PMATH 945 Project
On Combinatorial Species

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Abstract

In efforts of trying to provide a unified understanding of various combinatorial structures via generating series, André Joyal introduced combinatorial theory of species. In his masterpiece, André Joyal has provided a simple yet elegant solution to understand and answer various problems that arises from the study of combinatorics. In this paper, we will see how combinatorial species interacts with category theory.

1 Introduction

First introduced by André Joyal in 1981 in his paper "Une théorie combinatoire des series formalles"[4], combinatorial species is the study of combinatorial structures (usually discrete) in a abstract yet systematic way. Topics of interest usually includes but not limited to combinatorial structures such as formal power series, permutations, graphs, etc. Usually, these combinatorial structures are defined by their unique generating functions; hence by analysing the generating functions, we can count the number of combinatorial structures that shares some properties.

Since the last decade, significant progress has been made by various mathematicians on this area of combinatorics. As mentioned by François Bergeron, Gilbert Labelle, and Pierre Leroux in their book "Introduction to the Theory of Species of Structures"[2], various mathematicians has made contributions to the theory of combinatorial species; among all, Rota's work in Möbius inversion, and Stanley's binomial posets and exponential structures have made great contribution to the theory of combinatorial species. Many enumerationists have also underlined the importance of these methods developed from combinatorial species in solving enumerative problems; in particular, Ian P. Goulden and a David M. Jackson, the former dean of mathematics and a professor from the Combinatorics and Optimization department of University of Waterloo respectively, are among these enumerationists.

During the same period, this discipline of combinatorics has been greatly studied and enriched due to its interaction with algebra and theoretical computer science; former is self-explanatory, i.e. due to extensive use of algebraic techniques in enumerative problems in combinatorics, while latter is by virtue of importance of combinatorial structures in algorithms designs. Of particular importance is the study of formal power series, as well as combinatorial structures such as graph and matroid theory. The combinatorial theory of species, in general, provides an in-depth understanding of the extensive use of generating series in combinatorics. In particular, its ability to transform recursive definition of tree-like structures into functionals has greatly improved computer scientists' ability to work with these abstract structures.

Category theory provides us an abstract way of looking at combinatorial species. Although not desperately needed in the combinatorial theory of species, category theory has shed light upon the fundamental understanding of combinatorial species. We will see below that many definitions defined in combinatorial species corresponds well to definitions in category theory; hence results proven in category theory in general can be applied to the translated setting in combinatorial species. On a more category theoretic stand point, we wish to unravel the definitions of combinatorial species in terms of categories, as well as how species interact in terms of category theory.

2 Basic Definitions

The main interests of combinatorial theory are mostly on finite structures. For instance, combinatorialists deals mostly with finite graphs, finite permutations, etc. So if not mentioned, please assume that we are working with finite sets and finite permutations.

Definition 2.1. Given a finite set U, we define a *structure* s to be the pair (γ, U), where γ is a construction on elements of U.

Example. Some example of structures on finite set $U = \{1, \cdots, n\}$.

1. An empty structure 0 induces the empty set $\emptyset$ for all finite set U.
2. A rooted tree $\gamma = (\{r\}, E)$ is a structure on U, where $E \subseteq U \times U$ is defined to be the set of edges of the tree.
3. If we define $\gamma = \{(i, i+1)\}_{i=1}^{n-1}$, then γ is the oriented cycle $1, 2, \cdots, n, 1$.
4. In a matroid $M = (E, \mathscr{I})$, where E is the ground set and $\mathscr{I}$ is the collection of independent set, the collection $\mathscr{I}$ defines a structure on the ground set E.

From the first two examples above, we know that the structure corresponds to the edges of the graphs respectively. In graph theory, two graphs $G = (V, E)$ and $G' = (V', E')$ are said to be isomorphic if we can relabel elements in the vertex sets V into V' such that $E = E'$ upto relabelling of vertices; we can think of it in the following way: if $V = \{1, \cdots, n\}$ and $\sigma \in S_n$, then $V' = \{\sigma(1), \cdots, \sigma(n)\}$ such that $ij \in E$ if and only if $\sigma(i)\sigma(j) \in E'$.

Similar to graphs, given a two finite set $U = \{1, \cdots, n\}$, $V = \{1, \cdots, m\}$, and structures γ on U and λ on V, we would like to have a bijection $\sigma : \gamma(U) \to \lambda(V)$ so that we can "transport" a structure γ on U to the structure λ on V. In fact, since we want a bijection $\gamma(U) \to \lambda(V)$, we may assume $m = n$, i.e. λ, γ are both structures on U. Notice that this corresponds to a category $\mathscr{F}$, where $\mathrm{Ob}(\mathscr{F})$ is the collection of tuples (U, γ) (henceforth by writing U we are abusing notation to mean (U, γ)), and morphisms $\sigma \in \mathrm{Hom}(U, V)$ being bijections between (U, γ) and (V, λ). One can check that the description above indeed defines a category, as the composition rule is just composition of bijections on structures of (possibly different) finite sets, which satisfies associativity, and the identity is given by map that maps to the same structure with the same underlying finite set.

Remark. 1. In the book "Introduction to the Theory of Species of Structures" [2], they use the notation (γ, U) as objects of $\mathscr{F}$ instead of (U, γ). I use the notation (U, γ) because it corresponds well to graph theory and matroid theory.

2. Notice that given a morphism $\sigma : (U, \gamma) \to (V, \lambda)$, one does not require the underlying set U, V to have bijection; it only requires there is a 1-1 correspondence between the structure γ on U and λ on V.

Since $\mathscr{F}$ is a category, we can look at the endofunctor $F : \mathscr{F} \to \mathscr{F}$. Given objects (U, γ), (V, λ), and a bijection $\sigma : (U, \gamma) \to (V, \lambda)$, we have

1. $F[U]$ for all finite structures on the finite set FU
2. $F[\sigma] : F[(U, \gamma)] \to F[(V, \lambda)]$

such that the properties of a functor hold. This motivates the following definitions.

Definition 2.2. Given the category $\mathscr{F}$ and an endofunctor $F : \mathscr{F} \to \mathscr{F}$, we call $s = (FU, F\gamma)$ an *F-structure* if $F\gamma$ defines a structure with underlying set FU. The map $F[\sigma]$ is called the *transport* of F-structure with σ.

Definition 2.3. Let $\mathscr{F}$ be the category of finite sets, with morphisms being the bijections between structures of its underlying finite sets. A *species of structures* (more often *species*) is a functor $F : \mathscr{F} \to \mathscr{F}$ such that F yields an F-structure on A for all $A \in \mathrm{Ob}(\mathscr{F})$, along with its transports $F[\sigma]$ on F-structures along with σ.

In short, a species of structures on the collection of finite set is an endofunctor $F : \mathscr{F} \to \mathscr{F}$. This is good news, as functor is one of the fundamental object of study in category theory. It is not hard to see that $F(\sigma)$ also defines a bijection between objects of $\mathscr{F}$ by definition of F being a functor.

Example. Below are some example on species of interests.

1. Given a finite set U, the species of sets E is just $E[U] = \{U\}$, i.e. it has no additional structure on top of the set U.
2. Given a set $U = \{1, \cdots, n\}$, the species of permutation, denoted $\mathcal{S}$, is just

$$\mathcal{S}[U] = \{\sigma(i) : 1 \leq i \leq n, \sigma \in S_n\}$$

3. For the set $V = \{1, \cdots, n\}$ of vertices, the species of simple graphs, denoted G, is just

$$G[V] = \{(V, E) : E \in V \times V, \text{ elements in } E \text{ are unique}\}$$

One useful tool that lets us compare mathematical objects is via isomorphisms. Given structures (U, γ) and (V, λ), if the structure γ on U is isomorphic on the structure λ on V, it makes sense that we would like to have them "equal" in some way under the species F.

Definition 2.4. Given $\mathscr{F}$ the category of finite structures on finite sets, and a species F, if s_1, s_2 are two F-structures, we say that s_1 is isomorphic to s_2 if and only if $s_2 = F[\sigma](s_1)$ for some bijection σ, a morphism of $\mathscr{F}$. We say that the structures s_1, s_2 have the same isomorphism type. If $s_1 = s_2$, then it is called an automorphism of s_1.

What are some well-known species of structures? Bergeron, Labelle, and Leroux [2] has listed some commonly used species in their book. Below is a table of some well known species. We will take the underlying finite set as $U = \{1, \cdots, n\}$.

Species Name	Symbol of Species	Structure
Empty Species	0	$\emptyset$
Species of Sets	E	$\{U\}$
Species of Elements	$\mathcal{E}$	U
Species of Rooted Trees	$\mathcal{A}$	(Edges of graph, $\{r\}$)
Species of Simple Graphs	$\mathcal{G}$	$\{uv : u, v \in V(G)\}$
Species of Permutations	$\mathcal{S}$	$\{\sigma(i) : i \in U, \sigma \in S_n\}$

More details can be found in the book by Bergeron, Labelle, and Leroux [2].

3 Generating Series

Like a lot of mathematical objects, there are operations that we can do with species. However, to be to talk about operations on species of structures, we need to talk about formal power series; in particular, we need the generating series of the species F. Recall that given a field $\mathbb{F}$, a formal power series with indeterminate t is $f(t) = \sum_{n\geq 0} a_n t^n$ where $a_n \in \mathbb{F}$.

Definition 3.1. Let F be a species of structures. The *generating series* of F is a formal power series $\sum_{n\geq 0} f_n \frac{x^n}{n!}$, where $f_n := \big|F[n]\big|$, the number of F-structures on n-elements set.

Notice that when $f_n = 1$ for all n, then $F(x) = e^x$; hence $F(x)$ is sometimes referred as exponential power series. Given a formal power series $G(x) = \sum_{n\geq 0} g_n x^n$, we use $[x^n]$ to denote the n^{th} coefficient of $G(x)$, i.e. $[x^n]G(x) := g_n$. Now for as an exponential power series, if we multiply by $n!$, we get

$$n![x^n]F(x) = f_n = \frac{d^n}{dx^n}F(x)\bigg|_{x=0}$$

that is, the n^{th} Taylor coefficient evaluated at the origin. So given indeterminate $x_1, x_2, \cdots$, if we have a formal power series

$$H(x_1, x_2, x_3, \cdots) = \sum_{n_1, n_2, \cdots} h_{n_1, n_2, \cdots} \frac{x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots}{c_{n_1, n_2, n_3, \cdots}}$$

where $c_{n_1,\cdots}$ is a family of non-zero scalars, then $c_{n_1,\cdots}[x_1^{n_1} x_2^{n_2} \cdots]H(x_1, \cdots) = h_{n_1,\cdots}$.

Given a species, it is possible to calculate the generating series. For instance, for the empty species, the structure on $U = \{1, \cdots, n\}$ is 0 for all n, since it induces the empty set. Hence $0(x) = 0$. Perhaps a less trivial example would be the generating functions for species of permutation. Given $U = \{1, \cdots, n\}$, recall that

$$\mathcal{S}[U] = \{\sigma(U) : \sigma \in S_n\}$$

where $\sigma(U) := \{\sigma(i) : i \in U\}$. So $f_n = |\mathcal{S}[U]| = |S_n| = n!$. So the generating series of $\mathcal{S}$ is

$$\mathcal{S}(x) = \sum_{n\geq 0} \frac{f_n}{n!} x^n = \sum_{n\geq 0} x^n = \frac{1}{1-x}$$

Also a slightly less trivial example is $\mathcal{G}(x)$. Given U, the possible $\mathcal{G}$-structures are the set of all possible graphs on the set of n vertices. So $\mathcal{G}(x) = \sum_{n\geq 0} 2^{\binom{n}{2}} \frac{x^n}{n!}$. It is easy to see that for E and $\mathcal{E}$, f_n are 1 and n respectively. Below is a table of species mentioned earlier:

Species	f_n	Generating series
Empty Species, 0	0	0
Species of Sets, E	1	e^x
Species of Elements, $\mathcal{E}$	n	xe^x
Species of Simple Graphs, $\mathcal{G}$	$2^{\binom{n}{2}}$	$\sum_{n\geq 0} 2^{\binom{n}{2}} \frac{x^n}{n!}$
Species of Permutations, $\mathcal{S}$	$n!$	$\frac{1}{1-x}$

3.1 Type Generating Series

It is possible for two species to have the same generating series. In this section, we will introduce another useful characteristic of species, called the type generating series. Together with the generating series, these two series are the characteristics that defines a series.

We will first show that generating series do not uniquely define the species type; to do this, we will look at two particular species, namely the permutation species $\mathcal{S}$, and the linear species (also known as the species of total order) L.

The linear species on a finite set U has the structure $(U, \leq)$ where $\leq$ is a total order, i.e. for all $x, y \in U$, either $x \leq y$ or $y \leq x$, on top of the axioms of a partial order. It is easy to see that there are $n!$ such total order by enumerating the elements; hence the generating series for linear species is also $L(X) = \frac{1}{1-x} = \mathcal{S}(x)$.

Recall that if s_1, s_2 are two F-structures such that $s_1 = F[\sigma](s_2)$, then s_1, s_2 have the same isomorphism types. We may assume that $U = [n] := \{1, \cdots, n\}$ for simplicity.

Definition 3.2. We define $\sim$ to be the relation on the set $F[n]$ by $s \sim t$ if and only if s and t have the same isomorphic type.

Proposition 3.3. *The relation $\sim$ as defined above is an equivalence relation.*

Proof. Let $s, t, u \in F[n]$. To prove reflexivity, pick σ to be an automorphism of s; for symmetry, use the definition of isomorphism of s and t i.e. if $F[\sigma](s) = t$, then σ^{-1} such that $F[\sigma^{-1}](t) = s$ is given by undoing the permutation σ; for transitivity just use the composition of permutations, i.e. if $\sigma, \tau : [n] \to [n]$ such that $F[\sigma](s) = t$ and $F[\tau](t) = u$, then $F[\tau \circ \sigma](s) = u$. ☺

The equivalence classes of $\sim$ are called the *unlabelled F-structures of order n*. They are basically the same structures up to permutation. Let us denote by $T(F_n) = F[n]/\sim$, and

$$T(F) := \sum_{n \geq 0} T(F_n) = \sum_{n \geq 0} F[n]/\sim$$

Similar to species of labelled structures, one can define generating series of the unlabelled structures.

Definition 3.4. The *(isomorphism) type generating series* is the formal power series of a species F, defined as below

$$\widetilde{F}(x) = \sum_{n \geq 0} \widetilde{f}_n x^n$$

where $\widetilde{f}_n = |T(F_n)|$ is the number of unlabelled structures of order n.

Now we can see that for the linear species, $\widetilde{f}_n = 1$, since all total order are the same up to permutation. So $\widetilde{L}(x) = \frac{1}{1-x}$. On the other hand, for the permutation species, we can show with the aide of Burnside's lemma (also Cauchy-Frobenius lemma) that $\widetilde{\mathcal{S}}(x) = \prod_{k \geq 1} \frac{1}{1-x^k}$; details are theorem 1.13 and example 1.14 in Bergeron, Labelle, and Leroux [2]. So $\widetilde{L}(x) \neq \widetilde{\mathcal{S}}(x)$. One should also point out that $\widetilde{E}(x) = \widetilde{L}(x) = \frac{1}{1-x}$, so the type generating series alone is not enough to define a species.

Definition 3.5. For two species F, G, we say that F is isomorphic to G if there exists a family of bijections $\alpha_U : F[U] \to G[U]$ such that the following diagram commutes

$$\begin{tikzcd} F[U] \arrow[r, "\alpha_U"] \arrow[d, "F[\sigma]"] & G[U] \arrow[d, "G[\sigma]"] \\ F[V] \arrow[r, "\alpha_V"] & G[V] \end{tikzcd}$$

In categorical point of view, the two species (functors) are isomorphic if and only if there exist bijective natural transformations between F and G.

We will end this section with a fact: if F and G are two isomorphic species, written $F \simeq G$, then the generating series and the type generating series are equal, i.e. $F(x) = G(x)$, and $\widetilde{F}(x) = \widetilde{G}(x)$. Therefore generating series and type generating series uniquely defines a species.

4 Operations on Species of Structures

Often in combinatorics problem, one defines operations on the object of interest. For instance, in enumeration, one might want to talk about the addition and multiplication of two formal power series; in graph theory, one might be interested in the the tensor product of graphs, Cartesian product of graphs, and the strong graph products. These operations on combinatorial objects allows us to study how two combinatorial objects interacts with each other in details.

The motivation behind this is to mimic how formal power series interact in enumeration of a string; by investigating how respective formal power series reacts, we are able to tell how two strings interact and what should the resulting string when we do these operations.

Similar to other combinatorial objects, we can define operations on species of structures. By defining operations such as addition and multiplication, we will see how these species interact with one another; in particular, how their generating series interacts. As a result, when we look at the species of finite graphs $\mathcal{G}$, we are able to tell what the characteristics of the resulting finite graph should have.

4.1 Addition of Species of Structure

Definition 4.1. Let F and G be two species of structures, and $U = [n]$. We define the *sum of species* on U, written as $(F + G)[U]$, to be the F-structure or the G-structure of U.

Remark. The "or" statement above is exclusive; that is, either $(F+G)[U]$ has an $F[U]$-structure but not $G[U]$-structure or vice versa.

We will write $(F + G)[U] = F[U] + G[U]$, where "+" is to be understood as the disjoint union. Quick example: Consider the species of finite connected graph $\mathcal{G}^c$ and the species of finite disconnected graph $\mathcal{G}^d$, we can then write the species of finite graph as $\mathcal{G} = \mathcal{G}^c + \mathcal{G}^d$. If given two species of structures F and G such that $F[U] \cap G[U] \neq \emptyset$, then it is customary to think of $F + G$ contains two copy of this structure; we can think about it as $(F+G)[U] = F[U]\times\{0\}+G[U]\times\{1\}$.

One good consequence by defining the addition operation this way is that the empty species 0 acts as an identity element. Given any species F, we have

$$(F + 0)[U] = (0 + F)[U] = F[U] + \emptyset = F[U]$$

This is useful as we can write $(F + F)[U] = 2F[U]$, since $2F[U]$ contains exactly two copies of each structures in $F[U]$. Moreover, we also have the following proposition.

Proposition 4.2. *Let $U = [n]$, and F, G are two (possibly same) species of structures. The following holds:*

1. $(F + G)(x) = F(x) + G(x)$;
2. $(\widetilde{F + G})(x) = \widetilde{F}(x) + \widetilde{G}(x)$

So the addition operation is well defined.

Proof. Suppose $F(x) = \sum_{n\geq 0} f_n \frac{x^n}{n!}$, $G(x) = \sum_{n\geq 0} g_n \frac{x^n}{n!}$. Then by defining $h_n = f_n + g_n$ for all n, it is easy to see that

$$(F + G)(x) = \sum_{n\geq 0} h_n \frac{x^n}{n!} = F(x) + G(x)$$

they are a consequence of the definition of $|(F + G)[n] = |F[n]| + |G[n]|$ being disjoint union.

Similarly, suppose $\widetilde{F}(x) = \sum_{n\geq 0} \widetilde{f}_n x^n$, and $\widetilde{G}(x) = \sum_{n\geq 0} \widetilde{g}_n x^n$. By defining

$$\widetilde{h}_n = \widetilde{f}_n + \widetilde{g}_n \ \forall n$$

we can see that $|(F+G)[n]/\sim| = |F[n]/\sim| + |G[n]/\sim|$, so $(\widetilde{F+G})(x) = \widetilde{F}(x) + \widetilde{G}(x)$. ☺

This leads to the definition of summable species of structures, namely F is summable if $F = F_0 + F - 1 + F_2 + \cdots$ for species F_n, $n \geq 0$, and the canonical decomposition of species, namely the F_n's. These are useful constructions that many combinatorialists are interested in.

4.2 Multiplication of Species of Structure

As mentioned earlier, various products are considered in combinatorial objects such as graphs and matroids. In order to study sombinatorial species properly, we need to define similar operation on species of structuers.

Definition 4.3. Let F and G be two species, with finite set $U = [n]$. We define the *product of F and G* to be the $(F\cdot G)$-structure (or sometimes FG-structure) on U to be the ordered pair $s = (s_1, s_2)$ such that the following conditions hold:

- s_1 is an F-structure on $U_1 \subseteq U$;
- s_2 is an G-structure on $U_1 \subseteq U$;
- $U = (U_1, U_2)$ is a partition of U, i.e. $U = U_1 \sqcup U_2$, $U_1 \cap U_2 = \emptyset$.

Simply put, if we write $U = (U_1, U_2)$ to denote the partition U_1, U_2 of U, then

$$(FG)[U] = \sum_{(U_1,U_2)=U} F[U_1] \times G[U_2]$$

where $\times$ here denotes the Cartesian product, that is, $FG[U]$ denotes the sum over all partitions of U of the respective Cartesian products. Suppose σ is a morphism of structures of FG-structure $s = (s_1, s_2)$ on U, then we can write

$$FG[\sigma](s) = (F[\sigma_1](s_1), G[\sigma_2](s_2))$$

where $\sigma_i := \sigma|_{U_i}$ for $i = 1, 2$. What is this saying? We can think of FG as the species on the set U to contain all the possible combination of ordered pairs of F and G structures over the partition of U. Just like addition, it would be great if we can write the generating series and the type generating series of the multiplication of two species in terms the generating series and type generating series of the two species.

Proposition 4.4. *Suppose $U = [n]$, and F, G are two species of structures. Consider the multiplication of two species $F \cdot G$, we have*

1. $(F \cdot G)(x) = F(x)G(x)$;
2. $(\widetilde{F \cdot G})(x) = \widetilde{F}(x)\widetilde{G}(x)$

Therefore multiplication is also well-defined.

Proof. Similar to addition, and use $h_n := \sum_{k+l=n} \binom{n}{k} f_k \cdot g_l$, and $\widetilde{h}_n := \sum_{k+l=n} \widetilde{f}_k \cdot \widetilde{g}_l$ instead. ☺

Similar to addition, the multiplication operation on species also has identities. Let 1 be the 1-species, i.e. $1[U] = \{U\}$ if $U = \emptyset$, and $\emptyset$ otherwise. So the 1-species induces the empty structure for any non-empty set U, and induces the identity structure on empty sets. Note that the 0-species and 1-species are different, since $0(x) = x$, and $1(x) = 1$. Then 1 is the multiplicative identity of species of structures i.e. $F \cdot 1 = 1 \cdot F = F$. Also, $F \cdot 0 = 0 \cdot F = 0$. The multiplication operation is associative and commutative up to isomorphism; however, $F \cdot G$ and $G \cdot F$ are not identical in general.

4.3 Composition & Derivative of Species

Motivated by theory for formal power series, once can define compositions and derivative on species of structures. The idea is to compose the respective generating series and taking derivative of generating series, as one would do with formal power series. Notice that this is only possible because the generating series and type generating series can also be viewed as formal power series.

To be able to define composition and derivative of species, one turns to cycle index series of species F, denoted $Z_F(x_1, x_2, \cdots)$, which is well-studied by various combinatorialists including Joyal and Bergeron. Details of defining composition and derivatives on species is in Bergeron et al. [2]. One should mention that one can also study the addition and multiplication of species in terms of cycle index series, and that it is in some cases easier to compute the generating series and type generating series through cycle index series. To see a succinct construction of these operations, please refer to [3] §2.

5 Connection to Module Structures

Given the species of structures and the addition, multiplication operation defined, one can easily see that this defines a semi-ring. As one would expect, it is possible to define a ring S on the class of species. More concretely, the construction is given by Bergeron et al. in the book [1].

As mentioned earlier, given a species F, we write $F + F = 2F$. This is an analogue to the natural numbers, where the n-fold sum of 1 is n, i.e. $\underbrace{1 + \cdots + 1}_{n \text{ copies}} = n$. This leads to $mF + nF = lF$ if and only if $m + n = l$. This leads to a limited form of subtraction of species, namely $lF - nF = mF$ if and only if $l - n = m$. We wish to extend this result to arbitrary summable species.

5.1 Construction of the Ring $\mathbb{S}$

To define a ring on the class of species of structures, one require the additive inverse of each species to exists. To do this, we need to define the virtual ring of species.

Definition 5.1. Let $(S, +, \cdot)$ be the semi-ring of species. We define the *ring of virtual species* to be $V = S \times S / \sim_v$, where

$$(F, G) \sim_v (H, K) \iff F + G = H + K$$

Remark. It is easy to see that $\sim_v$ is an equivalence relation. We write the equivalence class of (F, G) under $\sim_v$ as $F - G$.

Now, any species F in the semi-ring S can be written as $F - 0$ under $\sim_v$ described above. Then we can define the additive inverse to be $0 - F$. In other words,

$$(F - 0) + (0 - F) = F - F = 0 - 0$$

is the additive identity in the ring of virtual species. On the other hand, since 1 is the multiplicative identity in the semi-ring of species, so it follows that $1 - 0$ is the multiplicative identity of the ring of virtual species. We can now define the addition and multiplication in the ring of virtual species, namely

Addition $(F - G) +_v (H - K) = (F + H - (G + K)$

Multiplication $(F - G) \cdot_v (H - K) = (F \cdot H + G \cdot K) - (F \cdot K + G \cdot H)$

for all species F, G, H, K, and "+" and "$\cdot$" denotes the addition and multiplication of the species. To simplify the notation we will just refer to "$+_v$" and "$\cdot_v$" as "+" and "$\cdot$" instead.

Unsurprisingly, by defining the subtraction in this way, we get that $(F-G)(x) = F(x)-G(x)$, $\widetilde{F-G}(x) = \widetilde{F}(x) - \widetilde{G}(x)$. This can be extended to the cycle index series too; we can see that $Z_{F-G} = Z_F - Z_G$. Thus we can complete the class of virtual species by extending these result onto species operations like multiplication, composition, and derivation. If we take $\mathbb{S}$ to be the ring of virtual species, V, then we can see that the composition of species actually forms an $\mathbb{S}$-vector space like structure, i.e. an $\mathbb{S}$-module.

6 Conclusion and Future Work

Category theory provides a abstract algebra setting to combinatorial objects. in effort of trying to understand combinatorial objects in depth in an abstract way gives rise to the theory of combinatorial species. Although category theory is absolutely necessary required to understand combinatorial species, it has proven to be a useful tool to posses.

It is surprising to me that how combinatorial species is related to the category of $\mathbb{S}$-modules; by building a ring $\mathbb{S}$, one can understand these species via studying the category of $\mathbb{S}$-modules, which latter is well-studied by ring theorists.

It is yet unknown to me how various R-module structures corresponds to species theory; for instance, how tensor products interact with species, in particular with $\mathbb{S}$-modules. On one end, it is exciting to see how species theory can be used to answer various questions in enumeration and graph theory; on the other end, it is thrilling to see how species theory evolved, and what theories developed on top of category of R-modules tells us about combinatorial species.

Finally, in memory of Gian-Carlo Rota and his contribution in combinatorial species, I shall end this project with a quote from his forewords for the book "Combinatorial Species & Tree-like Structures" [1]:

> *"I dare make a prediction on the future of acceptance of this book. At first, the old fogies will pretend the book does not exists. This pretense will last sufficiently many younger combinatorialists publish papers in which interesting problems are solved using the theory of species. Eventually, a major problem will be solved in the language of species, and from that time on everyone will have to take notice."*

References

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