Cubical Agda: A Dependently Typed Programming Language with Univalence and Higher Inductive Types

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Cubical Type Theory

- Cubical Type Theory: a constructive interpretation of the univalence axiom. Cyril Cohen, Thierry Coquand, Simon Huber, Anders Mörtberg
  TYPES '15 post-proceedings.
- On Higher Inductive Types in Cubical Type Theory
  Thierry Coquand, Simon Huber, Anders Mörtberg, LICS '18.

Types	Spaces
$x \equiv_A y$	There is a path from x to y in A
$i : \mathbb{I} \vdash t : A$	path represented as a map $[0, 1] \to A$

- Univalence is a theorem and transports along it compute.
- HITs have eliminators that compute on both point and path constructors.

Cubical Agda

An extension of Agda with support for Cubical Type Theory.

In addition it supports:

- Nested pattern matching with HITs.
- Bisimilarity as equality for coinductive types.
- Interval $\mathbb{I}$, and restriction $A\,[\,r \mapsto u\,]$ types.
- ...

Growing library on github: http://github.com/agda/cubical

17 contributors, 115 pull requests.

Cubical Agda

Interval I has endpoints i0 : I and i1 : I.

$$\mathsf{PathP} : (A : \mathsf{I} \to \mathsf{Set}\ \ell) \to A\ \mathsf{i0} \to A\ \mathsf{i1} \to \mathsf{Set}\ \ell$$

Given $t : (i : \mathsf{I}) \to A\ i$, we have $\lambda i \to t\ i : \mathsf{PathP}\ A\ (t\ \mathsf{i0})\ (t\ \mathsf{i1})$

Given $p : \mathsf{PathP}\ A\ a_0\ a_1$ and $r : \mathsf{I}$, we have $p\ r : A\ r$.

We have usual β and η rules and also:

$$p\ \mathsf{i0} = a_0$$
$$p\ \mathsf{i1} = a_1$$

In the rest of the talk

$$\_\equiv\_ : \{A : \mathsf{Set}\ \ell\} \to A \to A \to \mathsf{Set}\ \ell$$
$$\_\equiv\_\ \{A = A\}\ x\ y = \mathsf{PathP}\ (\lambda\ \_ \to A)\ x\ y$$

Coinductive Types

```
record Stream (A : Set) : Set where
  coinductive; constructor _,_
  field
    head : A
    tail : Stream A

mapS : (A → B) → Stream A → Stream B
mapS f xs .head = f (xs .head)
mapS f xs .tail = mapS f (xs .tail)

mapS-id : (xs : Stream A) → mapS (λ x → x) xs ≡ xs
mapS-id xs i .head = xs .head
mapS-id xs i .tail = mapS-id (xs .tail) i
```

Bisimilarity

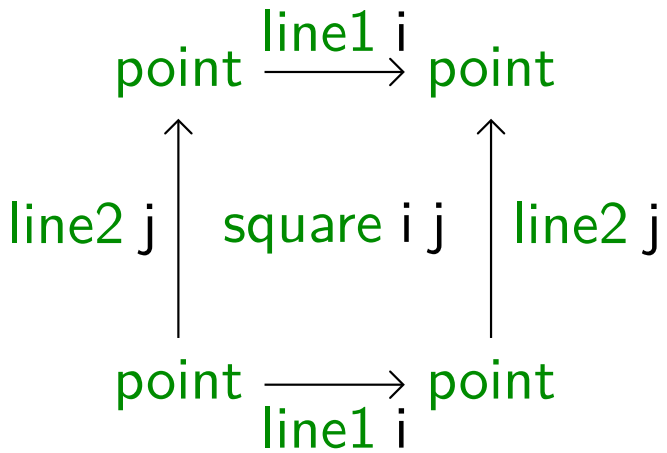
```
record _≈_ (xs ys : Stream A) : Set where
  coinductive
  field
    ≈head : xs .head ≡ ys .head
    ≈tail : xs .tail ≈ ys .tail

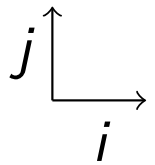
bisim : ∀ {xs ys : Stream A} → xs ≈ ys → xs ≡ ys
bisim xs≈ys i .head = xs≈ys .≈head i
bisim xs≈ys i .tail = bisim (xs≈ys .≈tail) i

path≡bisim : ∀ {xs ys : Stream A} → (xs ≡ ys) ≡ (xs ≈ ys)
```

Synthetic Homotopy Type Theory

```
data Torus : Set where
  point  : Torus
  line1  : point ≡ point
  line2  : point ≡ point
  square : PathP (λ i → line1 i ≡ line1 i) line2 line2
```





Torus $\simeq S^1 \times S^1$

```
data S¹ : Set where
  base : S¹
  loop : base ≡ base
```

```
t2c : Torus → S¹ × S¹
t2c point       = (base   , base)
t2c (line1 i)   = (loop i , base)
t2c (line2 j)   = (base   , loop j)
t2c (square i j) = (loop i , loop j)
```

```
c2t : S¹ × S¹ → Torus
c2t (base , base)   = point
c2t (loop i , base) = line1 i
c2t (base , loop j) = line2 j
c2t (loop i , loop j) = square i j
```

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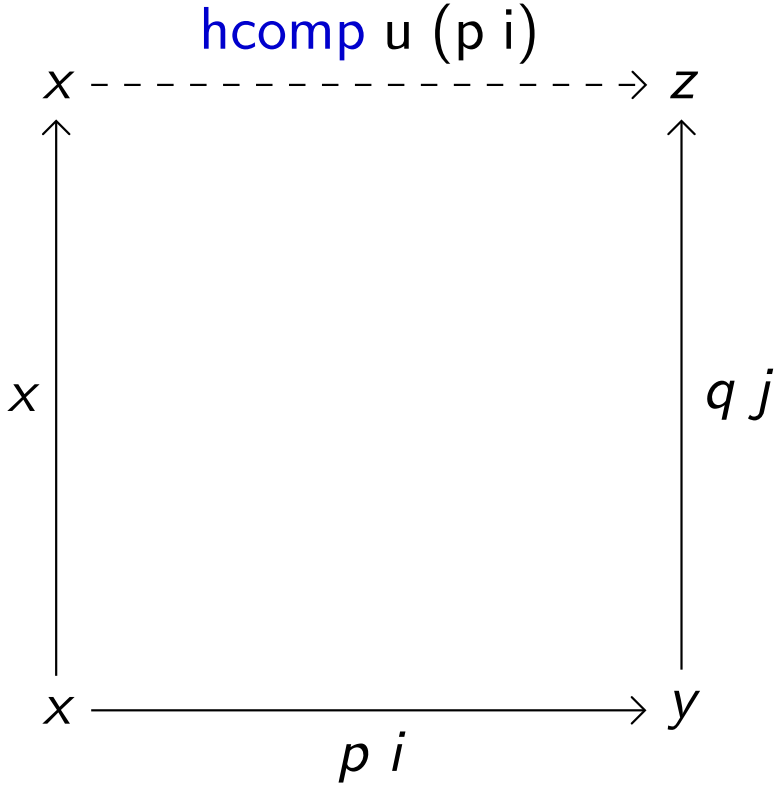
c2t : S¹ × S¹ → Torus
c2t (base , base)   = point
c2t (loop i , base) = line1 i
c2t (base , loop j) = line2 j
c2t (loop i , loop j) = square i j
```

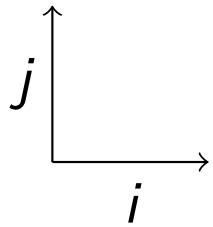
```
c2t-t2c : (t : Torus) → c2t (t2c t) ≡ t
c2t-t2c point        = refl
c2t-t2c (line1 _)    = refl
c2t-t2c (line2 _)    = refl
c2t-t2c (square _ _) = refl
```

hcomp

Given $p : x \equiv y$ and $q : y \equiv z$,

$$\mathsf{u} : (j : \mathsf{I}) \to \mathsf{Partial}\ (\sim i \vee i)\ A$$
$$\mathsf{u} = \lambda\, j \to \lambda\, \{\ (i = \mathsf{i0}) \to x;\ (i = \mathsf{i1}) \to q\ j\ \}$$





hcomp: computation

General form:

$$\text{hcomp}\ (\lambda\ j \to \lambda\ \{\ (r = \text{i1}) \to u\ j\ \})\ u_0 : A$$

It reduces according to the type A.

For Higher Inductive Types it behaves like a constructor:

$$\text{hcomp}\ r\ u\ u_0$$

but obeying

$$\text{hcomp i1}\ u\ u_0 = u\ \text{i1}\ \text{1=1}$$

Pattern Matching

$$
\begin{aligned}
&\mathsf{c2t} : \mathsf{S}^1 \times \mathsf{S}^1 \to \mathsf{Torus} \\
&\mathsf{c2t}\ (\mathsf{base}\ ,\ \mathsf{base}) &&= \mathsf{point} \\
&\mathsf{c2t}\ (\mathsf{loop}\ i\ ,\ \mathsf{base}) &&= \mathsf{line1}\ i \\
&\mathsf{c2t}\ (\mathsf{base}\ ,\ \mathsf{loop}\ j) &&= \mathsf{line2}\ j \\
&\mathsf{c2t}\ (\mathsf{loop}\ i\ ,\ \mathsf{loop}\ j) &&= \mathsf{square}\ i\ j
\end{aligned}
$$

Missing hcomp cases, discovered by coverage checking.

$$
\begin{aligned}
&\mathsf{c2t}\ (\mathsf{hcomp}\ r\ u\ u_0\ ,\ y) &&= ?0 \\
&\mathsf{c2t}\ (\mathsf{base}\ ,\ \mathsf{hcomp}\ r\ u\ u_0) &&= ?1 \\
&\mathsf{c2t}\ (\mathsf{loop}\ i\ ,\ \mathsf{hcomp}\ r\ u\ u_0) &&= ?2
\end{aligned}
$$

Inferring cases

$$\mathsf{c2t}(\mathsf{loop}\ i, \mathsf{hcomp}\ r\ u\ u_0) = ?2$$

$$?2 := \mathsf{hcomp}\ (\lambda\ j \to \lambda \left\{ \begin{array}{l} (r = \mathsf{i1}) \to \mathsf{c2t}\ (\mathsf{loop}\ i, u\ j\ 1{=}1) \\ (i = \mathsf{i0}) \to \mathsf{c2t}\ (\mathsf{base}, \mathsf{hcomp}\ r\ u\ u_0) \\ (i = \mathsf{i1}) \to \mathsf{c2t}\ (\mathsf{base}, \mathsf{hcomp}\ r\ u\ u_0) \end{array} \right\})$$
$$(\mathsf{c2t}\ (\mathsf{loop}\ i, u_0))$$

Conclusions

Cubical Agda: part of Agda 2.6.0.1, `--cubical` flag.

cubical library: `github.com/agda/cubical`

Further work:

- Support for general inductive families.
- Translation to eliminators.
- Compilation.
- Decidability of typechecking proof.
- Stealing from redtt: extension types, ...