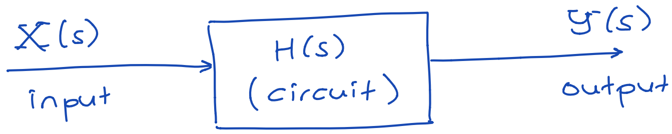
CHAPTER 11 - NETWORK FUNCTION

$$\text{Network Function} = \frac{\mathcal{L}\{\text{zero-state response of output}\}}{\mathcal{L}\{\text{input}\}}$$

i.e., use zero-initial conditions.

$$H(s) = \frac{Y(s)}{X(s)} \implies \boxed{Y(s) = H(s) \cdot X(s)}$$

called "transfer function"



* If we know H(s), we can find the output (zero-state response) when the input is specified.

$$\frac{\text{poly. in } s}{\text{poly. in } s} \cdot \frac{\text{poly. in } s}{\text{poly. in } s} = \underbrace{\frac{\text{poly. in } s}{\text{poly. in } s}}_{N+M \text{ poles}}$$

$$Y(s) = \underbrace{H(s)}_{N \text{ poles}} \cdot \underbrace{X(s)}_{M \text{ poles}}$$

partial fraction expansion

$$= \underbrace{\sum_{i=1}^{N} \frac{k_i}{s-p_i}}_{\substack{\text{natural poles} \\ \text{from the circuit} \\ (H(s))}} + \underbrace{\sum_{\ell=1}^{M} \frac{k_\ell}{s-p_\ell}}_{\substack{\text{forced poles} \\ \text{from the input} \\ (X(s))}}$$

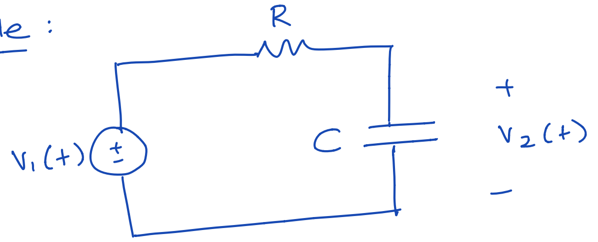
Then,

$$y(t) = \underbrace{\sum_{i=1}^{N} k_i \cdot e^{p_i t}}_{\text{Natural response}} + \underbrace{\sum_{\ell=1}^{M} k_\ell e^{p_\ell t}}_{\text{forced response}}$$

(decays to zero in a stable circuit)

(the part that may not decay $\Rightarrow$ called steady-state response)

Example :



$R = 10\ k\Omega$
$C = 10\mu F$

a) Find the transfer function $H(s) = \dfrac{V_2(s)}{V_1(s)}$

b) Find the zero-state response when
$v_1(t) = 10\,(1 - e^{-5t})\,u(t)$ V.

a) $$H(s) = \frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{sC}}{\frac{1}{sC} + R} \quad \text{(voltage divider)}$$

$$= \frac{1}{sRC+1} = \frac{1/RC}{s + 1/RC} = \frac{\frac{1}{10^4 \cdot 10^{-5}}}{s + \frac{1}{10^4 \cdot 10^{-5}}} = \boxed{\frac{10}{s+10}}$$

b) $v_1(t) = 10\,u(t) - 10 \cdot e^{-5t}\,u(t)$ V

$$V_1(s) = \underset{(s+5)}{\frac{10}{s}} - \underset{(s)}{\frac{10}{s+5}} = \frac{50}{s(s+5)}$$

So, $$V_2(s) = H(s) \cdot V_1(s) = \frac{500}{s(s+5)(s+10)}$$

$$= \underbrace{\frac{k_1}{s} + \frac{k_2}{s+5}}_{\text{forced poles from } V_1(s)} + \underbrace{\frac{k_3}{s+10}}_{\text{natural pole from } H(s)}$$

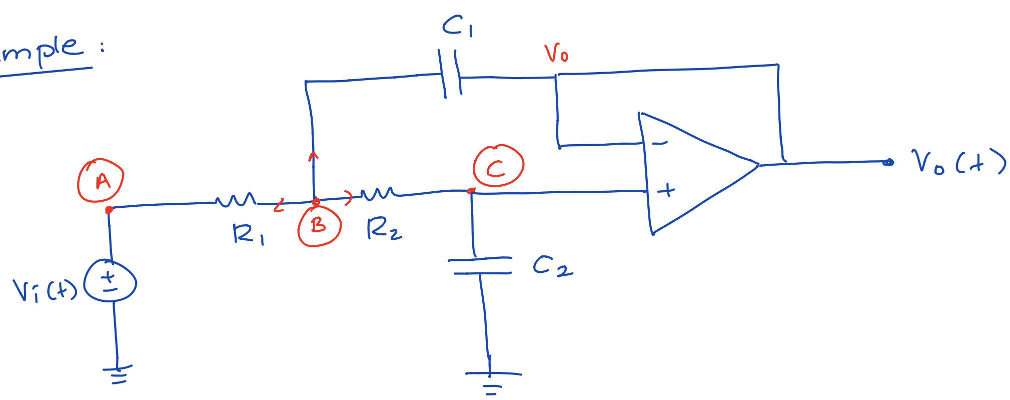
$$k_1 = s\,V_2(s)\Big|_{s=0} = \frac{500}{5 \cdot 10} = 10$$

$$k_2 = (s+5)\,V_2(s)\Big|_{s=-5} = \frac{500}{-5 \cdot 5} = -20$$

$$k_3 = (s+10)\,V_2(s)\Big|_{s=-10} = \frac{500}{-10 \cdot (-5)} = 10$$

So,
$$v_2(t) = \underbrace{\Big[\underbrace{10 - 20e^{-5t}}_{\text{forced response}} + \underbrace{10 \cdot e^{-10t}}_{\text{natural response}}\Big]}_{\text{zero-state response}}\,u(t) \quad \text{Volts}$$

Example:



* Find $H(s) = \dfrac{V_o(s)}{V_i(s)}$

* If $R_1 = R_2 = 1\Omega$, $C_1 = 1F$, $C_2 = \frac{1}{2}F$ and $v_i(t) = u(t)$ V. Find $v_o(t)$, assuming zero initial conditions.

* Assume OpAmp is ideal and not saturated.

$$V_+ = V_- \implies V_c = V_o$$

KCL at B:
$$\underset{(R_2)}{\frac{V_B - V_i}{R_1}} + \underset{(R_1)}{\frac{V_B - \overset{V_c}{V_o}}{R_2}} + \underset{(R_1 . R_2)}{(V_B - V_o) . sC_1} = 0$$

$$V_B(R_1 + R_2 + sC_1R_1R_2) = R_2 V_i + (R_1 + sC_1R_1R_2)V_o$$

KCL at C:
$$V_c = V_o = \frac{\frac{1}{sC_2}}{\frac{1}{sC_2} + R_2} . V_B = \frac{1}{sC_2R_2 + 1} . V_B$$

$$V_B = (1 + sC_2R_2)V_o$$

* Insert into the first equation:

$$(1 + sC_2R_2)(R_1 + R_2 + sC_1R_1R_2)V_o = R_2 V_i + (R_1 + sC_1R_1R_2)V_o$$

$$\left[(\cancel{R_1} + R_2) + \cancel{sC_1R_1R_2} + sC_2R_2(R_1+R_2) + s^2C_1C_2R_1R_2^2 - \cancel{R_1} - \cancel{sC_1R_1R_2}\right]V_o = R_2V_i$$

$$\left[\cancel{R_2} + sC_2\cancel{R_2}(R_1+R_2) + s^2C_1C_2R_1R_2^{\cancel{2}}\right]V_o = \cancel{R_2}V_i$$

$$\left[1 + sC_2(R_1+R_2) + s^2C_1C_2R_1R_2\right]V_o = V_i$$

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{s^2C_1C_2R_1R_2 + sC_2(R_1+R_2) + 1}$$

$$H(s) = \frac{1}{s^2 \cdot \frac{1}{2} + s \cdot \frac{1}{2}\, 2 + 1} = \boxed{\frac{2}{s^2 + 2s + 2}}$$

* $V_i(s) = \frac{1}{s}$

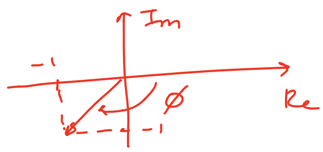
$$V_o(s) = H(s) \cdot V_i(s) = \underbrace{\frac{2}{s^2 + 2s + 2}}_{\substack{s_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2} \\ = \underbrace{-1 \pm j = -\alpha \pm j\beta}_{\text{natural poles}}}} \cdot \underbrace{\frac{1}{s}}_{s=0 \text{ (forced pole)}}$$

$$= \frac{k_1}{s} + \frac{k}{s-(-1+j)} + \frac{k^*}{s-(-1-j)}$$

$$k_1 = s\, V_o(s) \Big|_{s=0} = \frac{2}{2} = 1$$

$$k = (s-(-1+j))\, V_o(s) \Big|_{s=-1+j} = \frac{2}{(s-(-1-j))\, s} \Bigg|_{s=-1+j}$$

$$= \frac{\cancel{2}}{\cancel{2}j\,(-1+j)} = \frac{1}{-1-j} = \frac{1}{\sqrt{2}\, . e^{-j3\pi/4}} = \frac{1}{\sqrt{2}} e^{j3\pi/4} = |k| . e^{j\theta}$$



* so,

$$V_o(t) = \Big[\underbrace{1}_{\text{forced response}} + \underbrace{\overbrace{\sqrt{2}}^{2|k|} . e^{-t} . \cos(t + \overbrace{\tfrac{3\pi}{4}}^{\theta})}_{\text{natural response}}\Big]\, u(t) \quad \text{Volts}$$

Network Function and Impulse Response

* Let input be an impulse. The zero-state response in this case is called "impulse response".

$$Y(s) = H(s) \cdot \underbrace{\mathcal{L}\{\delta(t)\}}_{=1} = H(s) \cdot 1 = H(s)$$

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \text{impulse response in time-domain}$$

* $$H(s) = \frac{\text{poly. in } s}{\text{poly. in } s} = \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \cdots + \frac{k_N}{s-p_N}$$

$$= \sum_{i=1}^{N} \underbrace{\frac{k_i}{s-p_i}}_{\text{natural poles}}$$

(impulse does not introduce any forced poles)

* Then,

$$h(t) = \underbrace{\left[\sum_{i=1}^{N} k_i \cdot e^{p_i t}\right] u(t)}_{\text{natural response}}$$

If circuit is stable, $h(t)$ eventually decays to zero.

<u>Important</u>: $h(t)$ contains <u>all the information</u> needed to determine the circuit response to any other input.

$$y(t) = \mathcal{L}^{-1}\{H(s) \cdot X(s)\}$$

Step Response:

* The zero-state solution when the input is $u(t)$ is called "step response".

<u>Remember</u>: $U(s) = \frac{1}{s}$

* $$Y(s) = H(s) \cdot U(s) = \frac{1}{s} H(s)$$

$$\Big\downarrow \mathcal{L}^{-1}\{\cdot\}$$

$$y(t) = \int_0^t h(\tau)\, d\tau$$

step response = time integral of impulse response

$$\dot{y}(t) \Leftrightarrow h(t)$$

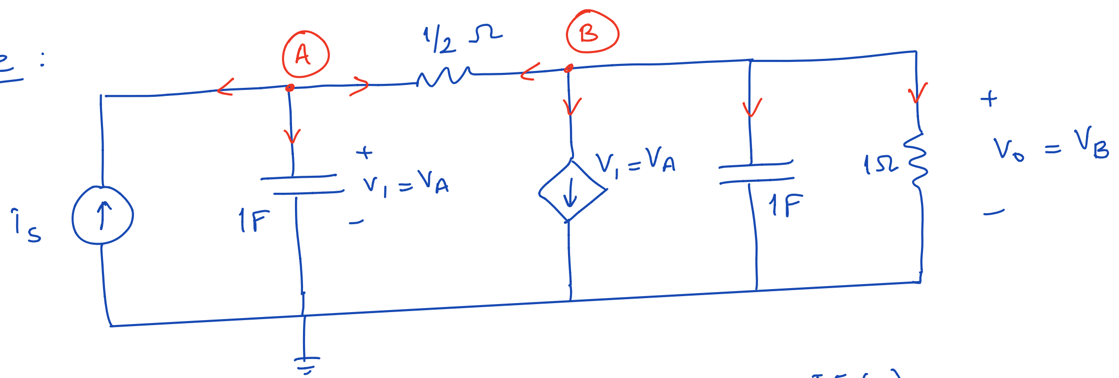
$$\frac{d}{dt}(\text{step response}) = \text{impulse response} \quad (\text{except at discontinuities})$$

$$* \quad Y(s) = \frac{1}{s} \cdot H(s) = \underbrace{\frac{k_0}{s}}_{\text{forced pole}} + \underbrace{\sum_{i=1}^{N} \frac{k_i}{s - p_i}}_{\text{natural poles}}$$

$$y(t) = \underbrace{k_0 \cdot u(t)}_{\text{forced response}} + \underbrace{\left[\sum_{i=1}^{N} k_i \cdot e^{p_i t}\right] u(t)}_{\text{natural response}}$$

(called DC steady-state response)

Example :



* Find the transfer function : $H(s) = \dfrac{V_o(s)}{I_s(s)}$

* Find the impulse response $h(t)$.

* Find the step response.

KCL at A : $-I_s + \dfrac{V_A}{\frac{1}{s \cdot 1}} + \dfrac{V_A - V_B}{\frac{1}{2}} = 0$

$$(s+2)V_A - 2V_B = I_s$$

KCL at B : $\dfrac{V_B - V_A}{\frac{1}{2}} + V_A + \dfrac{V_B}{\frac{1}{s \cdot 1}} + \dfrac{V_B}{1} = 0$

$$-V_A + (s+3)V_B = 0$$

$$V_A = (s+3)V_B$$

* Insert into first equation :

$$(s+2)(s+3)V_B - 2V_B = I_s$$

$$(s^2+5s+4)V_B = I_s$$

$$H(s) = \frac{V_o}{I_s} = \frac{V_B}{I_s} = \boxed{\frac{1}{s^2+5s+4}}$$

* $h(t) = \mathcal{L}^{-1}\{H(s)\}$

$$H(s) = \frac{1}{s^2+5s+4} = \frac{1}{(s+1)(s+4)} = \frac{k_1}{s+1} + \frac{k_2}{s+4}$$

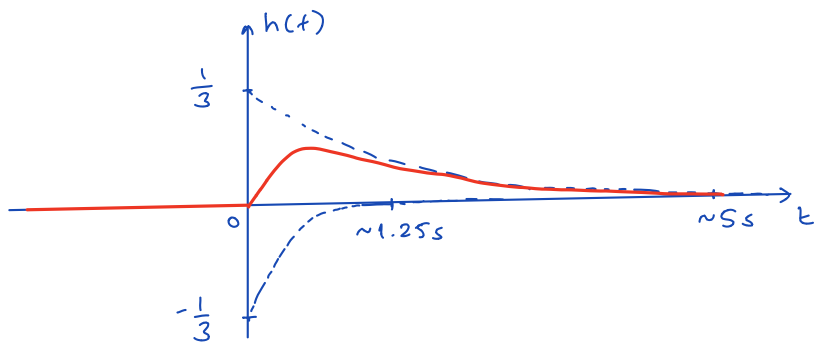
$$k_1 = (s+1)H(s)\Big|_{s=-1} = \frac{1}{3}$$

$$k_2 = (s+4)H(s)\Big|_{s=-4} = -\frac{1}{3}$$

Then,

$\tau_1 = 1s$ (dominant) $\qquad \tau_2 = \frac{1}{4}s = 0.25\ s$

$$\underset{\text{impulse response}}{h(t)} = \underbrace{\left[\frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t}\right]}_{\text{natural response}} u(t)$$



* $i_s(t) = u(t)$

$$I(s) = \frac{1}{s}$$

$$V_o(s) = H(s)\cdot I_s(s) = \frac{1}{s(s^2+5s+4)} = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+4}$$

$$k_1 = s\,V_o(s)\Big|_{s=0} = \frac{1}{4}$$

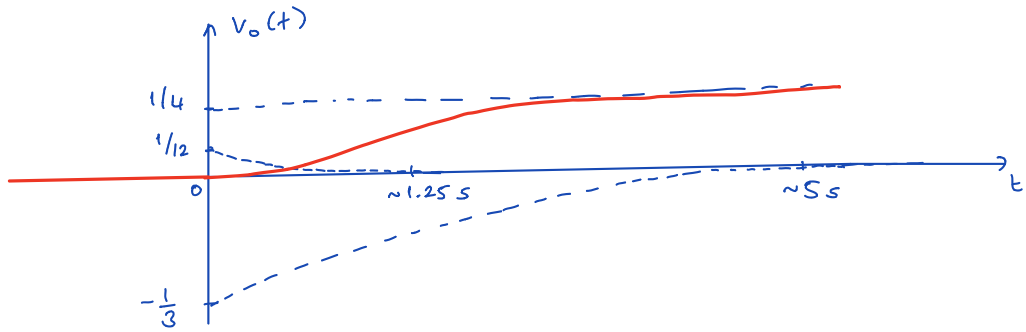
$$k_2 = (s+1)V_o(s)\Big|_{s=-1} = \frac{1}{-1\,.\,3} = -\frac{1}{3}$$

$$k_3 = (s+4)\,V_o(s)\Big|_{s=-4} = \frac{1}{-4\,(-3)} = \frac{1}{12}$$

Then,

$\tau_1 = 1s$ $\tau_2 = \frac{1}{4}s = 0.25\ s$

$$\underset{\text{step response}}{V_o(t)} = \left[\underbrace{\frac{1}{4}}_{\text{forced resp.}} \underbrace{- \frac{1}{3}e^{-t} + \frac{1}{12}e^{-4t}}_{\text{natural response}}\right]u(t)$$



* Doudle check :

$$\frac{dV_o}{dt} = \left[-(1).\left(-\frac{1}{3}\right)e^{-t} - \frac{4}{12}e^{-4t}\right]u(t)$$

$$= \left[\frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t}\right]u(t) = h(t)\ \checkmark$$

Impulse Response and Convolution

$$Y(s) = H(s)\,X(s)$$

$$y(t) = \mathcal{L}^{-1}\{H(s)\cdot X(s)\}$$

(computed in s-domain, followed by inverse Laplace transform)

* This corresponds to convolution in time-domain:

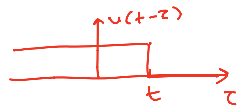
$$y(t) = \int_{0^-}^{t} h(t-\tau)\,x(\tau)\,d\tau = h(t) * x(t)$$

(computed in time-domain)

Proof:

$$\mathcal{L}\{y(t)\} = \mathcal{L}\left\{\int_{0^-}^{t} h(t-\tau)\,x(\tau)\,d\tau\right\}$$

$$= \mathcal{L}\left\{\int_{0^-}^{\infty} h(t-\tau)\,\overbrace{u(t-\tau)}^{\text{zeo if } \tau > t}\,x(\tau)\,d\tau\right\}$$



$$= \int_{0^-}^{\infty}\left[\int_{0^-}^{\infty} h(t-\tau)\,u(t-\tau)\,x(\tau)\,d\tau\right] e^{-st}\,dt$$

$$= \int_{0^-}^{\infty}\underbrace{\left[\int_{0^-}^{\infty} h(t-\tau)\,u(t-\tau)\,e^{-st}\,dt\right]}_{H(s)\cdot e^{-s\tau}\ \text{(time-domain translation)}} x(\tau)\,d\tau$$

$$= \int_{0^-}^{\infty} H(s)\,e^{-s\tau}\,x(\tau)\,d\tau$$

$$= H(s)\underbrace{\int_{0^-}^{\infty} x(\tau)\,e^{-s\tau}\,d\tau}_{X(s)}$$

$$= H(s)\cdot X(s)$$

Example : $h(t) = 2e^{-t}\,u(t)$ $\Rightarrow$ impulse response

$x(t) = e^{-2t}\,u(t)$ $\Rightarrow$ input

Find the zero-state response.

* $Y(s) = H(s)\,X(s)$

$$H(s) = \frac{2}{s+1}, \qquad X(s) = \frac{1}{s+2}$$

$$Y(s) = \frac{2}{(s+1)(s+2)} = \underbrace{\frac{k_1}{s+1}}_{\text{natural pole}} + \underbrace{\frac{k_2}{s+2}}_{\text{forced pole}}$$

$$k_1 = (s+1)\,Y(s)\Big|_{s=-1} = \frac{2}{1} = 2$$

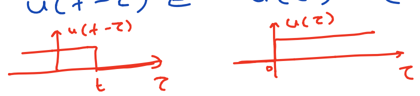
$$k_2 = (s+2)\,Y(s)\Big|_{s=-2} = \frac{2}{-1} = -2$$

$$y(t) = \mathcal{L}^{-1}\{H(s)\,X(s)\} = \underbrace{\underbrace{2e^{-t}}_{\text{natural resp.}} \; \underbrace{-\,2e^{-2t}}_{\text{forced resp.}}}_{\text{zero-state response}}, \qquad t \geq 0$$

* Alternatively, we can use convolution :

$$y(t) = \int_{0^-}^{t} h(t-\tau)\,x(\tau)\,d\tau = \int_0^t 2e^{-(t-\tau)}\,u(t-\tau)\,e^{-2\tau}\,u(\tau)\,d\tau$$

always 1 within the limits of the integral



$$= \int_0^t 2e^{-(t-\tau)}\,e^{-2\tau}\,d\tau = 2e^{-t}\int_0^t e^{\tau}\cdot e^{-2\tau}\,d\tau$$

$$= 2e^{-t}\int_0^t e^{-\tau}\,d\tau = 2e^{-t}\left[\frac{e^{-\tau}}{-1}\right]\Bigg|_0^t$$

$$= 2e^{-t}\left(-e^{-t}+1\right) = -2e^{-2t} + 2e^{-t}$$

$$= 2e^{-t} - 2e^{-2t}, \qquad \text{for } t \geq 0$$

$\Longrightarrow$ same result as before

Relation with Sinusoidal Steady State

$$H(s) = \frac{Y(s)}{X(s)} \quad \Longrightarrow \quad Y(s) = H(s)\cdot X(s) \qquad \text{(zero-state response)}$$

* Assume that the input $x(t)$ is sinusoidal:

$$x(t) = X_m \cdot \cos(\omega t + \phi) = \mathrm{Re}\left\{X_m \cdot e^{j\phi} \cdot e^{j\omega t}\right\} = \mathrm{Re}\left\{\hat{X} \cdot e^{j\omega t}\right\}$$

$$= \frac{1}{2}\hat{X} \cdot e^{j\omega t} + \frac{1}{2}\underbrace{\hat{X}^* e^{-j\omega t}}_{\text{complex conjugate}} \quad , \text{ where } \underbrace{\hat{X}}_{\text{phasor}} = X_m \cdot e^{j\phi}$$

$$\mathcal{L}\{x(t)\} = \frac{1}{2}\hat{X}\frac{1}{s-j\omega} + \frac{1}{2}\hat{X}^*\frac{1}{s+j\omega} = X(s)$$

$$Y(s) = H(s)\cdot X(s)$$

$$= \underbrace{H(s)}_{N \text{ natural poles}} \cdot \underbrace{\left\{\frac{1}{2}\hat{X}\frac{1}{s-j\omega} + \frac{1}{2}\hat{X}^* \cdot \frac{1}{s+j\omega}\right\}}_{2 \text{ forced poles}}$$

partial fraction expansion

$$= \underbrace{\sum_{i=1}^{N} \frac{k_i}{s-p_i}}_{\text{natural poles from } H(s)} + \underbrace{\frac{k}{s-j\omega} + \frac{k^*}{s+j\omega}}_{\text{forced poles from } X(s)}$$

$$k = (s-j\omega)\,Y(s)\Big|_{s=j\omega} = H(s)\cdot\left\{\frac{1}{2}\hat{X} \cdot \frac{\cancel{s-j\omega}}{\cancel{s-j\omega}} + \frac{1}{2}\hat{X}^* \cdot \frac{s-j\omega}{s+j\omega}\right\}\Bigg|_{s=j\omega}$$

$$= H(j\omega)\cdot\left\{\frac{1}{2}\hat{X} + \cancel{\frac{1}{2}\hat{X}^* \cdot \frac{0}{2j\omega}}\right\}$$

$$= \frac{1}{2} H(j\omega)\,\hat{X}$$

So,

$$y(t) = \underbrace{\sum_{i=1}^{N} k_i \cdot e^{p_i t}}_{\text{natural resp.}} + \underbrace{\overbrace{\overbrace{\frac{1}{2}H(j\omega)\hat{X}}^{k} \cdot e^{j\omega t}}^{A} + \overbrace{\overbrace{\frac{1}{2}H^*(j\omega)\hat{X}^*}^{k^*} \cdot e^{-j\omega t}}^{A^*}}_{\text{forced resp.}}$$

$$= \sum_{i=1}^{N} k_i \cdot e^{p_i t} + Re\{ \underbrace{H(j\omega)\hat{X}}_{\text{output phasor}} e^{j\omega t} \}$$

forced resp.
sinusoidal steady-state resp.

where $H(j\omega) = |H(j\omega)| \cdot e^{j\angle H(j\omega)}$

$$= \sum_{i=1}^{N} k_i \cdot e^{p_i t} + Re\{ |H(j\omega)| e^{j\angle H(j\omega)} \cdot X_m \cdot e^{j\phi} \cdot e^{j\omega t} \}$$

$$= \quad '' \quad + Re\{ |H(j\omega)| X_m \cdot e^{j(\omega t + \phi + \angle H(j\omega))} \}$$

$$= \underbrace{\quad '' \quad}_{\text{natural resp.}} + \underbrace{|H(j\omega)| X_m \cdot \cos(\omega t + \phi + \angle H(j\omega))}_{\text{sinusoidal steady-state resp.}}$$

(this will decay down to zero for stable case)

(same as in phasor analysis)

* Evaluate $H(s)$ at $j\omega$ to find the transfer function in phasor domain.

Output phasor $= H(j\omega) \cdot \hat{X}$

Transfer function in phasor domain $= \dfrac{\text{output phasor}}{\text{input phasor}} = \dfrac{H(j\omega) \cdot \hat{X}}{\hat{X}}$

$$= \boxed{H(j\omega) = H(s)\Big|_{s=j\omega}}$$

* So, phasor analysis reveals the steady-state (i.e., forced) response for the case of sinusoidal inputs.