# CHAPTER 7 : FIRST AND SECOND ORDER CIRCUITS

Here, order refers to the number of capacitors and inductors.

* FIRST ORDER CIRCUITS: contain only one capacitor or only one inductor.

* First Order RC Circuits : Such circuits contain independent and dependent sources, resistors, one capacitor

For a capacitor: $\boxed{i_c = C \dfrac{dV_c}{dt}}$

$$\dot{V}_c = \frac{dV_c}{dt} = \frac{i_c}{C}$$

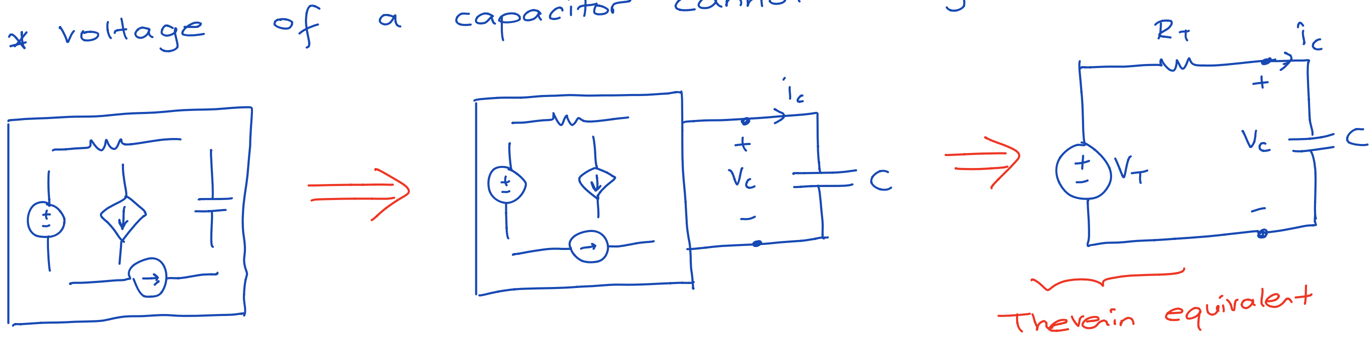
$$\int_{t_0}^{t} \dot{V}_c(\tau)\,d\tau = \frac{1}{C}\int_{t_0}^{t} i_c(\tau)\,d\tau$$

$$V_c(t) - V_c(t_0) = \frac{1}{C}\int_{t_0}^{t} i_c(\tau)\,d\tau$$

$$\boxed{V_c(t) = V_c(t_0) + \frac{1}{C}\int_{t_0}^{t} i_c(\tau)\,d\tau}$$

Also, $E(t) = \frac{1}{2} C V_c^2(t)$

* voltage of a capacitor cannot change instantaneously.



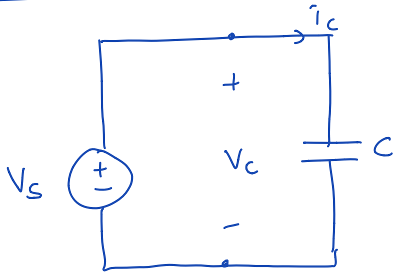

KVL : $V_T = R_T . i_c + V_c$ , $i_c = C \dfrac{dV_c}{dt}$

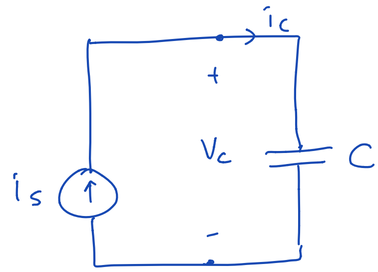
$$V_T = R_T . C \frac{dV_c}{dt} + V_c$$

$$\boxed{\frac{dV_c}{dt} + \frac{1}{R_T . C} V_c = \frac{1}{R_T . C} V_T}$$

* The result is a first order ordinary differential equation (ODE).

## Two degenerate cases




$V_c = V_s$

$i_c = C \frac{dV_s}{dt}$

constant $V_s$ ⟶ zero current

$i_c = i_s$

$V_c(t) = V_c(t_0) + \frac{1}{C} \int_{t_0}^{t} i_s(\tau) d\tau$

constant $i_s$ → growing voltage

**First Order RL Circuits:** Such circuits contain independent and dependent sources, resistors, one inductor.

For inductor: $\boxed{V_L = L \frac{di_L}{dt}}$

$$\dot{i}_L(t) = \frac{di_L}{dt} = \frac{V_L}{L}$$

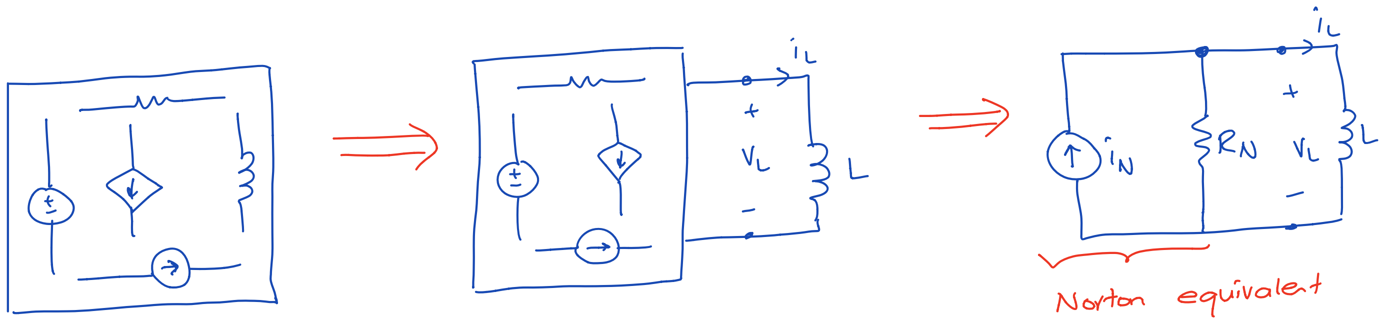
$$\int_{t_0}^{t} \dot{i}_L(\tau) d\tau = \frac{1}{L} \int_{t_0}^{t} V_L(\tau) d\tau$$

$$i_L(t) - i_L(t_0) = \frac{1}{L} \int_{t_0}^{t} V_L(\tau) d\tau$$

$$\boxed{i_L(t) = i_L(t_0) + \frac{1}{L} \int_{t_0}^{t} V_L(\tau) d\tau}$$

Also, $E(t) = \frac{1}{2} L i_L^2(t)$

* Current of an inductor cannot change instantaneously.



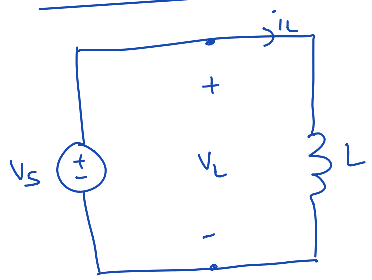
KCL : $i_N = \frac{V_L}{R_N} + i_L$ , $V_L = L\frac{di_L}{dt}$

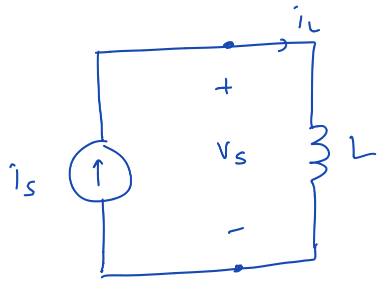
$$i_N = \frac{L}{R_N}\frac{di_L}{dt} + i_L$$

$$\boxed{\frac{di_L}{dt} + \frac{R_N}{L} i_L = \frac{R_N}{L} i_N}$$ , $G_N = \frac{1}{R_N}$

* The result is a first order ODE.

Two degenerate cases :




$V_L = V_s$

$$i_L(t) = i_L(t_0) + \frac{1}{L}\int_{t_0}^{t} V_s(\tau)\,d\tau$$

Constant $V_s$ ⟶ growing current

$i_L = i_s$

$$V_L = L\frac{di_s}{dt}$$

constant $i_s$ ⟶ zero voltage

## Step Response of First Order RC and RL Circuits

Assume we have DC sources ; i.e., $V_T$ and $i_N$ are constants.

$$\frac{dV_c}{dt} + \frac{1}{R_T C} V_c = \frac{1}{R_T C} V_T \quad , \quad \frac{di_L}{dt} + \frac{R_N}{L} i_L = \frac{R_N}{L} i_N$$

Then, these two equations can be combined into a single equation :

$$\frac{dx}{dt} + \frac{1}{\tau} x = \frac{1}{\tau} x_\infty$$

* For RC circuits : $x = V_c$ , $\tau = R_T . C$ , $x_\infty = V_T$ ($V_{final}$)

* For RL circuits : $x = i_L$ , $\tau = \frac{L}{R_N} = G_N L$ , $x_\infty = i_N$ ($i_{final}$)

* $\tau$ : Time constant (seconds)

Solution of ODE : Since $x_\infty$ is a constant, $\frac{dx_\infty}{dt} = 0$

$$\frac{d(x - x_\infty)}{dt} + \frac{1}{\tau}(x - x_\infty) = 0$$

* Define a new variable : $y(t) = x(t) - x_\infty$

New ODE : $\frac{dy}{dt} + \frac{1}{\tau} y = 0$

$$\frac{dy}{dt} = -\frac{y}{\tau} \Rightarrow \int_{y_0}^{y} \frac{dy}{y} = \int_{t_0}^{t} -\frac{dt}{\tau}$$

$$\ln \frac{y}{y_0} = -\frac{(t-t_0)}{\tau} \Rightarrow y = y_0 \cdot e^{-\frac{(t-t_0)}{\tau}}$$

Solution is : $y(t) = y(t_0) \cdot e^{-\frac{(t-t_0)}{\tau}}$

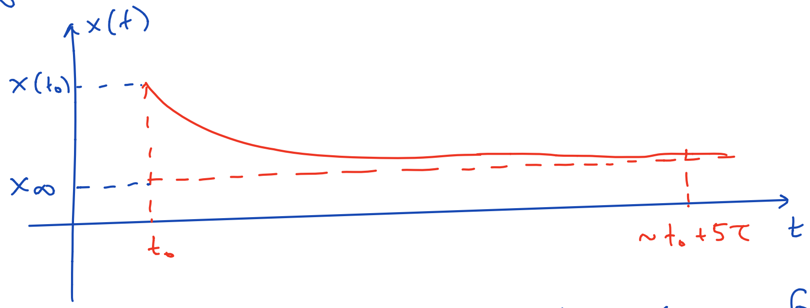
$$x(t) - x_\infty = (x(t_0) - x_\infty)\, e^{-\frac{(t-t_0)}{\tau}}$$

$$x(t) = x_\infty + (x(t_0) - x_\infty)\, e^{-\frac{(t-t_0)}{\tau}}$$

* For RC case : $V_C(t) = V_T + (V_C(t_0) - V_T)\, e^{-\frac{(t-t_0)}{\tau}}$

* For RL case : $i_L(t) = i_N + (i_L(t_0) - i_N)\, e^{-\frac{(t-t_0)}{\tau}}$

* $\lim_{t \to \infty} x(t) = x_\infty + \lim_{t \to \infty} (x(t_0) - x_\infty)\, e^{-\frac{(t-t_0)}{\tau}} = x_\infty$



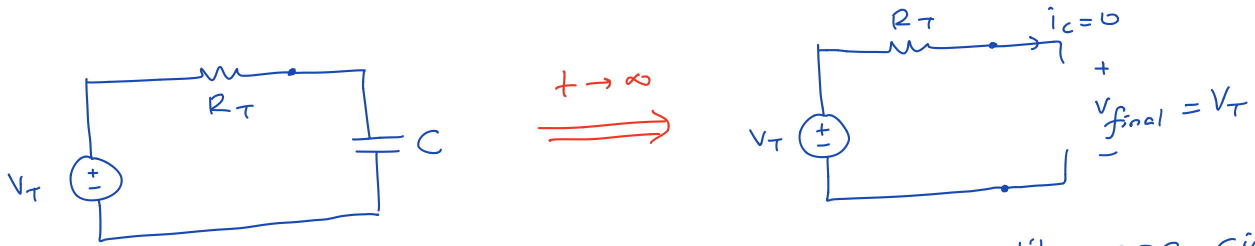

Practically, we have $x(t) = x_\infty$ for $(t - t_0) > 5\tau$.

$\Longrightarrow$ This is called steady state.

$$x(t) - x_\infty = (x(t_0) - x_\infty)\, e^{-5} \approx 0.006\, (x(t_0) - x_\infty)$$

* For RC case: $\lim_{t\to\infty} V_C(t) = V_T$ ← a constant

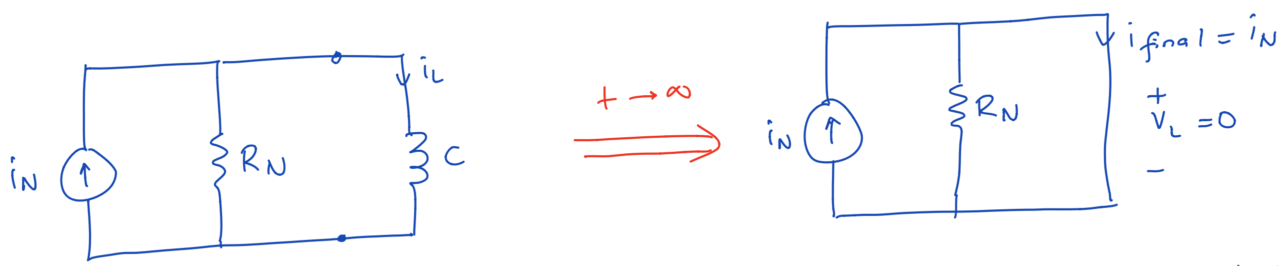
$i_C = C\dfrac{dV_C}{dt}$ , $\lim_{t\to\infty} i_C(t) = 0$




In the DC steady state, capacitor behaves like open circuit.

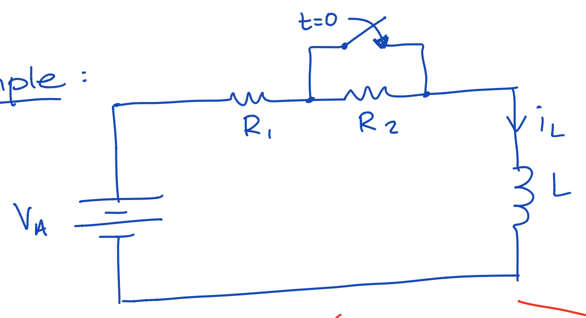
* For RL case: $\lim_{t\to\infty} i_L(t) = i_N$ ← a constant

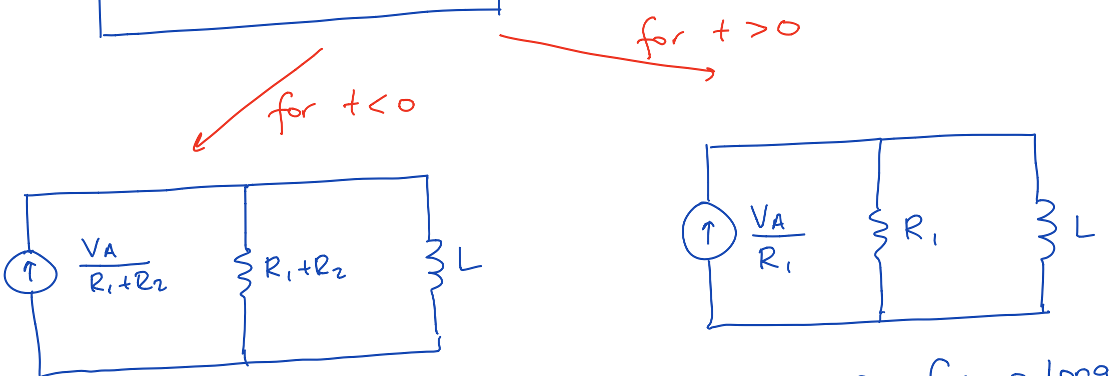
$V_L = L\dfrac{di_L}{dt}$ , $\lim_{t\to\infty} V_L(t) = 0$




In the DC steady state, inductor behaves like a short circuit.

Example:




* For $t<0$, since the switch was open for a long time, has reached steady state we can assume that inductor

→ short circuit.

$$i_L(0^-) = \frac{V_A}{R_1+R_2} = i_L(0^+) = i_L(0)$$

Since inductor current is continuous

* For $t > 0$, the switch is closed. If we wait long enough, we can assume that inductor will reach steady state
$\longrightarrow$ short circuit

$$i_L(\infty) = x_\infty = i_N = \frac{V_A}{R_1}$$

$$\tau = \frac{L}{R_N} = \frac{L}{R_1}$$

$$i_L(t) = x_\infty + \left(x(t_0) - x_\infty\right) e^{-\frac{(t-t_0)}{\tau}}$$

$$= \frac{V_A}{R_1} + \left(\frac{V_A}{R_1+R_2} - \frac{V_A}{R_1}\right) e^{-t/\tau}, \quad t \geq 0$$

$$V_L(t) = L\frac{di_L}{dt} = -\frac{L}{\tau}\left(\frac{V_A}{R_1+R_2} - \frac{V_A}{R_1}\right) e^{-t/\tau}, \quad t \geq 0$$

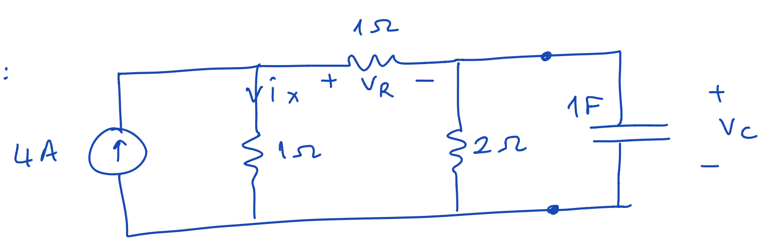
$$= -R_1\left(\frac{V_A}{R_1+R_2} - \frac{V_A}{R_1}\right) e^{-t/\tau}, \quad t \geq 0$$

Check : $V_L(t) \to 0$ as $t \to \infty$

* After finding $v_c(t)$, $i_c(t)$, $V_L(t)$, $i_L(t)$, how can we find the remaining voltages and currents ?

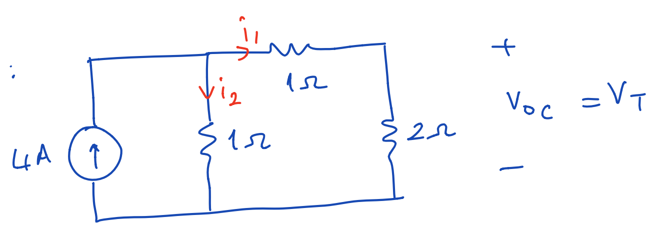
By substitution : i.e., replace the capacitor / inductor by a voltage source or current source with the found solution (like a time-varying voltage / current source)

Example :




Assume $V_c(0) = 1V$. Find $V_c(t)$ for $t \geq 0$.

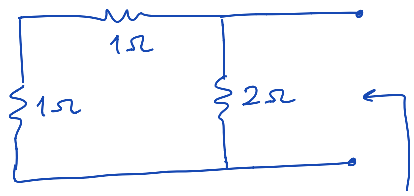
To find $V_{TH}$ :




$$i_1 = \frac{1}{1+3} \cdot 4A = 1A$$

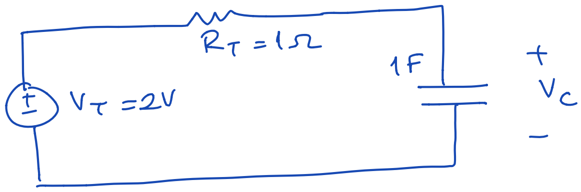
$$V_{oc} = V_T = 1A \cdot 2\Omega = 2V$$

To find $R_T$ :




$$R_T = (1+1) // 2 = 1\Omega$$

So, the circuit simplifies to:



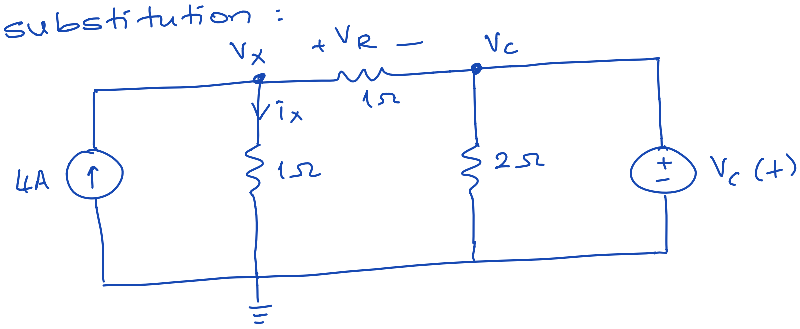

$\tau = R_T C = 1$ sec , $\quad V_c(\infty) = V_T = 2V$

$$V_c(t) = V_c(\infty) + \left(V_c(0) - V_c(\infty)\right) e^{-\frac{(t-t_0)}{\tau}}$$

$$= 2 + (1-2) e^{-t/\tau} = 2 - e^{-t} \text{ V} , \quad t \geq 0$$

$$i_c(t) = C \frac{dV_c(t)}{dt} = e^{-t} \text{ A} , \quad t \geq 0$$

* If we want to find $i_x$ and $V_R$, we can apply substitution :




KCL at x : $\quad -4 + \frac{V_x}{1} + \frac{V_x - V_c}{1} = 0$

$$V_x = \frac{V_c + 4}{2} = \frac{2 - e^{-t} + 4}{2} = 3 - 0.5 e^{-t} \text{ V}$$

$$i_x(t) = \frac{V_x}{1\Omega} = 3 - 0.5 e^{-t} \text{ A} , \quad t \geq 0$$

$$V_R(t) = V_x - V_c = 3 - 0.5 e^{-t} - (2 - e^{-t}) = 1 + 0.5 e^{-t} \text{ V} ,$$

$t \geq 0$

# Step Response of First-Order Circuits

## Alternative Approach:

$$\frac{dx}{dt} + \frac{1}{\tau}x = \frac{1}{\tau}x_\infty$$

$V_T$ or $i_N$

$$x(t) = \underbrace{x_N(t)}_{\text{natural response}} + \underbrace{x_F(t)}_{\text{forced response}}$$

* Natural response (homogeneous solution): $\frac{dx_N}{dt} + \frac{1}{\tau}x_N = 0 \Rightarrow x_N(t) = K \cdot e^{-t/\tau}$

to be determined later from initial condition

* Forced response (particular solution): $\frac{dx_F}{dt} + \frac{1}{\tau}x_F = \frac{1}{\tau}x_\infty \Rightarrow x_F(t) = V_F$

constant

Then, $0 + \frac{1}{\tau}V_F = \frac{1}{\tau}x_\infty \Rightarrow x_F(t) = V_F = x_\infty$

So,

$$x(t) = x_N(t) + x_F(t) = Ke^{-t/\tau} + x_\infty$$

* Use initial condition: $x(t_0) = Ke^{-t_0/\tau} + x_\infty$

$$K = (x(t_0) - x_\infty)\, e^{t_0/\tau}$$

So,

$$x(t) = (x(t_0) - x_\infty)\, e^{t_0/\tau} \cdot e^{-t/\tau} + x_\infty$$

$$= \underbrace{x_\infty}_{\text{forced response}} + \underbrace{(x(t_0) - t_\infty))\, e^{-\frac{(t-t_0)}{\tau}}}_{\text{natural response (decaying exponential)}}, \quad t \geq t_0$$

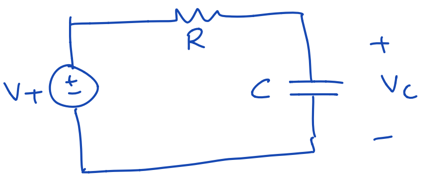
* Re-arranging terms:

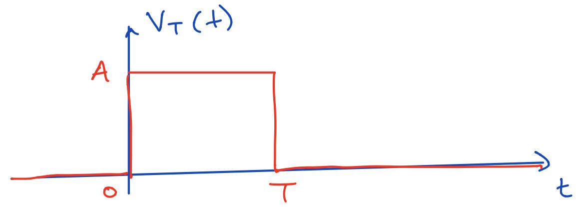
$$x(t) = \underbrace{x(t_0) \cdot e^{-\frac{(t-t_0)}{\tau}}}_{\text{zero-input response}} + \underbrace{x_\infty\left(1 - e^{-\frac{(t-t_0)}{\tau}}\right)}_{\text{zero-state response}}, \quad t \geq t_0$$

zero-input response due to initial conditions (produced by inputs before $t = t_0$)

zero-state response occurs when the initial condition (state) is zero. (produced by inputs after $t = t_0$)

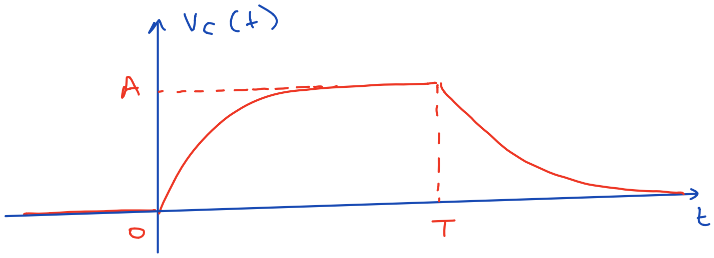
**Response to a Pulse:** The solution will depend on the relation between $\tau$ and $T$.



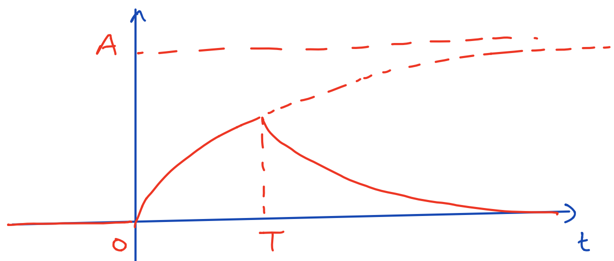




* if $T >> \tau$, $\quad v_c(T) \approx v_c(\infty) = A$.

otherwise, $\quad v_c(T) \neq v_c(\infty) = A$




$\tau << T$




$\tau$ is comparable with $T$

**Example:** $R = 1k\Omega$, $C = 1\mu F$, $A = 10V$, $T = 10ms$, $V_c(0) = 0V$.

$\tau = RC = 1\ ms << T$

* $0 < t < T \Rightarrow v_c(t) = 10 - 10e^{-t/\tau}\ V$

$V_c(10) = 10 - \cancel{10e^{-10}} \approx 10V$

* $t > 10\ ms$, $\quad v_c(t) = 10e^{-(t-10)/\tau}\ V$

**Example:** $R = 10k\Omega$, $C = 1\mu F$, $A = 10V$, $T = 10\ ms$, $V_c(0) = 0V$

$\tau = RC = 10ms = T$

* $0 < t < T \Rightarrow V_c(t) = 10 - 10e^{-t/\tau}\ V$

$V_c(10) = 10 - 10e^{-1} = 6.32\ V$

* $t > 10\ ms \Rightarrow V_c(t) = 6.32\ e^{-(t-10)/\tau}\ V$

## Response to Exponential Inputs for First-Order Circuits :

$$\frac{dx}{dt} + \frac{1}{\tau}x = \frac{1}{\tau}x_\infty \quad , \quad x_\infty = V_A . e^{-\alpha t}$$

add

* Natural response : $\frac{dx_N}{dt} + \frac{1}{\tau}x_N = 0 \Rightarrow x_N(t) = K.e^{-t/\tau}$
* Forced response : $\frac{dx_F}{dt} + \frac{1}{\tau}x_F = \frac{1}{\tau}x_\infty = \frac{1}{\tau}V_A . e^{-\alpha t}$

$$x_F(t) = V_F . e^{-\alpha t}$$

Insert into ODE :

$$-\alpha V_F . e^{-\alpha t} + \frac{1}{\tau}V_F e^{-\alpha t} = \frac{1}{\tau}V_A e^{-\alpha t}$$

$$V_F(-\alpha\tau + 1) = V_A \Rightarrow V_F = \frac{V_A}{1-\alpha\tau}$$

* The actual solution is a superposition of $x_N$ and $x_F$ :

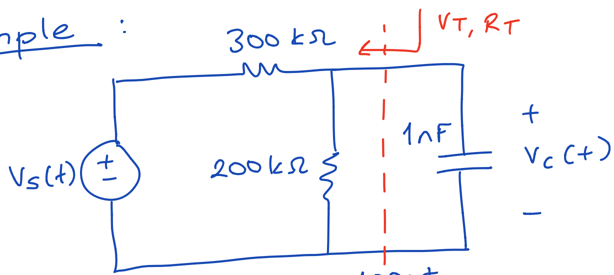
$$\frac{d(x_N + x_F)}{dt} + \frac{1}{\tau}(x_N + x_F) = \frac{1}{\tau}x_\infty$$

So, $x(t) = x_N(t) + x_F(t)$

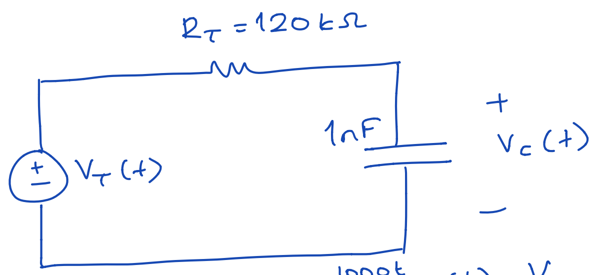
$$x(t) = K e^{-t/\tau} + \frac{V_A}{1-\alpha\tau}e^{-\alpha t}$$

will be determined by the initial conditions.

Example :




$\Rightarrow$




$$V_T(t) = 40.e^{-1000t}.u(t) \ V$$

$$V_S(t) = 100.e^{-1000t}.u(t) \ V$$

$$V_T(t) = \frac{200k}{300k + 200k}.V_S(t) = 40.e^{-1000t}.u(t) \ V$$

$$R_T = 200k // 300k = 120 \ k\Omega$$

* For $t<0$, $v_S(t) = 0 \Rightarrow$ so, $V_c(t) = 0$ for $t \leq 0$.
$V_c(0) = 0$

* For $t>0$,
$x_N(t) = K.e^{-t/\tau}$ , where $\tau = R_T.C = 120.10^3.10^{-9}$
$= 120 \ \mu s = 0.12 \ ms$

$$x_F(t) = V_F \cdot e^{-1000t}$$

$$\frac{dx_F}{dt} + \frac{1}{\tau} x_F = \frac{1}{\tau} x_\infty = \frac{1}{\tau} V_T(t)$$

$$-1000 \cdot V_F \cdot \cancel{e^{-1000t}} + \frac{1}{0.12\times10^{-3}} \cdot V_F \cdot \cancel{e^{-1000t}} = \frac{1}{0.12\times10^{-3}} \cdot 40 \cdot \cancel{e^{-1000t}}$$

$(0.12\times10^{-3})$

$$(-0.12 + 1)V_F = 40 \quad \Rightarrow \quad V_F = \frac{40}{0.88} = 45.5 \text{ V}$$

* So, $V_c(t) = x_N(t) + x_F(t)$

$$= K \cdot e^{-t/0.12\times10^{-3}} + 45.5\, e^{-1000t} \text{ V} \quad , \quad t \geq 0$$

* <u>Now</u>, use initial condition :

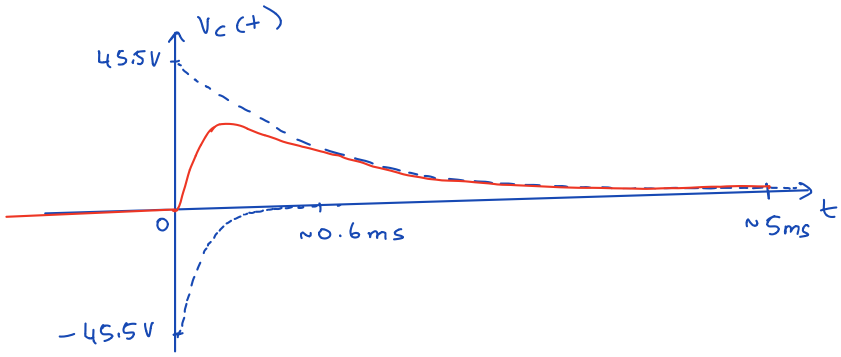
$V_c(0) = 0$

$V_c(0) = K + 45.5 = 0 \quad \Rightarrow \quad K = -45.5$

* So,

$$V_c(t) = -45.5\, e^{-t/\tau} + 45.5\, e^{-1000t} \text{ V} \quad , \quad t \geq 0$$

$\tau = 0.12\text{ms}$ $\quad\quad$ $\underbrace{\tau_2 = 1\text{ms}}_{\text{dominant exponential}}$




## Response to Sinusoidal Inputs for First Order Circuits

* Here, the source term in the Thevenin/Norton equivalent circuit is sinusoidal.

$$\frac{dx}{dt} + \frac{1}{\tau} x = \frac{1}{\tau} x_\infty \quad , \quad x_\infty = V_A \cdot \cos(\omega t)$$

$$x(t) = x_N(t) + x_F(t)$$

* Natural Response (homogeneous solution) : $\quad \dfrac{dx_N}{dt} + \dfrac{1}{\tau} x_N = 0$

$$x_N(t) = K \cdot e^{-t/\tau}$$

* Forced Response : $\frac{dx_F}{dt} + \frac{1}{\tau} x_F = \frac{1}{\tau} x_\infty = \frac{1}{\tau} V_A . \cos(\omega t)$
(particular form)

$$x_F(t) = V_F . \cos(\omega t + \phi)$$
$$= V_F . \cos(\omega t) . \cos(\phi) - V_F . \sin(\omega t) . \sin(\phi)$$
$$= a . \cos(\omega t) + b . \sin(\omega t)$$

where $a = V_F . \cos(\phi)$ , $b = -V_F . \sin(\phi)$

Insert into ODE :

$$\frac{dx_F}{dt} + \frac{1}{\tau} x_F = \frac{1}{\tau} V_A . \cos(\omega t)$$

$$\left(b\omega + \frac{a}{\tau}\right)\cos(\omega t) + \left(-a\omega + \frac{b}{\tau}\right)\sin(\omega t) = \frac{V_A}{\tau} \cos(\omega t)$$

* For both sides to match at all $t$, cosine terms and sine terms must match on both sides.

$$b\omega + \frac{a}{\tau} = \frac{V_A}{\tau} \quad , \quad \underbrace{-a\omega + \frac{b}{\tau} = 0}_{b = \tau a \omega}$$

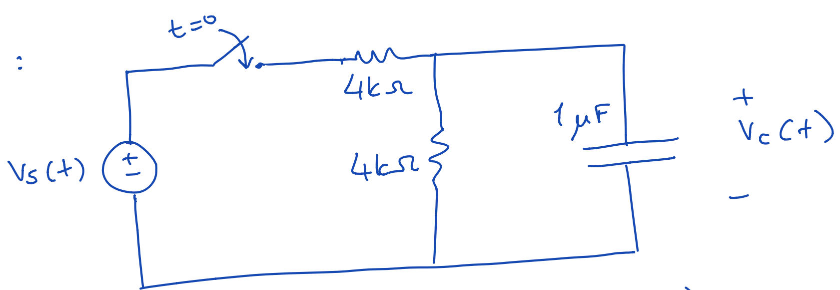
$$a\tau\omega^2 + \frac{a}{\tau} = \frac{V_A}{\tau}$$

$$a(\tau^2\omega^2 + 1) = V_A \Rightarrow a = \frac{V_A}{1 + (\omega\tau)^2} \quad , \quad b = \frac{\omega\tau . V_A}{1 + (\omega\tau)^2}$$

* Then , $V_F = \sqrt{a^2 + b^2}$
$$= \frac{V_A . \sqrt{1 + (\omega\tau)^2}}{1 + (\omega\tau)^2} = \frac{V_A}{\sqrt{1 + (\omega\tau)^2}}$$

$$\tan\phi = \frac{-b}{a} = -\omega\tau \Rightarrow \phi = \tan^{-1}(-\omega\tau)$$

<u>Example</u> :




$V_s(t) = 20 . \sin(1000t) V$

$$V_T(t) = \frac{4k}{4k + 4k} . V_s(t) = 10 . \sin(1000t) V$$

$$R_T = 4k // 4k = 2k\Omega$$

* For $t < 0$ , $V_c(t) = 0$ . $V_c(0) = 0$

* $\tau = R_T . C = 2.10^3 . 10^{-6} = 2\ ms = 2.10^{-3}\ s$

* $x_N(t) = K . e^{-t/\tau} = K . e^{-500t}$

* $x_F(t) = V_F . \cos(\omega t + \phi) = a\cos(\omega t) + b\sin(\omega t)$ , $\omega = 1000\ rps$

$\dot{x}_F(t) + \frac{1}{\tau} x_F = \frac{1}{\tau} V_A . \sin(\omega t)$

$\left(b\omega + \frac{a}{\tau}\right)\cos(\omega t) + \left(-a\omega + \frac{b}{\tau}\right)\sin(\omega t) = \frac{1}{\tau}\underbrace{V_A . \sin(\omega t)}_{V_T(t)}$

$b\omega + \frac{a}{\tau} = 0$ , $-a\omega + \frac{b}{\tau} = \frac{1}{\tau} V_A$

$1000b + 500a = 0$ , $-1000a + 500b = 500 \times 10 = 5000$

$a = -2b \implies 2000b + 500b = 5000$

$b = 2 \implies a = -4$

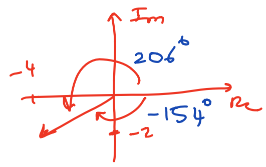
$x_F(t) = -4\cos(1000t) + 2\sin(1000t)\ V$

* Add : $V_c(t) = Ke^{-500t} - 4\cos(1000t) + 2.\sin(1000t)\ V$

$a = V_F . \cos\phi = -4$

$b = -V_F . \sin\phi = 2$

$V_F = \sqrt{a^2 + b^2} = 4.47$ , $\tan\phi = \frac{-b}{a} = \frac{-2}{-4}$




So, $V_c(t) = K . e^{-500t} + 4.47\cos(1000t - 154°)\ V$

* Use initial condition to find K :

$V_c(0) = 0 \implies V_c(0) = K - 4\cos(0) + 2\sin(0) = 0$

$K - 4 = 0 \implies K = 4$

$\tau = 2ms$ $\quad T = \frac{2\pi}{\omega} = \frac{2\pi}{1000} = 6.28\ ms$

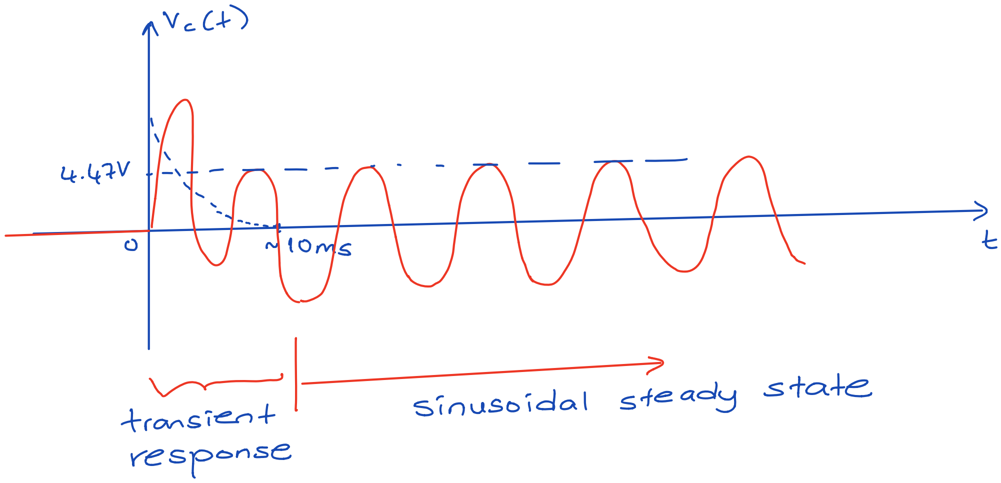
* Finally, $\boxed{V_c(t) = 4e^{-500t} + 4.47 . \cos(1000t - 154°)V}$ , $t \geq 0$

* Note that as $t \to \infty \implies V_c(t) = 4.47\cos(1000t - 154°)V$

This is called sinusoidal steady state.

Natural response decays to zero.

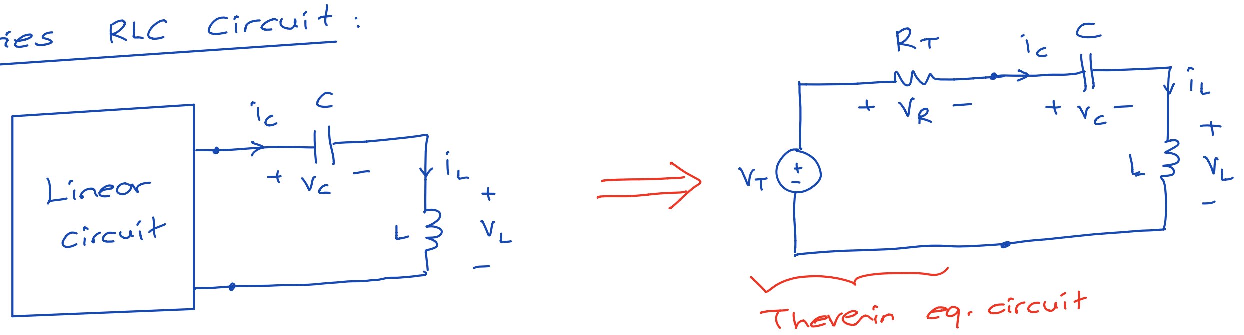
$\to$ can be found using phasor analysis.




<u>SECOND ORDER CIRCUITS</u> : contain a total of 2 capacitor/inductor combinations : 2 capacitors , 2 inductors, or 1 capacitor + 1 inductor.

* Two important cases : series and parallel RLC circuits

<u>Series RLC Circuit</u> :




<u>KVL</u> : $V_T = V_R + V_c + V_L$ , <u>KCL</u> : $i_R = i_c = i_L$

* $V_c$ and $i_L$ are state variables . We can use initial conditions for these variables, since they are continuous functions. $\Rightarrow$ Express circuit equations in terms of $V_c$ and $i_L$.

* $i_c = C \dfrac{dV_c}{dt} = i_L \implies \dfrac{dV_c}{dt} = \dfrac{i_L}{C}$ (Eqn. 1)

* $V_T = R_T \cdot i_R + V_c + V_L = R_T \cdot i_L + V_c + L \dfrac{di_L}{dt}$

$$\frac{di_L}{dt} = -\frac{1}{L} V_c - \frac{R_T}{L} i_L + \frac{1}{L} V_T \qquad \text{(Eqn. 2)}$$

* Eqn. 1 and Eqn. 2 are called state equations. They are first-order coupled ODE. They can be written in matrix form:

$$\frac{d}{dt}\begin{bmatrix} V_C \\ i_L \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & R_T/L \end{bmatrix}\begin{bmatrix} V_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} V_T$$

General form : $\dot{x} = Ax + bu$

matrix ($A$), vector ($b$), input ($u$), state variable vector ($x$)

* Alternative form: scalar second order ODE. Differentiate Eqn. 1, and use Eqn. 1 and 2 to eliminate $i_L$.

$$\ddot{V}_C = \frac{di_L}{dt}\cdot\frac{1}{C} \quad\Rightarrow\quad \frac{di_L}{dt} = C\ddot{V}_C$$

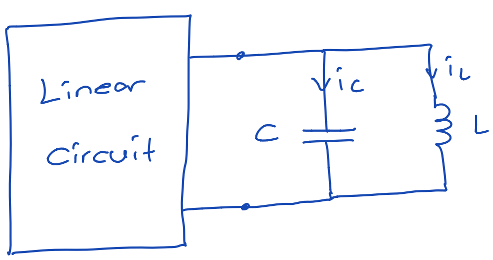
also, $i_L = C\dot{V}_C$

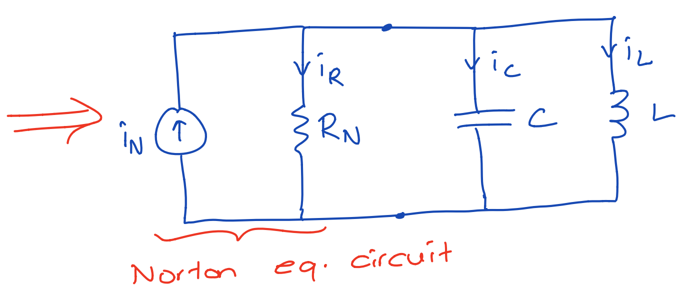
$$C\ddot{V}_C = -\frac{1}{L}V_C - \frac{R_T}{L}C\dot{V}_C + \frac{1}{L}V_T$$

$$\boxed{\ddot{V}_C + \frac{R_T}{L}\dot{V}_C + \frac{1}{LC}V_C = \frac{1}{LC}V_T}$$

* Usually, we are given $V_C(0)$ and $i_L(0)$. We need $V_C(0)$ and $\dot{V}_C(0)$. $\dot{V}_C(0)$ comes from Eqn. 1 : $\dot{V}_C(0) = \frac{i_L(0)}{C}$

Parallel RLC Circuit :




KCL : $i_N = i_R + i_C + i_L$ , KVL : $V_R = V_C = V_L$

* Again, $V_C$ and $i_L$ are state variables.

* $v_L = L\dfrac{di_L}{dt} = v_C \quad \Rightarrow \quad \dot{i}_L = \dfrac{v_C}{L}$ (Eqn. 1)

* $i_N = \dfrac{v_R}{R_N} + C\dot{v}_C + i_L = \dfrac{v_C}{R_N} + C\dot{v}_C + i_L$

$$\dot{v}_C = -\frac{1}{R_N C} v_C - \frac{1}{C} i_L + \frac{1}{C} i_N \qquad \text{(Eqn. 2)}$$

* Eqn. 1 and 2 are state equations. In matrix form:

$$\frac{d}{dt}\begin{bmatrix} v_C \\ i_L \end{bmatrix} = \begin{bmatrix} -1/R_N C & -1/C \\ 1/L & 0 \end{bmatrix}\begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 1/C \\ 0 \end{bmatrix} i_N$$

* Alternative form: scalar second order ODE. Differentiate Eqn. 1, and use Eqn. 1 and 2 to eliminate $v_C$.

$$\ddot{i}_L = \frac{\dot{v}_C}{L} \quad \Rightarrow \quad \dot{v}_C = L\,\ddot{i}_L$$

$$L\,\ddot{i}_L = -\frac{1}{R_N C}\cdot L\,\dot{i}_L - \frac{1}{C} i_L + \frac{1}{C} i_N$$

$$\boxed{\ddot{i}_L + \frac{1}{R_N C}\dot{i}_L + \frac{1}{LC} i_L = \frac{1}{LC} i_N}$$

* Usually, we are given $v_C(0)$ and $i_L(0)$.
We need $i_L(0)$ and $\dot{i}_L(0)$.
$\dot{i}_L(0)$ comes from Eqn. 1 : $\dot{i}_L(0) = \dfrac{v_C(0)}{L}$

* <u>Solution</u> = zero-input response + zero-state response

## Zero-Input Response : ($V_T = 0$ , $i_N = 0$)

In that case, the two equations can be written in the following form:

$$\ddot{x} + a\dot{x} + bx = 0$$

* Series RLC : $x = V_C$ , $a = \frac{R_T}{L}$ , $b = \frac{1}{LC}$

* Parallel RLC : $x = i_L$ , $a = \frac{1}{R_N \cdot C}$ , $b = \frac{1}{LC}$

* $x(0)$ and $\dot{x}(0)$ are given.

* Assume solution of the form: $x(t) = e^{st}$

$$s^2 . e^{st} + a.s.e^{st} + b.e^{st} = 0$$

$$(s^2 + as + b)e^{st} = 0 \Rightarrow s^2 + as + b = 0$$

* This is called the <u>characteristic polynomial</u> of the circuit. Here, $s$ could be real or complex.

* $s^2 + as + b = 0 \Rightarrow$ roots are $s_1$ and $s_2$

Roots : $s_{1,2} = \dfrac{-a \pm \sqrt{a^2 - 4b}}{2}$

## Case 1 : Distinct Roots, $s_1 \neq s_2$

Solution : $x(t) = K_1 . e^{s_1 t} + K_2 . e^{s_2 t}$

$K_1, K_2$ are constants (real or complex) to be found from the initial conditions.

$$x(0) = K_1 + K_2$$

$$\dot{x}(t) = K_1 . s_1 . e^{s_1 t} + K_2 s_2 . e^{s_2 t}$$

$$\dot{x}(0) = K_1 . s_1 + K_2 . s_2$$

2 equations
2 unknowns ($K_1, K_2$)

* $K_1 = \dfrac{s_2 x(0) - \dot{x}(0)}{s_2 - s_1}$ , $K_2 = \dfrac{s_1 x(0) - \dot{x}(0)}{s_1 - s_2}$

* What if the roots are complex?

$s = s_1 = \overline{s_2} = -\alpha + j\beta \implies K_1 = \overline{K_2} = \frac{r}{2} e^{-j\theta}$

Then,

$$x(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} = K_1 e^{s_1 t} + \overline{K_1} e^{\overline{s_1} t}$$

$$= 2\,\mathrm{Re}\left\{ K_1 . e^{s_1 t} \right\}$$

$$= 2\,\mathrm{Re}\left\{ \frac{r}{2} e^{-j\theta} . e^{(-\alpha + j\beta)t} \right\}$$

$$= 2 . \mathrm{Re}\left\{ \frac{r}{2} e^{-\alpha t + j(\beta t - \theta)} \right\}$$

$$x(t) = r . e^{-\alpha t} . \cos(\beta t - \theta)$$

* Need to find $r$ and $\theta$ by using initial conditions.

$$x(0) = r.\cos(\theta)$$

$$\dot{x}(t) = -\alpha . r . e^{-\alpha t} . \cos(\beta t - \theta) - r . e^{-\alpha t} . \beta . \sin(\beta t - \theta)$$

$$\dot{x}(0) = -\alpha r \cos(\theta) + \beta . r . \sin(\theta)$$

→ 2 equations, 2 unknowns $(r, \theta)$

$$r\cos(\theta) = x(0) \quad , \quad r.\sin(\theta) = \frac{\dot{x}(0) + \alpha x(0)}{\beta}$$

* Alternatively, we could find the solution as:

$$x(t) = r . e^{-\alpha t} . \cos(\theta) . \cos(\beta t) + r e^{-\alpha t} . \sin(\theta) . \sin(\beta t)$$

$$= C_1 . e^{-\alpha t} \cos(\beta t) + C_2 e^{-\alpha t} . \sin(\beta t)$$

where $C_1 = r.\cos(\theta)$, $C_2 = r.\sin(\theta)$

$$x(0) = C_1$$

$$\dot{x}(0) = -\alpha C_1 + \beta C_2 \implies C_2 = \frac{\dot{x}(0) + \alpha . x(0)}{\beta}$$

Then, $r = \sqrt{C_1^2 + C_2^2}$

$$\theta = \tan^{-1}\left(\frac{C_2}{C_1}\right)$$

**Case 2** : Repeated roots : $s_1 = s_2 = s$

**Solution** : $x(t) = K_1 e^{st} + K_2 . t . e^{st}$

$x(0) = K_1$

$\dot{x}(0) = s K_1 + K_2 \implies K_2 = \dot{x}(0) - s\,x(0)$

**Stability of Solutions** :

$s^2 + as + b = 0$

$$s_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Series RLC : $a = \frac{R_T}{L}$ , $b = \frac{1}{LC}$

Parallel RLC : $a = \frac{1}{R_N . C}$ , $b = \frac{1}{LC}$

**Note** : $R_T$ ($R_N$) can be negative if we have active components (dependent sources).

* $s_1 \neq s_2 \longrightarrow x_N(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$
* $s_1 = s_2 = s \longrightarrow x_N(t) = K_1 e^{st} + K_2 . t . e^{st}$
* If $Re\{s_i\} < 0 \implies x_N(t) \to 0$ as $t \to \infty$

$\implies$ **Stable case**

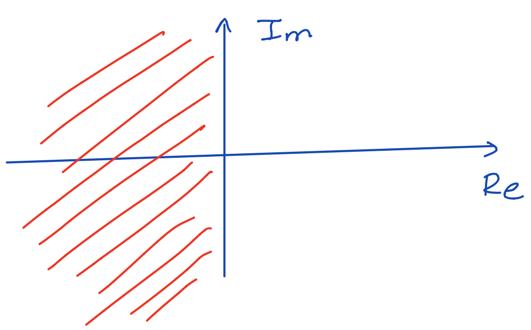
In this case, if the independent sources are bounded, solutions are bounded.

* If $Re\{s_i\} \geq 0 \implies x_N(t) \to \infty$ as $t \to \infty$

$\implies$ **Unstable case**

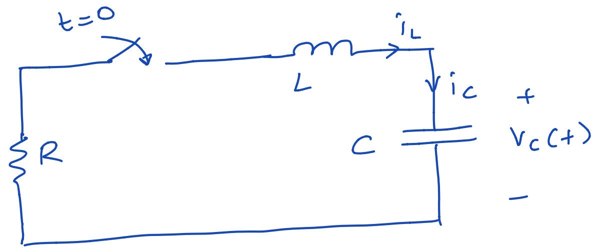
In this case, solutions are unbounded.

For practical reasons, we want stable systems.




The roots of the characteristic polynomial should be in the open left half of the complex plane for the **stable case**.

Example : we will further classification of the stable case.




$i_L(0) = 0$

$V_c(0) = 15\ V$

$C = 0.25\ \mu F$

$L = 1\ H$

Find the solution for : a) $R = 8.5\ k\Omega$, b) $R = 4k\Omega$, c) $R = 1k\Omega$

$$\ddot{v}_c + \frac{R_T}{L}\dot{v}_c + \frac{1}{LC}v_c = \frac{1}{LC}v_T = 0$$

* Characteristic polynomial:

$s^2 + as + b = 0$ , $a = \frac{R_T}{L}$ , $b = \frac{1}{LC}$

Roots : $s_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$

Case 1 : $R = 8.5\ k\Omega$

$a = \frac{R_T}{L} = 8500$ , $b = \frac{1}{LC} = 4.10^6$

$\Rightarrow s_1 = -500$ , $s_2 = -8000$

* Two distinct negative roots $\Rightarrow$ stable case

$v_c(t) = K_1 . e^{-500t} + K_2 . e^{-8000t}$

* $v_c(0) = K_1 + K_2 = 15$

* $C \frac{dv_c}{dt} = i_c = i_L$

$\dot{v}_c(0) = \frac{i_L(0)}{C} = 0 \Rightarrow -500K_1 - 8000K_2 = 0$

$K_1 = -16K_2$

Insert back : $-16K_2 + K_2 = 15 \Rightarrow K_2 = -1$ , $K_1 = 16$

* So,

$\tau_2 = \frac{1}{8000} = 0.125\,ms$

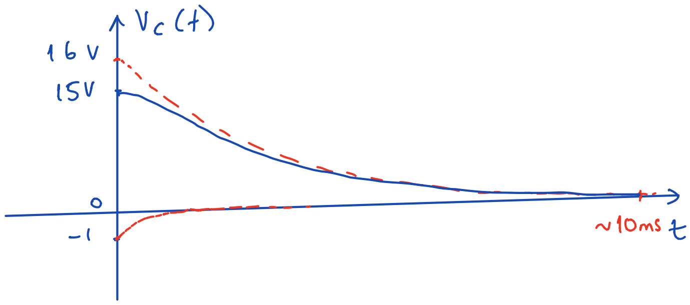
$v_c(t) = 16\,e^{-500t} - e^{-8000t}\ V$ , $t \geq 0$

$\tau_1 = \frac{1}{500} = 2ms$

* Then,

$C\frac{dv_c}{dt} = i_c(t) = i_L(t) = 0.25 \times 10^{-6} \cdot \left(-16 \times 500\,e^{-500t} + 8000\,e^{-8000t}\right)$

$i_L(t) = -2\,e^{-500t} + 2e^{-8000t}\ mA$ , $t \geq 0$




* This is called Overdamped Case : two distinct real negative roots

Case 2 : $R = 4k\Omega$

$$a = \frac{R_T}{L} = 4000 \quad , \quad b = \frac{1}{LC} = 4.10^6$$

$$s_1 = s_2 = -2000 \quad \text{(repeated negative roots)}$$

$$V_C(t) = K_1 \cdot e^{-2000t} + K_2 \cdot t \cdot e^{-2000t}$$

* $V_c(0) = K_1 = 15$

$\dot{V}_c(0) = \frac{i_L(0)}{C} = 0 \Rightarrow -2000K_1 + K_2 = 0 \Rightarrow K_2 = 30000$

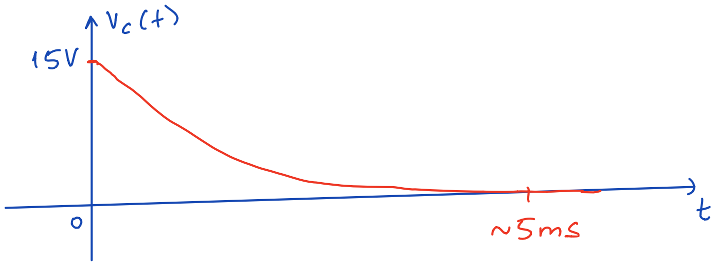
* $V_c(t) = (15 + 30000\,t)\,e^{-2000t}$ V , $t \geq 0$

$\tau = \frac{1}{2000} = 0.5$ ms

* $i_L(t) = i_c(t) = C\frac{dV_c}{dt}$

$$= 0.25 \times 10^{-6}\left[3.10^4 . e^{-2000t} - 2000\left(15 + 3.10^4 t\right)e^{-2000t}\right]$$

$$= -15 . t . e^{-2000t} \text{ A} \quad , \quad t \geq 0$$




* This is called the Critically Damped Case : two repeated real negative roots.

Case 3 : $R = 1k\Omega$

$$a = \frac{R_T}{L} = 1000 \quad , \quad b = \frac{1}{LC} = 4.10^6$$

$$s_{1,2} = -500 \pm j500\sqrt{15} = -\alpha \pm j\beta$$

$$\alpha = 500, \quad \beta = 500\sqrt{15}$$

$$V_c(t) = K.\, e^{-500t} \cos(500\sqrt{15}\, t - \phi)$$

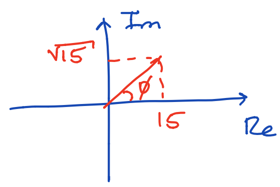
$$= K_1\, e^{-500t} \cos(500\sqrt{15}t) + K_2\, e^{-500t} . \sin(500\sqrt{15}\, t)$$

where $K_1 = K\cos(\phi)$ , $K_2 = K.\sin(\phi)$

* $V_c(0) = K_1 = 15$
* $\dot{V}_c(0) = \frac{i_L(0)}{C} = 0 \Rightarrow -500K_1 + 500\sqrt{15}\, K_2 = 0 \Rightarrow K_2 = \sqrt{15}$

* So,

$$V_c(t) = 15e^{-500t} \cos(500\sqrt{15}t) + \sqrt{15}\, e^{-500t} . \sin(500\sqrt{15}t) \text{ V} , \quad t \geq 0$$

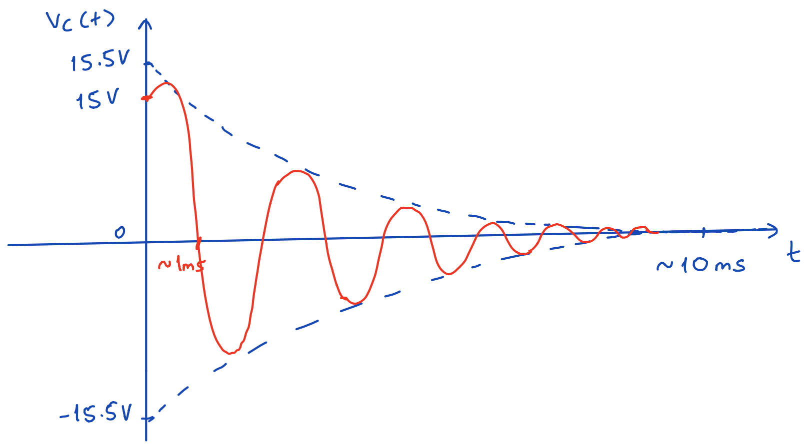



$$K\cos(\phi) = K_1 = 15$$
$$K.\sin(\phi) = K_2 = \sqrt{15}$$
$$K = \sqrt{K_1^2 + K_2^2} = 15.5 \quad , \quad \tan(\phi) = \frac{\sqrt{15}}{15}$$

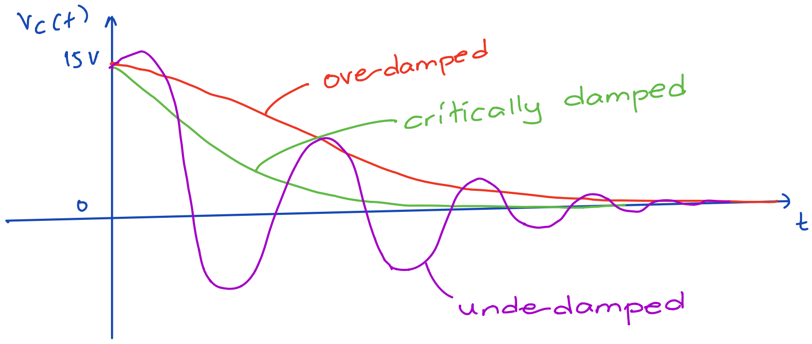
$$\phi = 0.25 \text{ rad} = 14.5^\circ$$

* So,

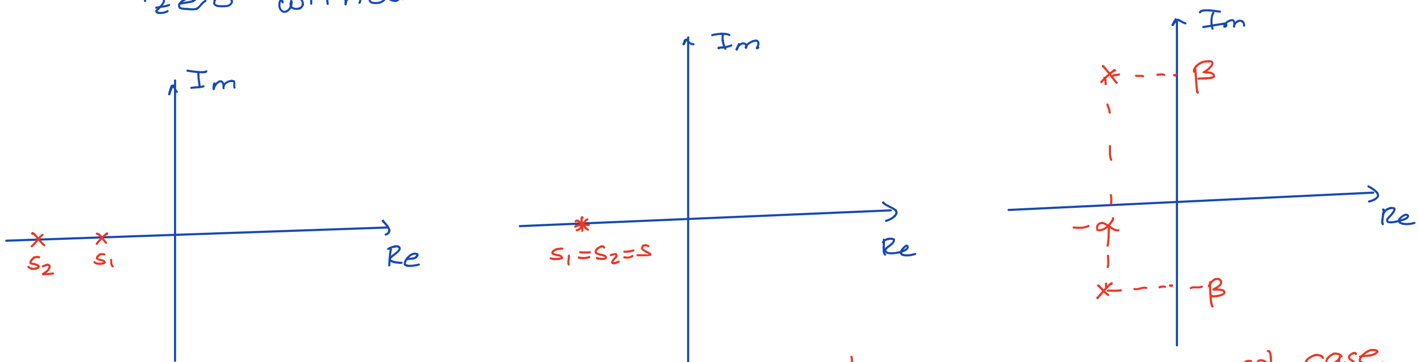
$$V_c(t) = 15.5\, \underbrace{e^{-500t}}_{\tau = \frac{1}{500} = 2\text{ ms}} . \cos(500\sqrt{15}\, t - 0.25) \text{ V} , \quad t \geq 0$$




* This is called the <u>Under-damped Case</u> : two complex conjugate roots with negative real parts.




* Critically damped case provides the quickest approach to zero without oscillations.




Overdamped    Critically Damped    Underdamped case

all three cases are stable solutions.

## Step Response of Second Order Circuits

* Sources are DC : $V_T$ and $i_N$ are constants

In this case, the ODE becomes :

$$\ddot{x}(t) + a\dot{x}(t) + b\,x(t) = A$$

* $x(t) = x_N(t) + x_F(t)$

* $x_N(t)$ : natural response, which can be found as before using the characteristic polynomial.

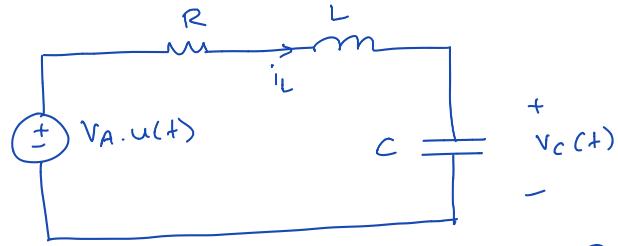
* $x_F(t)$ : forced response

Solution : $\quad x_F(t) = B$

$$\dot{x}_F = \ddot{x}_F = 0$$

Insert into ODE : $\quad bB = A \quad \Rightarrow \quad x_F = B = \dfrac{A}{b}$

Example :




$V_A = 10V$ , $C = 0.5 \mu F$ , $L = 2H$ , $R = 1k\Omega$

* for $t<0$ , $V_T = 0$ $\Rightarrow$ $i_L(t) = 0$ , $t<0$
$v_c(t) = 0$ , $t<0$

$v_c(0) = 0$ , $i_L(0) = 0$

* for $t \geq 0$ , $V_T = 10\ V$

$$\ddot{v}_c + \frac{R_T}{L}\dot{v}_c + \frac{1}{LC}v_c = \frac{1}{LC}V_T$$

$$\ddot{v}_c + 500\dot{v}_c + 10^6 v_c = 10^6 . 10 = 10^7$$

* Natural response : characteristic polynomial : $s^2 + 500s + 10^6 = 0$
Roots : $s_{1,2} = -250 \pm j968$ $\Rightarrow$ underdamped case

$$x_N(t) = K_1 e^{-250t} . \cos(968t) + K_2 e^{-250t} . \sin(968t)$$
$$= K . e^{-250t} . \cos(968t - \phi)$$

* Forced response : $x_F = B$
$10^6 . B = 10^7$ $\Rightarrow$ $x_F = B = 10\ V$

* So, $v_c(t) = x_N(t) + x_F$
$= 10 + K_1 e^{-250t} \cos(968t) + K_2 e^{-250t} . \sin(968t)\ V$, $t \geq 0$

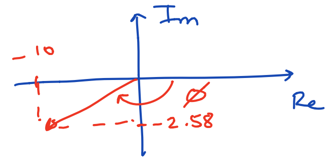
* Now , use initial conditions :
$v_c(0) = 10 + K_1 = 0$ $\Rightarrow$ $K_1 = -10$
$\dot{v}_c(0) = \frac{i_L(0)}{C} = 0$ $\Rightarrow$ $-250K_1 + 968K_2 = 0$
$\Rightarrow$ $K_2 = -2.58$

$$v_c(t) = 10 - 10e^{-250t}\cos(968t) - 2.58e^{-250t} . \sin(968t)\ V, \quad t \geq 0$$

$v_c(t) = 10 + K.\cos(968t - \phi)$

where $K_1 = K\cos(\phi) = -10$

$K_2 = K.\sin(\phi) = -2.58$




$K = \sqrt{K_1^2 + K_2^2} = 10.12$

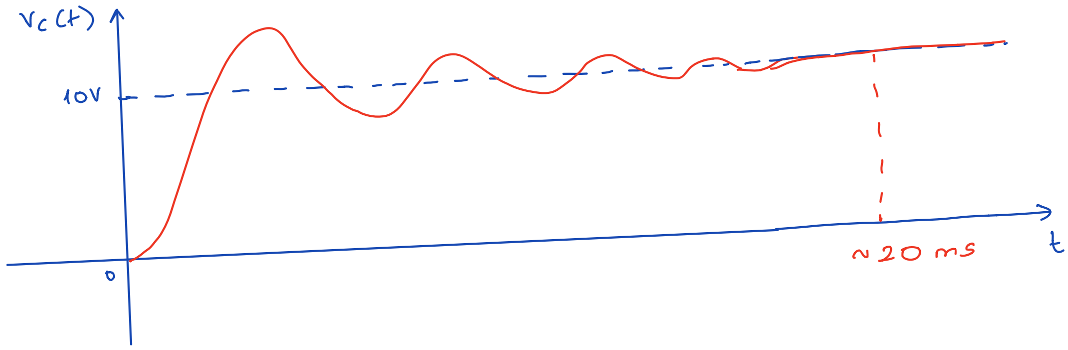
$\phi = \tan^{-1}\left(\frac{-2.58}{-10}\right)$

$= -165.5° = -2.89$ rad

$\tau = \frac{1}{250} = 4ms$

* So,

$v_c(t) = 10 + 10.12\, e^{-250t}.\cos(968t + 2.89)$ V, $t \geq 0$




<u>Note</u>: The characteristic polynomial can be written as:

$s^2 + as + b = 0$

$s^2 + 2\xi\omega_0 s + \omega_0^2 = 0$

( For example, for series RLC: $\omega_0 = \frac{1}{\sqrt{LC}}$, $\xi = \frac{1}{2}\sqrt{\frac{C}{L}}.R_T$ )

$\xi$: damping ratio, $\omega_0$: undamped natural frequency

* $s_{1,2} = \frac{-2\xi\omega_0 \pm \sqrt{4\xi^2\omega_0^2 - 4\omega_0^2}}{2} = \omega_0\left(-\xi \pm \sqrt{\xi^2 - 1}\right)$

* $\xi > 1 \Rightarrow$ Over-damped case (in series RLC, when $R_T$ is large)

* $\xi = 1 \Rightarrow$ Critically damped case (in series RLC, $R_T = 2\sqrt{\frac{L}{C}}$)

* $0 < \xi < 1 \Rightarrow$ under-damped case (in series RLC, when $R_T$ is small)

* $\xi = 0 \Rightarrow$ lossless case. In this case, the natural response is a pure sinusoid at $\omega_0$. (in series RLC, when $R_T = 0$)