Notes

Believe in yourself

1) a) $1 - P(\text{no rain}) = 1 - 0.3$
$\cancel{= 1 - 0.4 \cdot 0.5}$ $= 0.7 = P(A \cup B)$

b) $P(A \cap B) = P(A) + P(B) - P(A \cup B)$
$= 0.6 + 0.5 - 0.7$
$= 0.4$

c) $P(A \cap \bar{B}) = P(A \cup B) - P(B)$
$= 0.7 - 0.5 = 0.2$

d) $P(A \cup B) - P(A \cap B) = 0.3$

2) $S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$
$P(A) = 5/36$

3) $p$ - probability of girl (usually 1/2)

$P(\text{choosing a girl}) = \frac{1}{2}$ (dual) $= \sum_{r=1}^{n} \frac{r}{n} \, {}^{n}C_{r} \, p^{r} (1-p)^{n-r}$

$= \sum_{r=1}^{n} {}^{n-1}C_{r-1} \, p^{r} (1-p)^{n-r}$

$= p \, \sum {}^{n-1}C_{r-1} \, p^{r-1} (1-p)^{n-r}$

$= p$

$s_2$ $P = \frac{1 \cdot {}^nC_n\, h^n}{h}$

$= h^{n-1}$

$h = \frac{1}{2}$ $P = \frac{1}{2^{n-1}}$

4) $X \begin{cases} X_d & \text{wp } h \\ X_c & \text{wp } 1-h \end{cases}$

1) $F_X(x) = P(X < x) = h \cdot P(X_d < x)$
$1-h \cdot P(X_c < x)$

$= h\, F_d(x) + (1-h)\, F_c(x)$

2) $f_X(x) = h\, f_d(x) + (1-h)\, f_c(x)$

3) $E[X] = \int x\, f_X(x)\, d(x)$
$= h\, E[X_d] + (1-h)\, E[X_c]$

4) $\text{Var } X = E[X^2] - E^2[X]$

$= \int x^2 f_X(x)\, dx = h\, \text{Var}[X_d] + (1-h)\, \text{Var}[X_c] +$
$h(1-h)(E(X_d) - E(X_c))^2$

5) $Z = 1 + X + XY^2$, $W = 1 + X$

$\text{Cov}(Z, W) = E(ZW) - E(Z)\,E(W)$
$= E[(1+X+XY^2)(1+X)] - E[1+X+XY^2]\,E[1+X]$
$= E[1+X+XY^2 + X + X^2 + X^2Y^2]$
$-1$ $1$ $= 3 - 1 = 2$

6) $P(\geq 1 \text{ offer}) = 1 - P(\text{no offer})$

$= 1 - (0.8)^4$ (Assuming independence)

$= \cancel{0.4096}\ 0.59$

$P(A \cup B \cup C \cup D) \leq \Sigma P(X_i)$

$\leq 0.2 \times 4 = 0.8$

So $P(\geq 1 \text{ offer}) \leq 0.8$

So he is wrong.

7) $\dfrac{S_n - n\mu}{\sqrt{n}\,\sigma} \to N(0,1)$ as $n$ is large.

$\mu = p = 0.1 \qquad \sigma^2 = p(1-p) = 0.1 \times 0.9$

$\sigma^2 = 0.09$

$\sigma = 0.3$

$S_n \geq 120 \Rightarrow S_n \geq 121$

$N(0,1) \geq \dfrac{121 - 100}{\sqrt{1000}\ 0.3} = 2.21$

$\approx 0.0136$

13.6 % chance

8) CLT again $\quad \mu = 1, \ \sigma = \dfrac{1}{\sqrt{2}}$

& for 64 $\Rightarrow \quad \mu = 64, \ \sigma = \sqrt{32} = 5.66$

$z = 1.645$ for 95 %, $64 + 1.645 \times 5.66$

$= 74$ sandwich

9) 1) 0, 0

2) 1, 1

3) $\rho$, $\rho$

10) $E[X \mid Y = y] = \mu_x + \dfrac{\mathrm{cov}(X, Y)}{\mathrm{Var}\, Y}(y - \mu_y)$

$= 1 + \dfrac{1}{3}(y - 2)$

$\mathrm{Var}[X \mid Y = y] = \sigma_x^2 - \dfrac{\mathrm{cov}^2(X, Y)}{\sigma_Y^2}$

$= 4 - \dfrac{1}{3} = \dfrac{11}{3}$

$N\left(1 + \dfrac{1}{3}(y-2), \dfrac{11}{3}\right)$

11) X, Y are jointly Gaussian, their linear combⁿ is Gaussian too.

$E[Z] = 0$ $\quad$ ~~$\mathrm{Var}[Z] = 9\,\mathrm{Var}[X] + 4\,\mathrm{Var}[Y] = 9$~~

$\mathrm{Var}[Z] = 9\,\mathrm{Var}[X] + 4\,\mathrm{Var}[Y] - 2\,\mathrm{cov}(3X, 2Y)$

$= 9 + 16 - 2 \times 3 \times 2 = 13$

$Z \sim N(0, 13)$

$\rho = \dfrac{E[ZX] - \overbrace{E[Z]}^{\to 0} E[X]}{\sqrt{\sigma_Z^2 \sigma_X^2}} = \dfrac{E(3X^2 - 2XY)}{\sqrt{13}}$

Notes
Believe in yourself

$$= \frac{3 \times 1 - 2}{\sqrt{13}} = \frac{1}{\sqrt{13}}$$

12) $\cancel{\mu_x} = \mu_x +$

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

$$\Sigma \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{array}{ccc} & \boxed{4}\ \Sigma_{11} & \boxed{2 \quad 1}\ \Sigma_{12} \\ \Sigma_{21} & \boxed{\begin{matrix} 2 \\ 1 \end{matrix}} & \boxed{\begin{matrix} 3 & -1 \\ -1 & 5 \end{matrix}}\ \Sigma_{22} \end{array}$$

$$\bar{\mu} = \mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (a - \mu_2)$$

$$a = \begin{pmatrix} y \\ z \end{pmatrix} \qquad \Sigma_{22}^{-1} = \frac{1}{14} \begin{pmatrix} 5 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \end{pmatrix} \frac{1}{14} \begin{pmatrix} 5 & 1 \\ 1 & 3 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 11 & 5 \end{pmatrix}$$

$$\frac{1}{14} [11 (y - 0) + 5 (z - 0)]$$

$$= \frac{1}{14} [11 y + 5z]$$

$$\bar{\mu} = \frac{0 + 11y + 5z}{14} = \frac{11y + 5z}{14}$$

$$\Sigma = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

$$= 4 - \frac{1}{14}\begin{pmatrix} 11 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$Var(X) = 4 - \frac{27}{14}$$

$$= \frac{29}{14}$$

$$N\left(\frac{11y + 5z}{14}, \frac{29}{14}\right) \equiv X \text{ is ans.}$$

4) 4) $E[X^2] - E^2[X]$

$$= h\,E[X_d^2] + (1-h)E[X_c^2]$$
$$- h^2 E^2[X_d] - (1-h)^2 E[X_c^2]$$
$$- 2h(1-h)E[X_d]E[X_c]$$

$$= h\,Var[X_d] + (1-h)Var(X_c) + h\,E^2[X_d] + (1-h)E^2[X_c]$$
$$- h\,E^2[X_d]$$

$$E^2[X_d]\,h(1-h) + h(1-h)E^2(X_c) - 2h(1-h)E[X_d]E[X_c]$$
$$= h(1-h)(E[X_d] - E[X_c])^2 + h\,Var(X_d] + (1-h)Var(X_c)$$