Reasoning under uncertainty

David “davidad” Dalrymple
davidad@alum.mit.edu

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http://espr.cf

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Reasoning without uncertainty

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3. If a synapse is not intercellular, it is not an electrical synapse
4. Are all synapses intercellular (yes or no)?

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2. All three labels are incorrect; each belongs on a different box.
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4. How can you label the boxes correctly?

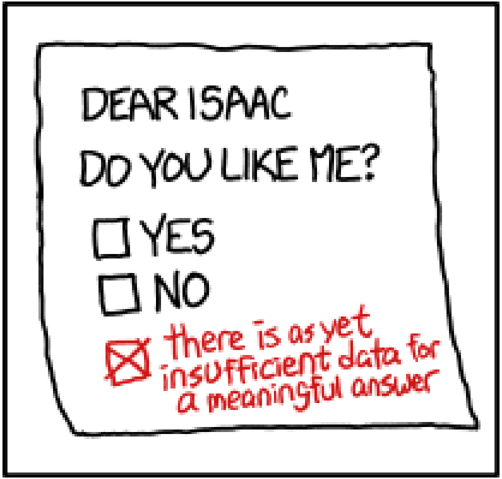
Reasoning with uncertainty

1. I have a full deck of 52 playing cards, and take the top card.

Reasoning with uncertainty

1. I have a full deck of 52 playing cards, and take the top card.
2. Is it a queen?

What's the next best thing?



http://xkcd.com/1448; reference is to “The Last Question” by Isaac Asimov

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Different framings of this question:

1. **Betting strategy**
2. **Subjective belief**
3. **Objective frequency**
4. **Many (deterministic) worlds**

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   - Consider hypothetically repeating a random experiment (infinitely) many times, and counting the number of outcomes where a proposition is true.
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   - Any given world would have enough information for deduction, but we don't know which world we're in. Count the number of worlds where a proposition is true.

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Note: there are other framings that justify different theories, like Dempster-Shafer theory, but this concordance is still surprising.

Boolean algebra: operators and notation

Logic		Algebra		Sets	
conjunction	$\wedge$	multiplication	$\cdot$	intersection	$\cap$
disjunction	$\vee$	addition	$+$	union	$\cup$
negation	$\neg$	negation	$\neg$	complement	$\Omega\backslash$

Boolean algebra: axioms

$\top$ means true; $\perp$ means false

$$A \vee \perp = A \qquad A \wedge \top = A$$

$$A \vee B = B \vee A \qquad A \wedge B = B \wedge A$$

$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C) \qquad A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$$

$$A \vee \neg A = \top \qquad A \wedge \neg A = \perp$$

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- Comma is sometimes used for $\wedge$, e.g. $\mathbb{P}(A,B|\Omega) := \mathbb{P}(A \wedge B|\Omega)$.

Derivations from each justification

In each framing/justification, I'll state the specific laws of probability, then go over the argument.

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The goal as Agent is to set your probabilities in such a way that Bookie cannot *guarantee* that they will make a profit from you.

Betting strategy: opponent strategies

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How can you guarantee a profit as Bookie?

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  - If A turns out to be true, Agent must pay Bookie an additional dollar, but in either case Bookie has made a profit.

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- Bookie sells this bet to Agent. Agent must pay strictly more than 1 dollar up front. Since it's a tautology, Bookie must return 1 dollar, but still makes a strictly positive profit.

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- Bookie buys Agent's bet for less than 1 dollar, and Agent must pay 1 dollar.

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  - Why do A and B have to be disjoint?
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  - Swap buying with selling.

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  - Why do A and B have to be disjoint?
- What if Agent's $\mathbb{P}(A \vee B|\Omega) < \mathbb{P}(A|\Omega) + \mathbb{P}(B|\Omega)$?
  - Swap buying with selling.
- This can be generalized to arbitrarily many disjoint events.

Betting strategy: opponent strategies

Suppose that if evidence B is observed, Agent will update to a $\mathbb{P}(A|B,\Omega) \neq \frac{\mathbb{P}(A \wedge B|\Omega)}{\mathbb{P}(B|\Omega)}$.
How can you guarantee a profit as Bookie?

Betting strategy: opponent strategies

Suppose that if evidence B is observed, Agent will update to a $\mathbb{P}(A|B,\Omega) < \frac{\mathbb{P}(A \wedge B|\Omega)}{\mathbb{P}(B|\Omega)}$. How can you guarantee a profit as Bookie?

- Bet 1. Bookie sells Agent a bet for $A \wedge B$; Agent pays $\mathbb{P}(A \wedge B|\Omega)$.
- Bet 2. Bookie sells Agent $\frac{\mathbb{P}(A \wedge B|\Omega)}{\mathbb{P}(B|\Omega)}$ of a bet for $\neg B$; Agent pays $\mathbb{P}(A \wedge B|\Omega)\,\frac{\mathbb{P}(\neg B|\Omega)}{\mathbb{P}(B|\Omega)}$.
- Bet 3. Bookie sells Agent $\frac{\mathbb{P}(A \wedge B|\Omega)}{\mathbb{P}(B|\Omega)} - \mathbb{P}(A|B,\Omega)$ of a bet for B; Agent pays $\mathbb{P}(A \wedge B|\Omega) - \mathbb{P}(A|B,\Omega)\,\mathbb{P}(B|\Omega)$.
- If B is false, then all the bets settle. Bookie pays back $\frac{\mathbb{P}(A \wedge B|\Omega)}{\mathbb{P}(B|\Omega)}$ for Bet 2. Due to countable additivity, Bets 1 and 2 exactly cancel out and the price for Bet 3 remains as profit.
- If B is true, then Bets 2 and 3 are settled, with Bookie paying back $\frac{\mathbb{P}(A \wedge B|\Omega)}{\mathbb{P}(B|\Omega)} - \mathbb{P}(A|B,\Omega)$ for Bet 3.
- Bet 4. Bookie then buys from Agent a bet for A at Agent's new probability of $\mathbb{P}(A|B,\Omega)$.
- At this point, whether or not A turns out to be true, Bookie and Agent owe each other a net 0 dollars, and Bookie has made a net profit.

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4. $\mathbb{P}(\neg A|\Omega) = N[\mathbb{P}(A|\Omega)]$ for some N.

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Subjective belief: proof

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- It turns out Cox's Theorem is wrong!
  - Factoid of the hour: "Cox's Theorem is false; Halpern in 1999 provided a counterexample"
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- But the laws of probability I showed here *do* follow from the premises I showed here, in a modern proof by Terenin and Draper

Subjective belief: laws of probability

1. (Bounding) $0 = \mathbb{P}(\bot|\Omega) \leq \mathbb{P}(A|\Omega) \leq \mathbb{P}(\Omega|\Omega) = 1$
2. (Negation) $\mathbb{P}(A|\Omega) = 1 - \mathbb{P}(\neg A|\Omega)$
3. (Product rule) $\mathbb{P}(A \wedge B|\Omega) = \mathbb{P}(A|B,\Omega) \cdot \mathbb{P}(B|\Omega)$
4. (Countable additivity) For arbitrary disjoint A_i,
   $\mathbb{P}\left(\bigvee_{i=1}^{\infty} A_i \middle| \Omega\right) B = \sum_{i=1}^{\infty} \mathbb{P}(A_i|\Omega)$

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  - The number of outcomes in which one of many mutually exclusive alternatives is true is precisely the number of outcomes in which at least one is true.

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  - So

$$\mathbb{P}(A|B) = \frac{N_{A \wedge B}}{N_B} = \frac{N_{A \wedge B}/N}{N_B/N} = \frac{\mathbb{P}(A \wedge B)}{\mathbb{P}(B)}$$

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- Formally, they behave the same and blend together.

Sample space

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- The subjective-belief and betting-strategy frameworks are sufficiently abstract that they don't come with a concrete sample space, but a canonical one can be constructed (via the Loomis-Sikorski representation theorem).

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Why is this useful?

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Generating good candidate hypotheses is much of the difficulty in most real inference problems.

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  - $\mathbb{P}(\omega\leqq x|\Omega)$ itself, also a function of x, is known as a **cumulative distribution function** (or simply **distribution function**), and uniquely specifies the probability of all events depending on ω.

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These follow from the linearity of the Lebesgue integral.

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- Similar arguments can be applied to density functions in continuous cases, usually to justify uniform density over an interval (scaled so that the integral of the density is 1, as required by the laws of probability).

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3. A meter-stick is cut at a uniformly sampled point along its length. What is the expected value of the larger piece?
   - Bonus puzzle: if the stick is instead cut simultaneously at two uniformly sampled points, what is the expected value of the largest piece?