Vector and Matrix Basics

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http://wangshusen.github.io/

Vector and Matrix

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Matrix ($m \times n$) $$\mathbf{A} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ columns}} \left.\vphantom{\begin{bmatrix} a \\ a \\ \vdots \\ a \end{bmatrix}}\right\} m \text{ rows}$$

Additions

Vector Addition

- Given $n \times 1$ vectors: $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^n$.
- Vector addition: $\underline{\mathbf{c}} = \underline{\mathbf{a} + \mathbf{b}} \in \mathbb{R}^n$.

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Initialization: $\mathbf{c} \leftarrow [0, 0, \cdots, 0]$.

For $i = 1$ to n:

$$c_i \leftarrow a_i + b_i.$$

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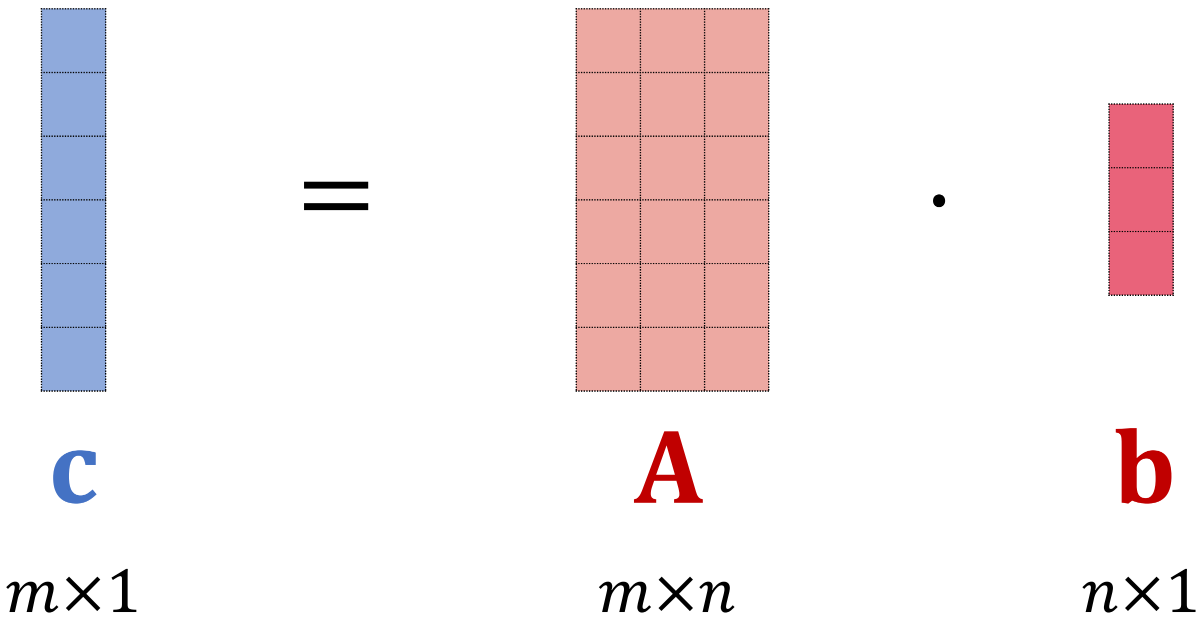

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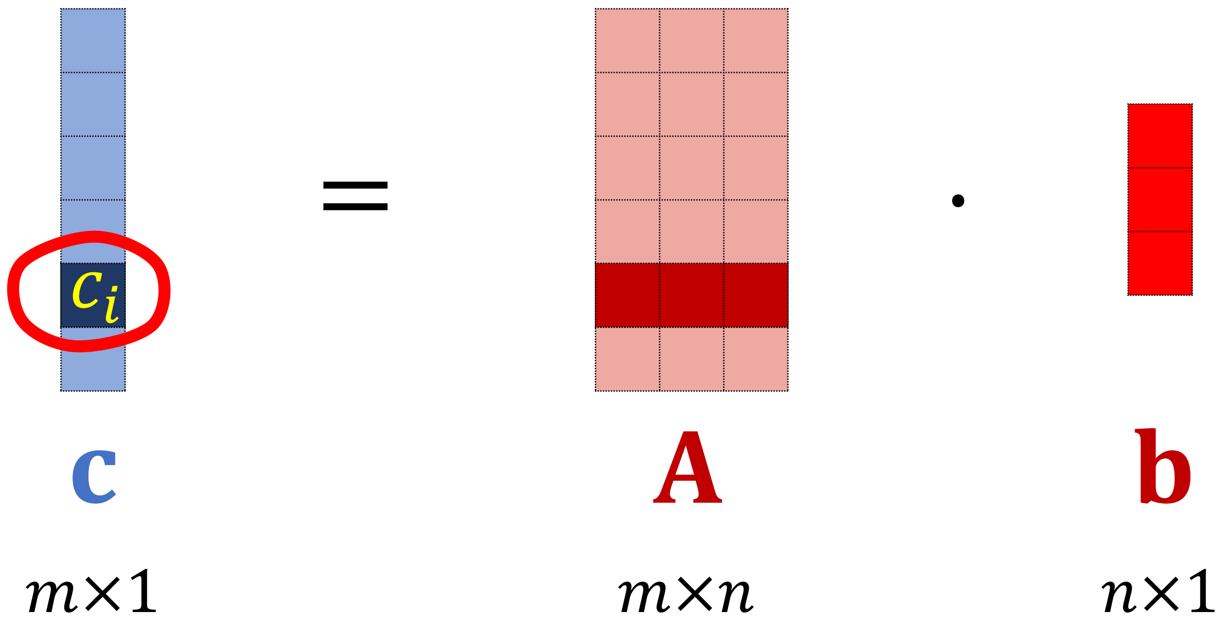

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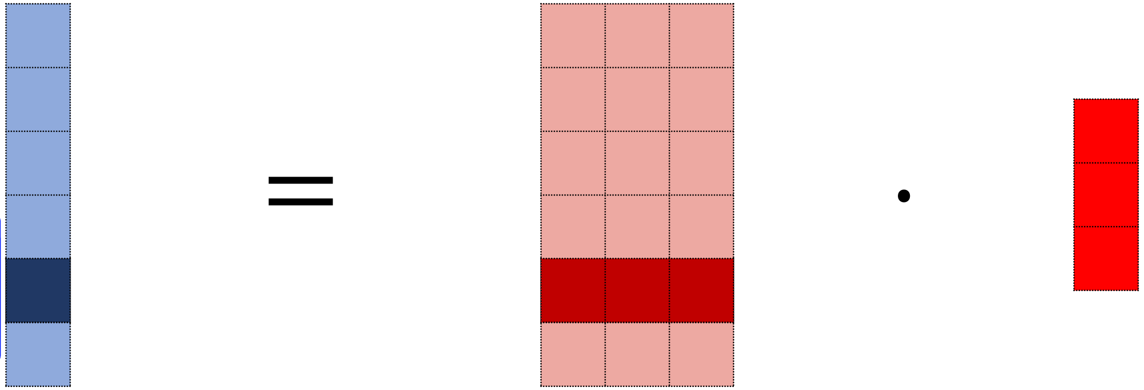

$$\mathbf{c} = \mathbf{A} \cdot \mathbf{b}$$

$m \times 1$ $m \times n$ $n \times 1$

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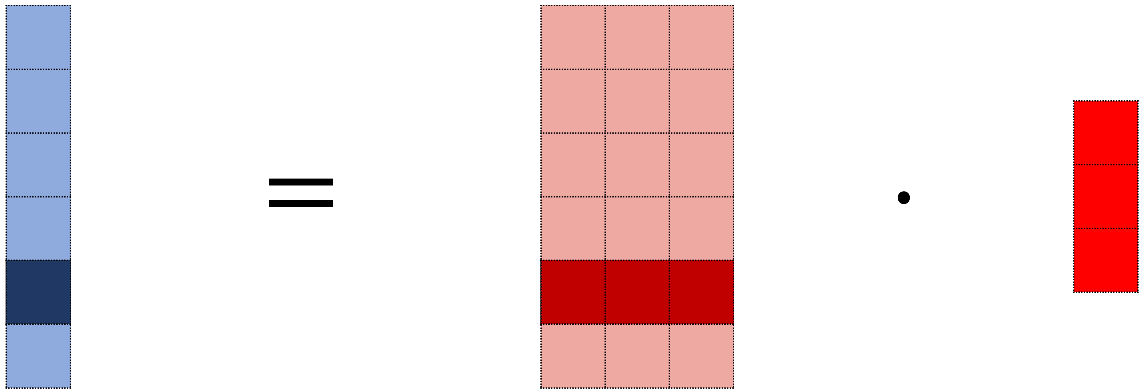

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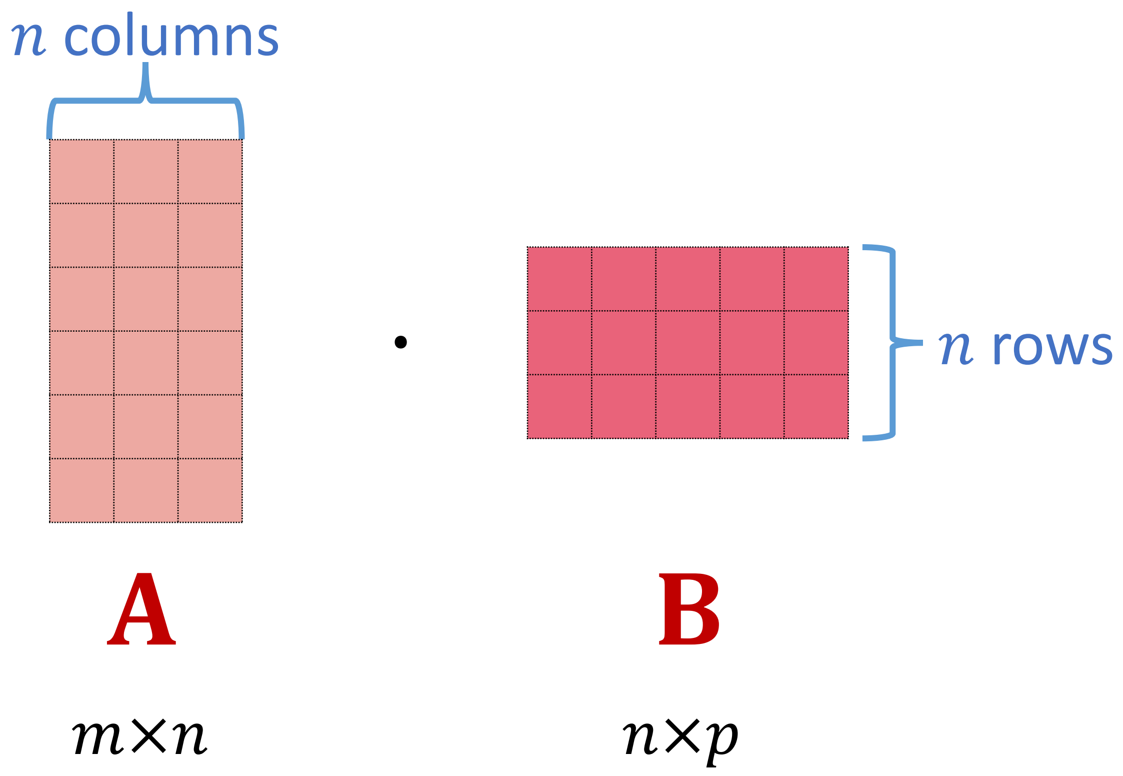

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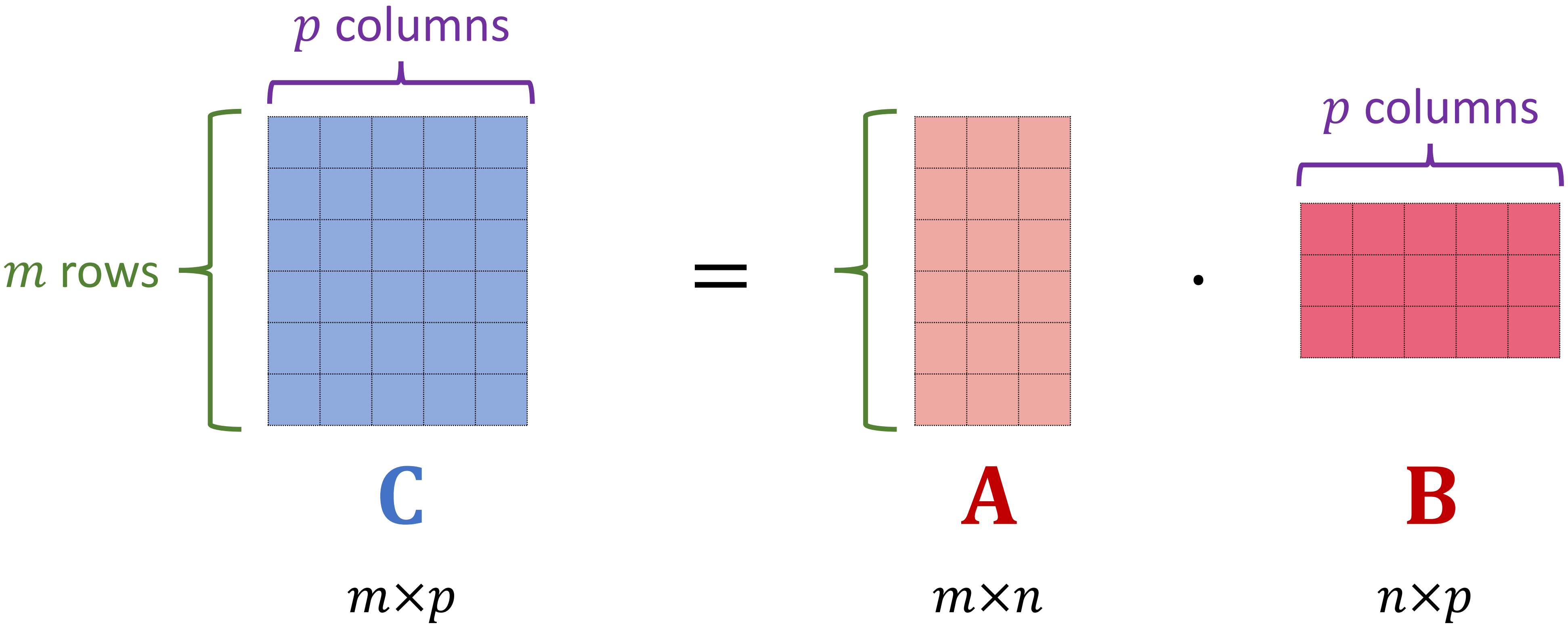

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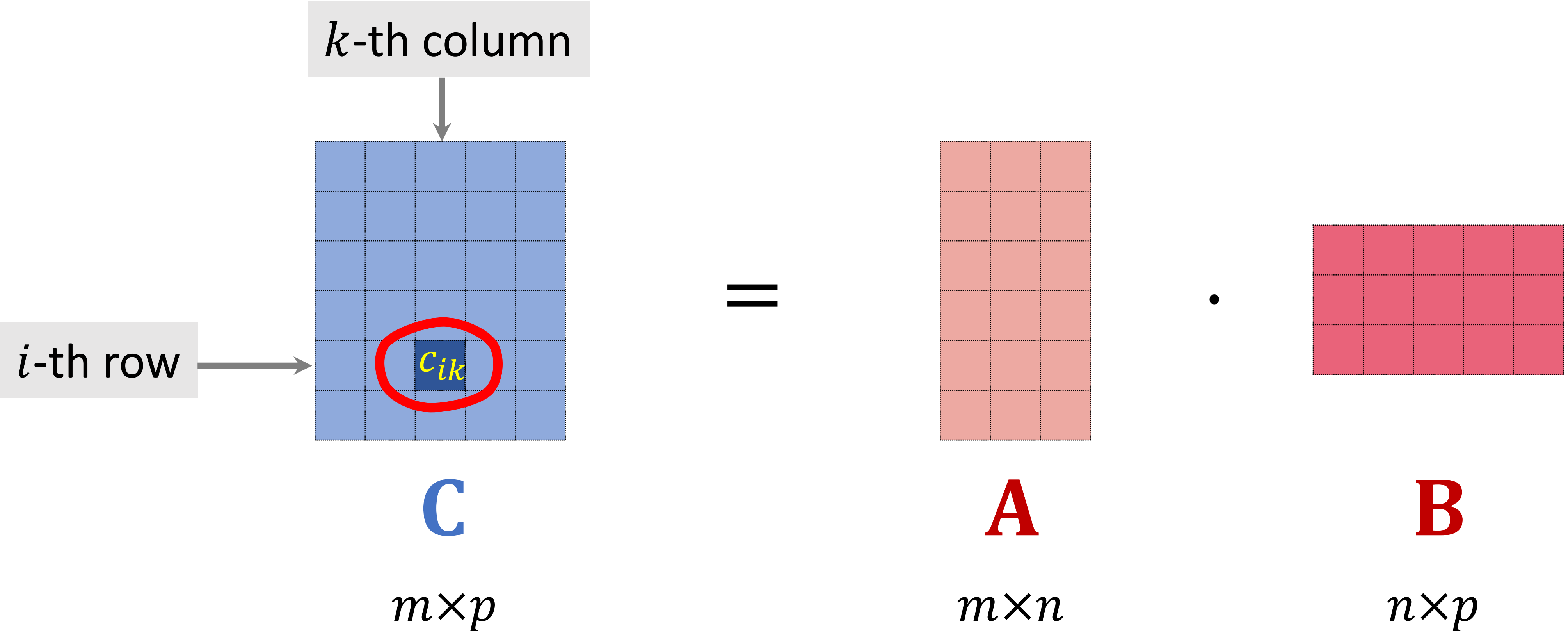

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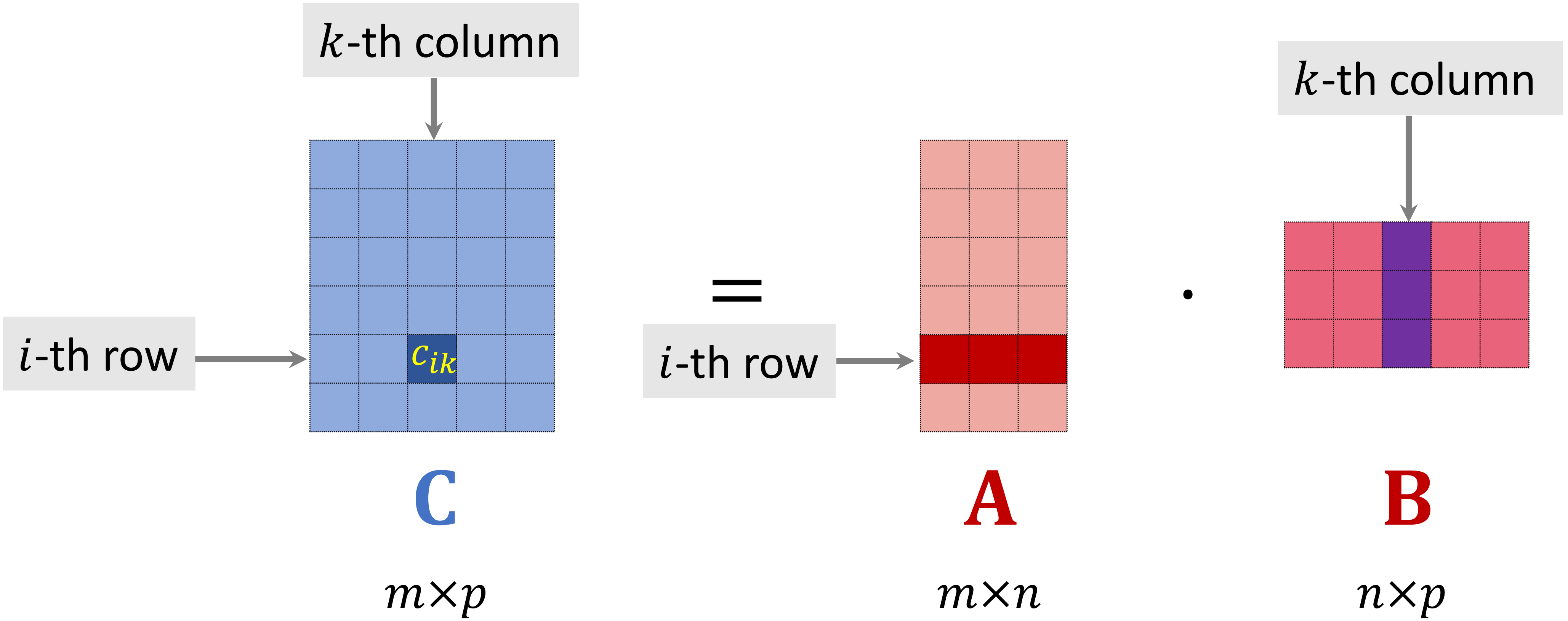

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Time complexity: $O(mnp)$.

Summary

Addition

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- Given matrix $\boxed{\mathbf{A} \in \mathbb{R}^{n \times n}}$ and matrix $\boxed{\mathbf{B} \in \mathbb{R}^{n \times n}}$.

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- **Vector addition:** $\boxed{\mathbf{c} = \mathbf{a} + \mathbf{b}}$.
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- **Matrix-matrix product:** $\boxed{\mathbf{C} = \mathbf{A}\,\mathbf{B}}$.
- **Time complexity:** $\boxed{O(n^3)}$.

Questions

Vector and Matrix Norms

- Given $n \times 1$ vector $\mathbf{a}$ and $m \times n$ matrix $\mathbf{B}$.

Question: What are the costs of computing the following norms?

- Vector ℓ_1-norm: $||\mathbf{a}||_1 = |a_1| + |a_2| + \cdots + |a_n|$.
- Vector ℓ_2-norm: $||\mathbf{a}||_2 = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$.
- Matrix Frobenius norm: $||\mathbf{B}||_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij}^2}$.

Thank You!