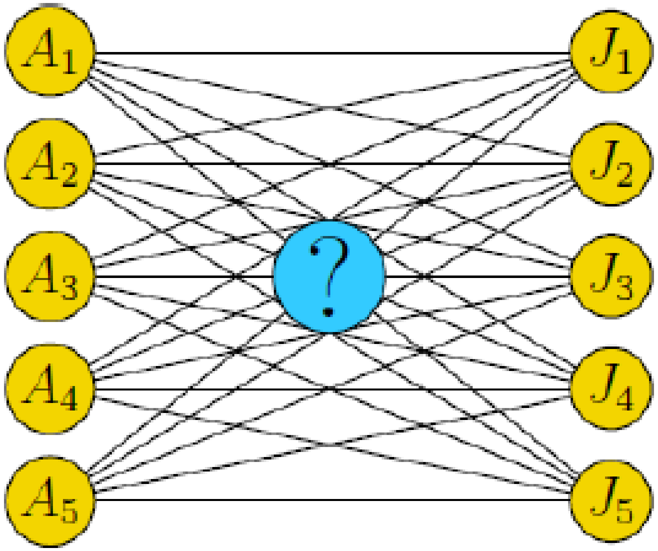
EE8212 Optimization Techniques for Engineers

Assignment Problem



Introduction

The assignment problem is a special type problem, where the objective is to minimize the cost or time of completing a number of jobs by a number of persons. In other words, when the problem involves the allocation of different facilities to different tasks, it is often termed as an assignment problem.

The model's primary usefulness is for planning. The assignment model is useful in solving problems such as, assignment of persons/machines to jobs, assignment of salesmen to sales regions etc.

Formulation of an Assignment Problem

Suppose a company has n persons of different capacities available for performing each different job in the concern, and there are the same number of jobs of different types. One person can be given one and only one job. The objective of this assignment problem is to assign n persons to n jobs, so as to minimize the total assignment cost. The cost matrix for this problem is given below.

	Job				
Worker	j_1	j_2	----	j_n	a_i
i_1	c_{11} x_{11}	c_{12} x_{12}	----	c_{1n} x_{n1}	1
i_2	c_{21} x_{21}	c_{22} x_{22}	----	c_{2n} x_{2n}	1
----	----	----	----	----	--
i_n	c_{n1} x_{n1}	c_{n2} x_{n2}	----	c_{nn} x_{nn}	1
b_j	1	1	----	1	

To formulate the assignment problem in mathematical programming terms, we define the activity variables as,

x_{ij} = 1 if job j is performed by the person i , 0 otherwise

for i= 1,2,3,……….,n and j= 1,2,………..,n

In the above table, C_{ij} is the cost of performing j^{th} job by the i^{th} person.

The optimization model

Minimize $c_{11}x_{11} + c_{12}x_{12} + ------- + c_{nn}x_{nn}$

subject to

$x_{i1} + x_{i2} + \ldots\ldots\ldots + x_{in} = 1$ $\qquad$ $i = 1, 2, \ldots\ldots, n$

$x_{1j} + x_{2j} + \ldots\ldots\ldots + x_{nj} = 1$ $\qquad$ $j = 1, 2, \ldots\ldots, n$

$x_{ij} = 0$ or 1

Characteristics

- Number of jobs is equal to the number of persons.
- Each person is assigned with only one job.
- Each person is independently capable of handling any job to be done.
- Assigning criteria is clearly specified (minimizing cost or maximizing profit).

Hungarian Method

It is an efficient method for solving assignment problems . This method is based on the following principle:

If a constant is added to, or subtracted from, every element of a row and/or a column of the given cost matrix of an assignment problem, the resulting assignment problem has the same optimal solution as the original problem.

Algorithm

The objective of this section is to examine a computational method - an algorithm - for deriving solutions to the assignment problems. The following steps summarize the approach:

1. Identify the minimum element in each row and subtract it from every element of that row.
2. Identify the minimum element in each column and subtract it from every element of that column.

3. Make the assignments for the reduced matrix obtained from steps 1 and 2 in the following way:

   a) For each row or column with a single zero value cell that has not assigned or eliminated, box that zero value as an assigned cell.

   b) If for a row and a column, there are two or more zeros and one cannot be chosen by inspection, then you are at liberty to choose the cell arbitrarily for assignment.

   c) For every zero that becomes assigned, cross out (X) all other zeros in the same row and the same column.

   d) The above process may be continued until every zero cell is either assigned or crossed (X).

4. An optimal assignment is found, if the number of assigned cells equals the number of rows (and columns). In case you have chosen a zero cell arbitrarily, there may be alternate optimal solutions. If no optimal solution is found, go to step 5.

5. Draw the minimum number of vertical and horizontal lines necessary to cover all the zeros in the reduced matrix obtained from step 3 by adopting the following procedure:

i. Mark all the rows that do not have assignments.
ii. Mark all the columns (not already marked) which have zeros in the marked rows.
iii. Mark all the rows (not already marked) that have assignments in marked columns.
iv. Repeat steps 5 (i) to (iii) until no more rows or columns can be marked.
v. Draw straight lines through all unmarked rows and marked columns.

6. Select the smallest element from all the uncovered elements. Subtract this smallest element from all the uncovered elements and add it to the elements, which lie at the intersection of two lines. Thus, we obtain another reduced matrix for fresh assignment.
7. Go to step 3 and repeat the procedure until you arrive at an optimal assignment.

Ex 1: Galle Software company has four developers available for four new projects. Only one developer can work on a project. Estimated number of days each developer takes to complete individual projects are given in the following table. How can these projects assigned to the developers in order to minimise the total time take?

Jobs

	A	B	C	D
Raja	20	25	22	28
Nimal	15	18	23	17
Hamza	19	17	21	24
Nalini	25	23	24	24

Step 1

	A	B	C	D
Raja	0	5	2	8
Nimal	0	3	8	2
Hamza	2	0	4	7
Nalini	2	0	1	1

1. Identify the minimum element in each row and subtract it from every element of that row.

	A	B	C	D
Raja	20	25	22	28
Nimal	15	18	23	17
Hamza	19	17	21	24
Nalini	25	23	24	24

Step 2

	A	B	C	D
Raja	0	5	1	7
Nimal	0	3	7	1
Hamza	2	0	3	6
Nalini	2	0	0	0

2. Identify the minimum element in each column and subtract it from every element of that column.

	A	B	C	D
Raja	0	5	1	7
Nimal	0	3	7	1
Hamza	2	0	3	6
Nalini	2	0	0	0

Step 3

	A	B	C	D
Raja	0	5	1	7
Nimal	$\cancel{0}$	3	7	1
Hamza	2	0	3	6
Nalini	2	$\cancel{0}$	0	$\cancel{0}$

3. Make the assignments for the reduced matrix obtained from steps 1 and 2 in the following way:

   a) For each row or column with a single zero value cell that has not assigned or eliminated, box that zero value as an assigned cell.

   b) If for a row and a column, there are two or more zeros and one cannot be chosen by inspection, then you are at liberty to choose the cell arbitrarily for assignment.

   c) For every zero that becomes assigned, cross out (X) all other zeros in the same row and the same column.

   d) The above process may be continued until every zero cell is either assigned or crossed (X).

Step 4

An Optimal solution is not found.

the number of assigned cells are not equal to the number of rows (and columns).

	A	B	C	D
Raja	0	5	1	7
Nimal	0	3	7	1
Hamza	2	0	3	6
Nalini	2	0	0	0

Step 5

	A	B	C	D
Raja	0	5	1	7
Nimal	0	3	7	1
Hamza	2	0	3	6
Nalini	2	0	0	0

5. Draw the minimum number of vertical and horizontal lines necessary to cover all the zeros in the reduced matrix obtained from step 3 by adopting the following procedure:

i. Mark all the rows that do not have assignments.

ii. Mark all the columns (not already marked) which have zeros in the marked rows.

iii. Mark all the rows (not already marked) that have assignments in marked columns.

iv. Repeat steps 5 (i) to (iii) until no more rows or columns can be marked.

v. Draw straight lines through all unmarked rows and marked columns.

	A	B	C	D
Raja	0	5	1	7
Nimal	0	3	7	1
Hamza	2	0	3	6
Nalini	2	0	0	0

6. Select the smallest element from all the uncovered elements. Subtract this smallest element from all the uncovered elements and add it to the elements, which lie at the intersection of two lines. Thus, we obtain another reduced matrix for fresh assignment.

Step 6

	A	B	C	D
Raja	0	4	0	6
Nimal	0	2	6	0
Hamza	3	0	3	6
Nalini	3	0	0	0

	A	B	C	D
Raja	0	4	0	6
Nimal	0	2	6	0
Hamza	3	0	3	6
Nalini	3	0	0	0

Go to Step 3 again for cell assignments

	A	B	C	D
Raja	0	4	0	6
Nimal	0	2	6	0
Hamza	3	0	3	6
Nalini	3	0	0	0

Optimum Solution is Found

Raja ---- Project A ---- 20
Nimal ---- Project D ---- 17
Hamza ---- Project B ---- 17
Nalini ---- Project C ---- 24

Total time = 78

Ex 2: Gamage publishing company hires typists on an hourly basis. There are five typists and their charges and speed are given in the table below. One job is given only to one typist and the typist is paid for the full hour even if he/she works for a fraction of an hour. Find the least cost allocation for the following data.

Typist	**Rate per hr**	**Pages per hr**
Amila	5	12
Bindu	6	14
Chanaka	3	8
Dinesh	4	10
Eranga	4	11

Job	**No of Pages**
A	199
B	175
C	145
D	198
E	178

The following matrix gives the cost incurred if the jobs are assigned to different typists.

Typist	A	B	C	D	E
Amila	85	75	65	85	75
Bindu	90	78	66	90	78
Chanaka	75	66	57	75	69
Dinesh	80	72	60	80	72
Eranga	76	64	56	72	68

Identify minimum element in each row and subtract it from every element in that row.

Typist	A	B	C	D	E
Amila	20	10	0	20	10
Bindu	24	12	0	24	12
Chanaka	18	9	0	18	12
Dinesh	20	12	0	20	12
Eranga	20	8	0	16	12

Identify minimum element in each column and subtract it from every element in that column.

Typist	A	B	C	D	E
Amila	2	2	0	4	0
Bindu	6	4	0	8	2
Chanaka	0	1	0	2	2
Dinesh	2	4	0	4	2
Eranga	2	0	0	0	2

Make the assignments for the reduced matrix. (Yellow colour- Assignments)

Typist	A	B	C	D	E
Amila	2	2	0	4	0
Bindu	6	4	0	8	2
Chanaka	0	1	0	2	2
Dinesh	2	4	0	4	2
Eranga	2	0	0	0	2

The number of assigned cells is not equal to the number of rows/columns. Hence we draw the minimum number of vertical and horizontal lines necessary to cover all the zeros in the reduced matrix. Red lines (cells) indicates vertical and horizontal lines.

Typist	A	B	C	D	E
Amila	2	2	0	4	0
Bindu	6	4	0	8	2
Chanaka	0	1	0	2	2
Dinesh	2	4	0	4	2
Eranga	~~2~~	~~0~~	~~0~~	0	~~2~~

Repeating the usual process as explained in the previous example, we get the following matrix.

Typist	A	B	C	D	E
Amila	2	2	2	4	0
Bindu	4	2	0	6	0
Chanaka	0	1	2	2	2
Dinesh	0	2	0	2	0
Eranga	2	0	2	0	2

Make the assignments again for the reduced matrix. (Yellow colour- Assignments)

Typist	A	B	C	D	E
Amila	2	2	2	4	0
Bindu	4	2	0	6	0
Chanaka	0	1	2	2	2
Dinesh	0	2	0	2	0
Eranga	2	0	2	0	2

The number of assigned cells is not equal to the number of rows/columns. Hence we draw the minimum number of vertical and horizontal lines necessary to cover all the zeros in the reduced matrix. Red lines (cells) indicates vertical and horizontal lines.

Typist	A	B	C	D	E
Amila	2	2	2	4	0
Bindu	4	2	0	6	0
Chanaka	0	1	2	2	2
Dinesh	0	2	0	2	0
Eranga	2	0	2	0	2

Repeating the usual process as explained in the previous example, we get the following matrix.

Typist	A	B	C	D	E
Amila	2	1	2	3	0
Bindu	4	1	0	5	0
Chanaka	0	0	2	1	2
Dinesh	0	1	0	1	0
Eranga	3	0	3	0	3

Make the assignments again for the reduced matrix. (Yellow colour- Assignments)

Typist	A	B	C	D	E
Amila	2	1	2	3	0
Bindu	4	1	0	5	0
Chanaka	0	0	2	1	2
Dinesh	0	1	0	1	0
Eranga	3	0	3	0	3

Since the number of assignments is equal to the number of rows/columns, this is the optimal solution. Substituting the values from original table, the total cost is,
80 + 66 + 66 + 72 + 75 = Rs 359

Typist	A	B	C	D	E
Amila	85	75	65	85	75
Bindu	90	78	66	90	78
Chanaka	75	66	57	75	69
Dinesh	80	72	60	80	72
Eranga	76	64	56	72	68