrace{\frac{s+\alpha}{(s+\alpha)^2+\beta^2} \quad \text{OR} \quad \frac{\beta}{(s+\alpha)^2+\beta^2}}_{\text{poles at } s=-\alpha\pm j\beta}$$

$$F(s) = \cdots + \underbrace{\frac{k_1}{s-(-\alpha+j\beta)} + \frac{k_2}{s-(-\alpha-j\beta)}}_{\text{the terms in their sum need to be real valued}} + \cdots$$

$$k_2 = k_1^* \quad , \quad k_1 = |k|.e^{j\theta} \quad , \quad k_2 = |k|.e^{-j\theta}$$

$$f(t) = \cdots + |k|.e^{j\theta}.e^{(-\alpha+j\beta)t} + |k|.e^{-j\theta}.e^{(-\alpha-j\beta)t} + \cdots$$

$$= \cdots + |k|.e^{-\alpha t}.e^{j(\beta t+\theta)} + |k|.e^{-\alpha t}.e^{-j(\beta t+\theta)} + \cdots$$

$$= \cdots + \underbrace{2|k|.e^{-\alpha t}\cos(\beta t+\theta)}_{\text{damped cosine}} + \cdots$$

Example : $F(s) = \dfrac{20(s+3)}{(s+1)(s^2+2s+5)}$

$s=-1$; $s = \dfrac{-2\pm\sqrt{2^2-4.5}}{2} = -1\pm j2$

$$= \frac{k_1}{s+1} + \frac{k}{s-(-1+j2)} + \frac{k^*}{s-(-1-j2)}$$

assigned to pole $-\alpha+j\beta$, where $\beta>0$

$$k_1 = (s+1)F(s)\Big|_{s=-1} = \frac{20(s+3)}{s^2+2s+5}\Big|_{s=-1} = \frac{20.2}{1-2+5} = 10$$

$$k = (s-(-1+j2)).F(s)\Big|_{s=-1+j2} = \frac{20(s+3)}{(s+1).(s-(-1-j2))}\Big|_{s=-1+j2}$$

$$= \frac{20(2+j2)}{j2.4j} = -5(1+j) = 5.e^{j\pi}.\sqrt{2}.e^{j\pi/4} = 5\sqrt{2}\,e^{j5\pi/4}$$

$$= |k|.e^{j\theta}$$

$$f(t) = 10 \cdot e^{-t} \cdot u(t) + \underbrace{10\sqrt{2}}_{2|k|} \cdot e^{-t} \cdot \cos(2t + \underbrace{5\pi/4}_{\theta}) \cdot u(t)$$

Multiple Poles : repeated poles

$$F(s) = \frac{p(s)}{(s-p_1)^m \cdot (\cdots\cdots)} = \frac{k_{11}}{(s-p_1)^m} + \frac{k_{12}}{(s-p_1)^{m-1}} + \cdots + \frac{k_{1m}}{(s-p_1)} + \underbrace{\cdots\cdots}_{\text{from other poles}}$$

pole repeated m times

$$(s-p_1)^m F(s) = k_{11} + k_{12}(s-p_1) + k_{13}(s-p_1)^2 + \cdots + k_{1m}(s-p_1)^{m-1} + (s-p_1)^m \underbrace{[\cdots\cdots]}_{\text{from other poles}}$$

$$k_{11} = (s-p_1)^m F(s) \Big|_{s=p_1}$$

$$k_{12} = \left\{ \frac{d}{ds} (s-p_1)^m F(s) \right\} \Big|_{s=p_1} = \left\{ k_{12} + 2k_{13}(s-p_1) + \cdots + (m-1)k_{1m}(s-p_1)^{m-2} + \frac{d}{ds}\left((s-p_1)^m [\cdots\cdots]\right) \right\} \Big|_{s=p_1}$$

$$k_{13} = \frac{1}{2} \left\{ \frac{d^2}{ds^2} (s-p_1)^m F(s) \right\} \Big|_{s=p_1} = \frac{1}{2} \left\{ 2k_{13} + \cdots + (m-1)(m-2)k_{1m}(s-p_1)^{m-3} + \frac{d^2}{ds^2}\left((s-p_1)^m [\cdots]\right) \right\} \Big|_{s=p_1}$$

$\vdots$

$$\boxed{k_{1n} = \frac{1}{(n-1)!} \left\{ \frac{d^{n-1}}{ds^{n-1}} (s-p_1)^m F(s) \right\} \Big|_{s=p_1}}$$

* How to find the inverse transform?

Remember : $\mathcal{L}\{t^n u(t)\} = \dfrac{n!}{s^{n+1}}$

$$\mathcal{L}\{t^n e^{pt} u(t)\} = \frac{n!}{(s-p)^{n+1}}$$

$$\mathcal{L}\{t^{n-1} e^{pt} u(t)\} = \frac{(n-1)!}{(s-p)^n}$$

$$\frac{1}{(n-1)!}\,\mathcal{L}\left\{t^{n-1}e^{pt}u(t)\right\} = \frac{1}{(s-p)^n}$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{k}{(s-p)^n}\right\} = \frac{k}{(n-1)!}\,e^{pt}\,t^{n-1}\,u(t)}$$

Then,

$$f(t) = \Bigg[\frac{k_{11}}{(m-1)!}e^{p_1 t}t^{m-1} + \frac{k_{12}}{(m-2)!}e^{p_1 t}\cdot t^{m-2} + \dots + k_{1m}e^{p_1 t} + \underbrace{\dots\dots}_{\text{from other poles}}\Bigg]\cdot u(t)$$

<u>Example</u> : $F(s) = \dfrac{4(s+3)}{s(s+2)^2} = \dfrac{k_{11}}{(s+2)^2} + \dfrac{k_{12}}{(s+2)} + \dfrac{k_2}{s}$

$s=0$ (simple pole) $\quad$ $s=-2$ (repeated pole)

$$k_{11} = (s+2)^2 \cdot F(s)\Big|_{s=-2} = \frac{4(s+3)}{s}\Big|_{s=-2} = \frac{4\cdot 1}{-2} = -2$$

$$k_{12} = \left\{\frac{d}{ds}(s+2)^2 F(s)\right\}\Big|_{s=-2} = \left\{\frac{d}{ds}\,\frac{4(s+3)}{s}\right\}\Big|_{s=-2}$$

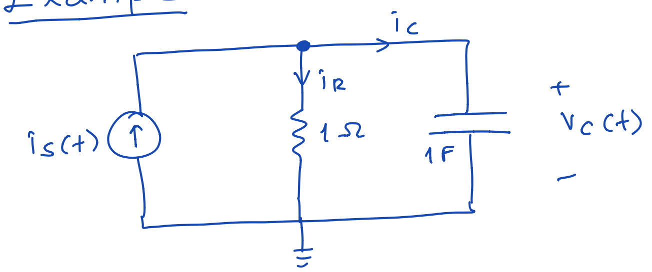
$$= \frac{4s - 4(s+3)}{s^2}\Big|_{s=-2} = \frac{-8-4}{4} = -3$$

$$k_2 = s\,F(s)\Big|_{s=0} = \frac{4(s+3)}{(s+2)^2}\Big|_{s=0} = \frac{4\cdot 3}{4} = 3$$

Then,

$$f(t) = \left[-2te^{-2t} - 3e^{-2t} + 3\right]\cdot u(t)$$

Example :



$v_c(0) = 1\,V$

$i_s(t) = \cos(t)$ A

Find $v_c(t)$ for $t \geq 0$

KCL : $i_s = i_R + i_c$

$$i_s = \frac{v_c}{1} + c\frac{dv_c}{dt}$$

Laplace transform

$$I_s(s) = V_c(s) + C\cdot\left[sV_c(s) - v_c(0^-)\right]$$

$$I_s(s) = \frac{s}{s^2+1}$$

So,

$$\frac{s}{s^2+1} = V_c(s) + sV_c(s) - 1$$

$$V_c(s)(s+1) = \frac{s}{s^2+1} + 1 = \frac{s^2+s+1}{s^2+1}$$

$$V_c(s) = \frac{s^2+s+1}{\underbrace{(s+1)}_{s=-1}\underbrace{(s^2+1)}_{s=\pm j}} = \frac{k_1}{s+1} + \frac{k}{s-j} + \frac{k^*}{s+j}$$

$s = -\alpha \pm j\beta$, where $\alpha = 0$, $\beta = 1$

$$k_1 = (s+1)V_c(s)\Big|_{s=-1} = \frac{s^2+s+1}{s^2+1}\Big|_{s=-1} = \frac{1-1+1}{1+1} = \frac{1}{2}$$

$$k = (s-j)V_c(s)\Big|_{s=j} = \frac{s^2+s+1}{(s+1)(s+j)}\Big|_{s=j} = \frac{-1+j+1}{(j+1)\cdot 2j} = \frac{\cancel{j}}{(j+1)\cdot 2\cancel{j}}$$

$$= \frac{1}{2(1+j)} = \frac{1}{2\cdot\sqrt{2}\cdot e^{j\pi/4}} = \frac{1}{2\sqrt{2}}e^{-j\pi/4} = |k|\cdot e^{j\theta}$$

* Then,

$$v_c(t) = \underbrace{\frac{1}{2}e^{-t}}_{\text{natural response}} + \underbrace{\overbrace{\frac{1}{\sqrt{2}}}^{2|k|}\cos\Big(t \overbrace{-\frac{\pi}{4}}^{\theta}\Big)}_{\text{forced response}}\ V, \quad \text{for } t \geq 0$$

(could be found using phasor analysis)