FALL 2016-2017

— MIDTERM #2 SOLUTIONS —

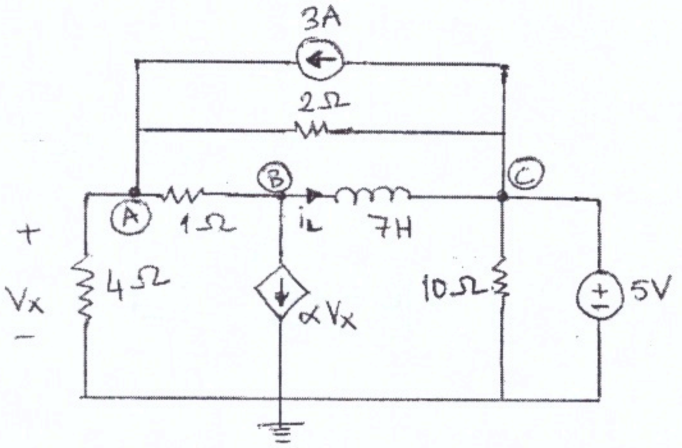
Question 1. [25 points]

Consider the following circuit. Let $i_L(0) = 3\text{A}$.

a) Find the value of α so that the time constant τ of the circuit is $\tau = 3$ *sec*. (If this is not possible, explain why.)

b) Let $\alpha = 0.5$. Find $i_L(t)$.

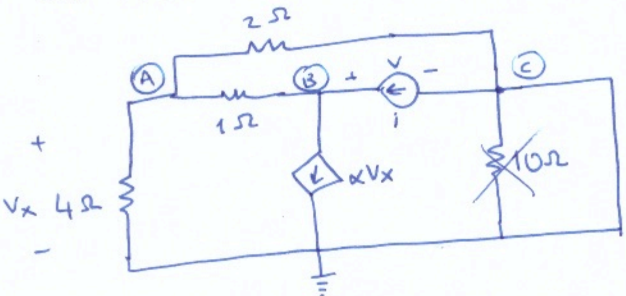
b) Again for $\alpha = 0.5$, find $v_x(t)$.



a) Find resistance seen by the inductor. Kill all independent sources.

$V_c = 0$, $V_x = V_A$

$V = V_B - V_C = V_B$



KCL at node A : $\frac{V_A}{4} + \frac{V_A - V_B}{1} + \frac{V_A}{2} = 0$ $\quad (4)\ (2)$

$7V_A = 4V_B \Rightarrow V_A = \frac{4}{7}V_B$

KCL at node B : $\alpha V_x + \frac{V_B - V_A}{1} - i = 0$

$\alpha \cdot \frac{4}{7}V_B + V_B - \frac{4}{7}V_B = i$

$(3 + 4\alpha)V_B = 7i$

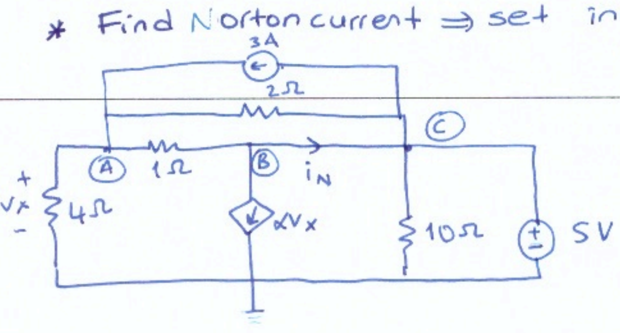
$R_{eq} = \frac{V}{i} = \frac{V_B}{i} = \boxed{\frac{7}{3+4\alpha}}$

* For $\tau = \frac{L}{R_{eq}} = \frac{7}{R_{eq}} = 3\,s \Rightarrow R_{eq} = \frac{7}{3}\,\Omega = \frac{7}{3+4\alpha} \Rightarrow \boxed{\alpha = 0}$

b) if $\alpha = 0.5 \Rightarrow R_{eq} = \frac{7}{3+2} = \frac{7}{5}\,\Omega \Rightarrow \tau = \frac{L}{R} = \frac{7}{7/5} = \boxed{5\text{ sec}}$

* Find Norton current $\Rightarrow$ set inductor to short circuit.

$V_B = V_C = 5V$, $V_x = V_A$



KCL at A : $\frac{V_A}{4} + \frac{V_A - 5}{1} + \frac{V_A - 5}{2} - 3 = 0$ $\quad (4)\ (2)\ (4)$

$7V_A - 20 - 10 - 12 = 0$

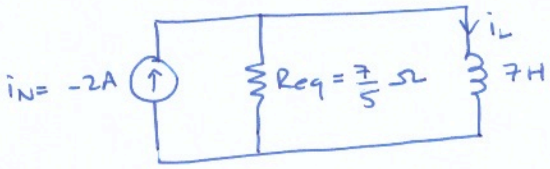
$7V_A = 42$

$\boxed{V_A = 6V}$

KCL at B: $i_N + \alpha V_x + \frac{V_B - V_A}{1} = 0$

$i_N + 0.5 \times 6 + 5 - 6 = 0 \Rightarrow \boxed{i_N = -2A}$

So, the circuit is simplified to:



$i_L(t) = i(\infty) + [i(0) - i(\infty)] e^{-t/\tau}$

$= -2 + (3 - (-2)) e^{-t/5}$ A

$\boxed{i_L(t) = -2 + 5e^{-t/5} \text{ A}}$

c) use KCL at B: $i_L + 0.5 V_x + \frac{V_B - V_x}{1} = 0$

$V_x = 2(i_L + V_B)$

$V_B = V_C + V_L = 5 + L\frac{di_L}{dt} = 5 + 7.5 \cdot \left(-\frac{1}{5}\right) \cdot e^{-t/5} = 5 - 7 \cdot e^{-t/5}$ V

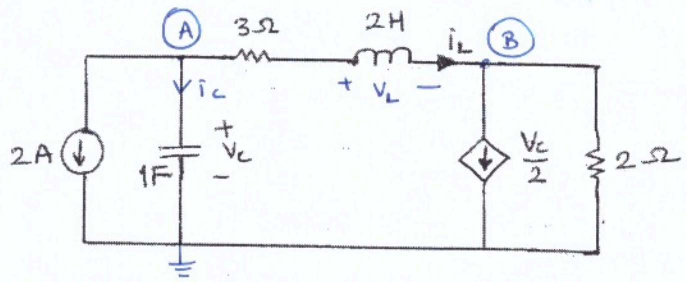
$V_x = 2\left[-2 + 5e^{-t/5} + 5 - 7 \cdot e^{-t/5}\right] = \boxed{6 - 4e^{-t/5} \text{ V}}$

Question 2. [25 points]

Consider the following circuit.

a) Find a second-order differential equation for i_L.

b) Find $i_L(t)$ for $t > 0$, if $i_L(0) = 0$ A and $v_C(0) = 1$ V.



a) KCL at A : $2 + i_c + i_L = 0$

$\boxed{i_c = -2 - i_L \quad = C\,\dot{V}_c = \dot{V}_c}$ (Eq. 1)

KCL at B : $-i_L + \frac{V_c}{2} + \frac{V_B}{2} = 0$

$-2i_L + V_c + V_B = 0$

* Also, $V_A - V_B = 3i_L + V_L$

$V_B = V_c - 3i_L - V_L$

$-2i_L + V_c + V_c - 3i_L - V_L = 0$

$2V_c - V_L - 5i_L = 0$

$\boxed{2V_c - 2\dot{i}_L - 5i_L = 0}$ (Eq. 2)

* Take derivative of Eq.2 :

$2\dot{V}_c - 2\ddot{i}_L - 5\dot{i}_L = 0 \xRightarrow{\text{use Eq. 1}} 2(-2-i_L) - 2\ddot{i}_L - 5\dot{i}_L = 0$

$\boxed{2\ddot{i}_L + 5\dot{i}_L + 2i_L = -4}$

b) * First, find the natural response : $2\ddot{i}_L + 5\dot{i}_L + 2i_L = 0$

Characteristic polynomial : $2s^2 + 5s + 2 = 0$

$(2s+1)(s+2) = 0$

$\boxed{s_1 = -\frac{1}{2}, \quad s_2 = -2}$

So, $\boxed{i_{L,N}(t) = K_1 . e^{-\frac{1}{2}t} + K_2 . e^{-2t}}$

* Next, find the forced response : Assume $i_{L,F} = A$ (a constant)

using $2\ddot{i}_L + 5\dot{i}_L + 2i_L = -4 \Rightarrow 2A = -4 \Rightarrow A = -2 \Rightarrow \boxed{i_{L,F} = -2A}$

* So, overall $\boxed{i_L(t) = -2 + K_1 e^{-\frac{1}{2}t} + K_2 e^{-2t}}$

$i_L(0) = 0 \Rightarrow -2 + K_1 + K_2 = 0 \Rightarrow K_1 + K_2 = 2$

From Eq.2 above, $2V_c(0) - 2\dot{i}_L(0) - 5i_L(0) = 0$

$2 - 2\dot{i}_L(0) = 0 \Rightarrow \dot{i}_L(0) = 1$

$\Rightarrow$

$$\dot{i}_L(t) = -\frac{1}{2}K_1 e^{-\frac{1}{2}t} - 2K_2 \cdot e^{-2t}$$

$$\dot{i}_L(0) = -\frac{1}{2}K_1 - 2K_2 = 1 \quad \Rightarrow \quad K_1 + 4K_2 = -2$$

$$\begin{array}{r} K_1 + 4K_2 = -2 \\ -\quad K_1 + K_2 = 2 \\ \hline 3K_2 = -4 \end{array}$$

$$K_2 = -\frac{4}{3}, \quad K_1 = 2 + \frac{4}{3} = \frac{10}{3}$$

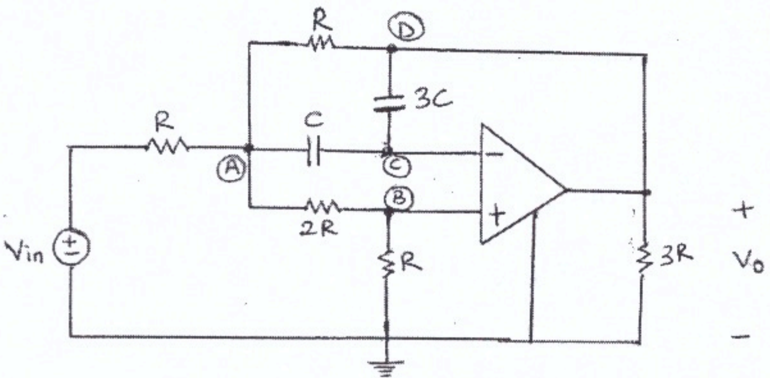
so, $\boxed{i_L(t) = -2 + \frac{10}{3}e^{-\frac{1}{2}t} - \frac{4}{3}e^{-2t} \quad A}$ for $t > 0$

Question 3. [25 points]

Consider the circuit below. Assume that the OPAMP is ideal and operating in linear mode. Also assume that the circuit is in sinusoidal steady state.

a) Find the transfer function $H(jw) = \frac{V_o}{V_{in}}$.

b) Assume that $R = 10\ k\Omega$, $C = 100\ \mu F$, and $v_i(t) = 2\cos(t - \frac{\pi}{6}) + 5\cos(\frac{10}{3}t)$ V. Find $v_o(t)$.



a) $V_B = V_C = V_A \cdot \frac{R}{2R+R} = \frac{V_A}{3}$, $V_D = V_o$

KCL at A : $\frac{V_A - V_{in}}{R} + \frac{V_A}{3R} + \frac{V_A - V_A/3}{\frac{1}{jwC}} + \frac{V_A - V_o}{R} = 0$

(3) (3R) (3)

$$7V_A - 3V_{in} + 2jwRC\, V_A - 3V_o = 0$$

$$(7 + 2jwRC)\, V_A = 3V_{in} + 3V_o \quad \text{--- (Eq.1)}$$

KCL at C : $\frac{V_A/3 - V_A}{\frac{1}{jwC}} + \frac{V_A/3 - V_o}{\frac{1}{3jwC}} = 0$

$$\frac{V_A}{3} - \cancel{V_A} + \cancel{V_A} - 3V_o = 0$$

$$V_A = 9V_o \quad \text{--- (Eq.2)}$$

Insert Eq.2 into Eq.1 :

$$(7 + 2jwRC)\, 9V_o = 3V_{in} + 3V_o$$

$$(60 + 18jwRC)\, V_o = 3V_{in} \quad \Rightarrow \boxed{H(jw) = \frac{V_o}{V_{in}} = \frac{1}{20 + 6jwRC}}$$

b) The input has two different frequencies. Apply superposition in time-domain.

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* For $V_{in,1}(t) = 2\cos(t - \frac{\pi}{6}) \Rightarrow w_1 = 1$, $V_{in,1} = 2 \cdot e^{-j\pi/6}$

$$H(jw_1) = \frac{1}{20 + 6j\, 10 \cdot 10^3 \cdot 100 \cdot 10^{-6}} = \frac{1}{20 + 6j} = \frac{1}{20.88 \cdot e^{j0.29}}$$

$$V_{out,1} = H(jw_1) \cdot V_{in,1} = \frac{2 \cdot e^{-j\pi/6}}{20.88 \cdot e^{j0.26}} = 0.096 \cdot e^{-j0.81}$$

$$\boxed{\begin{aligned} v_{out,1}(t) &= 0.096 \cdot \cos(t - 0.81)\ V \\ \text{OR,} \\ V_{out,1}(t) &= 0.096 \cdot \cos(t - 46.7^\circ)\ V \end{aligned}}$$

* For $v_{in,2}(t) = 5 \cdot \cos\left(\frac{10}{3} t\right)\ V \Rightarrow \omega_2 = \frac{10}{3}$, $V_{in,2} = 5$

$$H(j\omega_2) = \frac{1}{20 + 6j \cdot \frac{10}{3} \cdot 10 \cdot 10^3 \cdot 100 \cdot 10^{-6}} = \frac{1}{20 + j20} = \frac{1}{20\sqrt{2} \cdot e^{j\pi/4}}$$

$$V_{out,2} = H(j\omega_2) \cdot V_{in,2} = \frac{5}{20\sqrt{2} \cdot e^{j\pi/4}} = \frac{e^{-j\pi/4}}{4\sqrt{2}} = 0.18 \cdot e^{-j\pi/4}$$

$$\boxed{\begin{aligned} v_{out,2}(t) &= 0.18 \cdot \cos\left(\frac{10}{3} t - \frac{\pi}{4}\right)\ V \\ \text{OR,} \\ V_{out,2}(t) &= 0.18 \cdot \cos\left(\frac{10}{3} t - 45^\circ\right)\ V \end{aligned}}$$

* Finally,

$$V_{out}(t) = V_{out,1}(t) + V_{out,2}(t) = \boxed{0.096 \cdot \cos(t - 0.81) + 0.18 \cdot \cos\left(\frac{10}{3} t - 45^\circ\right)\ V}$$

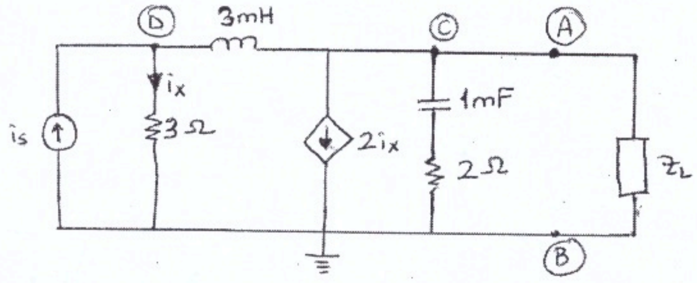
Question 4. [25 points]

Assume that the circuit below is in sinusoidal steady state. Here, Z_L represents an arbitrary load, and $i_s(t) = 2\ cos(1000t)$ A.

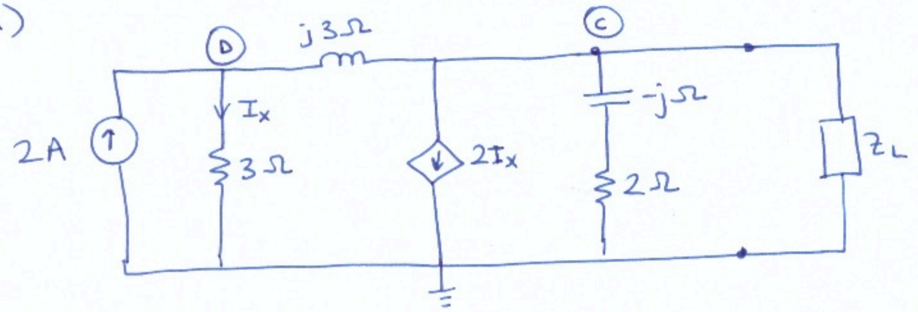
a) Draw the circuit in sinuoidal steady state by using the phasor values of $i_s(t)$ and indicating the impedance values of all circuit elements.

b) Assume that Z_L is replaced by a short circuit. Find the short circuit current phasor I_N between nodes A - B and evaluate $i_N(t)$.

c) Find the impedance value Z_L that maximizes the power transferred to the load. Find the average power dissipated in the load for that Z_L.



a)

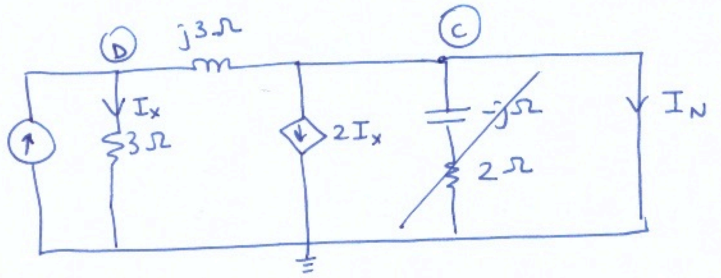


$\omega = 1000$ rps

$j\omega L = j10^3 . 3.10^{-3} = j3\Omega$

$\frac{1}{j\omega C} = \frac{1}{j10^3.10^{-3}} = -j\Omega$

b) .



$V_c = 0$

$I_x = \frac{V_D}{3}$

KCL at node D : $\frac{V_D}{3} + \frac{V_D}{j3} - 2 = 0$

$(-j) \quad (3)$

$\Rightarrow V_D(1-j) = 6$

$V_D = \frac{6\ (1+j)}{1-j\ (1+j)} = 3 + 3j$

KCL at node C : $\frac{-V_D}{j3} + 2I_x + I_N = 0$

$\frac{-V_D}{j3} + \frac{2V_D}{3} + I_N = 0$

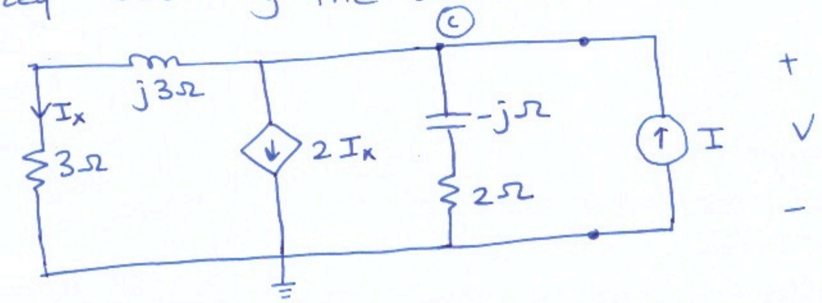
$(-j) \quad (3)$

$I_N = -\frac{V_D(2+j)}{3} = \frac{-(3+3j)(2+j)}{3} = -(1+j)(2+j)$

$I_N = -2 - 3j + 1 = -1 - 3j = \boxed{3.16.e^{-j1.89}\ A}$ $\quad -108.4^\circ$

$\boxed{i_N(t) = 3.16 . \cos(1000t - 1.89)\ A}$

c) Find Z_{eq} seen by the load. Kill independent sources.



$I_x = \dfrac{V_c}{3+3j}$, $V_c = V$, $Z_{eq} = \dfrac{V}{I}$

KCL at C: $\dfrac{V}{3+3j} + 2 \cdot I_x + \dfrac{V}{2-j} - I = 0$

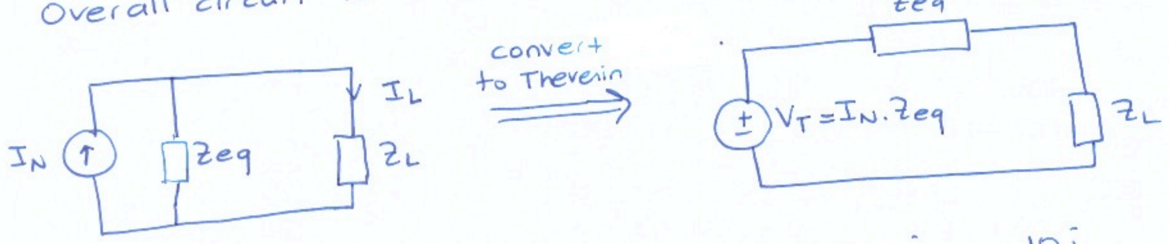
$$\frac{V}{3+3j} + \frac{2V}{3+3j} + \frac{V}{2-j} - I = 0$$

$$\underset{(2-j)}{\frac{V}{1+j}} + \underset{(1+j)}{\frac{V}{2-j}} = I$$

$$V \cdot \frac{3}{3+j} = I \Rightarrow Z_{eq} = \frac{V}{I} = \frac{3+j}{3} = \boxed{1 + \frac{j}{3}\ \Omega}$$

For maximum power transfer: $\boxed{Z_L = Z_{eq}^* = 1 - \frac{j}{3}\ \Omega}$

Overall circuit is



$$V_T = I_N \cdot Z_{eq} = (-1-3j)\left(1+\frac{j}{3}\right) = -1 + 1 - 3j - \frac{j}{3} = \frac{-10j}{3}$$

Then,

$$P_L = \frac{|V_T|^2}{8R_L} = \frac{\left(\frac{10}{3}\right)^2}{8.1} = \boxed{1.39\ W} = \frac{25}{18}\ W$$