# CHAPTER 2 : BASIC CIRCUIT ANALYSIS

## Connection Constraints : KCL, KVL

M elements $\longrightarrow$ M currents and M voltages $\Longrightarrow$ 2M unknowns

$\Rightarrow$ solve for constraints that relate these parameters.

## Kirchhoff's Current Law (KCL) based on Nodes

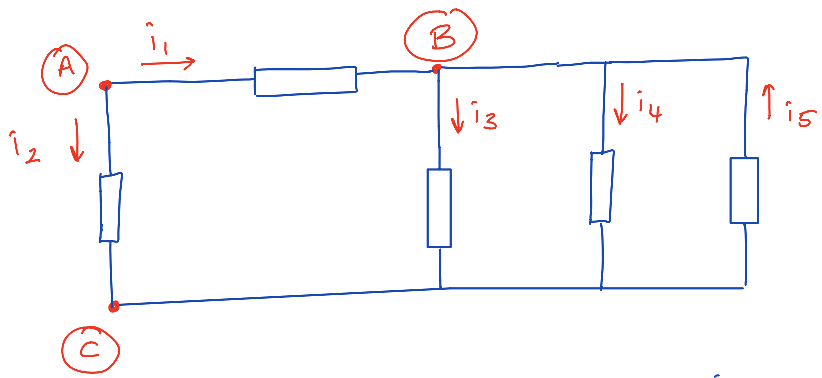
Algebraic sum of currents entering any node is zero.

Alternatively, sum of currents leaving any node is zero.

OR, sum of entering currents = sum of leaving currents.

Physical Reason : conservation of charge $\Rightarrow$ rate of change of total charge at any node is zero.

Example :




| | entering | leaving |
|---|---|---|
| A : | $-i_1 - i_2 = 0$ | $i_1 + i_2 = 0$ |
| B : | $i_1 - i_3 - i_4 + i_5 = 0$ | $-i_1 + i_3 + i_4 - i_5 = 0$ |
| C : | $i_2 + i_3 + i_4 - i_5 = 0$ | $-i_2 - i_3 - i_4 + i_5 = 0$ |

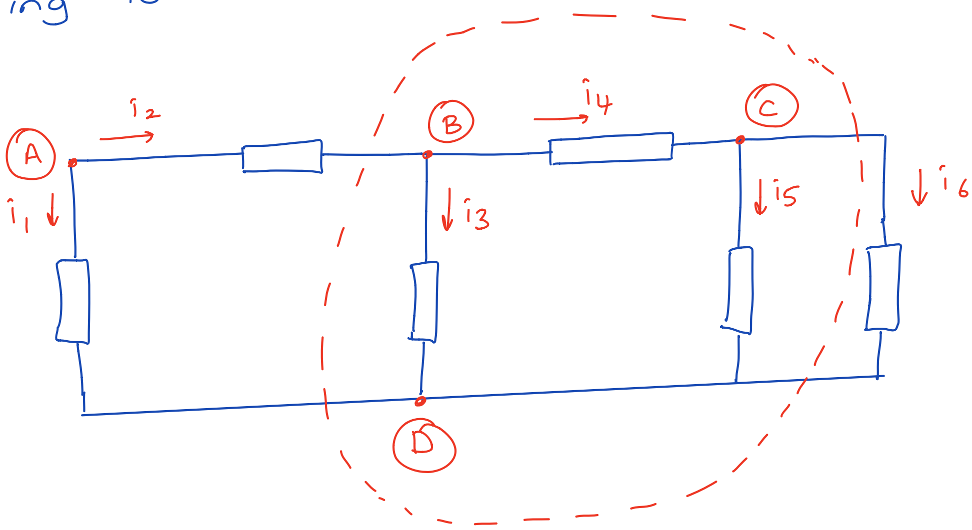
**Linear Dependence** : A set of equations are linearly dependent if a linear combination of them is exactly zero $\Rightarrow$ some equations can be derived from the others, and hence are unnecessary.

* In the example above, $A + B + C = 0$

**Conclusion** : All node equations together are linearly dependent.

* Any $(n-1)$ of KCL node equations are linearly independent, where $n$ is the # of nodes.

* An alternative view of KCL: Take any spherical volume, place it inside the circuit. Algebraic sum of currents entering to the volume is zero.




KCL: $i_1 + i_2 - i_6 + i_6 = 0$

Physical Reason: conservation of charge in any volume.

an shrink its size

* Place the volume around any node
$\Longrightarrow$ KCL node equations.

Matrix Formulation of KCL equations.

$$A\, i = 0$$

incidence matrix

vector containing all currents

$$\left.\begin{aligned} A &: -i_1 - i_2 = 0 \\ B &: i_1 - i_3 - i_4 + i_5 = 0 \\ C &: i_2 + i_3 + i_4 - i_5 = 0 \end{aligned}\right\} \quad \underbrace{\begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \end{bmatrix}}_{i} = 0$$

* Due to redundancy, we may delete any one row of matrix A $\Rightarrow$ equivalent to assigning one node as ground, and not writing KCL there.

# Kirchhoff's Voltage Law (KVL) based on Loops

**Loop** : A closed path formed by tracing through an ordered sequence of nodes without passing any node more than once

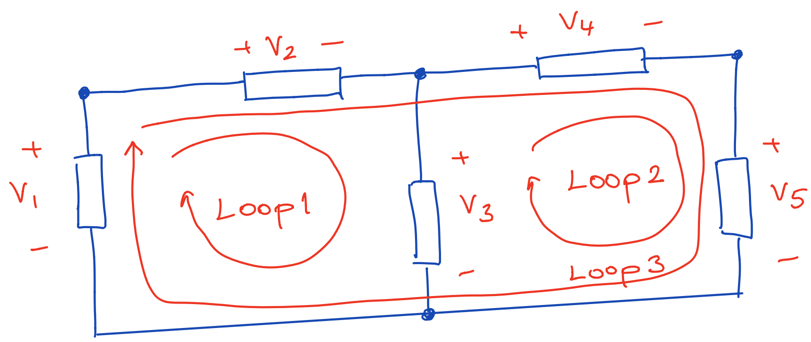
**Closed path** : starts and ends at the same node.

**Closed node sequence** : Same as loop, without the "passing any node more than once" requirement.
Union of loops.

**KVL** : Algebraic sum of voltages around any loop (or any closed node sequence) is zero.

**Physical Reason** : Due to conservative field, the potential difference between point A and A is zero.
The work done by moving a unit charge from point A to A is zero, no matter the path taken.

**Example** :




**Loop 1** : $V_2 + V_3 - V_1 = 0$

**Loop 2** : $V_4 + V_5 - V_3 = 0$

**Loop 3** : $V_2 + V_4 + V_5 - V_1 = 0$

$b = 5$
$n = 4$
$(b - n + 1) = 2$

**Question** : Do we need all of these loop equations?
Are they linearly independent?

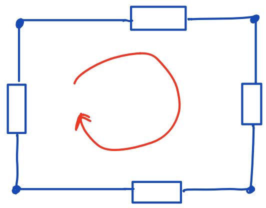
$$\text{Loop 1} + \text{Loop 2} - \text{Loop 3} = 0$$

* All loop equations together are linearly dependent.

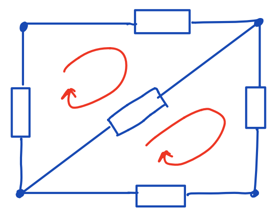
* $b$ : number of two terminal elements
  $n$ : number of nodes

  There are $(b-n+1)$ linearly independent loop equations.
  ( # of meshes, as we will see later).

<u>Examples</u> :



$b=4$
$n=4$
$(b-n+1)=1$



$b=5$
$n=4$
$(b-n+1)=2$

<u>Matrix Formulation</u> of KVL equations :

$$\underbrace{B}_{\text{loop matrix}} \underbrace{v}_{\text{vector containing all voltages}} = 0$$

<u>Loop 1</u> : $v_2 + v_3 - v_1 = 0$

<u>Loop 2</u> : $v_4 + v_5 - v_3 = 0$

<u>Loop 3</u> : $v_2 + v_4 + v_5 - v_1 = 0$

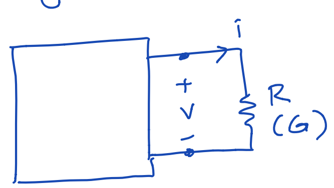
$$\underbrace{\begin{bmatrix} -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ -1 & 1 & 0 & 1 & 1 \end{bmatrix}}_{\substack{b-n+1 = 5-4+1=2 \\ \text{one row is redundant}}} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = 0$$

<u>Element Constraints</u> :

* Algebraic or differential relations between terminal voltage(s) and current(s) of a device.

* <u>Resistors</u> : algebraic relation between $v$ and $i$.

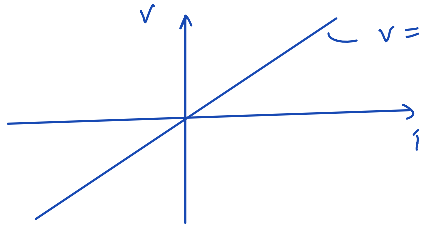
* <u>Linear resistors</u> : algebraic relation is linear : $av + bi = 0$




* $v = Ri$
  $v - Ri = 0$

  R : resistance in units of Ohm ($\Omega$)
  Ohm = V/A

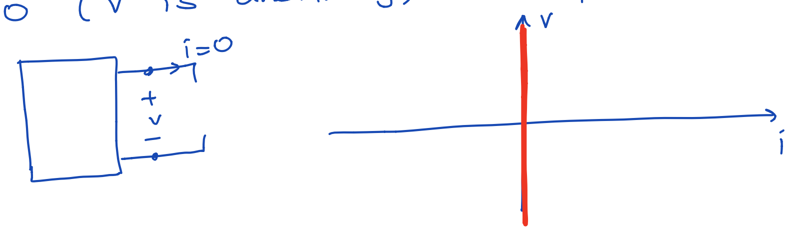
* $i = GV$
  $GV - i = 0$ } G : conductance in units of Siemens (S)
  Siemens = A/V



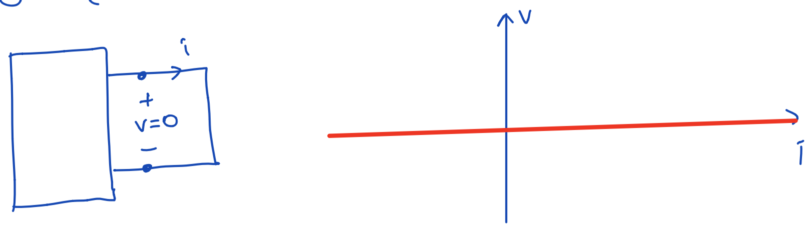

$R = \frac{1}{G}$

Special cases:

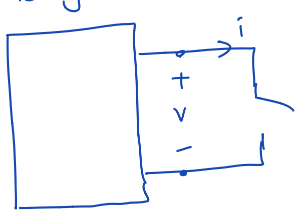
* $i = 0$ (v is arbitrary) → open circuit




* $v = 0$ (i is arbitrary) → short circuit

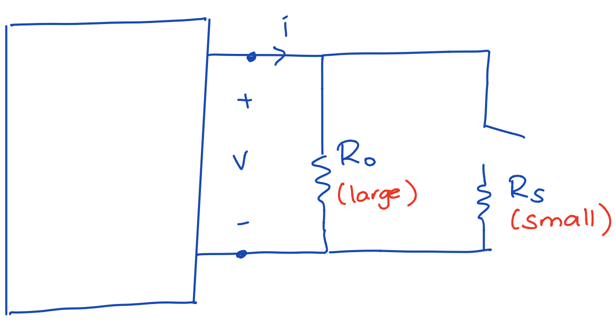



* Ideal Switch : changes between open and short circuits




* Practical Switch : has a small resistance when short and large resistance when open

model:

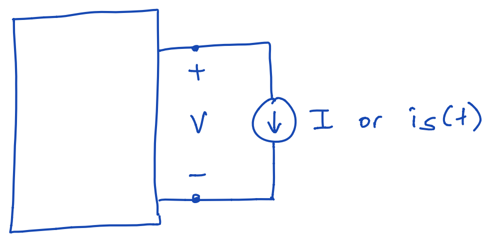


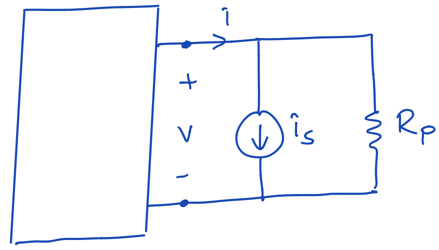

$R_{switch} = \begin{cases} R_o & , \text{open} \\ R_s // R_o \approx R_s & , \text{short} \end{cases}$

**Sources**: Generate power for the circuit

**ideal sources** | **practical sources**

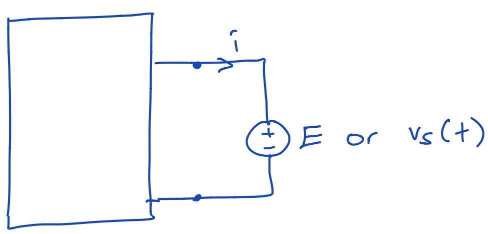
Current source

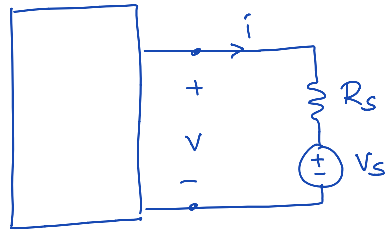



$$i = i_s + \frac{V}{R_p}$$

voltage source




$$V = R_s \cdot i + V_s$$

**Combined Constraints**:

* When we write KCL + KVL + Element constraints ⇒ Combined constraints

* b : number of 2-terminal elements
n : number of nodes

* Each element has its v and i are unknowns
⇒ 2b unknowns
⇒ we need 2b linearly independent equations

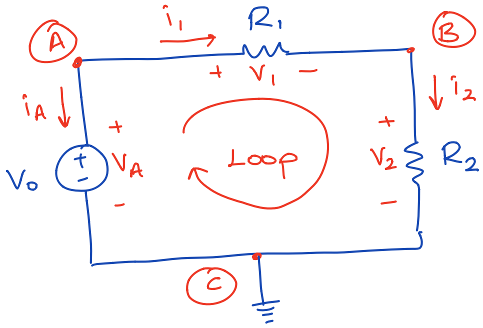
**How many equations do we have?**

KCL equations : $n-1$ (# of nodes, except ground node)
KVL equations : $b-n+1$ (# of meshes)
\+ Element Constraints : $b$

Total Equations : $2b$

* If all of these equations are linearly independent
⇒ circuit has a unique solution.

**Example :**




$n = 3$
$b = 3$

* Unknowns : $V_A,\ i_A,\ V_1,\ i_1,\ V_2,\ i_2 \Rightarrow 6$

* KCL equations : $\left.\begin{array}{l} A:\ i_A + i_1 = 0 \\ B:\ -i_1 + i_2 = 0 \end{array}\right\}$ (n-1) equations

* KVL equations : Loop : $\left. V_1 + V_2 - V_A = 0 \right\}$ (b-n+1) equations

* Element Constraints :
  - DA : $V_A = V_0$
  - D1 : $V_1 = R_1 i_1 \Rightarrow V_1 - R_1 i_1 = 0$
  - D2 : $V_2 = R_2 i_2 \Rightarrow V_2 - R_2 i_2 = 0$

**Tableau Equations :**

* write KCL + KVL + Element Constraints in matrix form :

* KCL : $Ai = 0$

* KVL : $Bv = 0$

* Element constraints : $Mv + Ni = u$

  depend on resistors

  vector that depends on sources

$$\begin{bmatrix} B & 0 \\ 0 & A \\ M & N \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix} \Longrightarrow T x = u_s$$

(b-n+1)xb, (n-1)xb, bxb, bxb, bx1, bx1, bx1

2bx2b, 2bx1, 2bx1

* If $T$ is invertible $\Rightarrow x = T^{-1} u_s$

* If $T$ is not invertible $\Rightarrow$ circuit has either no solution or has infinetely many solutions.

## Example : For the previous example :

$$
\begin{bmatrix}
-1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -R_1 & 0 \\
0 & 0 & 1 & 0 & 0 & -R_2
\end{bmatrix}
\begin{bmatrix} v_A \\ v_1 \\ v_2 \\ i_A \\ i_1 \\ i_2 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ V_0 \\ 0 \\ 0 \end{bmatrix}
$$

B, A, M, N, u

* KCL equations : A : $i_A + i_1 = 0$

  B : $-i_1 + i_2 = 0$

* KVL equations : Loop : $v_1 + v_2 - v_A = 0$

* Element Constraints : DA : $v_A = V_0$

  D1 : $v_1 = R_1 i_1 \Rightarrow v_1 - R_1 i_1 = 0$

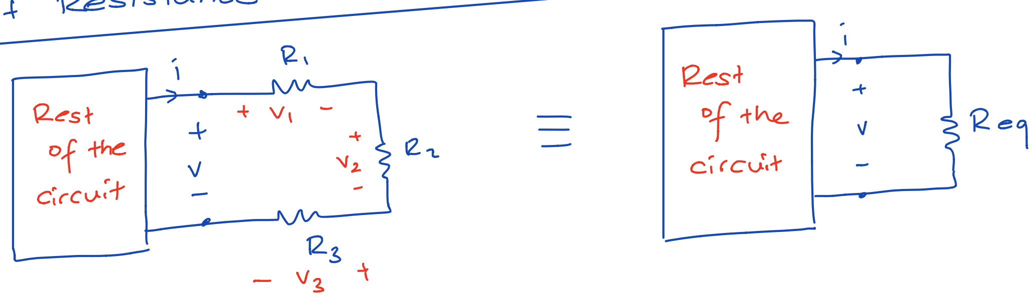
  D2 : $v_2 = R_2 i_2 \Rightarrow v_2 - R_2 i_2 = 0$

## Equivalent Circuits :

* Two circuits are equivalent if they have identical i - v relation between a specified pair of terminals.

* Result : The electrical behaviour inside the rest of the circuit will not change if we replace two identical circuits connected to it.

## Equivalent Resistance : Series Combination

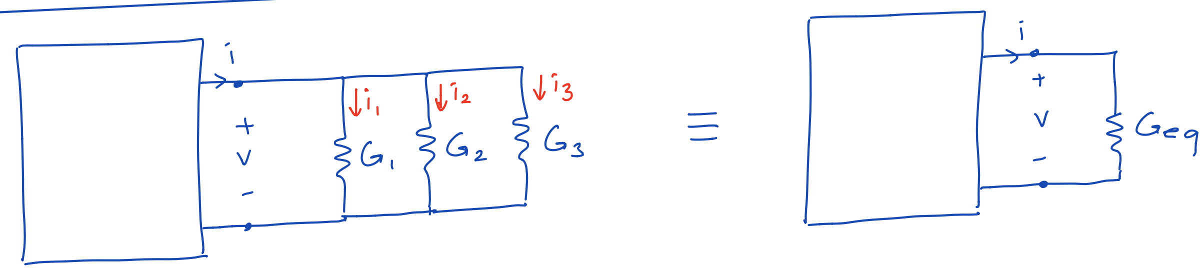



$$
\left.\begin{aligned} v &= v_1 + v_2 + v_3 \\ v_j &= R_j i_j = R_j i \end{aligned}\right\}
\qquad
\left.\begin{aligned} v &= (R_1 + R_2 + R_3) i \\ v &= R_{eq} \cdot i \end{aligned}\right\}
$$

compare

$R_{eq} = R_1 + R_2 + R_3$

## Equivalent Resistance : Parallel Combination




$$\left.\begin{aligned} i &= i_1 + i_2 + i_3 \\ i_j &= G_j v_j = G_j v \end{aligned}\right\} \quad \left.\begin{aligned} i &= (G_1 + G_2 + G_3)v \\ i &= G_{eq} \cdot v \end{aligned}\right\}$$

compare :

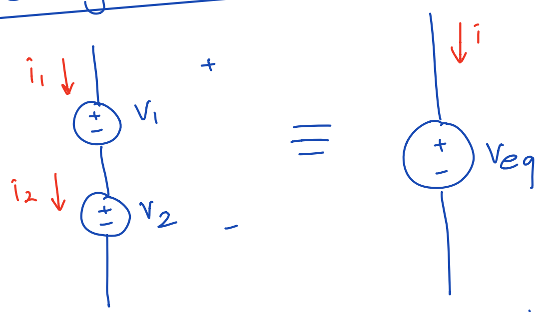
$$G_{eq} = G_1 + G_2 + G_3$$

OR :

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

* 2 parallel resistors : $\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$

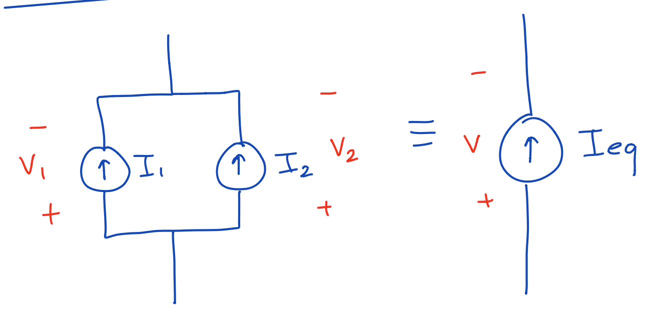
## Voltage Sources in series :




$$i = i_1 = i_2$$

$$V_{eq} = V_1 + V_2$$

## Current sources in parallel :

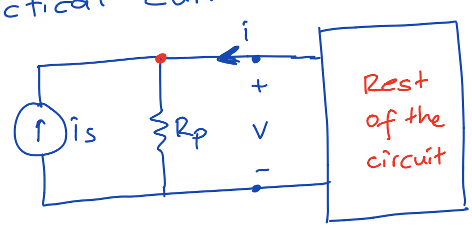



$$V = V_1 = V_2$$

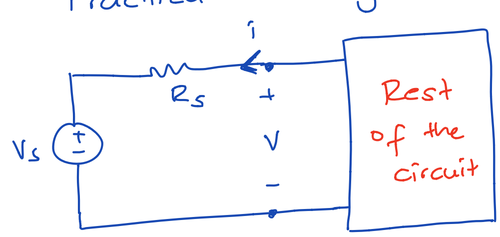
$$I_{eq} = I_1 + I_2$$

## Equivalence of Practical Sources (Source Transformation)

Practical current source :




Practical voltage source :

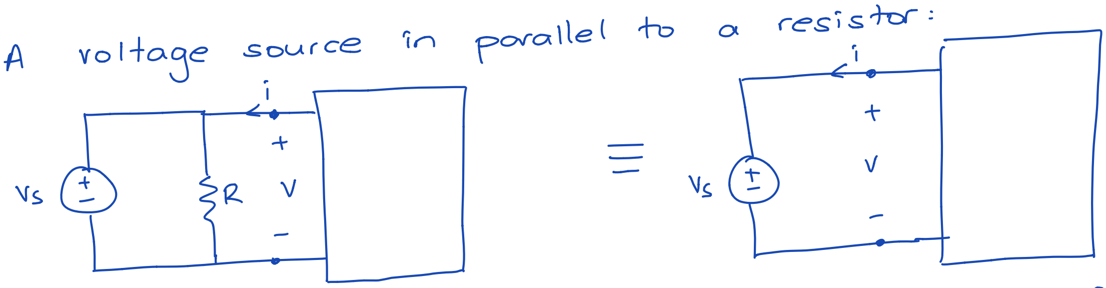



$$i = \frac{v}{R_p} - i_s \quad \xleftrightarrow{\text{compare}} \quad \begin{aligned} v &= R_s \cdot i + V_s \\ i &= \frac{v}{R_s} - \frac{V_s}{R_s} \end{aligned}$$

* Equivalence when $R_p = R_s$ and $v_s = R_s \cdot i_s$
(also, the relationship between Thevenin – Norton equivalent circuits)

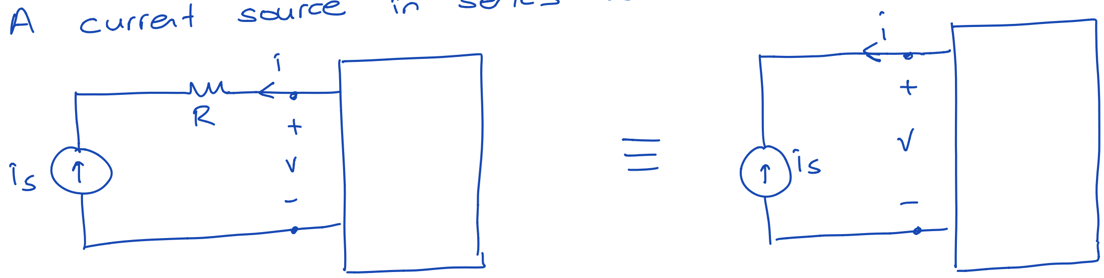
## more equivalent circuits :

* A voltage source in parallel to a resistor :



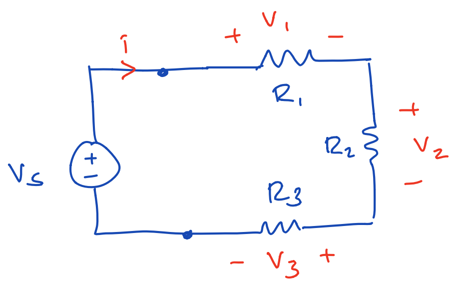

$v = v_s$ (independent of R)
$\Rightarrow$ Resistance can be omitted

* A current source in series with a resistor :




$i = -i_s$ (independent of R)
$\Rightarrow$ Resistance can be omitted

* Voltage Division :

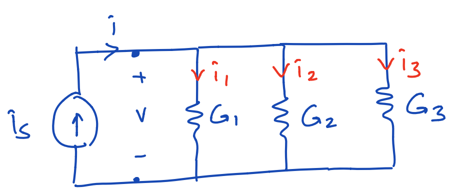



$$\left.\begin{aligned} v_s &= R_{eq} \cdot i = (R_1 + R_2 + R_3)\, i \\ v_j &= R_j \cdot i \end{aligned}\right\}$$

take ratio

$$v_j = \frac{R_j}{R_1 + R_2 + R_3} \cdot v_s$$

* Current Division :

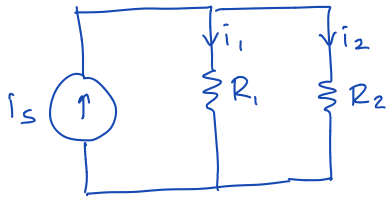



$$\left.\begin{aligned} i_s &= G_{eq} \cdot v = (G_1 + G_2 + G_3)\, v \\ i_j &= G_j \cdot v \end{aligned}\right\}$$

take ratio

$$i_j = \frac{G_j}{G_1 + G_2 + G_3} \cdot i_s$$

* For 2 resistors :



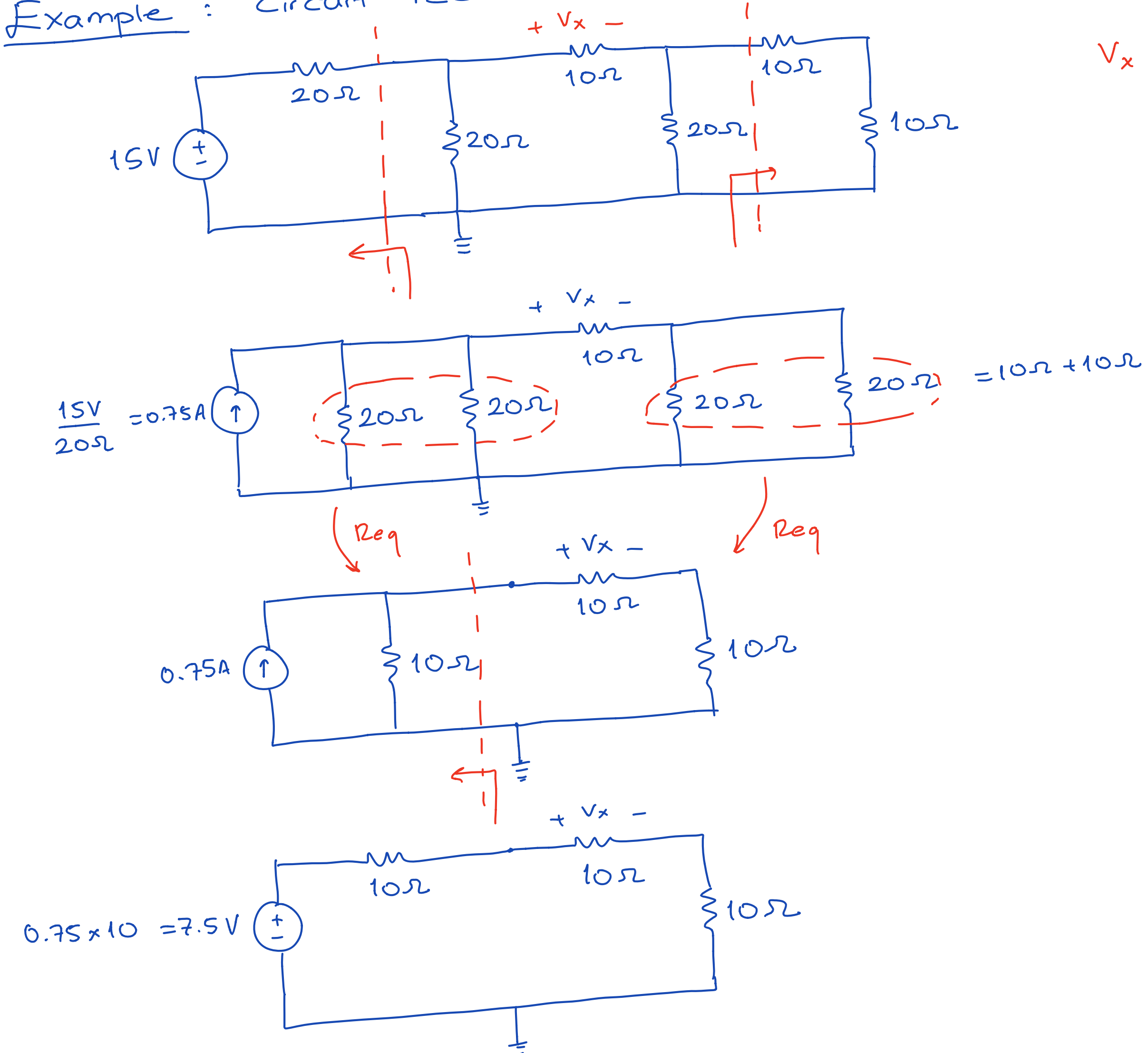

$$i_1 = \frac{G_1}{G_1 + G_2} \cdot i_s$$

$$i_1 = \frac{R_2}{R_1 + R_2} i_s$$

$$i_2 = \frac{R_1}{R_1 + R_2} i_s$$

Example : Circuit reduction

$V_x = ?$




From voltage divider: $V_x = \frac{10}{10 + 10 + 10} \cdot 7.5\text{ V} = 2.5\text{ V}$

* Note that not all resistors are linear. A typical example is a diode, which is a non-linear resistor.
  $i = I_S\,(e^{v/v_T} - 1)$. This model can be simplified (piecewise linear switch).

* A resistor is called <u>passive</u> if we have $p = vi \geq 0$ for all possible cases. Otherwise, it is called <u>active</u>.

* A resistor is called <u>time-varying</u>, if its i-v behaviour changes with time.

* A resistor is called <u>bilateral</u> if nothing changes when we replace the terminal connections
  $\Rightarrow$ its i-v relation is symmetric with respect to the origin.