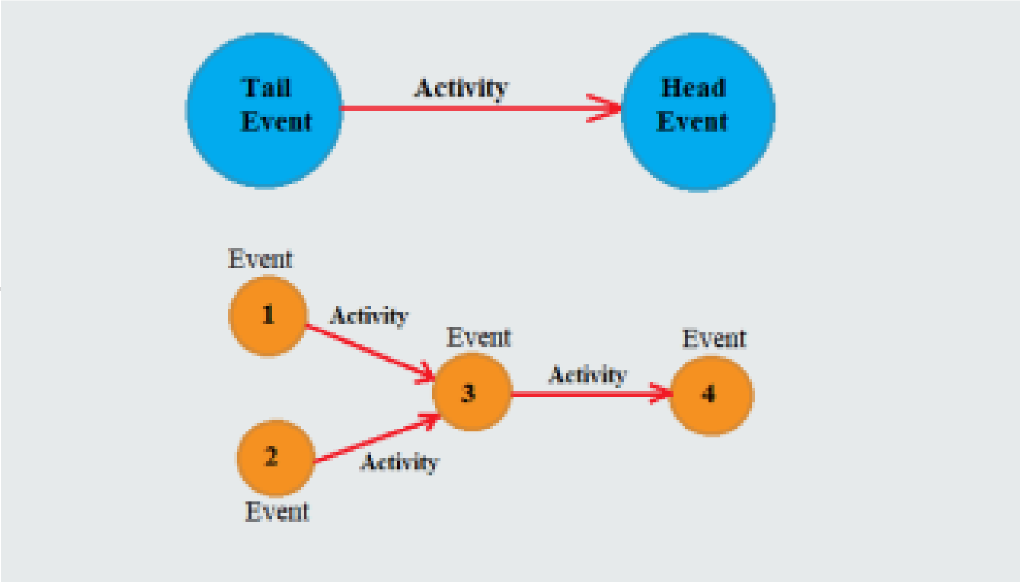
Network Techniques
Mathematical Modelling/ IS6303



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Reference

- **Schaum's Outline of Graph Theory, 1st Edition** by V. Balakrishnan

Introduction

- **Network technique** is a **technique** for planning, scheduling (programming) and controlling the progress of projects.
- This is very useful for projects which are complex in nature or where activities are subject to considerable degree of uncertainty in performance time.

Objectives of Network Techniques

- Can determine the project duration in advance.
- Can identify the inter-relationship of various activities and critical activities of project completion.
- Cost controlling.
- Time Reduction.
- Minimization of maintenance time.
- Avoid delays and interruptions.

Cont..

- Network can be used to model a variety of practical, problems arising in areas such as transportation, communications, job assignment and some graph models. It is natural to associate a real number called its weight with each edge or vertex of the graph. Depending on the model, the weight of an edge might represent the time or cost of constructing it, or its length or capacity, for example. A graph in which every edge (or vertex) is assigned a weight called a **network**.
- Networks are used in OR to solve a widely variety of problems, not only in the areas of communications and but also problems arising in the planning of production and construction projects, managing resources and planning the distribution of products and materials. In this chapter, we discuss three such problems and efficient network algorithms to solve them.

Applications

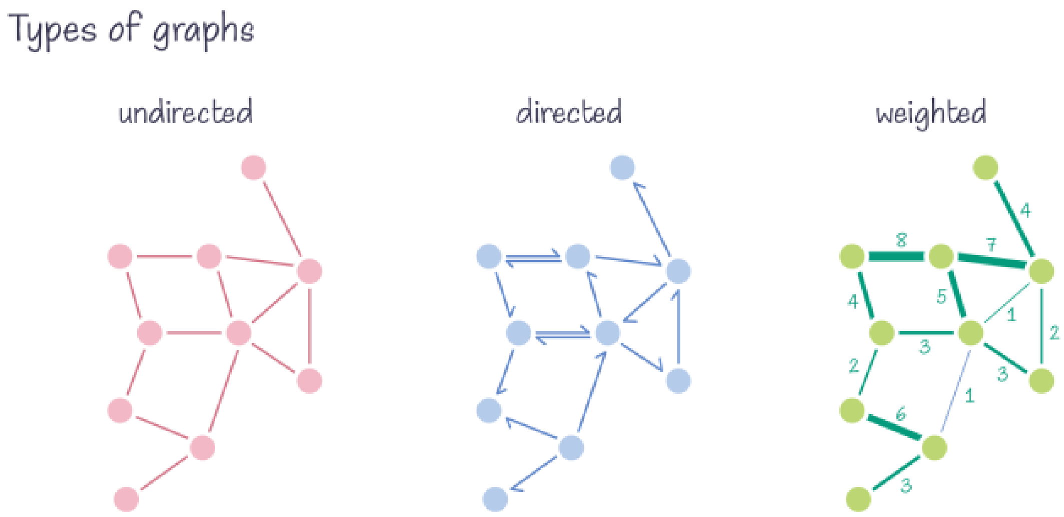
- Planning,
- Construction of buildings, bridges, highways, railways, stadiums, irrigation projects, factories, power projects etc.
- Assembly line scheduling,
- Development and launching of new products,
- Strategic and tactical military planning,
- Research and development,
- Market penetration programmes,
- Planning of political campaigns,
- Maintenance and overhauling of complicated or large machineries,
- Organizing big conferences etc.

Content

- The minimum connector problem
- Primr’s algorithm for finding a minimum weight spanning tree
- Acyclic directed networks
- Shortest Path Algorithm
- Longest Path Algorithm

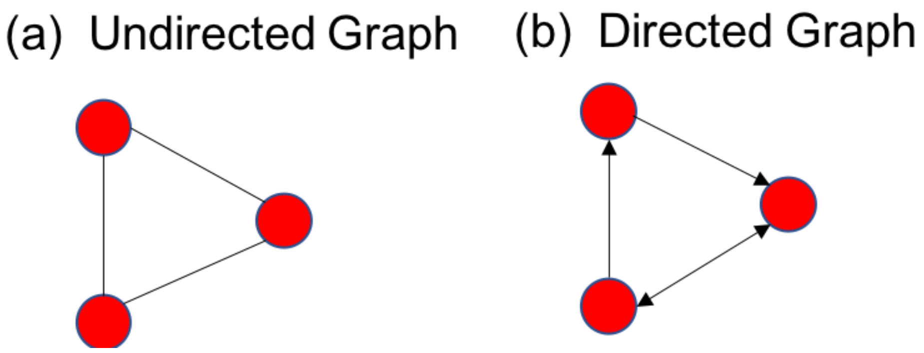
What is Graph Theory?

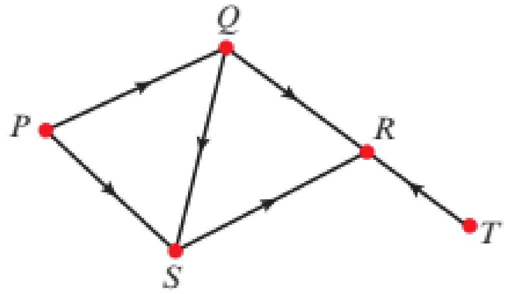
In Mathematics, networks are often referred to as graphs and the area of Mathematics concerning the study of graphs is called as **graph theory**.



Directed/ Undirected Graph

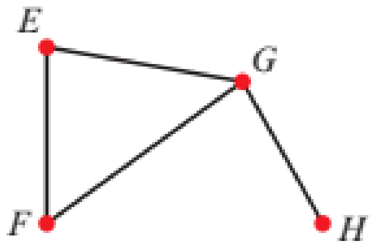
- If the edges of a graph have a direction associated with them they are known as directed edges and the graph is known as a directed graph.



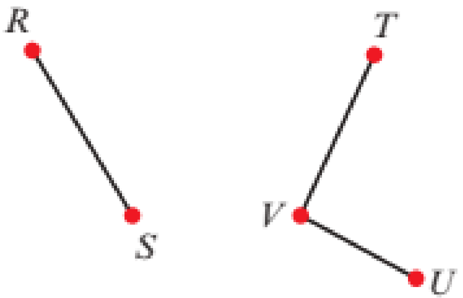


Connected Graph

- A graph is connected if all the vertices are connected.



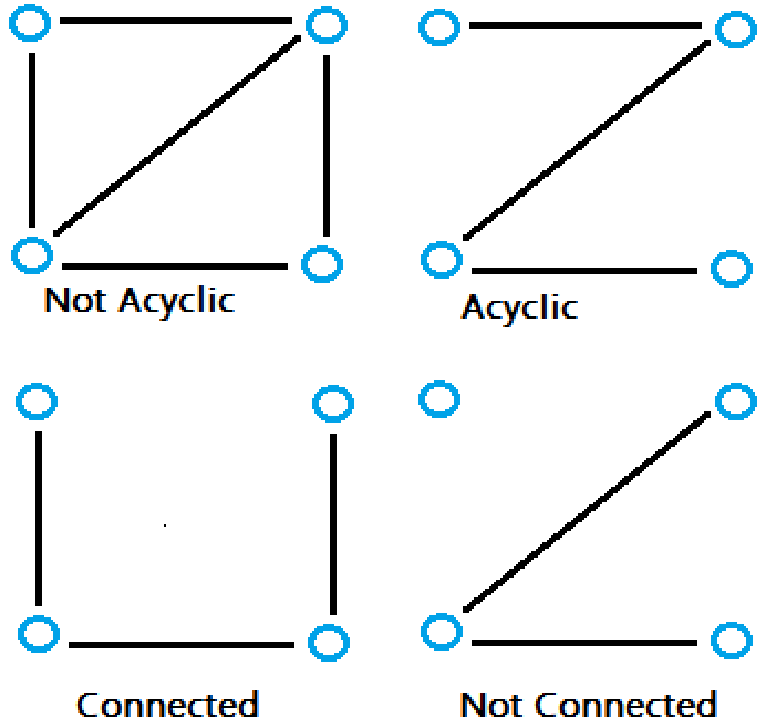
This is a **connected** graph.
A path can be found between any two vertices.



This graph is **not connected**.
There is no path from R to V, for example.

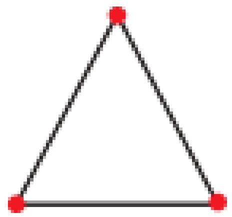
Acyclic Graph

- An acyclic graph is **a graph having no graph cycles**

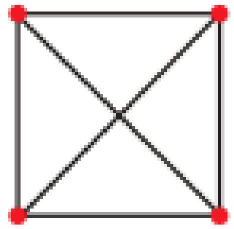


Complete graph

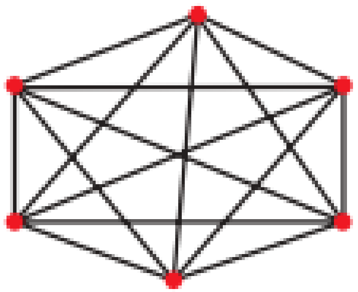
- If every vertex is directly connected by a single edge to ach of the other vertices, It is called as a complete graph.



K_3



K_4



K_6

Notation The complete graph with n vertices is written as K_n.

Subgraph of a graph

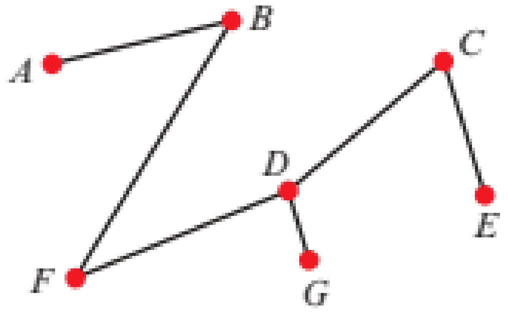
- A subgraph is a graph whose vertices and edges are **subsets of another graph**.

Example 1

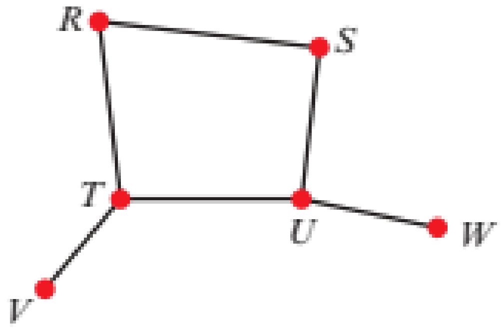
- The vertices in the network shown in Figure 1 represent eight off shore oil wells. The edges represent possible connections that can be made between the wells. The weight on each edge is proportional to the estimated cost of constructing that link. We want to design a network that will connect the wells at minimum cost.

What is a tree?

- A tree is a connected graph with no cycles.



This is a tree.

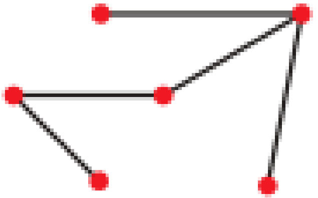


This is not a tree.
It contains cycle $RTUSR$.

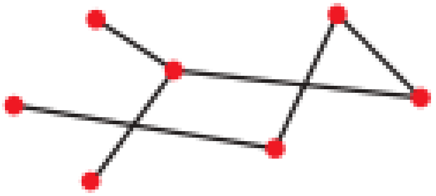
Question

1 State which of the following graphs are trees.

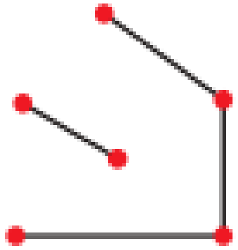
a



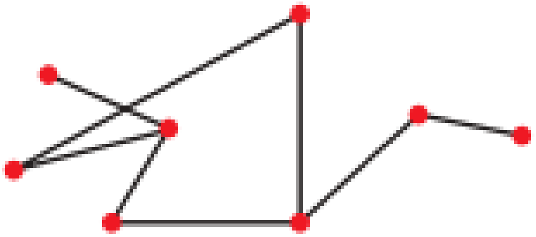
b



c



d



Spanning tree

A **spanning tree** of a graph ***G*** is a subgraph which includes all the vertices of ***G*** and is also a tree.

- **Minimum Spanning Tree**

A minimum spanning tree is a spanning tree in which the sum of the weight of the edges is as minimum as possible

Applications of Minimum Spanning Trees

- Minimum spanning trees can have direct applications in the design of networks, including computer networks,

❖ Telecommunication networks

❖ Transportation networks

❖ Water supply networks

❖ Electrical grids

❖ Cluster analysis

❖ Civil network planning

General Properties of Spanning Tree

- There may be more than one spanning tree to one graph. Following are few properties of a spanning tree.

❖ A connected graph G can have more than one spanning tree.

❖ All possible spanning trees of graph G, have the same number of edges and vertices.

❖ The spanning tree does not have any cycle (loops).

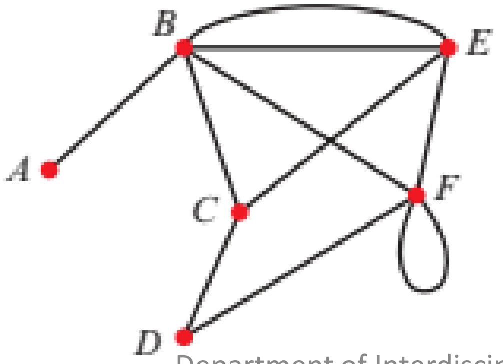
❖ The spanning tree is minimally connected. (Removing one edge from the spanning tree will make the graph disconnected)

❖ Adding one edge to the spanning tree will create a circuit or loop, i.e. the spanning tree is maximally acyclic.

Adjacency matrix

- You can use an **adjacency matrix** to represent a graph or network. The adjacency matrix provides information about the connections between the vertices in a graph.
- Each entry in an adjacency matrix describes the number of arcs joining the corresponding vertices.

Use an adjacency matrix to represent this graph.



Cont…

	A	B	C	D	E	F
A	0	1	0	0	0	0
B	1	0	1	0	2	1
C	0	1	0	1	1	0
D	0	0	1	0	0	1
E	0	2	1	0	0	1
F	0	1	0	1	1	2

Watch out You should be able to write down the adjacency matrix given a graph, and draw a graph given the adjacency matrix.

This indicates that there are 2 direct connections between B and E.

This indicates a loop from F to F. It could be travelled in either direction, and hence counts as 2.

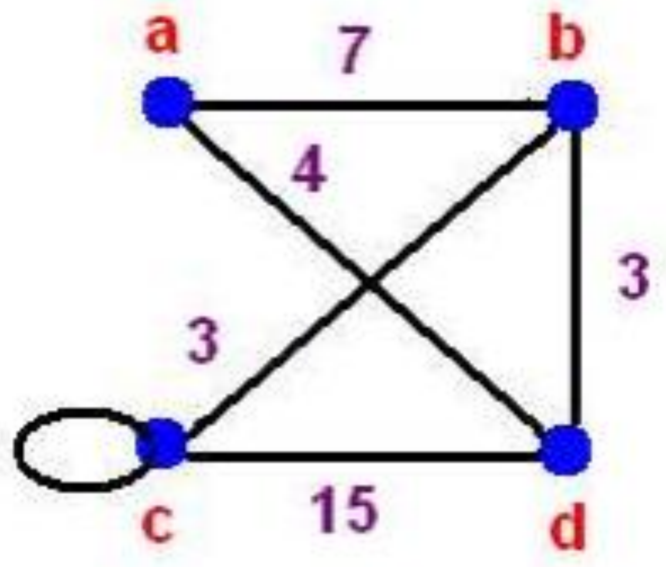
Distance matrix

- The **matrix** associated with a weighted graph is called a **distance matrix**.
  ■ In a distance matrix, the entries represent the weight of each arc, not the number of arcs.

Degree of a vertex

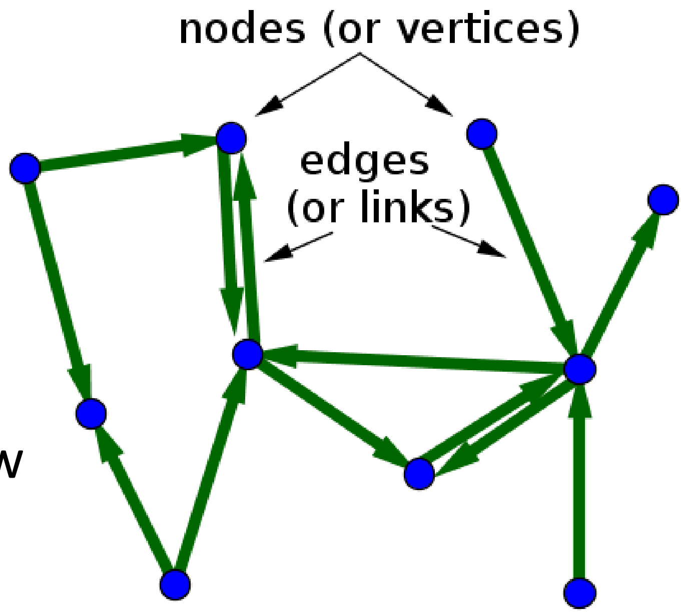
- Degree of a vertex in an un directed graph is the number of edges associated with it.
- Degree of vertex a is = 2
- Degree of vertex c is = 4

(if a vertex has a loop it contributes twice)



What is a network/ arrow diagram?

- a network is the graphical representation of logically and sequentially connected **arrows** and nodes representing activities and events of a project.
- In graphical representation finish of **an activi**ty or group of activities is shown by a **nodes/vertices** (refer them as events/ objects) and draw as points. Each and every activity has a point of time where it begins and a point where it ends.
- The connections between the nodes as edges and usually draw them as lines between points. (In a directed graph the activities are represented by the arrows)
- A network can represent all sort of systems in the real world.

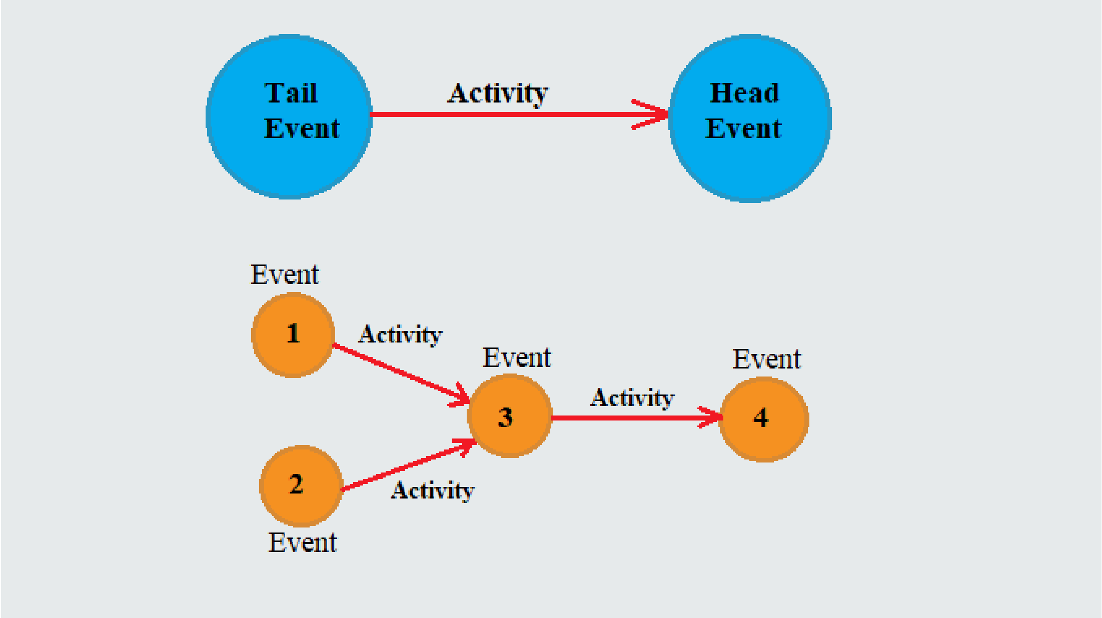


Cont…

Activity is represented in the network by an arrow as follows.

Activity – description/ duration

Events are represented by circular Node. There are several type of Events. Tail event, head event, merge event, burst event and dual event



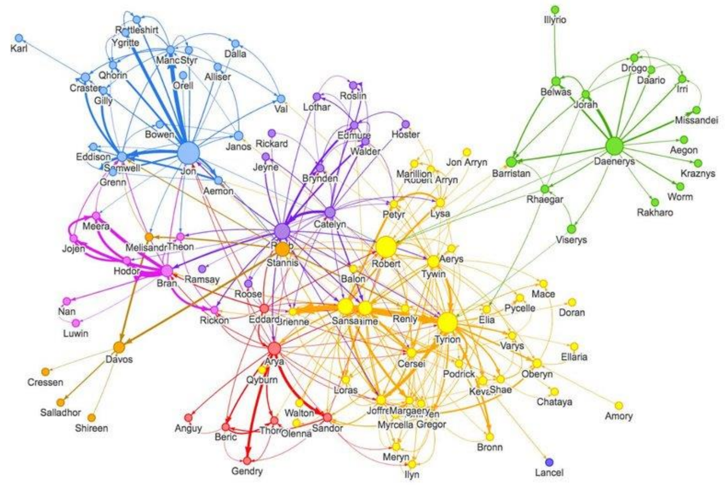
What is the difference between Graphs and Networks?

- **There's no difference**. They're just different words for the same thing, though “graph” tends to be more common in math and other formal areas, and “network” more common in more applied areas.

Cont..
Examples:-

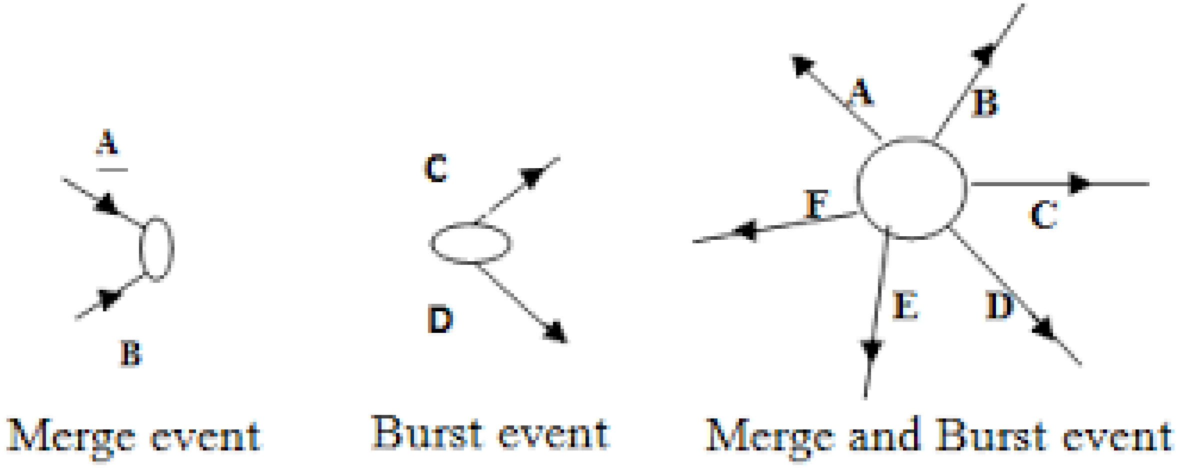
Internet (Computers and other devices are the nodes/ connection between these devices are as edges).

Word wide web (pages are the nodes and links are edges).



Merge and Burst events

- It is not necessary for an event to be the ending event of only one activity but can be the ending event of two or more activities. Such event is defined as a **Merge event**.
- If the event happens to be the beginning event of two or more activities it is defined as a **Burst event**.

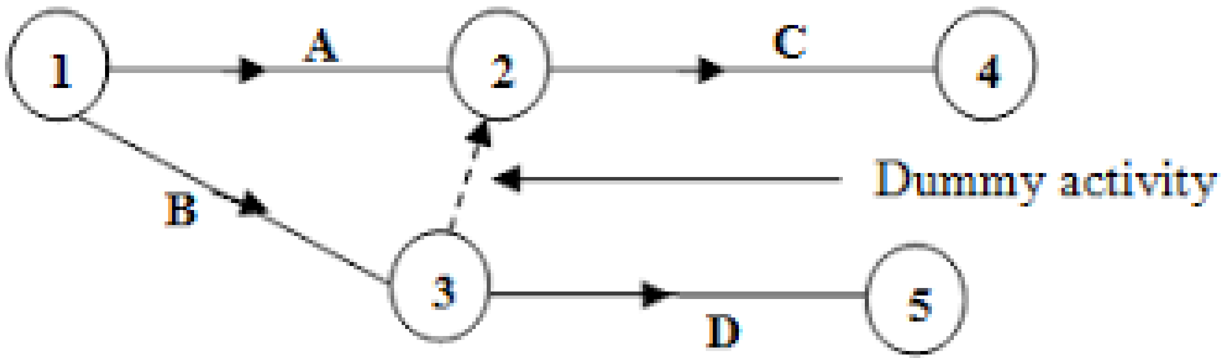


Dummy activity

- Certain activities which neither consumes time nor resources but are used simply to represent a connection or a link between the events are known as dummies. It is shown in the network by a dotted line. The purpose of introducing dummy activity is,

i. To maintain uniqueness in the numbering system as every activity may have distinct set of events by which the activity can be identified.

ii. To maintain a proper logic in the network.

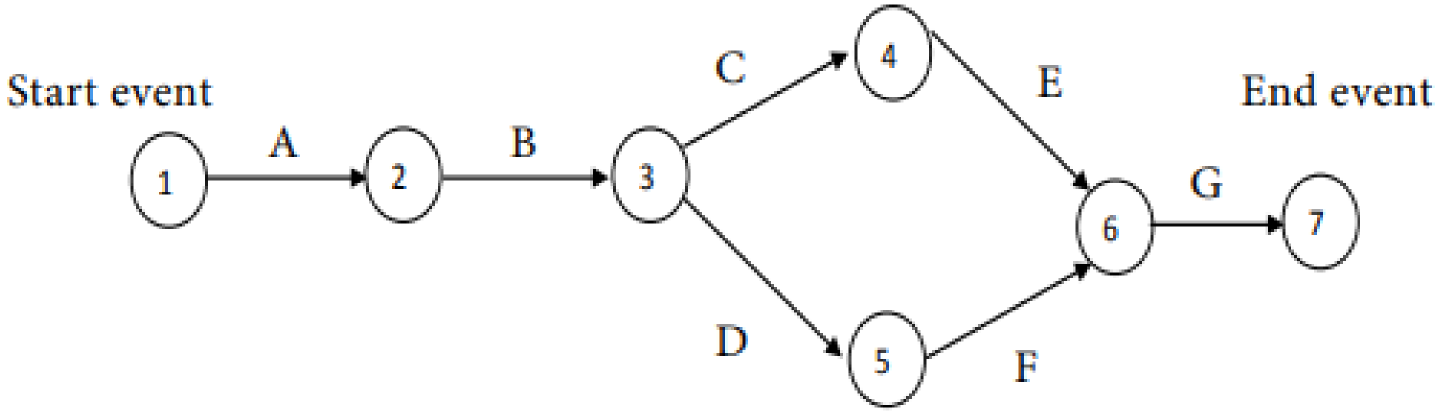

1
A
2
C
4
B
Dummy activity
3
D
5

Construction of Network

Construct a network for the project whose activities and their precedence relationships are as given in the following Table.

Example 1

Activity	Immediate Predecessor Activity
A	-
B	A
C, D	B
E	C
F	D
G	E, F



Activity	Immediate Predecessor Activity
A	-
B	-
C	A
D	B
E	A
F	C, D
G	E
H	E
I	F, G
J	H, I

Network Techniques

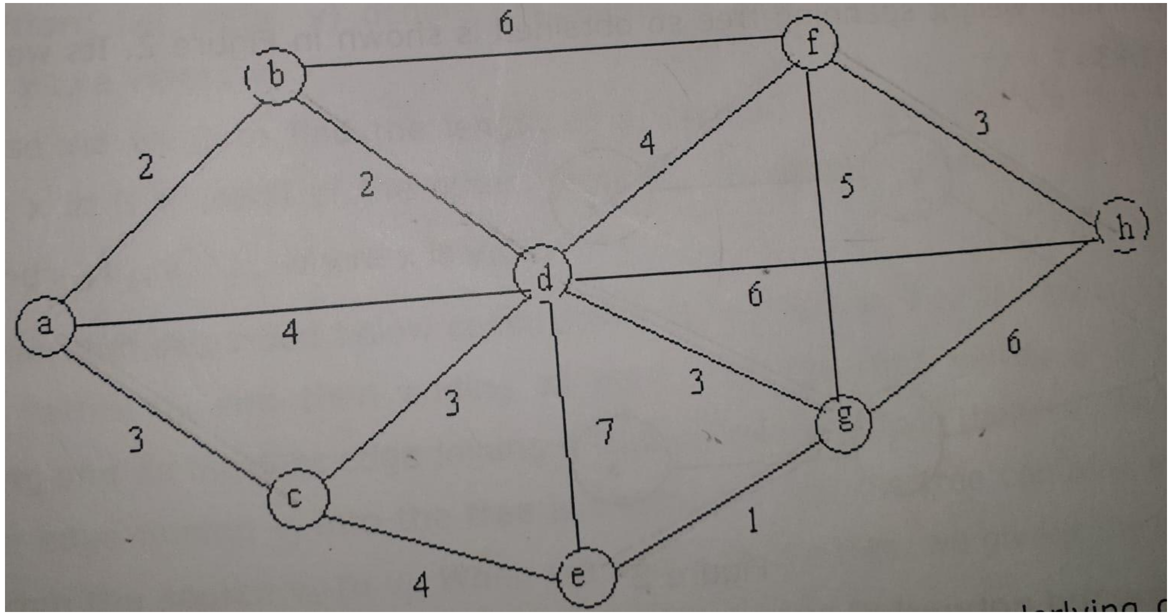
- The technique is a method of minimizing the trouble spots such as production, delays and interruptions, by determining critical factors and coordinating various parts of the overall job. There are two basic planning and control technique that utilize a network to complete a predetermined project or schedule. These are programme evaluation review techniques (PERT) and critical path method (CPM). A project is defined as a combination of interrelated activities all of which must be executed in a certain order for its completion.

The minimum connector problem

- A common practical problem that arises, particularly in the design of communications and transportation networks, is that of connecting a number of sites at minimum cost. The examples in the slides illustrates this kind of problems.

Cont…

- The vertices of the following network shows the eight off shore oil wells. The edges represents the possible connections that can be mad between the wells. The weight on each edge is proportional to the estimated cost of constructing that link. Design a minimum connector network that connect the wells at minimum cost.



Cont…

- We first consider the properties that a subgraph of the underlying graph of this network must possess in order to link the wells at minimum cost. Clearly, the subgraph must be connected and contain no unnecessary edges. Hence it must contain no cycles, and so be a tree. Further, the required subgraph must contain all the vertices of the network and hence it must be a spanning tree of the underlying graph.

What is the method to find minimum spanning tree?

- There are two algorithms.

1. Prim’s Algorithm
2. Kruskal’s Algorithm

Prim’s algorithm for finding a minimum weight spanning tree

- Given a connected network N. we build a subgraph T of N as follows.

1. (Initialization step). Select arbitrarily any vertex of N and put this vertex into T.
2. (Recursive step). Scan the edges joining a vertex of T to a vertex not in T, and select one such edge with minimum weight. Add this edge and its other end vertex to T.
3. (Stop condition). Stop when all the vertices of N are in T. The output T is a minimum weight spanning weight spanning tree of N.

(If you have large number of vertices, we can use matrices)

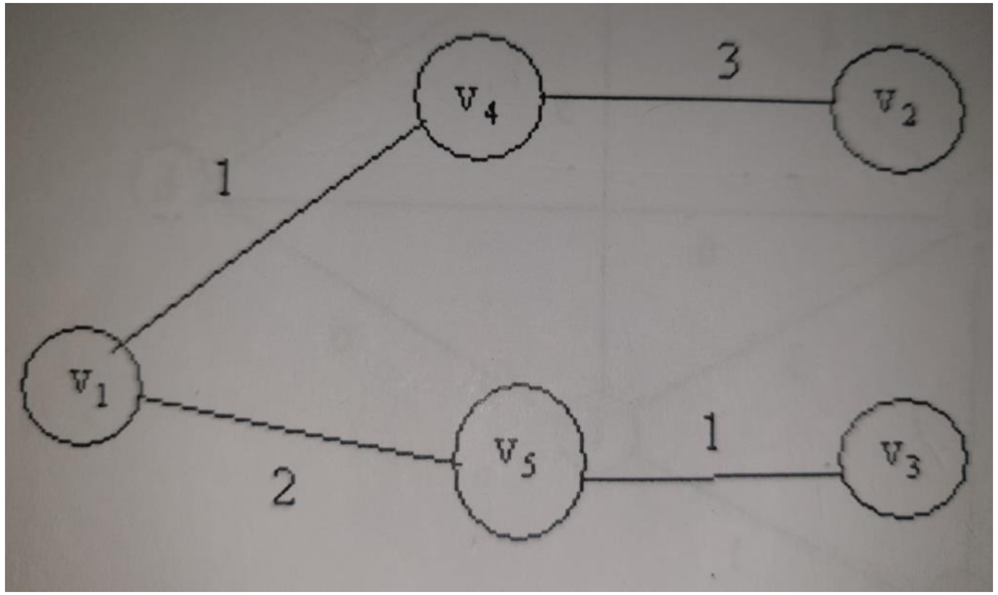
Example 3/ Applying Prim's Algorithm for a distance matrix

- He table below describes a network N with five vertices v1, v2, v3, v4, v5. We find a minimum weight spanning tree in N by applying Prim's algorithm directly to its table of weights. Fist complete the graph.

	v1	v2	v3	v4	v5	
V1		5	3	1	2	
V2	5		4	3	7	
V3	3	4		5	1	
V4	1	3	5		6	
v5	2	7	1	6		

The minimum weight spanning tree so obtained is shown in Figure 2. Its weight is 1+2+1+3=7

Answer



Question

- Apply Prim's Algorithm to find the minimum spanning tree.

	A	B	C	D	E
A	–	27	12	23	74
B	27	–	47	15	71
C	12	47	–	28	87
D	23	15	28	–	75
E	74	71	87	75	–

	1				
	A	B	C	D	E
~~A~~	~~–~~	~~27~~	~~12~~	~~23~~	~~74~~
B	27	–	47	15	71
C	12	47	–	28	87
D	23	15	28	–	75
E	74	71	87	75	–

Delete row A, and number column A.

	1				
	A	B	C	D	E
~~A~~	~~–~~	~~27~~	~~12~~	~~23~~	~~74~~
B	27	–	47	15	71
C	12	47	–	28	87
D	23	15	28	–	75
E	74	71	87	75	–

The first arc is AC.

The lowest undeleted entry in column A is 12, so circle it. The first arc is AC.

	1		2		
	A	B	C	D	E
~~A~~	–	27	12	23	74
B	27	–	47	15	71
~~C~~	12	47	–	28	87
D	23	15	28	–	75
E	74	71	87	75	–

The second arc is AD.

The new vertex is C. Delete row C, and number column C.

The lowest undeleted entry in columns A and C is 23, circle it. The second arc is AD.

	1		2	3	
	A	B	C	D	E
~~A~~	–	27	12	23	74
B	27	–	47	15	71
~~C~~	12	47	–	28	87
~~D~~	23	15	28	–	75
E	74	71	87	75	–

The third arc is DB.

The new vertex is D. Delete row D, and number column D.

The lowest undeleted entry in columns A, C and D is 15. Circle it. The third arc is DB.

	1	4	2	3	
	A	B	C	D	E
~~A~~	–	27	12	23	74
~~B~~	27	–	47	15	71
~~C~~	12	47	–	28	87
~~D~~	23	15	28	–	75
E	74	71	87	75	–

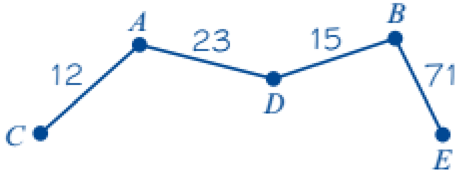
The fourth arc is BE.

The new vertex is B. Delete row B, and number column B.

The lowest undeleted entry in columns A, C, D and B is 71. Circle it. The fourth arc is BE.

	1	4	2	3	5
	A	B	C	D	E
~~A~~	–	27	12	23	74
~~B~~	27	–	47	15	71
~~C~~	12	47	–	28	87
~~D~~	23	15	28	–	75
~~E~~	74	71	87	75	–

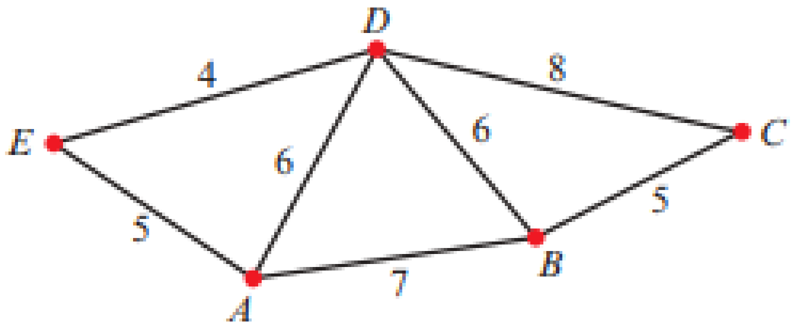
The minimum spanning tree is:



Its weight is 12 + 23 + 15 + 71 = 121

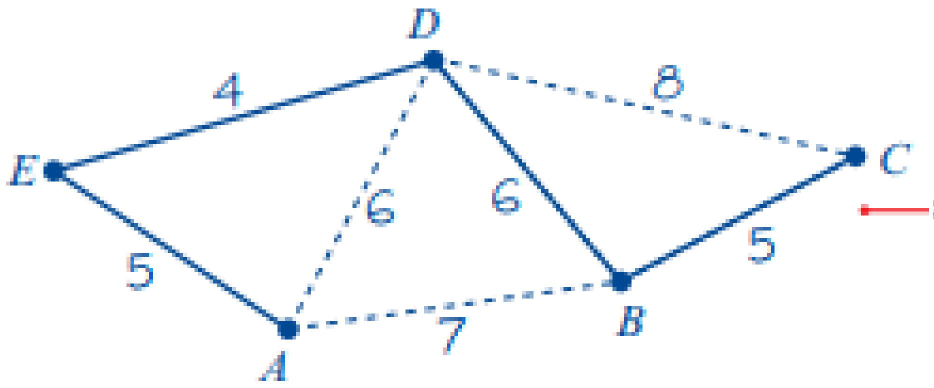
The arcs added were AC, AD, BD, BE.

Kruskal's Algorithm

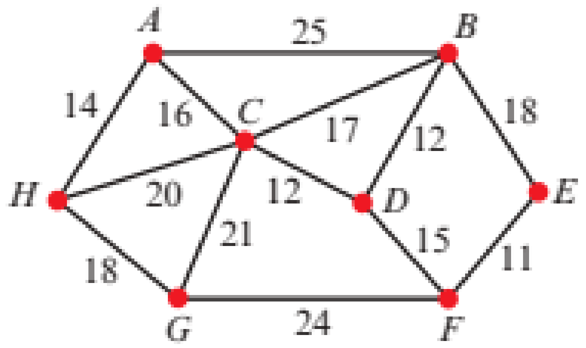


- 1st step – order the arcs with their weights in an ascending order.
- 2nd step – connect the nodes starting from the minimum weight to maximum until all the nodes are connected. **(avoid cycles**)
- Above network has 5 vertices. Minimum spanning tree is (AEDBC/ weight is 20)
- The order of the arcs DE(4), AE(5), BC(5), AD(6), BD(6), AB(7) and CD(8).

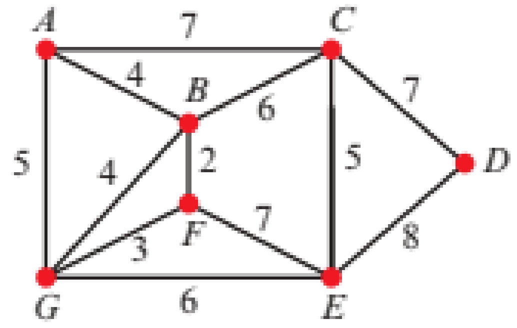
Answer


D
4
8
E
C
6
6
5
5
7
B
A

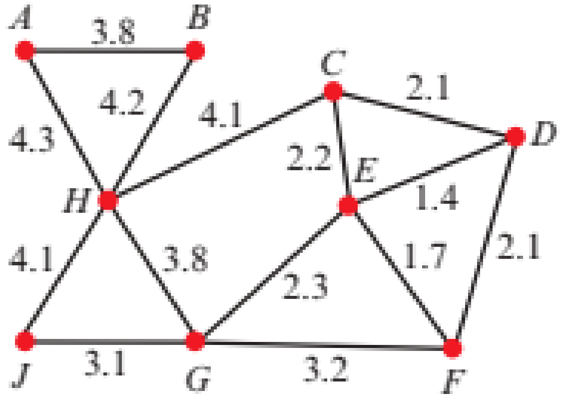
a


A
25
B
14
16
C
17
12
18
H
20
12
D
E
21
15
11
18
G
24
F

b


A
7
C
4
B
6
7
5
4
2
5
D
F
7
3
8
G
6
E

c


A
3.8
B
C
4.2
4.1
2.1
D
4.3
2.2
E
H
1.4
4.1
3.8
1.7
2.1
2.3
J
3.1
G
3.2
F

Is minimum spanning tree unique?

- No
- The minimum spanning tree is unique if all edge weights in a connected group are distinct.

More methods to find shortest path

- Nearest neighbor algorithm
- Dijkstra’s algorithm (pronounce as **dike-Stra**)

Acyclic directed network

- Definition 01

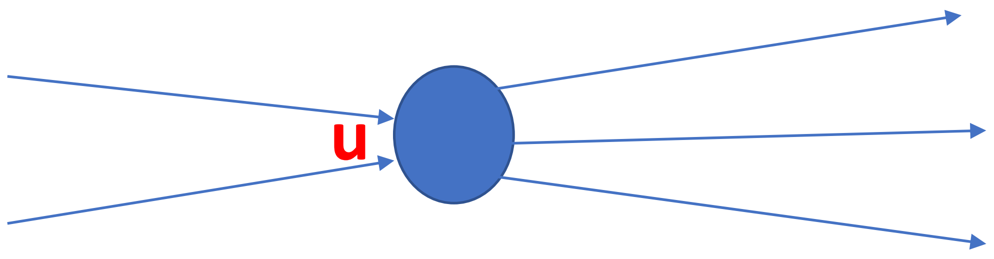
The number of arcs directed out of a vertex u is called the **out degree of u** and written deg +u, similarly, the Number of arcs directed into a vertex u is called the **in degree** of u and written deg –u. A vertex in a directed network that has in degree 0 is called a source. Thus every arc incident with a source v is directed out of v.

Walks, paths and cycles in graph have natural counterparts in diagraphs and directed networks, with the important extra condition that the direction of the arcs must be followed. A directed network containing no directed cycle is called acyclic. The network shown in Figure 3 is acyclic.

Cont…

Consider the vertex u

- out degree of u =3 (deg +u)
- In degree of u = 2 (deg -u)

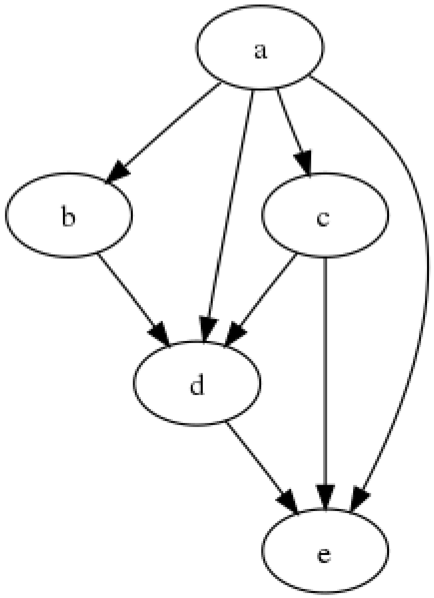


Source

If in degree of a vertex is zero it is a source.

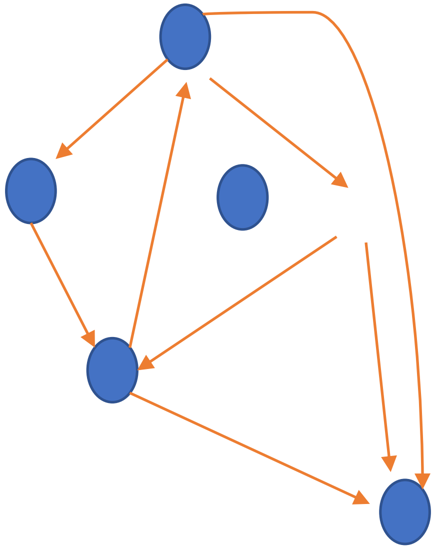
Directed Acyclic Graph

A directed network containing no directed cycle is called acyclic.



There are no any directed cycles in this network.

If there is a directed cycle it is not a acyclic network.



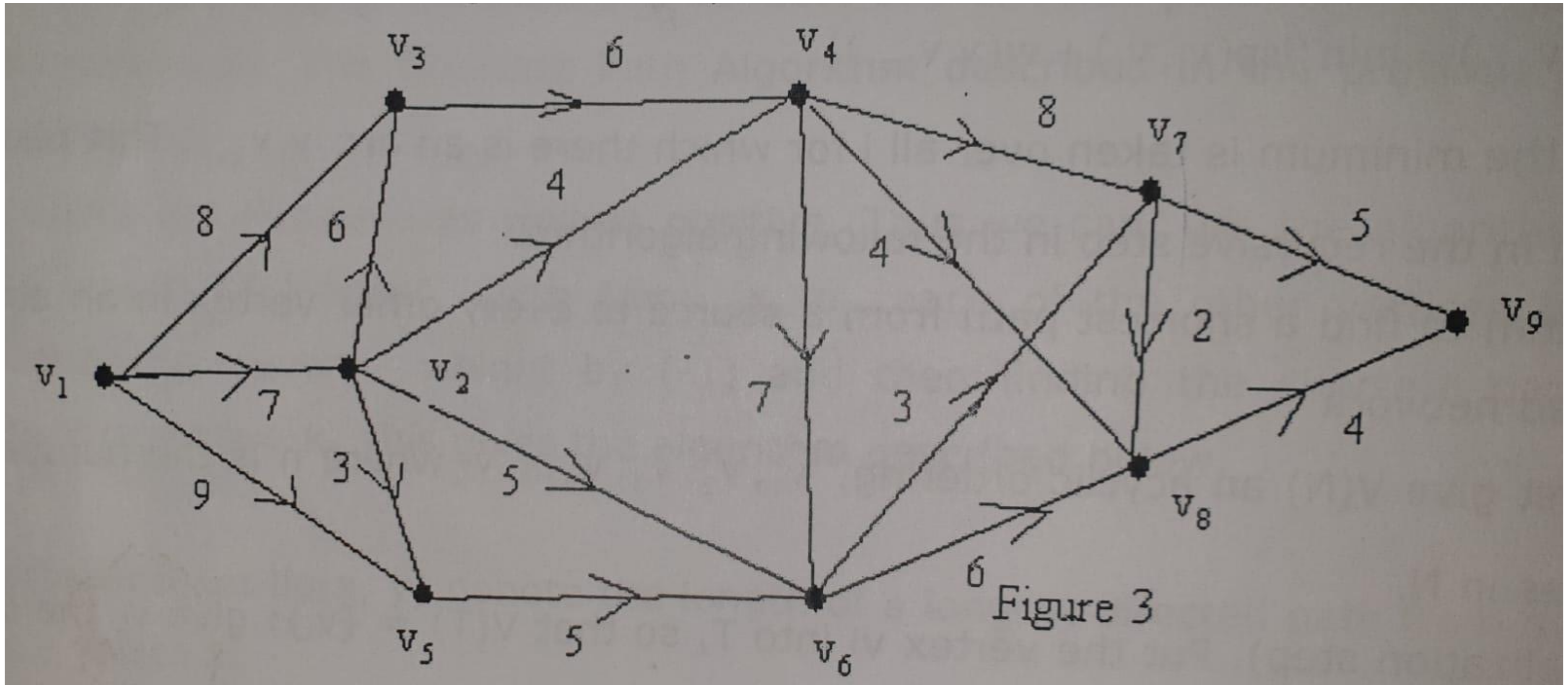


Figure 3

Shortest Path Algorithm

- An important problem in operational research is to find the shortest path in a network from a given point to any other point in the network. There is an efficient algorithm to solve this problem for all networks with non-negative weights. The below algorithm can be used to find the shortest path when the network is directed and acyclic, and the weights are any real numbers.

We denote the length of a <u>shortest directed path</u> from vertex x to a vertex y by $lsp(x, y)$.

The problem of finding the shortest path in a network is called a **shortest path problem**.

- Start from a vertex x.
- Start giving weights to each vertex.
- Add next vertex y by selecting the shortest path between x and y.

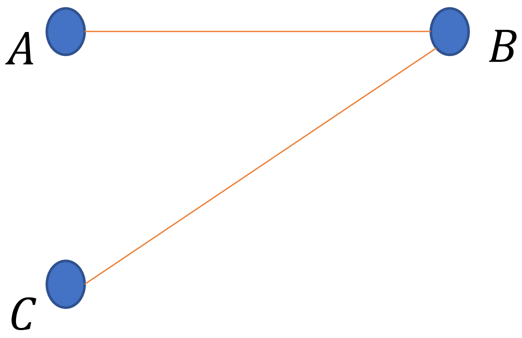
Note:

✓If A and B are adjacent if they are connected.

✓B is an incident since it is the end point.

✓Vertices join to incident B are neighborhood of B.

✓A and C are neighborhood.



Cont…

- Next step is to identify the direct connections from vertex y to other vertices.
- Add the new vertex z to the network by identifying the shortest distance of each vertex which are directly connected with previous vertices x and y.
- From the v_{k+1} to the tree and calculate $lsp(v_1, v_{k+1})$.
- The vertex we consider

Additional References

THANK YOU