Math 247, Advanced Calculus 3

Spring Term 2017

Essay Assignment

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This essay discusses the concept of *volume* for a subset of $\mathbb{R}^n$ and includes a list of six problems

Important Reminder: It is a tiny essay report based on the coursework

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A. Zero-content sets

Definition A1. We will use the name *half-open rectangle in* $\mathbb{R}^n$ in reference to sets of the form $P = (a_1, b_1] \times \cdots \times (a_n, b_n] \subseteq \mathbb{R}^n$, with $a_1 < b_1, \ldots, a_n < b_n$ in $\mathbb{R}$.

The collection of all half-open rectangles in $\mathbb{R}^n$ will be denoted by $\mathcal{P}_n$. (So note that $\mathcal{P}_n$ is a set of sets!)

We define a volume function $\mathrm{vol}_o : \mathcal{P}_n \to [0, \infty)$, where for $P = (a_1, b_1] \times \cdots \times (a_n, b_n] \in \mathcal{P}_n$ we put $\mathrm{vol}_o(P) := (b_1 - a_1) \cdots (b_n - a_n)$.

Definition A2. A set $S \subseteq \mathbb{R}^n$ is said to be a *zero-content set* when it has the following "(ε-Cover)" property:

$$(\varepsilon\text{-Cover}) \qquad \begin{cases} \text{For every } \varepsilon > 0 \text{ one can find } P_1, \ldots, P_\ell \in \mathcal{P}_n \text{ (for some } \ell \in \mathbb{N}) \\ \\ \text{such that } P_1 \cup \cdots \cup P_\ell \supseteq S \text{ and } \sum_{i=1}^{\ell} \mathrm{vol}_o(P_i) < \varepsilon. \end{cases}$$

Problem 1. Prove the following general properties of zero-content sets.

(a) If S is a zero-content subset of $\mathbb{R}^n$ and if T is a subset of S, then T also is a zero-content subset of $\mathbb{R}^n$.

(b) If $S_1, \ldots, S_k$ are zero-content subsets of $\mathbb{R}^n$, then the union $S := S_1 \cup \cdots \cup S_k$ also is a zero-content subset of $\mathbb{R}^n$.

(c) If S is a zero-content subset of $\mathbb{R}^n$, then the closure $\mathrm{cl}(S)$ also is a zero-content subset of $\mathbb{R}^n$.

Problem 2. Consider the set $X = [0, 1] \times [0, 1] \subseteq \mathbb{R}^2$, and let $f : X \to \mathbb{R}^3$ be a Lipschitz function. Prove that the image-set $Y = f(X) \subseteq \mathbb{R}^3$ is a zero-content subset of $\mathbb{R}^3$.

[Can you generalize this problem to higher dimensions?]

Problem 1 solution

(a) Since S is a zero-content subset of $\mathbb{R}^n$

Then $\forall\,\varepsilon > 0\,,\, \exists\, P_1, \ldots, P_\ell \in \mathcal{P}_n\ (\ell \in \mathbb{N})$ such that $P_1 \cup \cdots \cup P_\ell \supseteq S\,,\, \sum_{i=1}^{\ell} \mathrm{vol}_o(P_i) < \varepsilon$.

Since T is a subset of S, hence have that $T \subseteq S \subseteq P_1 \cup \cdots \cup P_\ell$

Hence $\forall\,\varepsilon > 0\,,\, \exists\, P_1, \ldots, P_\ell \in \mathcal{P}_n\ (\ell \in \mathbb{N})$ such that $P_1 \cup \cdots \cup P_\ell \supseteq T\,,\, \sum_{i=1}^{\ell} \mathrm{vol}_o(P_i) < \varepsilon$.

Therefore T also is a zero-content subset of $\mathbb{R}^n$.

(b) Since S_j is a zero-content subset of $\mathbb{R}^n$, for every $1 \le j \le k$, have that

$\forall\,\varepsilon > 0\,,\, \exists\, P_{j_1}, \ldots, P_{j_\ell} \in \mathcal{P}_n\ (j_\ell \in \mathbb{N})$ such that $P_{j_1} \cup \cdots \cup P_{j_\ell} \supseteq S_j\,,\, \sum_{i=j_1}^{j_\ell} \mathrm{vol}_o(P_i) < \frac{\varepsilon}{k}$.

Hence for all $\varepsilon > 0$, we get several half-open rectangles

$P_{1_1}, \ldots, P_{1_\ell}, P_{2_1}, \ldots, P_{2_\ell}, \ldots, \ldots, P_{k_1}, \ldots, P_{k_\ell}$ satisfying that

$$\begin{gathered} P_{1_1} \cup \cdots \cup P_{1_\ell} \supseteq S_1\,,\, \textstyle\sum_{i=1_1}^{1_\ell} \mathrm{vol}_o(P_i) < \frac{\varepsilon}{k} \\ P_{2_1} \cup \cdots \cup P_{2_\ell} \supseteq S_2\,,\, \textstyle\sum_{i=2_1}^{2_\ell} \mathrm{vol}_o(P_i) < \frac{\varepsilon}{k} \\ \ldots \\ P_{k_1} \cup \cdots \cup P_{k_\ell} \supseteq S_k\,,\, \textstyle\sum_{i=k_1}^{k_\ell} \mathrm{vol}_o(P_i) < \frac{\varepsilon}{k} \end{gathered}$$

S_i is a subset of The union of each group of half-open rectangles while the union of S_i is S.Hence by the property of set, S is exactly still a subset of the union of every group of these half-open rectangles. Which is $P_{1_1} \cup \ldots \cup P_{1_\ell} \cup P_{2_1} \cup \ldots \cup P_{2_\ell} \cup \ldots\ldots \cup P_{k_1} \cup \ldots \cup P_{k_\ell} \supseteq S$

Now we need to "filter" those identical half-open rectangles.

Let $P_r = \{\, P_1\,, \ldots,\, P_r \,\}$ be the collection of all these above half-open rectangles, where every two half-open rectangles are not identical. Still have that $P_1 \cup \ldots \cup P_r \supseteq S$ since we just filtered some "useless" half-open rectangles, which has no influence on the union.

Then consider the sum of the volume function of these half-open rectangles.

Have that $\forall\, P_i \in \{P_r\}\,,\, \sum_{i=1}^{r} \mathrm{vol}_o(P_i) < \sum_{j=1}^{k} \sum_{i=j_1}^{j_\ell} \mathrm{vol}_o(P_i) < \sum_{j=1}^{k} \frac{\varepsilon}{k} = \varepsilon$

Hence $\forall\,\varepsilon > 0\,,\, \exists\, P_1, \ldots, P_r \in \mathcal{P}_n\ (r \in \mathbb{N})$ such that $P_1 \cup \cdots \cup P_r \supseteq S\,,\, \sum_{i=1}^{r} \mathrm{vol}_o(P_i) < \varepsilon$.

Therefore the union S is also a zero-content subset of $\mathbb{R}^n$

(c) Since S is a zero-content subset of $\mathbb{R}^n$

Then $\forall\,\varepsilon > 0\,,\, \exists\, P_1, \ldots, P_\ell \in \mathcal{P}_n\ (\ell \in \mathbb{N})$ such that $P_1 \cup \cdots \cup P_\ell \supseteq S\,,\, \sum_{i=1}^{\ell} \mathrm{vol}_o(P_i) < \varepsilon$.

If $P_1 \cup \cdots \cup P_\ell \supseteq \mathrm{cl}(S) \supseteq S$, then we are done.

When $\mathrm{cl}(S) \supseteq P_1 \cup \cdots \cup P_\ell \supseteq S$

Since $\mathrm{cl}(S)$ is the smallest set containing S.

Hence can find a $P_\alpha \in \mathcal{P}_n$ such that $P_1 \cup \cdots \cup P_\ell \cup P_\alpha \supseteq \mathrm{cl}(S)$

Hence $\forall\,\varepsilon > 0\,,\, \exists\, P_1, \ldots, P_\ell, P_\alpha \in \mathcal{P}_n\ (\ell \in \mathbb{N})$ such that

$$\begin{gathered} P_1 \cup \cdots \cup P_\ell \cup P_\alpha \supseteq \mathrm{cl}(S) \\ \textstyle\sum_{i=1}^{\ell} \mathrm{vol}_o(P_i) < \mathrm{vol}_o(P_\alpha) \cdot \varepsilon, \text{ where } \mathrm{vol}_o(P_\alpha) \text{ is a positive constant in } \mathbb{R}. \end{gathered}$$

Therefore the closure $\mathrm{cl}(S)$ is a zero-content subset of $\mathbb{R}^n$

Problem 2 solution

Proof

Since f is Lipschiz. Hence $\exists\, c \geq 0$ s.t. $||\, f(\vec{x}_1) - f(\vec{x}_2)\,|| \leq c \cdot ||\, \vec{x}_1 - \vec{x}_2\,||, \forall\, \vec{x}_1\, \vec{x}_2 \in X$

For every $\varepsilon > 0$, let N be the smallest integer greater than $\frac{1}{\varepsilon}$

Let Δ be the division of X into N^2 squares of the form $P_{ij} = (\frac{i-1}{N}, \frac{i}{N}] \times (\frac{j-1}{N}, \frac{j}{N}], 1 \leq i, j \leq N$.When $i, j = 1$, P_{ij} is a closed interval containing 0. Then for any two points $\vec{x}^1_{ij}, \vec{x}^2_{ij}$ in P_{ij}, $||\, f(\vec{x}^1_{ij}) - f(\vec{x}^2_{ij})\,|| \leq c \cdot ||\, \vec{x}^1_{ij} - \vec{x}^2_{ij}\,|| \leq c \cdot \frac{\sqrt{2}}{N}$.This means the distance of any two points in P_{ij} is less than or equal to $c \cdot \frac{\sqrt{2}}{N}$, denote $d := c \cdot \frac{\sqrt{2}}{N}$. Hence we conclude that $f(P_{ij})$ is contained in a 3-dim ball of diameter d.

Hence $\sum_{1 \leq i,j \leq N} vol_o(f(P_{ij})) \leq N^2 \cdot d^3 \leq \frac{2\sqrt{2}}{N} \leq 2\sqrt{2}\varepsilon$

Also $f(P_{ij})$ can completely cover $f(X)$

Hence can find a series of hal-open rectangle $f(P_{ij}) \in \mathcal{P}_3$ such that

$$\bigcup_{1 \leq i,j \leq N} f(P_{ij}) \supseteq f(X) \text{ and } \sum vol_o(f(P_{ij})) < 2\sqrt{2}\varepsilon$$

Hence the image set $f(X)$ is a zero-content subset of $\mathbb{R}^3$.

The generalized problem can be stated as (the proof is similar)

Consider the set $X = [0,1] \times ... \times [0,1] \subseteq \mathbb{R}^n$, and let $f : X \to \mathbb{R}^{n+1}$ be a Lipschitz function. Prove that the image-set $Y = f(X) \subseteq \mathbb{R}^{n+1}$ is a zero-content subset of $\mathbb{R}^{n+1}$.

B. Jordan measurable sets

Definition B1. We will say that a set $A \subseteq \mathbb{R}^n$ is *Jordan measurable* when it satisfies the following two conditions (i) + (ii):

(i) A is a bounded set;

(ii) the boundary $\mathrm{bd}(A)$ is a zero-content set.

The collection of all Jordan measurable subsets of $\mathbb{R}^n$ will be denoted by $\mathcal{M}_n$. (So note that $\mathcal{M}_n$ is a set of sets!)

Problem 3. (a) Consider the open ball $D := \{\vec{x} \in \mathbb{R}^3 \mid \ ||\,\vec{x}\,|| < 1\}$ and the closed ball $C := \{\vec{x} \in \mathbb{R}^3 \mid \ ||\,\vec{x}\,|| \leq 1\}$. Prove that C and D are Jordan measurable subsets of $\mathbb{R}^3$.

(b) Let A be a subset of $\mathbb{R}^3$ which has the property that $D \subseteq A \subseteq C$, where D and C are as in parts (a) and (b) of the problem. Prove that A is a Jordan measurable subset of $\mathbb{R}^3$.

[Can you generalize this problem to higher dimensions?]

Definition B2. Let $\mathcal{A}$ be a set of subsets of $\mathbb{R}^n$. We say that $\mathcal{A}$ is a *field of subsets of* $\mathbb{R}^n$ when it has the following properties:

(FS-1) The empty set $\emptyset$ belongs to $\mathcal{A}$.

(FS-2) If $A, B \in \mathcal{A}$, then the set-difference $A \setminus B$ still belongs to $\mathcal{A}$.

(FS-3) If $A_1, \ldots, A_k \in \mathcal{A}$, then the union $A_1 \cup \cdots \cup A_k$ and the intersection $A_1 \cap \cdots \cap A_k$ still belong to $\mathcal{A}$.

Problem 4. Consider the collection $\mathcal{M}_n$ of Jordan measurable subsets of $\mathbb{R}^n$.

(a) Prove that $\mathcal{M}_n \supseteq \mathcal{P}_n$, where $\mathcal{P}_n$ is as in Definition A1.

(b) Prove that $\mathcal{M}_n$ is a field of subsets of $\mathbb{R}^n$, in the sense of Definition B2.

Problem 3 solution

(a) Can see that both C and D are bounded sets, so only need to verify condition(ii).

Since C is closed and D is open, must have that bd(C)=bd(D)=$\{\vec{x} \in \mathbb{R}^3 \mid ||\vec{x}|| = 1\}$.

Let $B = \{\vec{x} \in \mathbb{R}^2 \mid ||\vec{x}|| < 1\}$, $X := \{\vec{x} \in \mathbb{R}^2 \mid ||\vec{x}|| \leq 1\}$, $f : X \to \mathbb{R}^3$ defined by

$$f((x, y)) = \begin{cases} (x, y, \pm\sqrt{1 - x^2 - y^2}), & \text{if } (x, y) \in B \\ (x, y, 0) & \text{if } (x, y) \in X \setminus B \end{cases}$$

By the similar arguement in Problem 2, if f is Lipschiz, then the image of f, which is the boundary of C, D, is a zero-content subset of $\mathbb{R}^3$. So now need to show that f is Lipschiz. Consider any two points $\vec{x}, \vec{y} \in B$. If both points are in $X \setminus B$, then any $c \geq 1$ will do, since $||\, f(\vec{x}) - f(\vec{y})\, || = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + 0} = ||\, \vec{x} - \vec{y}\, || \leq c \cdot ||\, \vec{x} - \vec{y}\, ||$. If both points are in B, then

$$\begin{aligned} ||\, f(\vec{x}) - f(\vec{y})\, || &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (\sqrt{1 - x_1^2 - x_2^2} - \sqrt{1 - y_1^2 - y_2^2})^2} \\ &\leq \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} + \sqrt{(\sqrt{1 - x_1^2 - x_2^2} - \sqrt{1 - y_1^2 - y_2^2})^2} \\ &\leq ||\, \vec{x} - \vec{y}\, || + \sqrt{(\sqrt{1 - x_1^2 - x_2^2} - \sqrt{1 - y_1^2 - y_2^2})^2} \end{aligned}$$

can always find $c \geq 0$ to satisfy the inequality since $\sqrt{(\sqrt{1 - x_1^2 - x_2^2} - \sqrt{1 - y_1^2 - y_2^2})^2} \geq 0$. If one point is in B, the other is in $X \setminus B$, then

$$\begin{aligned} ||\, f(\vec{x}) - f(\vec{y})\, || &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (\sqrt{1 - x_1^2 - x_2^2} - 0)^2} \\ &\leq ||\, \vec{x} - \vec{y}\, || + \sqrt{1 - x_1^2 - x_2^2} \end{aligned}$$

can also find $c \geq 0$ since $\sqrt{1 - x_1^2 - x_2^2} \geq 0$.

Hence f is Lipschiz.

By the similar arguement in problem 2, can conclude that, the image of X, which is the boundary of C, D is a zero-content set.

Therefore C and D are Jordan measurable.

(b) Since $D \subseteq A \subseteq C$

Hence A can be expressed as $A := \{\vec{x} \in \mathbb{R}^3 \mid ||\vec{x}|| \leq 1, \text{but not all the equality holds}\}$

However the boundary of A is still bd$(A) = \{\vec{x} \in \mathbb{R}^3 \mid ||\vec{x}|| = 1\}$.

Hence by the same arguement in part (a), A is a Jordan measurable subset of $\mathbb{R}^3$.

The generalized porblem can be stated as

Consider the open ball $D := \{\vec{x} \in \mathbb{R}^n \mid \; ||\, \vec{x}\, || < 1\}$ and the closed ball $C := \{\vec{x} \in \mathbb{R}^n \mid ||\, \vec{x}\, || \leq 1\}$.Let A be a subset of $\mathbb{R}^n$ which has the property that $D \subseteq A \subseteq C$.

Prove that C, D and A are Jordan measurable subsets of $\mathbb{R}^n$.

Problem 4 Solution

(a) To show that all half-open rectangles P in $\mathcal{P}_n$ is a Jordan measurable subset of $\mathbb{R}^n$.

Let $r^2 := \sum_{i=1}^{n} \min\{a_i^2, b_i^2\}$, have that $||\vec{x}|| \leq r, \forall\, \vec{x} \in P$

Hence every half-open rectangles in $\mathcal{P}_n$ is bounded, which satisfying condition (i).

To prove that $\mathrm{bd}(P)$ is a zero-content set, try to use induction on dimension.

Basic case, when $n = 1$, $\mathrm{bd}(P_1)$ is two points a_1 and b_1, it is trivail to have that $\mathrm{bd}(P_1)$ is a zero-content set.

Suppse $\mathrm{bd}(P_r)$ is a zero-content, then consider the case of $\mathrm{bd}(P_{r+1})$.

$\forall\, \varepsilon > 0\,, \exists\, P_1, \ldots, P_\ell \in \mathcal{P}_r$ $(\ell \in \mathbb{N})$ such that $P_1 \cup \cdots \cup P_\ell \supseteq \mathrm{bd}(P_r)\,, \sum_{i=1}^{\ell} \mathrm{vol}_o(P_i) < \varepsilon$.

Constrcuting $\mathrm{bd}(P_{r+1})$ by adding the $r+1$-th dimensional interval $[a_{r+1}, b_{r+1}]$ to $\mathrm{bd}(P_r)$, dividing this interval into $r+1$ average parts, which is $(a_{r+1} + (j-1)d, a_{r+1} + j \cdot d\,], 1 \leq j \leq r+1$, where $d := \frac{b_{r+1} - a_{r+1}}{r+1}$, when $j = 1$, the interval is also left closed. Then get series new half-open rectagle, $P_{ij} = P_i \times (a_{r+1} + (j-1)\frac{b_{r+1}-a_{r+1}}{r+1}, a_{r+1} + j\frac{b_{r+1}-a_{r+1}}{r+1}]$, which can completely cover $\mathrm{bd}(P_{r+1})$, also $P_{ij} \in \mathcal{P}_{r+1}$. Then the sum of volume is

$$\begin{aligned}
\sum_{j=1}^{r+1}\sum_{i=1}^{r} vol_o(P_{ij}) &= (r+1) \cdot d \cdot \sum_{i=1}^{r} vol_o(P_i) \\
&< (r+1) \cdot d \cdot \varepsilon
\end{aligned}$$

Hence can conclude that $\mathrm{bd}(P_{r+1})$ is also a zero-content set, which satisfying condition (ii).

Therefore all half-open rectangles P in $\mathcal{P}_n$ is a Jordan measurable subset of $\mathbb{R}^n$.

Therefore $\mathcal{M}_n \supseteq \mathcal{P}_n$.

(b) Must show that $\mathcal{M}_n$ satisfies the three properties.

(FS-1) Want to show that $\emptyset$ is Jordan measurable.

$\emptyset$ is bounded by any number, and $\mathrm{bd}(\emptyset) = \emptyset$, which is a zero-content set.

Hence $\emptyset\, \in \mathcal{M}_n$

(FS-2) Suppose $A, B\, \in \mathcal{M}_n$

Then A, B is bounded sets, which leads to that $A \setminus B$ is still bounded since $A \setminus B \subseteq A$ leads to $\inf(A) \leq \inf(A \setminus B) \leq \sup(A \setminus B) \leq \sup(A)$.

Also $\mathrm{bd}(A)$ and $\mathrm{bd}(B)$ are zero-content sets.

Hence $\forall\, \varepsilon > 0\,, \exists\, P_1, \ldots, P_s \in \mathcal{P}_n$ $(s \in \mathbb{N})$ s.t. $P_1 \cup \cdots \cup P_s \supseteq \mathrm{bd}(A)\,, \sum_{i=1}^{s} \mathrm{vol}_o(P_i) < \varepsilon$.

$\exists\, Q_1, \ldots, Q_t \in \mathcal{P}_n$ $(t \in \mathbb{N})$ s.t. $Q_1 \cup \cdots \cup Q_t \supseteq \mathrm{bd}(B)\,, \sum_{i=1}^{t} \mathrm{vol}_o(Q_i) < \varepsilon$.

Can form $R_k = P_i \setminus Q_j$ such that these sets can cover $\mathrm{bd}(A \setminus B)$.

Must have that $\sum \mathrm{vol}_o(R_k) < \sum_{i=1}^{s} \mathrm{vol}_o(P_i) < \varepsilon$.

Hence $A \setminus B$ is also Jordan measurable, $A \setminus B\, \in \mathcal{M}_n$

(FS-3) Supoose $A_1, ...A_k$ are Jordan measurable. Then each set is bounded, which leads to that the union and the intersection of these sets are also bounded by set property.

$\forall\, \varepsilon > 0\,, \exists\, P_1^j, \ldots, P_r^j \in \mathcal{P}_n$ s.t. $P_1^j \cup \cdots \cup P_r^j \supseteq \mathrm{bd}(A_j)\,, \sum_{i=1}^{r} \mathrm{vol}_o(P_i) < \varepsilon\,, \forall\, 1 \leq j \leq k\,.$

Then $\bigcup_{j=1}^{k}\bigcup_{i=1}^{r} P_i^j \supseteq \bigcup_{j=1}^{k} \mathrm{bd}(A_j) \supseteq \mathrm{bd}(\bigcup_{j=1}^{k} A_j)$ by the set property.

Also $\sum_{j=1}^{k}\sum_{i=1}^{r} vol_o(P_i^j) < k \cdot \varepsilon$

Hence the union of these sets is still Jordan measurable, $A_1 \cup \cdots \cup A_k\, \in \mathcal{M}_n$

Since $A_1 \cap \cdots \cap A_k$ is a subset of $A_1 \cup \cdots \cup A_k$.

Then by Problem 1 (a), $A_1 \cap \cdots \cap A_k$ is also a zero-content set.

Hence $A_1 \cap \cdots \cap A_k$ is also in $\mathcal{M}_n$.

Therefore $\mathcal{M}_n$ is a field of subsets of $\mathbb{R}^n$.

C. Additive set-functions, and the concept of volume

Definition C1. Let $\mathcal{A}$ be a field of subsets of $\mathbb{R}^n$ (in the sense of Definition B2). A function $\mu : \mathcal{A} \to [0, \infty)$ is said to be an *additive set-function* when it has the property that:

$$\left(\begin{array}{c} A_1, \ldots, A_k \in \mathcal{A} \\ \text{with } A_i \cap A_j = \emptyset \text{ for } i \neq j \end{array} \right) \implies \Big(\mu(A_1 \cup \cdots \cup A_k) = \sum_{i=1}^{k} \mu(A_i) \Big).$$

Problem 5. Let $\mathcal{A}$ be a field of subsets of $\mathbb{R}^n$ and let $\mu : \mathcal{A} \to [0, \infty)$ be an additive function.

(a) Prove that $\mu(\emptyset) = 0$.

(b) Let A, B be two sets in $\mathcal{A}$, such that $A \subseteq B$. Prove that $\mu(A) \leq \mu(B)$.

(c) Let A, B be two sets in $\mathcal{A}$. Prove the following equality:

$$\text{(INC-EXC)} \qquad \mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B).$$

(d) Prove that the inequality

$$\text{(SUB-ADD)} \qquad \mu(A_1 \cup \cdots \cup A_k) \leq \sum_{i=1}^{k} \mu(A_i)$$

holds for every $k \geq 2$ and every $A_1, \ldots, A_k \in \mathcal{A}$ (where it is not assumed that one would have $A_i \cap A_j = \emptyset$ for $i \neq j$).

[Note: The equality (INC-EXC) of part (c) is sometimes called "the inclusion-exclusion" equality. The property of μ obtained in part (d) is called "sub-additivity".]

Problem 5 solution

(a) Since $\mathcal{A}$ is a field of subset of $\mathbb{R}^n$, then $\emptyset \in \mathcal{A}$
Then $\mu(\emptyset) = \mu(\bigcup_{i=1}^{k} \emptyset) = \sum_{i=1}^{k} \mu(\emptyset) = k \cdot \mu(\emptyset)$ since $\emptyset \cap \emptyset$ is still $\emptyset$.
Since the equality holds for any natrual number, hence must have $\mu(\emptyset) = 0$

(b) Let $C := B \setminus A$, hence have that $B = A \cup C, A \cap C = \emptyset$
Hence $\mu(B) = \mu(A) + \mu(C) \geq \mu(A)$, the equality holds when $A = B, C = \emptyset$

(c) When $A \cap B = \emptyset$. Then $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B) + 0 = \mu(A) + \mu(B)$
When $A \cap B \neq \emptyset$, denote $C := A \cap B$.

$$\begin{aligned} \text{Hence } \mu(A \cup B) + \mu(A \cap B) &= \mu\big((A \setminus C) \cup (B \setminus C) \cup C\big) + \mu(C) \\ &= \mu(A \setminus C) + \mu(B \setminus C) + \mu(C) + \mu(C) \\ &= \big(\mu(A \setminus C) + \mu(C)\big) + \big(\mu(B \setminus C) + \mu(C)\big) = \mu(A) + \mu(B) \end{aligned}$$

(d) Prove this statement by induction on k.
Basic case when $k = 2$, from part (c),
have that $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) - \mu(A_1 \cap A_2) \leq \mu(A_1) + \mu(A - 2)$
Supoose the hypothesis holds for k, consider the case of $k + 1$.

$$\begin{aligned} \mu(\textstyle\bigcup_{i=1}^{k+1}) = \mu(\bigcup_{i=1}^{k} \cup A_{k+1}) &\leq \mu(\textstyle\bigcup_{i=1}^{k}) + \mu(A_{k+1}) \text{ by the basic case.} \\ &\leq \textstyle\sum_{i=1}^{k} \mu(A_i) + \mu(A_{k+1}) \text{ by the hypothesis.} \\ &\leq \textstyle\sum_{i=1}^{k+1} \mu(A_i) \end{aligned}$$

Hence have that (SUB-ADD) holds for $k \geq 2$.

We will accept without proof the "proposition" part of the following "definition + proposition" concerning Jordan measurable sets.

Definition and Proposition C2. Let $A \subseteq \mathbb{R}^n$ be a Jordan measurable set. Consider the real non-negative numbers $V^-(A)$ and $V^+(A)$ defined as follows:

$$V^-(A) := \sup\Big\{ \sum_{i=1}^{r} \mathrm{vol}_o(P_i) \;\Big|\; \begin{array}{c} P_1, \dots, P_r \in \mathcal{P}_n, \text{ with } \cup_{i=1}^r P_i \subseteq A \\ \text{and with } P_i \cap P_j = \emptyset \text{ for } i \neq j \end{array} \Big\}.$$

$$V^+(A) := \inf\Big\{ \sum_{j=1}^{s} \mathrm{vol}_o(Q_j) \mid Q_1, \dots, Q_s \in \mathcal{P}_n, \text{ with } \cup_{j=1}^s Q_j \supseteq A \Big\}.$$

Then $V^-(A) = V^+(A)$. The *volume* of A is defined as

$$\mathrm{vol}(A) := V^-(A) = V^+(A).$$

[Note: In the case that there exists no half-open rectangle $P \in \mathcal{P}_n$ such that $P \subseteq A$, then one takes, by convention, $V^-(A) := 0$. In such a case, the proposition part of C2 assures us that $V^+(A) = 0$ as well, and we have $\mathrm{vol}(A) = 0$.]

Problem 6. Consider the function $\mathrm{vol} : \mathcal{M}_n \to [0, \infty)$ which was introduced in Definition C2.

(a) Prove that $\mathrm{vol}(P) = \mathrm{vol}_o(P)$, for every $P \in \mathcal{P}_n$. (That is: the function $\mathrm{vol} : \mathcal{M}_n \to [0, \infty)$ is an extension of the function $\mathrm{vol}_o : \mathcal{P}_n \to [0, \infty)$ from Definition A1.)

(b) Prove that $\mathrm{vol} : \mathcal{M}_n \to [0, \infty)$ is an additive set-function, in the sense of Definition C1.

(c) Suppose $\mu : \mathcal{M}_n \to [0, \infty)$ is an additive set-function, such that $\mu(P) = \mathrm{vol}_o(P)$ for every $P \in \mathcal{P}_n$. Prove that $\mu(A) = \mathrm{vol}(A)$ for every $A \in \mathcal{M}_n$.

Problem 6 Solution

(a) For every $P \in \mathcal{P}_n$, for every $P_1, \dots, P_r \in \mathcal{P}_n,$ with $\cup_{i=1}^r P_i \subseteq A\,, P_i \cap P_j = \emptyset$ for $i \neq j$
Must have that $\sum_{i=1}^r vol_o(P_i) \leq vol_o(A)$.
Hence have that $V^-(A) = \sup\{\sum_{i=1}^r vol_o(P_i)\} \leq vol_o(A)$.
For every $P \in \mathcal{P}_n$, for every $Q_1, \dots, Q_s \in \mathcal{P}_n,$ with $\cup_{j=1}^s Q_j \supseteq A\Big\}$.
Must have that $\sum_{j=1}^s vol_o(Q_j) \geq vol_o(A)$.
Hence have that $V^+(A) = \inf\{\sum_{j=1}^s vol_o(Q_j)\} \geq vol_o(A)$.
Now have that $\mathrm{vol}(P) = V^-(A) \leq vol_o(A) \leq V^+(A) = \mathrm{vol}(P)$.
Therefore $\mathrm{vol}(P) = \mathrm{vol}_o(P)$.

(b) For $A_1, ..., A_k \in \mathcal{A}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$.

Since $\mathcal{M}_n$ is a field of subsets of $\mathbb{R}^n$, from problem 4 (b).

Then the union $\bigcup_{i=1}^k A_i$ still belong to $\mathcal{M}_n$

Then $\exists$ some $P_1, ...P_r \in \mathcal{P}_n$ with $\bigcup_{j=1}^r P_j \subseteq \bigcup_{i=1}^k A_i$ and $P_i \cap P_j = \emptyset$ for $i \neq j$.

Since these P_j and A_i are both diajoint, reorder these P_j and fix some α_j such that $\bigcup_{i=\alpha_{j-1}+1}^{\alpha_j} P_i \subseteq A_j, \alpha_0 = 0, \alpha_k = r$, which means let α_j be the smallest number such that the union of the $\alpha_{j-1}+1$-th to the α_j-th P is a subset of A_j. Henc have that:

$$\begin{aligned}
vol(A_1 \cup ... \cup A_k) &= V^-(A_1 \cup ... \cup A_k) \\
&= \sup\{\sum_{i=1}^r vol_o(P_i) \,|\, P_1, ...P_r \in \mathcal{P}_n \,,\, \bigcup_{j=1}^r P_j \subseteq \bigcup_{i=1}^k A_i \,,\, P_i \cap P_j = \emptyset \; i \neq j\} \\
&= \sum_{j=1}^k \sup\{ \sum_{i=\alpha_{j-1}+1}^{\alpha_j} vol_o(P_i) \;|\; \bigcup_{i=\alpha_{j-1}+1}^{\alpha_j} P_i \subseteq A_j \,,\, P_i \cap P_j = \emptyset \; i \neq j\} \\
&= \sum_{j=1}^k vol(A_j)
\end{aligned}$$

Therefore $\mathrm{vol} : \mathcal{M}_n \to [0, \infty)$ is an additive set-function.

(c) Since $\mu(P) = vol_o(P)$ and $\mathrm{vol}(P) = vol_o(P)$ from part (a).

Then $\mu(P) = vol_o(P)$ for every $P \in \mathcal{P}_n$.

Then for every $A \in \mathcal{M}_n$ and μ a additive set-function,

$$\begin{aligned}
V_-(A) &= \sup\{\sum_{i=1}^r vol_o(P_i) \,|\, P_1, ...P_r \in \mathcal{P}_n \text{ with } \bigcup_{j=1}^r P_j \subseteq A \text{ and } P_i \cap P_j = \emptyset \text{ for } i \neq j\} \\
&= \sup\{\sum_{i=1}^r \mu(P_i) \,|\, P_1, ...P_r \in \mathcal{P}_n \text{ with } \bigcup_{j=1}^r P_j \subseteq A \text{ and } P_i \cap P_j = \emptyset \text{ for } i \neq j\} \\
&\leq \mu(A) \text{ since } \bigcup_{j=1}^r P_j \subseteq A
\end{aligned}$$

By the similar argument, can have that $V_+(A) \geq \mu(A)$

Hence $\mathrm{vol}(A) = V_-(A) \leq \mu(A) \leq V_+(A) = \mathrm{vol}(A)$.

Therefore $\mu(A) = \mathrm{vol}(A)$ for every $A \in \mathcal{M}_n$.