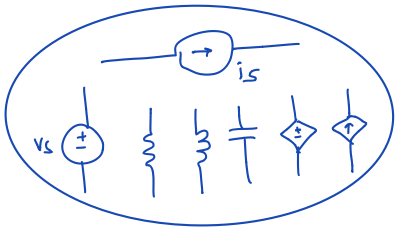
CHAPTER 8 - SINUSOIDAL STEADY STATE



All independent sources are sinusoidal.

$v_s(t) = V_A \cdot \cos(\omega t + \phi)$

* Any variable (voltage or current) in the circuit satisfies an ODE of the form:

$$\underbrace{y^{(n)}}_{n^{th}\text{ order derivative}} + a_{n-1} y^{(n-1)} + \dots + a_1 y^{(1)} + a_0 y = u$$

where u is sinusoidal and n is the order of the circuit.

* Solution : $y(t) = \underbrace{y_N(t)}_{\text{natural response}} + \underbrace{y_F(t)}_{\text{forced response is also sinusoidal}}$

* Natural Response : Assume $y_N(t) = e^{st}$

Insert into ODE : $s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 = 0$

characteristic polynomial with n roots : $s_1, \dots, s_n$

$y_N(t) = C_1 e^{s_1 t} + \dots + C_n e^{s_n t}$

OR , $y_N(t) = C_1 e^{s_1 t} + C_2 t e^{s_1 t} + \dots$ (if repeated roots exist)

Assume that $\text{Re}\{s_i\} < 0 \Rightarrow$ stable solution

* So, $\lim_{t \to \infty} y_N(t) = 0$ for stable circuits $\Rightarrow$ natural response decays down to zero.

i.e., $\lim_{t \to \infty} y(t) = y_F(t) \Rightarrow$ only forced response remains, which is sinusoidal.

This is called the sinusoidal steady state solution.

We will try to find this forced response when all sources are sinusoidal with the same frequency ω.

Phasor Representation:

We express sinusoids in cosine basis:

$$A_m \cdot \cos(\omega t + \phi)$$

where $A_m > 0$ by convention

e.g., $-A_m \cos(\omega t + \phi) = A_m \cdot \cos(\omega t + \phi + \pi)$

$A_m \cdot \sin(\omega t + \phi) = A_m \cdot \cos(\omega t + \phi - \pi/2)$

* $$A_m \cdot \cos(\omega t + \phi) = Re\{A_m \cdot e^{j(\omega t + \phi)}\} = Re\{\underbrace{A_m \cdot e^{j\phi}}_{A} \cdot e^{j\omega t}\}$$

$$A = \underbrace{A_m \cdot e^{j\phi}}_{\text{polar representation}} = \underbrace{A_m \cdot \cos\phi + j A_m \cdot \sin\phi}_{\text{Cartesian representation}}$$: phasor of sinusoid

$$\underbrace{A_m \cdot \cos(\omega t + \phi)}_{\substack{\text{signal waveform} \\ \text{in time domain}}} = Re\{\underbrace{A}_{\substack{\text{phasor} \\ \text{in phasor domain}}} \cdot e^{j\omega t}\}$$

fixed ω

* ω should be a given, known frequency.

Phasor: $A = \underbrace{A_m}_{\text{magnitude}} \cdot e^{j\phi} = A_m \angle \phi$

phase

Properties of Phasors:

$x(t) = X_m \cdot \cos(\omega t + \phi_x) \longleftrightarrow \mathbb{X} = X_m \cdot e^{j\phi_x}$

$y(t) = Y_m \cdot \cos(\omega t + \phi_y) \longleftrightarrow \mathbb{Y} = Y_m \cdot e^{j\phi_y}$

Uniqueness: iff $x(t) = y(t) \Longleftrightarrow \mathbb{X} = \mathbb{Y}$ for fixed ω

Linearity: $\alpha x(t) + \beta y(t) \Longleftrightarrow \alpha \mathbb{X} + \beta \mathbb{Y}$, $\alpha, \beta \in \mathbb{R}$

$$\alpha x(t) + \beta \cdot y(t) = \alpha \cdot Re\{\mathbb{X} \cdot e^{j\omega t}\} + \beta \cdot Re\{\mathbb{Y} \cdot e^{j\omega t}\}$$

$$= Re\{\alpha \mathbb{X} e^{j\omega t} + \beta \cdot \mathbb{Y} e^{j\omega t}\}$$

$$= Re\{\underbrace{(\alpha \mathbb{X} + \beta \mathbb{Y})}_{\text{phasor}} e^{j\omega t}\}$$ for fixed ω.

Differentiation Rule :

$$x(t) = X_m \cdot \cos(\omega t + \phi) \iff \mathbf{X} = X_m e^{j\phi}$$

$$\frac{d}{dt} x(t) = -\omega X_m \cdot \sin(\omega t + \phi)$$

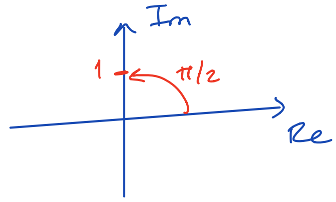
$$= \omega X_m \cdot \sin(\omega t + \phi + \pi)$$

$$= \omega X_m \cdot \cos(\omega t + \phi + \pi - \pi/2)$$

$$= \omega X_m \cdot \cos(\omega t + \phi + \pi/2)$$

$$= \mathrm{Re}\left\{ \underbrace{\omega X_m e^{j(\phi + \pi/2)}}_{\text{phasor of } \dot{x}(t)} \cdot e^{j\omega t} \right\}$$

$$\text{Phasor of } \dot{x}(t) = \omega X_m e^{j\phi} \cdot \underbrace{e^{j\pi/2}}_{j}$$



$$= j\omega \underbrace{X_m \cdot e^{j\phi}}_{\mathbf{X}} = j\omega \mathbf{X}$$

$$\boxed{\begin{aligned} x(t) &\iff \mathbf{X} \\ \frac{dx(t)}{dt} &\iff j\omega \mathbf{X} \end{aligned}}$$

Example : $x(t) = 3\cos(t) - 4\cos(t + \pi/4) + 2\sin(t + \pi/6)$ } same ω, $\omega = 1$ rps

write as $x(t) = A_m \cdot \cos(\omega t + \phi)$

$$x(t) = 3\cos(t) + 4\cos(t + \pi/4 + \pi) + 2\cos(t + \pi/6 - \pi/2)$$

$$\mathbf{X} = 3 \cdot e^{j0} + 4 \cdot e^{j(\pi/4 + \pi)} + 2 \cdot e^{j(\pi/6 - \pi/2)}$$

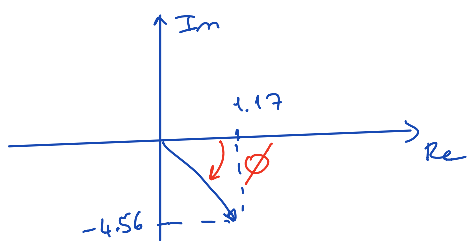
$$= 3 + 4e^{j5\pi/4} + 2e^{-j\pi/3}$$

using scientific calculator

$$= 1.17 - j4.56$$

$$= 4.71 \cdot e^{-j75.61^\circ}$$

$$= 4.71 e^{-j1.32}$$



$$x(t) = 4.71 \cos(t - 1.32)$$

Example : $\ddot{x} + 3\dot{x} + 2x = \sin(3t)$ $\quad \omega = 3 \text{ rps}$

$x(0) = 1, \quad \dot{x}(0) = 1$. Find $x(t)$ for $t \geq 0$.

$$x(t) = x_N(t) + x_F(t)$$

* Natural response : $\ddot{x} + 3\dot{x} + 2x = 0$

$$s^2 + 3s + 2 = 0$$

$$(s+1)(s+2) = 0 \Rightarrow s_1 = -1, \quad s_2 = -2$$

$$x_N(t) = c_1 e^{-t} + c_2 e^{-2t}$$

* Forced response : $x_F(t) = Re\left\{ X e^{j3t} \right\}$

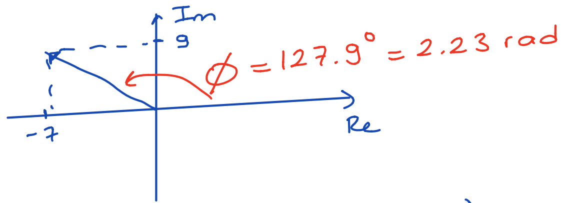
$$\ddot{x} + 3\dot{x} + 2x = \cos(3t - \pi/2)$$

convert to phasor domain

$$(j3)^2 X + 3(j3)X + 2X = e^{-j\pi/2}$$

$$(-9 + 9j + 2)X = e^{-j\pi/2}$$

$$X = \frac{e^{-j\pi/2}}{-7 + 9j} = \frac{e^{-j\pi/2}}{\sqrt{7^2 + 9^2} \angle 2.23} = \frac{1}{\sqrt{130}} \angle(-\pi/2 - 2.23)$$



$$x_F(t) = \frac{1}{\sqrt{130}} \cos(3t - \pi/2 - 2.23)$$

* Then,

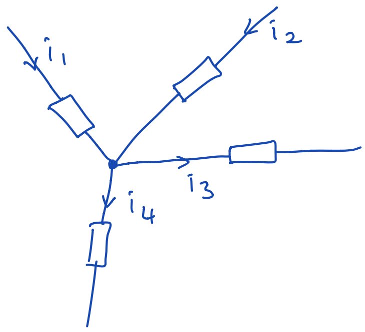
$$x(t) = x_N(t) + x_F(t)$$

$$= c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{\sqrt{130}} \cos(3t - \pi/2 - 2.23)$$

* Now, use initial conditions to find c_1 and c_2.

Circuit Equations in Phasor Form

KCL :



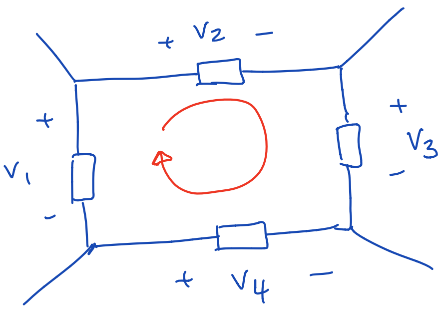
$$i_n(t) = Re\{\underbrace{I_n}_{\text{phasor}} . e^{j\omega t}\}$$

typical KCL : $-i_1(t) - i_2(t) + i_3(t) + i_4(t) = 0$

uniqueness & linearity of phasors

$$-I_1 - I_2 + I_3 + I_4 = 0$$

KVL :



$$V_n(t) = Re\{\underbrace{V_n}_{\text{phasor}} . e^{j\omega t}\}$$

typical KVL : $-V_1(t) + V_2(t) + V_3(t) - V_4(t) = 0$

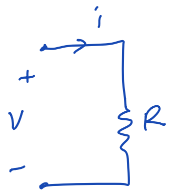
uniqueness & linearity

$$-V_1 + V_2 + V_3 - V_4 = 0$$

Result : KCL and KVL retain their form in phasor domain.

Element Relations :

Resistors :



in time-domain : $v(t) = R . i(t)$

$i(t) = G . v(t)$

in phasor-domain : $V = R . I$

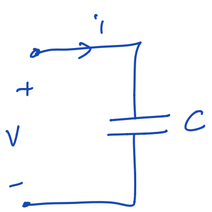
$I = G . V$

$$Z_R = \frac{V}{I} = R \quad , \quad Y_R = \frac{I}{V} = G = \frac{1}{R}$$

impedance of resistor

admittance of resistor

Capacitor :



in time-domain : $i(t) = C \frac{dV(t)}{dt}$

in phasor-domain : $I = C . j\omega V$

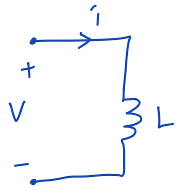
$I = j\omega C . V$

$$Z_C = \frac{V}{I} = \frac{1}{j\omega C} \quad , \quad Y_C = \frac{I}{V} = j\omega C$$

impedance of capacitor

admittance of capacitor

Inductor:



in time-domain: $v(t) = L \dfrac{di(t)}{dt}$

in phasor-domain: $\mathbb{V} = L \cdot j\omega \mathbb{I}$

$$\mathbb{V} = j\omega L \cdot \mathbb{I}$$

$$Z_L = \frac{\mathbb{V}}{\mathbb{I}} = j\omega L \quad , \quad Y_L = \frac{\mathbb{I}}{\mathbb{V}} = \frac{1}{j\omega L}$$

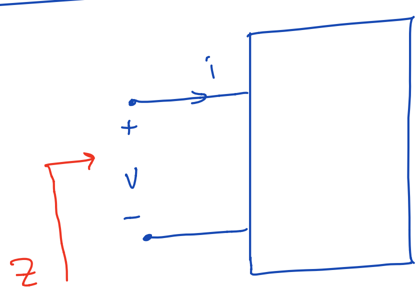
impedance of inductor

admittance of inductor

Result: The whole circuit equations, KCL + KVL + branch equations, are now linear, algebraic, but with complex coefficients.

Result: All of our analysis techniques: node, mesh, Thévenin, Norton, etc. can still be applied in phasor domain. The only difference is now we have to deal with complex coefficients.

Concept of Impedance and Admittance



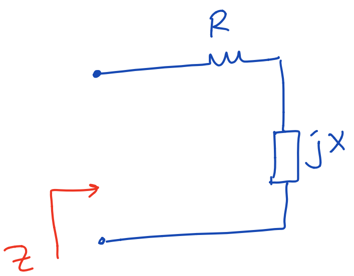
Impedance:

$$Z = \frac{\mathbb{V}}{\mathbb{I}} = \frac{|\mathbb{V}| \cdot e^{j\angle \mathbb{V}}}{|\mathbb{I}|\, e^{j\angle \mathbb{I}}} = \frac{|\mathbb{V}|}{|\mathbb{I}|} \cdot e^{j(\angle \mathbb{V} - \angle \mathbb{I})}$$

$$= |Z| \cdot e^{j\angle Z}$$

phase difference between voltage and current phasors

magnitude ratio between voltage and current phasors



$$Z = R + jX \quad [\text{in } \Omega]$$

resistance

reactance

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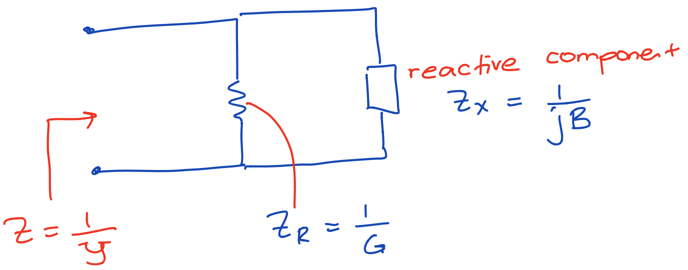
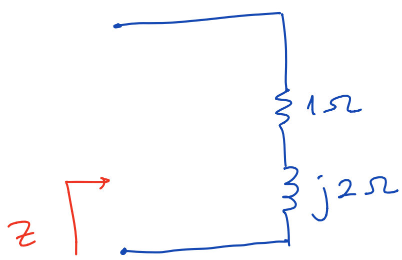
Admittance : $Y = \frac{I}{V} = \frac{1}{Z} = G + jB$ [in S]

conductance susceptance



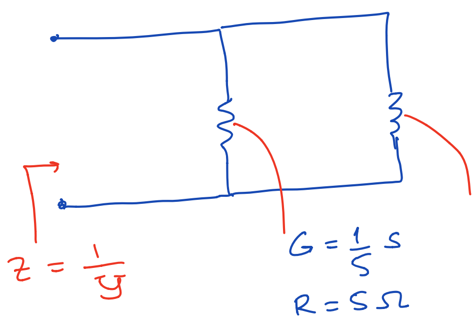
Example :

$Z = 1 + j2\ \Omega$



$Y = \frac{1}{Z} = \frac{1}{1+j2}\frac{(1-j2)}{(1-j2)} = \frac{1-j2}{5}$

$= \frac{1}{5} - j\frac{2}{5}\ S = G + jB$

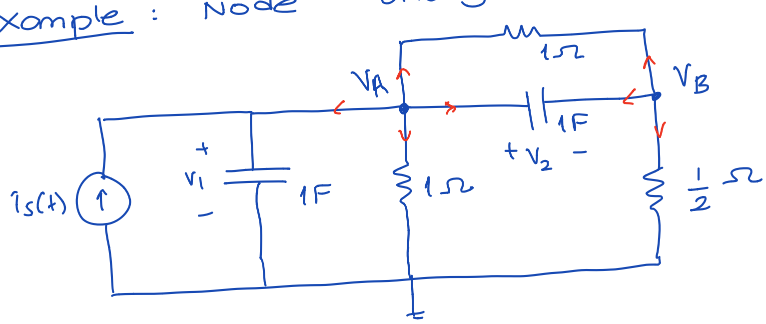


$Y_B = -j\frac{2}{5}$ S

$Z_B = \frac{1}{Y_B} = \frac{5}{-2j}\frac{(j)}{(j)} = j\frac{5}{2}\ \Omega$

* Series/parallel equivalences are the same as in resistive circuits.
* Voltage/current division work the same as in resistive circuits.

Example : Node analysis



$i_s(t) = 2\cos(t - 30°)$ A

$v_1(t) = ?$

$v_2(t) = ?$

$i_s(t) = 2\cos(t - \pi/6)$ A $\Longrightarrow$ $I_s = 2e^{-j\pi/6}$ A

$\omega = 1$ rps

KCL at A : $-I_s + \dfrac{V_A}{\frac{1}{j.1.1}} + \dfrac{V_A}{1} + \dfrac{V_A - V_B}{1} + \dfrac{V_A - V_B}{\frac{1}{j.1.1}} = 0$

$$(2+2j)V_A - (1+j)V_B = I_s$$

KCL at B : $\dfrac{V_B - V_A}{1} + \dfrac{V_B - V_A}{\frac{1}{j.1.1}} + \dfrac{V_B}{\frac{1}{2}} = 0$

$$-(1+j)V_A + (3+j)V_B = 0 \Rightarrow V_B = \frac{1+j}{3+j} V_A$$

Insert into the first equation:

$$\left[2+2j - (1+j)\frac{(1+j)}{3+j}\right] V_A = I_s$$

$(3+j)$

$$\left[\frac{4+8j - 2j}{3+j}\right] . V_A = I_s$$

$$V_A = \frac{3+j}{4+6j} I_s$$

$$V_B = \frac{1+j}{3+j} . V_A = \frac{1+j}{\cancel{3+j}} . \frac{\cancel{3+j}}{4+6j} . I_s = \frac{1+j}{4+6j} I_s$$

* Then, $V_1 = V_A = \dfrac{3+j}{4+6j} . I_s = \dfrac{3+j}{4+6j} 2.e^{-j\pi/6}$

$$= \frac{\sqrt{10} . e^{j\operatorname{atan}(1/3)}}{\sqrt{52} . e^{j\operatorname{atan}(6/4)}} . 2 . e^{-j\pi/6} = 0.87 . e^{-j1.17}$$

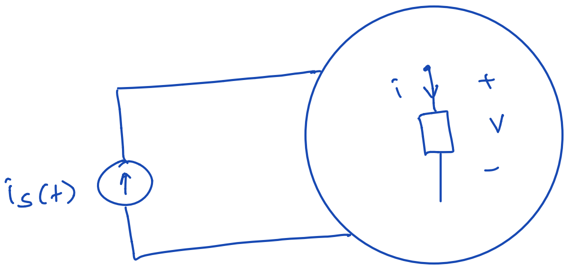
$$\boxed{v_1(t) = 0.87 \cos(t - 1.17)\ V}$$

* Then, $V_2 = V_A - V_B = \dfrac{3+j}{4+6j} I_s - \dfrac{1+j}{4+6j} I_s = \dfrac{2}{4+6j} I_s$

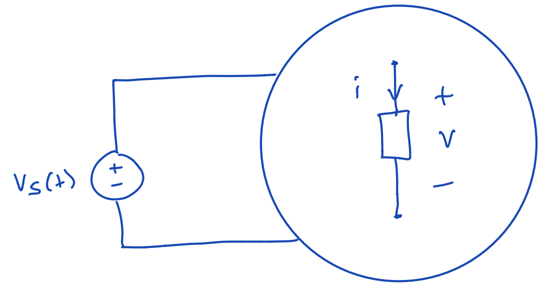
$$= \frac{2}{\sqrt{52} . e^{j\operatorname{atan}(6/4)}} . 2.e^{-j\pi/6} = 0.55\, e^{-j1.5}$$

$$\boxed{v_2(t) = 0.55 \cos(t - 1.5)\ V}$$

Concept of Network Function



OR



$$i_s(t) = \mathrm{Re}\{ I_s . e^{j\omega t} \}$$
$$i(t) = \mathrm{Re}\{ I . e^{j\omega t} \}$$

All sinusoidal

$$V_s(t) = \mathrm{Re}\{ V_s . e^{j\omega t} \}$$
$$v(t) = \mathrm{Re}\{ V . e^{j\omega t} \}$$

$$H = \frac{\text{output phasor}}{\text{input phasor}} \quad : \text{transfer function}$$

e.g., $H_1 = \dfrac{V}{I_s}$, $H_2 = \dfrac{I}{I_s}$, $H_3 = \dfrac{V}{V_s}$, ...

Importance : (output phasor) = H × (input phasor)

e.g., $V = H_1 . I_s$, let $I_s = I_{sm}\, e^{j\phi_s}$

$H_1 = |H_1| . e^{j\angle H_1}$

$$V = |H_1| . I_{sm} . e^{j(\angle H_1 + \phi_s)}$$

$$v(t) = \mathrm{Re}\{ V . e^{j\omega t} \} = \underbrace{|H_1| . I_{sm}}\, \cos(\omega t + \underbrace{\angle H_1 + \phi_s})$$

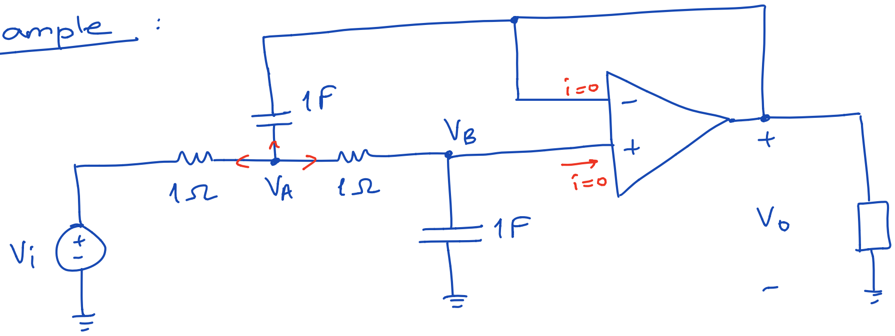
* output magnitude = (transfer fn. magnitude) × (input magnitude)
* output phase = (transfer fn. phase) + (input phase)

$$H(j\omega) = \frac{\text{polynomial in } (j\omega)}{\text{polynomial in } (j\omega)}$$

$$= \frac{a_m (j\omega)^m + a_{m-1} (j\omega)^{m-1} + \cdots + a_1 (j\omega) + a_0}{(j\omega)^n + b_{n-1} (j\omega)^{n-1} + \cdots + b_1 (j\omega) + b_0}$$

$$H(j\omega) = \underbrace{|H(j\omega)| . e^{j\angle H(j\omega)}}_{\substack{\text{both magnitude and phase} \\ \text{are functions of } \omega}}$$

Example :



Find $H(j\omega) = \dfrac{V_o}{V_i}$

Assume OPAMP is in linear region. $\Rightarrow$ $V_+ = V_-$

$V_B = V_o$

KCL at A : $\dfrac{V_A - V_i}{1} + \dfrac{V_A - V_B}{1} + \dfrac{V_A - V_o}{\frac{1}{j\omega \cdot 1}} = 0$

$$(2+j\omega) V_A - (1+j\omega) V_o = V_i$$

KCL at B : $V_B = \dfrac{\frac{1}{j\omega \cdot 1}}{1 + \frac{1}{j\omega \cdot 1}} \cdot V_A$ (voltage divider)

$$V_o = \frac{1}{1+j\omega} \cdot V_A \quad \Rightarrow \quad V_A = (1+j\omega) \cdot V_o$$

Insert into the first equation :

$$(2+j\omega)(1+j\omega) V_o - (1+j\omega) V_o = V_i$$

$$\left[(j\omega)^2 + 3(j\omega) + 2 - 1 - (j\omega) \right] V_o = V_i$$

$$\left[(j\omega)^2 + 2(j\omega) + 1 \right] V_o = V_i$$

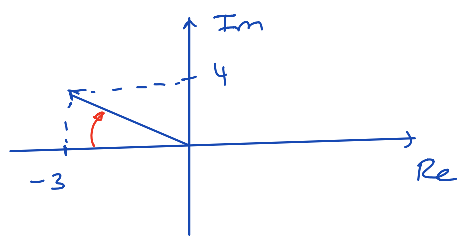
$$H(j\omega) = \frac{V_o}{V_i} = \frac{1}{(j\omega)^2 + 2(j\omega) + 1} = \frac{1}{-\omega^2 + 2j\omega + 1}$$

* Then, if $v_i(t) = 3\cos(2t)$ V , $\omega = 2$ rps

$$H(j2) = \frac{1}{-4 + j4 + 1} = \frac{1}{-3 + 4j} = \frac{1}{5 \cdot e^{j\,\mathrm{atan}\left(\frac{4}{-3}\right)}}$$

$$= \frac{1}{5 e^{j(\pi - \mathrm{atan}(\frac{4}{3}))}}$$

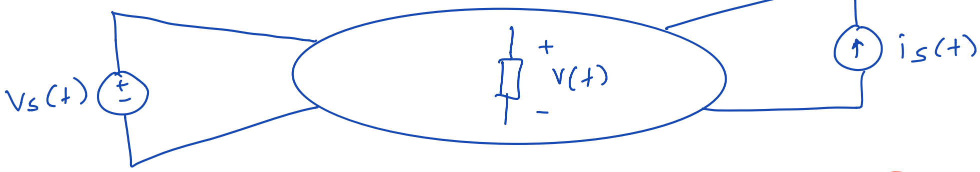
$$= \frac{1}{5} e^{-j2.21}$$



$$v_o(t) = 3 \cdot |H(j2)| \cdot \cos(2t + \angle H(j2)) \text{ V}$$
$$= \frac{3}{5} \cdot \cos(2t - 2.21) \text{ V}$$

Superposition:

Always applies in time domain.



$$\left.\begin{aligned} v_s(t) &= V_m \cdot \cos(\omega_1 t + \phi_1) \\ i_s(t) &= I_m \cdot \cos(\omega_2 t + \phi_2) \end{aligned}\right\} \quad \omega_1 \neq \omega_2$$

* $v_s(t)$ acting alone $\Rightarrow$ use phasor analysis with frequency ω_1
  $\Rightarrow$ Find $\bar{V}_1$ $\Rightarrow$ $v_1(t) = \text{Re}\{\bar{V}_1 \cdot e^{j\omega_1 t}\}$
* $i_s(t)$ acting alone $\Rightarrow$ use phasor analysis with frequency ω_2
  $\Rightarrow$ Find $\bar{V}_2$ $\Rightarrow$ $v_2(t) = \text{Re}\{\bar{V}_2 \cdot e^{j\omega_2 t}\}$

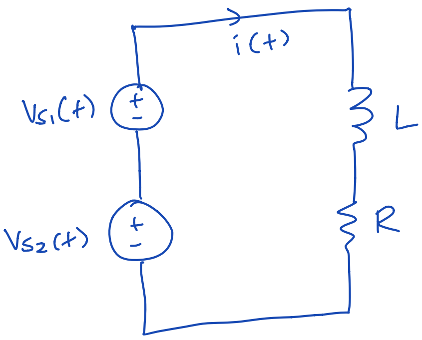
Superposition: $\boxed{v(t) = v_1(t) + v_2(t)}$ Added in time domain

*Note: If $\omega_1 = \omega_2$, then (and only then) we may also use superposition in phasor domain.

$$v(t) = v_1(t) + v_2(t) = \text{Re}\{\bar{V}_1 e^{j\omega_1 t}\} + \text{Re}\{\bar{V}_2 \cdot e^{j\omega_2 t}\}$$
$$= \text{Re}\{(\bar{V}_1 + \bar{V}_2) \cdot e^{j\omega t}\}, \quad \text{iff} \quad \omega = \omega_1 = \omega_2$$

Iff $\omega_1 = \omega_2$, we may first add $\bar{V}_1 + \bar{V}_2$ in phasor domain, then find $v(t)$ in time domain.

Simple Example:



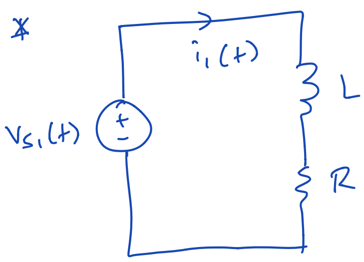
$R = 10\,k\Omega$, $\quad L = 200$ mH

$V_{s1}(t) = 24\cos(20000\,t)$ V

$V_{s2}(t) = 8\cos(60000\,t + 30^\circ)$ V

$i(t) = ?$

*



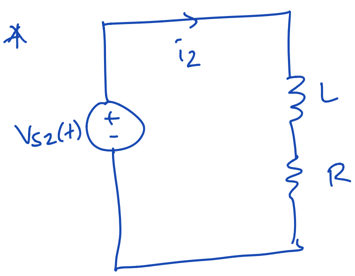
$Z_R = 10\,k\Omega$, $\quad \omega_1 = 20000$ rps

$Z_L = j\omega_1 L = j\,20\cdot10^3\cdot 200\times10^{-3} = j\,4\,k\Omega$

$$I_1 = \frac{V_{s1}}{Z_R + Z_L} = \frac{24\cdot e^{j0}}{(10 + j4)\,k\Omega} = 2.23\,e^{-j0.38}\ \text{mA}$$

$i_1(t) = 2.23\cos(20000\,t - 0.38)$ mA

*



$Z_R = 10\,k\Omega$, $\quad \omega_2 = 60000$ rps

$Z_L = j\omega_2 L = j\,60\cdot10^3\cdot 200\cdot10^{-3} = j\,12\,k\Omega$

$$I_2 = \frac{V_{s2}}{Z_R + Z_L} = \frac{8\cdot e^{j\pi/6}\ V}{(10 + j12)\,k\Omega} = 0.51\,e^{-j0.35}\ \text{mA}$$

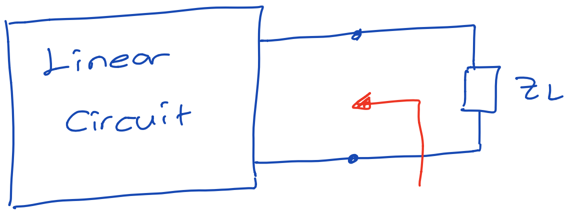
$i_2(t) = 0.51\cos(60000\,t - 0.35)$ mA

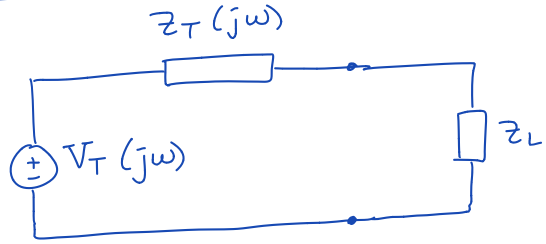
* Finally,

$i(t) = i_1(t) + i_2(t)$

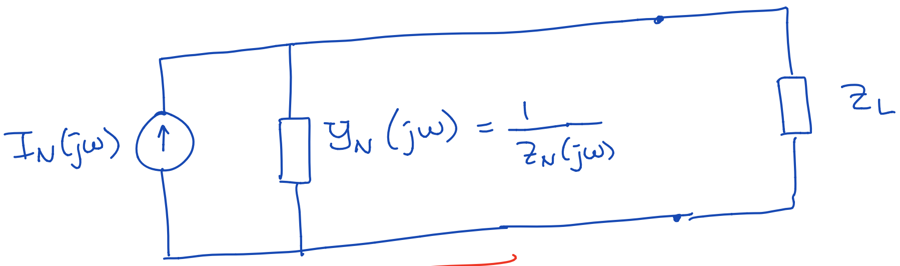
$= 2.23\cos(20000t - 0.38) + 0.51\cos(60000t - 0.35)$ mA

Thevenin – Norton Equivalent Circuits





Thevenin equivalent

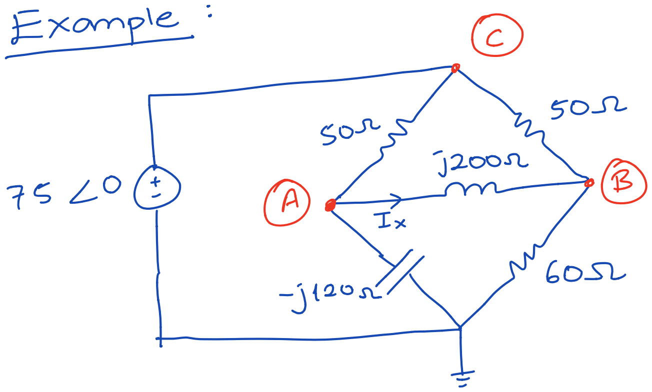


Norton equivalent

$$\boxed{I_N = \frac{V_T}{Z_T}} \quad , \quad \boxed{Z_T = Z_N} = \frac{1}{Y_N}$$

* Finding the equivalent circuit is the same as in resistive circuits, but with complex values in phasor domain.

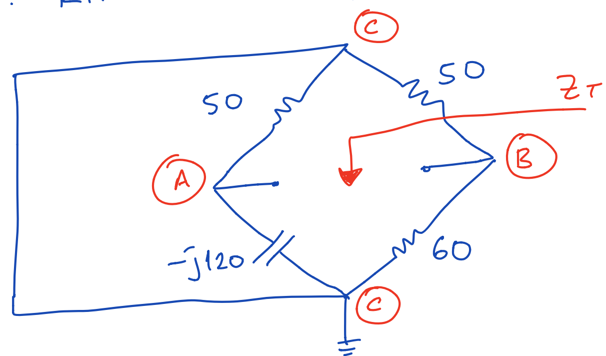
Example:

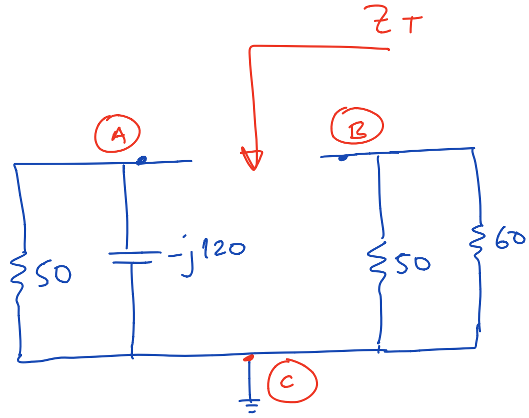


Find I_x in phasor domain by using Thevenin equivalent circuit.

So, first find Thevenin eq. seen by $j200\Omega$ inductor between A–B.

$\underline{Z_T}$: kill all independent sources

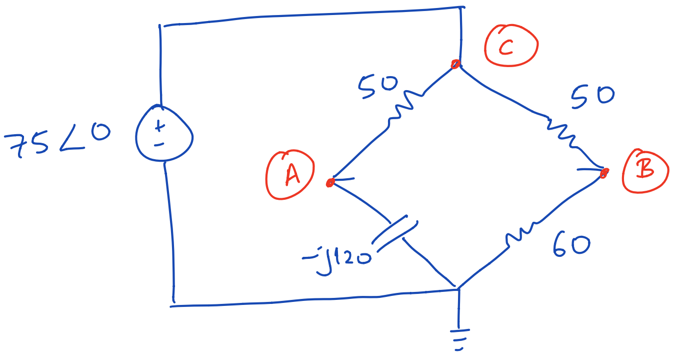




$$Z_T = [50 // (-j120)] + [50 // 60]$$

$$= \frac{50 \cdot (-j120)}{50 - j120} + \frac{50 \cdot 60}{50 + 60} = 69.9 - j17.8 \Omega$$

$\underline{V_T}$: open circuit voltage between A-B.



$$V_T = V_A - V_B$$

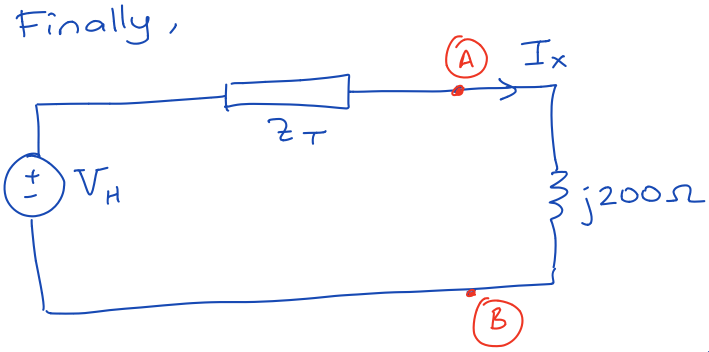
$$V_c = 75 \angle 0$$

$$V_A = \frac{-j120}{50 - j120} \cdot 75 \angle 0$$

$$V_B = \frac{60}{50 + 60} \cdot 75 \angle 0$$

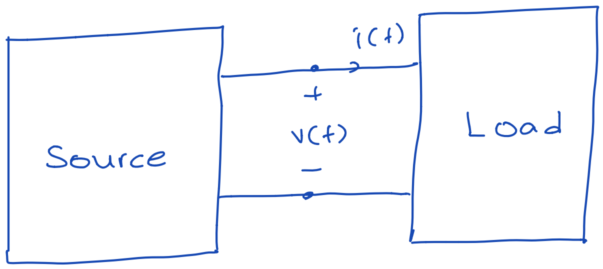
$$V_T = V_A - V_B = 23 - j26.6 \text{ V}$$

* Finally,



$$I_x = \frac{V_T}{Z_T + j200} = 0.18 \cdot e^{-j2.06} \text{ A} = 0.18 \, e^{-j118^\circ} \text{ A}$$

Power and Energy in Sinusoidal Steady State



$$v(t) = V_m \cdot \cos(\omega t + \measuredangle V)$$
$$i(t) = I_m \cdot \cos(\omega t + \measuredangle I)$$

$p(t) = v(t) \cdot i(t) \Rightarrow$ instantaneous power delivered to the load

$P_{ave} = \frac{1}{T} \int_0^T \underbrace{v(t) \cdot i(t)}_{p(t)} dt \Rightarrow$ average power delivered to the load

$T = \frac{2\pi}{\omega}$

$$p(t) = v(t) \cdot i(t) = V_m \cdot \cos(\omega t + \measuredangle V) \cdot I_m \cos(\omega t + \measuredangle I)$$
$$= \frac{1}{2} V_m \cdot I_m \left[\cos(2\omega t + \measuredangle V + \measuredangle I) + \cos(\measuredangle V - \measuredangle I) \right]$$

$$P_{ave} = \frac{1}{T} \int_0^T p(t)\, dt$$
$$= \cancel{\frac{1}{T} \int_0^T \frac{1}{2} V_m \cdot I_m \cdot \cos(\underbrace{2\omega t}_{T' = \frac{T}{2}} + \measuredangle V + \measuredangle I)\, dt} \rightarrow 0$$
$$+ \frac{1}{T} \int_0^T \underbrace{\frac{1}{2} V_m \cdot I_m \cdot \cos(\measuredangle V - \measuredangle I)}_{\text{constants}} dt$$

$\Rightarrow$ integrating over 2 periods of $\cos(2\omega t + \dots)$

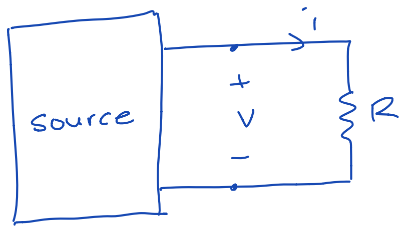
$$\boxed{P_{ave} = \frac{1}{2} V_m \cdot I_m \cdot \underbrace{\cos(\measuredangle V - \measuredangle I)}_{\text{called "power factor"}}}$$

$$I_{eff} = \frac{I_m}{\sqrt{2}} = I_{rms} \quad , \quad V_{eff} = \frac{V_m}{\sqrt{2}} = V_{rms}$$

Then,

$$\boxed{P_{ave} = I_{eff} \cdot V_{eff} \cdot \cos(\measuredangle V - \measuredangle I)}$$

For resistive load :



$$V = V_m . e^{j\measuredangle V}$$

$$I = I_m . e^{j\measuredangle I}$$

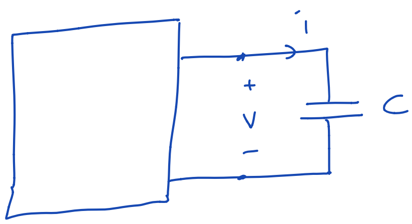
$$V = R I \Rightarrow V_m . e^{j\measuredangle V} = R . I_m . e^{j\measuredangle I}$$

$$V_m = R . I_m \quad , \quad \measuredangle V = \measuredangle I$$

$$P_{ave} = \frac{1}{2} V_m . I_m . \cos(\measuredangle V - \measuredangle I) = \frac{1}{2} V_m . I_m = \frac{1}{2} \frac{V_m^2}{R} = \frac{1}{2} I_m^2 . R$$

$$= \frac{V_{eff}^2}{R} = I_{eff}^2 . R$$

Capacitor :



$$I = j\omega C . V$$

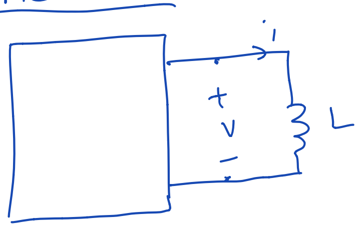
$$I_m . e^{j\measuredangle I} = \underset{\nwarrow \, e^{j\pi/2}}{j} \omega C . V_m . e^{j\measuredangle V}$$

$$I_m = \omega C V_m \quad , \quad \measuredangle I = \frac{\pi}{2} + \measuredangle V$$

$$P_{ave} = \frac{1}{2} V_m . I_m . \cos(\measuredangle V - \measuredangle I) = \frac{1}{2} \omega C V_m^2 . \cos(-\pi/2) = 0$$

$\Rightarrow$ capacitor does not dissipate power

Inductor :



$$V = j\omega L . I$$

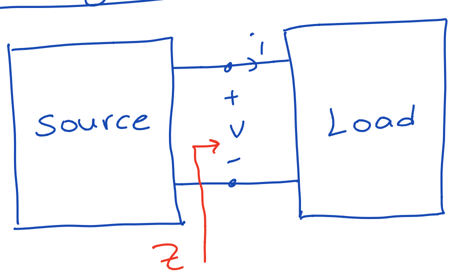
$$V_m . e^{j\measuredangle V} = \underset{\nwarrow \, e^{j\pi/2}}{j} \omega L . I_m . e^{j\measuredangle I}$$

$$V_m = \omega L I_m \quad , \quad \measuredangle V = \frac{\pi}{2} + \measuredangle I$$

$$P_{ave} = \frac{1}{2} V_m . I_m . \cos(\measuredangle V - \measuredangle I) = \frac{1}{2} \omega L I_m^2 \cos(\pi/2) = 0$$

$\Rightarrow$ inductor does not dissipate power

In general



$$Z = \frac{V}{I} = \frac{V_m . e^{j\measuredangle V}}{I_m . e^{j\measuredangle I}} = \frac{V_m}{I_m} . e^{j(\measuredangle V - \measuredangle I)}$$

$$Re\{Z\} = \frac{V_m}{I_m} . \underbrace{\cos(\measuredangle V - \measuredangle I)}_{\text{power factor}}$$

$$Y = \frac{1}{Z} = \frac{I}{V} = \frac{I_m}{V_m} \cdot e^{j(\measuredangle I - \measuredangle V)}$$

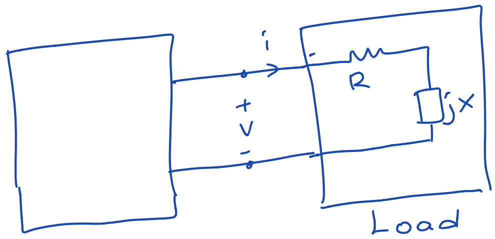
cosine is an even function

$$Re\{Y\} = \frac{I_m}{V_m} \underbrace{\cos(\measuredangle V - \measuredangle I)}_{\text{power factor}}$$

$$P_{ave} = \frac{1}{2} V_m \cdot I_m \cdot \cos(\measuredangle V - \measuredangle I)$$

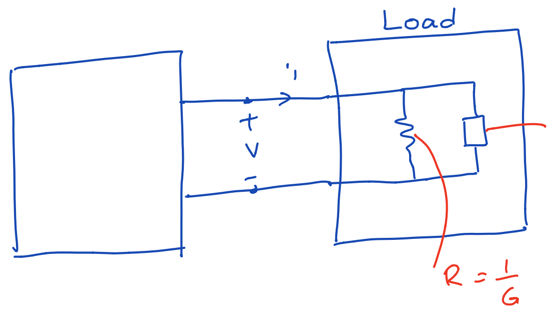
$$= \frac{1}{2} I_m^2 \cdot Re\{Z\} = I_{eff}^2 \cdot Re\{Z\} = I_{eff}^2 \cdot R$$

$Z = R + jX$



$$P_{ave} = \frac{1}{2} V_m^2 \cdot Re\{Y\} = V_{eff}^2 \cdot Re\{Y\} = V_{eff}^2 \cdot G$$

$Y = G + jB$



Alternative way of Calculating Average Power

$$V = V_m \cdot e^{j\measuredangle V} \quad , \quad I = I_m \cdot e^{j\measuredangle I}$$

$$S = \frac{1}{2} V \cdot I^* \quad \Rightarrow \text{complex power}$$

$$P_{ave} = Re\{S\} = Re\left\{\frac{1}{2} V \cdot I^*\right\}$$

$$= Re\left\{\frac{1}{2} V_m \cdot e^{j\measuredangle V} \cdot I_m \cdot e^{-j\measuredangle I}\right\} = Re\left\{\frac{1}{2} V_m \cdot I_m \cdot e^{j(\measuredangle V - \measuredangle I)}\right\}$$

$$= \frac{1}{2} V_m \cdot I_m \cdot \cos(\measuredangle V - \measuredangle I) \quad \Rightarrow \text{same result as before}$$

$$S = \frac{1}{2} V_m . I_m \, e^{j\underbrace{(\measuredangle V - \measuredangle I)}_{\theta}} = V_{rms} . I_{rms} . e^{j\theta}$$

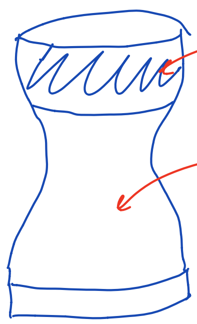
θ ← power factor angle

$$= \underbrace{\frac{1}{2} V_m . I_m . \cos(\theta)}_{P_{ave}} + j . \underbrace{\frac{1}{2} V_m . I_m . \sin(\theta)}_{Q}$$

P_{ave} : actual dissipated power (in units of Watts)

Q : "reactive power" (in units of Volt-Amperes Reactive (VAR))

Beer Analogy



foam $\Rightarrow$ reactive power

actual beer $\Rightarrow$ P_{ave}

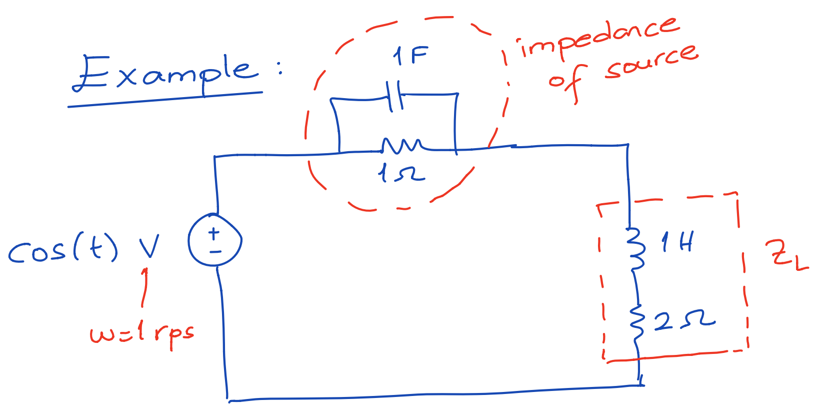
$|S| = \frac{1}{2} V_m . I_m$: called "apparent power" in units of Volt-Ampere (VA)

Then, power factor $= \cos(\measuredangle V - \measuredangle I) = \dfrac{P_{ave}}{\text{Apparent Power}}$

* We want Q (reactive power) to be close to zero
$\Rightarrow$ make $(\measuredangle V - \measuredangle I)$ as small as possible.

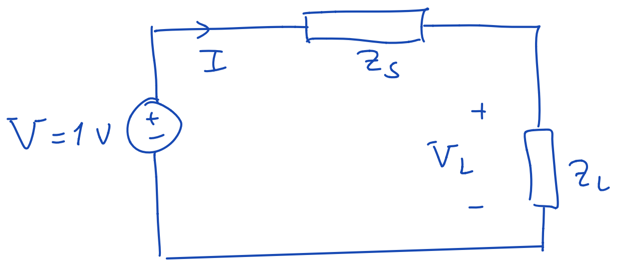
Typically, Q is positive, because most loads are inductive overall.

Example :



Find power dissipated in Z_L.

$\Downarrow$



$$Y_S = 1 + j\omega C = 1 + j \quad (S)$$

$$Z_S = \frac{1}{Y_S} = \frac{1}{1+j} = \frac{1-j}{2} \quad (\Omega)$$

$$Z_L = 2 + j\omega L = 2 + j \quad (\Omega)$$

$$Z_T = Z_S + Z_L = \frac{1-j}{2} + 2 + j = \frac{5+j}{2} \; \Omega$$

$$I = \frac{V}{Z_T} = \frac{1}{\frac{5+j}{2}} = \frac{2}{5+j} \; A$$

$$V_L = I \cdot Z_L = \frac{2(2+j)}{5+j} \; V$$

$$* \quad S = \frac{1}{2} V_L \cdot I^* = \frac{1}{\cancel{2}} \frac{\cancel{2}(2+j)}{5+j} \cdot \frac{2}{5-j} = \frac{2(2+j)}{26}$$

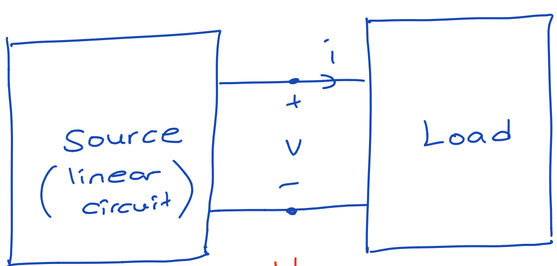
$$= \frac{2+j}{13} = \frac{2}{13} + j\frac{1}{13}$$

$$Re\{S\} = P_{ave} = \frac{2}{13} \; W$$

$*$ Looking at $2\,\Omega$ resistor only :

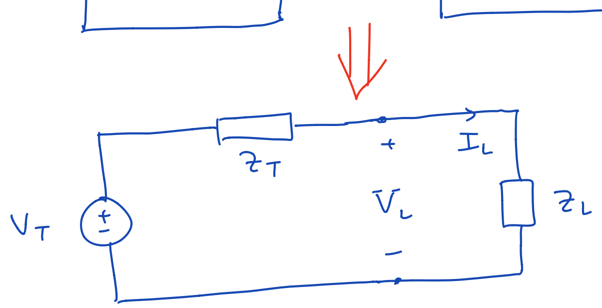
$$P_{ave} = \frac{1}{2} I_m^2 \cdot R = \frac{1}{\cancel{2}} \left| \frac{2}{5+j} \right|^2 \cdot \cancel{2} = \frac{4}{26} = \underbrace{\frac{2}{13} \; W}_{\text{same result}}$$

Maximum Power Transfer



source : fixed

Load : variable



Thevenin equivalent of source

$$Z_T = R_T + jX_T$$

$$Z_L = R_L + jX_L$$

* maximum power is transferred to the load when

$$\boxed{Z_L = Z_T^*}$$

* Then,

$$I_L = \frac{V_T}{Z_T + Z_L} = \frac{V_T}{2R_T}$$

$$P_{ave} = \frac{1}{2}|I_L|^2 \cdot R_T = \frac{1}{2}\frac{|V_T|^2}{4R_T^2} \cdot R_T = \boxed{\frac{|V_T|^2}{8R_T} = P_{max}}$$